



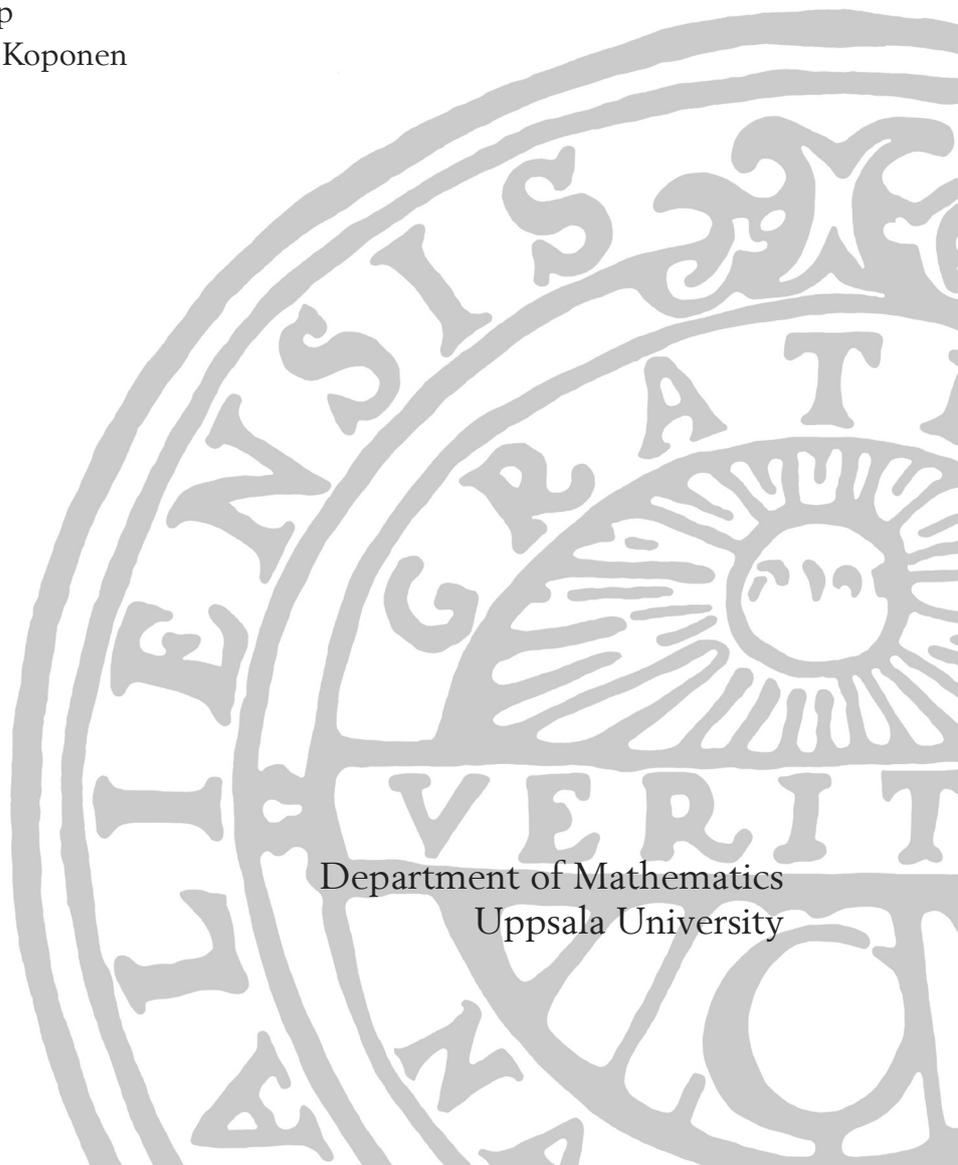
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Limit Laws for First Order Logic on Random Images

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A large, faint watermark of the Uppsala University seal is visible in the bottom right corner of the page. The seal features a sun with rays, a crown, and the Latin motto "ALMA MATER VERITAS".

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1 Introduction

The concept of *formal languages* is central to mathematical logic. A formal language consists of symbols which can be put together to form finite strings. There are rules which say which strings are permissible and which are not. Those which are, one usually calls formulas. This is what one calls the *syntax* of the language. The formulas may be interpreted in different ways, so that they carry a meaning. This is the *semantics* of the language. *Model theory* is the subject of the relationship between the syntax and semantics of a formal language. The most usual example of a formal language is the *first order logic*, and this will be our sole interest in this thesis. First order means that one is allowed to quantify over elements but not over subsets or relations. In a *structure* the symbols, which may be constants, relation symbols and function symbols (together usually called the vocabulary), are given a meaning on some universe of elements. We refer to the first few chapters of [10] for a more thorough clarification of the concepts just mentioned.

Finite model theory is the study of finite structures. For the purpose of this thesis we will study vocabularies with only relation symbols (no constants or function symbols). In finite model theory, a subject that has received much attention is the occurrence of limit laws, and in particular zero-one laws. One studies the probability for a finite structure to satisfy a given sentence or class of sentences. Of main interest then is the asymptotic properties of these structures, that is what happens to this probability when the size of the structures tends to infinity. If the probability converges one says that there is a limit law, and if the limit is always zero or one, one says that there is a zero-one law. For a strict definition, let \mathbf{A}_n , for each n , be a class of structures in a given vocabulary τ . We let the universe of each structure be $A := \{1, 2, \dots, n\}$. Let φ be any first order sentence in the vocabulary. The crucial question then is whether the limit

$$\lim_{n \rightarrow \infty} \frac{|\{\mathcal{A} : \mathcal{A} \models \varphi, \mathcal{A} \in \mathbf{A}_n\}|}{|\{\mathcal{A} : \mathcal{A} \in \mathbf{A}_n\}|}$$

exists or not, and if it does, what the value of the limit is. For any finite set M , by $|M|$ we mean the number of elements in the set. (We will usually abbreviate the denominator $|\{\mathcal{A} : \mathcal{A} \in \mathbf{A}_n\}|$ and write only $|\mathbf{A}_n|$ for it.) Structures of this kind are called *labelled* structures, because every element in the structures has been labelled with a natural number. In this way we count every individual structure once and we give all different structures the same weight or probability. Another way would be, instead of counting individual structures, to count *equivalence classes* of structures. The equivalence relation under consideration here is the isomorphism relation. If we denote the equivalence class of a structure \mathcal{A} by $[\mathcal{A}]$ we obtain another limit

$$\lim_{n \rightarrow \infty} \frac{|[\{\mathcal{A} : \mathcal{A} \models \varphi, \mathcal{A} \in \mathbf{A}_n\}]|}{|[\{\mathcal{A} : \mathcal{A} \in \mathbf{A}_n\}]|}$$

Structures of this kind are called *unlabelled*. It is easy to see that in most cases these two limits are not the same. It can be shown though, that they in fact are the same if we consider classes of structures which are almost surely rigid. Recall that rigid structures are structures where the only automorphism is the identity. Note that the two measures just defined are uniform, in the sense that *all* structures, or *all* isomorphism classes, are given the same weight. If one does not want only to consider uniform measures, and the limit laws that they induce, one can instead consider, for some given probability measure,

$$\lim_{n \rightarrow \infty} \Pr\{\mathcal{A} \models \varphi : \mathcal{A} \in \mathbf{A}_n\},$$

which gives rise to more general limit laws. This last type of measure is our main concern in this thesis. The notation does not spell out what specific probability measure we have in mind, but this will always be clear from the context.

The classes \mathbf{A}_n of structures that one considers may be the class of *all* structures of a certain kind or some subclass within a certain class. It is worth pointing out that we consider every \mathbf{A}_n as a sample space. We will study what we call *images*. An image is a graph where, in addition to the edge relation, the vertices are given some property, for example some binary colouring; a vertex may be black or it may be white. Our results extend, however, to structures with any number of unary properties on the vertices. In this thesis, a graph is always a finite, undirected graph, unless otherwise stated. The colours of the vertices in our models $\mathcal{A} \in \mathbf{A}_n$ are black with probability p and white with probability $1 - p$. The number p may either be a constant or some function $p : \mathbb{N} \rightarrow [0, 1]$. The probability measure under consideration here is over this p .

Images of the kind we are interested in, may arise in many different situations. The point of departure in this thesis is the paper [2], *A zero-one law for first order logic on random images*, by Coupier, Desolneux and Ycart. As they point out, it is known that when our human eye is confronted with an image, then the eye tend to focus first on properties which are remarkable or strange in some way, that is, features of the image that one would suppose to have a low probability of appearing if the image were completely random. This suggests that it may be of great interest to know what properties of a random image have a high or low probability of appearing. In image analysis, many theories try to find, in an image, geometric objects, like regions, contours, lines and so on. Therefore there is a close connection between logic, model theory, computer science and image analysis. We refer to [2] and [3] for a more thorough motivation of the subject. It is important to note here, that the "colouring" of the vertices may be considered to be just a metaphore, since our model can appear in other areas as well. We will keep, however, the term "image", though it is considered in a more abstract sense here.

The zero-one law for first order logic was proved, independently, by Glebskij, Logan, Lio-gonkij, Talanov, [6], and Fagin, [5]. It was extended to parametric classes by Oberschelp, [9], (we will define later what a parametric class is). Moreover, there are limit laws for different classes of graphs, for instance regular graphs, graphs with a given degree sequence and graphs with a bounded degree. See the paper by Koponen, [7]. Shelah and Spencer, [11], showed among other things that if the function $p(n)$, (which is the edge probability, that is, given any pair of vertices, $p(n)$ is the probability that there is an edge between them, and n is the order of the graph) is equal to $n^{-\alpha}$, where α is irrational, we have a zero-one law for first order logic on this kind of random graphs.

In [2] they show that there is a zero-one law on a certain class of structures they call finite lattices. These are essentially squares of $n \times n$ vertices either being black or white where in addition the upper border is identified with the lower border and the right hand side border is identified with the left hand side border, so that the vertices can be seen as lying on a torus. We denote them $\mathcal{I}_{n^2,p}$. A more thorough description of their model will follow later and we will also indicate the reason why they make this rather strange identification. They show that the limit

$$\lim_{n \rightarrow \infty} \Pr\{\mathcal{I}_{n^2,p} \models \varphi\}$$

exists and is either zero or one, if the function $p : \mathbb{N} \rightarrow [0, 1]$ is not a so called *threshold function* for φ . A threshold function can be seen as the border where the limit changes from zero to one. If the function p is larger than, or smaller than, the threshold function, the zero-one law holds, while if p is of the same size as the threshold function the limit is not zero or one. This thesis can be regarded as a generalisation of this result in a couple of directions. Firstly, we show that, though there is no zero-one law for first order logic on images of the same kind as in [2] if one does not identify the borders while holding p constant, there is indeed a limit law. By this we mean that $\lim_{n \rightarrow \infty} \Pr\{\mathcal{I}_{n^2,p} \models \varphi\}$ exists, though not necessarily zero or one. Secondly, we generalise the model of [2] to a more general underlying graph structure, and show that there is a limit law here. Thirdly, we derive some conditions for the existence of limit laws when the

underlying graph structure also is random. Fourthly, we characterise the limit theory of the first case. Fifthly, and finally, we make some comments on decidability of the limit theory.

2 Notation and terminology

As stated above, by a *graph* we mean a finite, undirected graph. The graphs are structures over the vocabulary $\sigma := \{E\}$, where E is the irreflexive and symmetric binary relation. By an *image* we mean a graph where the vertices have been given one out of two colours. The images are structures over the vocabulary $\tau := \{E, C\}$, where C is a unary predicate. If Cx holds we say that x is black, if $\neg Cx$ holds we say that x is white. Of course, we use the term "image" also in a more generalised way than it usually is. Ordinarily one thinks of a collection of pixels, ordered in a systematic way in a lattice. Still we will use this term since it is short and convenient.

Calligraphic letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots$ will denote structures. In particular, \mathcal{G} and \mathcal{H} will denote graphs, while \mathcal{I} and \mathcal{J} will denote images. Ordinary capital letters like A, B, C will denote *universes of structures*. Bold face letters, like $\mathbf{A}, \mathbf{B}, \mathbf{C}, \dots$ will denote *classes of structures*. Greek letters, like τ and σ , will denote vocabularies. All our vocabularies will be purely relational. It can be shown, that zero-one laws do not hold in general, if constant symbols and function symbols are allowed in the vocabulary, but other limit laws may hold anyway. The vocabularies of interest in this thesis are purely relational, however, since they are graphs or images. It is worth pointing out, that the model theoretic concept *substructure* is not the same thing as the graph theoretic concept *subgraph*. If $G = (V, E)$ is a graph, then *any* $G' = (V', E')$ with $V' \subseteq V$ and $E' \subseteq E$, with each edge in E' incident with vertices in V' , is called a *subgraph*. The subgraph G'' of G *induced* by a set $U \subseteq V$ is the graph G'' with vertices in U and an edge between v_1 and v_2 in G'' if and only if there is an edge between them in G . Therefore, when we speak about *subgraphs* in this thesis we always mean *induced subgraphs*, that is, the substructure induced by some set U . The notation $FO[\tau]$ stands for the class of first order sentences over the vocabulary τ . We will use lower case letters like f and g to denote functions, for instance isomorphisms. Greek letters, like φ, ψ and θ , are used for first order sentences (or formulas, if they contain free variables). When we write \bar{x} we mean the tuple (x_1, \dots, x_m) for some m .

3 Limit laws in general

The most general zero-one law on relational vocabularies states that the asymptotic fraction of finite structures that satisfy a certain sentence in first order logic is zero or one. This result may be proved using so-called extension axioms.

Definition 3.1 *An $(r + 1)$ -extension axiom is a sentence*

$$\chi_{\Phi} = \forall v_1 \dots \forall v_r \left(\bigwedge_{1 \leq i < j \leq r} v_i \neq v_j \rightarrow \exists v_{r+1} \left(\bigwedge_{1 \leq i \leq r} v_i \neq v_{r+1} \wedge \bigwedge_{\varphi \in \Phi} \varphi \wedge \bigwedge_{\varphi \notin \Phi} \neg \varphi \right) \right)$$

where

$$\Phi \subseteq \Delta_{r+1} = \{\varphi(v_1, \dots, v_{r+1}) : \varphi \text{ is } R\bar{x} \text{ for some } R \text{ in the vocabulary, and } v_{r+1} \text{ occurs in } \bar{x}\}.$$

Since it can be shown that any extension axiom holds in almost all finite structures, and since one can prove that either φ or $\neg \varphi$ follows from the set of extension axioms, the zero-one law holds for first order logic. This is the most general zero-one law, since it concerns *all* structures of some kind.

If one wants to study zero-one laws for a subclass of structures within some class of structures, one is lead to the concept of *parametric classes*. A parametric sentence is defined in the following way.

Definition 3.2 A first order sentence φ_0 is called parametric if it is the conjunction of sentences of the type

$$\forall \text{distinct } x_1, \dots, x_s \psi.$$

Here ψ is a boolean combination of formulas Ry_1, \dots, y_n with $\{y_1, \dots, y_n\} = \{x_1, \dots, x_s\}$. A class of structures \mathbf{K} is said to be parametric if $\mathbf{K} = \text{Mod}(\varphi_0)$ for some parametric sentence φ_0 . \square

Note that the sentence

$$\forall x \neg Rxx \wedge \forall \text{distinct } xy (Rxy \rightarrow Ryx),$$

which axiomatizes the class of undirected graphs, is parametric, while the sentence

$$\forall \text{distinct } xyz ((Rxy \wedge Ryz) \rightarrow Rxz),$$

expressing transitivity, is not. The problem is that, for example $\{x, y\} \neq \{x, y, z\}$. If k is the maximum arity of the relations involved, φ_0 is called *nontrivial* if it has models of cardinality exceeding k . We obtain the following theorem.

Theorem 3.3 If \mathbf{H} is any nontrivial parametric class, then \mathbf{H} satisfies both the labelled and unlabelled zero-one law for first order logic.

Proof. We refer to [4], chapter 4 for a proof. \square

Using this machinery, one can prove that, for instance, almost all finite graphs are non-planar, that almost all finite graphs are connected, and that almost all finite graphs are rigid. Of great importance in our case is to realize that this machinery however can not be used to prove the results we are after. We will show this in the next section.

A first step towards a solution of the problem at hand in [2] is the notion of the *Gaifman graph* of a structure. Let \mathcal{A} be a relational structure. We define a relation $E^{\mathcal{A}}$ on \mathcal{A} by

$$E^{\mathcal{A}} := \{(a, b) : a \neq b \text{ and there is a relation } R \text{ such that } R^{\mathcal{A}}\bar{c} \text{ and } a \text{ and } b \text{ occur in the tuple } \bar{c}\}.$$

The structure $\mathcal{G}(\mathcal{A}) := (\mathcal{A}, E^{\mathcal{A}})$ is called the *Gaifman graph* of \mathcal{A} . If the structure \mathcal{A} already is an undirected graph, we just identify $\mathcal{G}(\mathcal{A})$ with \mathcal{A} .

Note that two vertices in $\mathcal{G}(\mathcal{A})$ are adjacent if and only if they are related by any of the relations in the vocabulary we are studying. We also define a metric on any graph \mathcal{G} by $d(a, b) = r$ if the shortest path from a to b is of length r . It is easily checked that this function d actually is a metric in the usual sense. Accordingly we define the ball $B_r(a) = \{b \in A : d(a, b) \leq r\}$.

The last, but maybe the most important, item in this section is Gaifman's theorem. For this we need some more terminology. If ψ is any first order formula, with one free variable, say, then $\psi^{B_r(x)}$ is the *relativisation* of ψ to the ball $B_r(x)$. This simply means that all quantifiers in ψ are restricted to the ball $B_r(x)$. See [4] for a strict definition.

Definition 3.4 We call a sentence a basic local sentence if it is of the form

$$\exists x_1 \dots \exists x_m \bigwedge_{1 \leq i < j \leq m} (d(x_i, x_j) > 2r \wedge \psi^{B_r(x_i)}(x_i)).$$

for some integers m and r , and a first order formula ψ . A local sentence is a boolean combination of basic local sentences.

Now we are in a position to state Gaifman's theorem.

Theorem 3.5 *Every first order sentence is logically equivalent to a local sentence.*

Proof. The theorem can be proved using Gaifman graphs and the method of Ehrenfeucht-Fraïssé. We refer to [4], page 30, for a proof. \square

4 Limit laws of random images

Before entering into our main problem we will look at some tools which can be used for proving results about limit laws.

Example. This first example is a result that follows trivially from the machinery mentioned above (which also shows how powerful that machinery is). Let $\{\mathcal{A}_n\}_{n=1}^\infty$ be a sequence of structures in the vocabulary $\tau = \{E, C\}$ such that the universe A_n of each \mathcal{A}_n is $\{1, 2, \dots, n\}$, and

$$p = \text{probability for a vertex to be black,}$$

and

$$q = \text{probability for any pair of vertices to have an edge between them.}$$

It is important, in this specific example, that we keep these probabilities constant. We let \mathbf{K} denote this class of structures. It is axiomatised by the single sentence

$$\varphi_G := \forall x \neg Exx \wedge \forall \text{distinct } xy (Exy \rightarrow Eyx).$$

expressing the irreflexivity and symmetry of the edge relation, like in any undirected graph. Note that this is a parametric sentence, and that it is nontrivial (in the sense that it has models in cardinalities at least the arity of E , which is two). The extension axioms (which are compatible with φ_G) are given by

$$\{\forall \text{distinct } x_1 \dots x_n y_1 \dots y_l \exists z (\bigwedge_{i=1}^n Ex_i z \wedge \bigwedge_{i=1}^l (\neg Ey_i z \wedge y_i \neq z)) : n + l \geq 1\}$$

Hence \mathbf{K} is a nontrivial parametric class and so the following theorem follows. Let us call this set of axioms $T_{rand}(\varphi_G)$.

Theorem 4.1 *Let \mathbf{K} be as defined above. Then \mathbf{K} satisfies both the labelled and unlabelled zero-one law for first order logic.*

Proof. This follows from theorem 3.3. \square

Analysing the proof (for which we refer [4]) one finds that the conclusion is not guaranteed if the p or q are not constant, or if the underlying graph structure is more complicated, as is the case of [2] as well as for us in this thesis.

Now we turn to a brief (but hopefully thorough enough) review of the paper [2].

They study $n \times n$ random discrete images. Every vertex is coloured to be either black or white, with probability p or $1 - p$, respectively. The colour of any vertex is independent of all the others. This implies that the probability distribution is a product of n^2 Bernoulli distributions, each with parameter p . Hence the probability that a specific image appears in the square of vertices is $p(n)^{k_0} (1 - p(n))^{k_1}$, where k_0 is the number of black vertices and k_1 is the number of white vertices. The vocabulary is $\rho := \{C, U, R, D_1, D_2\}$. The unary predicate C denotes the

colour, that is, Cx means that x is black, while $\neg Cx$ means that x is white. U stands for "up", so that Uxy means that x lies directly above y . R stands for "right", so that Rxy means that x is directly to the right of y . D_1 and D_2 are the two possible diagonal relations. As noted in [2] the expressive power does not increase by introducing the two diagonals in the vocabulary, but it has the effect that all balls are squares, rather than the "diamonds" that would appear otherwise. In order to avoid special cases, they identify the borders of the square, so that $n + 1 = 1$ in their model. By this they mean that vertices at the top border of the square are identified with the vertices at the bottom border, while the vertices at the right hand border are identified with those on the left hand border. Consequently the structure can be seen as points lying on a two-dimensional torus. As stated by [2], though it seems unnatural for an image, it simplifies the solution procedure. If the number p is held fixed, then as n tends to infinity, any given subimage of fixed size will appear somewhere in the image with probability tending to 1, since the number of disjoint places where to "put" this given subimage tends to infinity as well. This result is rather trivial. In [2] they instead let p be a function of n , so that $p : \mathbb{N} \rightarrow [0, 1]$. Their main result is the following.

Theorem 4.2 *Let $p : \mathbb{N} \rightarrow [0, 1]$ and assume that $\forall k = 1, 2, \dots$*

$$\lim_{n \rightarrow \infty} n^{\frac{2}{k}} p(n) = 0 \text{ or } \infty$$

and

$$\lim_{n \rightarrow \infty} n^{\frac{2}{k}} (1 - p(n)) = 0 \text{ or } \infty$$

Let φ be any first order sentence in the vocabulary ρ defined above. Then

$$\Pr\{\mathcal{I}_{n^2, p} \models \varphi\} \rightarrow 0 \text{ or } 1 \text{ as } n \rightarrow \infty.$$

We will briefly review the proof of this theorem after making some interesting remarks. They point out that their results can be extended in a couple of different directions. First, the number of colours is not important. The result, and the proof, holds for any number of colours, like some grey-scale or palette of colours. Second, the dimension of the image does not have to be 2. In the theorem one can without causing any problems change the dimension to any positive integer d , replacing $n^{\frac{2}{k}}$ by $n^{\frac{d}{k}}$.

Now we turn to a short description of the solution procedure. It is included here since it is useful to understand why *we* cannot use the same procedure. The proof of the main theorem of [2] rests on Gaifman's theorem. Using Gaifman's theorem they are able to reduce all FO-sentences to basic local sentences, and in the next step they are able to reduce basic local sentences to so-called pattern sentences. A pattern sentence is a sentence of the form

$$\exists x_1, x_2, \dots, x_m \left(\bigwedge_{1 \leq i < j \leq m} d(x_i, x_j) > 2r \right) \wedge \left(\bigwedge_{1 \leq i \leq m} D_i(x_i) \right),$$

where m and r are given nonnegative integers, and for all $i = 1, \dots, m$, $D_i(x)$ is a complete description of the ball $B_r(x)$. By a complete description they mean a sentence which says everything there is to say (in first order logic) about this ball. If $\psi(x)$ is any formula with one free variable, and all other variables belong to the ball $B_r(x_i)$, we can write it as:

$$\psi(x) \leftrightarrow \bigvee_{D(x) \rightarrow \psi(x)} D(x).$$

If one replaces each $\psi^{B_r(x_i)}$ in the definition of a basic local sentence in this way, and rearranges terms, one finds that a basic local sentence is a finite disjunction of pattern sentences. Every satisfiable basic local sentence is a finite boolean combination of pattern sentences.

A crucial role is played by the concept of *threshold functions* that we mentioned in the introduction of this thesis. The reader may have noticed that the zero-one law does not hold for *every* function $p(n)$. There are functions that lie "in between". For an example consider the sentence "There is a black vertex". If the function $p(n)$ is of larger order than $\frac{1}{n^2}$ then its probability tends to 1, whereas its probability tends to 0 if $p(n)$ is of smaller order than $\frac{1}{n^2}$. If $p(n)$ actually is *equal* to $\frac{1}{n^2}$ its probability is

$$1 - \left(1 - \frac{1}{n^2}\right)^{n^2}$$

which tends to $1 - e^{-1}$ as $n \rightarrow \infty$. A precise definition is given here.

Definition 4.3 *Let φ be any first order sentence. A threshold function for φ is a function $r(n)$ such that*

$$\lim_{n \rightarrow \infty} \frac{p(n)}{r(n)} = 0 \text{ implies } \lim_{n \rightarrow \infty} \Pr\{\mathcal{I}_{n^2,p} \models \varphi\} = 0$$

and

$$\lim_{n \rightarrow \infty} \frac{p(n)}{r(n)} = \infty \text{ implies } \lim_{n \rightarrow \infty} \Pr\{\mathcal{I}_{n^2,p} \models \varphi\} = 1$$

A threshold function is of course not uniquely defined, it is defined up to multiplication of any constant $c > 0$. Note that the "number of places" inside an image, where to "put" a certain subimage is $O(n^2)$.

The first thing that we will do is to *not* identify the borders of the square lattice. At first we will keep p constant while letting n run to infinity. It is easy to see that the zero-one law of [2] no longer holds if one does not do this identification. Consider the sentence "There are four black vertices that have only one horizontal neighbour". This sentence has the constant probability p^4 of being satisfied in any image. We are going to show that, though there is no zero-one law, there certainly is a *limit law*. The actual limit is a function of p , of course, as the simple example just given indicates.

We will use the method of Ehrenfeucht-Fraïssé. Note that the solution procedure of [2] relies on the fact that the probability of boolean combinations of sentences, that tend to either zero or one, also tends to either zero or one. For probabilities different from zero or one, this is not the case: Let $\{\mathcal{A}_n\}_{n=1}^{\infty}$ be a sequence of random structures and let θ_1 and θ_2 be two first order sentences in some vocabulary. If $\Pr\{\mathcal{A}_n \models \theta_1\} \rightarrow 0$ or 1 and $\Pr\{\mathcal{A}_n \models \theta_2\} \rightarrow 0$ or 1 then $\Pr\{\mathcal{A}_n \models \theta_1 \wedge \theta_2\} \rightarrow 0$ or 1 , but if $\Pr\{\mathcal{A}_n \models \theta_1\}$ or $\Pr\{\mathcal{A}_n \models \theta_2\}$ tends to some number different from zero or one, then we cannot *directly* conclude that $\Pr\{\mathcal{A}_n \models \theta_1 \wedge \theta_2\}$ converges. Since this conclusion is vital in their solution procedure, we must find another way.

To distinguish our structures from the structures of [2] we will put a "prime" on the image symbol, so when we write $\mathcal{I}'_{n^2,p}$ we mean our class, where the borders are *not* identified. Our main theorem of this section follows here.

Theorem 4.4 *For any first order sentence φ , the limit of $\Pr\{\mathcal{I}'_{n^2,p} \models \varphi\}$, as n tends to infinity, exists.*

We let k be the quantifier rank of φ . Recall that the quantifier rank of a formula is the maximum number of nested quantifiers appearing in it. We refer to [4], page 7, for a strict definition. Recall also that by an r -ball we simply mean the subset of a structure which consists of all vertices at a distance of at most r from a certain vertex a , in notation $B_r(a)$. For the proof of the theorem we need to define three special kinds of balls that appear inside an image $\mathcal{I}'_{n^2,p}$. We use the term *balls* though they are, in our present case, actually rectangles or squares. (In subsequent sections of this thesis they will not necessarily be rectangles or squares).

Definition 4.5 Let r be any positive integer. By an interior r -ball we mean an r -ball of which the center is at a distance of more than r from any of the borders of the image. Further, by a border r -ball we mean an r -ball of which the center is at a distance of exactly r from a border, but not as close as r to any of the corners. There are four types of such balls. Last, by a corner r -ball we mean an r -ball of which the center is at a distance of exactly r to any of the corners. There are four types of such balls as well.

Definition 4.6 By an isomorphism type of ball (or any substructure, for that matter) we mean an equivalence class of substructures of a structure modulo the isomorphism relation. In other words, for example if \mathcal{A} and \mathcal{B} are two structures and $a \in \mathcal{A}$ and $b \in \mathcal{B}$, then two balls, $B_r^{\mathcal{A}}(a)$ and $B_r^{\mathcal{B}}(b)$, are of the same isomorphism type if $B_r^{\mathcal{A}}(a) \cong B_r^{\mathcal{B}}(b)$.

In order to proceed with our proof, we will introduce a special kind of division of the class of images into subclasses. Note that, according to definition 4.5, there are three different *kinds* of balls which may appear in any image. Note also, that if we study a ball of some radius r and the center of the ball is closer than r to any border or corner, we can see that it does not look the same as a ball which is in the *interior* of an image, by which we mean that the center is at least r away from any border or corner. Given some radius r , the interior balls always look the same, and they cannot be isomorphic to any balls which are closer than r to any border or corner. A special kind of sequence of numbers are involved here. As a short hand notation we set

$$m_i = 2^{k-i}, \text{ for } 1 \leq i \leq k.$$

This implies that, in particular, $m_1 = 2^{k-1}$. Why these numbers are central will be soon be clear.

Definition 4.7 We define classes of structures in the following way. We say that two images \mathcal{I} and \mathcal{J} belong to the same class \mathbf{A} if the following three conditions are fulfilled.

(i) \mathcal{I} and \mathcal{J} contain at least m_i disjoint copies of all possible isomorphism types of interior m_i -balls for each $i \leq k$.

(ii) \mathcal{I} and \mathcal{J} contain at least m_i disjoint copies of all possible isomorphism types of border m_i -balls for each $i \leq k$.

(iii) \mathcal{I} and \mathcal{J} share the same corner m_1 -balls.

The class of structures which do not belong to any of these classes, (which we denoted generically by \mathbf{A}) we call the garbage class and we denote it by \mathbf{A}_0 .

Remark. There are only finitely many classes, since there are only finitely many ways to combine the colours on the *corner* vertices, that is, the vertices within the distance m_1 from a corner. \square

Lemma 4.8 The notion garbage class, which we used for \mathbf{A}_0 is justified; almost all structures under consideration obey condition (i) and condition(ii) in definition 4.7.

Proof. The number of vertices in any r -ball inside an image, for which the center is at least at a distance of r from any border, is fixed (actually equal to $(r+1)^2$, but the exact number is irrelevant). The probability that a certain coloured isomorphism type of r -ball appears on a fixed place inside an image is $\tilde{p} = p^{h_0}(1-p)^{h_1} > 0$ (where h_0 and h_1 is the number of black and white vertices, respectively, and $h_0 + h_1 = (r+1)^2$). \tilde{p} is a fixed positive number, while the number of disjoint places inside any image, where such a ball may be placed, tends to infinity. Therefore we may conclude that the probability for an image *not* to have such a ball tends to zero as n tends to infinity. Note that the same reasoning may be applied to the case of

having any before-hand given number of disjoint such balls, too. Hence almost all structures obey condition (i) in definition 4.7. Practically the same reasoning goes through for *border* balls, too. They cannot be placed *anywhere* inside an image, but it is enough that the number of places where to put such a ball tends to infinity as n tends to infinity. Hence almost all structures obey condition (ii) as well. \square

Remark. One may comment here that there is a more general variant of the lemma just proved, saying that for certain classes of structures, almost all structures in any of the classes contain a given substructure, but this requires that the class of structures under consideration is axiomatized by a nontrivial parametric sentence, which is not the case here. \square .

The most important step on the way to proving our main result in this section, theorem 4.4, is the lemma below. First note the following relation between structures: We say that two structures, \mathcal{A} and \mathcal{B} , are k -equivalent, $\mathcal{A} \equiv_k \mathcal{B}$ in short, if they satisfy exactly the same sentences of quantifier rank up to k . That is to say that, for every φ with $qr(\varphi) \leq k$, we have

$$\mathcal{A} \models \varphi \Leftrightarrow \mathcal{B} \models \varphi.$$

Lemma 4.9 *If \mathcal{I} and \mathcal{J} belong to the same class \mathbf{A} , (which is not the garbage class \mathbf{A}_0) then $\mathcal{I} \equiv_k \mathcal{J}$.*

Proof. We will use the method of Ehrenfeucht-Fraïssé, that is, we will prove that the duplicator has a winning strategy for the k -step Ehrenfeucht-Fraïssé game (for a background to how this method works, see [4] chapter 2). In each round the spoiler starts by choosing an element from one of the structures \mathcal{I} or \mathcal{J} . Then the duplicator chooses an element from the other structure. If it is the case that, whatever choices made by the spoiler, the duplicator always can make choices, such that the sequence of elements chosen after k steps constitutes a partial isomorphism between the two structures, we say that the duplicator has a winning strategy. Then, by the Ehrenfeucht-Fraïssé theorem, we have proved $\mathcal{I} \equiv_k \mathcal{J}$. The proof will be by induction. Let i denote the number of steps of the game run so far. We will first present the strategy, then we will verify that it defines a partial isomorphism between the two structures. Since the problem of finding a strategy is symmetric in \mathcal{I} and \mathcal{J} we will assume that the spoiler chooses an element from \mathcal{I} . There is no loss of generality here, since the roles of \mathcal{I} and \mathcal{J} may be reversed in any step.

In any of the rounds of the game there are three cases to consider, depending on where in the structure the spoiler chooses an element (recall that we defined three different kinds of r -balls). Consider now the first round of the game.

(i) The spoiler chooses an element a_1 in the *interior* of \mathcal{I} , that is, at least m_1 from any border or corner. Then, by the assumption on \mathbf{A} , $B_{m_1}^{\mathcal{I}}(a_1) \cong B_{m_1}^{\mathcal{J}}(b_1)$ for some $b_1 \in \mathcal{J}$. The duplicator chooses such an element b_1 .

(ii) The spoiler chooses an element a_1 within a distance of less than m_1 from a border, but not a corner, of \mathcal{I} . Again, by the assumption on \mathbf{A} , $B_{m_1}^{\mathcal{I}}(a_1) \cong B_{m_1}^{\mathcal{J}}(b_1)$ for some $b_1 \in \mathcal{J}$ at the same border. The duplicator chooses such an element b_1 .

(iii) The spoiler chooses an element a_1 within a distance of less than m_1 from a corner of \mathcal{I} . Then, since by construction, the corners of \mathcal{I} and \mathcal{J} are identical, the duplicator chooses the corresponding element in \mathcal{J} (at the same corner).

Let the pairs $(a_1, b_1), \dots, (a_{i-1}, b_{i-1})$ denote the history of the game run so far, and consider round i . If the spoiler chooses an element in \mathcal{I} within a distance of m_i from an earlier chosen vertex a_j , where $j < i$, then the duplicator chooses b_i in $B_{m_j}^{\mathcal{J}}(b_j) \cong B_{m_j}^{\mathcal{I}}(a_j)$, so that also $B_{m_i}^{\mathcal{J}}(b_i) \cong B_{m_i}^{\mathcal{I}}(a_i)$ which is possible by how the numbers m_j and m_i were chosen. In fact, we

know that $B_{m_j}^{\mathcal{J}}(b_j) \cong B_{m_j}^{\mathcal{I}}(a_j)$. By assumption $d(a_i, a_j) \leq m_i = 2^{k-i}$. We want to reach the conclusion $B_{m_i}^{\mathcal{I}}(a_i) \subseteq B_{m_j}^{\mathcal{I}}(a_j)$. Take $v \in B_{m_i}^{\mathcal{I}}(a_i)$, so that $d(v, a_i) \leq m_i = 2^{k-i}$. Now, the triangle inequality gives $d(v, a_j) \leq d(v, a_i) + d(a_i, a_j) \leq 2^{k-i} + 2^{k-i} = 2^{k-i+1} \leq 2^{k-j}$, since $j < i$.

If the spoiler chooses an element *not* within m_i from any earlier chosen vertex and also *not* within m_i from any border or corner, then the duplicator chooses a vertex b_i with $B_{k-i}^{\mathcal{J}}(b_i) \cong B_{k-i}^{\mathcal{I}}(a_i)$ and disjoint from all the other $B_{k-1}^{\mathcal{J}}(a_1), \dots, B_{k-i+1}^{\mathcal{J}}(a_{i-1})$. This is possible by how \mathbf{A} was constructed.

If the spoiler chooses an element a_i within a distance of m_i from a border, but not within this distance from a corner, then the duplicator chooses a new m_i -border ball in \mathcal{J} disjoint from all other earlier balls, touching the same border in \mathcal{J} as a_i does in \mathcal{I} , and we let the element be b_i . By how \mathbf{A} was constructed, this is possible.

Finally, if the spoiler chooses an element within a distance of m_i from a corner in \mathcal{I} , then the duplicator chooses the corresponding element, in the same corner, in \mathcal{J} .

Let us denote the function defined by the strategy just presented by h , so that $b_i = h(a_i)$ for all i . Firstly, it is clear that h is injective, since the duplicator repeats an element if and only if the spoiler does. Secondly, $C^{\mathcal{I}}a_i$ if and only if $C^{\mathcal{J}}h(a_i)$ (where C is the colour predicate), by construction. Lastly, if $S \in \rho \setminus \{C\} = \{U, R, D_1, D_2\}$ and a_i is adjacent to some a_j by the relation S in \mathcal{I} , and j is the smallest such index, then we know that $B_{k-j}^{\mathcal{I}}(a_j) \cong B_{k-j}^{\mathcal{J}}(b_j)$, so that b_i is adjacent to b_j by S in \mathcal{J} .

Hence h defines a partial isomorphism between the two structures, and the proof is complete. \square

Now we are in a position to prove our main result of this section.

Proof of theorem 4.4 . In view of our lemmas above, and the remark that there only finitely many classes, it is enough to prove that $\lim_{n \rightarrow \infty} \Pr\{\mathcal{I}'_{n^2,p} \in \mathbf{A}\}$ exists for each class \mathbf{A} . The probability that $\mathcal{I}'_{n^2,p}$ belongs to a certain class, which is not the garbage class, is equal to the probability that $\mathcal{I}'_{n^2,p}$ has a certain isomorphism type of corner balls with radius m_1 (and this is constant for large enough n) times the probability that (i) and (ii) in definition 4.7 hold. The latter probability tends to 1 by lemma 4.8. (Recall that the probability that the colour of a vertex is black is independent of the colours of all the other vertices.) Hence $\Pr\{\mathcal{I}'_{n^2,p} \in \mathbf{A}\}$ converges. \square

5 Random images on a more general underlying graph structure

We now proceed to a discussion of a more general problem. Still we consider graphs having colours on its vertices, but now we allow sequences of more general underlying graph structure. First we will keep p to be a constant, later we will allow p to be a function $p : \mathbb{N} \rightarrow [0, 1]$. To reach the conclusion we desire we will assume that the graphs are of uniformly bounded degree. By this we mean that the maximum degree of the graphs we consider is bounded by one and the same number. We will still use the term "image", though it will be used in a more generalised sense from now on. The proofs of the main theorem and the lemmas of this section will be a bit more brief than in the previous section, since the ideas are often similar.

Theorem 5.1 *Let $\{\mathcal{G}_n\}_{n=1}^{\infty}$ be a sequence of graphs of uniformly bounded degree, where the index n denotes the number of vertices. Let \mathcal{D} be any graph that may appear as a subgraph (by which we mean induced subgraph, as we remarked earlier) in any of the \mathcal{G}_n 's (\mathcal{D} may be*

disconnected). Assume that the following condition is satisfied for each such \mathcal{D} : \mathcal{D} is a union of balls in \mathcal{G}_{n_1} if and only if \mathcal{D} is a union of balls in \mathcal{G}_{n_2} for all n_1 and n_2 large enough. Then there is a limit law for the sequence of coloured graphs, emerging if one randomly puts colours on the vertices. As before the vertices are black with probability p and white with probability $1 - p$. Using the notation we had before, $\lim_{n \rightarrow \infty} \{\mathcal{I}_{n,p} \models \varphi\}$ exists for all $\varphi \in FO[\tau]$.

Remark. In this section (and the next) we assume that we have a sequence of graphs where the sequence $1, 2, \dots, n, \dots$ denotes the order of the graphs. If, however, not *all* $n \in \mathbb{N}$ appear as the order of the graphs in the sequence, we encounter instead indices $n_1 < n_2 < \dots$. In [2] this corresponds to the sequence $1^2 < 2^2 < \dots < n^2 < \dots$. The result of this section (and the next) can be generalised to an arbitrary sequence of this kind, but for ease of notation we will keep the indices to be $1 < 2 < \dots < n < \dots$. \square

As in the previous section we fix $k = qr(\varphi)$. It is convenient to use the following definition. For this we need to introduce the notation $\mathcal{A} \uplus \mathcal{B}$ to be the *disjoint* union of two structures (graphs or images) \mathcal{A} and \mathcal{B} . This means that there are no edges between any pair of vertices not belonging to the same structure. If we take the disjoint union of a structure with *itself*, for instance m times, we mean that this structure appears m times.

Definition 5.2 For a possible induced subgraph \mathcal{D} of a graph \mathcal{G} we let $f_{\mathcal{D}}$ denote the maximum number of disjoint copies of \mathcal{D} in \mathcal{G} . By this we mean the following. Let $m = f_{\mathcal{D}}$. Then there is \mathcal{C}_m in \mathcal{G} such that $\mathcal{C}_m \cong \uplus_{i=1}^m \mathcal{D}$, but there is no \mathcal{C}_{m+1} in \mathcal{G} such that $\mathcal{C}_{m+1} \cong \uplus_{i=1}^{m+1} \mathcal{D}$. If $\{\mathcal{G}_n\}_{n=1}^{\infty}$ is a given sequence of graphs then $f_{\mathcal{D}}(n)$ denotes the number of disjoint copies of \mathcal{D} in \mathcal{G}_n . \square

Lemma 5.3 Under the condition of theorem 5.1, for each \mathcal{C} that appear as a union of balls in some \mathcal{G} in the sequence $\{\mathcal{G}_n\}_{n=1}^{\infty}$, $f_{\mathcal{C}}(n)$ converges to some number $l \in \mathbb{N} \cup \{\infty\}$ as $n \rightarrow \infty$.

Proof. The proof is by contradiction. Let \mathcal{C} be a subgraph for which $f_{\mathcal{C}}(n)$ does *not* converge. This assumption implies that for every number N there are $n_1, n_2 \geq N$ such that $f_{\mathcal{C}}(n_1) \neq f_{\mathcal{C}}(n_2)$. Let us assume that $m := f_{\mathcal{C}}(n_1) > f_{\mathcal{C}}(n_2)$. Now we take $\mathcal{D} = \uplus_{i=1}^m \mathcal{C}$. Then \mathcal{D} is also a union of balls, but \mathcal{D} is in \mathcal{G}_{n_1} and not in \mathcal{G}_{n_2} , which contradicts the condition of theorem 5.1. \square

Remark. Note that the assumption on uniformly bounded degree on the sequence $\{\mathcal{G}_n\}_{n=1}^{\infty}$ implies that there are, for each $r > 0$, only finitely many isomorphism types of r -balls in any \mathcal{G}_n , and the same is true for any $\mathcal{I}_{n,p}$. Note also that two isomorphism types of balls can intersect each other only in finitely many ways. \square

A word of caution is appropriate here. It is important to distinguish between substructures of any \mathcal{G}_n , which just have the edge relation, and substructures of any $\mathcal{I}_{n,p}$, which also have colours. For any element a in any structure and for any $r > 0$, $B_r^{\mathcal{G}}(a)$ is an induced *subgraph* of \mathcal{G} , while $B_r^{\mathcal{I}}(a)$ is an induced *subimage* of \mathcal{I} . Recall that we have fixed some positive integer k . The numbers $m_i = 2^{k-i}$ for $i \leq k$ which were defined before, will be used in this section, too.

Definition 5.4 Assume that the condition of theorem 5.1 is fulfilled. Balls \mathcal{B} , which may appear in graphs in the sequence $\{\mathcal{G}_n\}_{n=1}^{\infty}$, are said to be of category 1 if $f_{\mathcal{B}}(n) \rightarrow \infty$, and of category 2 otherwise.

Note that the assumptions imply that there are balls of category 1, but it is possible that there are no balls of category 2. The situation with no balls of category 2 is rather trivial, though. Remember that $m_1 = 2^{k-1}$.

Definition 5.5 For any \mathcal{G} in the sequence $\{\mathcal{G}_n\}_{n=1}^\infty$ we define

$$\mathcal{C}^{\mathcal{G}} = \bigcup_{v \in \mathcal{G}} \{B_{m_1}^{\mathcal{G}}(v) : B_{m_1}^{\mathcal{G}}(v) \text{ is of category 2}\}.$$

Lemma 5.6 Suppose that the conditions of theorem 5.1 are satisfied. Then, for all sufficiently large n_1 and n_2 , $\mathcal{C}^{\mathcal{G}_{n_1}} \cong \mathcal{C}^{\mathcal{G}_{n_2}}$.

Proof. Since $\mathcal{C}^{\mathcal{G}_{n_1}}$ and $\mathcal{C}^{\mathcal{G}_{n_2}}$ are finite unions of balls, it is easy to see that they have finitely many components, that they have the same number of components, and that they are component-wise isomorphic. \square

Definition 5.7 We say that two images \mathcal{I} and \mathcal{J} belong to the same class \mathbb{B} if the following two conditions are fulfilled.

(i) \mathcal{I} and \mathcal{J} contain at least k disjoint copies of all coloured m_1 -balls, for which the underlying graph m_1 -balls (not coloured) are of category 1.

(ii) Assume that \mathcal{G} and \mathcal{H} are the underlying graphs of \mathcal{I} and \mathcal{J} , respectively. According to lemma 5.6 we assume that $\mathcal{C}^{\mathcal{G}}$ and $\mathcal{C}^{\mathcal{H}}$ are isomorphic. Let $g : \mathcal{C}^{\mathcal{G}} \rightarrow \mathcal{C}^{\mathcal{H}}$ be the isomorphism. Then we demand that $v \in \mathcal{C}^{\mathcal{G}}$ and $g(v) \in \mathcal{C}^{\mathcal{H}}$ are given the same colour.

Just as before these classes do not exhaust the class of all structures under consideration. Therefore we once again define a garbage class to consist of all the rest of the structures, and we denote it by \mathbb{B}_0 .

Remark. Note that condition (i) in the definition, associated with the balls of category 1, corresponds to the interior and border balls in our previous problem, and that the elements in condition (ii) correspond to the corner balls. \square

The following lemma is also familiar.

Lemma 5.8 There are only finitely many different classes.

Proof. The isomorphism type of $\bigcup_{v \in \mathcal{G}} \{B_{m_1}^{\mathcal{G}}(v) : B_{m_1}^{\mathcal{G}}(v) \text{ is of category 2}\}$ stabilizes for $\mathcal{G} = \mathcal{G}_n$ if n is large enough, according to lemma 5.6. There are finitely many ways to put colours on a finite graph. \square

Just as before, we have the following lemma, which justifies the name "garbage class".

Lemma 5.9 $\Pr\{\mathcal{I}_{n,p} \in \mathbb{B}_0\} \rightarrow 0$ as $n \rightarrow \infty$.

Proof. We need to show that almost all structures under consideration satisfy condition (i) of definition 5.7. Let the m_1 -ball \mathcal{B} , which may appear in any of the graphs in the sequence, be of category 1. Then, by definition, $f_{\mathcal{B}}(n)$ approaches infinity as n approaches infinity. Note that p is held fixed, while n tends to infinity. This implies that almost all structures contain every possible isomorphism type of coloured m_1 -ball, in fact even k disjoint copies of such balls. \square

Lemma 5.10 For all \mathcal{I} and \mathcal{J} from the same class \mathbb{B} we have $\mathcal{I} \equiv_k \mathcal{J}$.

Proof. Again the proof is by showing that the duplicator has a winning strategy for the k -step Ehrenfeucht-Fraïssé game. Let \mathcal{G} and \mathcal{H} be the underlying graphs of \mathcal{I} and \mathcal{J} , respectively. Consider the i -th round of the game. If $a_i \in \mathcal{I}$ is within a distance of m_i from an earlier chosen vertex a_j , then $B_{m_i}^{\mathcal{I}}(a_i) \subseteq B_{m_j}^{\mathcal{I}}(a_j)$, and therefore there is a vertex $b_i \in \mathcal{J}$ such that $B_{m_i}^{\mathcal{J}}(b_i) \cong B_{m_i}^{\mathcal{I}}(a_i)$. Assume now that a_i is further than m_i away from any earlier vertex. Then

there are two cases to consider. Either $B_{m_i}^{\mathcal{G}}(a_i)$ is of category 1 or it is of category 2. In either case, however, by condition (i) and (ii) of definition 5.7, we are guaranteed that there is a b_i such that $B_{m_i}^{\mathcal{J}}(b_i) \cong B_{m_i}^{\mathcal{I}}(a_i)$. That the sequence $(a_1, b_1), (a_2, b_2), \dots, (a_k, b_k)$ defines a partial isomorphism between the two structures now follows as in the previous section. \square

Proof of theorem 5.1. Just as before we have, by the same reasoning as in the proof of theorem 4.4, for every class \mathbb{B} , $\Pr\{\mathcal{I}_{n,p} \in \mathbb{B}\}$ converges as n tends to infinity. Together with the fact that there are finitely many classes, and lemma 5.10, we obtain the convergence law. \square

Next we will generalise the results of [2] in another direction. In their model the underlying graphs for each n were completely determined by the relations in the vocabulary $\rho \setminus \{C\} = \{U, R, D_1, D_2\}$. We will allow a very general sequence of underlying graphs. Actually we have just a minimal requirement on these graphs. Analysing the proof of the zero-one law in [2], one finds that it essentially relies on the asymptotic relations between the functions $p : \mathbb{N} \rightarrow [0, 1]$ and the growth of the underlying graphs when $n \rightarrow \infty$. The following theorem expresses this fact. As a preliminary, for two functions $f, g : \mathbb{N} \rightarrow \mathbb{N}$ we say that they are *of the same magnitude* if

$$\frac{f(n)}{g(n)} \rightarrow c, \text{ as } n \rightarrow \infty, \text{ for some } c \neq 0.$$

Recall from definition 5.2 what we mean by $f_{\mathcal{D}}(n)$. The following inequality will be used in the proof of the theorem, and it will also be used in the next section.

Lemma 5.11 *Let a and b be two nonnegative real numbers and assume that $b < 1$. Then*

$$(1 - b)^a \leq \exp(-ab).$$

Proof. Since the exponential function is increasing, what we need to prove is

$$\log(1 - b) \leq -b,$$

because the a 's cancel. Let $\mu(x) = \log(1 - x) + x$.

$$\frac{d}{dx}\mu(x) = -\frac{1}{1-x} + 1,$$

which is ≤ 0 if $x < 1$. Now, since $\mu(0) = 0$, the proof is complete. \square

Theorem 5.12 *Let $\{\mathcal{G}_n\}_{n=1}^{\infty}$ be a sequence of graphs which fulfill the following conditions. For any graph \mathcal{D} that may be a subgraph of some of the \mathcal{G}_n 's, we demand that $f_{\mathcal{D}}(n) \rightarrow 0$ or ∞ as $n \rightarrow \infty$. Moreover, if $f_{\mathcal{D}}(n) \rightarrow \infty$, we demand that $f_{\mathcal{D}}$ is of the same magnitude as a certain given strictly increasing function $g : \mathbb{N} \rightarrow \mathbb{N}$. For the function $p : \mathbb{N} \rightarrow [0, 1]$ we assume that $g(n)p(n)^{k_0}(1-p(n))^{k_1} \rightarrow 0$ or ∞ as $n \rightarrow \infty$, for every pair $k_0, k_1 \in \mathbb{N}$. Then we have a zero-one law:*

$$\Pr\{\mathcal{I}_{n,p} \models \varphi\} \rightarrow 0 \text{ or } 1 \text{ as } n \rightarrow \infty$$

Proof. By Gaifman's theorem, and basic probability theory, it is enough to prove our theorem for basic local sentences. Accordingly, let φ be the sentence

$$\exists x_1 \dots \exists x_m \left(\bigwedge_{1 \leq i < j \leq m} d(x_i, x_j) > 2r \wedge \left(\bigwedge_{1 \leq i \leq m} \psi^{B_r(x_i)}(x_i) \right) \right)$$

where r and m are given positive integers, and ψ is a first order sentence. A basic local sentence essentially says that in a structure which satisfies it, there is a subset $\{x_1, x_2, \dots, x_m\}$ of m

elements where the distance between any two of them is more than $2r$ and $\psi^{B_r(x_i)}(x_i)$ is true for each $i \leq m$. To make $\psi^{B_r(x)}(x)$ true for some x there is some minimal requirement of the underlying graph structure of such an r -ball. Let us call it \mathcal{D} . Moreover, it demands that the vertices are correctly coloured. Let us assume that it demands k_0 to be black and k_1 to be white. First, if $f_{\mathcal{D}}(n) \rightarrow 0$ then

$$\Pr\{\mathcal{I}_{n,p} \models \varphi\} \leq f_{\mathcal{D}}(n)p(n)^{k_0}(1-p(n))^{k_1} \rightarrow 0.$$

Also, if $g(n)p(n)^{k_0}(1-p(n))^{k_1} \rightarrow 0$ then

$$\Pr\{\mathcal{I}_{n,p} \models \varphi\} \leq cg(n)p(n)^{k_0}(1-p(n))^{k_1} \rightarrow 0.$$

for some constant $c > 0$. Hence in both these cases $\Pr\{\mathcal{I}_{n,p} \models \varphi\} \rightarrow 0$.

So we may assume that both $f_{\mathcal{D}}(n) \rightarrow \infty$ and $g(n)p(n)^{k_0}(1-p(n))^{k_1} \rightarrow \infty$. Note the inequality $(1-b)^a \leq \exp(-ab)$. We obtain

$$\Pr\{\mathcal{I}_{n,p} \models \varphi\} \geq 1 - (1-p(n)^{k_0}(1-p(n))^{k_1})^{f_{\mathcal{D}}(n)} \geq 1 - \exp(-f_{\mathcal{D}}(n)p(n)^{k_0}(1-p(n))^{k_1}) \rightarrow 1$$

as $n \rightarrow \infty$. This establishes the zero-one law. \square

6 Random images on random underlying graphs

When studying random images one can let also the underlying graph structure be random. Above the sequence of underlying graphs was deterministic, that is, for each n there was only one underlying graph. Here we instead consider a sequence of *classes* of graphs. For simplicity we will in this section restrict ourselves to the case $p = 1/2$, though the results may be generalised to $p \neq 1/2$. This is notationally more complicated, however. The assumption $p = 1/2$, which we confine ourselves with, implies that all images are equally likely to appear on a certain graph. Let $\mathbf{G}_1, \mathbf{G}_2, \dots$ be a sequence of classes of graphs. The procedure in each step n is the following.

- 1) A graph \mathcal{G} is chosen uniformly from \mathbf{G}_n .
- 2) The graph \mathcal{G} is coloured uniformly, to produce an image \mathcal{I} .

Let $\mathbf{I}_1, \mathbf{I}_2, \dots$ denote the classes of images thus obtainable. In this section, when we write $\Pr\{\mathcal{A} \models \varphi : \mathcal{A} \in \mathbf{A}_n\}$, for some class \mathbf{A}_n , we mean the fraction of *structures* that satisfy φ . In other words, it is the *labelled* measure that we mean.

Theorem 6.1 *Assume that $\mathbf{G}_1, \mathbf{G}_2, \dots$ is a sequence which possesses a limit law for first order logic on the vocabulary $\sigma = \{E\}$, that is, for each sentence ψ , $\Pr\{\mathcal{G} \models \psi : \mathcal{G} \in \mathbf{G}_n\} \rightarrow c \in [0, 1]$ as $n \rightarrow \infty$. A sufficient condition for the corresponding classes of images, $\mathbf{I}_1, \mathbf{I}_2, \dots$ to possess a limit law also, is given by*

$$\frac{|\mathbf{G}_n|}{2^n} \rightarrow \infty \tag{1}$$

That is, if this condition holds, then for each first order sentence φ in the vocabulary $\tau = \{E, C\}$, there is a number $d \in [0, 1]$ such that

$$\Pr\{\mathcal{I} \models \varphi : \mathcal{I} \in \mathbf{I}_n\} \rightarrow d \text{ as } n \rightarrow \infty.$$

The condition in the theorem is obviously quite strong. This suggests that it is not a necessary condition. After the proof of the theorem we will give a (rather trivial) example where $|\mathbf{G}_n| = 2^{n-1}$ and yet a limit law holds. Before the proof we need some terminology.

Definition 6.2 Let φ be a sentence in the vocabulary $\tau = \{E, C\}$. Consider sentences ψ in the vocabulary $\sigma = \{E\}$. If for all \mathcal{G} such that $\mathcal{G} \models \psi$, \mathcal{G} can be coloured to obtain an image \mathcal{I} , such that $\mathcal{I} \models \varphi$, we say that ψ is colouring compatible with φ .

Example. We present sentences in plain english rather than in strict logical notation, since this makes them easier to read. Let φ be the sentence "There is a white triangle" in τ . A sentence ψ in σ , which is colouring compatible with φ , would be for instance "There is a triangle". There are many more, of course. "There are two triangles", "There are three triangles", "There is one triangle and one 4-cycle" and so on, are other examples of such sentences. \square

Proof of theorem 6.1. Let φ be a sentence in the vocabulary τ . We need to characterise those \mathcal{G} which can be coloured to obtain \mathcal{I} with $\mathcal{I} \models \varphi$. Let $\{\psi_1, \psi_2, \psi_3, \dots\}$ be an enumeration of all those sentences in σ that are colouring compatible with φ . Note that this collection is usually infinite, so we cannot take the disjunction of these sentences. By the original limit law in $\{\mathbf{G}_n\}_{n=1}^\infty$ we have $\Pr\{\mathcal{G} \models \psi_i : \mathcal{G} \in \mathbf{G}_n\} \rightarrow c_i$ as $n \rightarrow \infty$. For each j we have $d_j := \lim_{n \rightarrow \infty} \Pr\{\mathcal{G} \models \psi_1 \vee \psi_2 \vee \dots \vee \psi_j : \mathcal{G} \in \mathbf{G}_n\} \geq \max\{c_1, c_2, \dots, c_j\}$. $\{d_j\}_{j=1}^\infty$ is an increasing and bounded sequence, hence convergent, say to some d .

When $d = 0$ there is nothing to prove, so we assume that $d > 0$. This implies that there is a $\delta > 0$ such that at least the fraction δ of the graphs may be coloured in a way to produce an image which satisfies φ .

It is enough to show that, with probability 1, φ is true in those structures where any of ψ_i is true. The worst case scenario, in the colouring process, is that only 1 out of all the 2^n ways to colour the graph \mathcal{G} satisfies φ in \mathcal{I} . Hence the probability we are looking for is larger than or equal to $1 - (1 - \frac{1}{2^n})^{\delta|\mathbf{G}_n|} \geq 1 - \exp(-\frac{\delta|\mathbf{G}_n|}{2^n}) \rightarrow 1$ as $n \rightarrow \infty$. Here we have used the inequality $(1 - b)^a \leq \exp(-ab)$ again. \square

Corollary 6.3 If there is a zero-one law on $\{\mathbf{G}_n\}_{n=1}^\infty$, then there is also a zero-one law on $\{\mathbf{I}_n\}_{n=1}^\infty$.

Proof. Let $\varphi \in FO[\tau]$. The assumption is that $\Pr\{\mathcal{G} \models \psi : \mathcal{G} \in \mathbf{G}_n\} \rightarrow 0$ or 1 as $n \rightarrow \infty$ for all $\psi \in FO[\sigma]$. If there are any ψ which are colouring compatible with φ , then the number d in the proof of the theorem is greater than zero, hence must be equal to one. This implies that $\Pr\{\mathcal{G} \models \psi : \mathcal{G} \in \mathbf{G}_n\} \rightarrow 1$ for this sentence ψ . Hence $\Pr\{\mathcal{I} \models \varphi : \mathcal{I} \in \mathbf{I}_n\} \rightarrow 1$. Otherwise, if there are no such ψ , then $d = 0$ and accordingly $\Pr\{\mathcal{I} \models \varphi : \mathcal{I} \in \mathbf{I}_n\} \rightarrow 0$ as $n \rightarrow \infty$. \square

Example. We will give a simple example of how theorem 6.1 can be used for specific classes of underlying graphs. It is built on the following theorem of Bollobás, [1].

Theorem 6.4 Let Δ and n be natural numbers such that $\Delta n = 2m$ is even and assume that $\Delta \leq (2 \log n)^{1/2} - 1$. Then, as $n \rightarrow \infty$, the number of labelled Δ -regular graphs on n vertices is asymptotically equal to

$$e^{-\lambda - \lambda^2} \frac{(2m)!}{m! 2^m (\Delta!)^m},$$

where $\lambda = (\Delta - 1)/2$.

A moment's thought shows that the number of labelled regular graphs, given by theorem 6.4, is much larger than 2^n , so theorem 6.1 applies here. Therefore we may conclude that there is a limit law on the class of structures consisting of regular graphs when its vertices are given some binary colouring. \square

Now we turn to the example which we mentioned above.

Example. We will define a special class of graphs. Let $\{\mathbf{G}_n\}_{n=1}^\infty$ be the following sequence of classes of graphs. For each $\mathcal{G}_n \in \mathbf{G}_n$ we let the universe be $\{1, 2, \dots, n\}$. (That is, we label each vertex with a natural number). Vertex 1 can be connected to any other vertex, but no pair of the remaining $n - 1$ vertices can be connected. This implies that $|\mathbf{G}_n| = 2^{n-1}$, as promised. Now divide the graphs into equivalence classes. Let \mathbf{A}_j be the class of graphs which have exactly j vertices connected to vertex 1. This gives that $\mathcal{G}, \mathcal{H} \in \mathbf{A}_j$ implies $\mathcal{G} \cong \mathcal{H}$ (and hence $\mathcal{G} \equiv \mathcal{H}$). Let ψ be a sentence in the vocabulary $\sigma = \{E\}$ and assume that $k = qr(\psi)$. In order to proceed, we need the following basic lemma.

Lemma 6.5 *Let $\mathcal{G}, \mathcal{H} \in \mathbf{G}_n$. Assume that $n \geq 2k$. Then the following implication holds:*

$$\mathcal{G}, \mathcal{H} \in \bigcup_{j \geq k} \mathbf{A}_j \Rightarrow \mathcal{G} \equiv_k \mathcal{H}.$$

Proof. The lemma can be proved by showing that the duplicator has a winning strategy for the k -step Ehrenfeucht-Fraïssé game. \square

Example continued. Almost all graphs in our class have k or more vertices connected to vertex 1. Hence by the lemma, almost all graphs in the class are k -elementarily equivalent. This gives that a sentence ψ either is true in almost all or almost none of the structures. Hence we have a zero-one law on $FO[\sigma]$. The aim of the example was to arrive at a limit law for the *coloured* structures, that is on $FO[\tau]$. The procedure is almost the same, the only difference is that we now get a limit law, but not a zero-one law. We let \mathbf{I}_n denote the class of images obtainable from the class of graphs \mathbf{G}_n .

Corollary 6.6 *For almost all images \mathcal{I} and \mathcal{J} in \mathbf{I}_n defined above we have $\mathcal{I} \equiv_k \mathcal{J}$ if and only if they share the same colour on vertex 1.*

Proof. Almost all images have at least k black vertices and at least k white vertices connected to vertex 1, and these satisfy the same sentences of quantifier rank up to k if and only if the colour on vertex 1 is the same. \square

Remark. The possible limits of $\Pr\{\mathcal{I} \models \varphi : \mathcal{I} \in \mathbf{I}_n\}$ are $0, \frac{1}{2}$ and 1 . Hence the conclusion of the corollary may not be valid if the condition $\frac{|\mathbf{G}_n|}{2^n} \rightarrow \infty$ does not hold. \square

The condition in the theorem is, though quite crude, useful since it is easy to verify for any given class of graphs for which one knows how fast the number of graphs in the class grows when the size tends to infinity.

Example. This is an example that shows what can happen if the condition that there is a limit law in the underlying graph sequence is not fulfilled. Assume that $\mathbf{G}_1, \mathbf{G}_2, \dots$ does *not* have a limit law. We let every $\mathcal{G} \in \mathbf{G}_{2n}$ contain a triangle (that is, a 3-cycle), and every $\mathcal{G} \in \mathbf{G}_{2n+1}$ *not* contain a triangle. Let φ be the sentence "There is a white triangle" (that is, there is a 3-cycle with only white vertices). Then $\Pr\{\mathcal{I} \models \varphi : \mathcal{I} \in \mathbf{I}_{2n}\} \geq \frac{1}{8}$, while $\Pr\{\mathcal{I} \models \varphi : \mathcal{I} \in \mathbf{I}_{2n+1}\} = 0$. Hence $\lim_{n \rightarrow \infty} \Pr\{\mathcal{I} \models \varphi : \mathcal{I} \in \mathbf{I}_n\}$ does not exist. \square

7 The limit theory

In this section we let $p : \mathbb{N} \rightarrow [0, 1]$ be a function which is not a threshold function for any $\varphi \in FO[\tau]$. Such a function exists; since all threshold functions are of the form n^{-q} for some rational q one can avoid these by taking q *irrational* instead. This means that we have a zero-one law: $\Pr\{\mathcal{I}_{n,p} \models \varphi\} \rightarrow 0$ or 1 as $n \rightarrow \infty$ for every FO-sentence φ in the vocabulary τ . We let T^∞ be the limit theory of this measure, that is we let T^∞ consist of all φ such that $\Pr\{\mathcal{I}_{n,p} \models \varphi\} \rightarrow 1$ as $n \rightarrow \infty$.

Theorem 7.1 *T^∞ is consistent, that is, it has a model.*

Proof. We will use the compactness theorem. Let $\Sigma \subseteq T^\infty$ be any finite subset. Let us write it as $\Sigma = \{\psi_1, \psi_2, \dots, \psi_m\}$. This means that $\Pr\{\mathcal{I}_{n,p} \models \psi_i\} \rightarrow 1$ as $n \rightarrow \infty$ for every $i \leq m$. For every $\varepsilon > 0$ there is $N_{i,\varepsilon}$ such that $\Pr\{\mathcal{I}_{n,p} \models \psi_i\} \geq 1 - \varepsilon$ if $n \geq N_{i,\varepsilon}$. Now let $N_\varepsilon = \max\{N_{1,\varepsilon}, N_{2,\varepsilon}, \dots, N_{m,\varepsilon}\}$. We want to show that $\Pr\{\mathcal{I}_{n,p} \models \neg\psi_1 \vee \neg\psi_2 \vee \dots \vee \neg\psi_m\} < 1$. If $n \geq N_\varepsilon$, then $\Pr\{\mathcal{I}_{n,p} \models \neg\psi_i\} < \varepsilon$ for all $i \leq m$. Hence $\Pr\{\mathcal{I}_{n,p} \models \neg\psi_1 \vee \neg\psi_2 \vee \dots \vee \neg\psi_m\} < m\varepsilon$. Now we just need to take $\varepsilon < \frac{1}{m}$, to establish the fact that there is a model of $\psi_1 \wedge \psi_2 \wedge \dots \wedge \psi_m$. Every finite subset of T^∞ has a model, so by the compactness theorem, T^∞ has a model. \square

Proposition 7.2 *T^∞ is a complete theory.*

Recall that a complete theory is a theory T such that, for all φ in the vocabulary considered, either $\varphi \in T$ or $\neg\varphi \in T$. We refer to [10], page 36.

Proof. If φ is a sentence such that $\Pr\{\mathcal{I}_{n,p} \models \varphi\} \rightarrow 0$ as $n \rightarrow \infty$, then by ordinary laws of probability, $\Pr\{\mathcal{I}_{n,p} \models \neg\varphi\} \rightarrow 1$ as $n \rightarrow \infty$. Hence, if $\varphi \notin T^\infty$ then $\neg\varphi \in T^\infty$. \square

Proposition 7.3 *All models of T^∞ are elementarily equivalent, that is $T^\infty \models \mathcal{M}, \mathcal{N} \Rightarrow \mathcal{M} \equiv \mathcal{N}$.*

Proof. By a wellknown theorem of model theory, all models of a theory are elementarily equivalent if and only if the theory is complete. We again refer to [10], this time page 112. \square

One usually takes $\exists^{\geq n} x \theta(x)$ as an abbreviation for

$$\exists x_1 x_2 \dots x_n \left(\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j \right) \wedge \left(\bigwedge_{1 \leq i \leq n} \theta(x_i) \right)$$

saying that there are n or more elements having the property θ . Let φ be the sentence $\exists^{\geq n} x (x = x)$. This sentence is true in all structures with n elements or more, therefore true in almost all structures. Since $\Pr\{\mathcal{I}_{n,p} \models \varphi\} \rightarrow 1$ as $n \rightarrow \infty$, φ is in T^∞ . Hence T^∞ has only infinite models. In fact, by different versions of the Löwenheim-Skolem theorems, there is a countable model, and there are even models in every infinite cardinal.

If one adds some extra conditions on the underlying graphs, it is possible to say a bit more about the limit theory:

Theorem 7.4 *Let $\{\mathcal{G}_n\}_{n=1}^\infty$ be a sequence of connected graphs with bounded degree, and let $\{\mathcal{I}_{n,p}\}_{n=1}^\infty$ be the images thus obtained. Then the limit theory T^∞ is not \aleph_0 -categorical.*

Recall that a theory is said to be λ -categorical if it has, up to isomorphism, only one model of cardinality λ . Also recall that the diameter of a graph is the maximum distance between any two vertices. We need the following basic lemma.

Lemma 7.5 *If $\{\mathcal{G}_n\}_{n=1}^\infty$ is a sequence of connected graphs with bounded degree (and where the index denotes the order of the graphs in the sequence), then the diameter tends to infinity as n tends to infinity.*

Proof. From any vertex there is a bounded number of vertices reachable in one step, in two steps, and so on. Since every pair of vertices are connected, by taking n large enough, the distance between vertices can be made arbitrarily large. \square

We will use the following characterization of \aleph_0 -categorical theories.

Theorem 7.6 *A theory T is \aleph_0 -categorical if and only if there are only finitely many pairwise non-equivalent formulas $\varphi(\bar{x})$ modulo T for each \bar{x} .*

Proof. This is part of the Engeler - Ryll-Nardzewski - Svenonius Theorem. We refer to [10], theorem 13.3.1. \square

Proof of theorem 7.4.. Let \mathcal{M} be a model of T^∞ . Further, let ψ_m be the sentence $\exists xy(d(x, y) = m)$, expressing that there are at least two elements at exactly the distance m from each other. We have

$$\Pr\{\mathcal{I}_{n,p} \models \psi_m\} \rightarrow 1 \text{ as } n \rightarrow \infty,$$

since the diameter of \mathcal{G}_n tends to infinity by lemma. Hence ψ_m belongs to the limit theory for all $m \in \mathbb{N}$. In \mathcal{M} , for any $m_1 \neq m_2$, we have at least two elements x and y such that $d(x, y) = m_1$ but $d(x, y) \neq m_2$. This shows that $d(x, y) = m_1$ is not \mathcal{M} -equivalent to $d(x, y) = m_2$ for any $m_1 \neq m_2$. This holds for all $\mathcal{M} \models T^\infty$, and so $d(x, y) = m_1$ is not T^∞ -equivalent to $d(x, y) = m_2$. Therefore there are infinitely many nonequivalent formulas modulo T^∞ . By theorem this implies that T^∞ is *not* \aleph_0 -categorical. \square

An interesting question about the limit theory is whether it is decidable or not, that is, if for any φ it is decidable whether $\varphi \in T^\infty$. Reconsider the class of structures in the example in section 4, which we called \mathbf{K} . In the example we noticed that \mathbf{K} is axiomatizable by the set of formulas $T_{\text{rand}}(\varphi_G)$.

Theorem 7.7 *The limit theory of the class of structures \mathbf{K} is decidable.*

Proof. The set of axioms $T_{\text{rand}}(\varphi_G)$ we defined as the set of sentences axiomatizing undirected graphs and all extension axioms compatible with it. This set is clearly computable. Hence the limit theory of \mathbf{K} is a recursively axiomatizable complete theory. By a result in model theory this shows that the limit theory is decidable (see [8], page 242, for another example how to use this result). \square

For the more interesting and more complicated cases studied in this thesis there does not seem to be any easy way to derive any decidability results. To say something about decidability one must encode information about for instance the function $p : \mathbb{N} \rightarrow [0, 1]$ and the sequence $\{\mathcal{G}_n\}_{n=1}^\infty$. Apparently there is no straight forward way to do this.

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