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Valuation of American put options with exercise restrictions

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Valuation of American put options with exercise restrictions

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Domingos Celso Djindja

Abstract

In this work we price an American put with exercise restriction on weekends. The idea is to remove weekends and shrink the interval (useful days) and so we may have jumps. Thus, we hedge and price it both on analytical valuation and numerical one. On this last, we focus essentially on find the optimal exercise boundary. We describe a version of the finite difference method given in B. Kim et all [5] and we extend it to find the boundary of an American put with exercise restriction on weekends.

Contents

Introduction

An option which the holder has the right to sell or buy it at any time along the time life of it is called American option. Thus, pricing an American option consists on find the optimal time to exercise it (optimal time to stop and exercise) which corresponds to the optimal value. We will focus on American put with finite maurity. Such problems may rely on optimal stoping problems. Many authors have been writing on optimal stoping theory, for example, on G. Peskir [7], American problems on formulation of stopping and free-boundary problems are treated. We will focus on the problem of pricing an American put by using free boundary problem formulation. The conversion from an optimal stopping to a free-boundary problem for pricing American put is done on G. Peskir [7], for instance. Since we want to optimize our gains, the strategy of pricing an American put consists on wait if the option value is bigger than the pay-off and exercise when on the opposite case. Acting so, there will be a boundary (critical stock price) on which we will have this two different situations, either wait or exercise the option. Such boundary is also part of the solution, this becomes a free-boundary problem. Therefore, pricing an American put also leads to find the boundary.

It is well known that there is no closed formula for pricing an American put. Thus, numerical methods have been used. Since such methods are approximations it is good to refine them and increase the algorithms efficiency. Therefore, many authors have been focussed on such numerical methods to price an American put and find the corresponding boundary. Mostly they use finite difference method versions, binomial and Monte Carlo methods, for example on Paul Willmott [9] the first two methods are explained. They rely on solving the solving numerically the Black-Scholes partial differential equation (see chapter 7, 21 [1]) or solve numerically the analytic valuation formula presented, for instance on G. Peskir [7], which the American put is represented as sum of European put and a premium for exercise early the option. For example, on J. E. Zhang [10], the boundary and the American put is determined numerically by solving the corresponding integral equation (similar to the one on Peskir [7]) numerically.

On this work, we will focus on determine the critical stock price (exercise boundary) which is an important task to problems of pricing an American put. We will use a version of finite difference method (FDM) presented in B. Kim et all [5]. Thus, we will present a very close description to this algorithm present on that paper and produce a code in maple for numerical experiments. This is done on the first chapter. Then, we will study on chapter 2 how would be the critical stock price of an American put when it is not allowed to exercise the option during weekends. For simplicity, we will work with Bermudan options, regard that we can only exercise the option on discrete points (days). Therefore, we can remove the weekends and find the option value only on useful week days. Thus, on a more general case and at a first glimpse, we may have jumps on the stock price. Thus, questions about completeness of the market, hedging and how to choose a martingale measure may arise. However, this jumps may arise on known dates (weekends). Incidently, we will follow some ideas from authors who treated more general cases, when the jumps arrive on unknown dates, for example R. Cont, P. Tankov [2] and C. R. Gukhal [4]. Mainly, we will study the critical stock price to this problem both analytical and numerical valuation. On the numerical one, we will extend the algorithm presented on B. J. Kim et all [5] to find the critical stock price.

Chapter 1

Pricing an American put numerically. Finite difference method

We will determine the optimal exercise boundary for American put options by finite difference method given in [5], B. J. Kim et all. The method is based on a Lipschitz surface near the free boundary which is designed by Q, see the definition afterwards. The Lipschitz surface avoids the degeneracy of the solution surface near free boundary. This function produces a linearly converging algorithm to locate the free boundary.

We start by considering an financial market characterized by a risky asset in a risk neutral economy with a constant risk-free interest rate $r > 0$ and price process $\{S(t), t \in (0, T)\}$ over the option's life $[0, T]$ that is specified by constant volatility rate $\sigma > 0$. Let the market measure be denoted by P, let $\{W(t), t \in (0, T)\}$ be a P-Brownian motion. The stock dynamics of the Black-Scholes model follows the geometric Brownian motion:

$$dS = Srdt + \sigma SdW.$$

Let $P(t, S; T)$ denote the value of an American put option as a function of the current time t, the current stock price S, and the maturity date T. The critical stock price $\beta(t; T),\ t \in [0, T]$ is defined as the largest price S at which the American put option value $P(t, S; T)$ equals its exercise value $K - S$, where K is the strike price. The Black-Scholes partial differential equation is the following:

$$\begin{gathered} P_t + \frac{1}{2}\sigma^2 S^2 P_{SS} + SP_S - rP = 0 \\ \text{when } S \in (\beta(t, T), \infty),\ t \in (0, T) \end{gathered} \tag{1.1}$$

satisfying

$$P(T, S, T) = \max\{K - S,\ 0\} \quad \text{and} \quad \beta(T, T) = K.$$

and the boundary conditions

$$\lim_{S\uparrow\infty} P(t,S,T) = 0,$$
$$\lim_{S\downarrow\beta(t,T)} P(t,S,T) = K - \beta(t,T),$$
$$\lim_{S\downarrow\beta(t,T)} P_S(t,S,T) = -1. \tag{1.2}$$

Boundary value problems arising in option valuation usually require one terminal condition and two boundary conditions to guarantee an unique solution. Let

$$\Omega_e = \{(t,S) \in (0,T) \times (S_{min}, S_{max}),\ S \le \beta(t,T)\}$$
$$\Omega_c = \{(t,S) \in (0,T) \times (S_{min}, S_{max}),\ S > \beta(t,T)\}$$

be the exercise and continuous region, respectively, (S_{min}, S_{max}) the interval of the stock price. By setting $Q = \sqrt{P - K + S}$, we see that $Q = 0$ on the free boundary and Ω_e. Also Q has Lipschitz character with no-singular and non-degeneracy property near free boundary. The degeneracy of solution surface causes instability and slow convergence of numerical algorithm. Therefore, Q is a natural candidate for computation in continuation region.

Since the solution surface in the exercise region is a horizontal plane and in the continuation region it is an inclined surface, we can then find the boundary. We transform P to Q and we solve the equation for the boundary. In order to find P, we solve the backward time Black-Scholes equation. To get the boundary, we first chop the interval $[0,T]$ in N equal subintervals such that $t_n = n\Delta t$, $n = 0,1,...,N$, $\Delta t = \frac{T}{N}$. We do the same to the stock price, we chop the interval (S_{min}, S_{max}) in M equal subintervals, such that $S_i^n = S_i(t_n)$ $i = 0,1,2,...M$, and $\Delta S = \frac{S_{max}-S_{min}}{M}$,

$$S_i^n = \beta_n + \rho\Delta S + i\Delta S, \qquad n = N,\ N-1, ..., 0, \quad i = 0,1,...,M,$$

$0 < \rho < 1$, is a constant parameter and the option price $P(t_n, S_i^n, T) = P_i^n$. S_i^n is computed after the free boundary β_{n+1} be computed, and $\beta_N = K$.

1.1 The equation for the boundary

We will now derive the boundary equation. By using the analyticity of Q in the continuation region, we decompose it as Taylor series $(Q(t_n, S_0^n, T))$ at $\beta_n(t,T)$:

$$Q(t_n, S_0^n, T) = Q(t_n, \beta_n, T) + Q_S(t_n, \beta_n, T)(S_0^n - \beta_n) + \frac{1}{2}Q_{SS}(t_n, \beta_n, T)(S_0^n - \beta_n)^2 + \\ + O((S_0^n - \beta_n)^3). \tag{1.3}$$

We need $Q_S,\ Q_{SS}$, thus, differentiating $P = Q^2 + K - S$, We obtain

$$\begin{aligned} P_t = 2QQ_t, \quad P_S = 2QQ_{SS} - 1, \quad P_{SS} = 2Q_S^2 + 2QQ_{SS}, \\ P_{SSS} = 6Q_SQ_{SS} + 2QQ_{SSS}. \end{aligned} \tag{1.4}$$

Thus, the Black-Scholes equation (1.1) is transformed to

$$2QQ_t + \frac{1}{2}\sigma^2 S^2(2Q_S^2 + 2QQ_{SS}) + rS(2QQ_S - 1) - r(Q^2 + K - S) = 0.$$

Since $Q \to 0$ when $S \downarrow \beta(t,T)$, we have

$$\frac{1}{2}\sigma^2 S^2(2Q_S^2) + rS(-1) - r(K - \beta(t,T)) = 0.$$

Thus, when $S \to \beta^+$,

$$Q_S^2 \to \frac{rK}{\sigma^2\beta^2(t,T)}$$

and from the boundary condition (1.2) we have

$$Q_S = \frac{P_S + 1}{2\sqrt{P - K + S}} > 0$$

on the continuation region. Therefore,

$$\lim_{S\to\beta^+} Q_S = \frac{\sqrt{rK}}{\sigma\beta(t,T)}. \tag{1.5}$$

In order to find Q_{SS}, we differenciate the Black-Scholes partial differential equation with respect to S, we have

$$P_{tS} + \sigma^2 SP_{SS} + \frac{1}{2}\sigma^2 S^2 P_{SSS} + rP_S + rSP_{SS} - rP_S = 0.$$

By simplifying it, we get

$$P_{tS} + \sigma^2 SP_{SS} + \frac{1}{2}\sigma^2 S^2 P_{SSS} + rSP_{SS} = 0. \tag{1.6}$$

Moreover, differentiating the boundary condition $P(t,\beta(t,T),T) = K - \beta(t,T)$, we get

$$P_S(t,\beta,T)\cdot\beta' + P_t(t,\beta,T) = -\beta'(t,T).$$

Since $P_S(\beta,t,T) = -1$ on the free boundary, by condition (1.2), we get

$$P_t(t,\beta,T) = 0.$$

On other hand, we differentiate $P_S(t,\beta,T) = -1$ with respect to t and we obtain

$$P_{SS}(t,\beta(t,T),T)\cdot\beta'(t,T) + P_{St}(\beta,t,T) = 0$$

and hence

$$P_{tS}(\beta,t,T) = -P_{SS}(t,\beta,T)\cdot\beta'(t,T). \tag{1.7}$$

Thus, using this last relation, (1.6) becomes

$$-P_{SS}\beta' + \sigma^2 SP_{SS} + \frac{1}{2}\sigma^2 S^2 P_{SSS} + rSP_{SS} = 0.$$

By using (1.4) when $S \to \beta^+$, we have

$$P_{SSS} \to 6Q_S Q_{SS}, \qquad P_{SS} \to 2Q_S^2$$

and also from (1.5), we have

$$\begin{aligned}
&-\beta'\cdot 2Q_S^2 + \sigma^2\beta\cdot 2Q_S^2 + \frac{1}{2}\sigma^2\cdot\beta^2\cdot 6Q_SQ_{SS} + r\beta\cdot 2Q_S^2 = 0\\
\Leftrightarrow\ &-\beta'\cdot\frac{2rK}{\sigma^2\beta^2} + \sigma^2\beta\cdot\frac{2rK}{\sigma^2\beta^2} + 3\sigma^2\cdot\beta^2\cdot\frac{\sqrt{rK}}{\sigma\beta}Q_{SS} + r\beta\cdot\frac{2rK}{\sigma^2\beta^2} = 0\\
\Leftrightarrow\ &Q_{SS} = -\frac{2\sqrt{rK}}{3\sigma^3\beta^3}[-\beta' + (\sigma^2+r)\cdot\beta].
\end{aligned} \tag{1.8}$$

By substituting (1.5), (1.8) in (1.3), as $S \to \beta(t,T)$ we get

$$\begin{aligned}
Q(t_n, S_0^n, T) &= \frac{\sqrt{rK}}{\sigma\beta_n}(S_0^n - \beta_n) - \frac{\sqrt{rK}}{3\sigma^3\beta_n^3}[-\beta_n' + (\sigma^2+r)\cdot\beta_n]\cdot(S_0^n-\beta_n)^2 + O((\rho\Delta S)^3) =\\
&= \frac{\sqrt{rK}}{\sigma}\left(\frac{S_0^n}{\beta_n} - 1\right) - \frac{\sqrt{rK}}{3\sigma^3}\left[-\frac{\beta_n'}{\beta_n} + \sigma^2 + r\right]\cdot\left(\frac{S_0^n}{\beta_n} - 1\right)^2 + O((\rho\Delta)^3),
\end{aligned} \tag{1.9}$$

where β_n is considered a differentiable function of t_n. We make the following approximation of $\frac{\beta_n'}{\beta_n}$:

$$\begin{aligned}
\frac{\beta_n'}{\beta_n} = (\ln\beta_n)' &= \frac{\ln\beta_{n+1} - \ln\beta_n}{\Delta t} + O(\Delta t)\\
&= \frac{\ln S_0^n - \ln\beta_n - (\ln S_0^n - \ln\beta_{n+1})}{\Delta t} + O(\Delta t)\\
&= \frac{\ln\frac{S_0^n}{\beta_n} - \ln\frac{S_0^n}{\beta_{n+1}}}{\Delta t} + O(\Delta t).
\end{aligned}$$

Since $\ln \dfrac{S_0^n}{\beta_n} = \ln\left(1 + \dfrac{S_0^n}{\beta_n} - 1\right) \sim \dfrac{S_0^n}{\beta_n} - 1$ when $S_0^n \to \beta_n^+$, We have then

$$\frac{\beta_n'}{\beta_n} = \frac{\left(\frac{S_0^n}{\beta_n} - 1\right) - \ln \frac{S_0^n}{\beta_{n+1}}}{\Delta t} + O(\rho \Delta S \Delta t). \tag{1.10}$$

By replacing (1.10) in (1.9), we get

$$\begin{aligned} Q(t_n, S_0^n, T) = & \frac{\sqrt{rK}}{\sigma}\left(\frac{S_0^n}{\beta_n} - 1\right) + \\ & + \frac{\sqrt{rK}}{3\sigma^3} \cdot \left[\frac{\left(\frac{S_0^n}{\beta_n} - 1\right) - \ln \frac{S_0^n}{\beta_{n+1}}}{\Delta t} - (\sigma^2 + r)\right] \cdot \left(\frac{S_0^n}{\beta_n} - 1\right)^2 + \\ & + O((\rho\Delta S)^2 \cdot (\rho \Delta S + \Delta t)). \end{aligned}$$

By setting

$$\xi = \frac{S_0^n}{\beta_n} - 1,$$

we obtain then

$$Q(t_n, S_0^n, T) = \frac{\sqrt{rK}}{\sigma}\xi + \frac{\sqrt{rK}}{3\sigma^3} \cdot \left[\xi - \ln \frac{S_0^n}{\beta_{n+1}} \Delta t - (\sigma^2 + r)\right] \cdot \xi^2 + \; O((\rho\Delta S)^2 \cdot (\rho \Delta S + \Delta t)).$$

After some transformations, we obtain the following equation

$$\xi^3 - \left(\ln \frac{S_0^n}{\beta_{n+1}} + (\sigma^2 + r)\Delta t\right)\xi^2 - 3\sigma^2 \Delta t \xi - \frac{3\sigma^3 \Delta t}{\sqrt{rK}} Q(t_n, S_0^n, T) = O((\rho\Delta S)^2 \cdot (\rho \Delta S + \Delta t)). \tag{1.11}$$

We just proved the following theorem:

Theorem 1.1.1. *Suppose that $Q(t_n, S_0^n, T)$ is known, then ξ satisfies the approximate cubic equation* (1.11).

After solving the cubic equation (1.11), we get β_n from the relation

$$\beta_n = \frac{S_0^n}{\xi + 1}$$

and from β_{n+1}, (we do it recursively). Therefore, we need to know wether such ξ exists. The following lemma also in B. Kim et all [5], states the sufficient conditions for the existence of ξ.

Lemma 1.1.1. *Suppose that $\ln\left(1 + \rho \dfrac{\Delta S}{\beta_{n+1}}\right) < \left(\dfrac{9\sigma^4 Q^2(t_n, S_0^n, T) - (\sigma^2 + r)}{rK}\right)\Delta t$ with $\sigma > 0$, $r > 0$. Then, for sufficiently small ΔS and Δt,*

$$\xi^3 - \left(\ln \frac{S_0^n}{\beta_{n+1}} + (\sigma^2 + r)\Delta t\right)\xi^2 - 3\sigma^2 \Delta t \xi - \frac{3\sigma^3 \Delta t}{\sqrt{rK}} Q(t_n, S_0^n, T) = 0.$$

has a unique real solution on $(0, 1) \subset \mathbb{R}$.

1.2 Numerical algorithm

By using the time and stock price discretization as before, Q as defined before, the numerical algorithm should then follow the steps:

1. Determine the option price P_i^{n-1}, $i = 1, 2, ..., M$ explicitly from P_i^n. The price P_i^n of the American put option is a discrete solution to the discrete Black-Scholes equation:

$$\frac{P_i^n - P_i^{n-1}}{\Delta t} + \frac{1}{2}\sigma^2 S_i^2 \frac{P_{i+1}^n - 2P_i^n + P_{i-1}^n}{\Delta S^2} + rS_i \frac{P_{i+1}^n - P_i^n}{\Delta S} - rP_i^n = 0$$

$$\text{for } n = N,\ N-1,\ ..., 1, \qquad S_i = S_i^n, \quad i = 1, 2, ..., M-1,$$
$$P_i^N = 0 \qquad \text{for} \quad i = 0, 1, 2, ..., M.$$

2. Find P_0^{n-1} by solving:

$$\frac{P_0^n - P_0^{n-1}}{\Delta t} + \frac{1}{2}\sigma^2 S_0^2 \left(\frac{\frac{P_1^n - P_0^n}{\Delta S} - \frac{P_0^n - P_{-1}^n}{\rho \Delta S}}{\frac{\Delta S + \rho \Delta S}{2}} \right) + rS_0 \left(\frac{P_1^n - P_{-1}^n}{\Delta S + \rho \Delta S} \right) - rP_0^n = 0$$

$$P_{-1}^n = K - \beta_n.$$

   Assuming the computational domain is large, we impose zero boundary condition $P_M^n = 0, \quad n = N,\ N-1, ..., 0.$

3. Determine $\beta_{n-1} = \dfrac{S_0^{n-1}}{1+\xi}$, where ξ is the solution for the equation

$$\xi^3 - \left\{ \ln \frac{S_0^{n-1}}{\beta_n} + (\sigma^2 + r)\Delta t \right\} \xi^2 + 3\sigma^2 \Delta t \xi - \frac{3\sigma^2 \Delta t}{\sqrt{rK}} Q(t_{n-1}, S_0^{n-1}, T) = 0, \qquad (1.12)$$

   which has a unique real root $\xi \in (0, 1)$.

4. Change the values of S^{n-1} from old $S_i^{n-1} = \beta_n + \rho \Delta S + i \Delta S$ to new $S_i^{n-1} = \beta_{n-1} + \rho \Delta S + i \Delta S$ for $i = 0, 1, 2, ..., M$. Then we update the values of P_i^{n-1} too. If we finish step 4, then we repeat the running step 1 through 4 until t_0.

By using the software maple, see the code in appendix A, a numerical experiment with $K = 1$, $r = 0.05$, $T = 0.5$, $\sigma = 0.2$, $\rho = 0.4$, $N = M = 100$, we get:

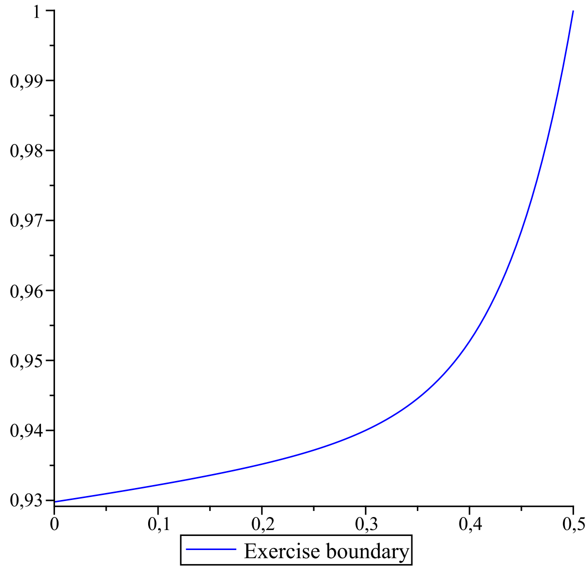


Fig. 1

Remark 1.2.1. For different values of the parameter ρ $(0 < \rho < 1)$ we may have slightly different boundaries but they tend to be the same as $\Delta S \to 0$ and $\Delta t \to 0$.

The corresponding plot for the option value at a fix time is given below.

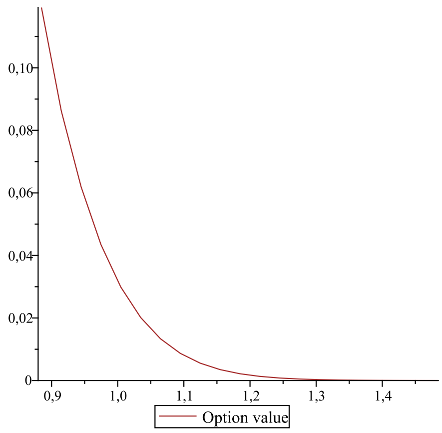


Fig. 2

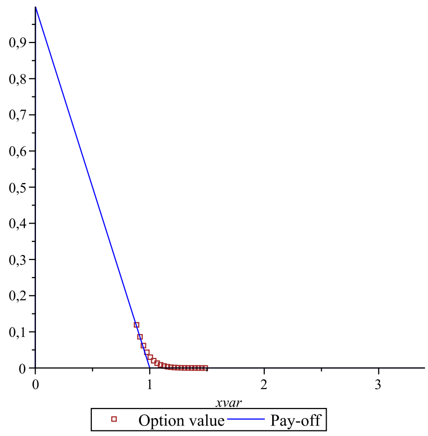


Fig. 3

Remark 1.2.2. It should be smoother for bigger values of M, i.e, when $\Delta S \to 0$.

Chapter 2

American put options with jumps

We start by presenting a very useful formula from Itô for jump-diffusion process, given in [2], Cont and Tankov (2004).

2.1 Itô formula for diffusion with jumps

Let X be a diffusion process with jumps defined as the sum of drift term, Brownian stochastic integral and a compound Poisson process:

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \sum_{i=1}^{N(t)} \Delta X_i,$$

where b_t and σ_t are continuous non-anticipating process with

$$E\left[\int_0^T \sigma_t^2 dt\right] < \infty$$

and the jumps on the stock price are given by ΔX_i.

Then, for any $C^{1,2}$ functions $f : [0,T]\times\mathbb{R} \to \mathbb{R}$, the process $Y_t = f(t, X_t)$ can be represented by

$$f(t, X_t) - f(0, X_0) = \int_0^t \left[\frac{\partial f}{\partial s}(s, X_s) + b_s \frac{\partial f}{\partial X}(s, X_s)\right] ds + \frac{1}{2}\int_0^t \sigma_s^2 \frac{\partial^2 f}{\partial x^2}(s, X_s) ds + \\ + \int_0^t \frac{\partial f}{\partial X}(s, X_s)\sigma_s dW_s + \sum_{i\geq 1,\ \tau_i \leq t} \left[f(\tau_i, X_{\tau_{i-}+\Delta X_i}) - f(\tau_i, X_{\tau_{i-}})\right]. \tag{2.1}$$

In differential notation:

$$dY_t = \frac{\partial f}{\partial t}(t, X_t)dt + b_t\frac{\partial f}{\partial x}(t, X_t)dt + \frac{\sigma_t^2}{2}\frac{\partial^2 f}{\partial X^2}(t, X_t)dt + \\ + \frac{\partial f}{\partial X}(t, X_t)\sigma_t dW_t + [f(t, X_{t_-} + \Delta X_t) - f(t, X_{t_-})]. \quad (2.2)$$

2.2 The stock price for the problem in analysis

We regard a problem of pricing an American put when we are not allowed to exercise the option during the weekends. We suppose that have a standard brownian motion $[W(t)]$ under a complete probability space $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)_{t\geq 0}$ is a filtration which satisfy the usual conditions. Furthermore, we consider a financial market with a constant risk-free interest rate $r > 0$ and a stock price $S(t)$ follows a geometric brown motion.

In order to deal with the exception of exercising the American put option during weekends, we suppose the following cases:

1. The stock price is traded continuously at any time;

2. The stock price cannot be traded during the weekends but during the week it is continuously traded.

Case 1: The stock price is traded continuously at any time.

Suppose that we the stock price is a geometric brownian motion, it is continuous and follows the dynamic:

$$dS(t) = \mu S dt + \sigma S dW(t), \qquad t \in [0, T],$$

where μ is the drift, σ is the volatility, $W(t)$ is a standard brownian motion.

Since we are not allowed to exercise the option during the weekends, we will remove the weekends and consider the stock price during the week. Therefore, we may have jumps from Friday to Monday since the price may change during the weekend. Since S is a geometric brownian motion, with σ, μ constants, by Bjork (Chapter 4, 2003) [1], we have

$$S(t_2) = S(t_{1_-}) \exp\left\{(\mu - \sigma^2/2)(t_2 - t_1) + \sigma[W(t_2) - W(t_1)]\right\},$$

for $t_1 < t_2$ and μ, σ constants.

Thus, if we regard τ_1 as a Friday and τ_2 as the next useful day (Monday), we have

$$S(\tau_2) = S(\tau_{1_-}) \cdot exp\left\{(\mu - \sigma^2/2)(\tau_2 - \tau_1) + \sigma[W(\tau_2) - W(\tau_1)]\right\}.$$

Therefore, the jump size of the stock price from τ_1 to τ_2 is given by

$$\Delta S = S(\tau_2) - S(\tau_{1_-}) = S(\tau_{1_-})\left[exp\{(\mu - \sigma^2/2)(\tau_2 - \tau_1) + \sigma(W(\tau_2) - W(\tau_1))\} - 1\right] = \\ = S(\tau_{1_-})(Y - 1),$$

where

$$Y = exp\{(\mu - \sigma^2/2)(\tau_2 - \tau_1) + \sigma[W(\tau_2) - W(\tau_1)]\}.$$

We regard now that there are n weekend over the interval $[0,\, T]$, and we order them as

$$(\tau_1,\, \tau_2), ..., (\tau_{2n-1},\, \tau_{2n}).$$

The jump size at each interval is given by

$$Y_i = exp\{(\mu - \sigma^2/2)(\tau_{2i} - \tau_{2i-1}) + \sigma[W(\tau_{2i}) - W(\tau_{2i-1})]\}.$$

By removing the weekends on the stock price, we will have the following dynamic

$$\frac{dS(t)}{S(t_-)} = \mu dt + \sigma dW(t) + dJ(t), \qquad t \in [0, T] \setminus \{\cup_{i=1}^{n} (\tau_{2i-1}, \tau_{2i})\},$$

where

$$J(t) = \sum_{i=1}^{n(t)} (Y_i - 1), \qquad n(t) \text{ is the number of weekends up to time } t$$

and

$$Y_i = exp\{(\mu - \sigma^2/2)(\tau_{2i} - \tau_{2i-1}) + \sigma[W(\tau_{2i-1}) - W(\tau_{2i-1})]\}.$$

In other to simplify notations and for calculations proposes, We regard that the stock price has the dynamics:

$$\frac{dS(t)}{S(t_-)} = \mu dt + \sigma dW(t) + [Y(t) - 1]dn(t),$$

where $n(t)$ is one if we have jump at time t (i.e. if $t = \tau_{2j}$ the time just after a weekend), it is zero otherwise and $S(t_-)[Y(t) - 1]$ represents the jump at time t. We suppose that dW and dn are independent. We will now solve the stochastic differential equation above by following the standard method for the case without any jump.

Therefore, by changing the variables $Z(t) = \ln S(t)$, by Itô we have

$$\begin{aligned} dZ &= \frac{1}{S} dS + \frac{1}{2}\left(-\frac{1}{S^2}\right)(dS)^2 \\ &= \frac{1}{S}\left(\mu S dt + \sigma S dW + S(Y-1)dn(t)\right) - \\ &\quad -\frac{1}{2S^2}\left(\mu S dt + \sigma S dW + S(Y-1)dn(t)\right)^2 \\ &= \mu dt + \sigma dW + (Y-1)dn(t) - \frac{\sigma^2}{2} dt, \end{aligned}$$

since

$$(dW)^2 = dt, \quad dt^2 = 0, \quad dtdw = 0,$$
$$dtdn(t) = 0, \quad [dn(t)]^2 = 0, \quad dWdn(t) = 0.$$

Integrating both sides we get:

$$\int_0^t dZ = \int_0^t [\mu - \sigma^2/2]ds + \int_0^t \sigma dWs + \sum_{i=1}^{n(t)} (Y_i - 1)$$

$$\ln S(t) = \ln S(0) + (\mu - \sigma^2/2)t + \sigma W(t) + \sum_{i=1}^{n(t)} (Y_i - 1)$$

$$S(t) = S(0) * exp\left[(\mu - \sigma^2/2)t + \sigma W(t) + \sum_{i=1}^{n(t)} (Y_i - 1)\right]. \tag{2.3}$$

In order to avoid arbitrage, We need the model to be a martingale. Without removing the weekends we have the standard Black-Scholes model which is complete and with a unique martingale measure. By removing the weekends and shrinking the time interval we may have a different scenario, i.e, some jumps on the stock price may appear at known dates. Since we know that $e^{-rt}S(t)$ is a martingale (in this case $\mu = r$ on (2.3)), it should be natural to it keep so even with the referred possible jumps since the stock price is continuously traded. Then, the jumps must be a martingale. So,

$$E[S(\tau_{i-})Y_i(t)|\mathcal{F}_s] = S(\tau_{i-}), \qquad s < t, \quad i = 1, ..., n(T),$$

$\{\mathcal{F}_t\}_{t\geq 0}$ is the information flow up to time t. The last formula is equivalent to

$$E[Y_i(t)|\mathcal{F}_s] = 1. \tag{2.4}$$

Thus,

$$e^{-rt}E[S(t)|\mathcal{F}_s] = e^{-rt} \cdot S(0) \cdot e^{rt} \cdot e^{\left[\sum_{i=1}^{n(t)} (E[Y_i|\mathcal{F}_s]-1)\right]} = S(0).$$

From condition (2.4), follows that the interest rate should be zero along the weekend. Therefore,

$$Y_i = exp\{-\sigma^2/2(\tau_{2i} - \tau_{2i-1}) + \sigma[W(\tau_{2i-1}) - W(\tau_{2i-1})]\}.$$

Remark 2.2.1. The market is still complete since the stock is traded continuously and the adjustments that we do only are made in order to compute the option price which cannot be exercised during weekends.

Case 2: The stock price cannot be traded during the weekends but during the useful week days it is continuously traded.

As before, we suppose that the stock price is a geometric brownian motion and has the following dynamic:

$$\frac{dS(t)}{S(t_-)} = \mu dt + \sigma dW(t), \qquad t \in [0, T] \setminus \{\cup_{i=1}^n (\tau_{2i-1}, \tau_{2i})\},$$

where (τ_{2i-1}, τ_{2i}) $i = 1, 2, ..., n(T)$ are weekends.

Since it is not traded during the weekends, after a weekend we may have a different value from the last one we end up with on the week before. There are many reasons that may be in the origin of this change, for instance, a political decision, a natural catastrophe, a terrorist attack, of course, depending on the asset on trading. Therefore, it is more convenient to consider that there will be a jump from Friday to Monday. Then, we have more random sources than traded assets. So, the market is incomplete and we may have then many martingale measures. However, if we suppose that the jumps are given by a stochastic variable which has log-normal distribution since the stock price has log normal distribution during the useful week days, we would have a bit similar case with the previous one. Thus, the value of the stock price just after a weekend is

$$S(\tau_{2i^+}) = S(\tau_{2i-1^-})e^{a+b\cdot Z(t)},$$

where a, b are constants and $Z(t) \sim N(0,1)$, i.e, $Z(t)$ has standard normal distribution.

Therefore, a similar argument as on the previous case to avoid arbitrage, we must have the jumps to be martingale

$$E[S(\tau_{2i-1^-})e^{a+b\cdot Z(t)}|\mathcal{F}_s] = S(\tau_{2i-1^-}), \qquad s < t$$

which implies that

$$E[e^{a+bZ(t)}|\mathcal{F}_s] = 1.$$

Consequently, we have $a = -b^2/2$. Since the jumps should reflect the stock price behavior during the weekend if it is traded along this time, then the natural value for b is the corresponding coefficient of a standard brownian motion that we have along the week which is

$$b = [W(t_{i+1}) - W(t_i)] \cdot \sigma, \quad t_i < t_{i+1}.$$

By introducing these possible jumps under martingale measure in $S(t)$, we have

$$S(t) = S(0) \cdot exp\left[(r - \sigma^2/2)t + \sigma W(t) + \sum_{i=1}^{n(t)}(e^{-b^2/2+bZ(\tau_{2i^+})} - 1)\right],$$

where

$$b = \sigma \cdot [W(\tau_{2i}) - W(\tau_{2i-1})], \quad i = 1, 2, .., n(T).$$

Remark 2.2.2. Both cases have similar (equal) formulas. However they are different, on the first one the market is complete and the second one not. Nevertheless, by choosing the martingale measure as we did on both cases, we have the same price process. Therefore, from now on we will treat them as a unique case, i.e, the stock price is given by

$$S(t) = S(0) \cdot exp\left[(r - \sigma^2/2)t + \sigma W(t) + \sum_{i=1}^{n(t)}(Y_i - 1)\right].$$

where

$$Y_i = exp\{(-\sigma^2/2)(\tau_{2i} - \tau_{2i-1}) + \sigma[W(\tau_{2i}) - W(\tau_{2i-1})]\}, \quad i = 1, 2, ..., n(T).$$

2.3 The pricing problem

We will now price an American option when the stock price is given by the last two cases and suppose that the strike price is K. We regard that $P(t,S)$ is the option price. By applying Itô formula (2.2), we have:

$$dP = \left(\frac{\partial P}{\partial t} + \mu S\frac{\partial P}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 P}{\partial S^2}\right)dt + \sigma S\frac{\partial P}{\partial S}dW(t) + [P(t,S(t)) - P(t,S(t_-))]dn(t).$$

Let us now make a Δ-hedged portfolio and we regard $\delta = \dfrac{\partial P}{\partial S}$:

$$\Pi(t) = P(t) - \delta \cdot S(t).$$

We have thus,

$$\begin{aligned} d\Pi(t) = dP - \delta dS &= \left(\frac{\partial P}{\partial t} + \mu S\frac{\partial P}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 P}{\partial S^2}\right)dt + \sigma S\frac{\partial P}{\partial S}dW(t) + \\ &+[P(t,S(t)) - P(t,S(t_-))]dn(t) - \frac{\partial P}{\partial S}S(\mu dt + \sigma dW(t) + (Y-1)dn(t)) \\ &= \left(\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 P}{\partial S^2}\right)dt + [\Delta P(t,S(t)) - \delta\cdot\Delta S]\,dn(t). \end{aligned}$$

In order to avoid arbitrage, the expected return of the hedged portfolio must be equal to the value of the portfolio invested at risk-free interest rate r. Therefore,

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 P}{\partial S^2} + E[\Delta P(t,S) - \delta\Delta S(t)|\mathcal{F}_s]\cdot I_{(\tau_{2i-1},\tau_{2i})}(t) = r(P - \delta S),$$

$s < t$, $I_{(a,b)}(t)$ is the indicator function of the interval (a,b). Since each $Y_i - 1$ is a martingale, we have

$$E[\delta\Delta S(t)|\mathcal{F}_s] = E[\delta S(\tau_{2i-1^-})(Y_i - 1)|\mathcal{F}_s] = \delta S(\tau_{2i-1^-})E[Y_i - 1|\mathcal{F}_s] = 0.$$

Thus, for the $e^{-rt}P(t,S)$ be a martingale, we impose the condition

$$E[\Delta P(t,S(t)\cdot I_{(\tau_{2i-1},\tau_{2i})}|\mathcal{F}_s] = 0, \quad i = 1,2,...,n(T),$$

which is equivalent to

$$E[P(\tau_{2i-1^-}, S(\tau_{2i-1^-})Y_i)|\mathcal{F}_s] = P(\tau_{2i-1^-}, S(\tau_{2i-1^-})), \quad i = 1,2,...,n(T).$$

Therefore, the pricing problem becomes

$$\begin{cases} \dfrac{\partial P}{\partial t} + \dfrac{1}{2}\sigma^2 S^2\dfrac{\partial^2 P}{\partial S^2} + rS\dfrac{\partial P}{\partial S} - rP = 0, \quad t \in (0,T) \\ E[P(\tau_{2i-1^-}, SY_i)|\mathcal{F}_s] = P(\tau_{2i-1^-}, S(\tau_{2i-1^-})), \quad i = 1,2,...,n(T), \\ P(T,S) = \max\{K - S,\, 0\}. \end{cases} \tag{2.5}$$

Remark 2.3.1. For the smooth pasting condition, we may use the result of a more general case in a jump-diffusion model. In H. Pham [8] for example, it was proved that for a jump-diffusion model with positive volatility ($\sigma > 0$) and finite jump intensity the value of an American put $P(t, S)$ is continuously differentiable with respect to the underlying asset on $[0, T] \times [0, \infty)$ and in particular the derivative is continuous across the exercise boundary:

$$\frac{\partial P}{\partial S}(t', S) \to -1, \qquad \text{when } (t', S) \to (t, S^*(t)),$$

for every t in $[0, T]$, where S^* is the critical stock price (exercise boundary).

We will have both analytical and numerical valuation for this problem.

2.3.1 Analytical valuation

We will now derive a formula for the option price under the stock price defined on the previous section. Consider an American put on the corresponding asset with strike price K and maturity time T. We consider the value of the American put at time $t = T - t'$ as $P_A(t',\ S)$, which is taken on the space $D = \{(t', S) : S \in (0, \infty),\ t' \in [0, T]\}$. There is a critical stock price S^* (exercise boundary) at each time $t \in [0, T]$ such that it is optimal to exercise the option when $S \leq S^*$ and it should continue otherwise. Thus, the American put can be written as

$$P_A(t', S) = \begin{cases} K - S(t), & \text{if } S(t) \leq S^*(t) \\ \\ P_A(t', S) > K - S(t), & \text{otherwise} \end{cases},$$

where $t = T - t'$.

We rewrite the pricing problem as

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0, \quad t \in (0, T) \tag{2.6}$$

satisfying

$$E[P(\tau_{2i-1^-}, SY_i)|\mathcal{F}_s] = P(\tau_{2i-1^-}, S(\tau_{2i-1^-})), \quad i = 1, 2, ..., n(T), \tag{2.7}$$

with the terminal and boundary conditions (also known as smooth-pasting conditions)

$$\begin{cases} P(T, S) = \max\{K - S,\ 0\} \\ \\ \lim\limits_{S \to \infty} P(t, S) = 0 \\ \\ \lim\limits_{S \to S^*} P(t, S) = K - S^* \\ \\ \lim\limits_{S \to S^*} \frac{\partial P}{\partial S}(t, S) = -1. \end{cases} \tag{2.8}$$

Let $F(t,S(t)) = e^{-rt}P(t,S(t))$, where P is the value of the American put option defined on the space D. By using the martingale measure for the stock price S and Itô formula (2.2), we have

$$\begin{aligned}
dF &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial S}(rSdt + \sigma SdW(t)) + \frac{\sigma^2}{2}\frac{\partial^2 F}{\partial S^2}dt + [F[(S(t_-) + \Delta S)] - F(S(t_-))] \\
\Longleftrightarrow dF &= e^{-rt}\left(-rP + \frac{\partial P}{\partial t} + rS\frac{\partial P}{\partial S} + \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial S^2}\right)dt + \\
&\quad + e^{-rt}\sigma S\frac{\partial P}{\partial S}dW(t) + e^{-rt}[P(S(t_-) + \Delta S) - P(S(t_-)].
\end{aligned}$$

By integrating along the interval $[0,T]$, we have

$$\begin{aligned}
F(T,S) &= F(0,S) + \int_0^T e^{-rt}\left(-rP + \frac{\partial P}{\partial t} + rS\frac{\partial P}{\partial S} + \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial S^2}\right)dt + \int_0^T e^{-rt}\sigma S\frac{\partial P}{\partial S}dW(t) + \\
&\quad + \sum_{i=1}^{n(T)} e^{-r\tau_{2i-1}}[P(Y_i S(\tau_{2i-1})) - P(S(\tau_{2i-1}))].
\end{aligned}$$

By substituting F we get:

$$\begin{aligned}
e^{-rT}P(T,S) &= P(0,S) + \int_0^T e^{-rt}\left(\frac{\partial P}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP\right)dt + \int_0^T e^{-rt}\sigma S\frac{\partial P}{\partial S}dW(t) + \\
&\quad + \sum_{i=1}^{n(T)} e^{-r\tau_{2i-1}}[P(Y_i S(\tau_{2i-1})) - P(S(\tau_{2i-1}))].
\end{aligned}$$

By using the critical stock price, we can rewrite $P(t,S)$ as follows

$$P(t,S) = I_{\{S>S^*\}}P(t,S) + I_{\{S\leq S^*\}}(K - S(t)),$$

where $I_{(a,b)}$ is the indicator function of the interval (a,b).

Using the above formula and boundary conditions, we get

$$e^{-rT}(K-S)^+ = P(0,S) + \int_0^T e^{-rt} \cdot I_{\{S>S^*\}} \left(\frac{\partial P}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP \right) dt -$$
$$-Kr\int_0^T e^{-rt} I_{\{S\le S^*\}}dt + \int_0^T e^{-rt}\sigma S\frac{\partial P}{\partial S}dW(t) +$$
$$+\sum_{i=1}^{n(T)} e^{-r\tau_{2i-1}} \cdot I_{\{S>S^*\}}[P(Y_i S(\tau_{2i-1})) - P(S(\tau_{2i-1}))] +$$
$$+\sum_{i=1}^{n(T)} e^{-r\tau_{2i-1}} \cdot I_{\{S\le S^*\}}[P(Y_i S(\tau_{2i-1})) - P(S(\tau_{2i-1}))].$$

Since on the continuation region the American put satisfies the Black-Scholes partial differential equation, then the first integral is zero. Thus, we get

$$e^{-rT}(K-S)^+ = P(0,S) - Kr\int_0^T e^{-rt} I_{\{S\le S^*\}}dt + \int_0^T e^{-rt}\sigma S\frac{\partial P}{\partial S}dW(t) +$$
$$+\sum_{i=1}^{n(T)} e^{-r\tau_{2i-1}} \cdot I_{\{S>S^*\}}[P(Y_i S(\tau_{2i-1})) - P(S(\tau_{2i-1}))] +$$
$$+\sum_{i=1}^{n(T)} e^{-r\tau_{2i-1}} \cdot I_{\{S\le S^*\}}[P(Y_i S(\tau_{2i-1})) - P(S(\tau_{2i-1}))].$$

Taking expectations both sides and considering that

$$E\left[\int_0^T e^{-rt}\sigma S\frac{\partial P}{\partial S}dW(t)\right] = 0$$

by the proposition 4.4 (in Björk [1]), we thus obtain

$$P_E = P_A - Kr\int_0^T e^{-rt} E(I_{\{S<S^*\}})dt + \sum_{i=1}^{n(T)} e^{-r\tau_{2i-1}} \cdot E[I_{\{S>S^*\}}[P(Y_i S(\tau_{2i-1})) - P(S(\tau_{2i-1}))]] +$$
$$+\sum_{i=1}^{n(T)} e^{-r\tau_{2i-1}} \cdot E[I_{\{S\le S^*\}}[P(Y_i S(\tau_{2i-1})) - P(S(\tau_{2i-1}))]],$$

where $P_E = P_E(S,T)$, $P_A = P_A(S,0)$ are the European and the America put options, respectively.

Since we want the American put, we have

$$P_A = P_E + Kr\int_0^T e^{-rt}E(I_{\{S\leq S^*\}})dt - \sum_{i=1}^{n(T)} e^{-r\tau_{2i-1}} \cdot E[I_{\{S>S^*\}}[P(Y_iS(\tau_{2i-1})) - P(S(\tau_{2i-1}))]]-$$
$$-\sum_{i=1}^{n(T)} e^{-r\tau_{2i-1}} \cdot E[I_{\{S\leq S^*\}}[P(Y_iS(\tau_{2i-1})) - P(S(\tau_{2i-1}))]].$$

This we can write as

$$\begin{aligned}
P_A = P_E + Kr\int_0^T e^{-rt}E(I_{\{S\leq S^*\}})dt- \\
-\sum_{i=1}^{n(T)} e^{-r\tau_{2i-1}} \cdot E[I_{\{S>S^*,YS>S^*\}}[P(Y_iS(\tau_{2i-1})) - P(S(\tau_{2i-1}))]]- \\
-\sum_{i=1}^{n(T)} e^{-r\tau_{2i-1}} \cdot E[I_{\{S>S^*,YS\leq S^*\}}[P(Y_iS(\tau_{2i-1})) - P(S(\tau_{2i-1}))]]- \\
-\sum_{i=1}^{n(T)} e^{-r\tau_{2i-1}} \cdot E[I_{\{S\leq S^*,\,YS\leq S^*\}}[P(Y_iS(\tau_{2i-1})) - P(S(\tau_{2i-1}))]]- \\
-\sum_{i=1}^{n(T)} e^{-r\tau_{2i-1}} \cdot E[I_{\{S\leq S^*,\,YS>S^*\}}[P(Y_iS(\tau_{2i-1})) - P(S(\tau_{2i-1}))]] \\
= P_E + Kr\int_0^T e^{-rt}E(I_{\{S\leq S^*\}})dt- \\
-\sum_{i=1}^{n(T)} e^{-r\tau_{2i-1}} \cdot E[I_{\{S>S^*,YS\leq S^*\}}[P(Y_iS(\tau_{2i-1})) - P(S(\tau_{2i-1}))]]- \\
-\sum_{i=1}^{n(T)} e^{-r\tau_{2i-1}} \cdot E[I_{\{S\leq S^*,\,YS>S^*\}}[P(Y_iS(\tau_{2i-1}-)) - P(S(\tau_{2i-1}))]].
\end{aligned}$$

Since on the exercise region the price is equal to the pay-off function, we have

$$
\begin{aligned}
P_A = P_E + Kr \int_0^T e^{-rt} E(I_{\{S \le S^*\}}) dt - \\
-\sum_{i=1}^{n(T)} e^{-r\tau_{2i-1}} \cdot E[I_{\{S>S^*, YS \le S^*\}}[K - YS(\tau_{2i-1^-}) - P(S(\tau_{2i-1}))]] - \\
-\sum_{i=1}^{n(T)} e^{-r\tau_{2i-1}} \cdot E[I_{\{S \le S^*, YS>S^*\}}[P(Y_i S(\tau_{2i-1^-})) - K + S(\tau_{2i-1^-})]].
\end{aligned}
$$

We need to simplify the early exercise premium, since it is taken only on useful week days. We may write it as

$$
Kr \int_0^T e^{-rt} E(I_{\{S \le S^*\}}) dt = Kr \left(\int_0^{\tau_1} e^{-rt} E(I_{\{S \le S^*\}}) dt + \sum_{i=1}^{n(T)} \int_{\tau_{2i}}^{\tau_{2i-1}} e^{-rt} E(I_{\{S \le S^*\}}) dt \right),
$$

here we suppose without loss of generality that T corresponds to a Friday. Thus, taking in consideration that the interest rate over the weekends is zero, we obtain

$$
\begin{aligned}
P_A = P_E + Kr \left(\int_0^{\tau_1} e^{-rt} E(I_{\{S \le S^*\}}) dt + \sum_{i=1}^{n(T)} \int_{\tau_{2i}}^{\tau_{2i-1}} e^{-rt} E(I_{\{S \le S^*\}}) dt \right) - \\
-\sum_{i=1}^{n(T)} E[I_{\{S>S^*, YS \le S^*\}}[K - YS(\tau_{2i-1^-}) - P(S(\tau_{2i-1}))]] - \\
-\sum_{i=1}^{n(T)} E[I_{\{S \le S^*, YS>S^*\}}[P(Y_i S(\tau_{2i-1^-})) - K + S(\tau_{2i-1^-})]].
\end{aligned}
$$

Remark 2.3.2. The solution is a sum of an European put, a premium for early exercise and some negative components. These last components correspond to the cases when the stock price jumps from the exercise region to continuation region and vice-versa without across the boundary. We can interpret it as a loss for not exercise the option at that time.

The critical stock price is given when $S(t) = S^*(t)$, thus it is the solution to the following equation

$$P_A(S^*,t) = P_E(S^*,t) + Kr\int_0^t e^{-r\xi}E(I_{\{S(t)\le S^*(t-\xi)\}})d\xi -$$

$$-\sum_{i=1}^{n(t)} E[I_{\{S>S^*,YS\le S^*\}}[K - YS^*(\tau_{2i-1}-) - P(S^*(\tau_{2i-1}))]]-$$

$$-\sum_{i=1}^{n(t)} E[I_{\{S\le S^*,YS>S^*\}}[P(Y_iS^*(\tau_{2i-1}-)) - K + S^*(\tau_{2i-1}-)]].$$

Since the first integral should be taken only on useful week days, we get:

$$K - S^*(t) = P_E(S_t^*,t) + Kr\left(\int_0^{\tau_1} e^{-r\xi}E(I_{\{S(t)\le S^*(t-\xi)\}})\cdot I_{\{t\le\tau_1\}}d\xi + \right.$$

$$\left. + \sum_{i=1}^{n(t)} \int_{\tau_{2i}}^{\tau_{2i-1}} e^{-r(\xi)}E(I_{\{S\le S^*\}})d\xi \right) -$$

$$-\sum_{i=1}^{n(t)} E[I_{\{S>S^*,YS\le S^*\}}[K - YS^*(\tau_{2i-1}-) - P(S^*(\tau_{2i-1}))]]-$$

$$-\sum_{i=1}^{n(t)} E[I_{\{S\le S^*,YS>S^*\}}[P(Y_iS^*(\tau_{2i-1}-)) - K + S^*(\tau_{2i-1}-)]],$$

where

$$P_E(S^*(t),t) = Kexp(-rt)N(-d_2) - S^*(t)N(-d_1),$$

$$d_1 = \frac{\ln(S^*/K) + (r+\sigma^2/2)t}{\sigma\sqrt{t}}, \qquad d_2 = d_1 - \sigma\sqrt{t}$$

and $N(x)$ is a cumulative distribution function for a standard normal random variable defined by

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} exp\left(-\frac{u^2}{2}\right)du.$$

2.3.2 Numerical valuation

We will rely on the numerical method presented in B. J. Kim (2013) [5] as before to price this American put option with jumps. We will use Bermudan options, regarding that it is only exercisable at discrete time points (days) which (τ_{2i-1}, τ_{2i}) $i = 1, 2, ..., n(T)$ correspond to weekends. First of all, we will make some transformations on the expectation of the price variation along the jumps. We have,

$$E[P(\tau_{2i-1}-, SY_i)|\mathcal{F}_s] = P(\tau_{2i-1}-, S(\tau_{2i-1}-)), \quad i = 1, 2, ..., n(T),$$

then if we suppose that the price just after the jump is equal to the one just before the jump we will have

$$P(\tau_{2i-1}-, SY_i) = P(\tau_{2i-1}-, S(\tau_{2i-1}-)), \quad i = 1, 2, ..., n(T).$$

Therefore, regarding as before $\beta(t, T)$ the critical stock price, we have the following free boundary problem:

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0, \quad t \in (0, T), \ S \in (\beta, \infty)$$

satisfying

$$P(\tau_{2i-1}-, SY_i) = P(\tau_{2i-1}-, S(\tau_{2i-1}-)), \quad i = 1, 2, ..., n(T),$$

moreover

$$P(T, S, T) = \max\{K - S, 0\}, \qquad \beta(T, T) = K,$$

and the boundary conditions

$$\lim_{S\uparrow\infty} P(t, S, T) = 0,$$
$$\lim_{S\uparrow\beta} P(t, S, T) = K - \beta(t, T),$$
$$\lim_{S\downarrow\beta} \frac{\partial P(t, S, T)}{\partial S} = -1.$$

For a numerical treatment, we need to generate the jumps. Since they contain brownian motion or normal distribution in their formula and a deterministic part, we only need to generate brownian motion paths and simulate normal distribution. We chop the interval $[0, T]$ in N equal subintervals such that $t_n = n\Delta t$, $n = 0, 1, ..., N$, $\Delta t = \frac{T}{N}$. We interpret each t_n as one day. By using the random walk construction in P. Glasserman (chapter 3.2, [3]), we have

$$W(t_{i+1}) = W(t_i) + \sqrt{t_{i+1} - t_i} Z_{i+1}, \qquad i = 0, ..., N-1,$$

where $0 = t_0 < t_1 < ... < t_n$, $W(0) = 0$, $Z_1, Z_2, ..., Z_n$ are independent normal distribution. Thus,

$$W(t_{i+1}) - W(t_i) = \sqrt{t_{i+1} - t_i} Z_{i+1}, \qquad i = 0, ..., N-1.$$

We have therefore,

$$
\begin{aligned}
Y_i &= exp[-\sigma^2/2(\tau_{2i} - \tau_{2i-1}) + \sigma\sqrt{\tau_{2i} - \tau_{2i-1}} Z_i] \\
&= exp\left[\frac{3T}{N}(-\sigma^2/2) + \sigma\sqrt{\frac{3T}{N}} \cdot Z_i\right], \qquad i = 1, 2, ..., N,
\end{aligned}
$$

where Z_1, Z_2, ..., Z_n are independent standard normal random distribution. At each weekend, we can simulate a sample of jumps and determine its mean. Then, we regard the sample mean as a jump at each weekend.

We follow then the algorithm for pricing a standard American put option (see chapter 1), but with some adjustments because of the possible jumps on the stock price. Let Ω_e, Ω_c be the exercise and continuous region, respectively, (S_{min}, S_{max}) the interval of the stock price. By setting $Q = \sqrt{P - K + S}$, we see that $Q = 0$ on the free boundary and Ω_e.

The method consists on solving the backward time Black-Scholes equation. We consider a variable weekdayforT, the starting useful day T as the number of days backward up to Sunday . We chop the interval of the stock price, $[S_{min}, S_{max}]$ in M equal subintervals, such that $S_i^n = S_i(t_n)$ $i = 0, 1, 2, ...M$, and $\Delta S = \frac{S_{max} - S_{min}}{M}$,

$$
S_i^n = \beta_n + \rho\Delta S + i\Delta S, \qquad n = N,\ N-1, ..., 0, \quad i = 0, 1, ..., M, \quad 0 < \rho < 1, \ (\text{constant})
$$

and the option price $P(t_n, S_i^n, T) = P_i^n$. S_i^n is computed after the free boundary β_{n+1} be computed, and $\beta_N = K$. Then we proceed by following the steps:

1. Determine the option price P_i^{n-1}, $i = 1, 2, ..., M$ explicitly from P_i^n. The price P_i^n of the American put option is a discrete solution to the discrete BlackŰ Scholes equation:

$$
\frac{P_i^n - P_i^{n-1}}{\Delta t} + \frac{1}{2}\sigma^2 S_i^2 \frac{P_{i+1}^n - 2P_i^n + P_{i-1}^n}{\Delta S^2} + rS_i \frac{P_{i+1}^n - P_i^n}{\Delta S} - rP_i^n = 0
$$

$$
\text{for } n = N,\ N-1,\ ..., 1, \qquad S_i = S_i^n, \quad i = 1, 2, ..., M-1,
$$
$$
P_i^N = 0 \qquad \text{for} \quad i = 0, 1, 2, ..., M.
$$

2. Find P_0^{n-1} by solving:

$$
\frac{P_0^n - P_0^{n-1}}{\Delta t} + \frac{1}{2}\sigma^2 S_0^2 \left(\frac{\frac{P_1^n - P_0^n}{\Delta S} - \frac{P_0^n - P_{-1}^n}{\rho\Delta S}}{\frac{\Delta S + \rho\Delta S}{2}}\right) + rS_0\left(\frac{P_1^n - P_{-1}^n}{\Delta S + \rho\Delta S}\right) - rP_0^n = 0
$$

$$
P_{-1}^n = K - \beta_n.
$$

Assuming the computational domain is large, we impose zero boundary condition $P_M^n = 0, \quad n = N,\ N-1, ..., 0$.

3. Determine $\beta_{n-1} = \dfrac{S_0^{n-1}}{1+\xi}$, where ξ is the solution for the equation

$$\xi^3 - \left\{ \ln \frac{S_0^{n-1}}{\beta_n} + (\sigma^2 + r)\Delta t \right\} \xi^2 + 3\sigma^2 \Delta t \xi - \frac{3\sigma^2 \Delta t}{\sqrt{rK}} Q(t_{n-1}, S_0^{n-1}, T) = 0,$$

which has a unique real root $\xi \in (0,1)$.

4. Change the values of S^{n-1} from old $S_i^{n-1} = \beta_n + \rho \Delta S + i \Delta S$ to new $S_i^{n-1} = \beta_{n-1} + \rho \Delta S + i \Delta S$ for $i = 0, 1, 2, ..., M$. If we archive a weekend day (Sunday), then we determine the jump (we simulate n jumps and take expectation) and we ignore Sunday and Saturday. Then, we determine the value on the previous useful day, Friday by

$$S_j^{n-3} = S_j^n / Y_n, \qquad j = 0, ..., M,$$

and we set the price on Friday by

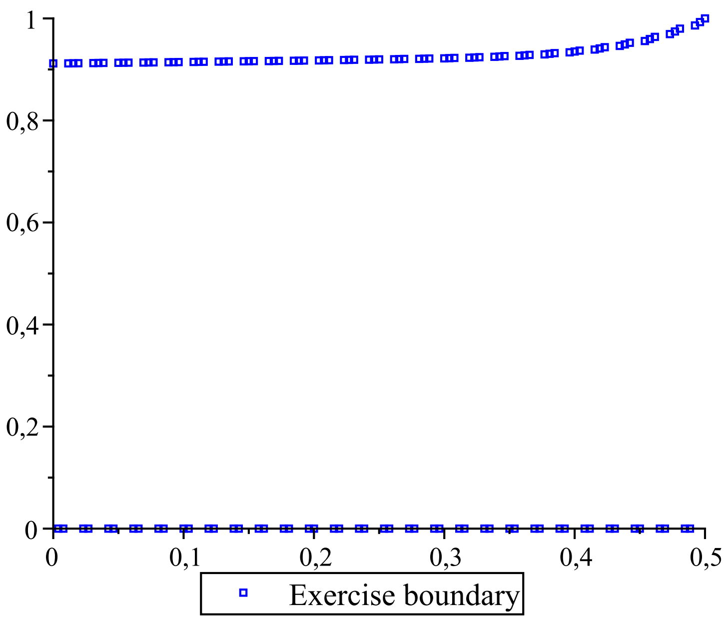
$$P_j^{n-3} = P_j^n$$

for $j = 0, ...M-1$. Then, we determine the value of $\beta[n-3]$ from $\beta[n]$ by solving the equation (1.12) and by setting

$$\beta_{n-3} = \frac{S_0^{n-3}}{1+\xi}.$$

If we finish step 4, then we repeat the running step 1 through 4 until t_0.

For a simulation by using the code in appendix B, with the following data: $N = 130$, $M1 = 130$, $T = 0.5$, $\sigma = 0.2$, $r = 0.05$, $K = 1$, $\rho = 0.4$, $MaxS = 30$ we get:



Remark 2.3.3. We have a continuous line during the useful week days and we set 0 along the weekends. When there is no jump, the line looks continuous. The jumps are random but they may arrive only on weekends (where we set the critical stock price as zero).

Conclusion

As already referred in B. Kim et all [5], the numerical algorithm presented on this paper is stable and with a good convergence. Furthermore, it is well known that finite difference method with explicit scheme is fact and simple to implement. Thus, this together with smooth pasting conditions and boundedness of the jumps intensities make the extended algorithm to our American put problem (exercise restrictions on weekends) also stable and with a good convergence.

This is an interesting practice problem since on some companies or markets have exercise restrictions during weekends or holidays. Moreover, this is a bit different version of jump models compared to the one presented on Merton [6], or Gukhal [4], for instance, where the jumps may occur at unknown dates. In fact, jump-diffusion models are very different from this one we presented here because they regard that the arrival of an additional information produces the jump on the stock on a random time (unknown time) whereas we obtained jumps by removing weekends and shrinking the time interval on a continuous stock price, obtaining then bounded jump intensities.

Incidently, we hedged the option price exposing ourselves to the jump risk since we supposed that the jumps are martingale and thus we do not make any strategy to cover the losses because of jumps. Thus, it would be interesting to try another types of hedging which can measure and minimize the risk of losses, for instance, superhedging or utility maximization referred in Cont and Tankov [2]. Of course, in a real market incompleteness is very natural and one should try to find good strategies to maximize the gains.

Appendix

A. Maple code for the critical stock price of an American put (standard case)

```
N := 100:
M1 := 100:
T := 0.5:  ### maturity time
sigma := 0.2:   ### volatility
r := 0.5e-1:    ### risk-free interest rate
K := 1:        ### strike price
rho := 0.4:     ### the constant parameter on Kim's algorithm
dt := T/N:
MaxV := 30:        ### suppose that Stock price maximum is 30
ds := MaxV/M1:
fbool := 0:
for i from 0 to N do
  P[i, M1] := 0:
end do:
beta[N] := K:
for i from 0 to M1 do
  P[N, i] := 0:
  S[0, i] := beta[N]+rho*ds:
  S[N, i] := beta[N]+rho*ds+i*ds:
end do:
for aux from 0 to N do
  i := N-aux:
   for j from 1 to M1-1 do
      x := P[i, j]:
      z := P[i, j+1]:
      w := P[i, j-1]:
      P[i-1, j] := solve((x-y)/dt+evalf((1/2)*sigma^2*S[i, j]^2*(z-2*x+w)/ds^2)
```

```
                +evalf(r*S[i, j]*(z-x)/ds)-r*x = 0, y):
   end do:
  AuxP[i] := K-beta[i]:
  P[i-1, 0] := solve((P[i, 0]-v)/dt+(1/2)*sigma^2*S[i, 0]^2*
               *evalf((P[i, 1]-P[i, 0])/ds-
               -(P[i, 0]-AuxP[i])/(rho*ds))/((1/2)*ds*(1+rho))
               +evalf(r*S[i, 0]*(P[i, 1]-AuxP[i])/((1+rho)*ds))-
               -r*P[i, 0] = 0, v):
  S[i-1, 0] := beta[i]+rho*ds:
  Q:= sqrt(max(P[i-1, 0]-K+S[i-1, 0], 0)):
    if AuxP[i] = P[i, 0] then
      t := i*T/N:
      count := i:
      fbool := 1:
    end if:
  l := solve(xi^3-(ln(evalf(S[i-1, 0]/beta[i]))+(sigma^2+r)*dt)*xi^2
       +3*sigma^3*dt*xi-3*sigma^3*dt*Q/sqrt(r*K) = 0, xi):
  for h from 1 to 3 do
    if Im(l[h] = 0) then
      reqone := l[h]:
    end if:
  end do:
  beta[i-1] := evalf(S[i-1, 0]/(1+reqone)):
  for j from 0 to M1 do
    S[i-1, j] := beta[i-1]+rho*ds+j*ds:
  end do:
end do:
Lx := [seq(beta[i], i = 0 .. N)]:
Ly := [seq(i*T/N, i = 0 .. N)]:
Lp := [seq(P[2, i], i = 0 .. 20)]:
Lb := [seq(K-S[2, i], i = 0 .. N)]:
Ls := [seq(S[2, i], i = 0 .. 20)]:
x1 := S[2, 2]:
y1 := K-S[2, 2]:
x2 := S[2, 20]:
y2 := K-S[2, 20]:
with(DynamicSystems):
DiscretePlot(Ly, Lx, style = line, legend = "Exercise boundary",
                       color = "blue", symbol = box);
with(plots):
DiscretePlot(Ls, Lp, style = line, legend = "Option value",
```

```
                        color = "brown", symbol = box);
p1 := DiscretePlot(Ls, Lp, style = point, legend = "Option value",
                         color = "brown", symbol = box):
p2 := plot(piecewise(0 < xvar and xvar < 1, (y2-y1)*(xvar-x1)/(x2-x1)+y1),
                          xvar = 0 .. 3.4, style = line, legend = "Pay-off",
                             color = "blue", symbol = box):
display({p1, p2}, axes = normal);
t; fbool;
```

B. Maple code for the critical stock price of an American put (for the main problem)

```
with(Statistics):
X := RandomVariable(Normal(0, 1)):
N := 130:
M1 := 130:
T := 0.5:         ### maturity time
sigma := 0.2:     ### volatility
r := 0.5e-1:      ### risk free interest rate
K := 1:           ### strike price
rho := 0.4:       ### safety parameter from the model
dt := T/N:        ### time variation on the discretization
MaxV := 30:       ### maximum value for the stock price
ds := MaxV/M1:
fbool := 0:
dec := 0:
weekends := 0:
weekdayforT := 2:       #### we start on a Tuesday
for i from 0 to N do
  P[i, M1] := 0:
end do:
beta[N] := K:
for i from 0 to M1 do
  P[N, i] := 0:
  S[0, i] := beta[N]+rho*ds:
  S[N, i] := beta[N]+rho*ds+i*ds:
end do:
for aux from 0 to N do
   i := N-aux-dec:
```

```
 for j from 1 to M1-1 do
     x := P[i, j]:
     z := P[i, j+1]:
     w := P[i, j-1]:
     P[i-1, j] := solve((x-y)/dt+evalf((1/2)*sigma^2*S[i, j]^2*(z-2*x+w)/ds^2)
                 +evalf(r*S[i, j]*(z-x)/ds)-r*x = 0, y):
 end do:
 AuxP[i] := K-beta[i]:
 P[i-1, 0] := solve((P[i, 0]-v)/dt+(1/2)*sigma^2*S[i, 0]^2*
              *evalf((P[i, 1]-P[i, 0])/ds-(P[i, 0]-AuxP[i])/(rho*ds))/((1/2)*
              *ds*(1+rho))+evalf(r*S[i, 0]*(P[i, 1]-AuxP[i])/((1+rho)*ds))
              -r*P[i, 0] = 0, v):
 S[i-1, 0] := beta[i]+rho*ds:
 Q := sqrt(max(P[i-1, 0]-K+S[i-1, 0], 0)):
  if AuxP[i] = P[i, 0] then
    t := i*T/N:
    count := i:
    fbool := 1:
  end if:
 l := solve(xi^3-(ln(evalf(S[i-1, 0]/beta[i]))+(sigma^2+r)*dt)*xi^2
     +3*sigma^3*dt*xi-3*sigma^3*dt*Q/sqrt(r*K) = 0, xi):
 for h from 1 to 3 do
   if Im(l[h] = 0) then
     reqone := l[h]:
   end if:
 end do:
beta[i-1] := evalf(S[i-1, 0]/(1+reqone)):
for j from 0 to M1 do
  S[i-1, j] := beta[i-1]+rho*ds+j*ds:
  end do:
 if `and`(i = N-weekdayforT-5*weekends, i >= 3) then
   Y := Statistics[Sample](X, N):
   for j from 0 to M1 do
     S[i-1, j] := 0:    ### Sunday ...we removed
     S[i-2, j] := 0:    ### Saturday ...we removed
     P[i-1, j] := 0:    ### Sunday ...we removed
     P[i-2, j] := 0:    ### Saturday ...we removed
     jump1:=exp(evalf(-(3*T*sigma^(2))/(2*N)+ sqrt((3*T)/(N))*sigma*X)):
                              ### generate jumps
     jump2:=Statistics[Sample](jump1,N):   ### create a sample of jumps
     jump:=Mean(jump2):    ### take a sample mean of the jumps
```

```
                              ###    at the corresponding weekend
      S[i-3,j]:=S[i,j]/jump : ### find the corresponding value of S after a jump
        if j < M1 then
          P[i-3, j] := P[i, j]:
        end if:
     end do:
  dec := dec+2:
  Q := sqrt(max(P[i-3, 0]-K+S[i-3, 0], 0)):
  l := solve(xi^3-(ln(evalf(S[i-3, 0]/beta[i]))
       +(sigma^2+r)*dt)*xi^2+3*sigma^3*dt*xi-3*sigma^3*dt*Q/sqrt(r*K) = 0, xi):
  for h from 1 to 3 do
     if Im(l[h] = 0) then
      reqone := l[h]:
     end if:
  end do:
  beta[i-3] := evalf(S[i-3, 0]/(1+reqone)):
  beta[i-2] := 0:
  beta[i-1] := 0:
  weekends := weekends+1:
  end if:
  if i = 0 then
    break
  end if:
end do:
Lx := [seq(beta[i], i = 0 .. N)]:
Ly := [seq(i*T/N, i = 0 .. N)]:
with(DynamicSystems):
DiscretePlot(Ly, Lx, style = point, legend = "Exercise boundary",
                                   color = "blue", symbol = box):
t; fbool;
```

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