UPPSALA
UNIVERSITET

Analytical Valuation of American-Style Asian Options under Jump-Diffusion Processes

Stefane Draiva Saize

Department of Mathematics
Uppsala University

UPPSALA UNIVERSITY
DEPARTMENT OF MATHEMATICS

Master Thesis

Theme:
ANALYTICAL VALUATION OF AMERICAN-STYLE ASIAN OPTIONS UNDER JUMP-DIFFUSION PROCESSES

Student: Stefane Draiva Saize
Nr: 840815-P192
Email: stefanesaize@gmail.com

Supervisor: Johan Tysk
Uppsala University, Sweden
Email: johan.tysk@math.uu.se

Uppsala, May 23, 2014

Contents

Chapter 1

Introduction

An Asian option is a financial derivative for which its payoff function is characterized by involvement of a stock price average. One case of this type of payoffs is when the underlying asset is defined to be an average and fixed strike price. The other case, the strike price is defined to be an average (floating strike). All types of averages are valid for the Asian option (discrete or continuous, arithmetic or geometric averages). The Asian options are most common in pricing a currency markets and commodities (eg. oil markets). These type of options reduce the risk of manipulations of the stock price at maturity and they are cheaper than standard European and American options. An Asian option can be classified as of European-style or American-style, depending of its time of exercise. In our studies, we will be dealing with the continuous overages (arithmetic and geometric cases). Hansen and Jorgensen 2000 [6], have studied the American-style Asian option with floating strike. They established the analytical solutions for this type of problem and they have found its numerical solutions based on the analytical ones. Also in Tomas Bokes [2], is his Phd thesis has studied the case of American-style Asian option with one or several underlying asset. Here he has studied the analytical valuation of the problem and its properties and, there are considered numerical methods using the analytical solutions.

In Merton 1976 [12] studies the case of European call option for a simple contract function (vanilla option) under jump-diffusion processes. In this paper is established the general form of the solution for vanilla option and the particular case, when the jump sizes follow the lognormal distribution. In a paper Huên Pham 1997 [15], is studied the American put option for a simple contract function under jump-diffusion model and there is stated the analytical solution to the problem, the exercise boundary and their properties. Also C. R. Gukal 2001 [5] has considered the problem of option pricing under jump diffusion model using the idea of Merton 1976 [12], and stated its analytical solutions.

In our studies, we will study the same problem in [6], but considering it under jump-diffusion process, instead. So, to achieve our results, we will use the results in [6], the theory established

by Merton 1976 [12] and the result of H. Pham 1997 [15] and other references. Here we will find the general analytical solution of American-style Asian option under jump diffusion process, for the case of floating strike and we will end by studying the particular cases, when the average is geometric and arithmetic. This thesis is organized as follows: In the second chapter we give some definition and properties of random variables and random processes. The third chapter concerns in some notes of stochastic processes and stochastic integrals. The key points of this chapter are diffusion and jump diffusion processes, Faynman-kac formula and Ito's lemma for diffusion and jump-diffusion models. The forth chapter treats about option pricing, where we give some concepts, the Black-Sholes formula for the European option, the analytic solution for the American option for diffusion and jump-diffusion models. In the fifth chapter we will present our investigation of the proposed problem. Here we start by transforming the problem into one-state variable problem. Then we will study this new problem, and to this problem, we will first investigate about its general analytical solution and then in the nest step we will consider the particular case when the average is geometric, for which we will investigate to figure out its analytical solution. Furthermore, we will study the geometric average case when the jump sizes are lognormally distributed. After the geometric average case, we will do the same investigation as in geometric average case, for the case of arithmetic average. By Hansen and Jorgensen 2000 [6], the dynamic of the new underlying asset isn't a geometric Brown motion so, first we will use the Wilkinson approximation (see P. Pirinen, [17]) in order to approximate it into a geometric Brownian motion and then to establish its solution. To end this chapter we have some numerical results, to compare the earliy exercise boundaries in a diffusion and jump-diffusion cases. At the end we have the conclusions chapter.

Chapter 2

Random Variable and random Processes

2.1 Basic definitions and properties

In this chapter we will set some definitions and properties of random variables. Also, we will define a random process and give some examples of random processes. In this chapter we will not get deeper with this theories, so for details see A. Klenke [10]. Before we get into the concept of random variable, let us give some definitions from measure theory (see M. Adams and V. Guillemin [1]).

Let Ω be a nonempty set and let $\aleph \subset 2^n$ (2^Ω is the set of all subsets of Ω) be a class of subsets of Ω.

Definition 1. A class of sets $\aleph \subset 2^\Omega$ is called a $\sigma - algebra$ if it satisfies the following properties:

1) $\Omega \subset \aleph$;

2) If $A \in \aleph$ then $A^c = \Omega \setminus A \in \aleph$ ($\aleph$ is closed under complements);

3) If $A_1, A_2, \ldots$ is a sequence of elements of $\aleph$, then $\bigcup_{n=1}^{\infty} A_n \in \aleph$.

Definition 2. Let $\aleph' \subset 2^\Omega$ be a class of sets. The class of sets $\sigma(\aleph') = \bigcap_{\aleph \subset 2^\Omega \ is \ a \ \sigma-algebra \ \ \aleph' \subset \aleph} \aleph$
is called $\sigma-$algebra generated by $\aleph'$ and, $\aleph'$ is called a generator. Moreover, this $\sigma-$algebga is the smallest $\sigma-$algebra containing $\aleph'$.

In the next definition, we will introduce the concept of measure. Fist of all, let us given $\aleph \subset 2^\Omega$ and $m : \aleph \longmapsto [0, \infty]$ a set function (a function which the arguments are sets).

Definition 3. A set function m is called a measure if it satisfies the following properties:

1) $m(\emptyset) = 0$;

2) If $A_1, A_2, \ldots$ is a sequence of elements of $\aleph$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$, then $m(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} m(A_n)$ (m is $\sigma-$aditive).

If $m(\Omega) = 1$, them m is called a probability measure. In this case we denote $m(A) = P(A)$ and A is called an event.

A set function m is said to be finite if $m(A) < \infty$, $\forall A \in \aleph$ and it is $\sigma-$finite if there exists a sequence $\Omega_1, \Omega_2, \ldots \in \aleph$, $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ and such that $m(\Omega_n) < \infty$ for all n.

Now, let $\Omega, \aleph, m$ as define above.

Definition 4. A pair $(\Omega, \aleph)$ is called a measurable space and $A \in \aleph$ is called measurable set. The triple $(\Omega, \aleph, m)$ is called measure space. If $m(\Omega) = 1$, then $(\Omega, \aleph, m)$ is called probability space and $A \in \aleph$ is called an event.

Let $m(\Omega) \neq 1$, then the normalized set function $\overline{m}(A) = m(A|\Omega) = \dfrac{m(A \cap \Omega)}{m(\Omega)} = \dfrac{m(A)}{m(\Omega)}$ (measure of A conditioned to the Ω) is a probability measure. Indeed, $\overline{m}(\Omega) = \dfrac{m(\Omega)}{m(\Omega)} = 1$.

From this idea, we can define the conditional probability as follows: let A and B two events such that $P(B) \neq 0$ then $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$.

2.2   Lebesgue Integral

This section is based on an introduction to the Lebesgue integral, and some properties. Here we do not go deeper on this, for more details we recommend the reader to see M. Adams and V. Guillemin [1]. Before we start discussing about the Lebesgue integral, let us begin with some definitions.

Definition 5. Let $(\Omega, \aleph)$ be a measurable space. Let $f : \Omega \longrightarrow R$ be a function such that $f^{-1}(B)$ (B is a set belonging to the $\sigma-$algebra generated by all open sets in R) is measurable, then f is said measured function.

Consider the measurable space $(\Omega, \aleph)$ and $s : \Omega \longrightarrow R$ be a measurable function. We say that s is a simple function if it takes on only finite number of values, let say $c_1, c_2, \ldots, c_n$.

If s takes values on the set $\{c_1, c_2, \ldots, c_n\}$ then let $E_i = s^{-1}(c_i) = \{x \in \Omega : \ s(x) = c_i\}$ $i = 1, 2, \ldots, n$. Thus, we can write s as follows

$$s(x) = \sum_{i=1}^{n} c_i \mathbf{1}_{E_i}(x),$$

where
$$\mathbf{1}_{E_i}(x) = \begin{cases} 1 & if \ x \in \ E_i \\ 0 & otherwise \end{cases}.$$

Definition 6. Let $s : \Omega \longrightarrow R$ be a nonnegative simple function and consider $E \in \aleph$. Let $c_1, c_2, \ldots, c_n$ be the distinct nonzero values of s and $E_i = s^{-1}c_i$, then we define the Lebesgue integral of s over E with respect to (w.r.t) m, as the sum

$$I_E(s) = \sum_{i=1}^{n} c_i m(E \cap E_i). \tag{2.1}$$

Now, let us extend this definition to any nonnegative function.

Definition 7. Let f be a nonnegative measurable function acting from Ω into nonnegative extended real numbers $(R_+ \cup \{+\infty\} = [0, +\infty])$ and let $E \in \aleph$. Then the Lebesgue integral of f on E w.r.t m is defined by

$$\int_E f \, dm = \sup\{I_E(s); \ 0 \leq s \leq f, \ s - simple\}. \tag{2.2}$$

Proposition 2.2.1. *Let $E, \ F \in \aleph$, f and g be nonnegative measurable functions. Then the following holds:*

1) If $f \leq g$ then $\int_E f \, dm \leq \int_E g \, dm$;

2) If $E \subset F$ them $\int_E f \, dm \leq \int_F g \, dm$;

3) If $m(E) = 0$ then $\int_E f\,dm = 0$;

4) If $\int_E f\,dm = 0$ then $f = 0$ almost surely on E.

5) If $E \cap F = \emptyset$ then $\int_{E \cup F} f\,dm = \int_E f\,dm + \int_F f\,dm$.

Now we are ready to define and discuss about a random variable.

2.2.1   Random variables

Definition 8. Let $(\Omega, \aleph, P)$ be a probability space and $(\Omega', \aleph')$ be a measurable space. Then, the function $X : \Omega \longmapsto \Omega'$ is called random variable acting from $(\Omega, \aleph)$ into $(\Omega', \aleph')$.

Given a random variable X. The probability measure $P_X := PoX^{-1}$ is called a distribution of the random variable X. In case of real random variable X, the map $F_X : x \longmapsto P[X \le x]$ is called a distribution function of the random variable X.

Let us give some examples of distribution of random variables.

Example 1. Let $p \in [0, 1]$, $P[X = 1] = p$ and $P[X = 0] = 1 - p$. So, this is a Bernoulli distribution with parameter p, denoted $Ber(p)$ and its distribution function is

$$F(x) = \begin{cases} 0 & if\ x < 0 \\ 1 - p & if\ x \in [0, 1) \\ 1 & if\ x \ge 1 \end{cases}.$$

Example 2. Let $\lambda \in [0, \infty)]$ and $X : \Omega \longrightarrow N_0$, be a random variable such that
$$P(X = n) = \lambda^n \frac{e^{-\lambda}}{n!}, \ \forall\ n \in N_0.$$

Then X has a Poisson distribution with parameter λ and we denote $X \sim Poi(\lambda)$.

Other important and most used distribution is described as follows:

Example 3. Let $\mu \in R$, σ^2 be a positive real number and X be a real random variable such that
$$P(X \le x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{(z-\mu)^2}{2\sigma^2}}\,dz, \text{ for all } x \in R.$$

Then the random variable X is normal distributed with parameters μ and σ. Symbolically denoted by $X \sim N(\mu, \sigma^2)$.

Example 4. Let λ be a positive real number an X be a nonnegative real random variable such thut

$$P[X \leq x] = \int_0^x e^{-\lambda z} \lambda dz$$

then we say that X follows the exponential distribution with parameter λ.

Definition 9. A collection $(X_i)_{i \in I}$ (I an index set) of random variables is said to be identical distributed if $P_{X_i} = P_{X_j}$ for all $i, j \in I$.

Let $(A_k)_{k \in I}$ be a collection of events of $\aleph$. Then we say that a collection $(A_k)_{k \in I}$ is independent if for any subset I' of I, we have $P(\bigcap_{k \in I'} A_k) = \prod_{k \in I'} P(A_k)$.

Definition 10. A collection $(X_i)_{i \in I}$ (I an index set) of random variables is said to be independent if the collection of $\sigma-$algebras $(\sigma(X_i))_{i \in I}$ (these σ algebra are called filtrations) is independent. The collection $(X_i)_{i \in I}$ of random variables is said to be independent identical distributed (i.i.d.) if the collectin $(X_i)_{i \in I}$ is independent and $P_{X_i} = P_{X_j}$ for all $i, j \in I$.

Consider $(\Omega, \aleph, P)$ be the probability space.

Definition 11. Let X be a real valued random variable.

1) If X integrable, then we call $E[X] := \int X dP$ the expectation or mean of the random variable X;

2) If X is square integrable, then we call $Var[X] = E[X^2] - E[X]^2$ the variance of X. The number $\sigma = \sqrt{Var[X]}$ is called the standard deviation of the random variable X.

3) If X, Y are square integrable, then we define the covariance of x and Y by $Cov[X, Y] = E[(X - E[X])(Y - E[Y])]$.

4) If X and Y are uncorrelated (independent) then $Cov[X, Y] = 0$.

Theorem 2.2.2. *Let X, Y, $n \in N$, be real integrable random variables on $(\Omega, \aleph, P)$.*

1) If X and Y have the same distribution then $E[X] = E[Y]$;

2) $E[aX + bY] = aE[X] + bE[y]$, a, b real numbers. This property is called linearity;

3) If $X \geq 0$ a.s. then $E[X] = 0 \Leftrightarrow X = 0$ a.s.;

4) If $X \leq Y$ a.s. then $E[X] \leq E[Y]$.

If the random variables X and Y are independent then $E[XY] = E[X]E[Y]$.

Another concept which is of our interest is a conditional expectation which can be defined as follows

Definition 12. Let X and Y be two random variables and $\mathbf{F}$ be a filtration. We say that a random variable Y is a conditional expectation of X given $\mathbf{F}$ and we write

$$Y := E[X|\mathbf{F}]$$

if Y is $F-$measurable and for any $A \in \mathbf{F}$, $E[X\mathbf{1}_A] = E[Y\mathbf{1}_A]$.

By this definition we have the following proposition

Proposition 2.2.3. *Let X and Y be two square integrable random variables. Then*

$$E[Var[Y|X] + Var(E[Y|X]) = Var[Y]. \tag{2.3}$$

Proof. Using the definition of variance we have
$Var(Y|X) = E[Y^2|X] - (E[Y|X])^2$ then

$$E[Var[Y|X] = E[E[Y^2|X] - (E[Y|X])^2] = E[Y^2] - E[(E[Y|X])^2]. \tag{2.4}$$

In other hand

$$Var(E[Y|X]) = E[E[Y|X]^2] - E[Y]^2. \tag{2.5}$$

From (2.4) and (2.5) we have

$$E[Var[Y|X] + Var(E[Y|X]) = Var[Y].$$

Next we provide the definition of a random process (or stochastic process).

2.3 Random processes

In this we will have a brief consideration of random processes, as well called stochastic processes. For more details we suggest the reader to see A. Klenke [10].

Definition 13. A random process (or stochastic process) is a collection of random variables (X_t), $t \in I$, where I is an index set.

An example about random process which we will use in this text, is the Poisson process. We will study further about this process in the next section.

2.3.1 Poisson process

Our aim in this part of the text, is to define the Poisson process and to mention some of its properties. The definitions and properties which we will consider in this thesis, can either be found in [10]. So the Poisson process is defined as follows:

Definition 14. Let $(\tau_i)_{i \geq 1}$ be a sequence of an exponential random variables with parameter λ and $T_n = \sum_{i=1}^{n} \tau_i$. The process $\{N_t\} = \sum_{n \geq 1} 1_{t \geq T_n}$ is called a Poisson process with parameter λ.

From this definition one can mention the following properties of a Poisson processes

1) N(0)=0;

2) For $0 \leq t_1 < t_2 < \ldots$, the increments $N(t_1)$, $N(t_2) - N(t_1), \ldots$ are independent;

3) $\forall \ t > s \geq 0$, the increment $N(t) - N(s)$ is a Poisson process with parameter λ and

$$P(N(t) - N(s) = n) = e^{-\lambda(t-s)} \frac{(\lambda(t-s))^n}{n!};$$

4) For each $\omega \in \Omega$, $N(\omega, t)$ is continuous in t;

5) The expected value and the variance are equal, i.e. $E[N(t)] = Var[N(t)] = \lambda t$;

6) $P[$an event does not occur at the interval (t,t+h)$] = 1 - \lambda h + O(h)$;

7) $P[$an event occur once at the interval (t,t+h)$] = \lambda h + O(h)$;

8) $P[$an event occurs more than one time at the interval (t,t+h)$] = O(h)$.

Compound Poisson process

In order to set the definition of compound Poisson process, let us consider $(Z_k)_{k \geq 1}$ being a square integrable sequence of *i.i.d.* random variable with probability distribution $\nu(dy)$. Wherefore,

$$P(Z_n \in [a, b]) = \nu[a, b] = \int_a^b \nu(dy).$$

Definition 15. The process

$$Y_t = \sum_{k=1}^{N_t} Z_k, \quad t \in R_+$$

is called compound Poisson process.

Proposition 2.3.1. *The expectation and variance of a poisson process are given by*

$$E[Y_t] = \lambda t E[Z_1]$$

and $Var[Y_t] = \lambda t E[|Z_1|^2]$.

Proof: $E[Y_t] = E \left[\sum_{k=1}^{N_t} Z_k \right] = \sum_{n=0}^{\infty} \frac{(\lambda t)^n e^{\lambda t}}{n!} E \left[\sum_{k=1}^{N_t} Z_k | N_t = n \right]$

$= \sum_{n=0}^{\infty} \frac{(\lambda t)^n e^{\lambda t}}{n!} E \left[\sum_{k=1}^{n} Z_k \right] = \lambda t E[Z_1]$. Here we have used the fact that $Z_k's$ are *i.i.d* random variables and independent to N_t. To calculate the variance of Y_t, we use the proposition (2.2.3) and the previous calculations. So,

$$Var[Y_t] = E[Var[Y_t|N_t]] + Var(E[Y_t|N_t]) = E[N_t Var[Z_1]] + Var[N_t E[Z_1]]$$
$$= \lambda t E[Z_1] + \lambda t E[Z_1]^2 = \lambda t E[Z_1^2]. \ \square$$

We end this section by setting the following definition.

Definition 16. Let T_t be a compound Poisson process with mean $\lambda t E[Z_1]$ with Z_1 defined above. The process $M = Y_t - \lambda t E[Z_1]$, is called compensated compound Poisson process. Furthermore, the process M_t is a martingale.

Chapter 3

Stochastic Differential Equations

3.1 Definitions and properties

In this section are presented some concepts, definitions and properties of diffusion processes. We will not give much details. In the case of details we refer the reader to see Bernt Öksendal [13]. Therefore, diffusion process S is a stochastic process such that its increment can be approximated by the stochastic difference equation,

$$S(t + \Delta t) - S(t) = \mu(t, S(t))\Delta t + \sigma(t, S(t))Z(t), \qquad (3.1)$$

where $Z(t)$ is a normal random variable (the disturbance term) which is independent of all information up to time t. The functions μ and σ are deterministic and, μ is a locally drift and σ the diffusion term.

Definition 17. The stochastic process W is called Wiener process (or Brownian motion) if it satisfies the following properties:

1) $W(0) = 0$;

2) The process W has independent increments i. e. $0 \leq t_1 \leq t_2 \leq \ldots$, then $W(t_1)$, $W(t_2) - W(t_1), \ldots$ are independent stochastic variables;

3) For $s \leq t$ the increment $W(t) - W(s)$ has normal distribution $N(0, t - s)$;

4) The stochastic process W has continuous trajectories.

If we replace the process $Z(t)$ in (3.1) by $\Delta W = W(t + \Delta t) - W(t)$ and taking $\Delta t \longrightarrow 0$, we can rewrite the difference equation (3.1) as follows

$$dS(t) = \mu(t, S(t))dt + \sigma(t, S(t))dW(t) \tag{3.2}$$
$$S(0) = s. \tag{3.3}$$

In a stochastic calculus we have the following proparties for the increments dt and dW:

1) $(dt)^2 = 0$;

2) $dtdW(t) = 0$;

3) $(dW(t))^2 = dt$.

Definition 18. Let X be a random variable. We say that F_t^X is a filtration generated by X if F_t^X is a $\sigma-$algebra generated by all the information of X up to time t. If $Y_t \in F_t^X$, we say that Y_t is adapted to the filtration F_t^X.

Let g be a process satisfying the following conditions:

1) $\displaystyle\int_a^b E[g^2(s)]ds < \infty$;

2) The process g is F_t^W adapted.

Then,

$$E\left[\int_a^b g(s)dW(s)\right] = 0, \quad E\left[\left(\int_a^b g(s)dW(s)\right)^2\right] = \int_a^b E[g^2(s)]ds.$$

Let given a random process S. The process S is said to be a F_t-martingale, if it satisfies:

1) The process S is an adapted process to the filtration $\mathbf{F}_t$;

2) $\forall \, t$, $E[||S(t)||]$ is finite;

3) $\forall \, s$ and t such that $s \leq t \ E[S(t)|F_s] = S(s)$.

If $\forall \, s \leq t$, S satisfies $E[X(t)|F_s] \leq S(s)$ ($E[S(t)|F_s] \geq S(s)$) then S is called a super-martingale (submartingale).

Theorem 3.1.1. *(**Ito's lemma**) Let S be a stochastic process satisfying the stochastic differential equation (SDE)*

$$dS(t) = \mu(S(t), t)dt + \sigma(S(t), t)dW(t). \tag{3.4}$$

where μ and σ are adapted processes and let F be a $C^{1,2}-$function. Let $F = F(t, S(t))$, then the stochastic differential equation for F is given by

$$dF = \left(F_t + \mu(t, S(t))F_s + \frac{1}{2}\sigma^2(t, S(t)F_{ss} \right) dt + \sigma(t, S(t))F_s dW(t). \tag{3.5}$$

3.1.1   Partial differential equations

Let $\mu(t, s)$, $\sigma(t, s)$ and $G(s)$ be a deterministic functions, and let F be a function satisfying the following boundary problem on $[0, T] \times R$

$$F_t + \mu(t, S(t))F_s + \frac{1}{2}\sigma^2(t, S(t))F_{ss} - rF = 0 \tag{3.6}$$

$$F(T, s) = G(s), \tag{3.7}$$

where S satisfies the SDE

$$dS(t) = \mu(t, S(t))dt + \sigma(t, S(t))dW(t).$$

Applying Ito's lemma to F we get

$$F(T, S(T)) - F(t, S(t)) = r \int_t^T F(\tau, S(\tau))d\tau + \int_t^T \sigma(\tau, S(\tau))F_s dW(\tau)d\tau.$$

Taking expectation value conditioned to $S(t) = s$, we have

$$E_{t,s}[F(T, S(T)) - F(t, S(t))] = r \int_t^T E_{t,s}[F(\tau, S(\tau))]d\tau.$$

Let

$$y(T) = E_{t,s}[F(T, S(T))$$

then,

$$y(T) - y(t) = r \int_t^T y(\tau)d\tau.$$

From this we get the following initial value problem

15

$$y'(T) = ry(T), \qquad y(T)|_{T=t} = y(t). \tag{3.8}$$

Solving this problem we have,

$$y(T) = y(t)e^{r(T-t)}.$$

Substituing $y(T)$ and $y(t)$ we get the well known Feynman-Kăc formula

$$F(t, S(t)) = e^{-r(T-t)} E_{t,s}[G(S(T))]. \tag{3.9}$$

Definition 19. We say that the diffusion process $S(t)$ is a geometric Brownian motion if it satisfies the following SDE

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t), \tag{3.10}$$
$$S(0) = s_0. \tag{3.11}$$

Let $z = \ln S(t)$ then, using Ito's lemma we have

$$dZ = [\mu(t) - \frac{1}{2}\sigma^2(t)]dt + \sigma(t)dW(t),$$

with solution

$$Z(t) - z_0 = \int_0^t [\mu(s) - \frac{1}{2}\sigma^2(s)]ds + \int_0^t \sigma(s)dW(s).$$

Therefore, $S(t)$ will be presented by the following formula

$$S(t) = s_0 e^{\{\int_0^t [\mu(s) - \frac{1}{2}\sigma^2(s)]ds + \int_0^t \sigma(s)dW(s)\}}. \tag{3.12}$$

If μ and σ are constant, then the solution (3.12) becomes

$$S(t) = s_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W(t)}. \tag{3.13}$$

16

3.1.2 Jump diffusion process

Definition 20. A stochastic process S is called a Jump diffusion process, if it satisfies the following stochastic differential equation

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t) + (X-1)dN_t, \quad S(0) = s, \tag{3.14}$$

where

1) $\mu(t)$ is a drift of the process;

2) σ the volatility of the stock price;

3) $W(t)$ is a standard Brownian motion;

4) N_t is a Poisson process with parameter λt;

5) X is a jump size in stock, if the a jump in the process N_t occurs;

6) X are *i.i.d.* random variables and $X-1$ is an impulse function producing a finite jump in S to XS (see Merton (1976) [12]);

7) $W(t)$, $N(t)$, X are mutually independent.

In this text, we will always consider the case that if there is a jump at time t then the value of a price is determined after the jump. This leads us to have a right continuous stock price $S(t)$. Merton in his paper of 1997 [12], has considered that if the jump process is included, then $dN = 1$ and if it is not included, then $dN = 0$. In our studies we we will be dealing with such type of events.

Let suppose that in interval $[0,t]$ the jump process does not occur so, the dynamics of the stock price will have the following form:

$$dS(t) = \mu(t)S(t)dt + \sigma(t)S(t)dW(t). \tag{3.15}$$

By the solution (3.12), $S(t)$ will be presented by

$$S(t) = s_0 e^{\{\int_0^t [\mu(s) - \frac{1}{2}\sigma^2(s)]ds + \int_0^t \sigma(s)dW(s)\}}. \tag{3.16}$$

Now, let us suppose that in the interval $[t, t+h]$, the jump process has occurred, then (see Merton (1976) [12]),

$$S(t+h) - S(t) = (X-1)S(t).$$

Therefore, $S(t+h) = XS(t)$. The solution of (3.14) is as follows:

$$S(T) = S(t) \exp\left\{ \int_t^T \mu(s)ds + \int_t^T \sigma(s)dW(s) - \frac{1}{2}\int_t^T |\sigma(s)|^2 ds \right\} \prod_{k=N_t+1}^{N_T} X_k. \qquad (3.17)$$

In particular, if μ and σ are constant, then (3.17) will take the form,

$$S(T) = S(t) \exp\left\{ (\mu - \frac{1}{2}\sigma^2)(T-t) + \sigma(W(T) - W(t)) \right\} \prod_{k=N_t+1}^{N_T} X_k. \qquad (3.18)$$

In order to make $e^{-rt}S(t)$ a martingale, let us choose $\mu = r - E[X-1]$, where r is a risk-free rate. Suppose that in the interval $[0,t]$ N_t jumps has occurred, then the solution (3.18) becomes,

$$S(t) = S(0) \exp\{(r - \frac{1}{2}\sigma^2)t + \sigma W(t) - \lambda E[X-1]t\} \prod_{k=1}^{N_t} X_k. \qquad (3.19)$$

and it can be presented as follows

$$S(t) = S(0) \exp\{(r - \frac{1}{2}\sigma^2)t + \sigma W(t) - \lambda E[X-1]t + \sum_{k=1}^{N_t} \ln X_k\}. \qquad (3.20)$$

Here, N_t is a Poisson process with parameter λ and, independent to N_t, X and $W(t)$.

Next we will set the Ito's formula for a jump diffusion processes. Let us consider the F be $C^{1,2}$-function such that $F = F(t, S(t))$. Then, from P. Tankov [4], we have the following proposition:

Proposition 3.1.2. *Let $S(t)$ be a diffusion process with jumps defined by*

$$dS(t) = \mu(t, S(t))dt + \sigma(t, S(t))dW(t) + (X-1)dN_t,$$

where $\mu(t, S(t))$ and $\sigma(t, S(t))$ are continuous and nonantecipating processes with

$$\int_0^T \sigma^2(\tau, S(\tau))d\tau < \infty.$$

Then, for any $C^{1,2}-function$ $F : [0,T] \times R_+ \rightarrow R$, the process $F(t) = F(t, S(t))$ can be represented as,

$$dF = \left(F_t + \mu(t, S(t))F_s + \frac{1}{2}\sigma^2(t, S(t))F_{ss} \right) dt + \sigma(t, S(t))F_s dW(t)$$
$$+ [F(t_- + \Delta t, S(t_- + \Delta t)) - F(t_-, S(t_-))].$$

(3.21)

Chapter 4

Options

In this chapter we will introduce some concepts and definitions about financial instruments. We will consider the Black-Scholes model and to give the Black-Scholes equation. Using the risk-neutral valuation formula, we will write a solution to the terminal value problem involving the Black-Scholes equation. In the end of this chapter we will derive the pricing equation under jump diffusion processes.

4.1 Financial derivatives and the Black-Scholes formula

Before we get into a financial derivatives study, let us first state some definition, starting with the following definition:

Definition 21. An underlying asset is a financial instrument (e.g. stock, commodity, future) on which a price of the derivative is based.

Definition 22. A contingent claim (financial derivative) is a stochastic variable Π of the form $\Pi = G(Z)$, where Z is a stochastic variable driving the stock price process.The function G is called contract function.

The classical examples of a financial derivatives are well known as European options and American options which are defined bellow.

Definition 23. An European option is a contingent claim written on an underlying asset $S(t)$, with strike price (exercise price) L at the maturity time (exercise time) T, with the following property:

The holder of the contract has the right but not the obligation to buy (sell) one share of the underlying asset, exactly at time of maturity, at the price L. An American option gives

the holder the right but not the obligation to buy (sell) one share of the underlying asset at any time before (exactly) the maturity time T, at the price L.

Suppose we have a market consisting on some financial assets. So, the collection of these financial assets is called a portfolio.

Definition 24. A contingent claim Π is said to be reachable if there exists a portfolio h such that $V_t^h = \Pi$ with probability one. Here V_t^h is the value of the portfolio at time t.

If for the claim Π there exists a portfolio with the property on definition above, we say that this portfolio is hedging portfolio or replicating portfolio.

A market is said to be complete if all claims are reachable. This definition of complete market is equivalent to the following definition:

Definition 25. A market is complete if the number of risk assets is equal to the number of random resources.

4.1.1 Pricing equation for European options

Let us consider a financial market consisting on two assets, the bond (a bank account) price process $B(t)$ which is a risk free asset, and a stock with price process $S(t)$, defined by the following dynamics

$$dB(t) = rB(t)dt, \tag{4.1}$$
$$B(0) = 1, \tag{4.2}$$

where r is the risk free rate. Then $B(t) = e^{rt}$, and $S(t)$ following

$$dS(t) = \mu(t, S(t))S(t)dt + \sigma(t, S())S(t)dW(t), \quad S(0) = s, \tag{4.3}$$

where $\mu(t, S(t))$ is a drift, $\sigma(t, S(t))$ is a volatility. Now, consider the contingent claim of the form $\Pi = G(S(T))$ with price process $F(t)(t) = V(t, S(t))$. This claim is called simple claim (see T. Björk, [3]) and V is some smooth function. Using Ito's lemma and by risk free arguments leads to the flowing pricing equation (see again T. Björk, [3])

$$\begin{cases} V_t + rSV_s + \frac{1}{2}\sigma^2(t, S)S^2 V_{ss} - rV = 0 \\ \qquad\qquad\qquad\qquad\quad V(T, s) = G(s). \end{cases} \tag{4.4}$$

Applying the Feynman-Kăc formula to the problem (4.4), we get the following risk-neutral valuation formula:

$$V(t, s) = e^{-r(T-t)} E_{t,s}^Q [G(S(T))]. \tag{4.5}$$

4.1.2 Black-Scholes formula

Let us consider the the problem (4.4) when the parameters μ and σ are constant and take a contract function of the form $G(S(T)) = [S(T) - L]^+ = max(S(T) - L, 0)$. In this conditions, we know from chapter 3.1 that

$$S(T) = s \exp\{(r - \frac{1}{2}\sigma^2)(T - t) + \sigma(W(T) - W(t))\}$$

and using the formula (4.5) we have,

$$V(t, s) = e^{-r(T-t)} \int_{-\infty}^{\infty} [S(T) - L]^+ f(z) dz.$$

And, from this we get the following solution:

$$V(t, s) = s\Phi(d_1(t, s)) - e^{-r(T-t)} L\Phi(d_2(t, s)), \tag{4.6}$$

where $\Phi(x) = \dfrac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{z^2}{2\sigma^2}} dz$,

$f(z) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2\sigma^2}}$

$d_1(t, s) = \dfrac{1}{\sqrt{T-t}} [\ln(\dfrac{s}{L}) + (r + \dfrac{1}{2}\sigma^2)(T - t)]$

and

$d_2(t, s) = d_1(t, s) - \sigma\sqrt{T - t}$.

4.1.3 Optimal stopping problem and American options

In this section we will discuss more or less about the optimal stopping problem and in the end of the section we will introduce an American options.

First of all, let us consider the following definition

Definition 26. A nonnegative random variable τ is called a stopping time with respect to the filtration $\mathbf{F}$ if it satisfies the condition $\{\tau \leq t\} \in \mathbf{F}_t$ for all $t \geq 0$.

Now let us consider the problem of the form $\max\limits_{0\le\tau\le T} E[Z_\tau]$. Taking τ over the set of all stopping times, this problem is optimal if

$$E_{\widehat{\tau}}[Z_{\widehat{\tau}}] = \sup_{0\le\tau\le T} E[Z_\tau]. \tag{4.7}$$

Suppose that all stopping times belong to the interval $[t, T]$, then we can define the optimal value function by

$$V_t = \sup_{t\le\tau\le T} E[Z_\tau]. \tag{4.8}$$

Now consider the diffusion process

$$dS(t) = \mu(t, S(t))S(t)dt + \sigma(t, S())S(t)dW(t), \quad S(0) = s$$

and the contract function $G(t, S(t))$. Our objective is to study the optimal stopping problem

$$\max_{0\le\tau\le T} E[G(t, S(t))].$$

Fix $(t, s) \in [0, T]xR_+$ and for each stopping time define $E_{t,s}[G(\tau, S_\tau)]$. Then according to the previous ideas, the optimal value function $V(t, s)$ is define by

$$V(t, s) = \sup_{t\le\tau\le T} E_{t,s}[G(\tau, S_\tau)]. \tag{4.9}$$

Assume that the function V is at least $C^{1,2}$ function, all other processes are enough integrable and for each (t, s) there exists an optimal stopping time $\widehat{\tau}$. Then $V(t, s)$ satisfies the following properties

1) It is optimal to stop iff $V(t, s) = G(t, s)$, where $V_t + rSV_s + \dfrac{1}{2}\sigma^2(t, S)S^2V_{ss} - rV < 0$;

2) It is optimal to continue iff $V(t, t) > G(t, s)$, where $V_t + rSV_s + \dfrac{1}{2}\sigma^2(t, S)S^2V_{ss} - rV = 0$.

Therefore we can define the continuation region by $C = \{(t, s)|V(t, s) > G(t, s)\}$.

Now we are ready to discuss further about American options. So, from the definition of American option, we know that the holder has the right but not the obligation to buy or sell one share of an underlying asset at price K at any time before (exactly) the expiry date T. Thus,

23

the problem in this case, is to determine when is optimal to buy or sell, in order to maximize the profit. Let us consider the problem with contract function $G(T, s) = [\rho(S(T) - K)]^+ = max(\rho(S(T) - K), 0)$, where $\rho = \pm 1$. So, the problem is reduced to solve the optimal stopping problem

$$\max_{0 \leq \tau \leq T} E^Q[e^{-r\tau}[\rho(S(T) - L)]^+] \tag{4.10}$$

In the case when $\rho = 1$, we are dealing with the American call option. In this case, the function $Z_t = e^{-r\tau}[S(T) - L]^+$ is a $Q-$submartingale, then the problem is optimal to stop when $\tau = T$, which coincides with the European call option (see T.Bjork, [3]).

If $\rho = -1$, then under risk-neutral measure Q, the optimal value function is given by

$$V(t, s) = \sup_{t \leq \tau \leq T} E_{t,s}[e^{-r(\tau-t)}[L - S(T)]^+]. \tag{4.11}$$

To end this subsection we set an proposition from the text book of T. Bjork, [3] given below

Proposition 4.1.1. *Assume that a sufficiently regular function $V(t, s)$ and an open set $C \subset R_+ \times R_+$, satisfies the following conditions:*

1) *C has a continuously differentiable boundary $b(t)$;*

2) *V satisfies the PDE $V_t + rSV_s + \frac{1}{2}\sigma^2(t, S)S^2V_{ss} - rV = 0$. $(t, s) \in C$;*

3) *V satisfies the final boundary condition $V(T, s) = \max(L - s, 0)$, $s \in R_+$;*

4) *V satisfies the inequality $V(t, s) > max(L - s, 0)$, $(t, s) \in C$;*

5) *V satisfies $V(t, s) = max(K - s, 0)$, $(t, s) \in C^c$;*

6) *V satisfies the smooth fit condition*
   $$\lim_{s \downarrow b(t)} \frac{\partial V}{\partial s} = -1, \quad 0 \leq t < T.$$

   Then

   - *V is the optimal value function and it has the form (5.10);*

   - *C is a continuation region;*

- *The stopping time is given by $\widehat{\tau} = \inf\{t \geq 0 | s(t) = b(t)\}$*

Let $G(S(T)) = max\{L - S(T), 0\}$ then the solution to the American put option is given by

$$V(t, s) = e^{-r(T-t)} E_{t,s}[G(S(T))] + rL \int_t^T e^{-r(u-t)} Q_{t,s}[S(u-t) \leq b(u-t)]du. \qquad (4.12)$$

The proof of this can be found on G. Peskir and A. Shiryaev [14].
Another type of option and which is of our interest is a so called Asian option. This type of options, they can be of European style or American style. The single characteristic of these options is that the payoff function is of the form

$$G(T) = \begin{cases} [\rho(A(T) - K)]^+ & \text{fixed strike price case} \\ [\rho(S(T) - A(T))]^+ & \text{floating strike price case} \end{cases}. \qquad (4.13)$$

where $\rho = \pm 1$, which means that if $\rho = 1$ then we have a call option else we get a put option and,

$$A(t) = \begin{cases} \dfrac{1}{t} \displaystyle\int_0^t S(\tau)d\tau, & \text{in the arithmetic average case} \\ exp\{\dfrac{1}{t} \displaystyle\int_0^t \ln S(\tau)d\tau\} & \text{in the geometric average case} \end{cases}.$$

In next section we will derive the pricing partial differential equation, when the underlying asset returns are discontinuous.

4.2 Pricing equations under jump-diffusion processes

Instead of considering an financial market with one random source, here we will take in consideration one more random process. This process will cause jumps in the underlying asset, making it discontinuous. As in the previous section, let us consider a financial market consisting on two assets, the bond $B(t)$ which is a risk free asset, and a stock with price process $S(t)$, defined by the dynamics

$$dB(t) = rB(t)dt, \qquad (4.14)$$
$$B(0) = 1, \qquad (4.15)$$

where r is the risk free rate. Then $B(t) = e^{rt}$, and $S(t)$ satisfies following the stochastic differential equation,

$$dS(t) = \mu S(t)dt + \sigma S(t)dW(t) + (X - 1)dN_t, \quad S(0) = s, \tag{4.16}$$

as defined in previous section, assuming μ and σ to be constant.

Let $V(S(t), t)$ be the option price at time t. Applying Ito's lemma we have

$$dV = (V_t + \mu S V_s + \frac{1}{2}\sigma^2 S^2 V_{ss})dt + \sigma V_s dW + [\Delta V]dN.$$

Now let $\mu = r - E[X - 1]$, (this is to make $e^{-rt}S(t)$ a martingale), where r is a "risk-free rate" under measure Q. By using hedging arguments, let $\delta = V_s$ and denote by $\Pi = V - \delta S$ the $\Delta - hedged$ portfolio, such that under risk-free measure $d\Pi = r\Pi dt$. Therefore,

$$d\Pi = (V_t + (r - E[X-1])SV_s + \frac{1}{2}\sigma^2 S^2 V_{ss})dt + \sigma V_s dW + [V(S(t_- + \Delta t), t_- + \Delta t) - V(S(t_-, t_-))]dY$$

$$-\delta S[(r - E[X - 1])dt + \sigma dW] - \delta \Delta S dN.$$

Then we have,

$$d\Pi = (V_t + \frac{1}{2}\sigma^2 S^2 V_{ss})dt + [V(S(t_- + \Delta t), t_- + \Delta t) - V(S(t_-, t_-))]dY - \delta \Delta S dN.$$

So we have eliminated the dW term. Now taking the expectation value over the random variable X to this last, we get the following expected variation on the portfolio:

$$d\Pi = (V_t - \lambda E[X - 1]SV_s + \frac{1}{2}\sigma^2 S^2 V_{ss} + \lambda E[V(S(t_- + \Delta t), t_- + \Delta t) - V(S(t_-, t_-))])dt. \tag{4.17}$$

Since $d\Pi = r(V - SV_s)dt$ then,

$$(V_t - \lambda E[X - 1]SV_s + \frac{1}{2}\sigma^2 S^2 V_{ss} + \lambda E[V(S(t_- + \Delta t), t_- + \Delta t) - V(S(t_-, t_-))] = r(V - V_s S).$$

Finally we have the pricing equation,

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{ss} + (r - \lambda E[X - 1])SV_s + \lambda E[\Delta V] - rV = 0, \tag{4.18}$$

where $\Delta V = V(S(t_- + \Delta t, t_- + \Delta t) - V(S(t_-, t_-))$.

Since the number of random sources is greater than the number of risk assets, then the market is incomplete so, the $\Delta - hedging$ strategy will not eliminate the random sources at all. We still have one random source in the pricing integro-partial differential equation.

Now let us consider the standard European call option $V(T, S(T)) = [S(T) - L]^+$ and $K = E[X - 1]$, where $V(t, S(t))$ satisfies the the following integro-partial differential equation:

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{ss} + (r - \lambda K])SV_s + \lambda E[\Delta V] - rV = 0. \tag{4.19}$$

This problem was already studied by Merton [12], and its solution is given by

$$
\begin{aligned}
V(t, s) &= \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} e^{-r(T-t)} E_{t,s}[G(S(T))|N_t = n] \\
&= \sum_{n=0}^{\infty} \frac{(\lambda(T-t))^n e^{-\lambda(T-t)}}{n!} E_n[H(s\varepsilon_n e^{-K\lambda(T-t)}, \sigma, r, T, t)],
\end{aligned}
\tag{4.20}
$$

where $\varepsilon_n = \prod_{k=1}^n X_k$, and where $H(s, \sigma, r, t)$ is the standard Black-Schole's formula as in (4.6). In the same way is obtained the solution for an European put option.

Recall from (3.20) that,

$$S(T) = S(t)\exp\{(r - \frac{1}{2}\sigma^2 - \lambda K)(T-t) + \sigma(W(t) - W(t)) + \sum_{k=1}^{N_t} \ln X_k\}. \tag{4.21}$$

If $\ln X_n$, $i = 1, 2, \ldots, n$ are normal i.i.d. random variables with mean a and variance b^2 then, the sum $\sum_{k=1}^n \ln X_k$ will follow the normal distribution with mean na and variance nb^2. Then,

$$(r - \frac{1}{2}\sigma^2 - \lambda K)(T-t) + \sigma(W(T) - W(t)) + \sum_{k=1}^n \ln X_k \sim N((r - \frac{1}{2}\sigma^2 - \lambda K)(T-t) + na, \sigma^2(T-t) + nb^2).$$

Thus,

$$(r - \frac{1}{2}\sigma^2 - \lambda K)(T - t) + na + \sqrt{\sigma^2 + \frac{nb^2}{T-t}}(W(T) - W(t)) \sim$$

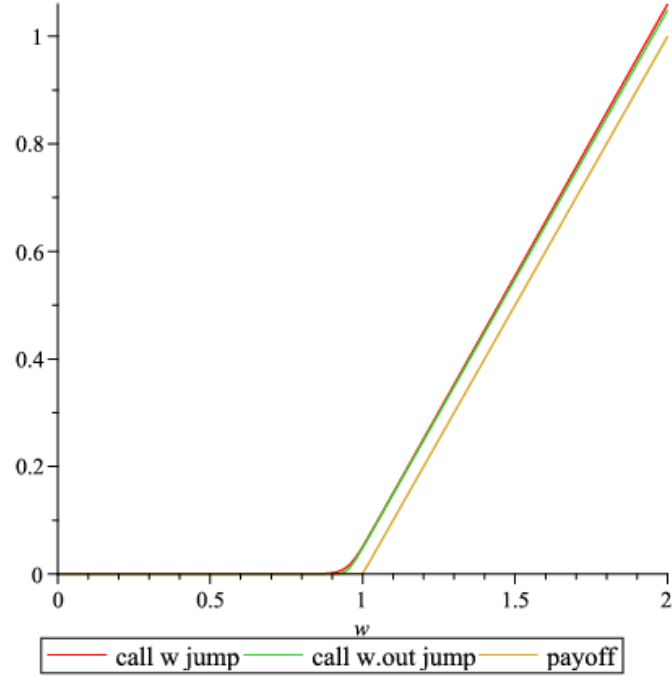$$N((r - \frac{1}{2}\sigma^2 - \lambda K)(T - t) + na, \sigma^2(T - t) + nb^2).$$

And therefore,

$$E_{t,s}[G(S(T))|N_t = n] = E_{t,s}[S(t)\exp\{(r - \frac{1}{2}\sigma^2 - \lambda K)(T - t) + na + \sqrt{\sigma^2 + \frac{nb^2}{T - t}}(W(T) - W(t))\}]$$

$$= E_{t,s}[S(t)\exp\{(r - \frac{1}{2}\sigma^2 - \frac{nb^2}{2(T - t)} + \frac{nb^2}{2(T - t)} - \lambda K + \frac{na}{T - t})(T - t) + \sqrt{\sigma^2 + \frac{nb^2}{T - t}}(W(T) - W(t))\}]$$

$$= E_{t,s}[S(t)\exp\{(r_n - \frac{1}{2}\sigma_n^2)(T - t) + \sigma_n(W(T) - W(t))\}].$$

Here $\sigma_n^2 = \sigma^2 + \frac{nb^2}{2(T - t)}$, $r_n = r - \lambda K + \frac{nb^2}{T-t} + \frac{na}{T-t} = r - \lambda K + \frac{n}{T-t}(a^2 + \frac{b^2}{2}) = r - \lambda K + \frac{n}{T-t}\ln(1 + K)$.

And thus, the solution $V(t, s)$ will be defined by

$$V(s, t) = \sum_{n=0}^{\infty} \frac{(\lambda'(T - t))^n e^{-\lambda'(T-t)}}{n!} H(s, \sigma_n, r_n, T, t), \qquad (4.22)$$

where $\lambda' = \lambda(1 + K)$.

28

In the case of American put options under jump-diffusion processes, the solution $V(t, s)$ satisfying the conditions of the Proposition 4.1.1 has the following form

$$V(s, t) = \sum_{n=0}^{\infty} \frac{(\lambda(T - t))^n e^{-\lambda(T-t)}}{n!} E_n \left[H_p(s\varepsilon_n e^{-\lambda K(T-t)}, \sigma, r, T, t) \right]$$

$$+ \sum_{n=0}^{\infty} \int_t^T \frac{(\lambda(\tau - t))^n e^{-(\lambda+r)(\tau-t)}}{n!} E_t[rL \cdot 1_{\{S(\tau-t)\le b(\tau-t)\}}] \qquad (4.23)$$

$$- \lambda E_t^Q \left\{ \int_t^T E\left[g(X, S, b) \right] d\tau \right\},$$

where
$g(X, S(\tau), b(\tau)) = \widetilde{V}(XS(\tau), \tau) - (L - XS(\tau))1_{\{S(\tau-)\le b(\tau), XS(\tau-)>b(\tau)\}}$ H_p is the corresponding solution of the European put option when there is no jumps and $b(t)$ is the exercise boundary.

If $\ln X_n$ are i.i.d. normal random variables as above, then the first part of the earlier exercise premium will be

$$e^{-r(\tau-t)}E_t[rL \cdot 1_{\{S(\tau-t)\leq b(\tau-t)\}}] = rLe^{-r(\tau-t)}Q([s_0 \exp\{(r_n - \frac{1}{2}\sigma_n^2)(\tau-t) + \sigma_n\sqrt{\tau-t}Z\} \leq b(\tau-t)]$$

$$= rLe^{-r(\tau-t)}\Phi(\frac{\ln\frac{b(\tau-t)}{s_0} - (r_n - \frac{\sigma_n^2}{2})(\tau-t)}{\sigma_n\sqrt{\tau-t}}).$$

So, the solution of an American put options under jump diffusion processes when the jump size follows the lognormal distribution, takes the form,

$$V(s,t) = \sum_{n=0}^{\infty} \frac{(\lambda'(T-t))^n e^{-\lambda'(T-t)}}{n!}[H_p(s, \sigma_n, r_n, T, t)]$$

$$+rL\sum_{n=0}^{\infty}\int_t^T \frac{(\lambda(\tau-t))^n e^{-(\lambda+r)(\tau-t)}}{n!} \cdot \Phi\left(\frac{\ln\frac{b(\tau-t)}{s} - (r_n - \frac{\sigma_n^2}{2})(\tau-t)}{\sigma_n\sqrt{\tau-t}}\right) d\tau$$

$$-\lambda E_t\left\{\int_t^T E[g(X, S(\tau), b(\tau))] d\tau\right\}.$$

Chapter 5

General valuation of the American-style Asian options under jump-diffusion processes

As in Hansen and Jorgensen (2000) [6], our goal is to give an analytical solution $V(t,s)$ to the free boundary problem, where the contract function is given below by the formula (5.1). In this article, is considered an American-style Asian options with floating strike, where the contracts are initialized at time zero and their pay-off's functions at time t are of the form defined by equation (4.13), but concretely of the form

$$pay - off = [\rho(S(t) - A(t)]^+. \tag{5.1}$$

Robert.C. Merton, in his paper of (1976),([12]) provides a method to solve the option pricing problems when the underlying stock returns are discontinuous. In paper of Hansen and Jorgensen (2000) [6] is given an analytical valuation for American-style Asian options. So, we will connect these two theories in order to find an analytical valuation of American-style Asian options when underlying stock returns are discontinuous .

By the result from Karout and Karatzas [9] and H. Pham (2001) [15], we have that the solution of the free boundary problem is given

$$V(t) = ess \sup_{\tau \in \Gamma_{t,T}} E_t^Q \left\{ [\rho(S(\tau) - A(\tau))]^+ \right\}, \tag{5.2}$$

$$\tag{5.3}$$

where $\Gamma_{t,T}$ is a set of all stopping times taking values in $[t, T]$.

Now, let

$$\xi(t) = e^{-rt}\frac{S(t)}{S(0)} = exp\{-\frac{1}{2}\sigma^2 t + \sigma W(t) - \lambda E[X-1]t\}\prod_{k=1}^{N_t} X_k. \tag{5.4}$$

We know from (3.19) that $\xi(t)$ defined by (5.4) is a martingale. Therefore, by Girsanov theorem (see T. Björk, [3] p. 164), let us introduce a new equivalent measure Q' such that $dQ' = \xi(T)dQ$, thus, the process $W^{Q'} = W^Q - \sigma t$ [6] and [9], is a standard Brownian motion under Q' and the stock price satisfies the stochastic differential equation

$$dS(t) = (r + \sigma^2 - \lambda K)Sdt + \sigma dW^{Q'}(t) + (X-1)dY(t), \tag{5.5}$$

where $K = E[X-1]$.

As in [6] let us transform (5.2) changing the measure Q into the equivalent measure Q'. Whence,

$$V(t) = ess \sup_{\tau \in \Gamma_{t,T}} E_t^Q \left\{ e^{-r(\tau-t)}[\rho(S(\tau) - A(\tau))]^+ \right\}$$

$$= ess \sup_{\tau \in \Gamma_{t,T}} E_t^{Q'} \left\{ \frac{\xi(t)}{\xi T} e^{-r(\tau-t)}[\rho(S(\tau) - A(\tau))]^+ \right\}$$

$$= ess \sup_{\tau \in \Gamma_{t,T}} E_t^{Q'} \left\{ \frac{S(t)}{e^{rt}} e^{-r(\tau-t)}[\rho(S(\tau) - A(\tau))]^+ E_\tau^{Q'}\left[\frac{e^{rT}}{S(T)}\right] \right\}$$

$$= ess \sup_{\tau \in \Gamma_{t,T}} E_t^{Q'} \left\{ \frac{S(t)}{e^{rt}} e^{-r(\tau-t)}[\rho(S(\tau) - A(\tau))]^+ E_\tau^{Q'}\left[\frac{e^{rT}}{S(T)}\right] \right\}$$

$$= ess \sup_{\tau \in \Gamma_{t,T}} E_t^{Q'} \left\{ \frac{S(t)}{e^{rt}} \frac{e^\tau}{S(\tau)} e^{-r(\tau-t)}[\rho(S(\tau) - A(\tau))]^+ \right\}$$

$$= ess \sup_{\tau \in \Gamma_{t,T}} E_t^{Q'} \left\{ S(t)[\rho(1 - \frac{A(\tau)}{S(\tau)})]^+ \right\}.$$

Therefore, we have reduced (5.2) into

$$V(t) = ess \sup_{\tau \in \Gamma_{t,T}} E_t^{Q'} \left\{ S(t)[\rho(1 - x(\tau))]^+ \right\} \tag{5.6}$$

32

where $x(\tau) = \frac{A(\tau)}{S(\tau)}$. According to Harrisson and Kreps (1979) [7], $\frac{V(t)}{S(t)}$ is a martingale.

Our next step is to derive the dynamic of $x(t)$ in order to continue with our studies. Thus, applying the Ito's formula for a jump diffusion process given by proposition (3.1.2) we have:

$$dx(t) = \frac{dA(t)}{S(t)} - \frac{(r + \sigma^2 - \lambda K)A(t)}{S^2(t)}dt + \frac{\sigma^2 A(t)}{S^2(t)}dt - \sigma\frac{A(t)}{S(t)}dW^{Q'} + \left[\frac{A(t_- + \Delta t)}{S(t_- + \Delta t)} - \frac{A(t_-)}{S(t_-)}\right]$$

$$dx(t) = x(t)\left[\left(\frac{d\ln A(t)}{dt} - r - \sigma^2 + \lambda K\right)dt - \sigma dW^{Q'}(t) + \sigma^2 dt + \frac{\Delta[A(t)] + (1 - X)A(t)}{XS(t)}\right].$$

Since we know that $\Delta t dN_t = 0$ then, $\frac{\Delta[A(t)]}{A(t)}dN_t = 0$. Therefore,

$$dx(t) = x(t)\left[\mu(t, x(t))dt - \sigma dW^{Q'}(t) + \frac{1 - X}{X}dN_t\right],$$

where $\mu(t, x(t)) = \left(\frac{d\ln A(t)}{dt} - r + \lambda K\right)$.

Hence,

$$dx(t) = x(t)\left[\mu(t, x(t))dt - \sigma dW^{Q'}(t) + \frac{1 - X}{X}dN_t\right]. \tag{5.7}$$

Going ahead with our studies, let us denote by $\widetilde{V}(t)$ the expression $\frac{V(t)}{S(t)}$, then problem (5.2) is reduced to the following one state variable problem (lets call it a dual problem) with strike price 1,

$$\widetilde{V}(t) = ess\sup_{\tau \in \Gamma_{t,T}} E_t^{Q'}\left\{[\rho(1 - x(\tau))]^+\right\}, \tag{5.8}$$

where $x(t)$ has dynamic defined by (5.29). The optimal stopping time for this problem is τ_t^* such that

$$\tau_t^* = \inf\{\tau \in [t, T] : x(\tau) = b(\tau)\},$$

where $b(\tau)$ is a boundary of the continuation (or the exercise) region. The regions has the following presentations:

33

1) Continuation region: $C = \{t \in [0, T] : \ \rho x(t) > \rho b(t)\}$;

2) Stopping region: $D = \{t \in [0, T] : \ \rho x(t) \leq \rho b(t)\}$.

From now on, we will study the dual problem (5.8), since $V(t) = S(t)\widetilde{V}(t)$. If there is no jump in the stock, then the problem is basically that was studied by A.T Hansen and P.L. Jorgensen [6] (1997), for which the solution of (5.8) is given by

$$\widetilde{V}(t) = \widetilde{v}(t) + \widetilde{e}(t), \tag{5.9}$$

where

$$\widetilde{v}(t) = E_t^{Q'}\left\{[\rho(1 - x(T))]^+\right\} \tag{5.10}$$

$$\widetilde{e}(t) = E_t^{Q'}\left\{\int_t^T \rho\mu_1(\tau, x(\tau))x(\tau)\mathbf{1}_D d\tau\right\} = \int_t^T E_t^{Q'}\left\{\rho\mu_1(\tau, x(\tau))x(\tau)\mathbf{1}_D\right\} d\tau. \tag{5.11}$$

and $\mu_1(x(t), t) = \mu(x(t), t) - \lambda K$. The first part of (5.9) in right hand side, is the corresponding solution for European put option and the second is a earlier exercise premium. Let us suppose that the jump process in the interval $[t, T]$ has occurred. Then by the results in C. R. Gukhal [5] (2001) or Huyen Pham [15] (1997), the solution of the dual problem (5.8) will be given as follow

$$\widetilde{V}(t) = \sum_{n=0}^{\infty} \int \left[\frac{e^{-\lambda(T-t)}(\lambda(T-t))^n}{n!}\widetilde{v}(t, x(t)Z_n e^{\lambda Kt}) + \widetilde{e}_J(t, x(t)Z_n e^{\lambda Kt})\right] F_n(dz) - $$
$$- \lambda E_t^{Q'}\left\{\int_t^T E\left[g(J, x(s), b(s))\right] ds\right\}, \tag{5.12}$$

where,

$$\widetilde{e}_J(t, x(t)Z_n e^{\lambda Kt}) = \int_t^T \frac{e^{-\lambda(\tau-t)}(\lambda(\tau-t))^n}{n!}E_{t, x(t)Z_n e^{\lambda Kt}}^{Q'}\left\{\rho\mu_1(\tau, x(\tau))x(\tau)\mathbf{1}_D\right\} d\tau,$$
$$g(J, x(s), b(s)) = \widetilde{V}(Jx(s), s) - (1 - Jx(s))1_{\rho\{x(s) \leq \rho b(s), \rho Jx(s) > \rho b(s)\}},$$

and $\widetilde{v}(t)$ is defined by (5.10), F_n is a distribution function of

$$Z_n = \prod_{k=1}^{n}\frac{1}{X_k} = \prod_{k=1}^{n} J_k.$$

34

Let us adopt the following notation,

$$\int \widetilde{v}(t, x(t)Z_n e^{\lambda Kt})F_n(dz) = E_n[\widetilde{v}(t, x(t)Z_n e^{\lambda Kt})]$$

and

$$\int \widetilde{e}_J(t, x(t)Z_n e^{-\lambda Kt})F_n(dz) = E_n[\widetilde{e}_J(t, x(t)Z_n e^{\lambda Kt})].$$

So we have the following result:

Theorem 5.0.1. *The solution to the dual problem* (5.8) *when the underlying stock returns are discontinuous, is given by*

$$\widetilde{V}(t) = \sum_{n=0}^{\infty}\left[\frac{e^{-\lambda(T-t)}(\lambda(T-t))^n}{n!}E_n[\widetilde{v}(t, x(t)Z_n e^{\lambda Kt})] + E_n[\widetilde{e}_J(t, x(t)Z_n e^{\lambda Kt})]\right] - $$
$$-\lambda E_t^{Q'}\left\{\int_t^T E\left[g(J, x(s), b(s))\right]ds\right\}. \tag{5.13}$$

where the first part in the right hand side, is the value of the corresponding European option with jumps, the second two terms correspond to the earlier exercise premium (the bonus by exercising the option before the maturity time T). The earlier exercise premium is composed by two terms, the first of the last two terms is a current value of the premium and the last one is the rebalancing cost due to jumps from the exercise region into continuation region (see C.R. Gukhal [5] (2001)). The last part of the right hand side, there is no an explicit form of it.

Proof: Since we know that $\widetilde{V}$ is a martingale under the measure Q', then in the continuation region $C = \{t \in [0, T] : \rho x(t) > \rho b(t)\}$ the function $\widetilde{V}$ must satisfy the equation

$$d\widetilde{V} = \widetilde{V}_t dt + \widetilde{V}_x dx + \widetilde{V}_{xx}(dx)^2. \tag{5.14}$$

Therefore, from H. Pham [15] it is shown that, in a continuation region,

$$\widetilde{V} = \sum_{n=0}^{\infty}\frac{e^{\lambda(T-t)}(\lambda(T-t))^n}{n!}E_n[\widetilde{v}(t, x(t)Z_n e^{\lambda Kt})], \tag{5.15}$$

and, R. C. Merton [12] (1997), have proved that the expression (5.15) is a solution to the problem, and by the martingale property in the continuation region

$$d\widetilde{V} = dM_1^{Q'}, \tag{5.16}$$

35

where M_1 is a martingale under measure Q'.

In other hand, if x belongs to the stopping region $D = \{t \in [0, T] : \rho x(t) \leq \rho b(t)\}$, then $\widetilde{V}(t) = \rho(1 - x(t))$, hence by H. Pham [15],

$$d\widetilde{V} = -\rho\mu_1(x(t), t)x(t)dt + \lambda E[\widetilde{V}(Jx, t) - (1 - Xx)1_{\{Jx > b(t)\}}]dt. \qquad (5.17)$$

From (5.16) and (5.17) we have

$$d\widetilde{V} = \{-\rho\mu_1(x(t), t)x(t)dt + \lambda E[\widetilde{V}(Jx, t) - (1 - Jx)1_{\{Jx > b(t)\}}]dt\}_{\{x(t) \leq b(t)\}} + dM^{Q'}, \qquad (5.18)$$

where $M^{Q'}$ is a martingale part under measure Q', and then the result follows. $\square$

From G. Peskir [14] or T. Björk [3], we know that in the exercise region $\widetilde{V}(t) = \rho(1 - x(t))$, then the the exercise boundary must satisfy the following free boundary equation:

$$\rho(1 - b(t)) = \sum_{n=0}^{\infty} \left[\frac{e^{-\lambda T(-t)}(\lambda(T - t))^n}{n!} E_n[\widetilde{v}(t, b(t)Z_n e^{\lambda K t})] + E_n[\widetilde{e}_J(t, b(t)Z_n e^{\lambda K t})] \right] - \qquad (5.19)$$

$$-\lambda E_{t,b(t)}^{Q'} \left\{ \int_t^T E[g(J, x(s), b(s))]\, ds \right\}.$$

When the jump intensity λ is small enough or if the jump sizes have a small mean then, it will cause very small chances in the American option. So, the cost term in the solution of American option should be very small (see Kou et all (2005), [11]). Therefore, in this circumstances, the cost term is negligenciable.

5.1 Pricing the American-style Asian options under jump-diffusion processes, in the case of geometric average

To study this case, we will follow the previous theory taking in consideration that

$$A(t) = e^{\frac{1}{t} \int_0^t \ln S(\tau)d\tau}.$$

In this case, $\dfrac{dA(t)}{A(t)} = \left(-\dfrac{1}{t^2} \int_0^t \ln S(\tau)d\tau - \ln S(t) \right) = -\dfrac{1}{t} \ln x(t)dt$. Therefore, the dynamics of the underlying asset $x(t)$ will be defined by

$$dx(t) = x(t) \left[\mu_g(t, x(t))dt - \sigma dW^{Q'}(t) + (J-1)dN_t \right], \tag{5.20}$$

where $[\mu_g(t, x(t)) = -\frac{1}{t}\ln x(t) - r + \lambda K$. If the jump process does not occur then the dynamics os $x(t)$ becomes,

$$dx(t) = x(t) \left[\mu_g(t, x(t))dt - \sigma dW^{Q'}(t) \right]. \tag{5.21}$$

Before we proceed with our studies, let us give the following lemmas from the paper of Hansen ad Jorgensen 1997 [6].

Lemma 5.1.1. *(Hansen and Jorgensen (2000) [6]) For $u > t$*

$$\ln x(t) \sim N(\alpha_g(t, u), \beta_g^2(t, u), \tag{5.22}$$

where

$$\alpha_g(t, u) = \frac{t}{u}\ln x(t) - \frac{u^2 - t^2}{2u}(r + \frac{1}{2}\sigma^2) \tag{5.23}$$

and

$$\beta_g^2(t, u) = \frac{\sigma^2}{3u^2}(u^3 - t^3). \tag{5.24}$$

Lemma 5.1.2. *(Hansen and Jorgensen (2000) [6]) Let $\ln V \sim N(\alpha, \beta^2)$ and define $\gamma = \frac{\alpha + \beta^2 - \ln L}{\beta}$.*

Assuming $L > 0$ and letting $\Phi(\cdot)$ and $\phi(\cdot)$ denote the cumulative distribution and density functions, respectively, we have:

1) $E[V \cdot 1_{\{V \geq L\}}] = e^{\alpha + \frac{1}{2}\beta^2}\Phi(\gamma);$

2) $E[V \cdot 1_{\{V \leq L\}}] = e^{\alpha + \frac{1}{2}\beta^2}\Phi(-\gamma);$

3) $E[(V - L)^+] = e^{\alpha + \frac{1}{2}\beta^2}\Phi(\gamma) - L\Phi(\gamma - \beta);$

4) $E[(L-V)^+] = L\Phi(\beta - \gamma) - e^{\alpha + \frac{1}{2}\beta^2}\Phi(-\gamma);$

5) $E[V \ln V \cdot 1_{\{V \geq L\}}] = e^{\alpha + \frac{1}{2}\beta^2}(\beta\phi(\gamma) + (\alpha + \beta^2)\Phi(\gamma));$

6) $E[V \ln V \cdot 1_{\{V \leq L\}}] = e^{\alpha + \frac{1}{2}\beta^2}((\alpha + \beta^2)\Phi(-\gamma) - \beta\phi(\gamma)).$

Using Lemma 5.1.1 and Lemma 5.1.2 with $L = 1$ we have

$$\widetilde{v}(t,x) = E_t[\rho(1 - x(T))]^+$$

$$= \rho\left\{\Phi(-\rho(\beta_g(t,T) - \gamma_g(t,T))) - e^{\alpha_g(t,T) + \frac{1}{2}\beta_g^2(t,T)}\Phi(-\rho\gamma_g(t,T))\right\}$$

$$= \rho\left\{\Phi\left(-\rho\frac{\alpha_g(t,T)}{\beta_g(t,T)}\right) - e^{\alpha_g(t,T) + \frac{1}{2}\beta_g^2(t,T)}\Phi\left(-\rho\frac{\alpha_g(t,T) + \beta_g(t,T)^2}{\beta_g(t,T)}\right)\right\}$$

and,

$$\widetilde{e}(t,x) = \int_t^T \rho E[(-r - \frac{1}{\tau}\ln x(\tau))x(\tau) \cdot 1\{\rho x(\tau) \leq \rho b(\tau)\}]d\tau$$

$$= \int_t^T e^{\alpha_g(t,\tau) + \frac{1}{2}\beta_g^2(t,\tau)}[-\rho(\frac{\alpha_g(t,\tau) + \beta_g^2(t,\tau)}{\tau} + r)\Phi(-\rho\frac{\alpha_g(t,\tau) + \beta_g^2(t,\tau) - \ln b(\tau)}{\beta_g(t,\tau)})$$

$$+ \frac{\beta_g(t,\tau)}{\tau}\phi(\frac{\alpha_g(t,\tau) + \beta_g^2(t,\tau) - \ln b(\tau)}{\beta_g(t,\tau)})]d\tau.$$

Here $L = b(t)$.

Thus, Theorem 5.0.1, $\widetilde{v}(t)$ and $\widetilde{e}_1(t)$ prove the following Theorem

38

Theorem 5.1.3. *In the geometric average case, the solution to the problem (5.8), is given by*

$$\widetilde{V}(t) = \sum_{n=0}^{\infty} E_n \left\{ \frac{e^{-\lambda(T-t)}(\lambda(T-t))^n}{n!} \widetilde{v}_n(t) + \widetilde{e}_{Jgn}(t, xZ_n e^{\lambda K(\tau-t)}) \right\} - \lambda E_{t,x}^{Q'} \left\{ \int_t^T E\left[g(J, x(\tau), b(\tau)) \right] d\tau \right\},$$

where,

$$\widetilde{v}_n(t) = \widetilde{v}(t, xZ_n e^{\lambda K(\tau-t)}) \ \ and$$

$$\widetilde{e}_{Jgn}(t, xZ_n e^{\lambda K(\tau-t)}) =$$

$$= \int_t^T \frac{e^{-\lambda(\tau-t)}(\lambda(\tau-t))^n}{n!} \rho e^{\alpha_g(t,\tau)+\frac{1}{2}\beta_g^2(t,\tau)} \left[-\left(\frac{\alpha_g(t,\tau) + \beta_g^2(t,\tau)}{\tau} + r \right) \Phi\left(-\rho \frac{\alpha_g(t,\tau) + \beta_g^2(t,\tau) - \ln b(\tau)}{\beta_g(t,\tau)} \right) \right.$$

$$\left. + \frac{\beta_g(t,\tau)}{\tau} \phi\left(\frac{\alpha_g(t,\tau) + \beta_g^2(t,\tau) - \ln b(\tau)}{\beta_g(t,\tau)} \right) \right] d\tau, \ \ with \ \ x(t) = xZ_n e^{\lambda K(\tau-t)}$$

The case when jumps sizes are i.i.d. lognormal random variables

Here we will give the solution for the case of geometric average under lognormal jump sizes. So, recall from Chapter 3.1 that

$$S(t) = S(0) \exp\left\{ (r + \lambda K - \frac{1}{2}\sigma^2)(T - t) + \sigma(W(T) - W(t)) + \sum_{k=1}^{N_t} \ln X_k \right\}, \qquad (5.25)$$

where $\ln X_k \sim N(a, b^2)$, , $k = 1, 2, \ldots, N_t$. So, if in the interval $[t, T]$ we have exactly n jumps, then we know that

$$(r + \frac{1}{2}\sigma^2 - \lambda K)(T - t) + na + \sqrt{\sigma^2 + \frac{nb^2}{T - t}}(W(T) - W(t)) \sim$$
$$N((r + \frac{1}{2}\sigma^2 - \lambda K)(T - t) + na, \sigma^2(T - t) + nb^2).$$
Whence,

$$S(T) = S(t) \exp\left\{ (r + \frac{1}{2}\sigma^2 - \lambda K)(T - t) + na + \sqrt{\sigma^2 + \frac{nb^2}{T - t}}(W(T) - W(t)) \right\} \qquad (5.26)$$

and hence

$$\ln S(T) \sim N(\ln S(t) + (r + \frac{1}{2}\sigma^2 - \lambda K)(T - t) + na, (\sigma^2 + \frac{nb^2}{T-t})(T-t). \tag{5.27}$$

Therefore, given $A(T) = e^{\frac{1}{T}\int_0^T \ln S(u)du}$. Then,

$$\ln A(T) = \frac{1}{T}\int_0^T \ln S(u)du = \frac{1}{T}\int_0^t \ln S(u)du + \frac{1}{T}\int_t^T \ln S(u)du$$

$$= \frac{t}{T}\ln A(t) + \frac{1}{T}\int_t^T \left(\ln S(t) + (r + \frac{1}{2}\sigma^2 - \lambda K)(u - t) + na + \sqrt{\sigma^2 + \frac{nb^2}{u-t}}\int_t^T dW(\tau)\right)du$$

$$= \frac{t}{T}\ln A(t) + \frac{T-t}{T}\ln S(t) + ((r + \frac{1}{2}\sigma^2 - \lambda K)\frac{(T-t)^2}{2T} + \frac{na(T-t)}{T}$$

$$+ \frac{1}{T}\int_t^T \sqrt{\sigma^2 + \frac{nb^2}{u-t}}\int_t^u dW(\tau)du$$

$$= \frac{t}{T}\ln A(t) + \frac{T-t}{T}\ln S(t) + ((r + \frac{1}{2}\sigma^2 - \lambda K)\frac{(T-t)^2}{2T} + \frac{na(T-t)}{T}$$

$$+ \frac{1}{T}\int_t^T \int_\tau^T \sqrt{\sigma^2 + \frac{nb^2}{u-t}}dudW(\tau)$$

$$= \frac{t}{T}\ln A(t) + \frac{T-t}{T}\ln S(t) + ((r + \frac{1}{2}\sigma^2 - \lambda K)\frac{(T-t)^2}{2T} + \frac{na(T-t)}{T}$$

$$+ \frac{1}{T}\int_t^T \left(\theta(T,\tau,t) + T\sqrt{\sigma^2 + \frac{nb^2}{T-t}}\right)dW(\tau),$$

where,

$$\theta(T,\tau,t) = \int_\tau^T \sqrt{\sigma^2 + \frac{nb^2}{u-t}}du - T\sqrt{\sigma^2 + \frac{nb^2}{T-t}}$$

40

Hence it leads us to the following result,

Lemma 5.1.4. *Let $S(t)$ satisfying the jump diffusion equation*

$$dS(t) = (r + \lambda K + \sigma^2)S(t)dt + \sigma S(t)dW(t) + S(t)(X - 1)dY_t,$$

with $\ln X \sim N(a, b^2)$ and define $A(t) = e^{\frac{1}{t}\int_0^t \ln S(\tau)}$. Then for $T > t$, $\ln A(T)$ conditioned to F_t and n jumps follows a normal distribution with mean and variance given by

$$E[\ln A(t)] = \frac{t}{T}\ln A(t) + \frac{T-t}{T}\ln S(t) + ((r - \frac{1}{2}\sigma^2 - \lambda K)\frac{(T-t)^2}{2T} + \frac{na(T-t)}{T}$$

and

$$Var[A(t)] = \frac{1}{T^2}\int_t^T \left(\theta(T,\tau,t) + T\sqrt{\sigma^2 + \frac{nb^2}{T-t}}\right)^2 d\tau.$$

Now let us find the distribution of $\ln X(T)$. Since $X(T) = \frac{A(T)}{S(T)}$ then,

$$\ln X(T) = \ln A(T) - \ln S(T) = \frac{t}{T}\ln A(t) + \frac{T-t}{T}\ln S(t) + ((r - \frac{1}{2}\sigma^2 - \lambda K)\frac{(T-t)^2}{2T} + \frac{na(T-t)}{T}$$

$$+\frac{1}{T}\int_t^T \left(\theta(T,\tau,t) + T\sqrt{\sigma^2 + \frac{nb^2}{T-t}}\right)dW(\tau) - \ln S(t) - (r + \frac{1}{2}\sigma^2 - \lambda K)(T-t)$$

$$-na - \sqrt{\sigma^2 + \frac{nb^2}{T-t}}\int_t^T dW(\tau)$$

$$= \ln x(t) - \left(r - \lambda K + \frac{\sigma^2}{2}\right)\frac{(T^2 - t^2)}{2T} - \frac{nat}{T} + \frac{1}{T}\int_t^T \theta(T,\tau,t)dW(\tau).$$

And so, we have proved the following lemma,

Lemma 5.1.5. *Let $T > t$ then $\ln x(T)|(\boldsymbol{F}_t \wedge N_t = n)$, follows a normal distribution with mean and variance given by*

41

$$\alpha_n(T,t) = E[x(T)] = \ln x(t) - \left(r - \lambda K + \frac{\sigma^2}{2}\right)\frac{(T^2 - t^2)}{2T} - \frac{nat}{T}$$

and

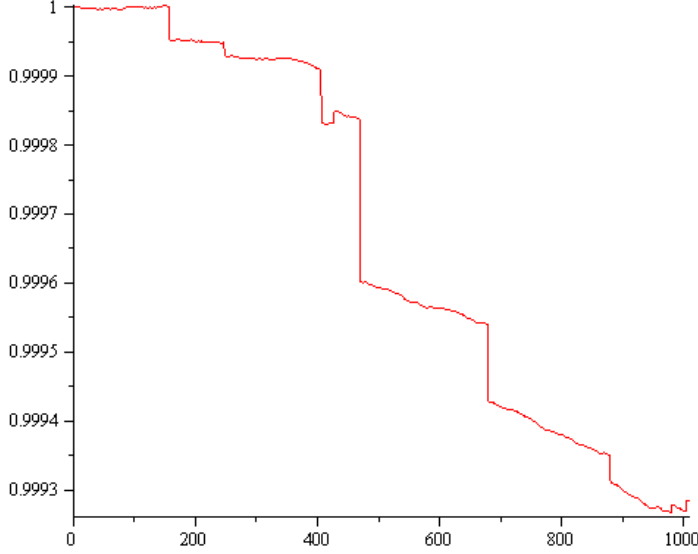$$\beta_n^2(T,t) = Var[x(T)] = \frac{1}{T^2}\int_t^T \theta^2(T,\tau,t)d\tau.$$

Figure 5.1: $x(t)$ with jumps, $\sigma = 0.2$, $T = \frac{7}{12}$, $r = 0.05$.

According to Lemma 5.1.2 and Lemma 5.1.5 the solution to the problem (5.8), in the geometric average case, under lognormal jump sizes, is given by

$$\widetilde{V}(t) = \sum_{n=0}^{\infty}\left\{\frac{e^{-\lambda(T-t)}(\lambda(T-t))^n}{n!}\widetilde{v}_n(t) + \widetilde{e}_{Jgn}(x,t)\right\} - \lambda E_t^{Q'}\left\{\int_t^T E\left[g(J,x(\tau),b(\tau))\right]d\tau\right\},$$

where,

$$\widetilde{v}_n(t) = \rho\left\{\Phi\left(-\rho\frac{\alpha_{g,n}(t,T)}{\beta_{g,n}}\right) - e^{\alpha_{g,n}(t,T)+\frac{1}{2}\beta_{g,n}^2(t,T)}\Phi\left(-\rho\frac{\alpha_{g,n}(t,T) + \beta_{g,n}^2(t,T)}{\beta_{g,n}(t,T)}\right)\right\},$$

42

and,

$$\widetilde{\overline{e}}_{Jgn}(t) = \int_t^T \frac{e^{-\lambda(\tau-t)}(\lambda(\tau-t))^n}{n!} \exp\{\alpha_{g,n}(t,\tau) + \frac{1}{2}\beta_{g,n}^2(t,\tau)\} \times$$

$$\times [-\rho \left(\frac{\alpha_{g,n}(t,\tau) + \beta_{g,n}^2(t,\tau)}{\tau} + r \right) \Phi\left(-\rho\gamma_{g,n}(t,\tau)\right) + \frac{\beta_{g,n}}{\tau}\phi\left(\gamma_{g,n}(t,\tau)\right)]d\tau.$$

5.2 Pricing the American-style Asian options under jump-diffusion processes, in the case of arithmetic average

In this section we will discuss in details the case when the average $A(t)$ is arithmetic. In this case $A(t)$ is given by $A(t) = \frac{1}{t}\int_0^t S(\tau)d\tau$ and $\frac{dA(t)}{A(t)} = \frac{1}{t}(\frac{1}{x(t)} - 1)dt$.

Therefore, the dynamics of $x(t)$ becomes

$$dx(t) = x(t)\left[\mu_a(t,x(t))dt - \sigma dW^{Q'}(t) + \frac{1-X}{X}dN_t\right], \tag{5.28}$$

where $\mu_a(t) = \frac{1}{t}(\frac{1}{x(t)} - 1) - r + \lambda K$.

Let us consider the case when the jump process does not occur, so the dynamics of $x(t)$ will be

$$dx(t) = x(t)\left[\mu_{a1}(t,x(t))dt - \sigma dW^{Q'}(t)\right]. \tag{5.29}$$

In this case, the distribution of $x(t)$ is in unknown. There is considered an approximation of $x(t)$ by $\hat{x}(t)$ which follows a lognormal distribution. This approximation is well done using the Wilkinson approximation (see P. Pirinen, [17]). From Hansen and Jorgensen [6], we have that for $T > t$ $\ln \hat{x}(T)|\mathbf{F}_t$ follows a normal distribution with mean and variance given by

$$\alpha_a(T,t) = 2E_t^{Q'}\{x(T) - \frac{1}{2}\ln E_t^{Q'}\{x^2 T\} \tag{5.30}$$

$$\beta_a^2(T,t) = \ln E_t^{Q'}\{x^2(T)\} - 2\ln E_t^{Q'}\{x(T)\}. \tag{5.31}$$

Here,

$$E_t^{Q'}\{x(T)\} = \frac{t}{T}x(t)e^{-r(T-t)} + \frac{1}{rT}(1 - e^{-r(T-t)})$$

and,

$$E_t^{Q'}\{x^2(T)\} = (\frac{t}{T})^2 x^2(t) e^{-(2r-\sigma^2)(T-t)} + \frac{2(r-\sigma^2) - (4r - 2\sigma^2)e^{-r(T-t)} + 2re^{-(2r-\sigma^2)(T-t)}}{T^2 r(2r-\sigma^2)(r-\sigma^2)}$$

$$+ x(t)\frac{2te^{-r(T-t)}}{T^2(r-\sigma^2)}(1 - e^{-(r-\sigma^2)(T-t)}).$$

Now, using Theorem 5.0.1, Lemma 5.1.2, (5.30) and (5.31) we prove the following result:

Theorem 5.2.1. *Let $A(t)$ be the arithmetic average. Then the approximated solution of* (5.8) *under jump diffusion process is given by*

$$\hat{\widetilde{V}}(t) = \sum_{n=0}^{\infty} E_n\{\frac{e^{-\lambda(T-t)}(\lambda(T-t))^n}{n!}\hat{\widetilde{v}}_{an}(t, Z_n\hat{x}e^{K(T-t)}) + \hat{\widetilde{e}}_{Jan}(t, Z_n\hat{x}e^{K(T-t)})\} - \lambda E_{t,\hat{x}}^{Q'}\left\{\int_t^T E[g(J\hat{x}(\tau))]d\tau\right\},$$

where,

$$g(J\hat{x}(\tau)) = [\widetilde{V}(J\hat{x}(\tau), \tau) - (1 - J\hat{x}(\tau))]1_{\{\rho\hat{x}(\tau-)\leq\rho b(\tau), \rho J\hat{x}(\tau-)>\rho b(\tau)\}},$$

$$\hat{\widetilde{v}}_a(t, Z_n\hat{x}e^{\lambda K(T-t)}) =$$
$$= \rho\left\{\Phi\left(\frac{\alpha_a(t, T, \hat{x}Z_n e^{K(T-t)})}{-\rho\beta_a(t, T)}\right) - e^{\alpha_a(t, T, \hat{x}Z_n e^{\lambda K(T-t)}) + \frac{1}{2}\beta_a^2(t, T)}\Phi\left(\frac{\alpha_a(t, T, \hat{x}Z_n e^{\lambda}K(T-t)) + \beta_a^2(t, T)}{-\rho\beta_a(t, T)}\right)\right\}$$

and

$$\hat{\widetilde{e}}_{Jan}(t, \hat{x}Z_n e^{K(T-t)}) =$$
$$\rho\int_t^T \frac{e^{-\lambda(\tau-t)}(\lambda(\tau-t))^n}{n!}\{\frac{1}{\tau}\Phi(\rho(\beta_a(t,\tau) - \gamma_a(t,\tau, \hat{x}Z_n e^{K(\tau-t)}))) - (r + \frac{1}{\tau})e^{\alpha_a(t,\tau,\hat{x}Z_n e^{K(\tau-t)}) + \frac{1}{2}\beta_a^2(t,\tau)} \times$$

$$\times\Phi(-\rho\gamma_a(t, \tau, \hat{x}Z_n e^{K(\tau-t)}))\}d\tau.$$

If the jump sizes are lognormal i.i.d random variables, then we know that $\ln S(T)$ has normal distribution given by will be given by (5.27). Here our aim is to determine the mean and variance of the approximated process $\hat{x}(t)$ in this particular case. Since $x(T) = \frac{\frac{1}{T}\int_0^T S(\tau)d\tau}{S(T)}$, then

$$E_t^{Q'}[x(T)] = \frac{1}{T} \int_0^T \frac{S(\tau)}{S(t)} d\tau E_t^{Q'}[\frac{S(t)}{S(T)}] + \frac{1}{T} \int_t^T E_{t,x}^{Q'}[\frac{S(\tau)}{S(T)}] d\tau.$$

So, $E_{t,x}^{Q'}[\frac{S(t)}{S(T)}] = \exp\{(r - \lambda K)(t - T) - \sigma^2 T - nb^2 + \sqrt{(\sigma^2 + \frac{nb^2}{t})(\sigma^2 + \frac{nb^2}{T})t}\} = \nu(t,T)$.
Therefore,

$$E_t^{Q'}[x(T)] = \frac{t}{T} x(t) \nu(t,T) + \frac{1}{T} e^{-(r - \lambda K + \sigma^2)T - nb^2} \int_t^T \nu(\tau, T) d\tau. \tag{5.32}$$

And,

$$E_t^{Q'}\left[x^2(T)\right] = E_t^{Q'}\left[\left(\frac{\frac{1}{T}\int_0^T S(\tau) d\tau}{S(T)}\right)^2\right]$$

$$= E_t^{Q'}[\left(\frac{1}{T} \int_0^T \frac{S(\tau)}{S(t)} d\tau E_t^{Q'}[\frac{S(t)}{S(T)}] + \frac{1}{T} \int_t^T E_{t,x}^{Q'}[\frac{S(\tau)}{S(T)}] d\tau\right) \times$$

$$\times \left(\frac{1}{T} \int_0^T \frac{S(u)}{S(t)} du E_t^{Q'}\left[\frac{S(t)}{S(T)}\right] + \frac{1}{T} \int_t^T E_t^{Q'}\left[\frac{S(u)}{S(T)}\right] du\right)]$$

$$= \frac{1}{T^2} \int_0^T \frac{S(\tau)}{S(t)} d\tau E_t^{Q'}\left[\frac{S(t)}{S(T)}\right] \int_0^T \frac{S(u)}{S(t)} du E_t^{Q'}\left[\frac{S(t)}{S(T)}\right]$$

$$+ \frac{2}{T^2} \int_0^T \frac{S(\tau)}{S(t)} d\tau E_t^{Q'}\left[\frac{S(t)}{S(T)}\right] \int_t^T E_t^{Q'}\left[\frac{S(u)}{S(T)}\right] du + \frac{1}{T^2} \int_t^T \int_t^T E_t^{Q'}\left[\frac{S(\tau)S(u)}{S(T)S(T)}\right] du d\tau.$$

Let $a(t) = \sqrt{\sigma^2 + \frac{nb^2}{t}}$ then,

$\frac{S(\tau)S(u)}{S(T)S(T)} = e^{(r - \lambda K + \sigma^2/2)(\tau + u - 2T) + a(\tau)W^{Q'}(\tau) + a(u)W^{Q'}(u) - 2a(T)W^{Q'}(T)}$.

If we suppose that $u \leq \tau \leq T$ then, let

$$\widetilde{\sigma}^2 = \sigma^2(\tau + u + 4T) + 3nb^2 + a(\tau)a(u)u - 4a(\tau)a(T)\tau - 4a(u)a(T)u.$$

Hence,

$$\frac{S(\tau)S(u)}{S(T)S(T)} = e^{(r - \lambda K + \frac{\sigma^2}{2})(\tau + u - 2T) + \widetilde{\sigma}Y}, \text{ where } Y \sim N(0,1).$$

Wherefore,

$$E_t^{Q'}\left[\frac{S(\tau)S(u)}{S(T)S(T)}\right] = e^{(r-\lambda K+\frac{\sigma^2}{2})(\tau+u-2T)-\frac{1}{2}\tilde{\sigma}^2} = \tilde{\nu}(u,\tau,T). \text{ Thus,}$$

$$E_t^{Q'}\left[x^2(T)\right] = \left(\frac{t}{T}\right)^2 x^2(t)\tilde{\nu}(t,t,T) + \frac{2t}{T^2}x(t)\nu(t,T)\int_t^T \nu(\tau,T)d\tau + \frac{1}{T^2}\int_t^T\int_t^T \tilde{\nu}(u,\tau,T)d\tau du. \tag{5.33}$$

The approximated process $\hat{x}(T)$ will have the mean and variance defined by (5.30) and (5.31), with $E_t^{Q'}[x(T)]$ and $E_t^{Q'}[x^2(T)]$ defined by (5.32) and (5.33), respectively.

Thus, the approximated solution to the dual problem (5.8) will be
$$\hat{\tilde{V}}(t) = \sum_{n=0}^{\infty}\frac{e^{-\lambda(T-t)}(\lambda(T-t))^n}{n!}\hat{\tilde{v}}_{an}(t,\hat{x}) + \sum_{n=0}^{\infty}\hat{\tilde{e}}_{Jan}(\hat{x},t,\tau) - \lambda E_{t,\hat{x}}^{Q'}\left\{\int_t^T E[g(J\hat{x}(\tau))]d\tau\right\},$$
where,

$$\hat{\tilde{\tilde{v}}}_{an}(\hat{x},t) =$$
$$= \rho\left\{\Phi\left(\frac{\alpha_{an}(t,T)}{-\rho\beta_{an}(t,T)}\right) - e^{\alpha_{an}(t,T)+\frac{1}{2}\beta_{an}^2(t,T)}\Phi\left(\frac{\alpha_{an}(t,T)+\beta_{an}^2(t,T)}{-\rho\beta_{an}(t,T)}\right)\right\}$$

and

$$\hat{\tilde{\tilde{e}}}_{Jan}(\hat{x},t) =$$
$$\rho\int_t^T\frac{e^{-\lambda(\tau-t)}(\lambda(\tau-t))^n}{n!}\left\{\frac{1}{\tau}\Phi(\rho(\beta_{an}(t,\tau)-\gamma_{an}(t,\tau))) - (r+\frac{1}{\tau})e^{\alpha_{an}(t,\tau)+\frac{1}{2}\beta_{an}^2(t,\tau)}\Phi(-\rho\gamma_{an}(t,\tau))\right\}d\tau.$$
Here,

$$\alpha_{an}(t,T) = 2E_t^{Q'}\{x(T)\} - \frac{1}{2}\ln E_t^{Q'}\{x^2(T)\} \tag{5.34}$$

$$\beta^2(t,T) = \ln E_t^{Q'}\{x^2(T)\} - 2\ln E_t^{Q'}\{x(T)\} \tag{5.35}$$

$$\gamma_{an}(t,T) = \frac{\alpha_{an}(t,T) + \beta^2(t,T) - \ln b(t)}{\beta_{an}(t,T)}, \tag{5.36}$$

with $E_t^{Q'}[x(T)]$ and $E_t^{Q'}[x^2(T)]$ defined respectively, by (5.32) and (5.33).

5.3 Free boundary and stopping region

The free boundary $b(t)$ is a smooth function (see Pham [15]) satisfying the following properties,

1)
$$\widetilde{v}(t, b(t)) + \widetilde{e}(t, b(t)) = \rho(1 - b(t)), \quad \forall t \in (0, T];$$

2) If $\rho x(t) \leq \rho b(t)$ (stopping region) then $\widetilde{v}(t, x(t))) + \widetilde{e}(t, x(t)) = \rho(1 - x(t))$;

3) If $\rho x(t) > \rho b(t)$ then solution $\widetilde{V}(t, x(t))$ satisfies the equation (2) in Proposition 4.1.1 and $\widetilde{V}(t, x(t)) > \rho(1 - x(t))$;

4) For a fixed time t, for all $\lambda_1, \lambda_2 \geq 0$ such that, $\lambda_1 < \lambda_2$ then $\rho b_{\lambda_1}(t) \geq \rho b_{\lambda_2}(t)$ (see for example, Figure 5.2);

5) For a fixed time t, for all a_1, a_2 such that, $a_1 < a_2$ then $\rho b_{a_1}(t) \geq \rho b_{a_2}(t)$ (see for example, Figure 5.3);

6) Contrary to the standard American options, in the American-style Asian options, $b(t)$ can be greater than the strike price (see Hansen and Jorgensen 2000, [6]).

5.4   Numerical results

First of all, let us consider the case when the jump process does not occur. So, by Kallast and Kivinukk 2003, [8], dividing the interval $[0, T]$ into n subintervals of length $\Delta t = t_k - t_{k-1}$, $k = 1, 2, \ldots, n$. At the maturity time T, the value of the boundary $b(T)$ is given by 1. Since $t_n = T$ then $b(t_n) = 1$ (see [8] or [18]). So, given $b(t_n)$, we can calculate $b(t_{n-1})$ by

$$\rho(1 - b(t_{n-1})) = \widetilde{v}(b(t_{n-1}), t_k) + \int_{t_{n-1}}^{t_n} f(b(t_{n-1}, b(\tau), \tau)) d\tau, \tag{5.37}$$

where, $\widetilde{v}(x(t), t)$ is a solution to the corresponding European option, $f(x(t), b(t), t, \tau)$ is an integrand in the earlier exercise premium.
In order to get the solution $b(t_{n-1})$ we use the trapezoidal rule to compute the integral in (5.37). Then we get,

$$\rho(1 - b(t_{n-1})) = \widetilde{v}(b(t_{n-1}), t_{n-1}, T) + \frac{\Delta t}{2}[f(b(t_{n-1}), b(t_n), t_{n-1}, T) + f(b(t_{n-1}), b(t_{n-1}), t_{n-1}, t_{n-1})],$$

and solving this equation, we get the value of $b(t_{n-1})$. For $k = n - 2, n - 3, \ldots, 1, 0$ we can compute the values of $b(t_k)$ by the trapezoidal rule as follows,

$$\rho(1 - b(t_k)) = \widetilde{v}(b(t_k), t_k, T)+$$
$$+\frac{\Delta t}{2}\left[f(b(t_k), b(t_n), t_k, T) + 2\sum_{j=k+1}^{n-1} f(b(t_k), b(t_j), t_k, t_j) + f(b(t_k), b(t_k), t_k, t_k)\right].$$

Here $f(y, y, t, t) = \lim_{\tau \longrightarrow t^+} f(y, y, t, \tau) = -\rho y \Phi\left(-\rho\frac{\frac{\ln y}{t} + r - \frac{1}{2}\sigma^2}{\sigma^2}\right)\left(\frac{\ln y}{t} + r\right).$

Now let us consider the case of jump-diffusion process. For simplicity, we will consider the case when the jump intensity λ is small enough, so the cost term will be negligenciable. In this case, in the exercise boundary $b(t)$ will satisfy the following equation,

$$\rho(1 - b(t)) = \sum_{n=0}^{\infty}\frac{e^{-\lambda t}(\lambda t)^n}{n!}E_n[\widetilde{v}(t, b(t)Z_n e^{\lambda K t})] + \sum_{n=0}^{\infty}\frac{e^{-\lambda t}(\lambda t)^n}{n!}E_n[\widetilde{e}(t, b(t)Z_n e^{\lambda K t})]. \quad (5.38)$$

So, numerically we have, $b(t_n) = 1$ and to obtain $b(t_{n-1})$ we have to solve the following equation,

$$\rho(1 - b(t_{n-1})) = \sum_{i=0}^{\infty}\frac{[\lambda(T - t_{n-1})]^i}{i!}e^{-\lambda(T-t_{n-1})}\widetilde{v}(b(t_{n-1}), t_{n-1}, T)$$
$$+\sum_{i=0}^{\infty}\int_{t_{n-1}}^{T} h(b(t_{n-1}), b(t_{n-1}), t_{n-1}, t_{n-1}, i),$$

and we obtain $b(k)$, $k = n - 2, n - 3, \ldots, 0$ from the following equations:

$$\rho(1 - b(t_k)) = \sum_{i=0}^{\infty}\frac{[\lambda(T - t_{n-1})]^i}{i!}e^{-\lambda(T-t_{n-1})}\widetilde{v}(b(t_k), t_k, T, i)+$$
$$+\sum_{i=0}^{\infty}\frac{\Delta t}{2}\left[h(b(t_k), b(t_n), t_k, T, i) + 2\sum_{j=k+1}^{n-1} h(b(t_k), b(t_j), t_k, t_j, i) + h(b(t_k), b(t_k), t_k, t_k, i)\right].$$

Here, $k = n - 2, n - 1, \ldots, 0$, $\widetilde{v}(x(t), t, i)$ plays the rule of a Black-Scholes formula for the corresponding European option and $h(x(t), b(t), t, \tau, i)$ is an integrand in the earlier exercise premium without the cost term. The values of $h(x, x, t, t, i)$ are determined by the following limit:

$$h(x, x, t, t, i) = \lim_{\tau \longrightarrow t^+} h(x, x, t, \tau, i) = \begin{cases} 0 & \text{if } i > 0 \\ -\rho x \Phi\left(-\rho\frac{\frac{\ln x}{t} + r - \frac{1}{2}\sigma^2}{\sigma^2}\right)\left(\frac{\ln x}{t} + r\right) & \text{if } i = 0 \end{cases}.$$

In general for American-style Asian put options (call options) with floating strike, the exercise boundary in a jump diffusion process is greater or equal (less or equal) to the exercise boundary

in a diffusion process case. This property gives the investor to have no much hope to get good profit at the start of time, but the hope increases when the time increase and from certain time, the investor starts again to lose a hope of getting a good profit. This occur because at time 0 (zero) the earlier exercise boundary is near to the strike and when time run, it increases in a put options (decreases in a call options) and it reaches its maximum (minimum) and then starts to decrease (increase) approaching again to the strike price.

The following figure is the result from the simulation of $b(t_k)$, by setting ρ to be equal to -1. In this case we have a call option for the dual problem (5.8) which corresponds to the American-style Asian put options with floating strike, under diffusion processes. It shows us the behaviour of the earlier exercise boundary when the values of the parameter λ change.
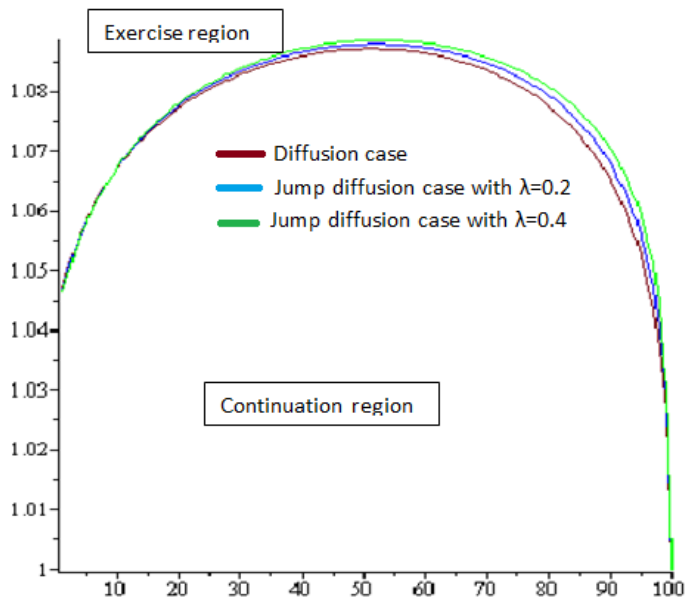

Figure 5.2: Exercise boundary for diffusion and jump diffusion cases, for an American-style Asian put option with $\sigma = 0.2$, $T = 7/12$, $r = 0.05$.

So it is possible to see that, the exercise boundary as a function of jump intensity λ, is a non-decreasing function in the case of American-style Asian put option and nonincreasing function for an American-style Asian call option, with floating strike price.

The picture bellow, shows how does the exercise boundary changes as the average of the jump sizes change.
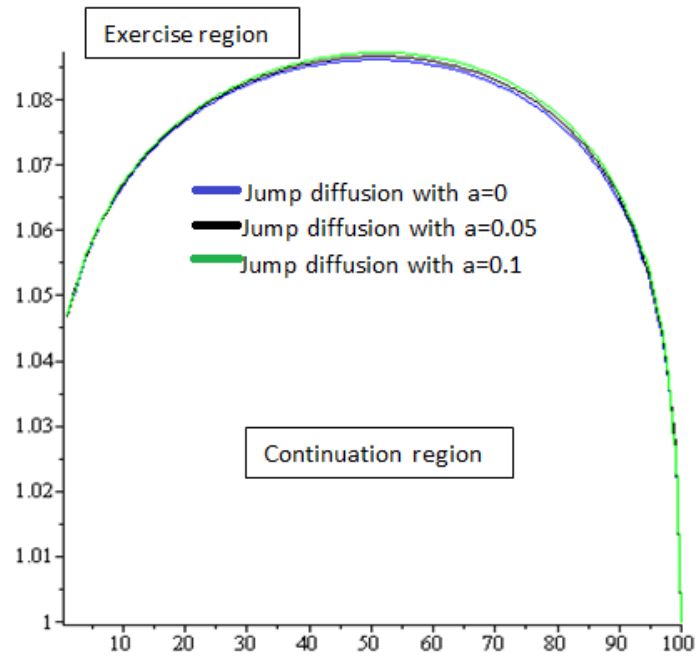
Figure 5.3: Exercise boundary for jump diffusion case, for an American-style Asian put option with $\sigma = 0.2$, $T = 7/12$, $r = 0.05$.

From this picture we can see that, the exercise boundary as a function of the jump-size is nondecreasing (nonincreasing) function for the American-style Asian put (call) option with floating strike price.

Chapter 6

Conclusions

In this thesis we extend the Analytical valuation of American-style Asian option under diffusion processes studied by Hansen and Jorgensen 2000 [6], to the case of jump-diffusion processes. This extension have never been considered anywhere before.

In our studies we derive the general solution for the American-style Asian options under jump-diffusion processes by solving the dual problem. For the case of geometric and arithmetic averages, respectively, we find analytical solutions. In the geometric average case we find that the one-state variable is a geometric Brownian motion and directly using the results of Merton 1976, Hansen and Jorgensen 2000, and Pham 1997, we derive its analytical solution. In the case of lognormal jumps, we derive a simplified solution to the problem. For the case of arithmetic average we find that the one-state variable is not a geometric Brownian motion and we approximate to a geometric Brownian motion using a Wilkinson approximation (see P. Pirinen, [17]). From this approximation, we derive its approximative analytical solution. Furthermore, we derive the approximative analytical solution for the case when the jump sizes are lognormally distributed. At the end of Chapter 5 we solve numerically the free boundary problem for the early exercise boundary, both for diffusion and jump-diffusion processes. We find that the continuation region C increases in jump-intensity λ and jump-size.

Bibliography

[1]   Adams, M., Guillemin, V., Measure theory and probability. Birkhauser, Boston (1996)

[2]   Bokes, T., Probabilistic and analytic methods for pricing American style of Asian options. Dissertation Thesis. Bratislava (2011)

[3]   Björk, T., Arbitrage Theory in contiuous time, third edition. New York, (2009)

[4]   Cont, R., Tankov, P., Financial modelling with jump processes. Chapman & HallICRC, USA (2004)

[5]   Gukhal, C. R., Analytical Valuation of American Options on Jump-Diffusin Pprocesses. Mathematical Finance, Vol. 11, No. 1 (January 2001), p.97-115.

[6]   Hansen, A.T., Jorgensen, P.L., Analytical Valuation of American-style Asian Options. Management Science, Vol. 46, No. 8 (Aug., 2000), pp. 1116-1136 .

[7]   Harrison and Kreps, MArtingales and Arbitradge in Multperiod Securities Markets. Jornal of economic theory, 20:381-408.

[8]   Kallast, S., Kivinukk, A., Pricing and Hedging American options Using Approximations by Kim integral Equation. Eurropean Finance Review 7:361-388, 2003.

[9]   Karout, N.E., Karatzas, I., The Optimal Stopping Problem for a General American Option

[10]   Klenke, A. Probability Theory. A comprehensive course. Springer-Verlag London Limited (2008)

[11]   Kou, S., Petrella, G., Wang, H., Pricing Path-dependent Options With Jump Risk Via Laplace Transform. The Tokyo Economic Review, 74(1):1-23 (June 2005)

[12]   *Merton, R. C., Option Pricing when the underlying Stock Returns are Discontinuous.* Journal of Financial Economics 3 125-144 (1976).

[13]   Øksendal, Bernt , Stochastic Differential Equations. Springer-Varlag, Berlin Heidelberg, New York, 3 edition

[14]   *Peskir, G., and Shiryaev, A., Optimal Stopping and Free-Boundary Problems. Birkhauser Verlag, Switzerland (2006)*

[15]   *Pham, H., Optimal Stopping, Free Boundary, and American Option in a Jump-Diffusion Model. Appl Math Optim 35:145-164 (1997).*

[16]   *Privault, N., An Elementary Introduction To Stochastic Interest Rate Modeling. World Scientific, 2012, 242 pages, ISBN: 978-981-4390-85-9*

[17]   *Pirinen, P., Statistical power sum analysis for noidentically distributed correlated lognormal signals. Telecommunication Laboratory and Centre for Wireless Communications, Finland.*

[18]   *Zhang, J. E., Pricing and Hedging American Options Analytically: A perturbation Method. Mathematical finance, Vol.20. No 1(January 2010) p.59-87.*