Generating Solutions in General Relativity using a Non-Linear Sigma Model

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Abstract

This report studies the generation of new solutions to Einstein’s field equations in general relativity by the method of sigma models. If, when projected from four to three dimensions, the relativistic action decouples into a gravity term and a non-linear sigma model term, target space isometries of the sigma model can be found that correspond to generating new solutions. We give a self-contained description of the method and relate it to the early articles through which the method was introduced. We discuss the virtues of the method and how it is used today. We find that it is a powerful technique of finding new solutions and can also give insight to the general features of the theory. We also identify some possible further developments of the method.

Summary in Swedish

Den allmänna relativitetsteorin skapades av Albert Einstein i början av 1900-talet. Teorin bygger på att man finner lösningar till fältekvationerna $R_{ab} - \frac{1}{2}Rg_{ab} = T_{ab}$. Dessa fältekvationer kan ekvivalent beskrivas med en så kallad Lagrangian på formen $L = R + L_M$, där $R$ står för geometrin och $L_M$ står för materieinnehållet.

Eftersom Einsteins fältekvationer är svåra att lösa utnyttjas ofta olika tekniker för att generera lösningar. I denna rapport studerar vi en metod som började utvecklas runt 1960 och som i sin moderna tolkning beskrivs med vad som kallas för en sigmamodell. I metoden utgår man från en känd lösning med vissa symmetriegenskaper och skapar utifrån den en uppsättning nya lösningar. Det första steget är att projicera lösningens fyra rumtidsdimensioner till tre dimensioner. Lagrangianen $L = R + L_M$ kommer då att kunna skrivas på formen $L = P + L_m$. $P$ står för geometrin i tre dimensioner och $L_m$ utgörs av bidrag från både $R$ och $L_M$. Termen $L_m$ kan i gynnsamma fall skrivas som en sigmamodell, ett uttryck på formen $L_m = G_{AB}\partial^i\varphi^A\partial_i\varphi^B$ där $\varphi^A$ är en uppsättning funktioner som beskriver vår lösning.

Genom att utnyttja den speciella formen för $L_m$ kan vi nu ändra funktionerna $\varphi^A$ till nya funktioner $\tilde{\varphi}^A$, på ett sådan sätt att $\tilde{L}_m = L_m$. Därmed kommer inte Lagrangianen $L = P + L_m$ att påverkas vilket gör att de nya funktionerna också kommer att beskriva en lösning. Går vi sedan tillbaka till fyra dimensioner får vi att den nya lösningen är på formen $L = \tilde{R} + \tilde{L}_M$, det vill säga vi har fått både en ny geometri och ett nytt materieinnehåll.

I rapporten studeras de artiklar där denna metod utvecklades för att länka samman hur olika författare beskrivit den. Vi visar också varför metoden fungerar och beskriver hur den används idag. Två intressanta områden att studera vidare är fallet med en nollskild kosmologisk konstant och fallet där man projicerar till två dimensioner i stället för tre.
1 Introduction

General relativity is the current theory of gravitation, which is considered as one of four fundamental forces of nature. The theory was first suggested by Albert Einstein in 1915 and it was an entirely new way of viewing the gravitational force. Before the theory was launched the world was described by the theory of special relativity, which considers a flat four-dimensional spacetime called Minkowski space. An event at time \( t \) and position \( \vec{x} \) corresponds to a point \( p \) in Minkowski space with coordinates \((t, \vec{x})\)\(^1\). In this theory free bodies follow straight lines, and bodies subject to any net force (including gravity) will deviate from their course.

The origin of the gravitational force is matter, which is described by the stress-energy tensor \( T_{ab} \). However, instead of having the tensor field \( T_{ab} \) defined on flat Minkowski space Einstein suggested that this matter field interacts with the space-time itself by changing its geometry. The flat Minkowski space is replaced by four-dimensional curved spacetime. Gravity is then not considered as a force but instead as a consequence of the curvature. A free body will follow a straight line in the curved space-time, which would in the projection to flat space-time look as if a force of gravity acts on the body to bend its trajectory.

The interaction between matter and geometry is described by the Einstein field equations

\[
R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 8\pi T_{ab},
\]

(1.1)

where the quantities \( g_{ab}, R_{ab} \) and \( R \) describe the geometry of space-time and \( \Lambda \) is the cosmological constant.\(^2\)

The theory was immediately successful since it was able to explain the precession of the perihelion of Mercury \([1, 2]\), which is not predicted by Newtonian gravity. Nowadays, corrections from general relativity are used to make the Global Positioning System (GPS) work properly \([3]\). The theory has also contributed to the understanding of our world, since it predicts the existence of black holes and is the foundation of the theory of Big bang.

Einstein’s equations are a system of ten coupled partial differential equations on some four space-time coordinates. There is no canonical choice of coordinates, not even of time coordinate. The terms \( R_{ab}, g_{ab} \) and \( T_{ab} \) are symmetric tensors with ten independent entries. \( g_{ab} \) is called the metric and it is analogous to the Lorentz metric \( \eta_{ab} \) in the Minkowski space of special relativity. Since \( R_{ab}, g_{ab} \) and \( R \) are functions of \( g_{ab} \) and its derivatives only, finding a solution to Einstein’s equations is equivalent to finding a metric \( g_{ab} \) fulfilling (1.1). This is generally a very hard problem. Examples of solutions are the Minkowski space of special relativity and Schwarzschild’s solution describing black holes.

This report will study a method, first described in 1957, to generate new solutions given an existing one. In the modern approach it is considered as an example of application of a sigma model, a geometrical method using differential geometry and calculus of variations. The method is considered an important tool in studying solutions to Einstein’s equations \([4, p. 182]\). The sigma model theory itself is used in different parts of theoretical physics \([5]\).

To perform the method we need a solution that is symmetric along a certain direction, given by a so called Killing vector field. We can then reduce the four-dimensional problem into three dimensions together with an extra term that constitutes the sigma model. The sigma model term will be the site of the generation of new solutions.

\(^1\)We use natural units for general relativity where \( c = G = 1 \).

\(^2\)We will not consider solutions where \( \Lambda \neq 0 \) until later in this text. Thus we will for now refer to Einstein’s equations as the equation

\[
R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab}.
\]
2 Method

This work will consist of a literature study which aims to give a self-contained presentation of the method of generating solutions, as well as a description of its development and current use. In the end we discuss the method and some possible further development.

We start by introducing the mathematical prerequisite, differential geometry. This subject is well-known but it is nevertheless important to be included, since we simultaneously present useful formulas and establish notation and conventions. Since the subject has many equivalent presentations differing in notation and conventions we will mainly follow textbooks in general relativity, such as [4] and [6]. In section 4 we proceed with describing sigma models. This section is made self-contained and is directly built on the previous section. In section 5 we present some relevant features of the theory of general relativity. We prove Einstein’s field equations from a Lagrangian formulation and present some known solutions. We focus on solutions that appear later in the text and in particular on the Schwarzschild solution [2] which is a good example of a solution that admits Killing vector fields.

In section 6 we present the details of the method of generating solutions. This section is based on studying various articles, but the subject is presented independently. We establish a notation that can be used for all relevant solutions. We describe how the solution is projected onto three dimensions as it is done in [7] and then present two alternative views of the generation method. One is used by the contributors under the period the method was developed, and the other one is what we can see as the proper viewpoint, using sigma models. Our presentation includes proving that the generated solution indeed is a solution to the theory.

In section 7 we outline the historical development of the theory, dating back to Ehlers [8], Harrison [9] and Geroch [7]. We then study how the method is connected to sigma models described by Sanchez [10] and some modern applications of the method in its sigma model formulation [11]. We finish by reviewing the most recent development considering the method, analysing the situation where we have a non-zero cosmological $\Lambda$, which is done by Leigh et.al. [12].

2.1 Problem formulation

This work will try to answer the following questions. How can sigma models be used for generating solutions in general relativity? How can the different formulations of the generation method be linked together? What further development is possible?

3 Differential geometry

The central concepts in this work are the notion of manifolds and tensors. A manifold is a generalisation of ordinary $n$-dimensional spaces, allowing it to be not only flat but also curved. For example, the Euclidean plane $\mathbb{R}^2$ is a two-dimensional flat manifold, whereas the ordinary sphere $S^2$ is a two-dimensional curved manifold.

On manifolds we define vectors and tensors. A tensor is a generalisation of a vector, allowing it to have not only a single column of coordinates, but a multidimensional array of coordinates. For example matrices constitute two-dimensional tensors.

Differential geometry is the subject of tensor calculus on manifolds and is the mathematical framework of general relativity. For a more thorough presentation we refer the reader to textbooks in differential geometry and tensor analysis, or to comprehensive textbooks in general relativity.

3.1 Manifolds and tangent spaces

A manifold $\mathcal{M}$ is a topological space that can be described locally by a coordinate system. The elements of $\mathcal{M}$ are called points, and we will denote the coordinates of the point $p$ by
The number \( n \) is called the dimension of the manifold. When needed, we will refer to the coordinate functions collectively as a map \( \psi : \mathcal{U} \to U \subset \mathbb{R}^n \) with \( \mathcal{U} \subset \mathcal{M} \) and \( \psi(p) = (x^1, \ldots, x^n) \).

A function is a smooth map \( f : \mathcal{M} \to \mathbb{R} \). We note that each coordinate \( x^\alpha \) is a function on \( \mathcal{M} \). Functions will sometimes be called scalar fields.

We shall now define vectors, one-forms and tensors on manifolds. When working in flat manifolds, such as in ordinary 3-space \( \mathbb{R}^3 \), we can freely move vectors around regardless of where they are situated. Since the general manifold is not flat it is essential to which point a vector belongs. For each point \( p \in \mathcal{M} \) we define a space \( V_p = T^1_0(p) \) whose objects are vectors. \( V_p \) is called the tangent space and is a vector space with same dimension as the manifold itself. In analog to curvilinear coordinates in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \), we will have a basis of the tangent vector space derived from the coordinates of the manifold around \( p \). The basis vector \( \partial/\partial x^\mu \) will denote the basis vector in \( V_p \) directed along the coordinate curve of \( x^\mu \).

The general vector \( X \) defined at \( p \) can then be written as \( X = X^\alpha \partial/\partial x^\alpha \).

This choice of basis vectors gives rise to a useful interpretation as linear operators. For example, let \( \gamma : [a, b] \to \mathcal{M} \) be a curve in \( \mathcal{M} \) parametrised by \( t \). Then let \( T = (\partial/\partial t)^\gamma \) be the tangent vector at \( \gamma(t) \) directed along \( \gamma \), whose action on the function \( f \) is \( T f = \partial f/\partial t \). In our coordinate system we have \( T = \partial x^\mu / \partial t \), so the components are \( T^\mu = \partial x^\mu / \partial t \).

A one-form, also called a dual vector, \( \omega \) at \( p \in \mathcal{M} \) is a linear map \( \omega : V_p \to \mathbb{R} \). The vector space of dual vectors at \( p \) will be denoted \( V_p^* = T_0^1(p) \). Given the coordinate basis in \( V_p \), the dual basis in \( V_p^* \) can be shown to be \( dx^a \) [6], where the action of \( \omega = \omega_b dx^b \) on \( T = T^\beta (\partial/\partial x^\beta) \) is

\[
(\omega, T) = \omega_a T^a_\beta \partial/\partial x^\beta = \omega_a T^a b \delta^a_b = T^a \omega_a. \quad (3.1)
\]

A tensor of type \((r, s)\) is an element of

\[
T^a_\alpha \cdots T^c_\beta \cdots V_p^* \cdots V_p^* \cdots V_p \cdots V_p, \quad \text{for } s \text{ times} \quad \text{and} \quad \text{r times}, \quad (3.2)
\]

where \( \otimes \) denotes the tensor product. An equivalent characterisation is that an \((r, s)\) tensor is a multilinear map \( V^*_p \times \cdots \times V^*_p \otimes V_p \times \cdots \times V_p \to \mathbb{R} \) [13].

We will mainly follow the abstract index notation described in [4, pp. 23–26]. The notion \( X^a \) (latin index) will be interpreted as the vector \( X = X^a \partial/\partial x^a \), independent of coordinate system, whereas \( X^\alpha \) (greek index) will denote the \( a \)-th component (a number) of \( X \) in the basis \( \{ \partial/\partial x^a \} \). Equivalently \( \omega_b \), with one lower index, is an element of \( V^* \), i.e. a one-form.

An \((r, s)\) tensor will be denoted by \( A^{a_1 \cdots a_r}_{b_1 \cdots b_s} \), which should be interpreted as the tensor

\[
A = A^{a_1 \cdots a_r}_{b_1 \cdots b_s} \partial/\partial x^{a_1} \otimes \cdots \otimes \partial/\partial x^{a_r} \otimes dx^{b_1} \otimes \cdots \otimes dx^{b_s}. \quad (3.3)
\]

It follows that when changing coordinates from \( x^a \) to \( y^\alpha \), the tensor components transform according to

\[
A^{a_1 \cdots a_r}_{b_1 \cdots b_s} = \frac{\partial y^{a_1}}{\partial x^{b_1}} \cdots \frac{\partial y^{a_r}}{\partial x^{b_s}} A^{d_1 \cdots d_r}_{c_1 \cdots c_s} A^{c_1 \cdots c_r}_{d_1 \cdots d_s}. \quad (3.4)
\]

If we demand that the coordinate change functions \( \psi' \circ \psi^{-1} : U \to U' \) be of class \( C^\infty \), we call the manifold smooth. We regard all manifolds we consider as smooth, but will not refer to it explicitly, and all theorems hold for class \( C^k \) for sufficiently large \( k \).

We follow the Einstein summation convention, where repeated indices will be summed over. For example

\[
T^{ab} \cdots U^c = \sum_{a=1}^{m} \sum_{c=1}^{n} T^{ab} \cdots U^c. \quad (3.5)
\]

\( U \otimes V \) is the vector space with basis \( \{ u_i \otimes v_j \}_{i=1, j=1}^{m, n} \), where \( \{ u_i \}_{i=1}^{m} \) and \( \{ v_j \}_{j=1}^{n} \) are bases of \( U \) and \( V \) respectively. Note that \( \dim (U \otimes V) = \dim U \cdot \dim V \).
3.2 Diffeomorphisms and maps

Let $\mathcal{M}$ and $\mathcal{N}$ be two manifolds. A map $\phi: \mathcal{M} \rightarrow \mathcal{N}$ is called a smooth map between manifolds if the corresponding map between coordinates of $\mathcal{M}$ and $\mathcal{N}$ respectively is $C^\infty$. It is called a diffeomorphism if it is bijective and its inverse is again a smooth map between manifolds.

The set of all diffeomorphisms $\phi: \mathcal{M} \rightarrow \mathcal{M}$ form a group acting on $\mathcal{M}$, since the composition of two diffeomorphisms is also a diffeomorphism. If a set of diffeomorphisms can be parametrised by the number $t \in \mathbb{R}$, such that $\phi_s \circ \phi_t = \phi_{s+t}$, $\phi_0 = \text{id}$, we speak of a one-parameter group of diffeomorphisms. If the group axioms are valid only for $|t| < \varepsilon$, we speak of a local group.

A smooth map $\phi: \mathcal{M} \rightarrow \mathcal{N}$ makes it possible to relate the tangent spaces of $p \in \mathcal{M}$ and $\phi(p) \in \mathcal{N}$. Let $f: \mathcal{N} \rightarrow \mathbb{R}$ be a function defined on $\mathcal{N}$. Then we make the following definitions:

1. The vector push-forward $\phi_*: T^1_p(\phi) \rightarrow T^1_{\phi(p)}(\phi)$, defined by demanding that the action of $\phi_*X$ on $f$ follow the rule $(\phi_*X)(f) = X(f \circ \phi)$. Note that the left-hand side is evaluated at $\phi(p)$, whereas the right-hand side is evaluated at $p$. The push-forward is sometimes called the differential or Jacobian.

2. The one-form pull-back $\phi^*: T^0_p(\phi(p)) \rightarrow T^0_{\phi(p)}(\phi)$, defined by demanding that the action of $\phi^*\omega$ on the vector $X \in T^1_p(p)$ follow the rule $\langle \phi^*\omega, X \rangle = \langle \omega, (\phi_*)X \rangle$. Now the left-hand side is evaluated at $p$ and the right-hand side at $\phi(p)$.

If $\phi$ is a diffeomorphism, we can define the pull-back and push-forward on arbitrary tensor fields, by identifying $\phi_* = (\phi^*)^{-1}$ and letting $\phi_*$ and $(\phi^*)^{-1}$ act on the factors $V^s_{\phi(p)}$ and $V^s_{\phi(p)}$ respectively in the tensor product in equation (3.1). This gives the following maps:

- Push-forward $\phi_*: T^s_p(\phi) \rightarrow T^s_{\phi(p)}(\phi)$, \hspace{1cm} (3.5)
- Pull-back $\phi^*: T^0_{\phi(p)}(\phi(p)) \rightarrow T^0_p(p)$. \hspace{1cm} (3.6)

3.3 Derivatives on manifolds

The notion of fields on manifolds is straightforward. A tensor field is the association of a tensor $T^{a...\gamma}_{\delta...\zeta}(p) \in T^s_p(p)$ at each point $p \in \mathcal{M}$, in a way such that the components $T^{a...\gamma}_{\delta...\zeta}(p)$ in the coordinate basis are functions (i.e. smooth maps $\mathcal{M} \rightarrow \mathbb{R}$). We will denote the space of tensor fields on $\mathcal{M}$ by $T^s_\mathcal{M}$, for example $T^1_\mathcal{M}$ is the space of vector fields on $\mathcal{M}$.

To perform calculus on tensor fields one has to be aware of that a general manifold is not flat. Since derivatives are limits of differences, we must simultaneously consider the tangent spaces at two different points, i.e. $V_p$ and $V_{p'}$. Since there is generally no unique way to relate different tangent spaces, the notion of difference between vectors at $V_p$ and vectors at $V_{p'}$ is not well-defined and hence there are several ways to take the derivative.

The most naïve way to differentiate is the coordinate derivative, $\partial_\alpha$. It works component-wise and is completely dependent of the choice of coordinate system. The tensor components can be regarded as functions of the coordinates, and the action of $\partial_\alpha$ is simply differentiation of the tensor components with respect to $x^\alpha$:

$$\partial_\alpha T^{a...\gamma}_{\delta...\zeta} = \frac{\partial T^{a...\gamma}_{\delta...\zeta}}{\partial x^\alpha} (x^1, \ldots, x^n).$$ \hspace{1cm} (3.7)

On flat manifolds, such as Minkowski spacetime, $\partial_\alpha$ is the natural derivative. A common notation in literature is $T^{b...d}_{c...g} = \partial_\alpha T^{b...d}_{c...g}$.

The covariant derivative, $\nabla_\alpha$, is a generalisation of the coordinate derivative, but together with the metric it will become the most natural way of differentiating on curved manifolds. Its action on vector fields $X^a$ is given by

$$\nabla_\alpha X^b = \partial_\alpha X^b + \Gamma^b_{ac} X^c,$$ \hspace{1cm} (3.8)
where \( T^b_{ac} \) consists of \( n^3 \) functions on \( \mathcal{M} \). However, despite its appearance \( T^b_{ac} \) is not the components of a tensor. Both \( \partial_a \) and \( T^a_{bc} \) depend on the choice of coordinate system, but \( \nabla_a X^b \) does not, and is thus a proper tensor. On a general tensor \( T^{b_1...b_{c_1...c_s}} \) the covariant derivative is given by

\[
\nabla_a T^{b_1...b_{c_1...c_s}} = \partial_a T^{b_1...b_{c_1...c_s}} + \Gamma^b_{da} T^{db_2...b_{c_1...c_s}} + \ldots + \Gamma^{b_d}_{ad} T^{b_1...d_{c_1...c_s}} - \Gamma^{c_d}_{ac} T^{b_1...b_{c_2...c_s}} - \ldots - \Gamma^{e_d}_{ae} T^{b_1...b_{e_2...c_s}}. \tag{3.9}
\]

We can interpret the covariant derivative by noting that \( T^a \nabla_a X \) is a tensor of the same type as \( X \), whose components is the change in components of \( X \) in the direction of the vector \( T^a \).

Note that for functions \( \phi \) we have

\[
\nabla_a \phi = \partial_a \phi, \tag{3.10}
\]

which will frequently be used. In literature we often see the notation \( T^{b_1...b_{d_1...d_f}}_{c_1...g_1} = \nabla_a T^{b_1...b_{d_1...d_f}}_{c_1...g_1} \).

The Lie derivative is another way to differentiate tensors fields. We consider a vector field \( v^a \) and look at its integral curves. That is, if \( x^a(t) \) are the coordinates of the integral curve at \( t \), then \( x^a(t) \) are given by solving the system of linear differential equations

\[
\frac{\partial x^a}{\partial t} = v^a(x^1, \ldots, x^n). \tag{3.11}
\]

The integral curves of \( v^a \) give rise to a local one-parameter group of diffeomorphisms \( \phi_t \), where \( \phi_t(p) \) is the point on the integral curve through \( p \) at distance \( t \) [4, p. 18].

The Lie derivative \( \mathcal{L}_X T^{a...c}_{d...f} \) of the tensor \( T^{a...c}_{d...f} \) along \( X \) is given by

\[
\mathcal{L}_X T^{a...c}_{d...f} \bigg|_p = \lim_{t \to 0} \frac{\phi_t^* \left( T^{a...c}_{d...f} \bigg|_{\phi_t(p)} \right) - T^{a...c}_{d...f} \bigg|_p}{t}, \tag{3.12}
\]

where \( \phi_t \) is the local one-parameter group of diffeomorphisms generated by \( X \). The reason for the name Lie derivative comes from form it takes when \( T \) is a vector. In this case we have \( \mathcal{L}_X T = [X, T] \), where [4, p. 440]

\[
[X, T]^a = X^b \frac{\partial T^a}{\partial x^b} - T^b \frac{\partial X^a}{\partial x^b}. \tag{3.13}
\]

It can be noted that for any covariant derivative \( \nabla_a \), including \( \partial_a \), we have \([X, T]^a = X^b \nabla_b T^a - T^b \nabla_b X^a\) [4, p. 31]. For general tensors we have

\[
\mathcal{L}_X T^{b_1...b_{c_1...c_s}} = X^a \nabla_a T^{b_1...b_{c_1...c_s}} - T^{ab_2...b_{c_1...c_s}} \nabla_a X^{b_1} - \ldots - T^{b_1...b_{c_2...c_s}} \nabla_a X^{b_r} + T^{b_1...b_{ac_2...c_s}} \nabla_{c_1} X^a + \ldots + T^{b_1...b_{ac_1...a}} \nabla_{c_1} X^a. \tag{3.14}
\]

### 3.4 Metric, geodesics and curvature

A metric \( g_{ab} \) is a \((0, 2)\) symmetric, non-degenerate tensor field on \( \mathcal{M} \). Locally at a point \( p \in \mathcal{M} \) the metric works as an inner product \( V_p \times V_p \to \mathbb{R} \). On manifolds its role is even more important since the metric will give all information of the curvature of the manifold.

Often the metric will be specified in the so called line element form, from which the metric components can be read off:

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{3.15}
\]

The metric will be used to lower and raise indices of tensors, e.g. if \( T^{abc} \) is a \((3, 1)\) tensor, then \( T^a_{bc} = g_{ae} g_{bf} T^{efb} \) (where \( g^{ab} = [g_{ab}]^{-1} \) as a matrix), should be regarded as another...
representation of the same tensor with first and third indices lowered and fourth index raised. Note that $g^{ab}g_{bc} = \delta_c^a$.

The definition of covariant derivative above leaves the choice of $\Gamma^a_{bc}$ arbitrary. By introducing the metric there is a unique natural choice of covariant derivative that goes “as straight as possible”, with respect to the metric$^8$. The components of $\Gamma^a_{bc}$ is given by:

$$\Gamma^a_{\beta\gamma} = \frac{1}{2} g^{a\mu} \left( \frac{\partial g_{\beta\mu}}{\partial x^\gamma} + \frac{\partial g_{\gamma\mu}}{\partial x^\beta} - \frac{\partial g_{\beta\gamma}}{\partial x^\mu} \right).$$

(3.16)

From now on $\nabla_a$ refer to this particular choice of functions $\Gamma^a_{bc}$, and we will call $\Gamma^a_{bc}$ the Christoffel symbols.

Geometrically we relate $\nabla_a$ to the concept of parallel transport. Let $\gamma$ be a curve with tangent vector $T^a$. Then the tensor $X^{a...d...f}$ is parallel transported along $\gamma$ if

$$T^a \nabla_a X^{b...d...f}_{e...g} = 0.$$  

(3.17)

This notation gives the characterisation of $\nabla_a$ as the covariant derivative that parallel transports the metric along any direction. Hence we have the equation

$$\nabla_a g_{bc} = 0,$$  

(3.18)

i.e. that $\nabla_a$ and $g_{bc}$ commute in equations.

A geodesic is a curve $\gamma$ that goes straight with respect to the metric. If $T^a$ is the tangent vector field of $\gamma$ we demand that the change in $T^a$ along $\gamma$ be tangent to $\gamma$, i.e. $T^a \nabla_a T^a = kT^a$. We will assume so called affine parametrisation, where $k = 0$, and hence

$$T^b \nabla_a T^a = 0.$$  

(3.19)

It can be shown [4, p. 41] that the coordinates of $\gamma$ satisfy the geodesic equation

$$\frac{d^2 \gamma^a}{dt^2} = -\Gamma^a_{\beta\gamma} \frac{d\gamma^\beta}{dt} \frac{d\gamma^\gamma}{dt}.$$  

(3.20)

By straightforward calculations it can be shown that the expression $2 \nabla_{[a} \nabla_{b]} \equiv \nabla_a \nabla_b - \nabla_b \nabla_a$ $^7$ defines a linear map $T^b_a(\mathcal{M}) \rightarrow T^b_a(\mathcal{M})$. We can thus introduce a $(1,3)$ tensor field $R^{a}_{b c d}$ on $\mathcal{M}$ called the Riemann curvature tensor and defined by the equation

$$R^{a}_{b c d} \omega^b = (\nabla_a \nabla_b - \nabla_b \nabla_a) \omega_c.$$  

(3.21)

We also define the Ricci tensor $R_{ac} = R^{b}_{a b c}$ and scalar curvature $R = R^{a}_{a}$ from the Riemann curvature tensor.

It is sometimes useful to express the action of $2 \nabla_{[a} \nabla_{b]}$ on arbitrary tensors in terms of the Riemann curvature tensor:

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) T^{c_1...c_r}_{d_1...d_s} = - R^{a}_{b c_1...c_r d_1...d_s} + ... - R^{a}_{b c_1...c_r d_1...d_s} + R^{a}_{b d_1...d_s c_1...c_r} + ... + R^{a}_{b d_1...d_s c_1...c_r}.$$  

(3.22)

By (3.21) $R^{a}_{b c d}$ describes the noncommutativity of the covariant derivatives. If a vector is parallel transported along a small loop back to its starting point, $R^{a}_{b c d}$ is related to the change in direction caused by the local curvature (for details, c.f. [4, p. 37–38]).

There is no fixed convention for defining the Riemann tensor and hence there are different ways of writing it in terms of the Christoffel symbols. We follow [4, p. 48], where $R^{a}_{b c d}$ is given by

$$R^{a}_{b c d} = \partial_b \Gamma^a_{c d} - \partial_d \Gamma^a_{b c} + \Gamma^e_{b c} \Gamma^a_{e d} - \Gamma^e_{b d} \Gamma^a_{e c}.$$  

(3.23)

$^6$In mathematical terminology this choice of covariant derivative is called the Levi-Civita connection.

$^7$Square brackets $[a...b]$ and parentheses $(a...b)$ denotes the antisymmetric and symmetric parts respectively, normalised by $\frac{1}{2}$. E.g. $T_{[a}U_{c]d} = \frac{1}{2} (T_{ab}U_{cd} + T_{ba}U_{dc} - T_{ca}U_{bd} - T_{ac}U_{bd} - T_{cb}U_{da} - T_{bc}U_{da}).$
By contracting this equation the Ricci tensor is given by

\[ R_{ab} = R_{abc}^c = \partial_c R_{ab}^c - \partial_a R_{cb}^c + R_{ac}^d R_{db}^c - R_{dc}^c R_{ab}^d, \]  

(3.24)

which in calculations can be simplified using [4, p. 48]

\[ \Gamma^a_{a\alpha} = \frac{\partial}{\partial x^\alpha} \ln \sqrt{\det g_{\mu\nu}}. \]  

(3.25)

3.5 Isometries and Killing vector fields

An isometry is a diffeomorphism \( \phi : M \rightarrow N \) that leaves the metric invariant, i.e.

\[ \phi_* \left( g_{ab} |_p \right) \big|_{\phi(p)} = g_{ab} |_{\phi(p)}. \]  

(3.26)

It is easy to see that the set of isometries \( \phi : M \rightarrow M \) forms a group acting on \( M \). An important special case is when we consider a one-parameter group of isometries generated by a vector field \( \xi^a \). We then have that \( g_{ab} \) is constant along integral curves to \( \xi^a \), which is equivalent to \( \mathcal{L}_\xi g_{ab} = 0 \). By using equation (3.14) we get

\[ 0 = \mathcal{L}_\xi g_{ab} = \xi^c \nabla_c g_{ab} + g_{cb} \nabla_a \xi^c + g_{ac} \nabla_b \xi^a = g_{bc} \nabla_a \xi^b + g_{ac} \nabla_b \xi^a, \]  

(3.27)

since \( \nabla_c g_{ab} = 0 \). By relabeling we arrive at Killing’s equation

\[ \nabla_a \xi_b + \nabla_b \xi_a = 0. \]  

(3.28)

Let \( x^a \) be the coordinates of a point in \( M \). Under action of an element \( \phi_\lambda \) in the isometry group the new coordinates will be given by [11]

\[ \phi_\lambda (x^a) = \exp(\lambda \xi) x^a, \]  

(3.29)

where \( \exp \) denotes the Taylor series of the exponential function.

3.6 Differential forms

A \( p \)-form is a completely antisymmetric \((0,p)\) tensor field \( \Omega \), i.e. \( \Omega_{a_1 \ldots a_p} = \Omega_{(a_1 \ldots a_p)} \). The set of all \( p \)-forms on \( M \) will be denoted \( \Lambda^p(M) \). The basis vectors in \( \Lambda^p(M) \) are the forms

\[ dx^{a_1} \wedge \cdots \wedge dx^{a_p} \equiv dx^{[a_1} \otimes \cdots \otimes dx^{a_p]} \]  

(3.30)

The exterior derivative is a map \( d : \Lambda^p(M) \rightarrow \Lambda^{p+1}(M) \), mapping the \( p \)-form \( \omega \) onto \( d\omega \) according to

\[ (d\omega)_{b_1 \ldots a_p} = \nabla_{[b} \omega_{a_1 \ldots a_p]} = \partial_{[b} \omega_{a_1 \ldots a_p]} \]  

(3.31)

Note that \( \nabla_b \) here can be any covariant derivative, including \( \partial_b \).

It can easily be shown that \( d^2 = 0 \), i.e. performing exterior derivative twice on any form gives the zero result. We call a \( p \)-form \( \omega \) closed if \( d\omega = 0 \). Then by the Poincaré lemma there exists locally a \((p-1)\)-form \( \Omega \) such that \( \omega = d\Omega \) [4, p. 429]. If such \( \Omega \) exists globally we say that \( \omega \) is an exact form.

A volume form, if any exists, is an \( n \)-form \( \epsilon \) on the \( n \)-dimensional manifold \( M \), such that

\[ \epsilon = f dx^1 \wedge \cdots \wedge dx^n, \]  

(3.32)

with \( f \) being a strictly positive function. If such form exists we say that \( M \) is orientable. On an orientable \( n \)-dimensional manifold we can integrate \( n \)-forms \( \alpha = \omega dx^1 \wedge \cdots \wedge dx^n \) on \( U \subset M \) by

\[ \int_U \alpha = \int_{\phi(U)} d^n x \alpha, \]  

(3.33)
where the right hand side is the integral over the coordinate region $\psi(\mathcal{U}) \subset \mathbb{R}^n$.

We can also, somewhat artificially, integrate functions on $\mathcal{M}$ if there exists a metric. We define the antisymmetric tensor $\varepsilon$ as the volume form $\sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n$ where $g = \det g_{\mu\nu}$ in the particular coordinate system, and let the integral of $\phi$ over $\mathcal{U}$ be defined as

$$\int_{\mathcal{U}} \phi \varepsilon = \int_{\psi(\mathcal{U})} d^n \phi \sqrt{|g|}. \quad (3.34)$$

We will often write $\mathcal{U}$ instead of $\psi(\mathcal{U})$ even in the right hand side integral.

Finally we introduce the Hodge star duality. Note that due to antisymmetry the spaces $\Lambda^p$ and $\Lambda^{n-p}$ have the same dimension $\binom{n}{p} = \binom{n}{n-p}$. There is, in fact, a natural isomorphism $* : \Lambda^p(\mathcal{M}) \rightarrow \Lambda^{n-p}(\mathcal{M})$ called the Hodge star duality. Its action on the basis forms is [14, p. 756]

$$* (dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_p}) = \frac{1}{(n-p)!} \varepsilon_{\alpha_1 \cdots \alpha_p \alpha_{p+1} \cdots \alpha_n} dx^{\alpha_{p+1}} \wedge \cdots \wedge dx^{\alpha_n}, \quad (3.35)$$

where $\varepsilon_{\alpha_1 \cdots \alpha_p \alpha_{p+1} \cdots \alpha_n} = g^{\alpha_1 \beta_1} \cdots g^{\alpha_p \beta_p} \varepsilon_{\beta_1 \cdots \beta_p \alpha_{p+1} \cdots \alpha_n}$ and $\varepsilon_{\alpha_1 \cdots \alpha_n}$ is the antisymmetric tensor. In the abstract index notation we have for $A = *B$

$$A_{\alpha_1 \cdots \alpha_{n-p}} = \frac{1}{(n-p)!} \varepsilon_{b_1 \cdots b_p \alpha_1 \cdots \alpha_{n-p}} B_{b_1 \cdots b_p}. \quad (3.36)$$

## 4 Sigma models

A sigma model is a theory considering certain functions $\varphi^A$, $A = 1, \ldots, d$ on an $n$-dimensional manifold $\mathcal{L}$ (script $\Sigma$). We will consider the functions to constitute coordinates of a $d$-dimensional manifold $\mathcal{F}$ (script $T$). We call $\mathcal{L}$ the domain manifold and $\mathcal{F}$ the target manifold. We can thus consider the $\varphi^A$ collectively as the coordinate maps corresponding to the smooth map

$$\Phi : \mathcal{L} \rightarrow \mathcal{F}. \quad (4.1)$$

### 4.1 Sigma model action

Suppose that there exists an action written

$$S = \int_{\mathcal{L}} \varepsilon L = \int_{\mathcal{L}} \varepsilon h^{ab} G_{AB} \partial_a \varphi^A \partial_b \varphi^B, \quad (4.2)$$

where $\varepsilon$ is the volume form and $h_{ab}$ is the metric on $\mathcal{L}$, and $G_{AB}$ are $d^2$ functions of the variables $\varphi^A$ such that $G_{AB} = G_{BA}$. If the $G_{AB}$ are non-constant we speak about a non-linear sigma model.

We shall now give an interpretation of the sigma model Lagrangian $L$ in (4.2). Consider a change of variables on $\mathcal{F}$, $\psi^B = \psi^B(\varphi^A)$. Then, for the quantities $\partial_a \varphi^a$, we have

$$\partial_a \psi^B = \frac{\partial \psi^B}{\partial \varphi^A} \frac{\partial \varphi^A}{\partial x^a} = \frac{\partial \psi^B}{\partial x^a} \partial_a \varphi^A. \quad (4.3)$$

Comparing with (3.4) we can see that the $\partial_a \varphi^A$ transform as vector components on $\mathcal{F}$. Thus the symmetric quantities $G_{AB}$ must transform as the components of a $(0, 2)$ symmetric tensor on $\mathcal{F}$. Accordingly we can see $G_{AB}$ as a metric on $\mathcal{F}$. However, $\partial_a \varphi^A$ should not be interpreted as vector components, but as the pull-back of a one-form basis vector. For the one-form basis on $\mathcal{F}$ is exactly the set $\{d\varphi^A\}_{A=1}^d$ and the pull-back of such one-form is

$$\Phi^* (d\varphi^A) = \partial_a \varphi^A dx^a, \quad (4.4)$$

8
which has components \( \partial_a \varphi^A \). Here \( \Phi^* : \mathcal{T}^0_1(\mathcal{T}) \longrightarrow \mathcal{T}^0_1(\mathcal{L}) \). Hence \( G_{AB} \partial_a \varphi^A \partial_b \varphi^B \) is the pull-back

\[
G_{AB} \partial_a \varphi^A \partial_b \varphi^B \, dx dx^b = \Phi^* \left( G_{AB} d\varphi^A d\varphi^B \right),
\]

i.e. the pull-back of the metric tensor. This is tensor of type \((0, 2)\) on \(\mathcal{L}\), which is contracted by the inverse metric \(h^{ab}\) into the scalar Lagrangian \(L = h^{ab} G_{AB} \partial_a \varphi^A \partial_b \varphi^B \). Finally this Lagrangian is integrated over \(\mathcal{L}\) using the volume form according to section 3.6.

The sigma model action of (4.2) is often written in alternative forms using different notation. If \(U = \psi(\mathcal{M})\) is the coordinate region that parametrises \(\mathcal{M}\) we can write:

\[
S = \int_{\mathcal{M}} \varepsilon G_{AB} h^{ab} \partial_a \varphi^A \partial_b \varphi^B = \int_U d^4 x \sqrt{h} G_{AB} h^{ab} \partial_a \varphi^A \partial_b \varphi^B
\]

\[
= \int_{\mathcal{M}} G_{AB} \langle d\varphi^A, d\varphi^B \rangle \varepsilon = \int_{\mathcal{M}} G_{AB} d\varphi^A \wedge \ast d\varphi^B.
\]

Here \(\langle d\varphi^A, d\varphi^B \rangle = (d\varphi^A)^a (d\varphi^B)_a\).

### 4.2 Isometries

Now suppose that there exists an isometry \(\chi_\lambda\) on \(\mathcal{T}\). Under the action of \(\chi_\lambda\) the Lagrangian \(L\) is invariant and hence is also the action \(S\). Assume that the functions \(\varphi^A\) solve any Lagrange problem, then for all \(\lambda\) the functions \(\psi^B = \chi_\lambda(\varphi^A)\) solve the same problem.

Let us illustrate with the special case \((d = 2)\) where we have the metric on \(\mathcal{T}\)

\[
dl^2 = dU^2 + dV^2 = dr^2 + r^2 d\theta^2,
\]

i.e. \(G_{UU} = G_{VV} = 1, G_{UV} = G_{VU} = 0\) and \(U = r \cos \theta, V = r \sin \theta\). A Killing vector of the metric is

\[
k = \frac{\partial}{\partial \theta} = \frac{\partial U}{\partial \theta} \frac{\partial}{\partial U} + \frac{\partial V}{\partial \theta} \frac{\partial}{\partial V} = -r \sin \theta \frac{\partial}{\partial U} + r \cos \theta \frac{\partial}{\partial V} = -V \partial_U + U \partial_V.
\]

Before applying the isometry we have

\[
L_0 = G_{AB} \partial_a \varphi^A \partial^a \varphi^B = \partial_a U \partial^a U + \partial_a V \partial^a V.
\]

By (3.29) we have

\[
\bar{U} = \exp(\lambda k) U = \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} \left( -V \frac{\partial}{\partial U} + U \frac{\partial}{\partial V} \right) U = U - \lambda V - U \frac{\lambda^2}{2} + V \frac{\lambda^3}{3!} + \ldots,
\]

so \(\bar{U} = U \cos \lambda - V \sin \lambda\) and similarly \(\bar{V} = V \cos \lambda + U \sin \lambda\). Then, using that \(\bar{G}_{AB} = G_{AB} = \text{diag}(1, 1)\) (in this particular case) we see that

\[
L_\lambda = \bar{G}_{AB} \partial_a \varphi^A \partial^a \varphi^B = \partial_a \bar{U} \partial^a \bar{U} + \partial_a \bar{V} \partial^a \bar{V}
\]

\[
= \partial_a (U \cos \lambda - V \sin \lambda) \partial^a (U \cos \lambda - V \sin \lambda)
\]

\[
+ \partial_a (V \cos \lambda + U \sin \lambda) \partial^a (V \cos \lambda + U \sin \lambda)
\]

\[
= \partial_a \partial^a U + \partial_a V \partial^a V = L_0,
\]

i.e. the Lagrangian is indeed invariant under this isometry.
5 General relativity

General relativity is essentially special relativity on manifolds instead of $\mathbb{R}^4$ together with Einstein’s field equations. A space-time is a pair $(\mathcal{M}, g_{ab})$ where $\mathcal{M}$ is a four-dimensional manifold with the metric $g_{ab}$ everywhere defined. The metric is not positive definite, but has instead the signature $(-, +, +, +)$, i.e. one negative and three positive eigenvalues. Such metric is called Lorentzian. The metric defines the covariant derivative and Riemann curvature tensor on $\mathcal{M}$, together with the Ricci tensor and scalar curvature.

We state the principle of general covariance, which says that the metric (and quantities deduced from the metric) is the only carrier of geometric information in the theory. There is also a principle of special covariance, which says that if the metric on two points on $\mathcal{M}$ is the same, then all laws of physics will apply equally there.

All non-gravitational fields and matter content of the universe is described by the same stress-energy tensor $T_{ab}$ as in special relativity. For example, a non-zero but traceless $T_{ab}$ describes electromagnetic fields without the presence of matter. Matter and geometry are linked together by Einstein’s field equations

$$R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab}. \quad (5.1)$$

We will deduce this equation from variational principles in section 5.1 below.

In general relativity there are no direct statements about gravity, it is instead included in the field equations. All free bodies follow geodesics on the space-time manifold unless they are affected by any (non-gravitational) force. The presence of matter only affects the body indirectly by creating curvature.

Since the metric is Lorentzian, vectors and curves will fall into different classes. A curve whose tangent vectors $T^a$ satisfy $g^{ab} T_a T_b < 0$ is called timelike. Similarly it is called spacelike if $g^{ab} T_a T_b > 0$ and null if $g^{ab} T_a T_b = 0$. Generally there is no globally defined coordinate system covering the whole of $\mathcal{M}$, so even if a coordinate curve is timelike, we may not always call it a time-axis.

In the case of weak gravity, i.e. $g_{ab} = \eta_{ab} + \gamma_{ab}$ for a small $\gamma_{ab}$, one can use the coordinate system of Minkowski space. It can be shown (confer e.g. [4, pp. 74-77]) that Einstein’s field equations for weak gravity and non-relativistic speeds reduce to the pre-relativistic equations

$$\vec{a} = -\vec{\nabla} \phi, \quad \vec{\nabla}^2 \phi = 4\pi \rho. \quad (5.2)$$

5.1 Variational methods

There is a Lagrangian formulation of the theory from which Einstein’s field equations can be derived. The relevant action is [6, p. 75]

$$S = \frac{1}{16\pi} \int d^{4}x \left( R - 2\Lambda + L_{M} \right) \sqrt{-g}, \quad (5.3)$$

where $g = \text{det}(g_{ab})$, $R$ is the scalar curvature, $\Lambda$ is the cosmological constant and $L_{M}$ is the Lagrangian describing matter. The integration is performed over a region $\mathcal{U} \subset \mathcal{M}$. We will derive the vacuum field equations, so we set $R_{ab} = 0$ and $\Lambda = 0$. This gives the so called Hilbert action (up to a numerical constant)

$$S = \int L_{H} d^{4}x, \quad L_{H} = \sqrt{-g}R. \quad (5.4)$$

Let us now perform variations of $g^{ab}$ parametrised by the variable $\lambda$. Following the notation of [4] we introduce the notation $\delta f = df/d\lambda$. Then we have $\delta S = 0$ if $\delta L_{H} = 0$.

$$\delta L_{H} = \delta \left( \sqrt{-g} g^{ab} R_{ab} \right) = \sqrt{-g} g^{ab} \delta R_{ab} + \sqrt{-g} R_{ab} \delta g^{ab} + \left( \sqrt{-g} R \right) \delta \left( \sqrt{-g} \right). \quad (5.5)$$

\textit{8}If factors $G$ and $\epsilon$ are restored in (5.1) the right hand side should be multiplied with a factor $G/c^4$, where $G$ is the known gravitational constant, hence we can see that (5.1) indeed are equations for gravity.  

\textit{9}Actually, $L_{M}$ is the Lagrangian density, which by integrating three dimensions gives the Lagrangian.
We start with the first term. By (3.24) we have\(^{10}\)
\[
\delta R_{ab} = 2\delta \left( \partial_a \Gamma^c_{cb} + \Gamma^c_{[ad]} \Gamma^d_{cb} \right) = 2\partial_a \delta \Gamma^c_{cb} + 2\delta \left( \Gamma^c_{[ad]} \right) \Gamma^d_{cb} + 2\Gamma^c_{[ad]} \delta \Gamma^d_{cb}. \tag{5.6}
\]

However, by using that, by (3.9),
\[
\nabla_a (\delta \Gamma^c_{cb}) = \partial_a (\delta \Gamma^c_{cb}) + \Gamma^c_{dc} \delta \Gamma^d_{cb} - \Gamma^c_{db} \delta \Gamma^d_{ca} - \Gamma^c_{db} \delta \Gamma^d_{ca} \tag{5.7}
\]
it can be shown \(^{15}\) that
\[
\delta R_{ab} = \nabla_a \delta \Gamma^c_{cb}. \tag{5.8}
\]

This derivative vanishes by Stokes’ theorem on the boundary of the region \(\mathcal{M}\) (for a more thorough discussion, c.f. [4, p. 454]).

The second term is already on the desired form. For the third term we use the matrix identity \(\det (\exp A) = \exp(\text{Tr} A)\). Letting \(B = \exp A\) and taking the logarithm we get
\[
\ln (\det B) = \text{Tr} (\ln B). \tag{5.9}
\]

Acting with \(\delta\) on each side we get
\[
\frac{1}{\det B} \delta \ln (\det B) = \text{Tr} \left( B^{-1} \delta B \right). \tag{5.10}
\]

Now take \(B = g^{ab}\). Then \(B^{-1} = g_{ab}\) and \(\det B = g^{-1}\). Using this we get
\[
\delta \left( \sqrt{-g} \right) = \delta \left( g^{-1} \right)^{-\frac{1}{2}} = \frac{1}{2} \left( g^{-1} \right)^{-\frac{3}{2}} \delta g^{-1} = \frac{1}{2} \sqrt{-g} \frac{1}{g^{-1}} \delta g^{-1} = \frac{1}{2} \sqrt{-g} \text{Tr} (g_{ab} \delta g^{bc}) = \frac{1}{2} \sqrt{-g} g_{ab} \delta g^{ab}. \tag{5.11}
\]

Thus we can write
\[
\delta L_H = 0 + \sqrt{-g} R_{ab} \delta g^{ab} - \frac{1}{2} R \sqrt{-g} g_{ab} \delta g^{ab} = \sqrt{-g} \left( R_{ab} - \frac{1}{2} R g_{ab} \right) \delta g^{ab} = 0, \tag{5.12}
\]
which gives us Einstein’s field equations in vacuum.

### 5.2 Solutions

A solution is a specification of \((\mathcal{M}, g_{ab})\) such that Einstein’s field equations are satisfied. Two solutions \((\mathcal{M}, g_{ab}), (\mathcal{M}', g'_{ab})\) are equivalent if there exists an isometry \(\phi: \mathcal{M} \rightarrow \mathcal{M}'\). We will regard equivalent solutions as equal and refer to their differences as being parametrised by different coordinate systems.

The trivial solution to Einstein’s equations is Minkowski space. In cartesian coordinates we have the metric \(g_{ab} = \eta_{ab} = \text{diag}(-1, 1, 1, 1)\). As a line element we have
\[
ds^2 = -dt^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \tag{5.13}
\]
with \((t, x^i) \in \mathbb{R}^4\).\(^{11}\) Minkowski space represents free space with no gravitational effects. Due to the smooth nature of manifolds this is also the zeroth order approximation of any solution; since \(g_{ab}|_p\) is symmetric and non-degenerate with Lorentz signature we can always choose a basis in \(V_p\) such that \((e_\mu)^a(e_\nu)_a = \eta_{\mu\nu}\).

An early breakthrough of the theory was the discovery of the Schwarzschild solution, the first exact solution not equivalent to Minkowski space. It was published 1916 by Karl Schwarzschild [2] and does, for \(R > 2M\), describe the exact gravitational field outside a body

\(^{10}\)Bars indicate lifting out the index from (anti)symmetrisation, i.e. \(T_{[abc]} = \frac{1}{2} \left( T_{abc} - T_{cba} \right)\).

\(^{11}\)In addition to having \(a,...,\) denoting abstract indices and \(\alpha,...,\mu,...\) denoting components, we will, when considering one out of four indices separately, let \(i,... = 1,2,3\) for the other three.
with mass $M$. It thus gave the exact answer to Einstein’s approximative solution of the perihelion precession of Mercury [1]. The metric is given by
\[
d s^2 = - \left(1 - \frac{2M}{R}\right) dt^2 + \left(1 - \frac{2M}{R}\right)^{-1} dR^2 + R^2 d\Omega^2,
\]
(5.14)
We will now briefly present how it was derived in [2]. We assume a vacuum solution, i.e. $R_{ab} = 0$, except for one possible singularity, presumably at $x^i = 0$ ($i = 1, 2, 3$) in Cartesian coordinates. We now look at Einstein’s field equations on a coordinate system where $g = \det g_{\mu\nu} = \text{const.} = -1$. Using (3.24) $R_{ab} = 0$ gives
\[
\partial_\gamma \Gamma^\gamma_{\alpha\beta} - \partial_\alpha \Gamma^\gamma_{\gamma\beta} + \Gamma^\gamma_{\alpha\beta} \Gamma^\delta_{\gamma\delta} = 0.
\]
(5.15)
Since $g$ is constant, the expression is simplified due to (3.25) into
\[
\partial_\gamma \Gamma^\gamma_{\alpha\beta} + \Gamma^\gamma_{\delta\beta} \Gamma^\delta_{\gamma\alpha} = 0.
\]
(5.16)
Furthermore we make the following assumptions
1. All metric components, and hence all Christoffel symbols, are independent of the coordinate $t = x^0$.
2. The metric have the components $g_{i0} = g_{0i} = 0$, i.e. it decomposes to a spatial part and a time part.
3. The spatial part of the metric is rotationally symmetric under the full $SO(3)$ rotation group.\footnote{SO(3) (the three-dimensional special orthogonal group) is the group of proper rotations in $\mathbb{R}^3$.}
4. At infinity, i.e. in the limit $r \to \infty$, the metric reduces to the Minkowski metric.

In Cartesian coordinates assumptions 1-3 give a metric on the general form
\[
d s^2 = -F dt^2 + G (dx^2 + dy^2 + dz^2) + H (x dx + y dy + z dz)^2,
\]
(5.17)
where $F, G, H$ are functions of $r = \sqrt{x^2 + y^2 + z^2}$. Changing coordinates to polar coordinates with determinant 1,
\[
x^1 = \frac{r^3}{3}, \quad x^2 = -\cos \theta, \quad x^3 = \phi,
\]
(5.18)
the line element becomes
\[
d s^2 = -f_1 dt^2 + f_1 (dx^1)^2 + f_2 \left(\frac{(dx^2)^2}{1 - (x^2)^2} + (1 - (x^2)^2) (dx^3)^2\right)
\]
(5.19)
with $f_1, f_2, f_4$ as functions of $x^1$. In this coordinate system $g = -1$ corresponds to $f_1 f_2 f_4 = 1$.

Multiplying the expression for the Christoffel symbol (3.16) by $g^\mu\nu$ we get
\[
g^\mu\nu \Gamma^\alpha_{\beta\gamma} = \frac{1}{2} \left( \frac{\partial g^\nu\alpha}{\partial x^\beta} + \frac{\partial g^\beta\alpha}{\partial x^\gamma} - \frac{\partial g^\gamma\alpha}{\partial x^\beta} \right).
\]
(5.20)
Since the metric in (5.19) is diagonal we can solve for the Christoffel symbols
\[
\Gamma^\alpha_{\beta\gamma} = \frac{1}{2g_{\alpha\alpha}} \left( \frac{\partial g_{\gamma\alpha}}{\partial x^\beta} + \frac{\partial g_{\beta\alpha}}{\partial x^\gamma} - \frac{\partial g_{\gamma\beta}}{\partial x^\alpha} \right) \quad \text{(No summation)}.
\]
(5.21)
The field equations (5.16) and the derivative of the determinant equation $g = -1$ now provide several equations that can be solved for $f_1, f_2, f_4$ to get
\[
f_1 = \frac{1}{R} \frac{1}{1 - \frac{\alpha}{R}}, \quad f_2 = R^2, \quad f_4 = 1 - \frac{\alpha}{R}.
\]
(5.22)
\( \alpha \) is an integration constant and \( R = \sqrt[3]{3 r_1 + \alpha^3} = \sqrt[3]{r^3 + \alpha^3} \) is a new variable, “Hilfsgröße”, which is introduced to simplify the metric \([2]\). In the modern approach \( R \) is introduced from the beginning; the spherical surfaces of constant distance from origin, i.e. the orbits\(^{13}\) of \( SO(3) \), are given the coordinate \( R \) such that their areas are \( 4\pi R^2 \). That means a spherical surface with area \( A \) has the \( R \) coordinate \( \sqrt{A/4\pi} \). Since we have a singularity at origin, \( R \) is the most proper choice of radial coordinate \([4, p. 120]\). \( \alpha \) is identified with \( 2M \), and by changing back to spherical angular coordinates we arrive at (5.14), where \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \).

The Schwarzschild solution shows some interesting features that are worth mentioning. Firstly the singularity at origin gives the solution a manifold structure of \( \mathbb{S}^1 \) rather than \( \mathbb{R}^4 \). We have \( R > 0 \), i.e. \( r > -2M \), with an essential singularity at \( R = 0 \). Secondly we have a coordinate singularity at \( R = 2M \). There are other coordinate systems where this singularity vanishes, showing that it is indeed only a coordinate singularity. The price of changing to such coordinate systems is that there is no longer any globally defined time coordinate (an illustrative description is given in \([4, pp. 148–156]\)). The interesting effects that follow the analysis of the Schwarzschild solution give birth to the idea of a black hole, and the radius \( R = 2M \) is called the Schwarzschild radius.

### 5.3 Classes of solutions

A solution is called static if there exists a timelike Killing vector field \( \xi^a \). Hence there is a one-parameter group of isometries with timelike orbits.

A solution is called axisymmetric if, in addition, there exist hypersurfaces orthogonal to the Killing vector field. Specify one hypersurface \( \mathcal{A} \), then any point on \( \mathcal{A} \) can be given the coordinates \((t, x^i)\), where \( t \) is the parameter value of \( p \) along an integral curve of \( \xi^a \) starting at a point \( p_0 \in \mathcal{A} \) with coordinates \( x^i \). Any static solution has the property that the metric decomposes in the same way as in assumption 2 in section 5.2.

A solution is called axisymmetric if it has a spacelike Killing vector field whose orbits are closed curves. In cylindrical and spherical coordinates we would have \( \xi^a = (\partial_a)^a \) as our spacelike Killing vector field.

The general form of an axisymmetric and stationary solution is \([11]\)

\[
\begin{align*}
\text{ds}^2 &= -e^{2U} (dt - \omega_i dx^i)^2 + e^{-2U} \gamma_{ij} dx^i dx^j, \\
&= (dx^0)^2 - 4x^3 dx^0 dx^2 + 2dx^1 dx^2 + 2(x^3)^2 (dx^2)^2 + (dx^3)^2.
\end{align*}
\] (5.23)

with killing vector \( \xi = \frac{\partial}{\partial t} \). All \( f, \omega_i, \gamma_{ij} \) are functions of \( x^i \).

One of the main reasons for studying axisymmetric solutions is that most of the known exact solutions are on this form. This comes from the fact that we can exclude the origin, as in the Schwarzschild solution. We can thus have a vacuum solution representing some matter at the origin (or close to the origin) and where the deviations from flat space-time emanates from this singularity. Therefore it was believed that the only vacuum solution without singularity with \( R_{ab} = 0 \) were Minkowski space. However, in 1962 the Ozsváth-Schücking metric was published, with the line element \([16]\)

\[
\begin{align*}
\text{ds}^2 &= (dx^0)^2 - 4x^3 dx^0 dx^2 + 2dx^1 dx^2 + 2(x^3)^2 (dx^2)^2 + (dx^3)^2.
\end{align*}
\] (5.24)

It has no singularities and vanishing Ricci tensor but still the Riemann tensor has non-zero components. This solution represent gravitational waves travelling in the \( x^1 \) direction.

\(^{13}\)When a group \( G \) acts on some set \( X \) we define for \( x \in X \) the orbit of \( x \) as \( \{ y \in X \mid y = gx \text{ for some } g \in G \} \).
6 Generating solutions

The method studied in this report starts with a solution $\mathcal{M}, g_{ab}$ that admits a Killing vector field. We describe the solution in terms of a coordinate system given by $\psi: \mathcal{M} \rightarrow U \subset \mathbb{R}^4$. This makes the metric a function of coordinates, i.e. $g_{ab} = g_{ab} \circ \psi^{-1}(x_1, \ldots, x_4)$. By a generated solution we will mean another solution $\tilde{\mathcal{M}}, \tilde{g}_{ab}$ given in the same coordinates, such that the metric is $\tilde{g}_{ab} = \tilde{g}_{ab} \circ \tilde{\psi}^{-1}(x_1, \ldots, x_4)$, where $\tilde{\psi}: \tilde{\mathcal{M}} \rightarrow U$.

We can obtain two situations. 1) The generated solution will coincide with the original, i.e. $\mathcal{M} = \tilde{\mathcal{M}}$. This means that we have merely performed a coordinate change and hence we have $g = \tilde{g}$. For example a metric on the form
\[
d s^2 = -d t^2 + dz^2 + d \rho^2 + \rho^2 d \phi^2 \tag{6.1}
\]
is the same solution as Minkowski space, but written in cylindrical coordinates. 2) The generated solution is not isometric to the old. It will still be a solution to Einstein’s equations but it will have a different metric and hence $\mathcal{M}$ will have different geometry. In this case we will call the generated solution new. The coordinates inherited from $\mathcal{M}$ might not be very useful, and a change of coordinates may be performed. The new solution can either be isometric to any previously known solution, or be a completely new one.

6.1 Projecting to three dimensions

The method of generating solutions will depend on the possibility to rewrite the metric in terms of a three-dimensional metric. We will follow a notation that is a compromise between the different authors we refer to.

Consider for example the Schwarzschild metric (5.14) and the stationary axisymmetric metric (5.23). They both possess timelike Killing vector fields, and thus we regard the spacelike part as our three-dimensional part. We will, however, not limit ourself to timelike Killing vectors. It turns out that the process of reducing to three dimensions works as well if the Killing vector is spacelike, but not if it is null [7].

In this section we will outline the projection to three dimensions as it is described by Geroch [7]. We will introduce a notation that makes it possible to follow both older and newer articles.

6.1.1 The metric

We will look at a metric on the form, which will refer to as our standard metric,
\[
d x^2 = \lambda (dx^0 + f_i dx^i)^2 + \lambda^{-1} h_{ij} dx^i dx^j. \tag{6.2}
\]
The metric has the Killing vector $\xi = \partial_0 = \frac{\partial}{\partial x^0}$, i.e. $\xi^a = \delta^a_0$. The function $\lambda$ is called the norm and is given by
\[
\lambda = \xi^a \xi_a. \tag{6.3}
\]
Since $\xi$ is either timelike or spacelike we must have that $\lambda$ has a determined sign. It is thus a conformal factor and will sometimes be written as $\lambda = \pm e^{2U}$.

In the case of $\xi$ being timelike we obtain axisymmetric stationary solutions. To obtain the metric (5.23) we put $\lambda = -e^{2U}$ and $h_{ij} = -\gamma_{ij}$. In this case $h_{ij}$ will be negative definite, a small disadvantage we must take to let the quantity $H_{ij} = \lambda^{-1} h_{ij}$ introduced below be positive definite.

If the one-form $f = f_i dx^i$ is zero the Killing vector $\xi$ is hypersurface-orthogonal, and we obtain static solutions.

In the case of $\xi$ being spacelike, $h_{ij}$ represents the components of a metric with signature $(-, +, +)$. An example of this is the Oszváth-Schücking metric (5.24), where we have $f = -2x^3 dx^2$ and
\[
h_{ij} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2(x^3)^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.4}
\]
6.1.2 The projection

The Killing vector field $\xi^a$ is everywhere defined on the manifold $\mathcal{M}$. We can therefore look at integral curves, or trajectories, to $\xi^a$ in accordance with (3.11). Since $\xi^a$ is a Killing vector field its integral curves generate a group of isometries, whose orbits are exactly the trajectories of $\xi^a$. For every point in $\mathcal{M}$ there will be a unique trajectory to $\xi^a$ passing through that point.

Define now $\mathcal{I}$ (script S) to be the set of all trajectories of $\xi^a$, i.e. the set of all orbits to the isometry group generated by $\xi^a$. We will consider $\mathcal{I}$ as a manifold on its own and define a smooth map $\mathcal{M} \rightarrow \mathcal{I}$ mapping the point $p$ onto the trajectory through $p$. If the Killing vector field were hypersurface-orthogonal, such as the case with the static metric, we could as well have chosen as our $x^a$ the coordinates $x^a$ of the hypersurface through $p$. However this is not the general case, which motivates our definition.

Geroch gives the following proposition written in the abstract index notation.

**Proposition 1.** [7] There is a one-to-one correspondence between tensor fields $T$ on $\mathcal{M}$ and tensor fields $T'$ on $\mathcal{I}$ if and only if

$$\xi_a T^{a...c}_{d...f} = 0, \quad \cdots, \quad \xi^j T^{a...c}_{d...f} = 0,$$

$$\xi^a T^{a...c}_{d...f} = 0.$$

A tensor field on $\mathcal{M}$ satisfying these equations will therefore be called a tensor field on $\mathcal{I}$.

With our definition of the norm $\lambda$ we can show that the following are tensor fields on $\mathcal{I}$:

$$H_{ab} = g_{ab} - \frac{\xi_a \xi_b}{\lambda}, \quad \varepsilon_{abcd} = \left| \lambda \right|^{-\frac{1}{2}} \varepsilon_{abcd} \xi^d,$$

where $\varepsilon_{abcd}$ is the antisymmetric tensor on $\mathcal{M}$. We will call $H_{ab}$ the projected metric.

In our case the Killing vector is assumed to be $\frac{\partial}{\partial x^a}$, and, as in [10], we can write

$$H_{ij} = g_{ij} - \frac{g_{0a}g_{0j}}{g_{00}}.$$  

This expression follows from (6.7) by $\lambda = g_{ab} \xi^a \xi^b = g_{0a}$, since $\xi^a = \delta^a_0$, and $\xi_a = g_{ab} \xi^b = g_{0a}$.

In order to adopt this convention to our standard metric we define the conformal projected metric

$$h_{ab} = \lambda H_{ab}.$$  

It can now be checked that our standard metric is indeed written on such form that $h_{ij}$ are the components of the conformal projected metric.

6.1.3 Abstract index notation or component notation

As can be understood from the arguments of the previous section there are two different formalisms for treating the three-dimensional metrics.

The abstract index notation is followed by [7,12] etc. Here we keep using indices $a,b,\ldots$ for tensor fields on $(\mathcal{I}, H_{ab})$. Indices can be raised and lowered by either $g_{ab}$ or $H_{ab}$. The projected covariant derivative $(\nabla_a)$ is given by

$$(\nabla_a)^{(3)} T_{b^1\ldots b_s c_1\ldots c_s} = H^d_a H^{b_1}_{e_1} \cdots H^{b_s}_{e_s} H^{f_1}_{c_1} \cdots H^{f_s}_{c_s} T_{d c_1\ldots c_s f_1\ldots f_s},$$

where $h^a_b = \delta^a_b - \lambda^{-1} \xi_a \xi^b$.

In the same manner the projected Ricci tensor $(\nabla_a) R_{ab}$ etc. can be calculated. For more details the reader is referred to the appendix of Geroch [7]. For practical calculations all projected quantities are later transformed into the conformal projected version. The notation for the conformal projected covariant derivative is $D_a$.

The component notation is followed by [8–11,17] etc. It uses the conformal projected metric $h_{ij}$ as a metric on $\mathcal{I}$, and has always $i,j,\ldots = 1,2,3$. This makes $(\mathcal{I}, h_{ij})$ a three-dimensional spacetime, on which we can define the derivative $\partial_i = \frac{\partial}{\partial x^i}$ and the conformal
projected covariant derivative $\nabla_i = D_i$ using the Christoffel symbols of $h_{ij}$. We can also define the conformal projected Ricci tensor $P_{ij}$ and its scalar curvature $P = P^i_i$, where indices are lowered and raised with $h_{ij}$.

It should be noted that the component notation goes directly into the conformal projected metric whereas the abstract index notation goes via the projected metric. In the conformal projected case we will use the same notation, i.e.

$$(\mathcal{F}, h_{ab}) \leftrightarrow (\mathcal{F}, h_{ij}), \quad D_a \leftrightarrow D_i = \nabla_i, \quad P_{ab} \leftrightarrow P_{ij}, \ldots \quad (6.11)$$

The important difference is that, when acting on functions we will write $\partial_i$ instead of $\nabla_i$, but we can not replace $D_a$ with $\partial_a$.

### 6.1.4 Norm and Twist

A solution is described completely by the ten independent components of the unprojected metric, that in turn depends on the four space-time coordinates. After a projection is performed we want to describe the solution depending on only the three coordinates of $\mathcal{S}$. This means that we have to give a complete description of the solution in terms of only tensor fields that fulfill the restrictions of proposition 1. In addition we seek a description that can be seen as a sigma model, which means that our tensor field should consist of one $(0, 2)$ tensor and $d$ scalar fields, i.e. $(H_{ab}, \varphi^1, \ldots, \varphi^d)$.

As seen above the projected metric fulfills our demands and what is left is to find scalar fields that together with $H_{ab}$ uniquely describe $g_{ab}$, where $g_{ab}$ is our standard metric (6.2).

The norm $\lambda$ is, as described above, given by $\lambda = \xi^a \xi_a$, or as in our case when the Killing vector is $\xi = \frac{\partial}{\partial x^0}$, $\lambda = g_{00}$.

The scalar twist $\omega$ is described differently by different authors, but the results are equivalent. Geroch [7] defines the one-form twist as

$$\omega_a = \varepsilon_{abcd} \xi^b \nabla^c \xi^d. \quad (6.12)$$

Then the exterior derivative of $\omega_a$ is zero for $R_{ab} = 0$, so by the Poincaré lemma there exists, locally, a function $\omega$ such that $\omega_a = \partial_a \omega = (d \omega)_a$. This $\omega$ is defined to be the scalar twist. That $\omega$ satisfies $L_{\xi} \omega = 0$ and hence proposition 1 is due to the fact that $L_{\xi} g_{ab} = 0$ and $L_{\xi} \xi^a = 0$.

Leigh et.al. [12] describe the same calculations but use explicitly the Hodge star duality for (6.12). Harrison [9] argues instead in terms of the three-dimensional quantities: Let $k_{ij} = \partial_j f_i - \partial_i f_j$. Then $k_{ij}$ is a two-form in three dimensions and hence the Hodge star dual of the one-form $z_l$ by

$$k_{ij} = \varepsilon_{ijk} h^{kl} z_l, \quad (6.13)$$

according to (3.36). $z_l$ is related to the twist by (7.6) below.

### 6.2 The classical method

In this section we outline the general procedure used by the early authors to generate solutions. In this formalism there is no obvious presence of the sigma model. The procedure goes as follows:

1. Choose a class of metrics with a Killing vector field for which there is at least one known solution.

2. Parametrise the metric in terms of functions $\varphi^A$ and the conformal projected metric $h_{ij}$. Common choices for functions are $\lambda$, or equivalently $U$, the scalar twist $\omega$ and any scalar fields giving rise to electromagnetic fields.

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14 In [12] we have not $R_{ab} = 0$, nevertheless it turns out that $d \omega = 0$. 

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3. Starting from Einstein’s field equations, derive equations for $\varphi^A$, which might include terms of the reduced projected metric or conformal projected Ricci tensor.

4. Find new functions $\tilde{\varphi}^A$, expressed in terms of $\varphi^A$ such that the equations of step 3 are fulfilled given that they are fulfilled for $\varphi^A$. The conformal projected metric remains invariant.

5. Find explicitly the new functions $\tilde{\varphi}^A$ from the $\varphi^A$ of a known solution on the form of step 1. Then a metric expressed in terms of the functions $\tilde{\varphi}^A$ is a generated solution of Einstein’s field equations.

### 6.3 The modern method

In the modern approach, used in by the more recent authors, we use the sigma model explicitly.

1. Choose a class of metrics admitting a Killing vector field. Also choose some additional matter fields with Lagrangian $L_M$. Then write down the action

$$S = \int_M d^4x \sqrt{-g} (R + L_M). \quad (6.14)$$

2. Parametrise the metric and its matter fields in terms of the conformal projected metric $h_{ij}$ and scalar fields $\varphi^A$.

3. Project the action to the corresponding action on the manifold $(\mathcal{S}, h)$,

$$S = \int_{\mathcal{S}} d^3x \sqrt{|h|} (P + L_m). \quad (6.15)$$

The term $R$ of (6.14) will contribute to both $P$ and $L_m$. We can view $(\mathcal{S}, h_{ij})$ as a three-dimensional theory of gravity, where $L_m$ are matter fields.

4. Now, by assumption, $L_m$ can be written on the form

$$L_m = G_{AB} h^{ij} \partial_i \varphi^A \partial_j \varphi^B \quad (6.16)$$

so that it constitutes a sigma model with domain space $\mathcal{S}$. Consider $G_{AB}$ as the metric on the target space $\mathcal{S}$. Find Killing vector fields $k^{(\alpha)}$, indexed by $(\alpha) = (1), \ldots, (m)$.

5. Perform isometries on $\mathcal{S}$ by exponentiating the Killing vectors. This gives new functions

$$\tilde{\varphi}^A = \exp \left( \lambda^{(\alpha)} k^{(\alpha)} \right) \varphi^A. \quad (6.17)$$

6. Find explicitly the new functions $\tilde{\varphi}^A$ from the $\varphi^A$ of a known solution on the form of step 1. Then a metric expressed in the $\tilde{\varphi}^A$ is a generated solution to Einstein’s field equations.

We will now show that the generated solution is indeed a solution to the field equations of the three-dimensional theory of gravity. First we derive the field equations and then we prove that they are fulfilled by the $\tilde{\varphi}^A$. 

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6.3.1 Field equations on $\mathcal{F}$

In we perform variations of $h^{ij}$ and $\varphi^A$ in the action

$$S = \int_{\mathcal{F}} d^3x \sqrt{|h|} \left( P + G_{AB}h^{ij}\partial_i\varphi^A\partial_j\varphi^B \right)$$

(6.18)

we get the following field equations on $\mathcal{F}$

$$P_{ij} = -G_{AB}\partial_i\varphi^A\partial_j\varphi^B,$$  

(6.19)

$$\nabla_i \partial^i \varphi^A = -\frac{1}{2}G^{AD} \left( \partial_C G_{DB} + \partial_B G_{DC} - \partial_D G_{BC} \right) \partial^i \varphi^B \partial_i \varphi^C = -\Sigma^B_{BC} \partial^i \varphi^B \partial_i \varphi^C,$$  

(6.20)

where $\Sigma^B_{BC}$ is the Christoffel symbols on $\mathcal{F}$ belonging to the metric $G_{AB}$ according to (3.16). We will denote (6.19) and (6.20) by the three-dimensional field equations, or field equations on $(\mathcal{F}, h_{ij})$. We will now show how these equations are derived.

Firstly, vary the action (6.18) with respect to $h_{ij}$. The first term yields $\sqrt{|h|}(P_{ij} - \frac{1}{2}Ph_{ij})$, according to section 5.1, where we have used the three-dimensional expressions $h_{ij}$, $P_{ij}$ and $P$ instead of $g_{ab}$, $R_{ab}$ and $R$. For the second term we have, since our $h^{ij}$-dependence lies only in $h$ and $h^{ij}$,

$$\delta \left( \sqrt{|h|}G_{AB}h^{ij}\partial_i\varphi^A\partial_j\varphi^B \right) = h_{ij}G_{AB}\partial_i\varphi^A\partial_j\varphi^B \delta\sqrt{|h|} + \sqrt{|h|}G_{AB}\partial_i\varphi^A\partial_j\varphi^B \delta h_{ij}.$$  

(6.21)

The second terms is already on the desired form. For the first one we use (5.11) to get that $\delta\sqrt{|h|} = -\frac{1}{2}h_{ij}\delta h^{ij}$. Thus, factoring out $\sqrt{|h|}$, we get the equations

$$P_{ij} - \frac{1}{2}Ph_{ij} + G_{AB}\partial_i\varphi^A\partial_j\varphi^B - \frac{1}{2}h_{ij}G_{AB}\partial_i\varphi^A\partial_j\varphi^B = 0,$$  

(6.22)

which are fulfilled if and only if $P_{ij} = -G_{AB}\partial_i\varphi^A\partial_j\varphi^B$, i.e. (6.19).

Now varying with respect to $\varphi^A$, the factor $\sqrt{|h|}$ and the term $P$ give no contributions. Thus we vary $G_{AB}\partial^i\varphi^A\partial_i\varphi^B$.

$$\delta \left( G_{AB}\partial^i\varphi^A\partial_i\varphi^B \right) = 2G_{AB}\partial^i\varphi^B \delta \left( \partial_i\varphi^A \right) + \partial_i\varphi^A \partial^i\varphi^B \delta G_{AB} = 2G_{AB}\partial^i\varphi^B \partial_i\left( \delta \varphi^A \right) + \partial_i\varphi^A \partial^i\varphi^B \partial C_{GAB} \delta \varphi^C,$$  

(6.23)

where $\partial_C = \frac{\partial}{\partial \varphi^C}$. Now integrate by parts\textsuperscript{15} in the first term and relabel in the second term to get

$$-2\nabla_i \left( G_{AB}\partial^i\varphi^B \right) \delta \varphi^A + \partial_i\varphi^C \partial^i\varphi^B \partial A_{GCB} \delta \varphi^A.$$  

(6.24)

To extremise the action we must have

$$0 = -2\nabla_i \left( G_{AB}\partial^i\varphi^B \right) + \partial_i\varphi^C \partial^i\varphi^B \partial A_{GCB} = -2G_{AB}\nabla_i \partial^i\varphi^B - 2\partial^i\varphi^B \partial C_{GAB} \partial \varphi^C + \partial_i\varphi^C \partial^i\varphi^B \partial A_{GCB}$$

$$= -2G_{AB}\nabla_i \partial^i\varphi^B - 2\partial C_{GAB} \partial A_{GCB} \partial^i\varphi^B \partial \varphi^C.$$  

(6.25)

In the second line we used that on scalar fields, $\nabla_i = \partial_i$. Now multiplying by $-\frac{1}{2}G^{AD}$ and using that indices $B$ and $C$ may be interchanged in the second term we get

$$0 = \nabla_i \partial^i\varphi^D + \frac{1}{2}G^{AD} \left( \partial C_{GAB} + \partial B_{GAC} - \partial A_{GBC} \right) \partial^i\varphi^B \partial_i\varphi^C = \nabla_i \partial^i\varphi^D + \Sigma^B_{BC} \partial^i\varphi^B \partial_i\varphi^C,$$  

(6.26)

by (3.16). This equation is exactly (6.20).

\textsuperscript{15}The integration by parts is performed on the whole expression including the factor $\sqrt{|h|}$, i.e. we get $\partial_i(\sqrt{|h|}2G_{AB}\partial^i\varphi^B)$. However, for a vector field $X^i$ we have

$$\partial_i \left( \sqrt{|h|}X^i \right) = \sqrt{|h|}\partial_i X^i + X^i \partial_i \sqrt{|h|} = \sqrt{|h|}\partial_i X^i + X^i \sqrt{|h|}\partial_i \ln \sqrt{|h|}$$

$$= \sqrt{|h|}\partial_i X^i + \sqrt{|h|} \Gamma^i_{jk} X^k = \sqrt{|h|}\nabla_i X^i,$$

where we used (3.25). Thus we get $\partial_i(\sqrt{|h|}2G_{AB}\partial^i\varphi^B) = \sqrt{|h|}\nabla_i(2G_{AB}\partial^i\varphi^B)$. 

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6.3.2 Proof

We will now show that the generated solution actually fulfills the three-dimensional field equations (6.19) and (6.20). We will consider only the case of infinitesimal isometries \( \chi_\epsilon : \mathcal{F} \rightarrow \mathcal{F} \). Look at new functions \( \bar{\varphi}^A \) according to (6.17) with the small group parameters \( \epsilon_0 = \epsilon_1 \) and denote by \( \epsilon k = \epsilon^C \partial_C \). The expression \( \epsilon_0 k^{(a)} \). We get

\[
\partial_i \bar{\varphi}^A = \partial_i \exp(\epsilon^C \partial_C)\varphi^A = \partial_i \varphi^A + \epsilon_0 (\epsilon^C \partial_C \varphi^A) + \mathcal{O} (\epsilon^2) = \partial_i \varphi^A + \epsilon_0 (\varphi^A) + \mathcal{O} (\epsilon^2). \tag{6.27}
\]

This can be rewritten further by \( \partial_i = \partial_i \varphi^C \partial_C \) to

\[
\partial_i \bar{\varphi}^A = \partial_i \varphi^A + \epsilon \partial_C k^A \partial_i \varphi^C + \mathcal{O} (\epsilon^2). \tag{6.28}
\]

For the metric we have that \( \bar{G} = \bar{G}_{AB} d \bar{\varphi}^A d \bar{\varphi}^B = (\chi_\epsilon)_* G \), i.e. \( \bar{G} \) is the push-forward of the metric \( G \) before the isometry. Its components are therefore, by (3.26), given by

\[
\bar{G}_{AB} = G_{AB}(\varphi^D) = G_{AB}(\varphi^D + \epsilon k^D) = G_{AB} + \epsilon \partial_D G_{AB} k^D + \mathcal{O}(\epsilon^2), \tag{6.29}
\]

where \( G_{AB} = G_{AB}(\varphi^A) \).

We are now ready to prove (6.19) using the above expressions for \( \bar{G}_{AB} \) and \( \bar{\varphi}^A \). Since \( P_{ij} \) is not affected by the transformation we shall prove that \( \bar{G}_{AB} \partial_i \bar{\varphi}^A \partial_j \bar{\varphi}^B = G_{AB} \partial_i \varphi^A \partial_j \varphi^B \) to first order in \( \epsilon \). Since each side is equal to the conformal projected Ricci tensor the expression is symmetric in \( i \) and \( j \), which we will use to reorganise the expression. Omitting higher order terms we get by (6.28) and (6.29)

\[
\bar{G}_{AB} \partial_i \bar{\varphi}^A \partial_j \bar{\varphi}^B = (G_{AB} + \epsilon \partial_D G_{AB} k^D) (\partial_i \varphi^A + \epsilon \partial_C k^A \partial_i \varphi^C) (\partial_j \varphi^B + \epsilon \partial_E k^B \partial_j \varphi^E)
\]

\[
= G_{AB} \partial_i \varphi^A \partial_j \varphi^B + \epsilon \partial_D G_{AB} k^D \partial_i \varphi^A \partial_j \varphi^B + 2 \epsilon G_{AB} \partial_i \varphi^A \partial_C k^A \partial_j \varphi^C
\]

\[
= G_{AB} \partial_i \varphi^A \partial_j \varphi^B + \epsilon (\partial_D G_{AB} k^D + 2 G_{AC} \partial_D k^C) \partial_i \varphi^A \partial_j \varphi^B. \tag{6.30}
\]

In the second line we used \( i \leftrightarrow j \) to collect two equal terms and in the third line we relabeled \( B \leftrightarrow C \) in the last term. We will now show that the \( \epsilon \) term equals zero. By interchanging \( i \) and \( j \) we note that we can symmetrise indices \( A \) and \( B \),

\[
2G_{C(A} \partial_B) k^C + \partial_D G_{(AB)} k^D = 2G_{C(A} \partial_B) k^C + k^D (\partial_D G_{(AB)} + \partial_i (G_{AB}) k^D - \partial_i (G_{AB}) d^D)
\]

\[
= 2G_{C(A} \partial_B) k^C + 2k^D \Sigma_{AB)} D^D
\]

\[
= 2 \left( G_{C(A} \partial_B) k^C + G_{C(A} \Sigma_{BC)} D^D k^D \right)
\]

\[
= 2G_{C(A} \Sigma_B k^C
\]

\[
= 2 \nabla_{(B k_A)}. \tag{6.31}
\]

In the first line we added a complicated zero, in the second line we identified the Christoffel symbols \( \Sigma_{ABC} \) on \( \mathcal{F} \) belonging to \( G_{AB} \), in the fourth line we identified the covariant derivative on \( \mathcal{F} \) and in the last line we used that \( \nabla_A G_{BC} = 0 \) and used Killing's equation (3.28). Hence we have proven that the generated fields satisfy (6.19).

We now proceed to proving that (6.20) are fulfilled after generation, i.e. that

\[
\nabla_i \partial^i \bar{\varphi}^A + \bar{\Sigma}^A_{BC} \partial^i \bar{\varphi}^B \partial_i \bar{\varphi}^C = 0 \tag{6.32}
\]

given that

\[
\nabla_i \partial_i \bar{\varphi}^A + \Sigma^A_{BC} \partial^i \varphi^B \partial_i \varphi^C = 0. \tag{6.33}
\]

Keeping terms up to first order in \( \epsilon \) we get

\[
\nabla_i \partial^i \bar{\varphi}^A + \Sigma^A_{BC} \partial^i \bar{\varphi}^B \partial_i \bar{\varphi}^C = \nabla_i \partial_i \bar{\varphi}^A + \Sigma^A_{BC} \partial^i \varphi^B \partial_i \varphi^C
\]

\[
+ \epsilon \left( \nabla_i \partial_i k^A + 2 \Sigma^A_{BC} \partial^i \varphi^B \partial_D k^C \partial_i \varphi^D
\]

\[
+ \partial^i \varphi^B \partial_i \varphi^C \partial_D \Sigma^A_{BC} k^D \right). \tag{6.34}
\]
The first two terms are cancelled by (6.33). We shall show that the \( \epsilon \) term is zero. Firstly

\[
\nabla^i \partial_j k^A = \nabla^i \left( \partial_B k^A \partial_i \varphi^B \right) = \partial^i \left( \partial_B k^A \right) \partial_i \varphi^B + \partial_C k^A \nabla^i \partial_j \varphi^C \\
= \partial_D \partial_B k^A \partial_i \varphi^B \partial_j \varphi^D - \partial_C k^A \Sigma^D_{BD} \partial_i \varphi^B \partial_j \varphi^D, \tag{6.35}
\]

where we in the second equality used that on functions \( \nabla^i = \partial^i \) and in the third equality used again (6.33). Thus the \( \epsilon \) term becomes, with some relabeling,

\[
\left( \partial_D \partial_B k^A - \Sigma^C_{BD} \partial_C k^A + 2 \Sigma_{\Delta \epsilon} \Delta_{\epsilon} \Sigma^C_{BD} + k^C \partial_C \Sigma^A_{BD} \right) \partial_i \varphi^B \partial_j \varphi^D. \tag{6.36}
\]

We proceed with the expression in parenthesis but may use \( B \leftrightarrow D \) due to the factor \( \partial^i \varphi^B \partial_j \varphi^D \). Using \( \Sigma_{\Delta \epsilon} \Delta_{\epsilon} k^C = \partial_D \left( \Sigma^A_{\Delta \epsilon} k^C \right) - k^C \partial_D \Sigma^A_{\Delta \epsilon} \) we have for the first term and half the third term

\[
\partial_D \partial_B k^A + \Sigma_{\Delta \epsilon} \Delta_{\epsilon} \partial_B k^C = \partial_D \left( \partial_B k^A + \Sigma_{\Delta \epsilon} \Delta_{\epsilon} k^C \right) - k^C \partial_D \Sigma^A_{\Delta \epsilon} = \partial_D \nabla_B k^A - k^C \partial_D \Sigma^A_{\Delta \epsilon}. \tag{6.37}
\]

Hence the parenthesis of (6.36) becomes

\[
\partial_D \nabla_B k^A - \Sigma^C_{BD} \partial_C k^A + \Sigma_{\Delta \epsilon} \Delta_{\epsilon} \partial_B k^C + k^C \left( \partial_C \Sigma^A_{\Delta \epsilon} - \partial_D \Sigma^A_{\Delta \epsilon} \right). \tag{6.38}
\]

By (3.23) we have the equation

\[
k^C \left( \partial_C \Sigma^A_{\Delta \epsilon} - \partial_D \Sigma^A_{\Delta \epsilon} \right) = k^C R^{A}_{\Delta \epsilon} - k^C \Sigma^E_{DB} \Sigma^A_{EC} + k^C \Sigma^E_{CB} \Sigma^A_{ED}, \tag{6.39}
\]

where \( R^{A}_{\Delta \epsilon} \) is the Riemann curvature tensor belonging to \( G_{AB} \). By (3.9) used twice we have the equation

\[
\nabla_D \nabla_B k^A = \partial_D \nabla_B k^A - \Sigma^E_{DB} \nabla_E k^A + \Sigma^A_{DE} \nabla_B k^E \\
= \partial_D \nabla_B k^A - \Sigma^C_{BD} \partial_C k^A - \Sigma^E_{DB} \Sigma^A_{EC} k^C + \Sigma^A_{DC} \partial_B k^C + \Sigma^A_{DE} \Sigma^E_{BC} k^C. \tag{6.40}
\]

Using simultaneously the substitutions of (6.39) and (6.40) and cancelling terms, the expression (6.38) becomes

\[
\nabla_D \nabla_B k^A + R^{A}_{\Delta \epsilon} k^C = \nabla_D \nabla_B \left( G^{AE} k_E \right) + G^{AE} R^{A}_{\Delta \epsilon} k^C \\
= G^{AE} \left( \nabla_D \nabla_B k_E + R^{A}_{BED} \right) k^C \\
= G^{AE} \left( \nabla_B \nabla_D k_E + \nabla_B \nabla_E k_D - \nabla_E \nabla_B k_D \right) \\
= G^{AE} \left( 2 \nabla_B \nabla_D k_E - \nabla_E \nabla_B k_D \right) = 0. \tag{6.41}
\]

In the second line we used that \( \nabla_A G^{BC} = 0 \) and that \( R^{A}_{\Delta \epsilon} = R_{\Delta \epsilon}^{-} \) (c.f. [4. p. 39]).

In the third line we used (3.21) for the Riemann tensor and in the third and fourth line we used the allowed relabeling \( B \leftrightarrow D \) described above. Finally the expression is equal to zero due to Killing’s equation (3.28). This finishes the proof that the generated solution fulfills the field equations.

### 7 Examples of generations

In this section we will study how different authors have used the method of sigma models to generate solutions. We will try to link the descriptions given by the authors to our two methods described in sections 6.2 and 6.3. Although we review the articles in chronological order we present not the full historical story but rather pieces of work that elucidate the development.

During our presentations we will follow our convention of notions and signs instead of the ones used by the authors.
7.1 Jürgen Ehlers

Jürgen Ehlers [8] looks first at static solutions, i.e. in our standard metric \( f_i = 0 \). We will here use the notation \( \lambda = e^{2U} \) and \( x^0 = t \).

\[
\text{ds}^2 = -e^{2U}dt - e^{-2U}h_{ij}dx^idx^j. \quad (7.1)
\]

This makes \( h_{ij} \) a negative definite metric. We consider \( h_{ij} \) as the metric on the manifold \( \mathcal{S} \) as described in section 6.1.2. Ehlers then shows that the corresponding field equations on \( (\mathcal{S}, h_{ij}) \) are

\[
P_{ij} = 2\partial_i U \partial_j U, \quad (7.2)
\]

\[
\nabla^i \partial_i U = 0. \quad (7.3)
\]

These equations indeed correspond to a sigma model with action \( -2\partial^i U \partial_i U \), but the isometries on this one-dimensional target space are trivial:

\[
\tilde{U} = U + \lambda, \quad \tilde{U} = -U. \quad (7.4)
\]

The case \( \tilde{U} = U + \lambda \) corresponds to rescaling the coordinates. The case \( \tilde{U} = -U \) corresponds to an earlier theorem of Hans Adolph Buchdahl [18].

Ehlers also shows that the field equations for two other metrics reduce to (7.2) and (7.3) if subject to certain additional constraints. One is a stationary solution with \( \lambda = (a \cosh(2U))^{-1} \) and \( f_i \) subject to \( a\varepsilon_{ijk} \partial^k U = \partial_{[j} f_{i]} \) \((a \text{ is a constant})\). The other is a solution describing rigid motion of incoherent matter.

Ehler’s conclusion is then that determining all static axisymmetric solutions is equivalent to determining all solutions of the other two forms described above. Thus he describes a method of generating solutions, but it is not, except for the theorem by Buchdahl, by using the sigma model method of isometries of a target space.

7.2 Kent Harrison

Kent Harrison [9] starts with a much larger class of metrics, equivalent to our standard metric. Up to signs he uses \( \lambda = e^{2U} \). The stress-energy tensor is pure electromagnetic i.e. \( T = T^a_o = 0 \).

The solution is parametrised in terms of \( h_{ij} \) and four scalar fields: \( U \), a twist function \( \phi \) and two electromagnetic fields \( A \) and \( B \). \( A \) and \( B \) are defined by \( F^{ij} = -(g)^{-1} e^{ijk} \partial_k A, F_{0a} = \partial_a B \), where \( F_{ab} \) is the electromagnetic field tensor\(^{16}\). Using the potentials \( A \) and \( B \) we write by using (6.13)

\[
k_{ij} = 2\partial_{[j} f_{i]} = \varepsilon_{ijk}h^{kl}z_l, \quad (7.5)
\]

with

\[
z_i = e^{-4U}(\partial_i \phi + 2(B\partial_i A - A\partial_i B)) = e^{-4U}(\partial_i \phi - 2R^2 \partial_i \theta). \quad (7.6)
\]

\( R, \theta \) are polar potentials substituting \( A, B \) by \( (A, B) = (R \cos \theta, R \sin \theta) \). \( A = B = 0 \) gives that \( \phi \) corresponds to our scalar twist \( \omega \).

Without explicitly referring to any three-dimensional manifold \( \mathcal{S} \), Harrison now writes five equations for \( P_{ij} \) and \( U, \phi, R, \theta \) corresponding to the three-dimensional field equations (6.19) and (6.20). We can now say that, keeping \( P_{ij} \) fixed, \( \text{any new functions} \tilde{U}, \tilde{\phi}, \tilde{R}, \tilde{\theta} \), \( \text{given in terms of the non-constant} U, \phi, R, \theta \) \( \text{such that the three-dimensional field equations are satisfied} \) \( \text{represent a generated solution to general relativity} \). However, for generating these new functions Harrison does not view the three-dimensional field equations as forming a sigma model. Instead solutions are generated by generations theorems. A theorem is derived by choosing which of \( U, \phi, R, \theta \) that be constant and then write expressions for \( \tilde{U}, \tilde{\phi}, \tilde{R}, \tilde{\theta} \) in terms of the non-constant \( U, \phi, R, \theta \).

\(^{16}F_{ab} \) is defined such that it gives the stress-energy tensor \( T_{ab} = \frac{1}{\pi^2} \left( F_{ac}F^c_b - \frac{1}{4} F_{cd}F^{cd} g_{ab} \right) \).
Harrison gives examples of three of 15 possible such theorems and uses them to generate new solutions from the Ozsváth-Schücking metric \((5.24), \xi = \partial_t\), to generate a twisted Melvin universe from the Minkowski space in cylindrical coordinates \((6.1), \xi = \partial_t\) and to generate the Brill space from the Schwarzschild metric \((5.14), \xi = \partial_t\).

Finally, Harrison notes that when applying the generation certain qualities of the solution seem to “compensate” for each other. For the Ozsváth-Schücking metric the gravitational wave is accompanied by an electromagnetic wave in the same direction, and for the other two solutions the geometrical twist and the electromagnetic field compensate each other.

### 7.3 Robert Geroch

From Ehlers we learned how to project the space-time onto the three-dimensional \(\mathcal{S}\). From Harrison we learned that pure electromagnetic solutions can be described by four scalar fields. Particularly, for stationary solutions vacuum solutions \((Harrison we learned that pure electromagnetic solutions can be described by four scalar fields. From Ehlers we learned how to project the space-time onto the three-dimensional \(\mathcal{S}\).)

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We now leave Geroch’s article for a moment and consider the setup from the modern point of view, using a sigma model. By comparing with the three-dimensional field equations \((6.19), (6.20)\) we conclude that \(7.7)\)–\(\tau.\) above come from the action

\[
S = \int_{\mathcal{F}} \text{d}^3x \sqrt{|h|} \left( P + \lambda^{-2} (\partial_t \omega \partial_j \omega + \partial_i \lambda \partial_j \lambda) h^{ij} \right) = \int_{\mathcal{F}} \text{d}^3x \sqrt{|h|} \left( P + 2(\tau - \bar{\tau})^{-2} \partial_t \bar{\tau} \partial_j \bar{\tau} h^{ij} \right).
\]

We identify the target space metric as \(G_{\lambda \lambda} = G_{\omega \omega} = \lambda^{-2}, G_{\lambda \omega} = G_{\omega \lambda} = 0, \) i.e. \(\text{d}^2 \lambda = \lambda^{-2} (\text{d} \lambda^2 + \text{d} \omega^2),\) and get the Killing vectors \[12\]

\[
k^{(1)} = \partial_\omega, \quad k^{(2)} = \omega \partial_\omega + \lambda \partial_\lambda, \quad k^{(3)} = (\lambda^2 - \omega^2) \partial_\omega - 2 \lambda \omega \partial_\lambda.
\]

These Killing vectors obey

\[
\left[ k^{(1)}, k^{(2)} \right] = k^{(1)}, \quad \left[ k^{(2)}, k^{(3)} \right] = k^{(3)}, \quad \left[ k^{(1)}, k^{(3)} \right] = -2k^{(2)},
\]

\footnote{\(SL(2, \mathbb{R})\) (the two-dimensional real special linear group) is the group of linear operators \(A : \mathbb{R}^2 \to \mathbb{R}^2\) with \(\det A = 1.\)}
and thus form a representation of the $sl(2,\mathbb{R})$ Lie algebra\(^{18}\) of the $SL(2,\mathbb{R})$ group, hence this corresponds directly to Geroch’s $SL(2,\mathbb{R})$. So what Geroch did was indeed a generation using sigma models with a target space isometry group equal to $SL(2,\mathbb{R})$.

However not all of the isometries of $SL(2,\mathbb{R})$ correspond to new solutions. It turns out [7] that there is a subgroup of pure gauge transformation, i.e. even though $(\tilde{\lambda}, \tilde{\omega}) \neq (\lambda, \omega)$ the solution is in an obvious way isometric to the old. For example $\tilde{\lambda} = \lambda \tilde{\omega}$ corresponds to rescale $\xi^a$ and $\tilde{\omega} = \omega + b$ leaves $\partial_a \omega = \tilde{\partial}_a \omega$.

One element of $SL(2,\mathbb{R})$ that gives a proper transform is $
abla \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$

which gives the following explicit form of the generated metric
\[
\tilde{g}_{ab} = [(\cos \theta - \omega \sin \theta)^2 + \lambda^2 \sin^2 \theta] g_{ab} + 2 \sin \theta \epsilon_{(a} [2\alpha_b \cos \theta - \beta_b \sin \theta] \\
+ \lambda [(\cos \theta - \omega \sin \theta)^2 + \lambda^2 \sin^2 \theta]^{-1} \sin^2 \theta (2\alpha_a \cos \theta - \beta_a \sin \theta) (2\alpha_b \cos \theta - \beta_b \sin \theta),
\]

where $\alpha_a$ and $\beta_a$ are solutions of $\nabla_{[\alpha} a_{\beta]} = \frac{1}{2} \epsilon_{abcd} \nabla^c \xi^d$ and $\nabla_{[\alpha} \beta_{\beta]} = 2\lambda \nabla_a \xi_b + \omega \epsilon_{abcd} \nabla^c \xi^d$

fulfilling $\xi^a \alpha_a = \omega$ and $\xi^a \beta_a = \omega^2 + \lambda^2 - 1$.

In his subsequent paper [19] Geroch considers the case with two commuting Killing vector fields $\xi^a$ and $\zeta^a$, i.e. $[\xi, \zeta]^a = 0$ by the means of (3.13). More details will be given in section 8.5. The idea in this paper is to perform the method if his first paper with respect to first one linear combination of the Killing vectors and then another linear combination. Under some restrictions the Killing vectors remain commuting Killing vectors through each transformation and we can thus iterate the method. The conclusion is that following this programme the generated solutions can be parametrised by two arbitrary functions of one variable.

Geroch puts up the following conjecture. Is it possible that all solutions with two commuting Killing vector fields, obeying these certain restrictions, can be generated by this method starting from another such solution? If so, then for example we could start with Minkowski space cylindrical coordinates $(\xi = \partial_t$ and $\zeta = \partial_\phi)$ and in principle generate all asymptotically flat stationary, axisymmetric vacuum solutions. Eight years later this conjecture was proven to be correct [20], but no explicit method of constructing such generation was given.

### 7.4 Norma Sanchez

Norma Sanchez [10] couples the generation of solutions to the formalism of sigma models, and particularly the $O(2,1)$ sigma model. In its canonical form, the sigma model Lagrangian is
\[
L_\sigma = \frac{1}{2} \sum_{k=1}^{3} \partial^A \sigma^A \partial_A \sigma^A,
\]

with $(\sigma^1)^2 + (\sigma^2)^2 - (\sigma^3)^2 = 1$. Now the coordinates $\sigma^A$ can be changed into
\[
V = \frac{1}{\sigma^1 + \sigma^2}, \quad \psi = \frac{\sigma^2}{\sigma^1 + \sigma^2},
\]

so that
\[
L = \frac{1}{V^2} \left( \partial^i V \partial_i V - 2 \partial^i \psi \partial_i \psi \right).
\]

The $SL(2,\mathbb{R})$ properties of the target space isometries are discussed. Then a solution in the form of our standard metric is analysed. The solution is described in terms of the norm, twist

\[\text{The Lie algebra } sl(2,\mathbb{R}) = \{ A \in \mathbb{R}^{2 \times 2} | \text{Tr} A = 0 \} = \text{span} \{ K^1, K^2, K^3 \}, \text{ with } K^1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, K^2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -1 \end{pmatrix} \text{ and } K^3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \]
and projected metric. By changing to the conformal projected metric the three-dimensional field equations are derived. Sanchez now finds that the three-dimensional field equations correspond exactly to the Lagrangian of equation (7.19) with $V = \lambda, \psi = i\omega$. Also a complex function corresponding to $\tau$ is introduced and the situation is analysed further.

7.5 Some modern works

Here we will present how the method in its modern description according to section 6.3 is used to generate new solutions. We will present two examples although other works done on the subject.

In [11] we start from stationary axisymmetric metric and consider the action

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left( -R + 2\partial_a \phi \partial^a \phi - e^{2\alpha} F_{ab} F^{ab} \right). \tag{7.20}$$

$\phi$ is the dilaton strength and $F_{ab}$ is the electromagnetic field tensor. When projecting to three dimensions we get the five functions $\{\varphi^A\} = \{\phi, \lambda, \omega, u, v\}$, where $\lambda, \omega$ are the norm and twist, $\phi$ is still the dilaton strength and $u, v$ are electromagnetic potentials. For the sigma model part of the three-dimensional action (6.16) we have a metric $G_{AB}$ with the line element

$$d^2 = 2d\phi^2 + \frac{1}{2\lambda^2} \left( d\lambda^2 + (d\omega + vdu - udv)^2 \right) - \frac{1}{\lambda} \left( e^{-2\alpha} du^2 + e^{2\alpha} dv^2 \right) \tag{7.21}$$

From this action five Killing vectors can be found that form a closed Lie algebra. However, for $a = 0$ and $a = \pm \sqrt{3}$ additional Killing vectors can be found. From the Killing vectors the new coordinates $\varphi^A$ are found and analysed. The method has been performed in the general setup, but three examples are given. The examples start from a vacuum solution ($u, v, \phi = 0$), a static dilaton black hole solution and the Kerr solution. The generated solutions correspond to previously known solutions.

The interesting feature of this article is that certain values of $a$ enlarge the isometry group. These certain values have previously been encountered when studying other aspects of the involved solutions, where they are the value for minimal coupling of some kind.

In [17] we start from the action

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left( R + 2\partial_a \phi \partial^a \phi + \frac{1}{2} e^{4\phi} \partial_a \kappa \partial^a \kappa - e^{-2\phi} F_{ab} F^{ab} - \frac{1}{2} \kappa \varepsilon^{abcd} F_{ab} F_{cd} \right) \tag{7.22}$$

from the study of electromagnetic dilaton-axion theory of gravity. The scalar field $\kappa$ arises from a dilaton-axion symmetry, and the rest of the fields are defined as in [11] described above. When projecting to three dimensions six functions parametrise the solution, $\{\varphi^A\} = \{\phi, \lambda, \omega, u, v, \kappa\}$. For the sigma model a ten-dimensional target space isometry group is identified. The finite transformations generated by the sigma model isometries are then analysed in an example. The example is non-trivial and involve many reparametrisations from electromagnetic dilaton-axion theory. The generations start in the general case from an arbitrary vacuum solution and in the special case from the vacuum Kerr-NUT solution. The obtained solutions have non-zero dilaton-axion fields.

The conclusions are that the generation of solutions partly corresponds to previously known dualities. A possible further development could be to look at solutions with two Killing vector fields in the same way as Geroch did, as described in section 7.3.

7.6 Non-zero cosmological constant

In everything above we have only considered the case where the cosmological constant $\Lambda = 0$. If this is not the case we must consider Einstein’s field equations (1.1) and the action (5.3) with non-zero $\Lambda$. A vacuum solution with $\Lambda \neq 0$ is called an Einstein space and the field equations have the form

$$R_{ab} = \Lambda g_{ab}. \tag{7.23}$$
Previously it was not studied how to perform the generation method using sigma models for Einstein spaces. However, Leigh et.al. [12] studied the case with non-zero $\Lambda$ using the setup and method of Geroch. The projection to $(\mathcal{S}, H_{ab})$ goes exactly as described in section 6.1.2 but the conformal metric is replaced by another three-dimensional metric $\hat{h}_{ab}$ given by

$$\hat{h}_{ab} = \frac{\lambda}{\kappa} H_{ab},$$

where $\lambda$ is as usual the norm but $\kappa$ is an additional scalar field. We will refer to $\hat{h}_{ab}$ as the reference metric, which is equal to the conformal projected metric if $\kappa = 1$ identically.

Leigh et.al. derive field equations on $(\mathcal{S}, \hat{h}_{ab})$ and find that they belong to the action

$$S = \int d^{3}x \sqrt{\hat{h}} \sqrt{-\kappa} \left( \hat{P} + \frac{\hat{D}^a \kappa \hat{D}_a \kappa}{2\kappa^2} + 2 \frac{\hat{D}^a \tau \hat{D}_a \bar{\tau}}{\tau - \bar{\tau}} - 4i \Lambda \frac{\kappa}{\tau - \bar{\tau}} \right).$$

(7.25)

We will consider the Lagrangian as consisting of two parts, one “matter potential”

$$V = \sqrt{-\kappa} \left( \hat{P} - 4i \Lambda \frac{\kappa}{\tau - \bar{\tau}} \right) = \sqrt{-\kappa} \left( \hat{P} - 2 \Lambda \frac{\kappa}{\lambda} \right)$$

(7.26)

and one “kinetic” term which we regard as our sigma model action. The kinetic term corresponds to a target space metric

$$d\ell^2 = \sqrt{-\kappa} \left( -\kappa^{-2} d\kappa^2 + \lambda^{-2} \left( d\lambda^2 + d\omega^2 \right) \right).$$

(7.27)

The Killing vectors are the ones given in (7.14) plus a fourth Killing vector

$$k^{(0)} = \frac{1}{2} \kappa \partial_{\omega}.$$ 

(7.28)

The isometry group would then be $\mathbb{R} \times SL(2, \mathbb{R})$. But we must also restrict the action (7.25) to be invariant under our isometries, which is true only when the isometries leave the matter potential (7.26) invariant. This reduces the transformations to the ones generated by $k^{(1)} = \partial_{\omega}$.

Hence we can only get coordinates by $\tilde{\varphi}^A = \exp(\lambda \partial_{\omega}) \varphi^A$, i.e. $\tilde{\omega} = \omega + \lambda$ and the others invariant. However, $\tilde{\omega} = \omega + \lambda$ will only correspond to a gauge transform.

Instead Leigh et.al. explicitly study the field equations on $\mathcal{S}$ in the special case $\mathcal{S} = \mathbb{R} \times \mathcal{L}$, $ds^2 = d\sigma^2 + d\Omega^2$, where $\mathcal{L}$ is a two-dimensional space (spherical, flat or hyperbolic) with metric denoted by $d\Omega^2$. The solution is assumed to be parametrised by the single coordinate $\sigma$. We obtain a Lagrangian as a function of $\sigma$ only together with an equation that works as a constraint.

The Lagrangian formalism is now changed into Hamiltonian formalism. The Killing vector Lie algebra of (7.15) is represented as Poisson brackets of phase-space functions. The equations are solved using the Hamilton-Jacobi method. The following remarks are made. The solution is parametrised by two functions $m$ and $n$. The cosmological constant $\Lambda$ works as a constant of motion. The function $\kappa$ plays a key role, and both $\Lambda$ and $\kappa$ vary in the $(m, n)$ plane, i.e. the solutions given in terms of different $(m, n)$ will have different values of $\Lambda$ and $\kappa$. 

25
8 Discussion and outlooks

In this section we discuss the presented material. We also present a possible generalisation of the sigma model that may be used as a way to generate solutions.

8.1 Generation of solutions

The notion of the sigma model was first introduced in 1960 by Murray Gell-Mann and Maurice Lévy as a part of a phenomenological model of beta decay [21]. The formalism of sigma models has since been generalised and is now used in other areas, such as different areas of the theory of supersymmetry [5].

The development of the generation method studied in this paper started even before the article by Gell-Mann and Lévy. Many authors refer to the method as introduced by Ehlers, Harrison and Geroch. Other important early contributors that have not been mentioned before in this text are Pascual Jordan and Achilles Papapetrou. In the beginning it seems that the results were considered as quite accidental. There is, as far as we know, not any full description of the historical development of the method and it is beyond the scope of this text to give such description, even though we have given some insight on the matter. If such work were to be continued, the most interesting historical phase to study should be when the early works were linked to the formalism of sigma models used in other fields.

There is a set of generation techniques in general relativity of which sigma models is only one. Firstly each of the known "standard solutions" does in itself contain symmetries since the solutions generally contain arbitrary constants. We could for example consider the Schwarzschild metric (5.14) as a solution generated from Minkowski space in spherical coordinates ($M = 0$). This simple example exhibits the feature encountered in our method that the generated solution is not isometric to the old solution and that it is in general not as well-behaved; the Schwarzschild solution has an essential singularity at the point $R = 0$, whereas the Minkowski space is everywhere defined. Many other solutions are in the same way generalisations of previously known solutions. The essential difference to the method of sigma models is that such generalisations are a posteriori constructions and do not follow any certain method.

Another technique for generating solutions is to apply Bäcklund transforms to general relativity, which was first done by Harrison [22] in 1978. The Bäcklund transform is a tool in the theory of partial differential equations and can in certain situations be applied to general relativity. In the article [22] the Bäcklund transform is applied in the Ernst equation, which in turn is a tool in generating solutions.

8.2 Generation using sigma models

We now return to discuss our generation technique, the sigma model. Due to the work of Geroch [7] the situation $R_{ab} = 0$, $A = 0$, one Killing vector is completely described. Even though the method is fully described, Geroch’s method can still be used on any solution discovered by some other method. If this solution has a Killing vector we can perform the method and find generated solutions. The hard work lies in the analysis of the generated solutions, i.e. to see if they are meaningful and to determine whether they are completely new or isometric to any other known solution. Geroch’s method is also very well understood from the sigma model point of view, as we have noted in section 7.3 where we compared with the paper of Leigh et.al. [12].

Geroch’s conjecture and the answer to it in the end of section 7.3 is of important theoretical interest, but the method of using commuting Killing vector fields and generate solutions with respect to first one and then the other vector field could also be used to explicitly generate solutions.

This simple kind of generation is however not so useful, since, due to the ansatz $R_{ab} = 0$, $A = 0$, the generated solution will also have these properties. It is meaningful to look for solutions that have certain electromagnetic or matter fields, so we would like a method that
can generate such solution. In this aspect, the situation where we have additional fields in the Lagrangian of \( \mathcal{M} \) (i.e. when \( L_M \neq 0 \) in (6.14)) is more promising. A good example of this is the article [17] considered in section 7.5. If we would like to produce a new solution with for example a dilaton field we add a dilaton term \( \partial_a \phi \partial^a \phi \) to \( L_M \). If now the method of section 6.3 is successful we would get \( \tilde{\phi} = \tilde{\phi}(\varphi^1, \ldots, \varphi^{d-1}, \phi) \). We can thus start from a solution without a dilaton field and still get generated solutions which have such field, for even if \( \phi = 0 \) then \( \tilde{\phi} \) is generally non-zero.

The disadvantage is that it is only in special cases that the scalar fields \( L_m \) of the projected metric actually form a sigma model, and thus it is hard to find successful generations. Despite this, the two examples of section 7.5 show that this kind of generations in fact can be used with interesting results.

One aspect that has not been much discussed is that many authors point out that some steps in the procedure of generations are only local. For example, the twist \( \omega \) was introduced in section 6.1.4 by stating that \( \omega_a \) is locally a gradient of a scalar field \( \omega \). In this text we have not discussed if there are any consequences of this. One possible such consequence could be that the generated solution contains singularities.

8.3 The nature of general relativity

The discussion above is mainly focused on the more direct use of the method, to find new solutions to the theory of general relativity. There is, however, also an outcome of more theoretical nature: to use the results from the method to gain more insight in the theory itself. Einstein’s field equations are, as described in the introduction, a complicated system of equations. Since the sigma model deals with continuous-parameter families of generations, the outcomes of the method can give information of the continuous symmetries of the field equations and their solution space.

The works by Geroch [19], Hauser and Ernst [20] and Leigh et.al. [12] are directed a bit more towards these kind of questions. By Hauser it is clear that all stationary axisymmetric vacuum solutions with \( \Lambda = 0 \) can be obtained by the kind of symmetries generated by sigma models. An open question is the status of other kinds of solutions \( \Lambda \neq 0 \). Here the work by Leigh et.al. [12] constitutes just the starting point in investigating these solutions.

Harrison’s discussion in the end of our section 7.2 should be considered as another good example of how this method gives insight to the theory in general. It is interesting to see how different features of the solutions seem to be related. The fact that for example the twist and the electromagnetic field “compensate” indicate that their description within the theory have common properties, which in turn could be exploited in other areas of general relativity.

8.4 Two-dimensional sigma models

In section 4 we studied only the torsion-free target space, i.e. we have the Lagrangian \( L = G_{AB} h^{ab} \partial_a \varphi^A \partial_b \varphi^B \) with \( G_{AB} = G_{BA} \). Remember that we consider a domain manifold \( \mathcal{M} \) with dimension \( n \) and a target manifold \( \mathcal{T} \) with dimension \( d \). \( a, b, \ldots = 1, \ldots, n \) are abstract indices and \( A, B, \ldots = 1, \ldots, d \) are viewed on \( \mathcal{M} \) as just enumeration. For \( n = 2 \) there is a generalisation of the Lagrangian given by [23]

\[
L + \hat{L} = G_{AB} \sigma^{ab} \partial_a \varphi^A \partial_b \varphi^B + B_{AB} \varepsilon^{ab} \partial_a \varphi^A \partial_b \varphi^B.
\]  

(8.1)

Here \( \sigma_{ab} \) is the two-dimensional metric on \( \mathcal{M} \), \( \varepsilon^{ab} \) is the antisymmetric tensor with its indices raised by \( \sigma^{ab} \) and \( B_{AB} = -B_{BA} \) are antisymmetric functions. The term \( \hat{L} \) is due to its antisymmetric appearance related to torsion. Note that \( B_{AB} \) has \( \frac{d(d-1)}{2} \) and \( G_{AB} \) has \( \frac{d(d+1)}{2} \) independent components.

The general form for \( \hat{L} \) with arbitrary \( n \) is

\[
\hat{L} = \frac{1}{n!} E_{A_1 \ldots A_n} \varepsilon^{a_1 \ldots a_n} \partial_{a_1} \varphi^{A_1} \ldots \partial_{a_n} \varphi^{A_n},
\]

(8.2)
where the expression for \( n = 2 \) is obtained by \( B_{AB} = 2E_{AB} \). Due to the nature of the field equations we do not expect any term \( \hat{L} \) to appear in the case \( n > 2 \) where we have more than two derivative factors. The form of (8.1) where \( n = 2 \) should however not be excluded and might occur if we consider a theory projected into two dimensions.

### 8.5 Projecting to two dimensions

In section 6.1.2 we presented Geroch’s description of projecting from four to three dimensions as it was given in [7]. In his subsequent paper [19] he describes an analogous projection onto two dimensions. The direct counterpart to proposition 1 goes as follows: Consider \( \mathcal{M} \) with two commuting Killing vector fields \( \xi^a \) and \( \zeta^a \), i.e. \([\xi, \zeta] = 0\) and let \( \mathcal{L} \) be the two-dimensional manifold whose elements are the two-dimensional orbits of isometries generated by \( \xi^a \) and \( \zeta^a \). Then there is a one-to-one correspondence between tensor fields \( T \) on \( \mathcal{M} \) and tensor fields \( T' \) on \( \mathcal{L} \) if and only if

\[
\begin{align*}
\xi_a T^{a...c...d...f} &= 0, \quad \cdots, \quad \xi_f T^{a...c...d...f} = 0, \\
\zeta_a T^{a...c...d...f} &= 0, \quad \cdots, \quad \zeta_f T^{a...c...d...f} = 0,
\end{align*}
\]

(8.3)

We define the norms \( \lambda_{00} = \xi^a \xi_a \), \( \lambda_{01} = \xi^a \zeta_a \), \( \lambda_{11} = \zeta^a \zeta_a \) and \( \tau = 2(\lambda_{01}^2 - \lambda_{00} \lambda_{11}) \). Then the two-dimensional metric \( \sigma_{ab} \) and antisymmetric tensor \( \varepsilon_{ab} \) defined by

\[
\sigma_{ab} = g_{ab} + \frac{2 \lambda_{01}}{\tau^2} \xi_a \zeta_b + \frac{2 \lambda_{00}}{\tau^2} \zeta_a \xi_b - \frac{4 \lambda_{01}}{\tau^2} \xi_a \zeta_b, \quad \varepsilon_{ab} = \sqrt{\frac{2}{\tau}} \varepsilon_{abcd} \xi^c \zeta^d
\]

(8.4)

are tensor fields on \( \mathcal{L} \). In his paper Geroch presents the formalism of projection but uses instead his three-dimensional sigma model generation. It is therefore interesting to seek a pure two-dimensional sigma model.

Norma Sanches [10] considers this situation explicitly. She starts from the canonical metric

\[
ds^2 = \lambda V (dx^1 + \omega dx^2)^2 + \frac{\gamma_{ij}}{V} dx^i dx^j, \quad \gamma_{ij} dx^i dx^j = e^2 \gamma \left[ (dx^3)^2 - \lambda(dx^4)^2 \right] + s^2 (dx^2)^2,
\]

(8.5)

with \( \lambda = \pm 1 \) and \( V, \omega, \gamma, s \) depending on \( x^3, x^4 \) only, which corresponds to many different physical situations admitting two Killing vectors. Two-dimensional field equations corresponding to (6.19) and (6.20) are derived. The solution is reparametrised in terms of complex functions and new solutions are generated by exploiting symmetries of the equation. The generated solutions are for example a complex Taub-NUT metric\(^1\) and a real Taub-NUT metric with signature \((2, 2)\) [10]. Unfortunately no explicit sigma model Lagrangian is given, but the generations use the so called \( O(2, 1) \) sigma model. This sigma model does not contain the torsion term \( \hat{L} \) and is thus not what we look for.

A more promising way to possibly achieve a two-dimensional projected manifold \( \mathcal{L} \) admitting a torsion term \( \hat{L} \) is to start from a three-dimensional spacetime, \( n = \dim \mathcal{M} = 3 \). The metric \( g_{ab} \) then has signature \((2, 1)\). Such gravitational theories are well studied. The electromagnetic field tensor \( F_{ab} = 2\partial_a A_b \) is defined as in \( n = 4 \) but it has only three linearly independent components, an electric 2-vector \( \vec{E} \) and a magnetic scalar \( B \). \( A_a \) is here a three-dimensional electromagnetic gauge field. The virtue of three-dimensional electromagnetic theory is that we can introduce the so called Chern-Simons potential

\[
L_{\text{CS}} = \frac{1}{2} \varepsilon^{abc} A_a \partial_b A_c - A_a J^a.
\]

(8.6)

This scalar field and the corresponding action is in fact gauge invariant, even though it explicitly contains \( A_a \)-dependent terms. This makes the so called Chern-Simons theory an interesting alternative to studying the conventional gauge theory. [25]

\(^1\)The Taub-NUT metric was first presented in 1963 [24].
For us, the antisymmetric appearance of the first term makes it a possible candidate for finding a torsion term in the sigma model action. We might find the following hypothetical scenario: We consider a three-dimensional spacetime $\mathcal{M}$ on which is defined a Chern-Simons potential and possibly some other fields. The solution is projected onto a two-dimensional gravity term plus some scalar fields. The scalar fields form a sigma model of the form (8.1), including the term $\hat{L}$ which comes form the antisymmetric part of $L_{CS}$.

We do not investigate here if there are any possibilities of finding such scenario, but it is an interesting open question to analyse in further research.

9 Conclusions

Sigma models are a useful tool in many fields of theoretical physics. The method discussed in this report is one of the most important to generate new solutions to Einstein’s field equations of general relativity. Besides the possibilities of finding interesting solutions the method can also give insight into general features of the theory.

Possible developments of the method are the case $\Lambda \neq 0$ and the possibility of having a torsion sigma model $\tilde{L}$ when reducing to two dimensions.
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References


