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Legendrian Approximations

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Abstract. We find a procedure for constructing a Legendrian approximation of an embedded oriented closed surface in $\mathbb{R}^5$ with the standard contact structure following methods similar to those that are commonly used to find a Legendrian approximation for knots.

Contents

1. Introduction

This introduction is aimed towards giving a background and motivation to the problem which will be dealt with in this paper. The reader who is already familiar with the subject or not interested in an introduction may skip this section.

Remark 1.0.1. Unless otherwise stated all manifolds and maps are assumed to be of class C^∞ and we will use the notation $U \overset{\text{open}}{\subset} X$ to denote that U is an open subset of X. A surface is an embedded 2-dimensional closed manifold.

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1.1. **The h-principle.** The general resultw on isotropic approximations of maps of manifolds into contact manifolds are the following:

Theorem 1.1.1 (Theorem 6.3.5 of [11])**.** *Let L be a k-dimensional manifold and $f_0\colon L \to M$ a map covered by a fibrewise injective complex bundle map $F_0\colon TL \otimes \mathbb{C} \to \xi$ where (M, ξ) is a $2n+1$-dimensional contact manifold. Then f_0 is homotopic to an isotropic immersion $f_1\colon L \to M$ and with $d^{\mathbb{C}} f_1$ homotopic to F_0 via fibrewise injective complex bundle maps. Moreover, the map f_1 and the homotopy between f_0 and f_1 can be chosen arbitrarily C^0-close to f_0.*

Proposition 1.1.2 (Proposition 6.3.6 of [11])**.** *Under the assumptions of the previous theorem, if f_0 is an embedding and L is closed, one may further achieve that f_1 is likewise an embedding isotopic to f_0 via an isotopy which can be chosen arbitrarily C^0-close to f_0.*

These are usually proven using the h-principle. For the case of knots, i.e. embeddings of $L = S^1$, in 3-dimensional contact manifolds the problem is well studied and proofs of the result are readily available without any knowledge of the h-principle.[1] In this paper we will prove this theorem for the special case of a orientable surface without using the h-principle.

In order to better understand the above results we begin by giving a quick overview of the h-principle. An understanding of jet spaces is necessary and will be assumed. We begin with some notation.

Definition 1.1.3. Let $p\colon X \to V$ be a fibre bundle. We denote the r-jet space of sections of this fibre bundle by $X^{(r)}$. We denote the projection $X^{(r)} \to X$ by p_0^r and the projection $X^{(r)} \to V$ by $p^r = p \circ p_0^r$. A section $\varphi\colon V \to X^{(r)}$ is called holonomic if it is the r-jet extension of a section $f\colon V \to X$, i.e. $\varphi = j^r f$. If $X = V \times W$ is a trivial fibre bundle we write $X^{(r)} = J^r(V, W)$. For $X = V \times \mathbb{R}$ we write $X^{(r)} = J^r(V)$.

Remark 1.1.4. Both $p_0^r\colon X^{(r)} \to X$ and $p^r\colon X^{(r)} \to V$ are fibre bundles.

Definition 1.1.5. A subset $\mathcal{R} \subset X^{(r)}$ is called a differential relation of order r. It is called an open differential relation if it is an open subset and, similarly, it is called a closed differential relation if it is a closed subset. A section $f\colon V \to X^{(r)}$ is called a formal solution to the differential relation $\mathcal{R}$ if $f(V) \subset \mathcal{R}$ and it is called a genuine solution if it is holonomic.[2] The space of genuine solutions is denoted by $Sol(\mathcal{R})$ and the space of formal solutions is denoted by $Sec(\mathcal{R})$.

Example 1.1.6. Let V and W be manifolds of dimension n and m respectively. A map $f\colon V \to W$ is called an immersion if $df\colon TV \to TW$ is fibrewise injective. Using this idea we can create the differential relation $\mathcal{R}_{imm} \subset J^1(V, W)$, called the immersion relation, in the following way:

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[1]We will look more closely into this special case in the next section.

[2]Such sections are sometimes called r-extended solutions, the term genuine solution reserved for sections $g\colon V \to X$ such that $j^r g(V) \subset \mathcal{R}$. Since there obviously is a bijection between such genuine solutions and r-extended solutions given by $f = j^r g$ we will not make such a distinction.

Consider a point $(v,w) \in V \times W$. The fibre $(p_0^1)^{-1}(v,w)$ then consists of all linear maps $T_vV \to T_wW$. In this way $\mathcal{R}_{imm}$ is also fibred over X as the subset of all monomorphism, i.e. fibrewise injective bundle maps. For $\dim(V) \leq \dim(W)$ this differential relation is open and dense. For $\dim(V) > \dim(W)$ we automatically have $\mathcal{R} = \emptyset$.

The differential relation $\mathcal{R}_{imm}$ explains the existence of a fibrewise injective bundle map which appears in Theorem 1.1.1.

Definition 1.1.7. A differential relation is said to satisfy the h-principle if every formal solution of $\mathcal{R}$ is homotopic in $Sec(\mathcal{R})$ to a genuine solution of $\mathcal{R}$.

There is also the following version of the h-principle for families of solutions:

Definition 1.1.8. A differential relation $\mathcal{R}$ is said to satisfy the one-parametric h-principle if every family of formal solutions $\{f_t\}_{t\in I}$, such that f_0, f_1 are genuine solutions, is homotopic in $Sec(\mathcal{R})$, keeping the endpoints f_0 and f_1 fixed, to a family of genuine solutions $\{\tilde{f}_t\}_{t\in I}$.

Lastly we need a local version of the h-principle:

Definition 1.1.9. Let $f\colon V \to X$ be a section. Let $U \overset{\text{open}}{\subset} X$ be a neighbourhood of $f(V)$. A differential relation $\mathcal{R} \subset X^{(r)}$ is said to satisfy the h-principle C^0-close to f if every section $\varphi_0\colon V \to \mathcal{R}$ such that $p_0^r \circ \varphi_0 = f$ is homotopic to a holonomic section φ_1 through sections $\varphi_t\colon V \to \mathcal{R} = (p_0^r)^{-1}(U) \cap \mathcal{R}$ for any such neighbourhood U of $f(V)$. The differential relation is called everywhere dense if the h-principle holds true C^0-close to any section $g\colon V \to X$.

There is also a parametric version of the local h-principle.

Now let (W,ξ) be a contact manifold of dimension $2n+1$ and V a manifold of dimension $\dim(V) \leq n$. We may then formulate a differential relation for isotropic immersions as the subset $\mathcal{R}_{isot} \subset J^1(V,W)$ of fibrewise injective bundle maps $TV \to \xi$ into Lagrangian subspaces of ξ. For $\dim(V) = n$, i.e. Legendrian immersions, we write $\mathcal{R}_{Leg}$.

Theorem 1.1.10 (Theorem 16.1.3 of [9])**.** *All forms of the h-principle hold for $\mathcal{R}_{Leg}$.*

Remark 1.1.11. The theorem is (of course) also true for $\mathcal{R}_{isot}$.

By considering the C^0-dense h-principle we now see that in order to find a Legendrian approximation of the map $f\colon V \to (W\xi)$ all we need is the existence of a fibrewise injective bundle map $\varphi_0\colon TV \to \xi$ covering the map f, i.e. such that $p_0^r \circ \varphi_0 = f$. Choosing a ξ-compatible complex structure J on ξ and considering the complexification of TV we arrive at the formulation of Theorem 1.1.1. To get from this to Proposition 1.1.2 one perturbs the approximating Legendrian immersion given by Theorem 1.1.1 to get an embedding.

1.2. **Results for knots and strategy for surfaces.** Having dealt with the general background we now describe how Proposition 1.1.2 can be proven for $L = S^1$ without the use of the h-principle. The proposition can be proven using either the front projection or the Lagrangian projection together with the following lemmas. The convention $\xi_0 = \ker(\alpha_0) = \ker(dz - y dx)$ is used for the standard contact structure on $\mathbb{R}^3$.

Lemma 1.2.1 (Lemma 3.2.6 of [11]). *Let* $\gamma\colon (a,b) \to (\mathbb{R}^3, \xi_0)$ *be a Legendrian immersion. Then the Lagrangian projection* $\gamma_L(s) = (x(s), y(s))$ *is also an immersed curve. The curve* γ *is recovered from* γ_L *via*

$$z(s_1) = z(s_0) + \int_{s_0}^{s_1} y(s)x'(s)ds.$$

A Legendrian immersion $\gamma\colon S^1 \to (\mathbb{R}^3, \xi_0)$ *has a Lagrangian projection that encloses zero area. Moreover,* γ *is embedded if and only if every loop in* γ_L *(except, in the closed case, the full loop* γ_L*) encloses a non-zero oriented area.*

Any curve (defined on an interval) immersed in the (x,y)*-plane is the Lagrangian projection of a Legendrian curve in* $(\mathbb{R}^3, \xi_0)$*, unique up to translation in the* z*-direction. A closed immersed curve* γ_L *in the* (x,y)*-plane, i.e. an immersion of* S^1 *in* $\mathbb{R}^2$*, lifts to a Legendrian immersion of* S^1 *in* $(\mathbb{R}^3, \xi_0)$ *precisely if* $\oint y dx = 0$.

Lemma 1.2.2 (Lemma 3.2.3 of [11]). *Let* $\gamma\colon (a,b) \to (\mathbb{R}^3, \xi_0)$ *be a Legendrian immersion. Then its front projection* $\gamma_F(s) = (x(s), z(s))$ *does not have any vertical tangencies. Away from cusp points,* γ *is recovered from its front projection via*

$$y(s) = \frac{z'(s)}{x'(s)} = \frac{dz}{dx},$$

i.e. $y(s)$ *is the slope of the front projection. The curve* γ *is embedded if and only if* γ_F *has only transverse self-intersections.*

By a C^∞*-small perturbation of* γ *we can obtain a generic Legendrian curve* $\tilde{\gamma}$ *isotopic to* γ*; by a* C^2*-small perturbation we may achieve that the front projection has only semi-cubical cusp singularities, i.e. around a cusp point at* $s = 0$ *the curve* $\tilde{\gamma}$ *looks like*

$$\tilde{\gamma}(s) = (\lambda s^2 + a, s + b, \lambda(2s^3/3 + bs^2) + c)$$

with $\lambda \neq 0$*. Any regular curve in the* (x,z)*-plane with semi-cubical cusps and no vertical tangencies can be lifted to a unique Legendrian curve in* $\mathbb{R}^3$.

We now give a description of how Proposition 1.1.2 can be proven using the front projection and the above lemma, roughly following the strategy of [11].

Consider an embedding $\gamma\colon S^1 \to (M, \xi)$. Choose a finite open cover by Darboux charts. Its front projection γ_F in each of the charts is then a curve in the (x,z)-plane defined on an interval, which we may choose to be closed. We want to approximate this front by a curve γ' of the type described in the last sentence of Lemma 1.2.2, making sure that the slope of the curve approximates the y-coordinate of the original curve γ. This can be done by adding semi-cubical cusps described in the lemma, creating a number of zig-zags. With proper care this can be done without introducing any non-trivial topological knotting or non-transverse self-intersections. If the interval is not already approximated by a Legendrian curve near the endpoints we may simply choose an appropriate slope. If it is already approximated by a Legendrian curve near the endpoints then we may choose the approximation in the interior of the interval to coincide with the approximation at the endpoints. This allows us to glue these local pieces of the knot together to form a global approximation.

The strategy used for orientable surfaces will be somewhat similar. The restriction to oriented surfaces guarantees the existence of Legendrian immersions since such surfaces always have even Euler characteristic and hence we are guaranteed, via the h-principle, that an approximation is possible. Note that surfaces of odd Euler characteristic (necessarily non-orientable) do not admit Legendrian immersions.[3]

We first want to find a suitable subdivision of our embedded surface M in $(\mathbb{R}^5, \xi_0)$. To do this we consider embeddings such that the x_2-coordinate is a Morse function. This gives us a subdivision of M into neighbourhoods of the critical points of the x_2-coordinate and the cylinders connecting them. We then construct Legendrian approximations around each critical point and on the cylinders connecting them separately. Lastly, these local approximations are glued together to form a global approximation.

2. The Maslov index

In this section we will look at the Maslov index and how it can be used to classify Legendrian curves. This will be used in later sections when we attempt to construct the global approximation out of the local approximations. The presentation here largely follows that of [1] but have been supplemented with preliminary results on symplectic linear algebra and various proofs have been supplied where such were missing in [1].

2.1. **Symplectic linear algebra.**

Definition 2.1.1. Let V be an even-dimensional real vector space. A symplectic linear structure on V is a skew symmetric bilinear form $\omega\colon V \times V \to \mathbb{R}$ such that if $\omega(u, v) = 0$ for all $v \in V$ then $u = 0$. A pair (V, ω) of an even-dimensional real vector space V and a symplectic linear structure ω on V is called a symplectic vector space.

Example 2.1.2. Consider $\mathbb{R}^{2n} = \{(x_1, \ldots, x_n, y_1, \ldots, y_n)\}$ and $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$. Then $(\mathbb{R}^{2n}, \omega_0)$ is a symplectic vector space. We will refer to this symplectic structure as the standard symplectic structure on $\mathbb{R}^{2n}$.

The above example will be the focus of the next part. For now let's continue the general discussion on symplectic linear algebra.

Using the symplectic form ω we define the symplectic complement W^ω of a subspace $W \subset V$ as

$$W^\omega := \{u \in V \,|\, \omega(u, v) = 0 \quad \forall v \in W\}.$$

We then say that a subspace is

- isotropic if $W \subset W^\omega$
- coisotropic if $W^\omega \subset W$
- symplectic if $W \cap W^\omega = \{0\}$
- Lagrangian if $W = W^\omega$

Note that a subspace W is isotropic if and only if $\omega|_W = 0$. Lagrangian subspaces are a special case of isotropic subspaces and will be of particular interest to us. To start studying subspaces of a symplectic vector space we will need the following lemma:

[3] See, for example, [3] p. 254 for this result and [6] for explicit constructions of Legendrian embeddings of any surface of genus $g > 0$ into $J^1(\mathbb{R}^2)$ and more.

Lemma 2.1.3 (Lemma 2.2 of [15]). *For any subspace $W \subset V$ of a symplectic vector space (V, ω) we have* $\dim(W) + \dim(W^\omega) = \dim(V)$ *and* $(W^\omega)^\omega = W$.

Proof. Define a linear map $\imath_\omega \colon V \to V^*$ by $\imath_\omega(u)(v) = \omega(u, v)$. Since ω is nondegenerate this map will be injective and therefore an isomorphism. The subspace W^ω is mapped to $W^\perp = \{f \in V^* | \, f|_W = 0\}$. We then have

$$\dim(W) + \dim(W^\omega) = \dim(W) + \dim(W^\perp) = \dim(V).$$

The second property follows using another application of the same idea. □

This lemma immediately allows us to identify one property of Lagrangian subspaces, namely that their dimension is half that of the symplectic vector space in which they are contained. This follows from their defining property by

$$\dim(W) + \dim(W^\omega) = 2\dim(W) = \dim(V) = 2n \Leftrightarrow \dim(W) = n.$$

We now want to define in what way we would consider two symplectic vector spaces to be "the same".

Definition 2.1.4. A linear isomorphism $f \colon V \to W$ between symplectic vector spaces $(V, \omega_1), (W, \omega_2)$ is called a linear symplectomorphism if $f^*\omega_2 = \omega_1$, i.e. if $\omega_2(f(u), f(v)) = \omega_1(u, v)$ for all $u, v \in V$.

Of particular importance to us will be the symplectomorphisms $f \colon V \to V$. These constitute a Lie group which we will denote $Sp(V, \omega)$. If $(V, \omega) = (\mathbb{R}^{2n}, \omega_0)$ we denote it simply by $Sp(2n)$. The following theorem will explain the focus on $(\mathbb{R}^{2n}, \omega_0)$.

Theorem 2.1.5 (Theorem 2.3 of [15]). *Let (V, ω) be a symplectic vector space of dimension $2n$. Then there exists a basis $u_1, \dots, u_n, v_1, \dots, v_n$ such that:*

- $\omega(u_j, u_k) = \omega(v_j, v_k) = 0 \quad \forall j, k$
- $\omega(u_j, v_k) = \delta_{jk} \quad \forall j, k$

Moreover, there exists a symplectomorphism $f \colon \mathbb{R}^{2n} \to V$ such that $f^*\omega = \omega_0$.

We call a basis of a symplectic vector space as above a symplectic basis.

Proof. The proof will, not surprisingly, be done by induction on the dimension.

Let (V, ω) be a $2n$-dimensional symplectic vector space. Since ω is non-degenerate there must exist two vectors u_1, v_1 such that $\omega(u_1, v_1) = 1$. The subspace W spanned by these two vectors is then a symplectic subspace. This 2-dimensional case also serves as our base case. Hence (W^ω, ω) will be a symplectic vector space of dimension $2n - 2$ and by the induction hypothesis this space will have a symplectic basis $u_2, \dots, u_n, v_2, \dots, v_n$. But then $u_1, \dots, u_n, v_1, \dots, v_n$ will be a symplectic basis of (V, ω), proving the first part of the theorem.

For the second part we consider the linear isomorphism $f \colon \mathbb{R}^{2n} \to V$ taking the standard symplectic basis of $(\mathbb{R}^{2n}, \omega_0)$, i.e. the basis $\dfrac{\partial}{\partial x_1}, \dots, \dfrac{\partial}{\partial x_n}, \dfrac{\partial}{\partial y_1}, \dots, \dfrac{\partial}{\partial y_n}$, to the above symplectic basis of (V, ω), i.e.

$$\frac{\partial}{\partial x_i} \mapsto u_i$$
$$\frac{\partial}{\partial y_j} \mapsto v_j$$

This will then be a linear symplectomorphism, as is easily seen. □

This theorem tells us that all symplectic vector spaces are linearly symplectomorphic and hence it suffices to consider the case $(\mathbb{R}^{2n}, \omega_0)$.

2.2. **The Lagrangian Grassmannian.** With the preliminaries done we can begin laying the foundations for the Maslov index. We will do this by studying the Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_0)$ which by Theorem 2.1.5 will also tell us about Lagrangian subspaces of any symplectic vector space.

We begin by introducing some additional structure on $\mathbb{R}^{2n}$.

Definition 2.2.1. A complex structure on a vector space V is a linear automorphism $J \in GL(V)$ such that $J^2 = -I$.

A complex structure can be used to turn a real vector space into a complex vector space via the isomorphism

$$\mathbb{C} \times V \to V, \quad (a + ib, v) \mapsto av + bJV.$$

Remark 2.2.2. A real vector space admitting a complex structure is necessarily even dimensional.

We will not be concerned with complex structures in general but we will state the following proposition:

Proposition 2.2.3 (Proposition 2.47 of [15]). *Let V be a $2n$-dimensional real vector space and let J be a complex structure on V. Then there is an isomorphism $f\colon \mathbb{R}^{2n} \to V$ such that*

$$f \circ J_0 = J \circ f$$

where

$$J_0 = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

is the standard complex structure on $\mathbb{R}^{2n}$.

It is this standard complex structure we will be interested in. Using it we can identify $\mathbb{R}^{2n}$ with $\mathbb{C}^n$ by $(x, y) \mapsto x + iy$ for $x, y \in \mathbb{R}^n$ and so J_0 corresponds to multiplication by i. We can then identify $GL(n, \mathbb{C}) \subset GL(2n, \mathbb{R})$ as the elements preserving this complex structure.

Finally we also have the standard Euclidean structure, given by the scalar product $\langle u, v \rangle$. As usual we denote the set of maps preserving this structure by $O(2n) \subset GL(2n, \mathbb{R})$.

These three structures, the symplectic, complex and Euclidean, on $\mathbb{R}^{2n}$ have a nice two-out-of-three property, i.e. a linear map preserving any two of the structures also necessarily preserves the third. This comes from the fact that

$$\langle u, v \rangle = \omega_0(u, J_0 v).$$

This clearly shows that a map preserving the symplectic and the complex structure must also preserve the Euclidean structure. The above equality also implies the following:

$$\langle J_0 u, v \rangle = \omega_0(J_0 u, J_0 v) = \omega_0(u, v)$$

which tells us that a map preserving the complex and the Euclidean structure must also preserve the symplectic structure. The above equalities also tells us that a map preserving both the symplectic and the Euclidean structure must also preserve the complex structure.

We thus conclude the following:

$$O(2n) \cap GL(n, \mathbb{C}) = GL(n, \mathbb{C}) \cap Sp(2n) = Sp(2n) \cap O(2n) = U(n).$$

Having investigated the relevant structures on $\mathbb{R}^{2n}$ we now turn to the subject of this part, the Lagrangian Grassmannian. We begin by defining it.

Definition 2.2.4. The set of all Lagrangian subspaces of $(\mathbb{R}^{2n}, \omega_0)$ is called the Lagrangian Grassmannian and is denoted by $\mathcal{L}(n)$.

Lemma 2.2.5 (Lemma 1.2. of [1])**.** *The group $U(n)$ acts transitively on $\mathcal{L}(n)$ with stabilizer $O(n)$.*

Proof. Let $\lambda \in \mathcal{L}(n)$ and let $v, w \in \lambda$. Since $\omega_0(u, v) = \langle J_0 u, v \rangle$ and λ is Lagrangian we must have that $J_0 \lambda \perp \lambda$. Now let $\lambda' \in \mathcal{L}(n)$ and choose orthonormal bases b, b' of λ, λ' respectively. Then the automorphism of $\mathbb{R}^{2n}$ carrying b to b' and $J_0 b$ to $J_0 b'$ is unitary. Furthermore, if $\lambda' = \lambda$ the automorphism reduces to a map of λ taking b to b' and so is in $O(n)$. □

We now know from Lie group theory that $\mathcal{L}(n) = U(n) \big/ O(n)$ and hence $\mathcal{L}(n)$ is a smooth manifold of dimension $\dfrac{n(n+1)}{2}$. This also gives us a fibration

$$O(n) \longrightarrow U(n) \longrightarrow \mathcal{L}(n).$$

Can we find more such fibrations? We have the two following fibrations:

$$SO(n) \xrightarrow{\subset} O(n) \xrightarrow{\det} S^0$$
$$SU(n) \xrightarrow{\subset} U(n) \xrightarrow{\det} S^1$$

Consider a Lagrangian plane $\lambda \in \mathcal{L}(n)$ and an automorphism f of $\mathbb{R}^{2n}$ taking the plane $x = 0$ to λ. This automorphism is uniquely determined up to some $A \in O(n)$. Since $\det(A) = \pm 1$ we then have that $\det^2(f)$ only depends on the Lagrangian plane λ itself. In this way we get a map

$$Det^2 \colon \mathcal{L}(n) \to S^1$$

Let $S\mathcal{L}(n) = (Det^2)^{-1}(1)$. We then get another fibration

$$S\mathcal{L}(n) \xrightarrow{\subset} \mathcal{L}(n) \xrightarrow{Det^2} S^1$$

Furthermore, since $SU(n)$ acts transitively on $S\mathcal{L}(n)$ we also get the fibration

$$SO(n) \xrightarrow{\subset} SU(n) \longrightarrow S\mathcal{L}(n)$$

Lastly we also have the fibration

$$S^0 \longrightarrow S^1 \xrightarrow{z^2} S^1$$

From all of this we can construct the following commutative diagram:

$$\begin{array}{ccccc}
SO(n) & \longrightarrow & O(n) & \xrightarrow{\det} & S^0 \\
\downarrow & & \downarrow & & \downarrow \\
SU(n) & \longrightarrow & U(n) & \xrightarrow{\det} & S^1 \\
\downarrow & & \downarrow & & \downarrow z^2 \\
S\mathcal{L}(n) & \longrightarrow & \mathcal{L}(n) & \xrightarrow{Det^2} & S^1
\end{array}$$

From these fibrations we can find $\pi_1(\mathcal{L}(n))$ if we know $\pi_1(SU(n))$. Thus we would first like to calculate $\pi_1(SU(n))$.

Lemma 2.2.6. *$\pi_1(SU(n)) \cong 0$ for $n \geq 1$, where 0 denotes the trivial group.*

Proof. This will follow from the fact that $SU(n+1)\big/SU(n) = S^{2n+1}$. This gives us a fibration

$$SU(n) \longrightarrow SU(n+1) \longrightarrow S^{2n+1}$$

from which we get a long exact homotopy sequence from theorem 4.41. of [13, p. 376]

$$\cdots \to \pi_k(SU(n)) \to \pi_k(SU(n+1)) \to \pi_k(S^{2n+1}) \to \pi_{k-1}(SU(n)) \to \cdots \to \pi_0(SU(n+1)) \to 0$$

The interesting part of this sequence is

$$\cdots \to \pi_2(S^{2n+1})) \to \pi_1(SU(n)) \to \pi_1(SU(n+1)) \to \pi_1(S^{2n+1}) \to \pi_0(SU(n)) = 0$$

Since $\pi_k(S^{2n+1}) = 0$ for $k = 1, 2$, $n \geq 1$ and $SU(n)$ is connected we have that this sequence reduces to

$$\cdots \to 0 \to \pi_1(SU(n)) \to \pi_1(SU(n+1)) \to 0$$

and by exactness we then have that $\pi_1(SU(n)) \cong \pi_1(SU(n+1))$ for $n \geq 1$. The proof can then be done by induction on n and so we only need to find $\pi_1(SU(1))$. But $SU(1) = 1 \in \mathbb{C}$ and so $\pi_1(SU(1)) = 0$, proving the lemma.

□

We are now ready to find $\pi_1(\mathcal{L}(n))$.

Proposition 2.2.7 (Lemma 1.4.1. of [1])**.** *The fundamental group of the Lagrangian Grassmannian is isomorphic to the integers, $\pi_1(\mathcal{L}(n)) \cong \mathbb{Z}$, and an isomorphism is induced by Det^2.*

Proof. Consider the leftmost fibration in the commutative diagram above. The end of the associated long exact homotopy sequence is

$$\cdots \to \pi_2(S\mathcal{L}(n)) \to \pi_1(SO(n)) \to \pi_1(SU(n)) \to \pi_1(S\mathcal{L}(n)) \to \pi_0(SO(n)) \cong 0$$

Since $\pi_1(SU(n)) \cong 0$ the last terms become

$$0 \to \pi_1(S\mathcal{L}(n)) \to 0$$

hence $\pi_1(S\mathcal{L}(n)) \cong 0$

Next consider the fibration at the bottom of the diagram. The end of its associated long exact homotopy sequence is

$$0 \cong \pi_1(S\mathcal{L}(n)) \to \pi_1(\mathcal{L}(n)) \to \pi_1(S^1)) \cong \mathbb{Z} \to \pi_0(S\mathcal{L}(n)) \cong 0$$

and from this sequence we see that $\pi_1(\mathcal{L}(n)) \cong \pi_1(S^1) \cong \mathbb{Z}$, the isomorphism being given by the map on homotopy level induced by Det^2.

□

Before going into the construction of the Maslov index itself we give another lemma which is needed in one formulation of the Maslov index. However, we will not be very interested in this approach.

Lemma 2.2.8. *The fundamental groups of $Sp(2n)$ and $U(n)$ are isomorphic to the integers, $\pi_1(Sp(2n)) \cong \pi_1(U(n)) \cong \mathbb{Z}$.*

Since we will not be very interested in the approach given from this lemma we will skip the proof. The lemma itself is a combination of propositions 2.22 and 2.23 of [15].

2.3. **The Maslov index.** We are now ready to begin looking at the Maslov index from various points of view. We begin with the functorial approach given in [15] which is also the one we will be least interested in.

Theorem 2.3.1 (Theorem 2.29 of [15])**.** *There exists a unique functor μ, called the Maslov index, which assigns an integer $\mu(\gamma)$ to each loop $\gamma\colon S^1 \to Sp(2n)$ satisfying the following:*

1. *Two loops are homotopic if and only if they have the same Maslov index.*
2. *For any two loops γ_1, γ_2 we have*
$$\mu(\gamma_1\gamma_2) = \mu(\gamma_1) + \mu(\gamma_2)$$
3. *If $n = n_1 + n_2$ we identify $Sp(2n_1) \oplus Sp(2n_2)$ with a subgroup of $Sp(2n)$ and we then have*
$$\mu(\gamma_1 \oplus \gamma_2) = \mu(\gamma_1) + \mu(\gamma_2)$$
4. *The loop $\gamma\colon S^1 \to U(1) \subset Sp(2)$ given by $\gamma(t) = e^{2\pi i t}$ has Maslov index 1.*

There is of course also a correpsonding Maslov index for $\mathcal{L}(n)$.

Theorem 2.3.2 (Theorem 2.35 of [15])**.** *There exists a unique functor μ, called the Maslov index, which assigns an integer $\mu(\Lambda)$ to each loop $\Lambda\colon S^1 \to \mathcal{L}(n)$ satisfying the following:*

(1) *Two loops are homotopic if and only if they have the same Maslov index.*
(2) *For any two loops* $\Lambda\colon S^1 \to \mathcal{L}(n)$ *and* $\gamma\colon S^1 \to Sp(2n)$ *we have*

$$\mu(\gamma\Lambda) = \mu(\Lambda) + 2\mu(\gamma)$$

(3) *If* $n = n_1 + n_2$ *and* $\mathcal{L}(n_1) \oplus \mathcal{L}(n_2)$ *is identified with a submanifold of* $\mathcal{L}(n)$ *we have*

$$\mu(\Lambda_1 \oplus \Lambda_2) = \mu(\Lambda_1) + \mu(\Lambda_2)$$

(4) *The loop* $\Lambda\colon S^1 \to \mathcal{L}(1)$ *given by* $\Lambda(t) = e^{\pi i t}\mathbb{R} \subset \mathbb{C} = \mathbb{R}^2$ *has Maslov index* 1*.*

Again, we will not be interested in this approach to the Maslov index and hence we skip the proofs of these theorems. Instead we move on to, for us, more interesting approaches. Our approach will also make an explicit connection to Lagrangian submanifolds. First we need another lemma.

Lemma 2.3.3 (Lemma 1.4.1. of [1])**.** *The following are isomorphic:*

$$\pi_1(\mathcal{L}(n)) \cong H_1(\mathcal{L}(n);\mathbb{Z}) \cong H^1(\mathcal{L}(n);\mathbb{Z}) \cong \mathbb{Z}$$

Proof. We have already proven that $\pi_1(\mathcal{L}(n)) \cong \mathbb{Z}$ and so we only need to prove that

$$H_1(\mathcal{L}(n);\mathbb{Z}) \cong H^1(\mathcal{L}(n);\mathbb{Z}) \cong \pi_1(\mathcal{L}(n)).$$

The isomorphism $\pi_1(\mathcal{L}(n)) \cong H_1(\mathcal{L}(n);\mathbb{Z})$ is given by the Hurewicz theorem[4]; since $H_1(\mathcal{L}(n);\mathbb{Z})$ is isomorphic to the abelianization of $\pi_1(\mathcal{L}(n))$ and $\pi_1(\mathcal{L}(n))$ already is abelian we have that $H_1(\mathcal{L}(n);\mathbb{Z}) \cong \pi_1(\mathcal{L}(n))$.

Since we now know that $H_1(\mathcal{L}(n);\mathbb{Z}) \cong \mathbb{Z}$ we have that

$$H^1(\mathcal{L}(n);\mathbb{Z}) \cong Hom_{\mathbb{Z}}(H_1(\mathcal{L}(n);\mathbb{Z}),\mathbb{Z}) \cong H_1(\mathcal{L}(n);\mathbb{Z}).$$

since $Ext^1_{\mathbb{Z}}(H_0(\mathcal{L}(n);\mathbb{Z}),\mathbb{Z}) \cong Ext^1_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \cong 0$. □

Now let $\gamma\colon S^1 \to \mathcal{L}(n)$. We can then compose this with $Det^2\colon \mathcal{L}(n) \to S^1$ to get a map $S^1 \to S^1$. Finally, taking the degree of this map gives us a way of assigning an integer to each closed curve in $\mathcal{L}(n)$. Since the degree is homotopy invariant this gives us a map $\pi_1(\mathcal{L}(n)) \to \mathbb{Z}$ and therefore also a map $H_1(\mathcal{L}(n);\mathbb{Z}) \to \mathbb{Z}$. Thus, by Lemma 2.3.3, this defines a generator $\mu \in H^1(\mathcal{L}(n);\mathbb{Z})$. This generator coincides with the Maslov index defined above but since we are not interested in that formulation we will not be investigating it further. Instead we will explore the newly defined cohomology class which we will also call the Maslov index. We begin with an example.

Example 2.3.4. Let $\lambda \in \mathcal{L}(n)$ and consider the automorphisms $e^{i\varphi}I \in U(n)$ where $I \in GL(2n,\mathbb{R})$ is the identity matrix. Since $e^{i\pi}I = -I$ the Lagrangian planes $e^{i\varphi}\lambda$ for $0 \le \varphi \le \pi$ form a closed curve in $\mathcal{L}(n)$. To find the value of μ on this curve $\gamma(\varphi) = e^{i\varphi}\lambda$ we first consider

$$\det(e^{i\varphi}I) = e^{in\varphi} \Rightarrow Det^2(e^{i\varphi}\lambda) = e^{2in\varphi}Det^2(\lambda) \Rightarrow \mu(\gamma) = n$$

since it is the degree of this map. For $n = 1$ this coincides with the normalization property (number 4) of the Maslov index from above.

To go further we now make a definition.

[4]For reference, see Theorem 2A.1. of [13, p. 166]

Definition 2.3.5. A Lagrangian submanifold $L \subset \mathbb{R}^{2n}$ of $(\mathbb{R}^{2n}, \omega_0)$ is a submanifold such that each tangent space T_pL is Lagrangian, i.e. $\omega|_{T_pL} = 0$ for all $p \in L$.

We remark that such a submanifold will always be n-dimensional since every tangent space needs to be n-dimensional.

The Maslov index can also be used to examine the topology of Lagrangian submanifolds of $(\mathbb{R}^{2n}, \omega_0)$; Since every tangent space of a Lagrangian submanifold L is Lagrangian any curve γ on L determines a curve in $\mathcal{L}(n)$ via the tangential mapping $\tau\colon\, p \mapsto T_pL$. Thus we can use τ to define the cohomology class $\mu^* = \tau^*\mu \in H^1(L;\mathbb{Z})$, also called the Maslov index, and its value on a curve γ is then the degree of the composition

$$S^1 \xrightarrow{\gamma} L \xrightarrow{\tau} \mathcal{L}(n) \xrightarrow{Det^2} S^1$$

We will now use a different technique from differential topology to calculate the Maslov index of regular curves on Lagrangian submanifolds. We begin with yet another definition.

Definition 2.3.6. The Maslov cycle $\Sigma(n)$ is the set of all Lagrangian planes λ having non-transversal intersection with the Lagrangian subspace $\sigma = \{0\}\times\mathbb{R}^n = \{(0,\dots,0,y_1,\dots,y_n)\} \subset \mathbb{R}^{2n}$.

As subsets of the Maslov cycle we single out the sets $\Sigma^k(n)$ of all planes $\lambda \in \mathcal{L}(n)$ having k-dimensional intersection with σ.

Definition 2.3.7. A closed subset S of a manifold M is called stratified if it can be written as $S = \bigcup_{i=0}^n S_i$ where each S_i is a locally closed submanifold of M, called the strata of S, such that for each k we have $\overline{S_k} = \bigcup_{i=k}^n S_i$. The dimension of a stratified set is the maximal dimension of its strata.

It turns out that the Maslov cycle can be stratified in the following way:

Lemma 2.3.8 (Lemma 3.2.1. of [1])**.** *For each k the set $\Sigma^k(n)$ is an open manifold of codimension $\dfrac{k(k+1)}{2}$ in the Lagrangian Grassmannian.*

Proof. Consider the Grassmannian manifold $G(n,k)$ of k-dimensional subspaces of $\mathbb{R}^n$. There is a natural map $\Sigma^k(n) \to G(n,k)$ given by considering the intersection of a plane $\lambda \in \Sigma^k(n)$ with σ. This map naturally determines a fibration over $\Sigma^k(n)$ with fibre $\Sigma^0(n-k)$, i.e. the set of all Lagrangian planes which are transverse to the complement of the given intersection with σ. Note that $\Sigma^0(n)$ is an open subset of $\mathcal{L}(n)$, hence is also a submanifold. Thus, $\Sigma^0(n-k)$ has dimension $\dfrac{(n-k)(n-k+1)}{2}$. From this fibration we find, since $\dim(G(n,k) = k(n-k)$,

$$\begin{aligned}\dim(\Sigma^k(n)) &= \dim(\Sigma^0(n-k)) + \dim(G(n,k)) \\ &= \frac{(n-k)(n-k+1)}{2} + k(n-k) \\ &= \frac{(n-k)(n+k+1)}{2} \\ &= \frac{n(n+1)}{2} - \frac{k(k+1)}{2} \\ &= \dim(\mathcal{L}(n)) - \frac{k(k+1)}{2}\end{aligned}$$

□

Next consider the Maslov cycle $\Sigma(n) = \overline{\Sigma^1(n)}$. This set is stratified as it can be written as

$$\overline{\Sigma^1(n)} = \bigcup_{k=1}^{n} \Sigma^k(n).$$

Its codimension is then equal to 1. It turns out that the Maslov cycle admits a coorientation and that a generic curve in the Lagrangian Grassmannian intersects only its highest strata. It thus seems natural to, given a fixed coorientation, consider the intersection number of such a curve with the Maslov cycle. It is immediately clear that this will then be a homotopy invariant of the curve. Furthermore it turns out that this is precisely the Maslov index, with the right coorientation. For this we need the following lemma:

Lemma 2.3.9 (Lemma 3.5.1. of [1])**.** *For any $\lambda \in \Sigma^1(n)$ the curve $\gamma\colon S^1 \to \mathcal{L}(n)$ given by $e^{i\theta} \mapsto e^{i\theta}\lambda$ is transversal to the cycle $\Sigma^1(n)$ at the point $\sigma = 0$.*

As a corollary of this lemma we get the following:

Lemma 2.3.10 (The fundamental lemma of [1])**.** *There is a continuous vector field in $\mathcal{L}(n)$ which is transversal to $\Sigma^1(n)$.*

This allows us to define a coorientation on $\Sigma^1(n)$ by defining the velocity vectors of the curves $e^{i\theta}\lambda$ to be pointing in the positive direction.

Using this we may now define the intersection number of oriented closed curves transversal to $\overline{\Sigma^1(n)}$. We denote it by $Ind(\gamma)$. This can then be represented as a cohomology class $Ind \in H^1(\mathcal{L}(n);\mathbb{Z})$. As before we can also use the tangential mapping to define the intersection number for curves on Lagrangian submanifolds of $(\mathbb{R}^{2n}, \omega_0)$.

By calculating the intersection index of the curve $\gamma(\theta) = e^{i\theta}\lambda$ for $\lambda \in \mathcal{L}(n)$ we find that this coincides with the previously defined Maslov index $\mu(\gamma)$. Thus using the fact that $H^1(\mathcal{L}(n);\mathbb{Z}) = \mathbb{Z}$ we get the following theorem:

Theorem 2.3.11 (Theorem 1.5. of [1])**.** *The cohomology class $\mu^* \in H^1(M;\mathbb{Z})$ coincides with the intersection number Ind on the Lagrangian submanifold M.*

This gives us a couple of different ways with which to calculate the Maslov index, which we shall henceforth denote only by μ.

2.4. **Applications to Legendrian knots.** We will now consider how we can use the Maslov index to give us information about Legendrian knots in $(\mathbb{R}^3, \xi_0)$. For clarity we will use $\xi_0 = \ker(dz - ydx) = \ker(\alpha_0)$. We will also consider the front projection π_F, projecting onto the (x, z)-plane, and the Lagrangian π_L projection, projecting onto the (x, y)-plane.

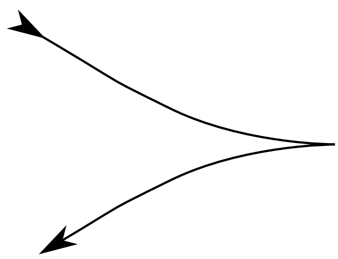
To get us started, recall Lemma 1.2.1 and Lemma 1.2.2. The first thing to note is that for a generic Legendrian knot, in the sense that the front projection of the knot only has the cusps as in the above lemma, it is quite simple to calculate the Maslov index. Note that $d\alpha_0 = \omega_0$ and that the Maslov cycle of $(\mathbb{R}^2, \omega_0)$ is simply the line $x = 0$. Since the Lagrangian projection of any immersed Legendrian curve is an immersed Lagrangian curve (the fact that it is Lagrangian is trivial since any curve in any 2-dimensional Lagrangian manifold is immediately Lagrangian for dimensional reasons but this is still true in higher dimensions) we can calculate the Maslov index of the Lagrangian projection. Thus we make the following definition:

Definition 2.4.1. The Maslov index of an immersed Legendrian curve $\gamma\colon S^1 \to (\mathbb{R}^3, \xi_0)$ is the Maslov index of the curve $\gamma_L\colon S^1 \to (\mathbb{R}^2, \omega_0)$. We denote the Maslov index of a Legendrian curve γ by $\mu(\gamma)$.

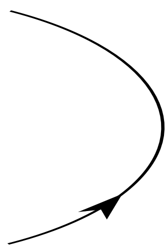
Thus, we can calculate the Maslov index of a Legendrian curve by looking for points where the tangent vector of its Lagrangian projection is vertical, i.e. parallel to $x = 0$. Alternatively we can look at the curve in the front projection. By the above lemma the vertical tangencies correspond precisely to the cusps of its front projection, since here we have $\dot{x} = 0$ and necessarily $\dot{y} \neq 0$ because its Lagrangian projection is an immersion. Thus the Maslov index of a generic Legendrian curve can be calculated by looking only at its front projection.

Since we have chosen a coorientation given by defining the curves $e^{i\varphi}\lambda$ to have Maslov index $+1$ we must first find which type of cusp has Maslov index 1. After this is done, calculating the Maslov index of a knot is straight forward.

Consider the below figures, where in both cases the horizontal direction is the x-direction.



(A) The front projection of a cusp.



(B) The Lagrangian projection of a cusp.

Clearly, the Lagrangian projection of such a cusp corresponds to Maslov index $+1$, hence we now know that the given cusp will have Maslov index $+1$. From this we may calculate the Maslov index of any Legendrian knot by simply taking the number of cusps traveled down and subtracting the number of knots traveled up. It is then clear that the Maslov index of a Legendrian knot corresponds to twice the rotation number of the knot and therefore also equal to twice the winding number of its Lagrangian projection.[5] From this it is already clear that the Maslov index is an invariant of Legendrian knots up to Legendrian isotopy but we give an explicit proof of this property without making reference to other invariants.

Proposition 2.4.2. *Let* $\gamma_0, \gamma_1 \colon S^1 \to (\mathbb{R}^3, \xi_0)$ *be Legendrian knots and let* φ_s *be a Legendrian isotopy between* γ_0 *and* γ_1*, i.e.* $\varphi_0(t) = \gamma_0(t)$ *and* $\varphi_1(t) = \gamma_1(t)$*. Then* $\mu(\gamma_0) = \mu(\gamma_1)$*.*

Proof. Since each curve φ_s is a Legendrian knot, the family of curves $\pi_L(\varphi_s)$ gives us a homotopy through Lagrangian immersions between $\pi_L(\gamma_0)$ and $\pi_L(\gamma_1)$ and can hence be lifted to a homotopy in $\mathcal{L}(1)$. Since the Maslov index is invariant under homotopy we must then have $\mu(\pi_L(\gamma_0)) = \mu(\pi_L(\gamma_1))$ and hence, by definition, $\mu(\gamma_0) = \mu(\gamma_1)$. □

Example 2.4.3. Consider the following oriented Legendrian knot.

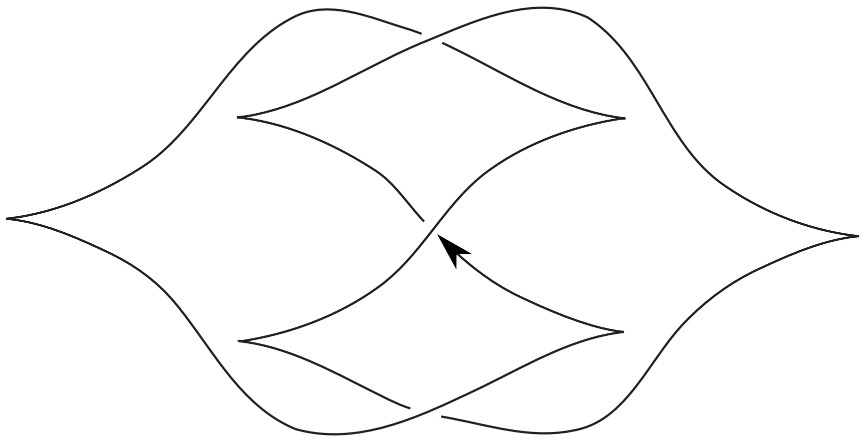


FIGURE 2. An oriented Legendrian trefoil knot.

Counting the cusps we see that it has four cusps traveled up (the ones in the center of the picture) and two cusps traveled down (the ones at the left and right edges of the picture). Thus the Maslov index of the knot is $2-4=-2$. This tells us that it is indeed not Legendrian isotopic to a Legendrian unknot. Reversing the orientation would change the sign of the Maslov index, making it $+2$.

It turns out however that the Maslov index here defined, since it coincides with twice the rotation number of knots, does not give a complete classification of Legendrian knots up to Legendrian isotopy, i.e. there are knots which are not Legendrian isotopic but have the same Maslov index. The situation changes when instead of considering knots up to Legendrian isotopy we instead consider knots up to Legendrian cobordism. In this setting the Maslov index does indeed provide such a complete invariant. This is due to the following result:

[5]See, for example, [10]

Theorem 2.4.4 ([3], p. 253)**.** *The group of cobordism classes of oriented Legendre immersions in $J^1(\mathbb{R})$ is isomorphic with the group of even numbers; the isomorphism associates with an immersion the number of cusps of the front, taking signs into account.*

Since, as we have shown, the Maslov index defined here counts cusps and takes signs into account this is then a complete invariant of Legendrian knots up to Legendrian cobordism. We will not give a complete proof of the above theorem here but we will give all relevant definitions and results in order to see exactly what the theorem means and how it can be used. We follow the path set out by [2] and [3] where proofs of the statements can also be found.

Definition 2.4.5. Two manifolds M_0, M_1 are called cobordant if there exists a manifold M such that ∂M is diffeomorphic to $M_0 \sqcup M_1$. The manifold M is called a cobordism between M_0 and M_1. A cobordism M is called cylindrical if $M = M_0 \times [0, 1]$. If M is a cobordism between M_0 and M_1 we will write $\partial M = M_1 - M_0$.

Manifolds and cobordisms between them constitute a category, the category of cobordisms, where objects are manifolds and a morphism between two manifolds is a cobordism between them. It is clear that cobordantness and cylindrical cobordantness are equivalence relations.

Now consider the fibre bundle $\pi\colon\ J^1(V) \to J^0(V)$ where V is a manifold with boundary. This is a contact manifold and a Legendrian bundle, i.e. the fibers $\pi^{-1}((x, z))$ are Legendrian submanifolds for all $(x, z) \in J^0(V) = V \times \mathbb{R}$. The image of a Legendrian submanifold $L \subset J^1(V)$ under the projection π is called the front of L. We get a map $\rho\colon\ J^1(V) \to J^1(\partial V)$ by $f \mapsto f|_{\partial V}$.

Let $\lambda \subset J^1(V)$ be a Legendrian submanifold transverse to the boundary $\pi^{-1}(\partial V) = \partial J^1(V)$. Then $\rho(\lambda \cap \partial J^1(V))$ is an immersed Legendrian submanifold of $J^1(\partial V)$. We call this the Legendrian boundary of the Legendrian submaniold λ and write $\partial\lambda = \rho(\lambda \cap \partial J^1(V))$.

Definition 2.4.6. Let V be a cobordism between $\partial V = V_1 - V_0$. A Legendrian submanifold $\lambda \subset J^1(V)$ with Legendrian boundary $\partial\lambda = L_1 - L_0$ is called a Legendrian cobordism between the immersed Legendrian submanifolds $L_0 \subset J^1(V_0)$ and $L_1 \subset J^1(V_1)$. It is called cylindrical if $V = V_0 \times [0, 1]$. If L_0 is cylindrical Legendrian cobordant with L_1 we write $L_0 \sim L_1$.

It is not clear from the definition that Legendrian cobordantness is an equivalence relation since there may be problems with transitivity, since the union of two Legendrian cobordisms $\partial\lambda_1 = L_1 - L_0$ and $\partial\lambda_2 = L_2 - L_1$ over $\partial W_1 = V_1 - V_0$ and $\partial W_2 = V_2 - V_1$ respectively may have a singularity or even a discontinuity over V_1. This can be overcome by inserting a vertical submanifold over V_1 connecting the two immersed copies of L_1 (this is Legendrian since we are working in a Legendrian bundle) and then smoothing along the corners. This then allows us to define the product of two Legendrian cobordisms $\partial\lambda_1 = L_1 - L_0$ and $\partial\lambda_2 = L_2 - L_1$ as $\partial\lambda = L_2 - L_0$. It is clear that the product of two cylindrical cobordisms is again a cylindrical cobordism. Hence both Legendrian cobordantness and cylindrical Legendrian cobordantness are equivalence relations. We denote the set of cylindrical Legendrian cobordism classes of oriented Legendrian submanifolds in $J^1(V)$ by $Leg(V)$.

For cylindrical Legendrian cobordisms one can prove the following:

Theorem 2.4.7 ([2], p. 174)**.** *If $L_0 \sim L_0'$ and $L_1 \sim L_1'$ then $L_0 \cup L_1 \sim L_0' \cup L_1'$.*

Using this we may turn $Leg(V)$ into a semigroup by defining the addition via $[L_0] + [L_1] := [L_0 \cup L_1]$. For $V = \mathbb{R}^n$ this can also be made into a group by considering also the orientation of the Legendrian submanifolds, each element $-[L]$ obtained from $[L]$ by reversing its orientation.

We will now restrict ourselves only to the group $Leg(\mathbb{R})$, i.e. the group of cylindrical Legendrian cobordism classes of oriented Legendrian curves in $J^1(\mathbb{R})$. Since the front of a generic Legendrian curve uniquely determines the original Legendrian curve we may reduce the problem of cobordisms between Legendrian curves to the problem of cobordisms between fronts of curves, by which we will mean a surface which is transversal to the boundary of the base when considered as a stratified manifold.

Theorem 2.4.8 ([3], p. 253)**.** *Each oriented front of an immersion in $J^1(\mathbb{R})$ is oriented cobordant with a sum of bow-ties.*

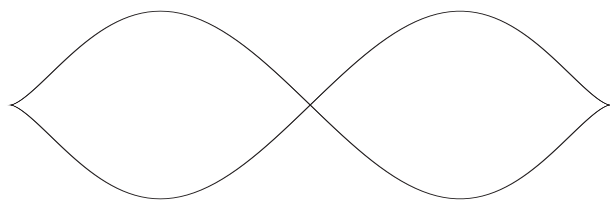


FIGURE 3. A bow-tie.

The Maslov index of such a bow-tie is ± 2 depending on which orientation is chosen, as is seen by orienting the bow-tie and counting the cusps with signs. From this Theorem 2.4.4 follows easily.

This gives us a condition for when it is possible to glue our local approximations together to form a global approximation, namely the Maslov index of the parts we are trying to glue together must coincide.

3. Morse functions and embeddings

We will only consider embeddings for which the x_2-coordinate is a Morse function, meaning that we will only have to deal with finitely many critical points. A map for which the x_2-coordinate is a Morse function will simply be called Morse in the x_2-direction. We motivate this restriction by showing that a generic embedding will have this property.

Definition 3.0.9. Let M, N be smooth manifolds and let $U \overset{\text{open}}{\subset} J^r(M,N)$. The Whitney C^r-topology (also called the strong or fine C^r-topology) on $C^\infty(M,N)$ is generated by the sets $B^r(U) = \{f \in C^\infty(M,N)\colon (j^r f)(M) \subset U\}$. The C^∞-topology is then $W = \bigcup_{r=0} W^r$.

Definition 3.0.10. A topological space is called a Baire space if for any countable collection $\{U_n\}$ of dense, open sets their intersection $\bigcap_n U_n$ is also dense. A subset V

of a topological space is called residual if it is the intersection of countably many dense, open sets. A property is called generic if it holds on a residual set.

Proposition 3.0.11. *$C^\infty(M,N)$ with Whitney topology is a Baire space and hence residual sets are dense.*

Definition 3.0.12. Let M_0 denote the zero section $M \to T^*M$. A function $f \in C^\infty(M)$ is called a Morse function if df is transverse to M_0.

We will need the following two propositions:

Proposition 3.0.13 (Theorem 3.2.8 of [14])**.** *Let M, N be C^∞ manifolds without boundary and let $A \subset J^r(M,N)$ be a C^∞-submanifold. Suppose $1 \leq r < s \leq \infty$. Then $\pitchfork^s(M,N; j^r, A)$ is residual and thus dense in $C^s_S(M,N)$ and open if A is closed.*

Proposition 3.0.14 (Theorem 2.2.13 of [14])**.** *Let M, N be C^r manifolds where $1 \leq r \leq \infty$ with $\dim(N) \geq 2\dim(M)+1$. If M is closed then embeddings are dense in $C^r_S(M,N)$.*

Remark 3.0.15. The topological spaces $C^k_S(M,N)$ referenced in the above propositions are the Whitney C^k-topologies.

Lemma 3.0.16. *Let M be a closed 2-dimensional manifold. Then the set of embeddings $f\colon M \to \mathbb{R}^5 = \{(x_1, x_2, y_1, y_2, z)\}$ for which the x_2-coordinate function is a Morse function is dense in the Whitney C^r-topology on $C^\infty(M, \mathbb{R}^5)$ for $r \leq 2 \leq \infty$.*

Proof. Consider the projections $\pi_1\colon J^1(M,\mathbb{R}^5) \to J^1(M)$ and $\pi_2\colon J^1(M) \to T^*M$, given by $\pi_1(j^1 f) = j^1(x_2 \circ f)$ and $\pi_2(j^1 g) = dg$ respectively, and let $\pi = \pi_2 \circ \pi_1\colon J^1(M,\mathbb{R}^5) \to T^*M$. Let M_0 be the embedding of the zero section in T^*M. Then a map $f\colon M \to \mathbb{R}^5$ is Morse in the x_2-direction if its 1-jet extension is transverse to $A = \pi^{-1}(M_0)$. Because π is a submersion and M_0 is closed in T^*M we have that A is a closed submanifold of $J^1(M,\mathbb{R}^5)$. By Proposition 3.0.13 the set of maps which are transverse to A form a dense open subset of $C^\infty(M,\mathbb{R}^5)$ in the Whitney C^s-topology for any $2 \leq s \leq \infty$. Using Proposition 3.0.14 the lemma now follows. ☐

It is immediately clear that this lemma naturally generalizes to $C^\infty(M,\mathbb{R}^m)$ where M is any closed n-dimensional manifold, $m \geq 2n+1$ and any direction.

The reason we want an embedding which is Morse in the x_2-direction can be summarized by the following two theorems:

Theorem 3.0.17 (Morse's Lemma, 6.1.1. of [14])**.** *Let $p \in M$ be a non-degenerate critical point of index k of a C^{r+2} map $f\colon M \to \mathbb{R}$, $1 \leq r \leq \omega$. Then there is a C^r chart (φ, U) at p such that*

$$f \circ \varphi^{-1}(u_1, \ldots, u_n) = f(p) - \sum_{i=1}^{k} u_i^2 + \sum_{i=k+1}^{n} u_i^2.$$

In particular, non-degenerate critical points are isolated. If M is compact then f can only have finitely many non-degenerate critical points. Since every critical point of a Morse function is non-degenerate this means that a Morse function on a compact manifold can only have finitely many critical points.

Theorem 3.0.18 (Theorem 6.2.2. of [14])**.** *Let $f\colon M \to [a,b]$ be a C^{r+1} map on a compact manifold with boundary, $1 \leq r \leq \omega$. Suppose f has no critical points and that $f(\partial M) = \{a,b\}$. Then there is a C^r diffeomorphism $F\colon f^{-1}(a) \times [a,b] \to M$ so that the diagram commutes. In particular all level surfaces of f are diffeomorphic.*

$$\begin{array}{ccc} f^{-1}(a) \times [a,b] & \xrightarrow{F} & M \\ & \searrow & \downarrow f \\ & & [a,b] \end{array}$$

These theorems allow us to divide the manifold into neighbourhoods of finitely many critical points of the x_2-direction and cylinders connecting them. The idea will then be to construct a "local" Legendrian approximation on each of these parts separately and then connecting them into a global approximation.

4. Legendrian approximations

We will now begin looking into constructing a Legendrian approximation of a surface.

4.1. **Partition of surfaces.** Let $f\colon M \to (\mathbb{R}^5, \xi)$ be an embedded surface. If this embedding is not such that its x_2-coordinate is a Morse function, begin by choosing such an approximation according to Proposition 3.0.14. Using Theorem 3.0.18 we may then partition this embedded surface into small neighbourhoods of a finite number of critical point and cylinders connecting them.

The appearance of the surface near the critical points is given by Theorem 3.0.17; let $p \in M$ be a critical point of the x_2-coordinate function. Then according to Theorem 3.0.17 we can find local coordinates (u,v) at p such that the x_2-coordinate can be written as

$$x_2(u,v) = \begin{cases} x_2(p) + u^2 + v^2 \\ x_2(p) - u^2 + v^2 \\ x_2(p) - u^2 - v^2 \end{cases}$$

depending on the index of the critical point.

The cylinders connecting the critical points are described by Theorem 3.0.18; Let $[a,b]$ be an interval containing no critical values of the x_2-coordinate. Then $x_2^{-1}(a)$ is diffeomorphic to a finite disjoint union of circles, since these are the only closed 1-dimensional manifolds up to diffeomorphism. By Theorem 3.0.18 the embedding over this interval is diffeomorphic to $x_2^{-1}(a) \times [a,b]$.

The goal for the rest of this section is to find Legendrian approximations of each of these pieces individually.

4.2. **Critical points and their approximation.** We begin by considering how to construct an approximation at a critical point of the x_2-coordinate. We first describe in words how this can be done.

Consider a critical point p of the x_2-coordinate and suppose that, for example, p is a regular point of both the x_1 and y_2 coordinates. Using the implicit function theorem we can then locally (i.e. in a neighbourhood of the image of p) view our surface as a graph over the (x_1, y_2)-plane, i.e. $y_1 = y_1(x_1, y_2)$, $x_2 = x_2(x_1, y_2)$ and $z = z(x_1, y_2)$. If we can then construct a Legendrian surface which is also a graph over the (x_1, y_2)-plane which coincides with the original surface at the point p this Legendrian surface can be used to make a C^0-close approximation of the original surface if we choose our neighbourhood small enough.

In order to view the original surface M as a graph over the (x_1, y_2)-plane we need to make sure that the point p is a regular point of both those coordinate functions. If this is not so then consider the set of critical points of both functions. Since the set of critical points of any smooth function has measure zero according to Sard's theorem their union also has measure zero. Hence in any neighbourhood of p we can find points which are regular for both x_1 and y_2 and we can then perturb those functions so that p is a regular point of both those functions. Thus our surface can locally be written as a graph over the (x_1, y_2)-plane.

We will now construct a Legendrian surface which we will take as our approximation. For this we will need the following theorem:

Theorem 4.2.1 (Corollary on p. 313 of [5])**.** *Every germ of a Legendrian submanifold in the* $2n+1$*-dimensional contact space with contact form* $dz - ydx$ *is given by one of the* 2^n *generating functions* S *according to the formulas*

$$\begin{aligned} y_I &= \frac{\partial S}{\partial x_I}, \\ x_J &= -\frac{\partial S}{\partial y_J}, \\ z &= S(x_I, y_J) + \langle x_J, y_J \rangle \end{aligned}$$

where (I, J) *is a partition of the set* $\{1, \dots, n\}$ *into non-intersecting subsets.*

The expression $\langle x_J, y_J \rangle$ is taken to mean $\sum_{j \in J} x_j y_j$.

We thus want to construct such a generating function $S = S(x_1, y_2)$, i.e. we have $I = \{1\}$ and $J = \{2\}$. We will also use the fact that the x_2-coordinate is a Morse function by introducing Morse coordinates at p, i.e. coordinates (u, v) on M such that $p = (0, 0)$ and $x_2(u, v) = \pm u^2 \pm v^2 + C$ where $C = x_2(p)$.

Consider first the case where p is a maximum. We then have $x_2(u,v) = u^2 + v^2 + C$. Since we want to be able to consider this Legendrian surface as a graph over the (x_1, y_2)-plane we make the following simple choice:

$$\begin{aligned} x_1(u,v) &= u + A \\ y_2(u,v) &= v + B \end{aligned}$$

where $A = x_1(p)$ and $B = y_2(p)$. Using the equation $x_j = -\dfrac{\partial S}{\partial y_j}$ from Theorem 4.2.1 we get the following:

$$x_2(u,v) = u^2 + v^2 + C = -\frac{\partial S}{\partial y_2}(u,v) = -\frac{\partial S}{\partial v}$$

Solving this equation gives us

$$S(u,v) = -\frac{1}{3}v^3 - u^2 v - Cv + f(u)$$

where f is some smooth function of the u-coordinate alone. We then get

$$\begin{aligned} y_1(u,v) &= \frac{\partial S}{\partial x_1} = \frac{\partial S}{\partial u} = -2uv + f'(u) \\ z(u,v) &= -\frac{1}{3}v^3 - u^2 v - Cv + f(u) + x_2 y_2 \\ &= -\frac{1}{3}v^3 - u^2 v - Cv + f(u) + u^2 v + u^2 B + v^3 + v^2 B + Cv + CB \\ &= \frac{2}{3}v^3 + Bu^2 + Bv^2 + f(u) + CB\,. \end{aligned}$$

In order for these to coincide with the original surface at p we hence need $y_1(0,0) = f'(0) = y_1(p)$ and $z(0,0) = f(0) + CB = z(p)$. The easiest way to do this is to choose $f(u) = au + b$ where $b = z(p) - CB$ and $a = y_1(p)$ and hence we get

$$\begin{aligned} y_1(u,v) &= -2uv + a \\ z(u,v) &= \frac{2}{3}v^3 + Bu^2 + Bv^2 + au + b + CB\,. \end{aligned}$$

In summary we now have the following surface:

$$(u,v) \mapsto \begin{cases} x_1(u,v) = u + A \\ x_2(u,v) = u^2 + v^2 + C \\ y_1(u,v) = -2uv + a \\ y_2(u,v) = v + B \\ z(u,v) = \dfrac{2}{3}v^3 + Bu^2 + Bv^2 + au + b + CB\,. \end{cases}$$

To verify that this does indeed give us a Legendrian surface we calculate the pull-back of the contact form $dz - y_1 dx_1 - y_2 dx_2$:

$$\begin{aligned}
dz &= \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv \\
&= (2Bu + a)du + (2v^2 + 2Bv)dv \\
y_1 dx_1 &= (-2uv + a)du \\
y_2 dx_2 &= (v + B)2udu + (v + B)2vdv \\
\Rightarrow \quad dz - y_1 dx_1 - y_2 dx_2 &= (2Bu + a)du + (2v^2 + 2Bv)dv + \\
&- (-2uv + a)du - (v + B)2udu - (v + B)2vdv \\
&= (2Bu + a + 2uv - a - 2uv - 2Bu)du + \\
&+ (2v^2 + 2Bv - 2v^2 - 2Bv)dv \\
&= 0
\end{aligned}$$

so this surface is indeed Legendrian. Its front is given by the following:

$$\begin{aligned}
x_1(u, v) &= u + A \\
x_2(u, v) &= u^2 + v^2 + C \\
z(u, v) &= \frac{2}{3}v^3 + Bu^2 + Bv^2 + au + b + CB
\end{aligned}$$

and can be seen in Figure 4.

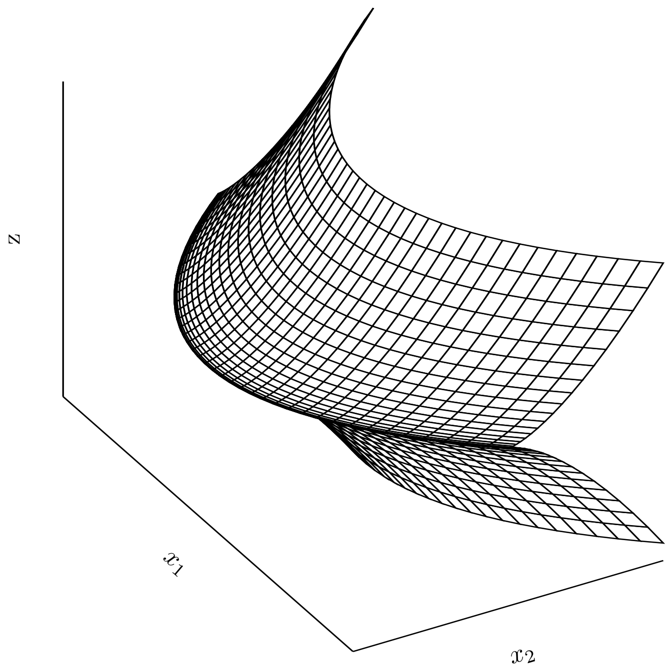


FIGURE 4. The front projection of the Legendrian approximation of a local maximum.

Fixing a regular value of $x_2 = r^2 + C$, i.e. intersecting this Legendrian surface with the hyperplane $\{(x_1, x_2, y_1, y_2, z) | \, x_2 = r^2 + C\}$, we have $x_2(u, v) = u^2 + v^2 + C$ and hence $u^2 + v^2 = r^2$, thus there is a straight forward parametrization of this curve given by introducing the polar coordinates

$$u = r\cos(t)$$
$$v = r\sin(t)$$

giving us the following expression for the curve:

$$x_1(u, v) = u + A = r\cos(t) + A$$
$$x_2(u, v) = r^2 + C$$
$$z(u, v) = \frac{2}{3}v^3 + Br^2 + au + b + CB = \frac{2}{3}r^3\sin^3(t) + Br^2 + ar\cos(t) + b + CB\,.$$

Omitting the x_2-coordinate we can think of this as the front of a Legendrian curve in $(\mathbb{R}^3, \xi_0)$ and hence calculate its Maslov index.

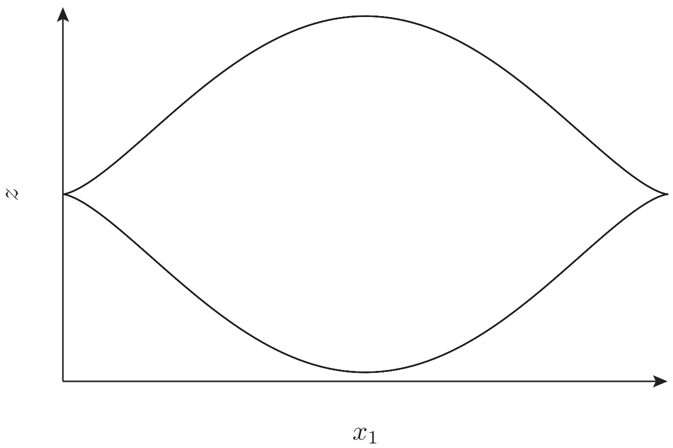


FIGURE 5. A slice of the approximation.

As is easily seen from Figure 5 the Maslov index of this curve is zero. Note that this is to be expected, since the curve is cylindrically Legendrian null-cobordant and hence according to Theorem 2.4.4 it must have Maslov index zero. Note also that from Theorem 3.0.17 and Theorem 3.0.18 it follows that for some small enough $\varepsilon > 0$ the preimage of $[C - \varepsilon, C + \varepsilon]$ consists of disjoint pieces of the surface, one of which contains the critical point in question and the others being cylinders. This model then gives us an approximation of the entire connected component containing the critical point for a small enough ε. This is an important property since it will allow us to glue our local approximations together to form a global approximation.

Suppose now that we instead have a local minimum, i.e. there are coordinates (u, v) at p such that $x_2(u, v) = -u^2 - v^2 + C$ where $C = x_2(p)$. Then the procedure above, changing the appropriate signs, gives us what we need.

Now consider the case where p is a saddle point of x_2. Again we choose Morse coordinates (u, v) at p so that we have $x_2(u, v) = -u^2 + v^2 + C$. Like before we want to consider the Legendrian surface as a graph over the (x_1, y_2)-plane and hence we make the following choice:

$$\begin{aligned} x_1(u, v) &= u + A \\ y_2(u, v) &= v + B \end{aligned}$$

where $A = x_1(p)$ and $B = y_{2(p)}$. As before we find

$$x_2(u, v) = -u^2 + v^2 + C = -\frac{\partial S}{\partial y_2} = -\frac{\partial S}{\partial v}$$

Solving this equation yields

$$S(u, v) = -\frac{1}{3}v^3 + u^2 v - Cv + f(u)$$

where f is some smooth function depending only on u. Using this we get

$$\begin{aligned} y_1(u, v) &= \frac{\partial S}{\partial x_1} = \frac{\partial S}{\partial u} = 2uv + f'(u) \\ z(u, v) &= S(u, v) + x_2(u, v) y_2(u, v) \\ &= -\frac{1}{3}v^3 + u^2 v - Cv + f(u) - u^2 v - u^2 B + v^3 + v^2 B + Cv + CB \\ &= \frac{2}{3}v^3 - Bu^2 + Bv^2 + f(u) + CB \end{aligned}$$

We once again require that $y_1(0, 0) = f'(0) = y_1(p)$ and $z(0, 0) = f(0) + CB = z(p)$. Making the easy choice $f(u) = au + b$ where $a = y_1(p)$ and $b = z(p) - CB$ this is satisfied. With this we now have

$$\begin{aligned} y_1(u, v) &= 2uv + a \\ z(u, v) &= \frac{2}{3}v^3 - Bu^2 + Bv^2 + au + b + CB \end{aligned}$$

In summary we have a surface given by the following:

$$(u, v) \mapsto \begin{cases} x_1(u, v) & = u + A \\ x_2(u, v) & = -u^2 + v^2 + C \\ y_1(u, v) & = 2uv + a \\ y_2(u, v) & = v + B \\ z(u, v) & = \frac{2}{3}v^3 - Bu^2 + Bv^2 + au + b + CB \end{cases}$$

We verify that this is indeed a Legendrian surface by calculating the pull-back of the contact form:

$$
\begin{aligned}
dz &= \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv \\
&= (-2Bu + a)du + (2v^2 + 2Bv)dv \\
y_1 dx_1 &= (2uv + a)du \\
y_2 dx_2 &= -(v + B)2udu + (v + B)2vdv \\
\Rightarrow \quad dz - y_1 dx_1 - y_2 dx_2 &= (-2Bu + a)du + (2v^2 + 2Bv)dv \\
&\quad - (2uv + a)du + (v + B)2udu - (v + B)2vdv \\
&= (-2Bu + a - 2uv - a + 2uv + 2Bu)du + \\
&\quad + (2v^2 + 2Bv - 2v^2 - 2Bv)dv \\
&= 0
\end{aligned}
$$

and we see that the surface we have constructed is Legendrian. The front projection of this surface is given by

$$
\begin{aligned}
x_1(u, v) &= u + A \\
x_2(u, v) &= -u^2 + v^2 + C \\
z(u, v) &= \frac{2}{3}v^3 - Bu^2 + Bv^2 + au + b + CB
\end{aligned}
$$

and can be seen in Figure 6.

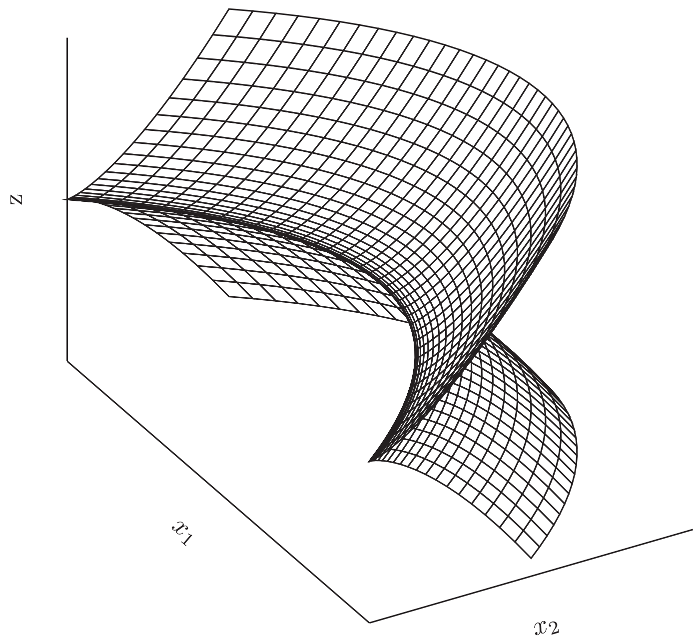


FIGURE 6. The front projection of the Legendrian approximation of a saddle point.

This gives us a way of finding an approximating Legendrian surface close to a critical point p of the x_2-coordinate. Since both the original surface and the approximating Legendrian surface are graphs over a small disk in the (x_1, y_2)-plane at p they are in particular isotopic[6]. As such we can remove a small disk around each critical point p and replace it with a Legendrian surface, thus producing a new C^0-close surface having exactly the same critical points in the x_2-direction as the original surface and being isotopic to it but such that a small neighbourhood of each critical point is Legendrian.

However, note that fixing a value $x_2(u, v) = R + C$ around an approximated saddle point does not give a closed curve. If $R \geq 0$ we may simply write $v = \pm\sqrt{R + u^2}$ so that each component of the curve becomes the graph of the function $f(u) = \pm(R + u^2)^{\frac{3}{2}}$. These curves never intersect the Maslov cycle. If $R < 0$ we get two oppositely oriented cusps pointing towards each other. The case $R = 0$ is somewhat special as we here get two oppositely oriented cusps which are connected at the cusps. Therefore, as opposed to the case of a local maximum or minimum, this method does not give us an approximation of the entire connected component of the surface over a small interval $[C - \varepsilon, C + \varepsilon]$ of x_2 values which is what we ultimately want to be able to construct our global approximation. Thus we need to deal with this issue.

To begin with, consider the slice of a Legendrian approximation of a saddle point at $x_2 = C$, for which we get two oppositely oriented cusps connected at their cusp points. For any $\varepsilon > 0$ the pre-image of the interval $[C - \varepsilon, C + \varepsilon]$ then contains each type of intersection curve, i.e. oppositely oriented cusps, oppositely oriented connected cusps and regular curves not intersecting the Maslov cycle. We would then like to find, for some $\varepsilon > 0$, an approximation of this entire interval. We can begin by approximating the complement of the curve of intersection of our saddle point approximation at $x_2 = C$ as a Legendrian curve in $(\mathbb{R}^3, \xi_0)$, ignoring the y_2-coordinate. This gives us an isotropic approximation of the curve of intersection with $x_2 = C$. Note that when doing this approximation we may choose its Maslov index. Therefore we will always choose this approximation to have Maslov index 0. This is a natural choice in our situation since this assures us that this component of the curve of intersection is Legendrian null-cobordant.[7] Next, we can take the y_2-coordinate at each point of this approximated complement and use it to flow outward in the x_2-direction such that $\frac{\partial z}{\partial x_2} = y_2$, i.e. we create a variation $\gamma(s, t)$ of the curve $\gamma(s)$ at $x_2 = C$ such that $z(s, t) = z(s, 0) + y_2(s, 0)x_2(s, t)$ and $x_2(s, t) = t + C$. Since this is smooth and the curve is C^0-close to the original surface at $x_2 = C$ this variation remains C^0-close to the original surface for $t \in [-\delta, \delta]$ for δ small enough.

Next we want to glue this together with the Legendrian approximation of the saddle point. Consider the front projection. By smoothness, for small enough ε both the endpoints of our approximated saddle point and the above constructed variation of the complement at $x_2 = C$, as well as their slopes, inside $[C - \varepsilon, C + \varepsilon]$ are as close as we

[6]To see this note that they are both isotopic to the disk over which they are defined, an explicit isotopy from the disk to the surface being given by $F(x_1, y_2, t) = (x_1, tx_2(x_1, y_2), ty_1(x_1, y_2), y_2, tz(x_1, y_2))$ for $t \in [0, 1]$.

[7]Non-zero choices are sometimes possible as well but extra care has to be taken so that, in the end, it all fits together.

want. Hence we can interpolate between these without changing the position or the slope (and hence the lifted y_2-coordinate) too much, allowing us to remain C^0-close to the original surface. This then gives us a Legendrian approximation of an entire interval $[C-\varepsilon, C+\varepsilon]$ of x_2-coordinates around a saddle point.

4.3. **Cylinders.** Having achieved an approximation around each critical point we now turn to the cylinders connecting those critical points.

We assume that we have an embedding $f\colon \gamma \times [a,b] \to (\mathbb{R}^5, \xi_0)$ where γ is some curve which is diffeomorphic to a disjoint union of finitely many circles and $x_2(s,t) = t$ where $s \in \gamma$ and $t \in [a,b]$. We may further assume that there is some $\varepsilon > 0$ such that the embedding restricted to $[a, a+\varepsilon) \cup (b-\varepsilon, b]$ is already Legendrian and Legendrian null-cobordant. We then want to construct a Legendrian approximation of this embedding which coincides with the already Legendrian endpoints of the original embedding f. To do this we will create a C^0-close cylindrical Legendrian cobordism using 1-parameter families of Legendrian curves. Essentially we will consider the surface as a 1-parameter family of curves, each curve $\gamma \times \{t\}$ contained in the $x_2 = t$ hyperplane, approximate each curve in a smooth fashion and then approximate the thus attained Legendrian surface in the x_2-direction separately.

We begin with general definitions and results concerning 1-parameter families of Legendrian curves.

Following [5] we consider a 1-parameter family of Legendrian maps, i.e. maps $F\colon L \times \mathbb{R} \to J^1(M)$ such that for each $t \in \mathbb{R}$ the map $F_t = F(\bullet, t)\colon L \to J^1(M)$ is a Legendrian embedding, composed with the front projection. We call $M \times \mathbb{R}$ space-time and the projection to the $\mathbb{R}$-component the time function. The union of all the fronts in space-time is called the big front. One can then prove the following proposition.

Proposition 4.3.1 ([5], p. 346)**.** *At every point the germ of the big front is the germ of the front of a Legendrian map to space-time.*

This leads us to the following definition, also from [5]:

Definition 4.3.2. A *bifurcation of a front* is a diagram of germs

$$\Sigma \xrightarrow{\imath} \mathbb{R}^{n+1} \xrightarrow{t} \mathbb{R}$$

where $\imath$ is the inclusion of an n-dimensional front and t is a smooth function whose differential at the point of interesest is non-zero. An equivalence of bifurcations is a commutative diagram

$$\begin{array}{ccccc} \Sigma_1 & \xrightarrow{\imath_1} & \mathbb{R}^{n+1} & \xrightarrow{t_1} & \mathbb{R} \\ \downarrow & & \downarrow & & \downarrow \\ \Sigma_2 & \xrightarrow{\imath_2} & \mathbb{R}^{n+1} & \xrightarrow{t_2} & \mathbb{R} \end{array}$$

where the horizontal arrows are bifurcations of fronts and the vertical arrows are diffeomorphisms. Two bifurcations are called strongly equivalent if the rightmost vertical arrow is a translation, i.e. $t_2 = t_1 + c$ for some constant c.

One can now consider the problem of classifying the possible bifurcations of generic 1-parameter families of Legendrian maps. They turn out to be locally strongly equivalent to germs of what is called special bifurcations. The details of this can be found in [5]. The bifurcations of generic 1-parameter families of Legendrian curves are depicted below. These can be found in [4].

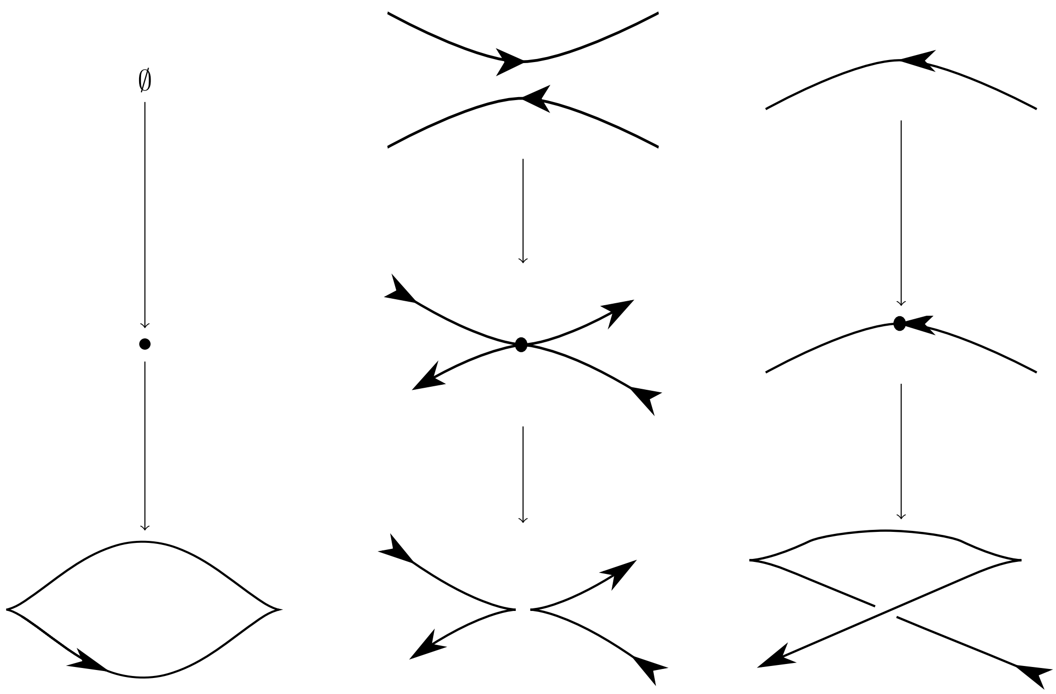


Figure 7. Bifurcations of generic 1-parameter families of Legendrian curves.

Note that the leftmost bifurcation is modeled by our approximation of a local maximum or minimum and the center bifurcation corresponds to our approximation of a saddle point, in both cases the x_2-coordinate considered as the time function of the bifurcation.

The idea is now to construct a Legendrian cobordism between the two already Legendrian endpoints of the cylinder using the above bifurcations. Additionally we can use the Legendrian Reidemeister moves, pictured below in the front projection.[8]

[8]See, for instance, [10].

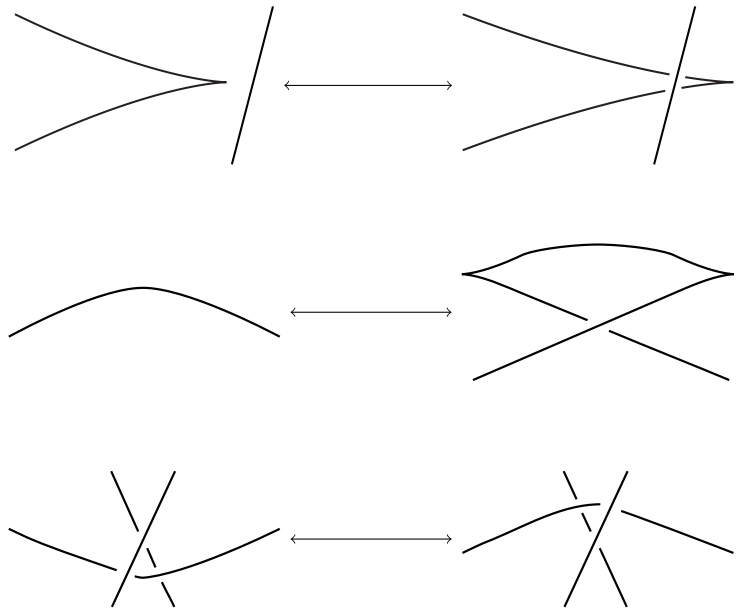


FIGURE 8. Legendrian Reidemeister moves.

However we do not want to construct just any Legendrian cobordism since we also want it to approximate the original cylinder. To achieve this we first make sure to approximate the curves of intersection with $x_2 = c$ for each $c \in [a, b]$ by an isotropic curve in a smooth fashion. This will give us a Legendrian cylinder for which $\frac{\partial z}{\partial x_1}$ approximates y_1 at each point but not necessarily for which $\frac{\partial z}{\partial x_2}$ approximates y_2 at each point. We then modify this cylinder further to achieve the approximation in the x_2-direction as well.

But first, let us again consider how the Legendrian approximation of a saddle point was constructed. Since the x_2-coordinate function is Morse we know[9] that the curves of intersection with an x_2 = constant-hyperplane will have one connected component on one side of the critical point and two connected components on the other side. Since the approximation was created so that the total Maslov index would be 0 we know that the curve with a single connected component will have Maslov index 0 and the curve with two connected components γ_0, γ_1 will have Maslov index $\mu(\gamma_i) = (-1)^i 2k$, $i = 0$, 1, for some integer k. Thus each of the curves γ_i are cylindrically Legendrian cobordant to k bow-ties of appropriate orientation. However this can be remedied by choosing the approximation correctly, i.e. by choosing an approprate approximation of the curve of intersection at the saddle point which we then use to create a variation. By adding cusps of one orientation to one side of the curve and cusps of the opposite orientation to the other side we can make sure that the two conected components both have Maslov index 0. Since this is also true of the curves created by a local maximum or minimum we may now assume that each connected component of the curves of intersection at the already

[9]See, for instance, theorem 6.3.1. of [14].

Legendrian endpoints of the cylinder all have Maslov index 0.

For definiteness, let $f|_{\gamma\times\{a\}} = \bigsqcup_{i=1}^n \gamma_i^a$ and $f|_{\gamma\times\{b\}} = \bigsqcup_{i=1}^n \gamma_i^b$. We know that $\mu(\gamma_i^a) = \mu(\gamma_i^b) = 0$ for each i hence the curves γ_i^a and γ_i^b can be directly connected by a cylindrical Legendrian cobordism. It thus suffices to construct an approximation of each such cylinder directly. In order to make sure that each curve of intersection in between them is appropriately approximated by an isotropic curve we need to add enough cusps to the front projection. This can be done using the first bifurcation as in the following diagram:

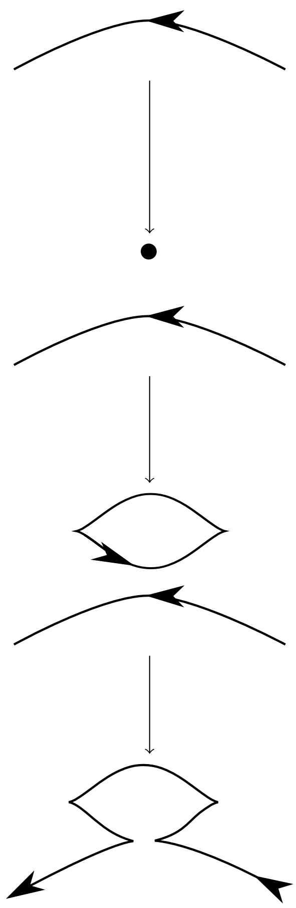


FIGURE 9. Adding cusps.

Note that this procedure is also local as it is given by our Legendrian approximation of a local maximum or minimum and hence can be done in any arbitrarily small neighbourhood. Thus we may freely add cusps while still remaining C^0-close to the original cylinder. Note also that cusps must appear in oppositely oriented pairs since they must have Maslov index 0. Fixing a tubular neighbourhood of the original cylinder we may also consider how many cusps the curves γ_i^c for each $c \in [a, b]$, $1 \leq i \leq n$ requires. We need to show that the maximum number of cusps needed is finite or we would be liable to get into trouble using this approach. Let $c \in (a, b)$, $\varepsilon > 0$ and consider the curve of intersection of the cylinder with the hyperplane $x_2 = c$. Make a C^0-approximation to within ε of this curve as a Legendrian curve in $\mathbb{R}^3 = \{(x_1, y_1, z)\}$, leaving the y_2-coordinate unchanged. Then this curve not only approximates the curve at $c \in (a, b)$ to within ε but in fact all curves in $(c - \delta_c, c + \delta_c)$ for some small $\delta_c > 0$, depending on c and ε, since varying c changes the curves of intersection smoothly. We may cover the entire interval $[a, b]$ by such intervals $(c_i - \delta_i, c_i + \delta_i)$, $i \in I$ where I is some index set, the approximation at c_i having n_i cusps. Since $[a, b]$ is compact there is a finite subcover, say $i = 1, \ldots, k$. Hence $\max_i(\{n_i\})$ exists and since ε was arbitrary this is true no matter how close we want our C^0-approximation, solving this issue.

Starting at $a \in [a, b]$ and working in the front projection may now we find the maximum number of cusps which will be needed in order to approximate every curve of intersection in $[a, b]$. We then add cusps, if needed, until we have reached at least this number using the above prescribed procedure. We can then pass over to the other side, approximating each curve in between, where we can then remove the added cusps using the above procedure in reverse until we get back to the already Legendrian part at $b \in [a, b]$. Note that the only critical points of the x_2-coordinate introduced by doing this are Legendrian local maxima or minima, hence $\frac{\partial z}{\partial x_2}$ can be used to lift the front of the cylinder to a Legendrian cylinder in $\mathbb{R}^5$. This is of course not necessarily an approximation of the original cylinder as we have not yet made any effort to control $\frac{\partial z}{\partial x_2}$. The final goal of this section will therefore be to achieve this.

We begin by considering a small piece of the surface not containing any cusps, i.e. the front of a curve of intersection in this small piece of the surface is regular. Without destroying the approximation in the x_1-direction we now want to add cusps in the x_2-direction of the front of this piece in some way so as to be able to move this piece up or down along the z-axis in order to manipulate the slope of the z-coordinate in the x_2-direction. Here our local models for a critical point again become useful. By using the local model of a saddle point we may "cut open" the surface inside this small piece, leaving the edges untouched, and then sow it together again a bit higher up or down using our model of a local maximum or minimum. This is precisely what the bifurcation in Figure 9 does. It is important to note that this does not affect the approximation in the x_1-direction as these new cusps can be added with any slope in the x_1-direction we want. Of course the procedure can also be used in reverse provided the necessary cusps exist and removing those cusps will not affect the approximation in the x_1-direction.

Next consider a small neighbourhood of a cusp in the x_1-direction. The procedure here is again to use the bifurcation from Figure 9. Adding such a bifurcation close to the cusp in question we may replace that cusp with one coming from the bifurcation which is then slightly above or below the original cusp, all without affecting the edges of the picture and without destroying the approximaton in the x_1-direction. The entire procedure will look like in Figure 10.

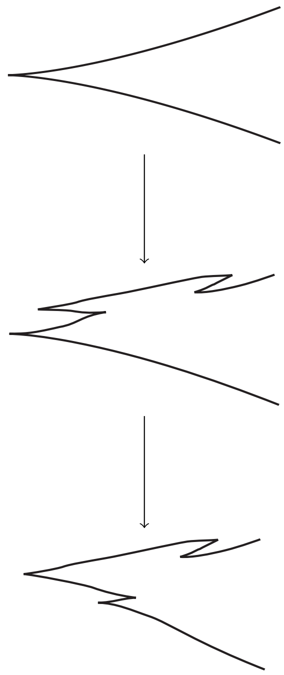


FIGURE 10. Approximating the slope in the x_2-direction of a cusp of a curve of intersection.

Again this procedure may also be done in reverse in order to achieve an approximation in the x_2-direction if we have cusps in the x_1-direction to spare.

Note however that both of the above procedures create four more cusps in the x_1-direction. Thus we need to show that we will only need to use finitely many of them in order to approximate the cylinder in the x_2-direction. The solution, again, uses the fact that the cylinder is compact. Given some $\varepsilon > 0$ and starting at one end of the cylinder, which we assume is already Legendrian and approximated in the x_1-direction, we can flow the front in the x_2-direction using the condition $\frac{\partial z}{\partial x_2} = y_2$ until any point of the cylinder is, say, $\frac{\varepsilon}{2}$ distance away from the original cylinder. This gives us an approximation of the interval $[a, \delta_1)$ of x_2-coordinates along the cylinder where a is the endpoint we started from. We then use either of the above procedures, depending on the typ of point, to change a neighbourhood of that point to get closer to the original cylinder. We may then continue flowing in the x_2-direction for some time until another point gets $\frac{\varepsilon}{2}$ distance away from the original cylinder. Continuing in this manner we reach the other endpoint $x_2 = b$ in finitely many steps due to the compactness of the cylinder. Thus we only need to use finitely many of the procedures above to approximate the entire cylinder. The approximation created in this way does not necessarily coincide with the already Legendrian part of the cylinder around $x_2 = b$ and hence we need to solve this problem as well. To do this we use the above procedure to flow to a point $x_2 = b - \delta$ for some $\delta > 0$ small enough such that the original cylinder at this point is already Legendrian. Doing each of the operations we have done while flowing across in reverse in some small enough distance in the x_2-direction we get back to the cylinder we had previously which was only approximated in the x_1-direction. From before this then connects nicely to the already Legendrian endpoint, giving us a Legendrian approximation of the entire cylinder which coincides with the already Legendrian endpoints.

4.4. **Global approximation.** Having constructed the local approximations of our surface we now consider the problem of turning these into a global approximation. With the setup we have used this is rather simple.

First, following Lemma 3.0.16, let us assume that we have a surface which is Morse in the x_2-direction. We then find all of the critical points of the x_2-coordinate and make Legendrian approximations of them according to their Morse index. Next we consider the cylindrical parts of the surface, all of which now have Legendrian ends coming from the Legendrian approximation at the critical points. We now cosntruct Legendrian approximations of these cylindrical parts as well as above. Since these are made to coincide with the already Legendrian endpoints we are now free to attach these cylinders back to their corresponding critical points, thus yielding a Legendrian approximation of the entire surface we started with.

Lastly we consider if the approximating Legendrian surface we have just created is isotopic to the original surface. To do this we check that none of the constructions used, such as the bifurcations of fronts, do not change the isotopy class of the surface.

We begin by considering the approximations at the critical points. It is immediately clear that the Legendrian approximation at a local minimum or maximum does not change the isotopy type, as was proven in a footnote following the construction. Similarly, it is clear that our Legendrian version of a saddle point is isotopic to the original saddle point by the same construction however we need to be a bit careful about how the complement of the saddle point is approximated. Choosing an approximating curve of the curve of intersection at the saddle point itself which is isotopic to the original curve of intersection guarantees that the entire approximation will be as well.

Next we consider the approximation for the cylinders. Here we need to make sure that the bifurcations and the Reidemeister moves used on the families of curves do not change the isotopy class of the curve itself. It is clear that the Reidemeister moves do not change the isotopy class since they do not even change the Legendrian isotopy class of the curves of intersection. To see that the bifurcations do not change the isotopy class either we note again that they can also be seen as using our Legendrian maxima/minima and saddle points. With this point of view it is easy to see that the process used to add cusps does not change the isotopy class of the surface.

Hence the Legendrian approximation we have created is also isotopic to the original surface.

5. Generalizations

Having dealt with the approximation problem for $\mathbb{R}^3$ and $\mathbb{R}^5$ one is naturally led to consider going further, even to the general case of $\mathbb{R}^{2n+1}$. However, there are obstacles in doing this stemming from the techniques we have used. While certain constructions lend themselves to generalization quite naturally, for example the local approximations of the critical points, other constructions prove more difficult to generalize. For instance the theory of generic 1-parameter families of Legendrian maps becomes difficult to work with for $n \geq 6$ owing to the nature of generic singularities of Legendrian submanifolds since for $n \geq 6$ the generic Legendrian maps are unstable and have moduli. To make matters worse, these moduli remain finite only for $n \leq 9$. For $n \geq 10$ the moduli become functional[10]. However we also saw that the needed bifurcations could also be modeled using the local models of critical points hence it might be possible to circumvent this problem.

As for the theory of oriented Legendrian cobordisms there are some results due to Eliashberg, see [8].

Theorem 5.0.1. *The group of oriented Legendrian cobordisms in $J^1(\mathbb{R}^n)$ is isomorphic to the stable homotopy group*

$$\lim_{k\to\infty} \pi_{n+k}(T\lambda_k)$$

where λ_k is the tautological bundle over the oriented Lagrangian Grassmannian and T denotes the Thom space.

To get the correct requirements for constructing such a cobordism, for example to connect the two endpoints of a cylinder, these would then have to be calculated and a complete invariant would have to be found for each of them. We have seen that the Maslov index acts as such a complete invariant for Legendrian cobordisms between knots and that the Maslov index is closely related to the classical invariant of Legendrain knots known as the rotation number. In [7] this classical invariant of knots is extended to an invariant of Legendrian submanifolds up to Legendrian isotopy. It thus seems natural to consider if this is also an invariant of Legendrian submanifolds up to cylindrical Legendrian cobordisms (alternatively exact Lagrangian cobordisms) and, if that is the case, if it is also a complete invariant.

One could also generalize to approximations of surfaces in general 5-dimensional contact manifolds. The methods developed here could then be used as a sort of local version. The groups of cylindrical Legendrian cobordisms in general contact manifolds are, of course, different than those of $(\mathbb{R}^5, \xi_0)$. For M a compact 2-dimensional manifold the groups of $J^1(M)$ are found in [2].

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