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Two Notions of Semantics of the Simple Theory of Types

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En populärvetenskaplig sammanfattning

Enkel typteori är ett formellt system för högre ordningens logik. Den version vi häri betraktar är den som studerades av Church och Henkin i [3] respektive [4]. I [7] studerade vi uppbyggnaden av formler i språket och det formella härledningssystemet, kortfattat uttryckt *syntaxen* för enkel typteori. Målet för denna magisteruppsats är att förtydliga och mer detaljerat redogöra för resultatet från Henkins artikel [4] som behandlar tolkningen av formler, det vill säga systemets *semantik*. Formler tolkas som objekt i *modeller*, och särskild vikt läggs vid de formler som däri tolkas som något av sanningsvärdena 0 (falskt) och 1 (sant), alltså påståenden. Två önskvärda egenskaper är då att allt som kan bevisas är sant i alla modeller (sundhet med avseende på semantiken) samt att allt som är sant i alla modeller också kan bevisas (fullständighet med avseende på semantiken). Vi definierar två varianter av semantik till enkel typteori, standardmodeller och generella modeller, och visar att båda är sunda, men att endast den senare är fullständig.

Alla formler i enkel typteori är tilldelade en *typ* (därav namnet), som avgör hur de får sammanfogas med andra formler. På samma sätt är modellerna (av båda slagen) indelade i *typdomäner*, och varje formel tolkas som ett objekt i typdomänen motsvarande formelns typ. Typerna är konstruerade på så sätt att det finns en typ ι för individer (de underliggande, primitiva objekten för vårt studium, e.g. naturliga tal), en typ o för påståenden (sanningsvärden), samt för alla par α, β av typer en typ $\langle\alpha\beta\rangle$ (eller $\alpha\beta$ kort och gott) för funktioner från typdomän β till typdomän α. Formler av typ $o\alpha$ kommer därvidlag att tolkas som (unära) predikat på (dvs. påståenden om) objekt av typ α, och kan således inte appliceras på objekt av någon annan typ (påståenden om individer kan exempelvis inte appliceras på andra påståenden). I standardmodeller utgör objekten av typ $\alpha\beta$ *alla* funktioner mellan β och α, medan generella modeller tillåter vilken mängd som helst av sådana funktioner, så länge den övergripande konstruktionen blir meningsfull och innehåller likhetspredikatet.

Formlernas uppbyggnad är enkel och elegant, inga termer finns som skymmer sikten: alla variabler x och konstanter c är formler; sammansättningen $[\varphi\psi]$ av en formel φ av typ $\alpha\beta$ med en formel ψ av typ β är en formel av typ α, och tolkas som det värde "funktionen" φ antar för input ψ; och λ-*abstraktionerna* $[\lambda x\varphi]$ av en formel φ är formler av typ $\alpha\beta$, där φ har typ α och x har typ β. Dessa senare tolkas, i både standard- och generella modeller, som den funktion som till ett objekt d av typ β tilldelar det objekt av typ α som erhålls om man tolkar φ med x utbytt mot d.

Vi kan exemplifiera skillnaden mellan standard- och generella modeller enligt följande: I en standardmodell kommer objekten av typ $o\iota$ att utgöra alla delmängder av individdomänen, medan så icke nödvändigtvis är fallet i en generell modell. Detta får till följd att en axiomatisering av de naturliga talen entydigt definierar dessa i standarmodeller. En följd av fullständighet med avseende på en semantik (givet vissa andra antaganden som här är uppfyllda) är dock att detta är omöjligt, då godtyckligt stora modeller av dessa kan skapas i en fullständig semantik. Alltså är standardsemantiken ofullständig.

Bevissystemet för enkel typteori har blott ett fåtal regler: vi får byta namn på bundna variabler (α-konvertering); byta ett utryck på formen $[[\lambda x\varphi]\psi]$ mot det uttryck som erhålls om alla x i φ byts mot ψ, och vice versa (β-konvertering); allkvantifiera en fri variabel i en formel (generalisering) eller ersätta densamma med vilken formel det vara må (substitution); samt från en implikation (e.g. $\varphi \to \psi$) och dess antecedent (φ) sluta oss till dess succedent (ψ) (modus ponens). Därutöver antas vissa formler som axiom (dessa är ur semantisk synvinkel rimliga, då de kommer att vara sanna i alla modeller). Dessutom får vi naturligtvis använda de utsagor som antagits som premisser för beviset.

Eftersom alla standardmodeller också är generella modeller räcker det att visa att en bevisbar formel är sann i alla generella modeller för att visa att båda typerna av semantik är sunda. Att alla axiom är sanna är en enkel sak att verifiera. Vidare tolkas formler likadant oavsett namn på de bundna variablerna, så α-konvertering bevarar sanning. Formler med (minst) en fri variabel anses vara sanna enbart om de är sanna för alla tolkningar av den fria variabeln i fråga, vilket är detsamma som att dess alltillslutning är sann; generalisering bevarar sanning. På liknande sätt följer att substitution bevarar sanning. Det är dessutom relativt enkelt att visa att en formell implikation tolkas som en (klassisk) implikation, varför även modus ponens bevarar sanning. Slutligen tolkas β-konvertering i princip som funktionsapplikation, vilket bevarar sanning.

Fullständighet bevisas till sist genom att en konsistent (motsägelsefri) mängd utsagor Γ utvidgas till en maximalkonsistent dito Δ (maximal m.a.p. $\subseteq$). Två slutna formler φ och ψ identifieras om likheten $\varphi \equiv \psi$ är bevisbar från Δ, och typdomäner konstrueras utifrån dessa för att precis motsvara tolkningen av sammansättning som funktionsapplikation, vilket ger en modell för Δ. Alltså har varje konsistent mängd utsagor en modell, så om en utsaga φ är sann i alla modeller till Γ, och $\Gamma \cup \{\neg\varphi\}$ därmed saknar modell, måste således den senare vara inkonsistent. Detta är möjligt enbart om φ kan bevisas från Γ.

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The simple theory of types, as formulated by Church, is considered from a semantic perspective. Two kinds of models are defined, standard and general, and the deductive system is proven sound with respect to both notions of semantics thus conceived. Proofs are given for the Model Existence Theorem and Completeness Theorem of general models, while the semantics of standard models is shown to be incomplete.

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Introduction

In a desire to better understand the foundations of mathematics, I, through the supervisor of my bachelor's thesis, came to consider the simple theory of types which, it could be argued, can fill this role. Said thesis ([7]), therefore, concerned the simple theory of types, principally as formulated by Church ([3]) but with a few alterations due to Henkin ([4]), though merely as a formal theory. However, due to time constraints, only the syntactic part of the theory was presented, though in great detail. Accordingly, this one-year master's thesis concerns the semantics of this theory, the results of which (most notably the Completeness (4.28) and Incompleteness (4.30) theorems) are primarily due to Henkin ([4]), though we will include a technical correction by Andrews ([1]).

The reader is encouraged to have access to a copy of [7], since a great many of the results and definitions therein are used, implicitly and explicitly, throughout this thesis. That being said, the most fundamental definitions and results thereof will be given in section 1. The other sections, meanwhile, contain the lion's share of the subject matter; in section 2 the basic definitions concerning models and interpretations, as well as some initial consequences thereof, are given, while section 3 contains the soundness theorem(s). Section 4, finally, treats the completeness and incompleteness results alluded to above. The thesis closes with an appendix containing a number of formal derivations, which would have made the, already slightly opaque, main text nearly unreadable had they been included.

For convenience, we will work in a standard mathematical setting, that is our meta theory will be ZFC. It would be of some interest to investigate whether the full strength of said theory is needed, since it seems plausible that we cannot do without neither choice nor replacement. Such an investigation, however, falls way outside the scope of this thesis.

Some Notational Remarks

For any $n, m \in \mathbb{N} \cup \{\omega\}$ we will use the following notations:

$$[n, m] = \{k \in \mathbb{N} \cup \{\omega\} \,|\, n \leq k \leq m\}$$
$$[n, m[= \{k \in \mathbb{N} \cup \{\omega\} \,|\, n \leq k < m\}$$
$$]n, m[= \{k \in \mathbb{N} \cup \{\omega\} \,|\, n < k < m\}$$
$$]n, m] = \{k \in \mathbb{N} \cup \{\omega\} \,|\, n < k \leq m\}.$$

We will also use the abbreviations $\neg$, $\wedge$, $\vee$, $\forall$, $\exists$, $\Rightarrow$, $\Leftarrow$ and $\Leftrightarrow$ in the meta language, corresponding to the intended meaning of the formal symbols $\neg$, $\wedge$, $\vee$, $\forall$, $\exists$, $\rightarrow$, $\leftarrow$ and $\leftrightarrow$ in the obvious and usual fashion. Furthermore we will use many of the usual conventions, which most readers will probably be familiar with, as well as the extra conventions that strings will be denoted by a, b, c, d, e, f, type symbols by lowercase Greek letters at the beginning of the alphabet, well formed formulae by lowercase Greek letters and assignments and sets of well formed formulae by uppercase Greek letters. Arbitrary variable symbols will be denoted by lowercase boldface letters towards the end of the alphabet, e.g. $\mathbf{x}, \mathbf{y}, \mathbf{z}$, while x, y, z are used to denote specific such (i.e. there is (for every type) a specific variable by the name x). Furthermore $\subset$ will mean strict and $\subseteq$ non-strict set containment. A statement like $a, b \in A$ (such as the one in the beginning of this section) is to be interpreted as $a \in A$ and $b \in A$. Concerning sequences and indexed families (finite or not), we will use the convention (as opposed to strings) of treating them as sets, that is we will write $a \in \{a_k\}_{k\in I} \subseteq A$ when we wish to convey that $a = a_k$ for some $k \in I$ and $a_k \in A$ for all $k \in I$. However, should the a_k come from different sets A_k, we will write $\{a_k\}_{k\in I} \in \prod_{k\in I} A_k$ as usual. For finite sequences (with a meta theoretically well-defined arity) we will not distinguish between sequence and tuple notation, e.g. $\{k\}_{k=1}^4 = (1, 2, 3, 4)$.

Some comments as to the notation concerning functions will also be in order. If A and B are sets, we will write $f : A \longrightarrow B$ to mean that f is a function from A to B; if in addition f happens to be injective we will denote this by $f : A \rightarrowtail B$. If f is only a partial function from A to B (that is, a function from some $D \subseteq A$ to B) we will write $f : A \rightharpoonup B$ instead. Finally, if $f : A \longrightarrow A$ is a function whose image is contained within its domain we will denote the iterations of f by

$$f^{n+1} = f \circ f^n$$

for all $n \in \mathbb{N}$.

Finally we will, for the benefit of readers acquainted with first order logic, use a terminology akin to the standard one in that area. Moreover, for convenience we will often refer to the simple theory of types as formulated herein (and in [3], [4] and [7]) by simply “type theory”, though this is usually taken to mean a much wider class of related theories.

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I would like to express my thanks to my supervisor Inger Sigstam, who accepted this role despite the numerous delays associated with my bachelor’s thesis.

1. Preliminaries: Syntax

For convenience we will here briefly recall the basics of the syntax of the simple theory of types, in the form that we shall use it. A (much) more thorough exposition of this subject is given in [7], which we shall refer to occasionally (some of the below definitions are in fact literally pasted from that work).

Formulae and Variables

The *type symbols* (or types) $\mathcal{T}$ are given by:

- $\mathsf{i}, \mathsf{o} \in \mathcal{T}$,
- $\forall \alpha, \beta \in \mathcal{T} : (\langle \alpha\beta \rangle \in \mathcal{T})$.

Here i and o are the types of individuals and propositions, respectively, whereas $\alpha\beta$ (short for $\langle\alpha\beta\rangle$) is the type of functions from objects of type β to those of type α. We will also have use for the shorthand notation $\alpha' = \langle\alpha\alpha\rangle\langle\alpha\alpha\rangle$, where $\alpha \in \mathcal{T}$.

The *primitive symbols* $\mathcal{S}$ of the language of type theory are:

- the *improper symbols* $\mathcal{K} = \{\lambda, [,]\}$;
- the *proper symbols* $\mathcal{B}$, consisting of
  - the *variables* $\mathcal{X}$,
  - the *logical constants* $\mathcal{L} = \{\mathsf{N}_{\mathsf{oo}}, \mathsf{A}_{\langle\mathsf{oo}\rangle\mathsf{o}}\} \cup \{\Pi_{\mathsf{o}\langle\mathsf{o}\alpha\rangle} \mid \alpha \in \mathcal{T}\} \cup \{\iota_{\alpha\langle\mathsf{o}\alpha\rangle} \mid \alpha \in \mathcal{T}\}$,
  - the (non-logical) *constant symbols* $\mathcal{C}$, an arbitrary set of symbols.

Here N is the formal negation, A the disjunction, $\Pi_{\mathsf{o}\langle\mathsf{o}\alpha}$ the universal quantifier and $\iota_{\alpha\langle\mathsf{o}\alpha\rangle}$ a choice (or description) operator, for the type α. As the subscripts of these might suggest, every proper symbol is assigned a type. This partitions the sets $\mathcal{X}$ and $\mathcal{C}$ into countable sets X_α and C_α, where $\alpha \in \mathcal{T}$. We postulate that $m_\alpha, n_\alpha, p_\alpha, q_\alpha, x_\alpha, y_\alpha, z_\alpha \in \mathsf{X}_\alpha$ denote distinct variables, for every $\alpha \in \mathcal{T}$.

The well-formed formulae W are constructed inductively from the primitive symbols:

- Proper symbols are well-formed formulae of the same type as the symbol.
- If $\varphi \in W$ has type α and $\boldsymbol{x} \in \mathcal{X}$ has type β, then $[\lambda \boldsymbol{x} \varphi] \in W$ and has type $\alpha\beta$.
- If $\varphi \in W$ has type $\alpha\beta$ and $\psi \in W$ has type β, then $[\varphi\psi] \in W$ and has type α.

In particular every well-formed formula has a type, given by the function $\mathsf{T} : W \longrightarrow \mathcal{T}$. That being said we will, in order to facilitate reading, often indicate the type of a formula by an index, as in N_{oo}. However, if the type is irrelevant, we will often simply omit it. We denote the set of all well formed formulae of type α by W_α.

The following abbreviations, where $\varphi, \psi \in W$ and $\boldsymbol{x} \in \mathcal{X}$, will ease the reading and interpretation of formulae:

- $(\neg\varphi_\mathsf{o}) = [\mathsf{N}_{\mathsf{oo}}\varphi_\mathsf{o}]$.

- $(\varphi_o \vee \psi_o) = [[A_{\langle oo\rangle o}\varphi_o]\psi_o]$.
- $(\varphi_o \wedge \psi_o) = (\neg((\neg\varphi_o) \vee (\neg\psi_o)))$.
- $(\varphi_o \to \psi_o) = ((\neg\varphi_o) \vee \psi_o)$.
- $(\varphi_o \leftrightarrow \psi_o) = ((\varphi_o \to \psi_o) \wedge (\psi_o \to \varphi_o))$.
- $(\forall \boldsymbol{x}_\alpha \varphi_o) = [\Pi_{o\langle o\alpha\rangle}[\lambda \boldsymbol{x}_\alpha \varphi_o]]$.
- $(\exists \boldsymbol{x}_\alpha \varphi_o) = (\neg(\forall \boldsymbol{x}_\alpha(\neg\varphi_o)))$.
- $(\imath \boldsymbol{x}_\alpha \varphi_o) = [\iota_{\alpha\langle o\alpha\rangle}[\lambda \boldsymbol{x}_\alpha \varphi_o]]$.
- $Q_{\langle o\alpha\rangle\alpha} = [\lambda x_\alpha[\lambda y_\alpha(\forall f_{o\alpha}([f_{o\alpha}x_\alpha] \to [f_{o\alpha}y_\alpha]))]]$.
- $(\varphi_\alpha \equiv \psi_\alpha) = [[Q_{\langle o\alpha\rangle\alpha}\varphi_\alpha]\psi_\alpha]$.
- $(\varphi_\alpha \not\equiv \psi_\alpha) = (\neg(\varphi_\alpha \equiv \psi_\alpha))$.
- $\bot = (\forall p_o p_o)$.
- $\top = (\neg\bot)$.
- $\mathrm{Id}_{\alpha\alpha} = [\lambda x_\alpha x_\alpha]$.
- $\pi_{\langle\alpha\beta\rangle\alpha} = [\lambda x_\alpha[\lambda y_\beta x_\alpha]]$.
- $0_{\alpha'} = [\lambda f_{\alpha\alpha}\,\mathrm{Id}_{\alpha\alpha}]$.
- $1_{\alpha'} = [\lambda f_{\alpha\alpha}[\lambda x_\alpha[f_{\alpha\alpha}x_\alpha]]]$.
- $2_{\alpha'} = [\lambda f_{\alpha\alpha}[\lambda x_\alpha[f_{\alpha\alpha}[f_{\alpha\alpha}x_\alpha]]]]$.
- $S_{\alpha'\alpha'} = [\lambda n_{\alpha'}[\lambda f_{\alpha\alpha}[\lambda x_\alpha[f_{\alpha\alpha}[[n_{\alpha'}f_{\alpha\alpha}]x_\alpha]]]]]$.
- $N_{o\alpha'} = [\lambda n_{\alpha'}(\forall f_{o\alpha'}([f_{o\alpha'}0_{\alpha'}] \to ((\forall x_{\alpha'}([f_{o\alpha'}x_{\alpha'}] \to [f_{o\alpha'}[S_{\alpha'\alpha'}x_{\alpha'}]])) \to [f_{o\alpha'}n_{\alpha'}])))]$.

These will be subject to the aforementioned convention of omitting types, when possible.

We define the sets $\mathsf{VF}(\varphi)$ and $\mathsf{VB}(\varphi)$ of free and bound variables of φ, respectively, by induction on φ as follows

- For every $s \in \mathcal{B}$,

$$\mathsf{VF}(s) = \begin{cases} \{s\} & \text{if } s \in \mathcal{X} \\ \varnothing & \text{otherwise} \end{cases} \qquad \text{and} \qquad \mathsf{VB}(s) = \varnothing.$$

- For every $\psi \in W$ and $\boldsymbol{x} \in \mathcal{X}$

$$\mathsf{VF}([\lambda \boldsymbol{x}\nu]) = \mathsf{VF}(\nu) \setminus \{\boldsymbol{x}\}$$

and

$$\mathsf{VB}([\lambda \boldsymbol{x}\nu]) = \mathsf{VB}(\nu) \cup \{\boldsymbol{x}\}.$$

- For all $\alpha, \beta \in \mathcal{T}$ and $\varphi_{\alpha\beta}, \psi_\beta \in W$

$$\mathsf{VF}([vw]) = \mathsf{VF}(v) \cup \mathsf{VF}(w)$$

and

$$\mathsf{VB}([vw]) = \mathsf{VB}(v) \cup \mathsf{VB}(w).$$

Thus, in short, variables are bound by λ. The operators $\mathrm{SF}(\mathbf{x})(\vartheta)$ and $\mathrm{SB}(\mathbf{x})(\mathbf{y})$ of substitution of free occurrences of $\mathbf{x} \in \mathcal{X}$ by $\vartheta \in W$ and bound occurrences of $\mathbf{x} \in \mathcal{X}$ by $\mathbf{y} \in \mathcal{X}$, respectively, are similarly defined, as is the operator $\mathrm{S}(\mathbf{x})(\vartheta)$, substitution of *all* occurrences of $\mathbf{x} \in \mathcal{X}$ by $\vartheta \in W$ (their composition). Worthy of note is also the notation $\overline{W} = \{\varphi \in W \mid \mathsf{VF}(\varphi) = \varnothing\}$ for the set of *closed* well-formed formulae, which we will partition by typing using indices, as before.

Formal Deduction

The deductive system of type theory is made up by *axioms* and *rules of inference*.

1.1 Definition (Rules of inference)**.** The rules of inference are the following:

α-conversion Let $\alpha, \beta \in \mathcal{T}$ and $\mathbf{x}, \mathbf{y} \in X_\alpha$. For any well-formed formulae φ_β, ψ_o such that $a\varphi b = \psi$ for some $a, b \in \mathcal{S}^*$ where $\mathbf{x}$ is not free in φ and $\mathbf{y}$ does not occur in φ, we may from ψ infer $a\,\mathrm{S}(\mathbf{x})(\mathbf{y})(\varphi)b$.

β-contraction Let $\alpha, \beta \in \mathcal{T}$. For any $\mathbf{x} \in X_\alpha$, any well-formed formulae φ_β, ψ_α and η_o such that neither $\mathbf{x}$ nor any free variable of ψ is a bound variable of φ, and $a[[\lambda\mathbf{x}\varphi]\psi]b = \eta$ for some $a, b \in \mathcal{S}^*$, we may from η infer $a\,\mathrm{S}(\mathbf{x})(\psi)(\varphi)b$.

β-expansion From any well-formed formula φ_o we may infer ψ_o, where ψ is any well-formed formula from which φ could be inferred by β-contraction.

Substitution Let $\alpha \in \mathcal{T}$. For any well-formed formulae $\varphi_{o\alpha}, \psi_\alpha$ and any variable $\mathbf{x}_\alpha$ such that $\mathbf{x}$ is not free in φ, we may from $[\varphi\mathbf{x}]$ infer $[\varphi\psi]$

Modus ponens For any well-formed formulae $\varphi_o, \psi_o \in W$ we may from $(\varphi \rightarrow \psi)$ and φ infer ψ.

Generalisation Let $\alpha \in \mathcal{T}$. For any well-formed formula $\varphi_{o\alpha}$ and any variable $\mathbf{x}_\alpha$ such that $\mathbf{x}$ is not free in φ, we may from $[\varphi\mathbf{x}]$ infer $[\Pi_{o\langle o\alpha\rangle}\varphi_{o\alpha}]$.

1.2 Definition (Axioms)**.** The following are the axioms of simple type theory:

1. $((p_o \vee p_o) \rightarrow p_o)$.
2. $(p_o \rightarrow (p_o \vee q_o))$.
3. $((p_o \vee q_o) \rightarrow (q_o \vee p_o))$.

4. $((p_o \to q_o) \to ((r_o \vee p_o) \to (r_o \vee q_o)))$.

5. For all $\alpha \in \mathcal{T}$: $([\Pi_{o\langle o\alpha\rangle} f_{o\alpha}] \to [f_{o\alpha} x_\alpha])$.

6. For all $\alpha \in \mathcal{T}$: $((\forall x_\alpha (p_o \vee [f_{o\alpha} x_\alpha])) \to (p_o \vee [\Pi_{o\langle o\alpha\rangle} f_{o\alpha}]))$.

7. $(\exists x_i (\exists y_i (x_i \not\equiv y_i)))$.

8. $([N_{oi'} x_{i'}] \to ([N_{oi'} y_{i'}] \to (([S_{i'i'} x_{i'}] \equiv [S_{i'i'} y_{i'}]) \to (x_{i'} \equiv y_{i'}))))$.

9. For all $\alpha \in \mathcal{T}$: $([f_{o\alpha} x_\alpha] \to ((\forall y_\alpha ([f_{o\alpha} y_\alpha] \to (x_\alpha \equiv y_\alpha))) \to [f_{o\alpha} [\iota_{\alpha\langle o\alpha\rangle} f_{o\alpha}]]))$.

10. For all $\alpha, \beta \in \mathcal{T}$: $((\forall x_\beta ([f_{\alpha\beta} x_\beta] \equiv [g_{\alpha\beta} x_\beta])) \to (f_{\alpha\beta} \equiv g_{\alpha\beta}))$.

10^o. $((x_o \leftrightarrow y_o) \to (x_o \equiv y_o))$.

11. For all $\alpha \in \mathcal{T}$: $([f_{o\alpha} x_\alpha] \to [f_{o\alpha} [\iota_{\alpha\langle o\alpha\rangle} f_{o\alpha}]])$.

We will not consider all of the above axioms as integral to our theory, but will mainly be concerned with 1–6 and 10–11. The rest are presented mainly for completeness.

1.3 Definition (Formal Proof)**.** Let $\Gamma \subseteq \overline{W}_o$, $\varphi \in W_o$ and $n \in \mathbb{N}$. A *formal proof* of length $n+1$ of φ on the assumptions Γ is a string $\mathfrak{P} \in (W_o)^{n+1}$ such that $\mathfrak{P}(n) = \varphi$, where for every $0 \leq k \leq n$, $\mathfrak{P}(k)$ is either a formal axiom, a formula of Γ, or can be inferred from $\{\mathfrak{P}(l) \,|\, 0 \leq 1 < k\}$. We write

$$\Gamma \vdash \varphi$$

when there is a formal proof (of any length) of φ on the assumptions Γ. If $\varnothing \vdash \varphi$ we call φ a *formal theorem* and write

$$\vdash \varphi$$

1.4 Theorem (The Deduction Theorem)**.** *In a simple theory of types with (at least) axioms 1–6, we have for every* $\Gamma \subseteq \overline{W}_o$, $\varphi \in \overline{W}_o$ *and* $\psi \in W_o$ *that*

$$\Gamma \cup \{\varphi\} \vdash \psi \Rightarrow \Gamma \vdash (\varphi \to \psi).$$

2. Models

As explained earlier, we will henceforth consider the simple theory of types with axioms 1–6, 10, 10^o and 11. The following definitions and results are due to Henkin [4]. However, since his exposition is quite brief, while brevity is not the aim of this thesis, we will introduce some minor auxiliary concepts. We begin by defining the structures forming the backbone of our models, and proceed by defining the two different classes of models as certain subclasses thereof. Finally we fix the notion of truth (in models), and show the validity of the axioms.

Frames

In order to systematically interpret well-formed formulae, we would need a set of each type in which to do so for the formulae of that type. We also need to have interpretations of the constants (logical or not) under consideration. The following definition makes these ideas precise.

2.1 Definition. By a *frame* we will mean a sequence $\mathfrak{D} = \{D_\alpha\}_{\alpha \in \mathcal{T}}$ of sets. Given $\alpha \in \mathcal{T}$ we will call D_α the *domain* (of $\mathfrak{D}$) of type α. A *structured frame* is a tuple $\mathfrak{M} = (\mathfrak{D}, \mathfrak{C})$ consisting of a frame $\mathfrak{D} = \{D_\alpha\}_{\alpha \in \mathcal{T}}$ and a sequence $\mathfrak{C} = \{k_c\}_{c \in \mathcal{L} \cup \mathcal{C}}$ such that

- $D_o = \{0, 1\}$
- For all $\alpha, \beta \in \mathcal{T}$, $D_{\alpha\beta} \subseteq D_\alpha^{D_\beta}$
- For every $\alpha \in \mathcal{T}$ and every $c_\alpha \in \mathcal{L} \cup \mathcal{C}$, $k_c \in D_\alpha$
- $k_N \in D_{oo}$ is such that

  $$k_N(n) = 1 - n$$

  for all $n \in \{0, 1\}$.
- $k_A : D_{\langle oo \rangle o}$ is such that

  $$k_A(n)(m) = 1 - (1 - n) \cdot (1 - m)$$

  for all $n, m \in \{0, 1\}$.[1]
- For all $\alpha \in \mathcal{T}$, $k_{\Pi_{o\langle o\alpha \rangle}} \in D_{o\langle o\alpha \rangle}$ is such that

  $$k_{\Pi_{o\langle o\alpha\rangle}}(f) = \begin{cases} 1 & \text{if } f(d) = 1 \text{ for all } d \in D_\alpha \\ 0 & \text{otherwise} \end{cases}$$

  for all $f \in D_{o\alpha}$.

[1] Thus D_{oo} contains at least the identity and constantly affirmative (if 1 and 0 are regarded as true and false, respectively, which is intended) function.

- For all $\alpha \in \mathcal{T}$, $k_{\iota_{\alpha\langle o\alpha\rangle}} \in D_{\alpha\langle o\alpha\rangle}$ is such that

$$f(k_{\iota_{\alpha\langle o\alpha\rangle}}(f)) = 1$$

for every $f \in D_{o\alpha}$ for which there is a $d \in D_\alpha$ such that $f(d) = 1$.[2]

We will call a frame or structured frame *degenerate* if $D_\alpha = \varnothing$ for some $\alpha \in \mathcal{T}$.

Considered as structures where we wish to interpret the well-formed formulae, however, non-degenerate structured frames does not really suffice, since we are not guaranteed that the interpretations of all well-formed formulae are contained in the respective type domain. This issue will be treated in the next subsection. Later, we will show that it is in fact enough to require that all *closed* well-formed formulae has an interpretation (4.21).

In [7] a general version of the following theorem was exhibited.

2.2 Theorem (Type-recursion)**.** *For every nonempty set* M*, every* $a, b \in M$ *and every function* $h : M \times M \longrightarrow M$*, there is a unique function* $f : \mathcal{T} \longrightarrow M$ *satisfying*

- $f(o) = a$,
- $f(i) = b$,
- $f(\alpha\beta) = h((f(\alpha), f(\beta)))$,

for all $\alpha, \beta \in \mathcal{T}$.

Indeed, we have an even stronger recursion property, which we will need to construct frames.

2.3 Theorem (Grounded Type-recursion)**.** *Let* H *be a (class) function and* a, b *be arbitrary (sets). There is a unique set* M *and function* $f : \mathcal{T} \longrightarrow M$ *such that* f *is surjective and*

- $f(o) = a$,
- $f(i) = b$,
- $f(\alpha\beta) = H((f(\alpha), f(\beta)))$,

for all $\alpha, \beta \in \mathcal{T}$.

Proof. Define $L : \mathcal{T} \longrightarrow \mathcal{P}(\mathcal{T})$ by recursion (2.2) as follows:

[2]Henkin defines the interpretations of the symbols in $\mathcal{L}$ directly when defining valuations (see below), where he proclaims $V(\Phi)(\iota_{\alpha\langle o\alpha\rangle})$ to be "some fixed function whose value for any argument f of $D_{o\alpha}$ is one of the elements of D_α mapped into T by f" (where T denotes the truth-value "Truth", which is 1 in our definition). Depending on how you interpret "fixed" (i.e. whether the interpretation is fixed with respect to Φ or not), the valuations described may or may not be uniquely defined. Therefore, we have chosen the corresponding definition from Andrews [2] for his essentially equivalent system, which corresponds to interpreting "fixed" as "uniquely determined by the structure".

- $L(o) = \{o\}$,
- $L(i) = \{i\}$,
- $L(\alpha\beta) = L(\alpha) \cup L(\beta)$,

for all $\alpha, \beta \in \mathcal{T}$.

Sublemma 2.3.1. $L(\alpha)$ *is finite for every* $\alpha \in \mathcal{T}$.

Proof. By induction.

- $L(o)$ and $L(i)$ are singletons, and thus finite.
- Assume $\gamma, \delta \in \mathcal{T}$ are such that $L(\gamma)$ and $L(\delta)$ are finite. Then $L(\gamma\delta) = L(\gamma) \cup L(\delta)$ is finite as well.

Hence $L(\alpha)$ is finite for every $\alpha \in \mathcal{T}$. □

Let $R = \{(\alpha, \beta) \in \mathcal{T}^2 \mid \alpha \in L(\beta)\}$, so that $L(\beta) = \{\alpha \in \mathcal{T} \mid \alpha R \beta\}$ for every $\beta \in \mathcal{T}$.

Sublemma 2.3.2. R *is wellfounded, i.e. every nonempty* $A \subseteq \mathcal{T}$ *has an* R*-minimal element, that is an element* $\alpha \in A$ *such that for every* $\beta \in A$ *for which* $\beta R \alpha$, $\beta = \alpha$.

Proof. Take $\gamma \in A$. Then $L(\gamma) \cap A \neq \varnothing$ but finite, whence there is an R-minimal $\alpha \in L(\gamma) \cap A$. Now if $\beta \in A$ is such that $\beta R \alpha$, then $\beta \in L(\gamma) \cap A$ and thus $\beta = \alpha$. Hence α is R-minimal in A. □

Sublemma 2.3.3. *There are functions* $p_1 : \mathcal{T} \setminus \{o, i\} \longrightarrow \mathcal{T}$ *and* $p_2 : \mathcal{T} \setminus \{o, i\} \longrightarrow \mathcal{T}$ *such that*

$$\gamma = \langle p_1(\gamma) p_2(\gamma) \rangle$$

for all $\gamma \in \mathcal{T} \setminus \{o, i\}$.

Proof. Let $\pi_1, \pi_2 : \mathcal{T}^2 \longrightarrow \mathcal{T}$ be the standard projections (i.e. $q = (\pi_1(q), \pi_2(q))$ for all $q \in \mathcal{T}^2$). Define $p : \mathcal{T} \longrightarrow \mathcal{T}^2$ by recursion as

- $p(o) = (o, o)$.
- $p(i) = (i, i)$. (These cases are highly irrelevant.)
- For all $\alpha, \beta \in \mathcal{T}$, $p(\alpha\beta) = (\alpha, \beta)$.

Now $p_1 = \pi_1 \circ p \restriction (\mathcal{T} \setminus \{o, i\})$ and $p_2 = \pi_2 \circ p \restriction (\mathcal{T} \setminus \{o, i\})$ has the desired properties. □

Let

$$F((A, B)) = \begin{cases} H(B(p_1(A)), B(p_2(A))) & \text{if } A \in \mathcal{T} \setminus \{o, i\} \text{ and} \\ & B \text{ is a function with } L(A) \subseteq \mathrm{dom}(B) \\ a & \text{if } A = o \\ b & \text{if } A = i \\ \varnothing & \text{otherwise} \end{cases}$$

so that F is a (class) function. By well-founded recursion (see for example [5]), there is a unique (class) function G on $\mathcal{T}$ such that

$$G(\alpha) = F((\alpha, G \restriction L(\alpha)))$$

for all $\alpha \in \mathcal{T}$. By the axiom of replacement $M = G[\mathcal{T}]$ is a set. Let $f : \mathcal{T} \longrightarrow M$ be defined by $f(\alpha) = G(\alpha)$ for all $\alpha \in \mathcal{T}$. Then

$$\begin{aligned} f(o) &= F((o, f \restriction \varnothing)) = a \\ f(i) &= F((i, f \restriction \varnothing)) = b \\ f(\alpha\beta) &= F((\alpha\beta, f \restriction L(\alpha\beta))) \\ &= H((f \restriction L(\alpha\beta)(p_1(\alpha\beta)), f \restriction L(\alpha\beta)(p_2(\alpha\beta)))) \\ &= H((f(\alpha), f(\beta))) \end{aligned}$$

as desired.

Now if g were any other such function, then $g(\alpha) = F((\alpha, g \restriction L(\alpha)))$, whence $g(\alpha) = G(\alpha) = f(\alpha)$, for all $\alpha \in \mathcal{T}$, by uniqueness of G. Thus $g[\mathcal{T}] = f[\mathcal{T}] = M$, whereby f and M above are unique. □

Remark 1. Given a frame $\mathfrak{D} = \{D_\alpha\}_{\alpha \in \mathcal{T}}$, since the set D_α is uniquely determined for every $\alpha \in \mathcal{T}$, the axiom of replacement also gives that $\{D_\alpha \mid \alpha \in \mathcal{T}\}$ is a set, whereby $\bigcup_{\alpha \in \mathcal{T}} D_\alpha$ is a set as well.

Thus, given a structured frame we can consider functions into its "total universe", so to speak. This is exactly what we need to define the interpretation, or evaluation, of formulae. Our first step along that road is the interpretation of variables, which we consider next.

2.4 Definition (Assignment)**.** Let $\mathfrak{M} = (\{D_\alpha\}_{\alpha \in \mathcal{T}}, \{k_c\}_{c \in \mathcal{L} \cup \mathcal{C}})$ be a structured frame. An *assignment* with respect to $\mathfrak{M}$ is a function $\Phi : \mathcal{X} \longrightarrow \bigcup_{\alpha \in \mathcal{T}} D_\alpha$ such that $\Phi(x_\alpha) \in D_\alpha$ for every $\alpha \in \mathcal{T}$. We denote by $\mathcal{A}_\mathfrak{M}$ the set of all assignments with respect to $\mathfrak{M}$.

Remark 2. Since $X_\alpha \neq \varnothing$ for all $\alpha \in \mathcal{T}$, $\mathfrak{M}$ is a non-degenerate structured frame if and only if $\mathcal{A}_\mathfrak{M} \neq \varnothing$. Moreover, if $\mathfrak{M} = (\mathfrak{D}, \mathfrak{C})$ and $\mathfrak{N} = (\mathfrak{E}, \mathfrak{B})$ are structured frames with the same underlying frame, then $A_\mathfrak{M} = A_\mathfrak{N}$.

2.5 Definition. Let $\mathfrak{M} = (\{D_\alpha\}_{\alpha\in\mathcal{T}}, \{k_c\}_{c\in\mathcal{L}\cup\mathcal{C}})$ be a non-degenerate structured frame, $\Phi \in \mathcal{A}_{\mathfrak{M}}$, $\alpha \in \mathcal{T}$, $d \in D_\alpha$ and $\mathbf{x} \in X_\alpha$. The assignment $\Phi^{\mathbf{x}}_{d} \in \mathcal{A}_{\mathfrak{M}}$ is defined via

$$\Phi^{\mathbf{x}}_{d}(\mathbf{y}) = \begin{cases} d & \text{if } \mathbf{y} = \mathbf{x} \\ \Phi(\mathbf{y}) & \text{otherwise} \end{cases}$$

for all $y \in \mathcal{X}$.

The following easy lemma will be of use later.

2.6 Lemma. *For all non-degenerate structured frames* $\mathfrak{M} = (\{D_\alpha\}_{\alpha\in\mathcal{T}}, \{k_c\}_{c\in\mathcal{L}\cup\mathcal{C}})$, $\Phi \in A_{\mathfrak{M}}$, $\alpha, \beta \in \mathcal{T}$, $d \in D_\alpha$, $e \in D_\beta$ *and distinct* $\mathbf{x}_\alpha, \mathbf{y}_\beta \in \mathcal{X}$*:*

1. $\Phi^{\mathbf{x}\,\mathbf{x}}_{d\,e} = \Phi^{\mathbf{x}}_{e}$.
2. $\Phi^{\mathbf{x}\,\mathbf{y}}_{d\,e} = \Phi^{\mathbf{y}\,\mathbf{x}}_{e\,d}$.

Proof. 1. Clearly

$$\begin{aligned} \Phi^{\mathbf{x}\,\mathbf{x}}_{d\,e}(\boldsymbol{z}) &= \begin{cases} e & \text{if } \boldsymbol{z} = \mathbf{x} \\ \Phi^{\mathbf{x}}_{d}(\boldsymbol{z}) & \text{otherwise} \end{cases} \\ &= \begin{cases} e & \text{if } \boldsymbol{z} = \mathbf{x} \\ \Phi(\boldsymbol{z}) & \text{otherwise} \end{cases} \\ &= \Phi^{\mathbf{x}}_{e}(\boldsymbol{z}) \end{aligned}$$

for all $\boldsymbol{z} \in \mathcal{X}$.

2. Similarly, since $\mathbf{x} \neq \mathbf{y}$,

$$\begin{aligned} \Phi^{\mathbf{x}\,\mathbf{y}}_{d\,e}(\boldsymbol{z}) &= \begin{cases} e & \text{if } \boldsymbol{z} = \mathbf{y} \\ \Phi^{\mathbf{x}}_{d}(\boldsymbol{z}) & \text{otherwise} \end{cases} \\ &= \begin{cases} e & \text{if } \boldsymbol{z} = \mathbf{y} \\ d & \text{if } \boldsymbol{z} = \mathbf{x} \\ \Phi(\boldsymbol{z}) & \text{otherwise} \end{cases} \\ &= \begin{cases} d & \text{if } \boldsymbol{z} = \mathbf{x} \\ \Phi^{\mathbf{y}}_{e}(\boldsymbol{z}) & \text{otherwise} \end{cases} \\ &= \Phi^{\mathbf{y}\,\mathbf{x}}_{e\,d}(\boldsymbol{z}) \end{aligned}$$

for all $\boldsymbol{z} \in \mathcal{X}$.

□

Having thus specified how to assign a meaning to variables, we can extend this to all formulae inductively. Recall, however, that we are not guaranteed that a valuation as below exist even in every non-degenerate structured frame. In the next section we will therefore limit our attention to those frames where this is possible.

2.7 Definition (Valuation)**.** Let $\mathfrak{M} = (\{D_\alpha\}_{\alpha \in \mathcal{T}}, \{k_c\}_{c \in \mathcal{L} \cup \mathcal{C}})$ be a non-degenerate structured frame. A *valuation for* $\mathfrak{M}$ is a function $V : \mathcal{A}_\mathfrak{M} \longrightarrow (\bigcup_{\alpha \in \mathcal{T}} D_\alpha)^W$ which has the following properties:

- For every $\Phi \in \mathcal{A}_\mathfrak{M}$, $\alpha \in \mathcal{T}$ and every $\varphi \in W_\alpha$ we have that $V(\Phi)(\varphi) \in D_\alpha$.
- For every $\Phi \in \mathcal{A}_\mathfrak{M}$ and every $\mathbf{x} \in \mathcal{X}$, we have that $V(\Phi)(\mathbf{x}) = \Phi(\mathbf{x})$.
- For every $\Phi \in \mathcal{A}_\mathfrak{M}$ and every $c \in \mathcal{L} \cup \mathcal{C}$, we have that $V(\Phi)(c) = k_c$.
- For every $\Phi \in \mathcal{A}_\mathfrak{M}$ and all $\alpha, \beta \in \mathcal{T}$, $\mathbf{x} \in X_\beta$ and $\varphi_\alpha \in W$, $V(\Phi)([\lambda \mathbf{x} \varphi]) \in D_{\alpha\beta}$ is such that

  $$V(\Phi)([\lambda \mathbf{x} \varphi])(d) = V(\Phi^{\mathbf{x}}_d)(\varphi)$$

  for all $d \in D_\beta$.
- For every $\Phi \in \mathcal{A}_\mathfrak{M}$ and all $\alpha, \beta \in \mathcal{T}$ and $\varphi_{\alpha\beta}, \psi_\beta \in W$, we have that

  $$V(\Phi)([\varphi_{\alpha\beta}\psi_\beta]) = V(\Phi)(\varphi_{\alpha\beta})(V(\Phi)(\psi_\beta)).$$

Given a valuation V for $\mathfrak{M}$, for each $\varphi \in W$ and $\Phi \in A_\mathfrak{M}$ we define the *evaluation of* φ *with respect to* Φ as the object

$$V(\Phi)(\varphi) \in D_{T(\varphi)}.$$

General and Standard Models

With a method of evaluating formulae now at our disposal, we will define the more general class of structures (whence the name) which will constitute models for our system.

2.8 Definition (General model)**.** A *general model* is a non-degenerate structured frame $\mathfrak{M} = (\{D_\alpha\}_{\alpha \in \mathcal{T}}, \{k_c\}_{c \in \mathcal{L} \cup \mathcal{C}})$ with the following properties:

- For every $\alpha \in \mathcal{T}$ there is a function $q \in D_{\langle o\alpha \rangle \alpha}$ such that

  $$q(a)(b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

  for all $a, b \in D_\alpha$[3].
- There is a valuation function for $\mathfrak{M}$.

2.9 Lemma. *For every general model* $\mathfrak{M}$ *there is a unique valuation function.*

[3]The existence of such a function is necessary to ensure that the formal equality is interpreted as the meta theoretical equality, which Andrews [1] pointed out (and proved). Without this requirement we will have models in which extensionally equal functions are considered distinct.

Proof. Let $\mathfrak{M} = (\{D_\alpha\}_{\alpha\in\mathcal{T}}, \{k_c\}_{c\in\mathcal{L}\cup\mathcal{C}})$ be a general model. We know there is at least one valuation function by definition, so it suffices to show that if V and V' are valuation functions, then $V = V'$. This follows from the fact that given $\varphi \in W$, $V(\Phi)(\varphi) = V'(\Phi)(\varphi)$ for all $\Phi \in A_{\mathfrak{M}}$, which will be proved by induction on φ.

- Consider $s \in \mathcal{B}$. Since $\mathcal{B} = \mathcal{L} \cup \mathcal{X} \cup \mathcal{C}$ we see that

$$\begin{aligned} V(\Phi)(s) &= \begin{cases} \Phi(s) & \text{if } s \in \mathcal{X} \\ k_s & \text{if } s \in \mathcal{L} \cup \mathcal{C} \end{cases} \\ &= V'(\Phi)(s) \end{aligned}$$

  for all $\Phi \in A_{\mathfrak{M}}$.

- Now consider $[\lambda \mathbf{x}_\alpha \psi_\beta]$, where $\alpha, \beta \in \mathcal{T}$, $\mathbf{x} \in X_\alpha$ and $\psi \in W_\beta$ is such that

$$V(\Phi)(\psi) = V'(\Phi)(\psi)$$

  for all $\Phi \in A_{\mathfrak{M}}$. Then, given such a $\Phi \in A_{\mathfrak{M}}$, we see that

$$\begin{aligned} V(\Phi)([\lambda \mathbf{x} \psi])(d) &= V(\Phi^{\mathbf{x}}_d)(\psi) \\ &= V'(\Phi^{\mathbf{x}}_d)(\psi) \\ &= V'(\Phi)([\lambda \mathbf{x} \psi])(d) \end{aligned}$$

  for all $d \in D_\alpha$. Thus $V(\Phi)([\lambda \mathbf{x}\psi]) = V'(\Phi)([\lambda \mathbf{x}\psi])$ for all $\Phi \in A_{\mathfrak{M}}$.

- Finally consider $[\psi_{\alpha\beta}\vartheta_\beta] \in W$, where $\alpha, \beta \in \mathcal{T}$ and $\psi, \vartheta \in W$ are such that

$$V(\Phi)(\eta) = V'(\Phi)(\eta)$$

  for all $\Phi \in A_{\mathfrak{M}}$ and both $\eta \in \{\psi, \vartheta\}$. Then

$$\begin{aligned} V(\Phi)([\psi\vartheta]) &= V(\Phi)(\psi)V(\Phi)(\vartheta) \\ &= V'(\Phi)(\psi)V'(\Phi)(\vartheta) \\ &= V'(\Phi)([\psi\vartheta]) \end{aligned}$$

  for all $\Phi \in A_{\mathfrak{M}}$.

Hence $V(\Phi)(\varphi) = V'(\Phi)(\varphi)$ for all $\Phi \in A_{\mathfrak{M}}$ as desired. □

Thus we will henceforth when speaking about "the valuation" in the context of a general model mean the unique valuation for that model. Unless stated otherwise, it will always be denoted by V.

Having defined a way of interpreting formulae, we can work our way towards the definition of truth in the next subsection. Along the way we will make a detour to single out the other class of models we will consider, namely standard models (2.12).

2.10 Lemma. *Let* $\mathfrak{M} = (\{\mathsf{D}_\alpha\}_{\alpha\in\mathcal{T}}, \{\mathsf{k}_c\}_{c\in\mathcal{L}\cup\mathcal{C}})$ *be a general model and* $\varphi \in W$. *If* $\Phi, \Psi \in \mathcal{A}_{\mathfrak{M}}$ *are such that* $\Phi \restriction \mathsf{VF}(\varphi) = \Psi \restriction \mathsf{VF}(\varphi)$, *then* $\mathsf{V}(\Phi)(\varphi) = \mathsf{V}(\Psi)(\varphi)$. *In particular, the evaluation of a closed well-formed formula is independent of the assignment.*

Proof. By induction on $\varphi \in W$.

- Let $s \in \mathcal{B}$. If $s \in \mathcal{X}$ then $\mathsf{VF}(s) = \{s\}$, wherefore

$$\mathsf{V}(\Phi)(s) = \Phi(s) = \Psi(s) = \mathsf{V}(\Psi)(s)$$

  for all $\Phi, \Psi \in \mathsf{A}_{\mathfrak{M}}$ such that $\Phi \restriction \mathsf{VF}(s) = \Psi \restriction \mathsf{VF}(s)$. Otherwise, $s \in \mathcal{L} \cup \mathcal{C}$, whence $\mathsf{VF}(s) = \varnothing$ and

$$\mathsf{V}(\Phi)(s) = \mathsf{k}_s = \mathsf{V}(\Psi)(s)$$

  for all $\Phi, \Psi \in \mathsf{A}_{\mathfrak{M}}$.

- Suppose $\alpha, \beta \in \mathcal{T}$, $\psi \in W_\alpha$ and $\mathbf{x} \in \mathsf{X}_\beta$ are such that

$$\mathsf{V}(\Phi)(\psi) = \mathsf{V}(\Psi)(\psi)$$

  for all $\Phi, \Psi \in \mathsf{A}_{\mathfrak{M}}$ such that $\Phi \restriction \mathsf{VF}(\psi) = \Psi \restriction \mathsf{VF}(\psi)$. Assume $\Phi, \Psi \in \mathsf{A}_{\mathfrak{M}}$ are such that $\Phi \restriction \mathsf{VF}([\lambda\mathbf{x}\psi]) = \Psi \restriction \mathsf{VF}([\lambda\mathbf{x}\psi])$. Thus

$$\Phi(\mathbf{y}) = \Psi(\mathbf{y})$$

  for all $\mathbf{y} \in \mathsf{VF}(\psi) \setminus \{\mathbf{x}\}$, so that

$$\begin{aligned}\Phi^{\mathbf{x}}_{\mathsf{d}}(\mathbf{y}) &= \begin{cases}\mathsf{d} & \text{if } \mathbf{y} = \mathbf{x}\\ \Phi(\mathbf{y}) & \text{otherwise}\end{cases}\\ &= \begin{cases}\mathsf{d} & \text{if } \mathbf{y} = \mathbf{x}\\ \Psi(\mathbf{y}) & \text{otherwise}\end{cases}\\ &= \Psi^{\mathbf{x}}_{\mathsf{d}}(\mathbf{y})\end{aligned}$$

  for all $\mathbf{y} \in \mathsf{VF}(\psi)$ and all $\mathsf{d} \in \mathsf{D}_\beta$. Consequently

$$\mathsf{V}(\Phi)([\lambda\mathbf{x}\psi])(\mathsf{d}) = \mathsf{V}(\Phi^{\mathbf{x}}_{\mathsf{d}})(\psi) = \mathsf{V}(\Psi^{\mathbf{x}}_{\mathsf{d}})(\psi) = \mathsf{V}(\Psi)([\lambda\mathbf{x}\psi])(\mathsf{d})$$

  for all $\mathsf{d} \in \mathsf{D}_\beta$, whence $\mathsf{V}(\Phi)([\lambda\mathbf{x}\psi]) = \mathsf{V}(\Psi)([\lambda\mathbf{x}\psi])$.

- Let $\alpha, \beta \in \mathcal{T}$ and $\psi_{\alpha\beta}, \vartheta_\beta \in W$ be such that

$$\mathsf{V}(\Phi)(\eta) = \mathsf{V}(\Psi)(\eta)$$

  for all $\Phi, \Psi \in \mathsf{A}_{\mathfrak{M}}$ with the property that $\Phi \restriction \mathsf{VF}(\eta) = \Psi \restriction \mathsf{VF}(\eta)$, for both $\eta \in \{\psi, \vartheta\}$. Then

$$\mathsf{V}(\Phi)([\psi\vartheta]) = \mathsf{V}(\Phi)(\psi)(\mathsf{V}(\Phi)(\vartheta)) = \mathsf{V}(\Psi)(\psi)(\mathsf{V}(\Psi)(\vartheta)) = \mathsf{V}(\Psi)([\psi\vartheta])$$

  for all $\Phi, \Psi \in \mathsf{A}_{\mathfrak{M}}$ such that $\Phi \restriction \mathsf{VF}([\psi\vartheta]) = \Psi \restriction \mathsf{VF}([\psi\vartheta])$, since we know that $\mathsf{VF}([\psi\vartheta]) = \mathsf{VF}(\psi) \cup \mathsf{VF}(\vartheta)$.

Hence $V(\Phi)(\varphi) = V(\Psi)(\varphi)$ for all $\Phi, \Psi \in A_{\mathfrak{M}}$ such that $\Phi \restriction VF(\varphi) = \Psi \restriction VF(\varphi)$, as claimed. □

This allows us to define the denotation of a closed formula as an object independent of the choice of assignment.

2.11 Definition (Denotation)**.** Let $\mathfrak{M}$ be a general model. For every $\varphi \in \overline{\mathcal{W}}$, the *denotation* of φ is the evaluation of φ with respect to some, and thus every, assignment $\Phi \in A_{\mathfrak{M}}$.

As pointed out earlier, general models will be our prime semantic objects. However, a certain subclass thereof seems to have particularly simple properties.

2.12 Definition (Standard model)**.** A *standard model* is a non-degenerate structured frame $\mathfrak{M} = (\{D_\alpha\}_{\alpha\in\mathcal{T}}, \{k_c\}_{c\in\mathcal{L}\cup\mathcal{C}})$ such that

$$D_{\alpha\beta} = D_\alpha^{D_\beta}$$

for all $\alpha, \beta \in \mathcal{T}$.

2.13 Lemma. *Every standard model is a general model.*

Proof. Let $\mathfrak{M} = (\{D_\alpha\}_{\alpha\in\mathcal{T}}, \{k_c\}_{c\in\mathcal{L}\cup\mathcal{C}})$ be a standard model; then $\mathfrak{M}$ is a non-degenerate structured frame. For every $\alpha \in \mathcal{T}$ the function $q : D_\alpha \longrightarrow D_o^{D_\alpha}$ defined by

$$q(a)(b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}$$

for all $a, b \in D_\alpha$ is an element of $(D_o^{D_\alpha})^{D_\alpha} = D_{o\alpha}^{D_\alpha} = D_{\langle o\alpha\rangle\alpha}$. Furthermore, we define $F : \mathcal{W} \longrightarrow (\bigcup_{\alpha\in\mathcal{T}} D_\alpha^{A_{\mathfrak{M}}})$ (our valuation to be, with arguments in reversed order) as the unique such function which satisfies the following criteria:

- For all $\alpha \in \mathcal{T}$ and every $\varphi \in \mathcal{W}_\alpha$, $F(\varphi) : A_{\mathfrak{M}} \longrightarrow D_\alpha$.
- For every $s \in \mathcal{B}$,

  $$F(s)(\Phi) = \begin{cases} \Phi(s) & \text{if } s \in \mathcal{X} \\ k_s & \text{if } s \in \mathcal{L} \cup \mathcal{C} \end{cases}$$

  for all $\Phi \in A_{\mathfrak{M}}$.
- For all $\alpha, \beta \in \mathcal{T}$, $\varphi_\alpha \in \mathcal{W}$ and $\mathbf{x} \in X_\beta$

  $$F([\lambda\mathbf{x}\varphi])(\Phi)(d) = F(\varphi)(\Phi_d^x)$$

  for all $\Phi \in A_{\mathfrak{M}}$ and $d \in D_\beta$.

- For all $\alpha, \beta \in \mathcal{T}$ and $\varphi_{\alpha\beta}, \psi_\beta \in W$ let

$$\mathsf{F}([\varphi\psi])(\Phi) = \mathsf{F}(\varphi)(\Phi)(\mathsf{F}(\psi)(\Phi)).$$

Such a function exists by Theorem 2.8 of [7]. Thus let $\mathsf{V} : \mathsf{A}_{\mathfrak{M}} \longrightarrow \left(\bigcup_{\alpha\in\mathcal{T}} \mathsf{D}_\alpha\right)^W$ be defined via

$$\mathsf{V}(\Phi)(\varphi) = \mathsf{F}(\varphi)(\Phi)$$

for all $\Phi \in \mathsf{A}_{\mathfrak{M}}$ and $\varphi \in W$, whence

$$\mathsf{V}(\Phi)(\varphi) \in \mathsf{D}_\alpha$$

for all $\Phi \in \mathsf{A}_{\mathfrak{M}}$, $\alpha \in \mathcal{T}$ and $\varphi \in W_\alpha$. Furthermore, for every $\Phi \in \mathsf{A}_{\mathfrak{M}}$:

$$\mathsf{V}(\Phi)(\mathsf{s}) = \mathsf{F}(\mathsf{s})(\Phi) = \begin{cases} \Phi(\mathsf{s}) & \text{if } \mathsf{s} \in \mathcal{X} \\ \mathsf{k}_\mathsf{s} & \text{if } \mathsf{s} \in \mathcal{L} \cup \mathcal{C} \end{cases}$$

for all $\mathsf{s}_\alpha \in \mathcal{B}$;

$$\mathsf{V}(\Phi)([\lambda\mathbf{x}\psi])(\mathsf{d}) = \mathsf{F}([\lambda\mathbf{x}\psi])(\Phi)(\mathsf{d}) = \mathsf{F}(\psi)(\Phi^\mathbf{x}_\mathsf{d}) = \mathsf{V}(\Phi^\mathbf{x}_\mathsf{d})(\psi)$$

for all $\alpha, \beta \in \mathcal{T}$, $\psi \in W_\alpha$, $\mathbf{x} \in \mathsf{X}_\beta$ and $\mathsf{d} \in \mathsf{D}_\beta$; and

$$\mathsf{V}(\Phi)([\psi\vartheta]) = \mathsf{F}([\psi\vartheta])(\Phi) = \mathsf{F}(\psi)(\Phi)(\mathsf{F}(\vartheta)(\Phi)) = \mathsf{V}(\Phi)(\psi)(\mathsf{V}(\Phi)(\vartheta))$$

for all $\alpha, \beta \in \mathcal{T}$ and $\psi_{\alpha\beta}, \vartheta_\beta \in W$. Hence V is a valuation for $\mathfrak{M}$, whereby $\mathfrak{M}$ is a general model, as desired. □

These two classes of models, general models and standard models, give rise to, as we shall see, two quite different notions of semantics; we will speak of *general semantics* and *standard semantics*, respectively. This difference is characterised by theorems 4.28 and 4.30, which also clarify why we do not restrict ourselves to the study of standard models, even though they seem to be, if anything, more well-behaved than the general. On the contrary, since everything true of all general models is true of all standard models in particular, we will henceforth mostly consider general models.

Truth

To simplify things later, and since it is worthy of note on its own merits, we now show that many of the aforementioned abbreviations behave largely as we would expect (and indeed intend) them to. The following lemma will thus be widely applicable, even though this will not always be made explicit.

2.14 Lemma. *Let $\mathfrak{M} = (\{\mathsf{D}_\alpha\}_{\alpha\in\mathcal{T}}, \{\mathsf{k}_\mathsf{c}\}_{\mathsf{c}\in\mathcal{L}\cup\mathcal{C}})$ be a general model. The valuation V for $\mathfrak{M}$ has the following properties, for every $\Phi \in \mathsf{A}_{\mathfrak{M}}$, $\varphi, \psi \in W$, $\alpha \in \mathcal{T}$, $\mathbf{x} \in \mathsf{X}_\alpha$ and $\mathsf{a} \in \mathsf{D}_\alpha$:*

1. $V(\Phi)((\neg\varphi_o)) = 1$ *if and only if* $V(\Phi)(\varphi_o) = 0$.

2. $V(\Phi)((\varphi_o \vee \psi_o)) = 1$ *if and only if* $V(\Phi)(\varphi_o) = 1$ *or* $V(\Phi)(\psi_o) = 1$.

3. $V(\Phi)((\varphi_o \wedge \psi_o)) = 1$ *if and only if* $V(\Phi)(\varphi_o) = 1$ *and* $V(\Phi)(\psi_o) = 1$.

4. $V(\Phi)((\varphi_o \rightarrow \psi_o)) = 1$ *if and only if* $V(\Phi)(\varphi_o) = 0$ *or* $V(\Phi)(\psi_o) = 1$ *if and only if* $V(\Phi)(\varphi_o) = 1 \Rightarrow V(\Phi)(\psi_o) = 1$.

5. $V(\Phi)((\varphi_o \leftrightarrow \psi_o)) = 1$ *if and only if* $V(\Phi)(\varphi_o) = V(\Phi)(\psi_o)$.

6. $V(\Phi)((\forall \mathbf{x}_\alpha \varphi_o)) = 1$ *if and only if* $V(\Phi^{\mathbf{x}}_{d})(\varphi_o) = 1$ *for all* $d \in D_\alpha$.

7. $V(\Phi)((\exists \mathbf{x}_\alpha \varphi_o)) = 1$ *if and only if* $V(\Phi^{\mathbf{x}}_{d})(\varphi_o) = 1$ *for some* $d \in D_\alpha$.

8. *If* $V(\Phi^{\mathbf{x}}_{d})(\varphi_o) = 1$ *for some* $d \in D_\alpha$ *and* $V(\Phi)((\iota \mathbf{x}_\alpha \varphi_o)) = a$, *then* $V(\Phi^{\mathbf{x}}_{a})(\varphi_o) = 1$.

9. $V(\Phi)(\varphi \equiv \psi) = 1$ *if and only if* $V(\Phi)(\varphi) = V(\Phi)(\psi)$.

10. $V(\Phi)(\varphi \not\equiv \psi) = 1$ *if and only if* $V(\Phi)(\varphi) \neq V(\Phi)(\psi)$.

11. $V(\Phi)(\bot) = 0$.

12. $V(\Phi)(\top) = 1$.

13. $V(\Phi)(\mathrm{Id}_{\alpha\alpha}) = \mathrm{id}_{D_\alpha}$

14. $V(\Phi)(0_{\alpha'})(g) = \mathrm{id}_{D_\alpha}$ *for all* $g \in D_{\alpha\alpha}$.

15. $V(\Phi)(S_{\alpha'\alpha'})(F)(g) = g \circ F(g)$ *for all* $F \in D_{\alpha'}$ *and* $g \in D_{\alpha\alpha}$.

16. $V(\Phi)(N_{o\alpha'})(V(\Phi)(0_{\alpha'})) = 1$.

17. $V(\Phi)(N_{o\alpha'})(V(\Phi)(S_{\alpha'\alpha'})(F)) = 1$ *for all* $F \in D_{\alpha'}$ *for which* $V(\Phi)(N_{o\alpha'})(F) = 1$.

Proof. 1.

$$\begin{aligned}
& V(\Phi)((\neg\varphi_o)) = 1 \\
\Leftrightarrow\ & V(\Phi)([N_{oo}\varphi_o]) = 1 \\
\Leftrightarrow\ & V(\Phi)(N_{oo})(V(\Phi)(\varphi_o)) = 1 \\
\Leftrightarrow\ & 1 - V(\Phi)(\varphi_o) = 1 \\
\Leftrightarrow\ & V(\Phi)(\varphi_o) = 0.
\end{aligned}$$

2.

$$\begin{aligned}
& V(\Phi)((\varphi_o \vee \psi_o)) = 1 \\
\Leftrightarrow\ & V(\Phi)([[A_{\langle oo\rangle o}\varphi_o]\psi_o]) = 1 \\
\Leftrightarrow\ & V(\Phi)([A_{\langle oo\rangle o}\varphi_o])(V(\Phi)(\psi_o)) = 1 \\
\Leftrightarrow\ & V(\Phi)(A_{\langle oo\rangle o})(V(\Phi)(\varphi_o))(V(\Phi)(\psi_o)) = 1 \\
\Leftrightarrow\ & 1 - (1 - V(\Phi)(\varphi_o)) \cdot (1 - V(\Phi)(\psi_o)) = 1 \\
\Leftrightarrow\ & (1 - V(\Phi)(\varphi_o)) \cdot (1 - V(\Phi)(\psi_o)) = 0.
\end{aligned}$$

Furthermore, $(1-V(\Phi)(\varphi_o)) \cdot (1-V(\Phi)(\psi_o)) = 0$ if and only if $1-V(\Phi)(\varphi_o) = 0$ or $1 - V(\Phi)(\psi_o) = 0$, if and only if $V(\Phi)(\varphi_o) = 1$ or $V(\Phi)(\psi_o) = 1$.

3. Since $(\varphi_o \wedge \psi_o) = (\neg((\neg\varphi_o) \vee (\neg\psi_o)))$ we have that $V(\Phi)((\varphi_o \wedge \psi_o)) = 1$ if and only if $V(\Phi)(((\neg\varphi_o) \vee (\neg\psi_o))) = 0$, if and only if $V(\Phi)((\neg\varphi_o)) = 0$ and $V(\Phi)((\neg\psi_o)) = 0$, if and only if $V(\Phi)(\varphi_o) = 1$ and $V(\Phi)(\psi_o) = 1$, by above.

4. Since $(\varphi_o \rightarrow \psi_o) = ((\neg\varphi_o)\vee\psi_o)$ we have that $V(\Phi)((\varphi_o \rightarrow \psi_o)) = 1$ if and only if $V(\Phi)((\neg\varphi_o)) = 1$ or $V(\Phi)(\psi_o) = 1$, if and only if $V(\Phi)(\varphi_o) = 0$ or $V(\Phi)(\psi_o) = 1$, if and only if $V(\Phi)(\varphi_o) = 1 \Rightarrow V(\Phi)(\psi_o) = 1$.

5. Since $(\varphi_o \leftrightarrow \psi_o) = ((\varphi_o \rightarrow \psi_o) \wedge (\psi_o \rightarrow \varphi_o))$, $V(\Phi)((\varphi_o \leftrightarrow \psi_o)) = 1$ if and only if $V(\Phi)((\varphi_o \rightarrow \psi_o)) = 1$ and $V(\Phi)((\psi_o \rightarrow \varphi_o)) = 1$, if and only if

$$V(\Phi)(\varphi_o) = 0 \quad \text{or} \quad V(\Phi)(\psi_o) = 1$$

and

$$V(\Phi)(\psi_o) = 0 \quad \text{or} \quad V(\Phi)(\varphi_o) = 1,$$

if and only if

$$V(\Phi)(\varphi_o) = 0 \quad \text{and} \quad V(\Phi)(\psi_o) = 0$$

or

$$V(\Phi)(\psi_o) = 1 \quad \text{and} \quad V(\Phi)(\varphi_o) = 1,$$

if and only if $V(\Phi)(\varphi_o) = V(\Phi)(\psi_o)$.

6. First

$$\begin{aligned} & V(\Phi)((\forall \mathbf{x}_\alpha \varphi_o)) = 1 \\ \Leftrightarrow\ & V(\Phi)([\Pi_{o\langle o\alpha\rangle}[\lambda \mathbf{x}_\alpha \varphi_o]]) = 1 \\ \Leftrightarrow\ & V(\Phi)(\Pi_{o\langle o\alpha\rangle})(V(\Phi)([\lambda \mathbf{x}_\alpha \varphi_o])) = 1 \\ \Leftrightarrow\ & k_{\Pi_{o\langle o\alpha\rangle}}(V(\Phi)([\lambda \mathbf{x}_\alpha \varphi_o])) = 1. \end{aligned}$$

Furthermore, $k_{\Pi_{o\langle o\alpha\rangle}}(V(\Phi)([\lambda \mathbf{x}_\alpha \varphi_o])) = 1$ if and only if $V(\Phi)([\lambda \mathbf{x}_\alpha \varphi_o])(d) = 1$ for all $d \in D_\alpha$, if and only if $V(\Phi^{\mathbf{x}_\alpha}_d)(\varphi_o) = 1$ for all $d \in D_\alpha$.

7. Since $(\exists \mathbf{x}_\alpha \varphi_o) = (\neg(\forall \mathbf{x}_\alpha(\neg\varphi_o)))$, we have that $V(\Phi)((\exists \mathbf{x}_\alpha \varphi_o)) = 1$ if and only if $V(\Phi)((\forall \mathbf{x}_\alpha(\neg\varphi_o))) = 0$, if and only if $V(\Phi^{\mathbf{x}_\alpha}_d)((\neg\varphi_o)) = 0$ for some $d \in D_\alpha$, if and only if $V(\Phi^{\mathbf{x}_\alpha}_d)(\varphi_o) = 1$ for some $d \in D_\alpha$.

8. Assume that $V(\Phi)((\imath \mathbf{x}_\alpha \varphi_o)) = a$ and that $d \in D_\alpha$ is such that $V(\Phi^{\mathbf{x}_\alpha}_d)(\varphi_o) = 1$. Then

$$V(\Phi)([\lambda \mathbf{x}_\alpha \varphi_o])(d) = V(\Phi^{\mathbf{x}_\alpha}_d)(\varphi_o) = 1.$$

Thus

$$\begin{aligned}
V(\Phi^{x_\alpha}_{a})(\varphi_o) &= V(\Phi)([\lambda x_\alpha \varphi_o])(a) \\
&= V(\Phi)([\lambda x_\alpha \varphi_o])(V(\Phi)((\imath x_\alpha \varphi_o))) \\
&= V(\Phi)([\lambda x_\alpha \varphi_o])(V(\Phi)([\iota_{\alpha\langle o\alpha\rangle}[\lambda x_\alpha \varphi_o]])) \\
&= V(\Phi)([\lambda x_\alpha \varphi_o])(V(\Phi)(\iota_{\alpha\langle o\alpha\rangle})(V(\Phi)([\lambda x_\alpha \varphi_o]))) \\
&= V(\Phi)([\lambda x_\alpha \varphi_o])(k_{\iota_{\alpha\langle o\alpha\rangle}}(V(\Phi)([\lambda x_\alpha \varphi_o]))) \\
&= 1
\end{aligned}$$

as well, by the definition of $k_{\iota_{\alpha\langle o\alpha\rangle}}$.

9. Let $a, b \in D_\alpha$. If $a = b$ let $d \in D_{o\alpha}$. Then $d(a) = d(b)$, i.e. either $d(a) = 0$ or $d(b) = 1$, while $V(\Phi^{x\,y\,f}_{a\,b\,d})([fx]) = d(a)$ and $V(\Phi^{x\,y\,f}_{a\,b\,d})([fy]) = d(b)$. Thus

$$V(\Phi^{x\,y\,f}_{a\,b\,d})(([fx] \to [fy])) = 1.$$

Since $d \in D_{o\alpha}$ was arbitrary, we have that

$$\begin{aligned}
V(\Phi)(Q_{\langle o\alpha\rangle\alpha})(a)(b) &= V(\Phi)([\lambda x[\lambda y(\forall f_{o\alpha}([fx] \to [fy]))]])(a)(b) \\
&= V(\Phi^{x\,y}_{a\,b})((\forall f_{o\alpha}([fx] \to [fy]))) \\
&= 1.
\end{aligned}$$

If on the other hand $a \neq b$, let $q \in D_{\langle o\alpha\rangle\alpha}$ be such that $q(a)(b) = 0$ but $q(a)(a) = 1$, as guaranteed in the definition of general models. Then $V(\Phi^{x\,y\,f}_{a\,b\,q(a)})([fx]) = q(a)(a) = 1$ and $V(\Phi^{x\,y\,f}_{a\,b\,q(a)})([fy]) = q(a)(b) = 0$, so that

$$V(\Phi^{x\,y\,f}_{a\,b\,q(a)})(([fx] \to [fy])) = 0.$$

Thus

$$\begin{aligned}
V(\Phi)(Q_{\langle o\alpha\rangle\alpha})(a)(b) &= V(\Phi)([\lambda x[\lambda y(\forall f_{o\alpha}([fx] \to [fy]))]])(a)(b) \\
&= V(\Phi^{x\,y}_{a\,b})((\forall x_\alpha([fx] \to [fy]))) \\
&= 0.
\end{aligned}$$

Hence

$$V(\Phi)(Q_{\langle o\alpha\rangle\alpha})(a)(b) = \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases}.$$

Since $V(\Phi)(\varphi \equiv \psi) = V(\Phi)(Q_{\langle o\alpha\rangle\alpha})(V(\Phi)(\varphi))(V(\Phi)(\psi))$, the claim follows.

10. $V(\Phi)(\varphi \not\equiv \psi) = 1 \Leftrightarrow V(\Phi)(\varphi \equiv \psi) = 0 \Leftrightarrow V(\Phi)(\varphi) \neq V(\Phi)(\psi)$.

11. Since $V(\Phi^{p_o}_{0})(p_o) = 0$, we have

$$V(\Phi)(\bot) = V(\Phi)((\forall p_o p_o)) = 0$$

by 6.

12. $V(\Phi)(\top) = V(\Phi)((\neg\bot)) = 1$.

13. For every $d \in D_\alpha$ we have that

$$V(\Phi)(\mathrm{Id}_{\alpha\alpha})(d) = V(\Phi)([\lambda x_\alpha x_\alpha])(d) = V(\Phi_d^{x_\alpha})(x_\alpha) = \Phi_d^{x_\alpha}(x_\alpha) = d.$$

Hence $V(\Phi)(\mathrm{Id}_{\alpha\alpha}) = \mathrm{id}_{D_\alpha}$.

14. For every $g \in D_{\alpha\alpha}$

$$V(\Phi)(0_{\alpha'})(g) = V(\Phi)([\lambda f_{\alpha\alpha}\,\mathrm{Id}_{\alpha\alpha}])(g) = V(\Phi_g^{f_{\alpha\alpha}})(\mathrm{Id}_{\alpha\alpha}) = \mathrm{id}_{D_\alpha}$$

by above.

15. Let $F \in D_{\alpha'}$ and $g \in D_{\alpha\alpha}$. Then

$$\begin{aligned}
V(\Phi)(S_{\alpha'\alpha'})(F)(g)(d) &= V(\Phi)([\lambda n_{\alpha'}[\lambda f_{\alpha\alpha}[\lambda x_\alpha[f_{\alpha\alpha}[[n_{\alpha'}f_{\alpha\alpha}]x_\alpha]]]]])(F)(g)(d) \\
&= V(\Phi_F^{n_{\alpha'}})([\lambda f_{\alpha\alpha}[\lambda x_\alpha[f_{\alpha\alpha}[[n_{\alpha'}f_{\alpha\alpha}]x_\alpha]]]])(g)(d) \\
&= V(\Phi_{F\ g}^{n_{\alpha'}f_{\alpha\alpha}})([\lambda x_\alpha[f_{\alpha\alpha}[[n_{\alpha'}f_{\alpha\alpha}]x_\alpha]]])(d) \\
&= V(\Phi_{F\ g\ d}^{n_{\alpha'}f_{\alpha\alpha}x_\alpha})([f_{\alpha\alpha}[[n_{\alpha'}f_{\alpha\alpha}]x_\alpha]]) \\
&= V(\Phi_{F\ g\ d}^{n_{\alpha'}f_{\alpha\alpha}x_\alpha})(f_{\alpha\alpha})(V(\Phi_{F\ g\ d}^{n_{\alpha'}f_{\alpha\alpha}x_\alpha})([[n_{\alpha'}f_{\alpha\alpha}]x_\alpha])) \\
&= \Phi_{F\ g\ d}^{n_{\alpha'}f_{\alpha\alpha}x_\alpha}(f_{\alpha\alpha})(V(\Phi_{F\ g\ d}^{n_{\alpha'}f_{\alpha\alpha}x_\alpha})([n_{\alpha'}f_{\alpha\alpha}])(V(\Phi_{F\ g\ d}^{n_{\alpha'}f_{\alpha\alpha}x_\alpha})(x_\alpha))) \\
&= g(V(\Phi_{F\ g\ d}^{n_{\alpha'}f_{\alpha\alpha}x_\alpha})(n_{\alpha'})(V(\Phi_{F\ g\ d}^{n_{\alpha'}f_{\alpha\alpha}x_\alpha})(f_{\alpha\alpha}))(\Phi_{F\ g\ d}^{n_{\alpha'}f_{\alpha\alpha}x_\alpha}(x_\alpha))) \\
&= g(\Phi_{F\ g\ d}^{n_{\alpha'}f_{\alpha\alpha}x_\alpha}(n_{\alpha'})(\Phi_{F\ g\ d}^{n_{\alpha'}f_{\alpha\alpha}x_\alpha}(f_{\alpha\alpha}))(d)) \\
&= g(F(g)(d)) \\
&= (g \circ F(g))(d)
\end{aligned}$$

for all $d \in D_\alpha$, whereby

$$V(\Phi)(S_{\alpha'\alpha'})(F)(g) = g \circ F(g)$$

as desired.

16. Notice first that for all $P \in D_{o\alpha'}$ and all $\Psi \in \mathcal{A}_{\mathfrak{M}}$, if $P(V(\Phi)(0_{\alpha'})) = 0$ and $\Psi(f_{o\alpha'}) = P$ then

$$V(\Psi)(([f_{o\alpha'}0_{\alpha'}] \to ((\forall x_{\alpha'}([f_{o\alpha'}x_{\alpha'}] \to [f_{o\alpha'}[S_{\alpha'\alpha'}x_{\alpha'}]])) \to [f_{o\alpha'}n_{\alpha'}]))) = 1,$$

since

$$\begin{aligned}
V(\Psi)([f_{o\alpha'}0_{\alpha'}]) &= V(\Psi)(f_{o\alpha'})(V(\Psi)(0_{\alpha'})) \\
&= \Psi(f_{o\alpha'})(V(\Psi)(0_{\alpha'})) \\
&= P(V(\Phi)(0_{\alpha'})) \\
&= 0
\end{aligned}$$

by Lemma 2.10 ($0_{\alpha'}$ is closed). Hence, if $P \in D_{o\alpha'}$ is such that $P(V(\Phi)(0_{\alpha'})) = 0$ then

$$V(\Phi^{n_{\alpha'}}_{V(\Phi)(0_{\alpha'})}{}^{f_{o\alpha'}}_{P})((([f_{o\alpha'}0_{\alpha'}] \to ((\forall x_{\alpha'}([f_{o\alpha'}x_{\alpha'}] \to [f_{o\alpha'}[S_{\alpha'\alpha'}x_{\alpha'}]])) \to [f_{o\alpha'}n_{\alpha'}]))) = 1.$$

If $P(V(\Phi)(0_{\alpha'})) = 1$, on the other hand, we have

$$\begin{aligned}
V(\Phi^{n_{\alpha'}}_{V(\Phi)(0_{\alpha'})}{}^{f_{o\alpha'}}_{P})([f_{o\alpha'}n_{\alpha'}]) &= V(\Phi^{n_{\alpha'}}_{V(\Phi)(0_{\alpha'})}{}^{f_{o\alpha'}}_{P})(f_{o\alpha'})(V(\Phi^{n_{\alpha'}}_{V(\Phi)(0_{\alpha'})}{}^{f_{o\alpha'}}_{P})(n_{\alpha'})) \\
&= \Phi^{n_{\alpha'}}_{V(\Phi)(0_{\alpha'})}{}^{f_{o\alpha'}}_{P}(f_{o\alpha'})(\Phi^{n_{\alpha'}}_{V(\Phi)(0_{\alpha'})}{}^{f_{o\alpha'}}_{P}(n_{\alpha'})) \\
&= P(V(\Phi)(0_{\alpha'})) \\
&= 1,
\end{aligned}$$

whence

$$V(\Phi^{n_{\alpha'}}_{V(\Phi)(0_{\alpha'})}{}^{f_{o\alpha'}}_{P})(((\forall x_{\alpha'}([f_{o\alpha'}x_{\alpha'}] \to [f_{o\alpha'}[S_{\alpha'\alpha'}x_{\alpha'}]])) \to [f_{o\alpha'}n_{\alpha'}])) = 1$$

and thereby

$$V(\Phi^{n_{\alpha'}}_{V(\Phi)(0_{\alpha'})}{}^{f_{o\alpha'}}_{P})((([f_{o\alpha'}0_{\alpha'}] \to ((\forall x_{\alpha'}([f_{o\alpha'}x_{\alpha'}] \to [f_{o\alpha'}[S_{\alpha'\alpha'}x_{\alpha'}]])) \to [f_{o\alpha'}n_{\alpha'}]))) = 1.$$

Thus

$$\begin{aligned}
&V(\Phi^{n_{\alpha'}}_{V(\Phi)(0_{\alpha'})}) \\
&\qquad ((\forall f_{o\alpha'}([f_{o\alpha'}0_{\alpha'}] \to ((\forall x_{\alpha'}([f_{o\alpha'}x_{\alpha'}] \to [f_{o\alpha'}[S_{\alpha'\alpha'}x_{\alpha'}]])) \to [f_{o\alpha'}n_{\alpha'}])))) \\
&=1,
\end{aligned}$$

and so

$$\begin{aligned}
&V(\Phi)(N_{o\alpha'})(V(\Phi)(0_{\alpha'})) \\
=&V(\Phi)([\lambda n_{\alpha'}(\forall f_{o\alpha'}([f_{o\alpha'}0_{\alpha'}] \to ((\forall x_{\alpha'}([f_{o\alpha'}x_{\alpha'}] \to [f_{o\alpha'}[S_{\alpha'\alpha'}x_{\alpha'}]])) \to [f_{o\alpha'}n_{\alpha'}])))]) \\
&\quad (V(\Phi)(0_{\alpha'})) \\
=&1,
\end{aligned}$$

confirming the claim.

17. Assume that $V(\Phi)(N_{o\alpha'})(F) = 1$ and let $P \in D_{o\alpha'}$. Then in particular

$$V(\Phi^{n_{\alpha'}}_{F}{}^{f_{o\alpha'}}_{P})(([f_{o\alpha'}0_{\alpha'}] \to ((\forall x_{\alpha'}([f_{o\alpha'}x_{\alpha'}] \to [f_{o\alpha'}[S_{\alpha'\alpha'}x_{\alpha'}]])) \to [f_{o\alpha'}n_{\alpha'}]))) = 1.$$

First assume $P(V(\Phi)(0_{\alpha'})) = 0$; then

$$\begin{aligned}
&V(\Phi^{n_{\alpha'}}_{V(\Phi)(S_{\alpha'\alpha'})(F)}{}^{f_{o\alpha'}}_{P}) \\
&\qquad ((([f_{o\alpha'}0_{\alpha'}] \to ((\forall x_{\alpha'}([f_{o\alpha'}x_{\alpha'}] \to [f_{o\alpha'}[S_{\alpha'\alpha'}x_{\alpha'}]])) \to [f_{o\alpha'}n_{\alpha'}]))) \\
&=1.
\end{aligned}$$

Now assume $P(V(\Phi)(0_{\alpha'})) = 1$. Then we must have that

$$V(\Phi_{F}^{n_{\alpha'}}{}_{P}^{f_{o\alpha'}})(((\forall x_{\alpha'}([f_{o\alpha'}x_{\alpha'}] \to [f_{o\alpha'}[S_{\alpha'\alpha'}x_{\alpha'}]])) \to [f_{o\alpha'}n_{\alpha'}])) = 1. \tag{1}$$

We consider cases according as

$$V(\Phi_{F}^{n_{\alpha'}}{}_{P}^{f_{o\alpha'}})((\forall x_{\alpha'}([f_{o\alpha'}x_{\alpha'}] \to [f_{o\alpha'}[S_{\alpha'\alpha'}x_{\alpha'}]]))) = 1 \tag{2}$$

or not. If (2) does *not* hold, then it follows that

$$V(\Phi_{V(\Phi)(S_{\alpha'\alpha'})(F)}^{n_{\alpha'}}{}_{P}^{f_{o\alpha'}})((\forall x_{\alpha'}([f_{o\alpha'}x_{\alpha'}] \to [f_{o\alpha'}[S_{\alpha'\alpha'}x_{\alpha'}]]))) = 0,$$

since $n_{\alpha'}$ is not free in $(\forall x_{\alpha'}([f_{o\alpha'}x_{\alpha'}] \to [f_{o\alpha'}[S_{\alpha'\alpha'}x_{\alpha'}]]))$, and hence that

$$V(\Phi_{V(\Phi)(S_{\alpha'\alpha'})(F)}^{n_{\alpha'}}{}_{P}^{f_{o\alpha'}})(((\forall x_{\alpha'}([f_{o\alpha'}x_{\alpha'}] \to [f_{o\alpha'}[S_{\alpha'\alpha'}x_{\alpha'}]])) \to [f_{o\alpha'}n_{\alpha'}])) = 1.$$

If (2) holds, on the other hand, then

$$V(\Phi_{F}^{n_{\alpha'}}{}_{P}^{f_{o\alpha'}}{}_{F}^{x_{\alpha'}})((([f_{o\alpha'}x_{\alpha'}] \to [f_{o\alpha'}[S_{\alpha'\alpha'}x_{\alpha'}]]))) = 1 \tag{3}$$

and also

$$V(\Phi_{F}^{n_{\alpha'}}{}_{P}^{f_{o\alpha'}})([f_{o\alpha'}n_{\alpha'}]) = 1,$$

by (1). Since

$$\begin{aligned} V(\Phi_{F}^{n_{\alpha'}}{}_{P}^{f_{o\alpha'}}{}_{F}^{x_{\alpha'}})([f_{o\alpha'}x_{\alpha'}]) &= V(\Phi_{F}^{n_{\alpha'}}{}_{P}^{f_{o\alpha'}}{}_{F}^{x_{\alpha'}})(f_{o\alpha'})(V(\Phi_{F}^{n_{\alpha'}}{}_{P}^{f_{o\alpha'}}{}_{F}^{x_{\alpha'}})(x_{\alpha'})) \\ &= P(F) \\ &= V(\Phi_{F}^{n_{\alpha'}}{}_{P}^{f_{o\alpha'}})(f_{o\alpha'})(V(\Phi_{F}^{n_{\alpha'}}{}_{P}^{f_{o\alpha'}})(n_{\alpha'})) \\ &= V(\Phi_{F}^{n_{\alpha'}}{}_{P}^{f_{o\alpha'}})([f_{o\alpha'}n_{\alpha'}]) = 1, \end{aligned}$$

we get that

$$\begin{aligned} &V(\Phi_{V(\Phi)(S_{\alpha'\alpha'})(F)}^{n_{\alpha'}}{}_{P}^{f_{o\alpha'}})([f_{o\alpha'}n_{\alpha'}]) \\ =&P(V(\Phi)(S_{\alpha'\alpha'})(F)) \\ =&V(\Phi_{F}^{n_{\alpha'}}{}_{P}^{f_{o\alpha'}}{}_{F}^{x_{\alpha'}})(f_{o\alpha})(V(\Phi_{F}^{n_{\alpha'}}{}_{P}^{f_{o\alpha'}}{}_{F}^{x_{\alpha'}})(S_{\alpha'\alpha'})(V(\Phi_{F}^{n_{\alpha'}}{}_{P}^{f_{o\alpha'}}{}_{F}^{x_{\alpha'}})(x_{\alpha'}))) \\ =&V(\Phi_{F}^{n_{\alpha'}}{}_{P}^{f_{o\alpha'}}{}_{F}^{x_{\alpha'}})([f_{o\alpha}[S_{\alpha'\alpha'}x_{\alpha'}]]) \\ =&1 \end{aligned}$$

by (3), whereby

$$V(\Phi_{V(\Phi)(S_{\alpha'\alpha'})(F)}^{n_{\alpha'}}{}_{P}^{f_{o\alpha'}})(((\forall x_{\alpha'}([f_{o\alpha'}x_{\alpha'}] \to [f_{o\alpha'}[S_{\alpha'\alpha'}x_{\alpha'}]])) \to [f_{o\alpha'}n_{\alpha'}])) = 1.$$

In any case (that is, whether $V(\Phi^{n_{\alpha'}}_{F}{}^{f_{o\alpha'}}_{P})((\forall x_{\alpha'}([f_{o\alpha'}x_{\alpha'}] \to [f_{o\alpha'}[S_{\alpha'\alpha'}x_{\alpha'}]]))) = 1$ or not),

$$V(\Phi^{n_{\alpha'}}_{V(\Phi)(S_{\alpha'\alpha'})(F)}{}^{f_{o\alpha'}}_{P})(((\forall x_{\alpha'}([f_{o\alpha'}x_{\alpha'}] \to [f_{o\alpha'}[S_{\alpha'\alpha'}x_{\alpha'}]])) \to [f_{o\alpha'}n_{\alpha'}])) = 1$$

whence

$$\begin{aligned}&V(\Phi^{n_{\alpha'}}_{V(\Phi)(S_{\alpha'\alpha'})(F)}{}^{f_{o\alpha'}}_{P})\\ &\qquad(([f_{o\alpha'}0_{\alpha'}] \to ((\forall x_{\alpha'}([f_{o\alpha'}x_{\alpha'}] \to [f_{o\alpha'}[S_{\alpha'\alpha'}x_{\alpha'}]])) \to [f_{o\alpha'}n_{\alpha'}])))\\ =&1.\end{aligned}$$

Hence we can conclude that

$$\begin{aligned}&V(\Phi^{n_{\alpha'}}_{V(\Phi)(S_{\alpha'\alpha'})(F)}{}^{f_{o\alpha'}}_{P})\\ &\qquad(([f_{o\alpha'}0_{\alpha'}] \to ((\forall x_{\alpha'}([f_{o\alpha'}x_{\alpha'}] \to [f_{o\alpha'}[S_{\alpha'\alpha'}x_{\alpha'}]])) \to [f_{o\alpha'}n_{\alpha'}])))\\ =&1\end{aligned}$$

for *all* $P \in D_{o\alpha}$. Consequently

$$\begin{aligned}&V(\Phi)(N_{o\alpha'})(V(\Phi)(S_{\alpha'\alpha'})(F))\\ =&V(\Phi)([\lambda n_{\alpha'}(\forall f_{o\alpha'}([f_{o\alpha'}0_{\alpha'}] \to ((\forall x_{\alpha'}([f_{o\alpha'}x_{\alpha'}] \to [f_{o\alpha'}[S_{\alpha'\alpha'}x_{\alpha'}]])) \to [f_{o\alpha'}n_{\alpha'}])))])\\ &\quad(V(\Phi)(S_{\alpha'\alpha'})(F))\\ =&V(\Phi^{n_{\alpha'}}_{V(\Phi)(S_{\alpha'\alpha'})(F)})\\ &\quad((\forall f_{o\alpha'}([f_{o\alpha'}0_{\alpha'}] \to ((\forall x_{\alpha'}([f_{o\alpha'}x_{\alpha'}] \to [f_{o\alpha'}[S_{\alpha'\alpha'}x_{\alpha'}]])) \to [f_{o\alpha'}n_{\alpha'}]))))\\ =&1\end{aligned}$$

as desired.

□

We are now in a position to define what we mean by a formula being satisfiable and true, respectively, and prove that the axioms always hold true.

2.15 Definition (Satisfiability)**.** Let $\Gamma \subseteq W_o$. Γ will be called *satisfiable* if there is a general model $\mathfrak{M}$ and an assignment $\Phi \in \mathcal{A}_{\mathfrak{M}}$ such that $V(\Phi)(\varphi) = 1$ for every $\varphi \in \Gamma$.

2.16 Definition (Truth in a general model)**.** Let $\Gamma \subseteq W_o$ and $\mathfrak{M}$ be a general model. If $\varphi \in W_o$ is such that $V(\Phi)(\varphi) = 1$ for every $\Phi \in \mathcal{A}_{\mathfrak{M}}$ we say that φ *is true in* $\mathfrak{M}$, or that $\mathfrak{M}$ *is a model of* φ, and write

$$\mathfrak{M} \models \varphi.$$

If $\mathfrak{M} \models \varphi$ for every $\varphi \in \Gamma$, then $\mathfrak{M}$ *is a model of* Γ, written

$$\mathfrak{M} \models \Gamma.$$

2.17 Definition. Let $\Gamma, \Delta \subseteq W_o$ and $\varphi \in W_o$. If $\mathfrak{M} \models \Gamma$ for every general model $\mathfrak{M}$ such that $\mathfrak{M} \models \Delta$, then Γ is a *consequence* of Δ, written

$$\Delta \models \Gamma.$$

If $\mathfrak{M} \models \varphi$ for every general model $\mathfrak{M}$ (that is, $\varnothing \models \varphi$), then φ is *valid in the general sense*, which we denote by

$$\models \varphi.$$

Finally, if $\mathfrak{M} \models \varphi$ for every standard model $\mathfrak{M}$, then φ is *valid in the standard sense.*

2.18 Proposition. *The axioms 1–6, 10, 10° and 11 of simple type theory (see definition 1.2) are valid in the general sense.*

Proof. Let $\mathfrak{M} = (\{D_\alpha\}_{\alpha \in \mathcal{T}}, \{k_c\}_{c \in \mathcal{L} \cup \mathcal{C}})$ be a general model and $\Phi \in A_{\mathfrak{M}}$.

1. If $V(\Phi)(p_o) = 1$ then $V(\Phi)(((p_o \vee p_o) \to p_o)) = 1$.
   If $V(\Phi)(p_o) = 0$ then $V(\Phi)((p_o \vee p_o)) = 0$, whereby $V(\Phi)(((p_o \vee p_o) \to p_o)) = 1$.

2. Assume $V(\Phi)(p_o) = 1$. Then $V(\Phi)((p_o \vee q_o)) = 1$, whence $V(\Phi)((p_o \to (p_o \vee q_o))) = 1$.

3. Assume $V(\Phi)((p_o \vee q_o)) = 1$, i.e. that $V(\Phi)(p_o) = 1$ or $V(\Phi)(q_o) = 1$. Then $V(\Phi)((q_o \vee p_o)) = 1$. Hence $V(\Phi)(((p_o \vee q_o) \to (q_o \vee p_o))) = 1$.

4. Assume that $V(\Phi)((p_o \to q_o)) = 1$, i.e. that $V(\Phi)(p_o) = 0$ or $V(\Phi)(q_o) = 1$. Likewise, assume that $V(\Phi)((r_o \vee p_o)) = 1$, i.e. that $V(\Phi)(r_o) = 1$ or $V(\Phi)(p_o) = 1$. Thus if $V(\Phi)(p_o) = 0$ then $V(\Phi)(r_o) = 1$, and if $V(\Phi)(p_o) = 1$ then $V(\Phi)(q_o) = 1$, so that in either case $V(\Phi)((r_o \vee q_o)) = 1$. Therefore

   $$V(\Phi)(((r_o \vee p_o) \to (r_o \vee q_o))) = 1$$

   Hence $V(\Phi)(((p_o \to q_o) \to ((r_o \vee p_o) \to (r_o \vee q_o)))) = 1$.

5. Let $\alpha \in \mathcal{T}$. Assume $V(\Phi)([\Pi_{o\langle o\alpha\rangle} f_{o\alpha}]) = 1$, i.e. that $V(\Phi)(f_{o\alpha})(d) = 1$ for all $d \in D_\alpha$. In particular

   $$V(\Phi)([f_{o\alpha} x_\alpha]) = V(\Phi)(f_{o\alpha})(V(\Phi)(x_\alpha)) = 1.$$

   Hence $V(\Phi)(([\Pi_{o\langle o\alpha\rangle} f_{o\alpha}] \to [f_{o\alpha} x_\alpha])) = 1$.

6. Let $\alpha \in \mathcal{T}$ and assume that $V(\Phi)((\forall x_\alpha (p_o \vee [f_{o\alpha} x_\alpha]))) = 1$, i.e. that

   $$V(\Phi^{x_\alpha}_d)((p_o \vee [f_{o\alpha} x_\alpha])) = 1$$

   for all $d \in D_\alpha$. Thus, given $d \in D_\alpha$ we have either

   $$V(\Phi^{x_\alpha}_d)(p_o) = 1$$

or

$$V(\Phi_d^{x_\alpha})([f_{o\alpha}x_\alpha]) = 1.$$

If $V(\Phi_d^{x_\alpha})([f_{o\alpha}x_\alpha]) = 1$ for all $d \in D_\alpha$, then, by Lemma 2.10, $V(\Phi)(f_{o\alpha})(d) = 1$ for all $d \in D_\alpha$ i.e.

$$V(\Phi)([\Pi_{o\langle o\alpha\rangle}f_{o\alpha}]) = 1.$$

Otherwise there is a $d \in D_\alpha$ such that $V(\Phi_d^{x_\alpha})([f_{o\alpha}x_\alpha]) = 0$. Then

$$V(\Phi)(p_o) = V(\Phi_d^{x_\alpha})(p_o) = 1$$

for this d. In either case

$$V(\Phi)((p_o \vee [\Pi_{o\langle o\alpha\rangle}f_{o\alpha}])) = 1.$$

Hence $V(\Phi)(((\forall x_\alpha(p_o \vee [f_{o\alpha}x_\alpha])) \rightarrow (p_o \vee [\Pi_{o\langle o\alpha\rangle}f_{o\alpha}]))) = 1$.

10. Let $\alpha, \beta \in \mathcal{T}$. Assume that $V(\Phi)((\forall x_\beta(f_{\alpha\beta}x_\beta \equiv g_{\alpha\beta}x_\beta))) = 1$, i.e. that

$$V(\Phi_d^{x_\beta})(([f_{\alpha\beta}x_\beta] \equiv [g_{\alpha\beta}x_\beta])) = 1$$

for all $d \in D_\beta$. Hence

$$\begin{aligned}
V(\Phi)(f_{\alpha\beta})(d) &= V(\Phi_d^{x_\beta})(f_{\alpha\beta})(V(\Phi_d^{x_\beta})(x_\beta)) \\
&= V(\Phi_d^{x_\beta})([f_{\alpha\beta}x_\beta]) \\
&= V(\Phi_d^{x_\beta})([g_{\alpha\beta}x_\beta]) \\
&= V(\Phi_d^{x_\beta})(g_{\alpha\beta})(V(\Phi_d^{x_\beta})(x_\beta)) \\
&= V(\Phi)(g_{\alpha\beta})(d)
\end{aligned}$$

for all $d \in D_\beta$, so that $V(\Phi)(f_{\alpha\beta}) = V(\Phi)(g_{\alpha\beta})$, whereby $V(\Phi)((f_{\alpha\beta} \equiv g_{\alpha\beta}))$. Thus $V(\Phi)(((\forall x_\beta(f_{\alpha\beta}x_\beta \equiv g_{\alpha\beta}x_\beta)) \rightarrow (f_{\alpha\beta} \equiv g_{\alpha\beta}))) = 1$.

10°. Observe that

$$\begin{aligned}
&V(\Phi)((x_o \leftrightarrow y_o)) = 1 \\
\Leftrightarrow\ &V(\Phi)(x_o) = V(\Phi)(y_o) \\
\Leftrightarrow\ &V(\Phi)((x_o \equiv y_o)) = 1.
\end{aligned}$$

Then in particular $V(\Phi)(((x_o \leftrightarrow y_o) \rightarrow (x_o \equiv y_o))) = 1$.

11. Let $\alpha \in \mathcal{T}$, and assume that $V(\Phi)([f_{o\alpha}x_\alpha]) = 1$, i.e. $V(\Phi)(f_{o\alpha})(V(\Phi)(x_\alpha)) = 1$. Thus

$$V(\Phi)([f_{o\alpha}[\iota_{\alpha\langle o\alpha\rangle}f_{o\alpha}]]) = V(\Phi)(f_{o\alpha})(k_{\iota_{\alpha\langle o\alpha\rangle}}(V(\Phi)(f_{o\alpha}))) = 1.$$

Hence $V(\Phi)(([f_{o\alpha}x_\alpha] \rightarrow [f_{o\alpha}[\iota_{\alpha\langle o\alpha\rangle}f_{o\alpha}]])) = 1$.

□

We conclude this section by showing the following noteworthy proposition, though it has a flavour of side remark.

2.19 Proposition. *There is a standard model in which the axioms of simple type theory (definition 1.2) hold true.*

Proof. Define D_α for every $\alpha \in \mathcal{T}$ recursively as follows:

$$D_o = \{0, 1\}$$
$$D_i = \mathbb{N}$$
$$D_{\alpha\beta} = D_\alpha^{D_\beta}$$

Since now D_α is uniquely determined for every $\alpha \in \mathcal{T}$ the axiom of replacement gives that $\{D_\alpha \mid \alpha \in \mathcal{T}\}$ is a set, whence both of

$$\mathfrak{D} = \{D_\alpha\}_{\alpha\in\mathcal{T}}$$
$$\mathfrak{U} = \bigcup_{\alpha\in\mathcal{T}} D_\alpha$$

are sets as well. Furthermore

$$|D_o| = 2$$
$$|D_i| = \aleph_0 \geq 2$$

and

$$|D_{\alpha\beta}| = |D_\alpha|^{|D_\beta|} \geq 2^2 \geq 2$$

for all α, β such that $D_\alpha, D_\beta \geq 2$. Hence

$$|D_\alpha| \geq 2$$

for all $\alpha \in \mathcal{T}$. In particular, for each $\alpha \in \mathcal{T}$ there is an $a \in D_\alpha \subseteq \mathfrak{U}$. By the axiom of choice there is a function $F : \mathcal{T} \longrightarrow \mathfrak{U}$ such that $F(\alpha) \in D_\alpha$ for every $\alpha \in \mathcal{T}$. We thus define

$$k_c = F(\alpha) \in D_\alpha$$

for every $\alpha \in \mathcal{T}$ and $c \in C_\alpha$. Furthermore we define $k_N \in D_{oo}$, $k_A \in D_{\langle oo\rangle o}$ and, for each $\alpha \in \mathcal{T}$, $k_{\Pi_{o\langle o\alpha\rangle}} \in D_{o\langle o\alpha\rangle}$ as in Definition 2.1 (this is possible since $D_{\alpha\beta} = D_\alpha^{D_\beta}$ for all $\alpha, \beta \in \mathcal{T}$). Finally, let $\alpha \in \mathcal{T}$ and consider $\iota_{\alpha\langle o\alpha\rangle}$. By the Axiom of choice there is a $g \in D_{\alpha\langle o\alpha\rangle}$ such that $f(g(f)) = 1$ for all $f \in D_{o\alpha}$ with $f(d) = 1$ for some $d \in D_\alpha$ ($g(f)$ is completely arbitrary for other $f \in D_{o\alpha}$). Hence (by the Axiom of choice again) there is an $h : \mathcal{T} \longrightarrow \bigcup_{\alpha\in\mathcal{T}} D_{\alpha\langle o\alpha\rangle}$ such that $h(\alpha) \in D_{\alpha\langle o\alpha\rangle}$ and

$$f(h(\alpha)(f)) = 1$$

for every $f \in D_{o\alpha}$ with $f(d) = 1$ for some $d \in D_\alpha$, for all $\alpha \in \mathcal{T}$. Thus we define $k_{\iota_{\alpha\langle o\alpha\rangle}} = h(\alpha)$ for all $\alpha \in \mathcal{T}$, and set $\mathfrak{C} = \{k_c\}_{c\in\mathcal{L}\cup\mathcal{C}}$. Then $\mathfrak{M} = (\mathfrak{D}, \mathfrak{C})$ is a standard model.

Furthermore, the axioms 1–6 10, 10^o and 11 are true in $\mathfrak{M}$ by Lemma 2.18. To check that the remaining axioms 7, 8 and 9 hold true in $\mathfrak{M}$, let $\Phi \in \mathcal{A}_{\mathfrak{M}}$.

7. Since $0, 1 \in D_i$ and $0 \neq 1$ we have that $V(\Phi^{x_i\,y_i}_{0\;1})((x_i \not\equiv y_i)) = 1$. Hence $V(\Phi^{x_i}_0)((\exists y_i(x_i \not\equiv y_i))) = 1$ and $V(\Phi)((\exists x_i(\exists y_i(x_i \not\equiv y_i)))) = 1$.

8. Now assume that $V(\Phi)([N_{oi'}x_{i'}]) = V(\Phi)([N_{oi'}y_{i'}]) = 1$. Also assume that $V(\Phi)(([S_{i'i'}x_{i'}] \equiv [S_{i'i'}y_{i'}])) = 1$. Then $V(\Phi)([S_{i'i'}x_{i'}]) = V(\Phi)([S_{i'i'}y_{i'}])$. To prove that $V(\Phi)(x_{i'}) = V(\Phi)(y_{i'})$ we first prove two sublemmata.

   Sublemma 2.19.1. *Let* $M_1 = \{F \in D_{i'} \,|\, \exists n \in \mathbb{N} : \forall f \in D_{ii} : F(f) = f^n\}$ *and* $M_2 = \{F \in D_{i'} \,|\, V(\Phi)(N_{oi'})(F) = 1\}$*. Then* $M_1 = M_2$*.*

   Proof. We first verify that, for every $n \in \mathbb{N}$, if $F \in M_1$ is given by $F(g) = g^n$ for all $g \in D_{ii}$ then $F \in M_2$, by induction.

   - Assume that $F \in M_1$ is such that
   $$F(g) = g^0 = \mathrm{id}_{D_i}$$
   for all $g \in D_{ii}$. Thus
   $$F = V(\Phi)(0_{i'})$$
   by Lemma 2.14, whence
   $$V(\Phi)(N_{oi'})(F) = V(\Phi)(N_{oi'})(V(\Phi)(0_{i'})) = 1$$
   by the same lemma. Consequently $F \in M_2$.
   - Let $k \in \mathbb{N}$ be such that if $F \in M_1$ is given by
   $$F(g) = g^k$$
   for all $g \in D_{ii}$, then $F \in M_2$. Consider $G \in M_1$ defined by
   $$G(g) = g^{k+1}$$
   for all $g \in D_{ii}$, that is
   $$G(g) = g \circ F(g)$$
   where $F(g) = g^k$ for all $g \in D_{ii}$, so $F \in M_2$ by induction hypothesis. By Lemma 2.14 again
   $$V(\Phi)(N_{oi'})(G) = V(\Phi)(N_{oi'})(V(\Phi)(S_{i'i'})(F)) = 1,$$
   that is
   $$G \in M_2.$$

Thus $M_1 \subseteq M_2$. Let $P : D_{i'} \to D_o$ be the characteristic function of M_1, that is

$$P(F) = \begin{cases} 1 & \text{if } F \in M_1 \\ 0 & \text{otherwise} \end{cases}$$

for all $F \in D_{i'}$. Clearly $P \in D_{oi'}$. Observe that

$$P(V(\Phi)(0_{i'})) = 1$$

and that if $F \in D_{i'}$ is such that $P(F) = 1$ (that is $F \in M_1$), then there is some $n \in \mathbb{N}$ such that $F(g) = g^n$ for all $g \in D_{ii}$, whereby $V(\Phi)(S_{i'i'})(F)(g) = g^{n+1}$ for all $g \in D_{ii}$, whence

$$P(V(\Phi)(S_{i'i'})(F)) = 1.$$

Now suppose $F \in M_2$. Then

$$\begin{aligned} 1 &= V(\Phi)(N_{oi'})(F) \\ &= V(\Phi)([\lambda n_{i'}(\forall f_{oi'}([f_{oi'}0_{i'}] \to ((\forall x_{i'}([f_{oi'}x_{i'}] \to [f_{oi'}[S_{i'i'}x_{i'}]])) \to [f_{oi'}n_{i'}])))])(F) \\ &= V(\Phi^{n_{i'}}_{F})((\forall f_{oi'}([f_{oi'}0_{i'}] \to ((\forall x_{i'}([f_{oi'}x_{i'}] \to [f_{oi'}[S_{i'i'}x_{i'}]])) \to [f_{oi'}n_{i'}])))) \end{aligned}$$

whereby in particular

$$V(\Phi^{n_{i'}}_{F}{}^{f_{oi'}}_{P})((([f_{oi'}0_{i'}] \to ((\forall x_{i'}([f_{oi'}x_{i'}] \to [f_{oi'}[S_{i'i'}x_{i'}]])) \to [f_{oi'}n_{i'}]))) = 1.$$

Since

$$\begin{aligned} V(\Phi^{n_{i'}}_{F}{}^{f_{oi'}}_{P})([f_{oi'}0_{i'}]) &= V(\Phi^{n_{i'}}_{F}{}^{f_{oi'}}_{P})(f_{oi'})(V(\Phi^{n_{i'}}_{F}{}^{f_{oi'}}_{P})(0_{i'})) \\ &= P((V(\Phi)(0_{i'}))) \\ &= 1 \end{aligned}$$

by Lemma 2.10, we must have that

$$V(\Phi^{n_{i'}}_{F}{}^{f_{oi'}}_{P})((((\forall x_{i'}([f_{oi'}x_{i'}] \to [f_{oi'}[S_{i'i'}x_{i'}]])) \to [f_{oi'}n_{i'}])) = 1.$$

Furthermore, seeing as every $G \in D_{i'}$ is such that

$$\begin{aligned} &V(\Phi^{n_{i'}}_{F}{}^{f_{oi'}}_{P}{}^{x_{i'}}_{G})([f_{oi'}x_{i'}]) = 1 \\ \Leftrightarrow &P(G) = 1 \\ \Rightarrow &P(V(\Phi)(S_{i'i'})(G)) = 1 \\ \Leftrightarrow &V(\Phi^{n_{i'}}_{F}{}^{f_{oi'}}_{P}{}^{x_{i'}}_{G})([f_{oi'}[S_{i'i'}x_{i'}]]) = 1 \end{aligned}$$

since $S_{i'i'}$ is closed, we infer that

$$V(\Phi^{n_{i'}}_{F}{}^{f_{o\alpha'}}_{P}{}^{x_{i'}}_{G})((([f_{oi'}x_{i'}] \to [f_{oi'}[S_{i'i'}x_{i'}]])) = 1$$

for all $G \in D_{i'}$, whereby

$$V(\Phi_{F}^{n_{i'}}{}_{P}^{f_{oi'}})((\forall x_{i'}([f_{oi'}x_{i'}] \to [f_{oi'}[S_{i'i'}x_{i'}]]))) = 1.$$

Thus we conclude that

$$P(F) = V(\Phi_{F}^{n_{i'}}{}_{P}^{f_{oi'}})([f_{oi'}n_{i'}]) = 1,$$

that is

$$F \in M_1.$$

So $M_2 = M_1$ as claimed. □

Sublemma 2.19.2. *For all $n, m \in \mathbb{N}$, if $f^n = f^m$ for all injections $f \in D_{ii}$, then $n = m$.*

Proof. By induction on n. Since $D_i = \mathbb{N}$, there is an injection $f \in D_{ii}$ which is not surjective. Then $\mathrm{id} = f^0 = f^{l+1} = f \circ f^l$ is a contradiction; the claim holds for $n = 0$. Now assume that $k \in \mathbb{N}$ is such that for all $m \in \mathbb{N}$, if $f^k = f^m$ for all injections $f \in D_{ii}$, then $k = m$. Let $m \in \mathbb{N}$ and assume that $f^{k+1} = f^m$ for all injections $f \in D_{ii}$. Then $m \neq 0$ by above, so $m = l + 1$ for some $l \in \mathbb{N}$. Hence $f \circ f^k = f^{k+1} = f^{l+1} = f \circ f^l$ for all injections $f \in D_{ii}$, so that $f^k = f^l$ for all injections $f \in D_{ii}$. By the induction hypothesis $k = l$, whence $k + 1 = l + 1 = m$. □

Since

$$V(\Phi)(N_{oi'})(\Phi(x_{i'})) = V(\Phi)([N_{oi'}x_{i'}]) = 1$$

and similarly for $y_{i'}$, by Sublemma 2.19.1 there are $n, m \in \mathbb{N}$ such that $\Phi(x_{i'})(f) = f^n$ and $\Phi(y_{i'})(f) = f^m$ for all $f \in D_{ii}$. Hence

$$\begin{aligned}
f^{n+1} &= f \circ \Phi(x_{i'})(f) \\
&= V(\Phi)(S_{i'i'})(V(\Phi)(x_{i'}))(f) \\
&= V(\Phi)([S_{i'i'}x_{i'}])(f) \\
&= V(\Phi)([S_{i'i'}y_{i'}])(f) \\
&= V(\Phi)(S_{i'i'})(V(\Phi)(y_{i'}))(f) \\
&= f \circ \Phi(y_{i'})(f) \\
&= f^{m+1}
\end{aligned}$$

for all $f \in D_{ii}$, and thus in particular for every injection $f \in D_{ii}$. Hence $m+1 = n+1$ by Sublemma 2.19.2, whence $m = n$. Therefore $V(\Phi)(x_{i'})(f) = V(\Phi)(y_{i'})(f)$ for all $f \in D_{ii}$, that is

$$V(\Phi)(x_{i'}) = V(\Phi)(y_{i'}).$$

Thus $V(\Phi)((x_{i'} \equiv y_{i'})) = 1$. Hence

$$V(\Phi)(([N_{oi'}x_{i'}] \to ([N_{oi'}y_{i'}] \to (([S_{i'i'}x_{i'}] \equiv [S_{i'i'}y_{i'}]) \to (x_{i'} \equiv y_{i'}))))) = 1$$

as desired.

9. Let $\alpha \in \mathcal{T}$. Assume that $V(\Phi)([f_{o\alpha}x_\alpha]) = 1$. As in the proof of Lemma 2.18, this implies that $V(\Phi)([f_{o\alpha}[\iota_{\alpha\langle o\alpha\rangle}f_{o\alpha}]]) = 1$. Thus

$$V(\Phi)(((\forall y_\alpha([f_{o\alpha}y_\alpha] \to (x_\alpha \equiv y_\alpha))) \to [f_{o\alpha}[\iota_{\alpha\langle o\alpha\rangle}f_{o\alpha}]])) = 1.$$

Hence $V(\Phi)(([f_{o\alpha}x_\alpha] \to ((\forall y_\alpha([f_{o\alpha}y_\alpha] \to (x_\alpha = y_\alpha))) \to [f_{o\alpha}[\iota_{\alpha\langle o\alpha\rangle}f_{o\alpha}]]))) = 1$ (which is hardly surprising, since axiom 9 is easily seen to be a consequence of axiom 11.).

So all axioms of 1.2 are true in $\mathfrak{M}$. In particular, they are satisfiable in a standard model. □

3. Soundness

Soundness, the property of (deduction in) a formal system to preserve truth, is in some sense *the* desirable property of such a system. While other properties, such as completeness, might very well be considered desirable, a system which may prove propositions that are not true is unlikely to be of any significant applicability.

It will hardly be surprising, therefore, to find that the simple theory of types is indeed sound, with respect to both standard and general semantics. To prove this fact we will need a few lemmata.

3.1 Lemma. *Let* $\mathfrak{M} = (\{D_\alpha\}_{\alpha\in\mathcal{T}}, \{k_c\}_{c\in\mathcal{L}\cup\mathcal{C}})$ *be a general model. Given* $\varphi \in W$*, suppose that* $\alpha \in \mathcal{T}$ *and* $\vartheta, \psi \in W_\alpha$ *are such that* ϑ *is not a single character string,* $\varphi = a\vartheta b$ *for some* $a, b \in \mathcal{S}^*$ *and* $V(\Phi)(\vartheta) = V(\Phi)(\psi)$ *for all* $\Phi \in \mathcal{A}_\mathfrak{M}$*. Then*

$$V(\Phi)(\varphi) = V(\Phi)(a\psi b)$$

for all $\Phi \in \mathcal{A}_\mathfrak{M}$[4]*.*

Proof. By induction on φ.

- Consider $s \in \mathcal{B}$. All $\vartheta \in W$ for which there are $a, b \in \mathcal{S}^*$ such that $s = a\vartheta b$ are single character strings, whence there is nothing to prove.

- Next consider $[\lambda x\tau]$ for $x \in X_\beta$ and $\tau \in W_\delta$, where $\beta, \delta \in \mathcal{T}$. Assume that τ is such that if there are $\gamma \in \mathcal{T}$, $\upsilon, \chi \in W_\gamma$ and $c, d \in \mathcal{S}^*$ such that $\tau = c\upsilon d$, υ is not a single character string and $V(\Psi)(\upsilon) = V(\Psi)(\chi)$ for all $\Psi \in \mathcal{A}_\mathfrak{M}$, then $V(\Psi)(\tau) = V(\Psi)(c\chi d)$ for all $\Psi \in \mathcal{A}_\mathfrak{M}$.

  Suppose there are $\alpha \in \mathcal{T}$ and $\vartheta, \psi \in W_\alpha$ such that ϑ is not a single character string, $V(\Psi)(\vartheta) = V(\Psi)(\psi)$ for all $\Psi \in \mathcal{A}_\mathfrak{M}$ and $[\lambda x\tau] = a\vartheta b$ for some $a, b \in \mathcal{S}^*$. Let $\Phi \in \mathcal{A}_\mathfrak{M}$. Since $\vartheta \neq x$, Lemma 2.9 of [7] assures that either $a = b = \diamond$ or there are $c, d \in \mathcal{S}^*$ such that $c\vartheta d = \tau$, $a = [\lambda xc$ and $b = d]$. In the first case $\vartheta = [\lambda x\tau]$ whereby

  $$V(\Phi)([\lambda x\tau]) = V(\Phi)(\vartheta) = V(\Phi)(\psi) = V(\Phi)(a\psi b),$$

  and in the second

  $$\begin{aligned} V(\Phi)([\lambda x\tau])(e) &= V(\Phi)([\lambda xc\vartheta d])(e) \\ &= V(\Phi^x_e)(c\vartheta d) \\ &= V(\Phi^x_e)(c\psi d) \\ &= V(\Phi)([\lambda xc\psi d])(e) \\ &= V(\Phi)(a\psi b)(e) \end{aligned}$$

  by induction hypothesis, for all $e \in D_\beta$. In either case $V(\Phi)([\lambda x\tau]) = V(\Phi)(a\psi b)$.

[4] Recall that, by Lemma 2.10 of [7], $a\psi b \in W_{T(\varphi)}$

- Finally consider $[\eta\tau]$, where $\eta_{\delta\beta}, \tau_\beta \in W$ for some $\beta, \delta \in \mathcal{T}$. Assume that, if there are $\gamma \in \mathcal{T}$, $\upsilon, \chi \in W_\gamma$ and $c, d \in \mathcal{S}^*$ such that $\xi = c\upsilon d$, υ is not a single character string and $V(\Psi)(\upsilon) = V(\Psi)(\chi)$ for all $\Psi \in \mathcal{A}_{\mathfrak{M}}$, then $V(\Psi)(\xi) = V(\Psi)(c\chi d)$ for all $\Psi \in \mathcal{A}_{\mathfrak{M}}$; for all $\xi \in \{\eta, \tau\}$.

  Suppose that $\alpha \in \mathcal{T}$ and $\vartheta, \psi \in W_\alpha$ are such that ϑ is not a single character string, $V(\Psi)(\vartheta) = V(\Psi)(\psi)$ for all $\Psi \in \mathcal{A}_{\mathfrak{M}}$ and $[\eta\tau] = a\vartheta b$ for some $a, b \in \mathcal{S}^*$. Let $\Phi \in \mathcal{A}_{\mathfrak{M}}$. By Lemma 2.9 of [7], either $a = b = \diamond$, in which case

$$V(\Phi)([\eta\tau]) = V(\Phi)(\vartheta) = V(\Phi)(\psi) = V(\Phi)(a\psi b),$$

  or there are $c, d \in \mathcal{S}^*$ such that $c\vartheta d = \eta$, $a = [c$ and $b = d\tau]$, or $c\vartheta d = \tau$, $a = [\eta c$ and $b = d]$. In the first (remaining) case $V(\Phi)(\eta) = V(\Phi)(c\psi d)$ by induction hypothesis, whence

$$\begin{aligned} V(\Phi)([\eta\tau]) &= V(\Phi)(\eta)(V(\Phi)(\tau)) \\ &= V(\Phi)(c\psi d)(V(\Phi)(\tau)) \\ &= V(\Phi)([c\psi d\tau]) \\ &= V(\Phi)(a\psi b), \end{aligned}$$

  and similarly for the last case.

Thus $V(\Phi)(\varphi) = V(\Phi)(a\psi b)$ for all $\Phi \in \mathcal{A}_{\mathfrak{M}}$, for all $\varphi \in W$, $\alpha \in \mathcal{T}$, $\vartheta, \psi \in W_\alpha$ and $a, b \in \mathcal{S}^*$ with $\varphi = a\vartheta b$ such that ϑ is not a single character string and $V(\Phi)(\vartheta) = V(\Phi)(\psi)$ for all $\Phi \in \mathcal{A}_{\mathfrak{M}}$. □

3.2 Lemma. *Let $\mathfrak{M} = (\{D_\alpha\}_{\alpha\in\mathcal{T}}, \{k_c\}_{c\in\mathcal{L}\cup\mathcal{C}})$ be a general model, $\alpha \in \mathcal{T}$ and $\mathbf{x}, \mathbf{y} \in X_\alpha$. For all $\varphi \in W$ we have that, if $\mathbf{y}$ does not occur in φ,*

$$V(\Phi)(S(\mathbf{x})(\mathbf{y})(\varphi)) = V(\Phi^{\mathbf{x}}_{V(\Phi)(\mathbf{y})})(\varphi)$$

for all $\Phi \in \mathcal{A}_{\mathfrak{M}}$.

Proof. By induction on φ.

- Consider $s \in \mathcal{B}$, such that $s \neq \mathbf{y}$. If $s = \mathbf{x}$, then

$$\begin{aligned} V(\Phi)(S(\mathbf{x})(\mathbf{y})(s)) &= V(\Phi)(\mathbf{y}) \\ &= \Phi^{\mathbf{x}}_{V(\Phi)(\mathbf{y})}(\mathbf{x}) \\ &= V(\Phi^{\mathbf{x}}_{V(\Phi)(\mathbf{y})})(s) \end{aligned}$$

  for all $\Phi \in \mathcal{A}_{\mathfrak{M}}$.

  Otherwise

$$V(\Phi)(S(\mathbf{x})(\mathbf{y})(s)) = V(\Phi)(s) = V(\Phi^{\mathbf{x}}_{V(\Phi)(\mathbf{y})})(s)$$

  for all $\Phi \in \mathcal{A}_{\mathfrak{M}}$.

- Now consider $[\lambda z\psi]$, where $\beta,\gamma \in \mathcal{T}$, $z \in X_\beta$ and $\psi \in W_\gamma$ is such that if y does not occur in ψ, then $V(\Psi)(S(x)(y)(\psi)) = V(\Psi^{x}_{V(\Psi)(y)})(\psi)$ for all $\Psi \in \mathcal{A}_{\mathfrak{M}}$. Assume that y does not occur in $[\lambda z\psi]$, that is $z \neq y$ and y does not occur in ψ. Let $\Phi \in \mathcal{A}_{\mathfrak{M}}$. If $z \neq x$ then

$$\begin{aligned} V(\Phi)(S(x)(y)([\lambda z\psi]))(d) &= V(\Phi)([\lambda z\, S(x)(y)(\psi)])(d) \\ &= V(\Phi^{z}_{d})(S(x)(y)(\psi)) \\ &= V(\Phi^{z\,x}_{d\,V(\Phi^{z}_{d})(y)})(\psi) \\ &= V(\Phi^{z\,x}_{d\,V(\Phi)(y)})(\psi) \\ &= V(\Phi^{x\,z}_{V(\Phi)(y)\,d})(\psi) \\ &= V(\Phi^{x}_{V(\Phi)(y)})([\lambda z\psi])(d) \end{aligned}$$

for all $d \in D_\beta$. If $z = x$, on the other hand, we have

$$\begin{aligned} V(\Phi)(S(x)(y)([\lambda z\psi]))(d) &= V(\Phi)([\lambda y\, S(x)(y)(\psi)])(d) \\ &= V(\Phi^{y}_{d})(S(x)(y)(\psi)) \\ &= V(\Phi^{y\,x}_{d\,V(\Phi^{y}_{d})(y)})(\psi) \\ &= V(\Phi^{x}_{V(\Phi^{y}_{d}(y))})(\psi) \\ &= V(\Phi^{x}_{d})(\psi) \\ &= V(\Phi^{x\,x}_{V(\Phi)(y)\,d})(\psi) \\ &= V(\Phi^{x}_{V(\Phi)(y)})([\lambda z\psi])(d) \end{aligned}$$

for all $d \in D_\beta$, by Lemma 2.6.

- Finally consider $[\psi\vartheta]$, where $\beta,\gamma \in \mathcal{T}$, $\psi_{\beta\gamma}, \vartheta_\gamma \in W$ are such that if y does not occur in η, then $V(\Psi)(S(x)(y)(\eta)) = V(\Psi^{x}_{V(\Psi)(y)})(\eta)$ for all $\Psi \in \mathcal{A}_{\mathfrak{M}}$, for both $\eta \in \{\psi,\vartheta\}$. Assume that y does not occur in $[\psi\vartheta]$, whence y does not occur in either of ψ and ϑ. Thus

$$\begin{aligned} V(\Phi)(S(x)(y)([\psi\vartheta])) &= V(\Phi)([S(x)(y)(\psi)\, S(x)(y)(\vartheta)]) \\ &= V(\Phi)(S(x)(y)(\psi))(V(\Phi)(S(x)(y)(\vartheta))) \\ &= V(\Phi^{x}_{V(\Phi)(y)})(\psi)(V(\Phi^{x}_{V(\Phi)(y)})(\vartheta)) \\ &= V(\Phi^{x}_{V(\Phi)(y)})([\psi\vartheta]) \end{aligned}$$

for all $\Phi \in \mathcal{A}_{\mathfrak{M}}$.

Hence every $\varphi \in W$ is such that if y does not occur in φ then

$$V(\Phi)(S(x)(y)(\varphi)) = V(\Phi^{x}_{V(\Phi)(y)})(\varphi)$$

for all $\Phi \in \mathcal{A}_{\mathfrak{M}}$. □

3.3 Lemma. *Let* $\mathfrak{M} = (\{D_\alpha\}_{\alpha\in\mathcal{T}}, \{k_c\}_{c\in\mathcal{L}\cup\mathcal{C}})$ *be a general model,* $\alpha \in \mathcal{T}$*,* $\boldsymbol{x} \in X_\alpha$ *and* $\psi \in W_\alpha$*. For all* $\varphi \in W$ *we have that, if no free variable of* ψ *except possibly* $\boldsymbol{x}$ *is bound in* φ*,*

$$V(\Phi)(\mathrm{SF}(\boldsymbol{x})(\psi)(\varphi)) = V(\Phi^{\boldsymbol{x}}_{V(\Phi)(\psi)})(\varphi)$$

for all $\Phi \in \mathcal{A}_{\mathfrak{M}}$*.*

Proof. By induction on φ.

- Consider $s \in \mathcal{B}$ (which has no bound variables). If $s = \boldsymbol{x}$ then

$$\begin{aligned} V(\Phi)(\mathrm{SF}(\boldsymbol{x})(\psi)(s)) &= V(\Phi)(\psi) \\ &= \Phi^{\boldsymbol{x}}_{V(\Phi)(\psi)}(s) \\ &= V(\Phi^{\boldsymbol{x}}_{V(\Phi)(\psi)})(s) \end{aligned}$$

  for all $\Phi \in \mathcal{A}_{\mathfrak{M}}$. If, on the other hand, $s \neq \boldsymbol{x}$

$$\begin{aligned} V(\Phi)(\mathrm{SF}(\boldsymbol{x})(\psi)(s)) &= V(\Phi)(s) \\ &= V(\Phi^{\boldsymbol{x}}_{V(\Phi)(\psi)})(s) \end{aligned}$$

  for all $\Phi \in \mathcal{A}_{\mathfrak{M}}$.

- Next consider $[\lambda \boldsymbol{y} \vartheta]$ for some $\beta, \gamma \in \mathcal{T}$, $\boldsymbol{y} \in X_\beta$ and $\vartheta \in W_\gamma$ such that if no free variable of ψ except possibly $\boldsymbol{x}$ is bound in ϑ, then $V(\Psi)(\mathrm{SF}(\boldsymbol{x})(\psi)(\vartheta)) = V(\Psi^{\boldsymbol{x}}_{V(\Psi)(\psi)})(\vartheta)$ for all $\Psi \in \mathcal{A}_{\mathfrak{M}}$. Assume that no free variable of ψ except possibly $\boldsymbol{x}$ is bound in $[\lambda \boldsymbol{y} \vartheta]$, and let $\Phi \in \mathcal{A}_{\mathfrak{M}}$. We consider cases according as $\boldsymbol{y} = \boldsymbol{x}$ or not.

  If $\boldsymbol{y} = \boldsymbol{x}$, then

$$\begin{aligned} V(\Phi)(\mathrm{SF}(\boldsymbol{x})(\psi)([\lambda \boldsymbol{y} \vartheta])) &= V(\Phi)([\lambda \boldsymbol{y} \vartheta]) \\ &= V(\Phi^{\boldsymbol{x}}_{V(\Phi)(\psi)})([\lambda \boldsymbol{y} \vartheta]). \end{aligned}$$

  If $\boldsymbol{y} \neq \boldsymbol{x}$, meanwhile, then no free variable of ψ, except possibly $\boldsymbol{x}$, is bound in ϑ, whereby

$$\begin{aligned} V(\Phi)(\mathrm{SF}(\boldsymbol{x})(\psi)([\lambda \boldsymbol{y} \vartheta]))(d) &= V(\Phi)([\lambda \boldsymbol{y}\, \mathrm{SF}(\boldsymbol{x})(\psi)(\vartheta)])(d) \\ &= V(\Phi^{\boldsymbol{y}}_{d})(\mathrm{SF}(\boldsymbol{x})(\psi)(\vartheta)) \\ &= V(\Phi^{\boldsymbol{y}\,\boldsymbol{x}}_{d\,V(\Phi^{\boldsymbol{y}}_{d})(\psi)})(\vartheta) \\ &= V(\Phi^{\boldsymbol{y}\,\boldsymbol{x}}_{d\,V(\Phi)(\psi)})(\vartheta) \\ &= V(\Phi^{\boldsymbol{x}\,\boldsymbol{y}}_{V(\Phi)(\psi)\,d})(\vartheta) \\ &= V(\Phi^{\boldsymbol{x}}_{V(\Phi)(\psi)})([\lambda \boldsymbol{y} \vartheta])(d) \end{aligned}$$

  for all $d \in D_\beta$.

- Finally consider $[\eta\vartheta]$, where $\beta, \gamma \in \mathcal{T}$ and $\eta_{\beta\gamma}, \vartheta_\gamma \in W$ are such that if no free variable of ψ, except possibly $\mathbf{x}$, is bound in τ, then $V(\Psi)(\mathrm{SF}(\mathbf{x})(\psi)(\tau)) = V(\Psi^{\mathbf{x}}_{V(\Psi)(\psi)})(\tau)$ for all $\Psi \in \mathcal{A}_{\mathfrak{M}}$, for both $\tau \in \{\eta, \vartheta\}$. Assume that no free variable of ψ except possibly $\mathbf{x}$ is bound in $[\eta\vartheta]$, whence no such is bound in any of η or ϑ. Thus

$$\begin{aligned} V(\Phi)(\mathrm{SF}(\mathbf{x})(\psi)([\eta\vartheta])) &= V(\Phi)([\mathrm{SF}(\mathbf{x})(\psi)(\eta)\,\mathrm{SF}(\mathbf{x})(\psi)(\vartheta)]) \\ &= V(\Phi)(\mathrm{SF}(\mathbf{x})(\psi)(\eta))(V(\Phi)(\mathrm{SF}(\mathbf{x})(\psi)(\vartheta))) \\ &= V(\Phi^{\mathbf{x}}_{V(\Phi)(\psi)})(\eta)(V(\Phi^{\mathbf{x}}_{V(\Phi)(\psi)})(\vartheta)) \\ &= V(\Phi^{\mathbf{x}}_{V(\Phi)(\psi)})([\eta\vartheta]) \end{aligned}$$

  for all $\Phi \in \mathcal{A}_{\mathfrak{M}}$.

Hence $V(\Phi)(\mathrm{SF}(\mathbf{x})(\psi)(\varphi)) = V(\Phi^{\mathbf{x}}_{V(\Phi)(\psi)})(\varphi)$ for all $\Phi \in \mathcal{A}_{\mathfrak{M}}$, for all $\varphi \in W$ such that no free variable of ψ, except possibly $\mathbf{x}$, is bound in φ, as desired. □

3.4 Lemma. *Let* $\mathfrak{M} = (\{D_\alpha\}_{\alpha\in\mathcal{T}}, \{k_c\}_{c\in\mathcal{L}\cup\mathcal{C}})$ *be a general model,* $\alpha \in \mathcal{T}$ *and* $\mathbf{x}, \mathbf{y} \in X_\alpha$*. For all* $\varphi \in W$ *such that* $\mathbf{y}$ *does not occur in* φ *we have that*

$$V(\Phi)(\mathrm{SB}(\mathbf{x})(\mathbf{y})(\varphi)) = V(\Phi)(\varphi)$$

for all $\Phi \in \mathcal{A}_{\mathfrak{M}}$*.*

Proof. By induction on φ.

- First consider $s \in \mathcal{B}$. Since $\mathrm{SB}(\mathbf{x})(\mathbf{y})(\varphi) = s$ we have

$$V(\Phi)(\mathrm{SB}(\mathbf{x})(\mathbf{y})(s)) = V(\Phi)(s)$$

  for all $\Phi \in \mathcal{A}_{\mathfrak{M}}$ trivially.

- Consider $[\lambda \mathbf{z}\psi]$, where $\beta, \gamma \in \mathcal{T}$, $\mathbf{z} \in X_\beta$ and $\psi \in W_\gamma$ is such that if $\mathbf{y}$ does not occur in ψ then $V(\Psi)(\mathrm{SB}(\mathbf{x})(\mathbf{y})(\psi)) = V(\Psi)(\psi)$ for all $\Psi \in \mathcal{A}_{\mathfrak{M}}$. Let $\Phi \in \mathcal{A}_{\mathfrak{M}}$ and assume that $\mathbf{y}$ does not occur in $[\lambda \mathbf{z}\psi]$. Thus if $\mathbf{z} = \mathbf{x}$ then

$$\begin{aligned} V(\Phi)(\mathrm{SB}(\mathbf{x})(\mathbf{y})([\lambda \mathbf{z}\psi]))(d) &= V(\Phi)([\lambda \mathbf{y}\,\mathrm{S}(\mathbf{x})(\mathbf{y})(\psi)])(d) \\ &= V(\Phi^{\mathbf{y}}_{d})(\mathrm{S}(\mathbf{x})(\mathbf{y})(\psi)) \\ &= V(\Phi^{\mathbf{y}\,\mathbf{x}}_{d\,V(\Phi^{\mathbf{y}}_{d})(\mathbf{y})})(\psi) \\ &= V(\Phi^{\mathbf{y}\,\mathbf{x}}_{d\,d})(\psi) \\ &= V(\Phi^{\mathbf{x}}_{d})(\psi) \\ &= V(\Phi)([\lambda \mathbf{z}\psi])(d) \end{aligned}$$

  for all $d \in D_\beta$, whereas if $\mathbf{z} \neq \mathbf{x}$ then

$$\begin{aligned} V(\Phi)(\mathrm{SB}(\mathbf{x})(\mathbf{y})([\lambda \mathbf{z}\psi]))(d) &= V(\Phi)([\lambda \mathbf{z}\,\mathrm{SB}(\mathbf{x})(\mathbf{y})(\psi)])(d) \\ &= V(\Phi^{\mathbf{z}}_{d})(\mathrm{SB}(\mathbf{x})(\mathbf{y})(\psi)) \\ &= V(\Phi^{\mathbf{z}}_{d})(\psi) \\ &= V(\Phi)([\lambda \mathbf{z}\psi])(d) \end{aligned}$$

  for all $d \in D_\beta$.

- Consider $[\psi\vartheta]$, where $\beta, \gamma \in \mathcal{T}$, $\psi_{\beta\gamma}, \vartheta_\gamma \in W$ are such that $V(\Psi)(\mathrm{SB}(\mathbf{x})(\mathbf{y})(\tau)) = V(\Psi)(\tau)$ for all $\Psi \in \mathcal{A}_{\mathfrak{M}}$ if $\mathbf{y}$ does not occur in τ, for both $\tau \in \{\psi, \vartheta\}$. Assume that $\mathbf{y}$ does not occur in $[\psi\vartheta]$. Then

$$\begin{aligned} V(\Phi)(\mathrm{SB}(\mathbf{x})(\mathbf{y})([\psi\vartheta])) &= V(\Phi)([\mathrm{SB}(\mathbf{x})(\mathbf{y})(\psi)\,\mathrm{SB}(\mathbf{x})(\mathbf{y})(\vartheta)]) \\ &= V(\Phi)(\mathrm{SB}(\mathbf{x})(\mathbf{y})(\psi))(V(\Phi)(\mathrm{SB}(\mathbf{x})(\mathbf{y})(\vartheta))) \\ &= V(\Phi)(\psi)(V(\Phi)(\vartheta)) \\ &= V(\Phi)([\psi\vartheta]) \end{aligned}$$

for all $\Phi \in \mathcal{A}_{\mathfrak{M}}$.

Hence $V(\Phi)(\mathrm{SB}(\mathbf{x})(\mathbf{y})(\varphi)) = V(\Phi)(\varphi)$ for all $\Phi \in \mathcal{A}_{\mathfrak{M}}$ and all $\varphi \in W$ such that $\mathbf{y}$ does not occur in φ, as claimed. □

3.5 Theorem (Soundness Theorem)**.** *Let* $\Gamma \subseteq \overline{W}_o$ *and* $\varphi \in W_o$ *be such that* $\Gamma \vdash \varphi$*. Then* $\Gamma \models \varphi$*.*

Proof. By complete induction on the steps of the proof. Let $n \in \mathbb{Z}^+$ and $\mathfrak{P}$ be a proof of φ on the assumptions Γ, of length n. We will prove that

$$\Gamma \models \mathfrak{P}(j)$$

for all $j \in [0, n[$. Thus let $k \in [0, n[$ and assume that

$$\Gamma \models \mathfrak{P}(l)$$

for all $l \in [0, k[$. Let $\mathfrak{M} = (\{D_\alpha\}_{\alpha\in\mathcal{T}}, \{k_c\}_{c\in\mathcal{L}\cup\mathcal{C}})$ be a general model of Γ, so that

$$\Gamma \models \mathfrak{P}(l)$$

for all $l \in [0, k[$. By definition, $\mathfrak{P}(k)$ is either one of the axioms 1–6, 10, 10^o or 11, a member of Γ or can be inferred from $\{\mathfrak{P}(l) \,|\, l \in [0, l[\}$. If $\mathfrak{P}(k)$ is one of the above axioms, then it is generally valid, whence $\mathfrak{M} \models \mathfrak{P}(k)$. If $\mathfrak{P}(k) \in \Gamma$ then $\mathfrak{M} \models \mathfrak{P}(k)$, by definition. If, finally, $\mathfrak{P}(k)$ can be inferred from $\{\mathfrak{P}(l) \,|\, l \in [0, l[\}$, we consider cases according to by which rule of inference this is possible:

α-conversion Assume that $\mathfrak{P}(k)$ can be inferred from $\mathfrak{P}(m)$ by α-conversion, where $m \in [0, k[$. Then there are $\alpha, \beta \in \mathcal{T}$, $\mathbf{x}, \mathbf{y} \in X_\alpha$, $a, b \in \mathcal{S}^*$ and $\vartheta \in W_\beta$ such that $\mathbf{x}$ is not free in ϑ, $\mathbf{y}$ does not occur in ϑ, $a\vartheta b = \mathfrak{P}(m)$ and $a\,\mathrm{S}(\mathbf{x})(\mathbf{y})(\vartheta)b = \mathfrak{P}(k)$.

If ϑ is a single symbol then $\mathbf{x}$ does not occur in ϑ, whence $\mathrm{S}(\mathbf{x})(\mathbf{y})(\vartheta) = \vartheta$. In that case

$$\mathfrak{P}(k) = a\,\mathrm{S}(\mathbf{x})(\mathbf{y})(\vartheta)b = a\vartheta b = \mathfrak{P}(m),$$

whereby $\mathfrak{M} \models \mathfrak{P}(k)$ by induction hypothesis.

Otherwise, let $\Phi \in \mathcal{A}_{\mathfrak{M}}$. By induction hypothesis

$$V(\Phi)(\mathfrak{P}(m)) = 1.$$

Since x is not free in ϑ,

$$S(x)(y)(\vartheta) = SB(x)(y)(\vartheta),$$

whence by Lemma 3.4 $V(\Phi)(S(x)(y)(\vartheta)) = V(\Phi)(\vartheta)$. By Lemma 3.1,

$$V(\Phi)(\mathfrak{P}(k)) = V(\Phi)(\mathfrak{P}(m)) = 1.$$

Hence $\mathfrak{M} \models \mathfrak{P}(k)$.

β-contraction Assume that $\mathfrak{P}(k)$ can be inferred from $\mathfrak{P}(m)$ by β-contraction, where $m \in [0, k[$. Then there are $\alpha, \beta \in \mathcal{T}$, $x, \in X_\alpha$, $a, b \in \mathcal{S}^*$ and $\psi_\alpha, \vartheta_\beta \in W$ such that neither x nor any free variable of ψ is a bound variable of ϑ, $\mathfrak{P}(m) = a[[\lambda x\vartheta]\psi]b$ and $\mathfrak{P}(k) = a\,S(x)(\psi)(\vartheta)b$. Let $\Phi \in \mathcal{A}_{\mathfrak{M}}$. By induction hypothesis

$$V(\Phi)(\mathfrak{P}(m)) = 1.$$

By Lemma 3.3 we have that

$$\begin{aligned} V(\Phi)([[\lambda x\vartheta]\psi]) &= V(\Phi)([\lambda x\vartheta])(V(\Phi)(\psi)) \\ &= V(\Phi^{x}_{V(\Phi)(\psi)})(\vartheta) \\ &= V(\Phi)(SF(x)(\psi)(\vartheta)) \\ &= V(\Phi)(S(x)(\psi)(\vartheta)). \end{aligned}$$

Since $[\lambda x\vartheta]$ not a single character, Lemma 3.1 assures that

$$V(\Phi)(\mathfrak{P}(k)) = V(\Phi)(\mathfrak{P}(m)) = 1.$$

Hence $\mathfrak{M} \models \mathfrak{P}(k)$.

β-expansion Assume that $\mathfrak{P}(k)$ can be inferred from $\mathfrak{P}(m)$ by β-expansion, where $m \in [0, k[$. Then there are $\alpha, \beta \in \mathcal{T}$, $x, \in X_\alpha$, $a, b \in \mathcal{S}^*$ and $\psi_\alpha, \vartheta_\beta \in W$ such that neither x nor any free variable of ψ is a bound variable of ϑ, $\mathfrak{P}(m) = a\,S(x)(\psi)(\vartheta)b$ and $\mathfrak{P}(k) = a[[\lambda x\vartheta]\psi]b$. Let $\Phi \in \mathcal{A}_{\mathfrak{M}}$. By induction hypothesis

$$V(\Phi)(\mathfrak{P}(m)) = 1.$$

It follows by lemmata 3.3 and 3.1 that $V(\Phi)(\mathfrak{P}(k)) = 1$ similarly to the previous case (read the equations backwards).

Hence $\mathfrak{M} \models \mathfrak{P}(m)$.

Substitution If, for some $m \in [0, k[$, $\mathfrak{P}(k)$ can be inferred from $\mathfrak{P}(m)$ by Substitution, there are $\alpha \in \mathcal{T}$, $x \in X_\alpha$ and $\vartheta_{o\alpha}, \psi_\alpha \in W$ such that x is not free in ϑ, $\mathfrak{P}(m) = [\vartheta x]$ and $\mathfrak{P}(k) = [\vartheta\psi]$. Let $\Phi \in \mathcal{A}_{\mathfrak{M}}$. By induction hypothesis and Lemma 2.10

$$\begin{aligned}
1 &= V(\Phi^x_{V(\Phi)(\psi)})(\mathfrak{P}(m)) \\
&= V(\Phi^x_{V(\Phi)(\psi)})(\vartheta)(V(\Phi^x_{V(\Phi)(\psi)})(x)) \\
&= V(\Phi)(\vartheta)(V(\Phi)(\psi)) \\
&= V(\Phi)([\vartheta\psi]) \\
&= V(\Phi)(\mathfrak{P}(k)).
\end{aligned}$$

Thus $\mathfrak{M} \models \mathfrak{P}(k)$ as desired.

Modus ponens If, for some $m, p \in [0, k[$, $\mathfrak{P}(k)$ can be inferred form $\mathfrak{P}(m)$ and $\mathfrak{P}(p)$ by Modus ponens, we can without loss of generality assume that

$$\mathfrak{P}(m) = (\mathfrak{P}(p) \to \mathfrak{P}(k)).$$

For every $\Phi \in \mathcal{A}_{\mathfrak{M}}$ we then have that

$$V(\Phi)([\mathfrak{P}(p) \to \mathfrak{P}(k)]) = 1$$

and

$$V(\Phi)(\mathfrak{P}(p)) = 1$$

by induction hypothesis, whence

$$V(\Phi)(\mathfrak{P}(k)) = 1$$

by Lemma 2.14.

Hence $\mathfrak{M} \models \mathfrak{P}(k)$.

Generalisation If, for some $m \in [0, k[$, $\mathfrak{P}(k)$ can be inferred form $\mathfrak{P}(m)$ by Generalisation, then there are $\alpha \in \mathcal{T}$, $x \in X_\alpha$ and $\vartheta \in W_{o\alpha}$ such that x is not free in ϑ, $\mathfrak{P}(m) = [\vartheta x]$ and $\mathfrak{P}(k) = [\Pi_{o\langle o\alpha\rangle}\vartheta]$. Let $\Phi \in \mathcal{A}_{\mathfrak{M}}$. Furthermore let $d \in D_\alpha$. By induction hypothesis

$$\begin{aligned}
1 &= V(\Phi^x_d)(\mathfrak{P}(m)) \\
&= V(\Phi^x_d)(\vartheta)(V(\Phi^x_d)(x)) \\
&= V(\Phi)(\vartheta)(d)
\end{aligned}$$

by Lemma 2.10. Since $d \in D_\alpha$ was arbitrary

$$\begin{aligned}
1 &= k_{\Pi_{o\langle o\alpha\rangle}}(V(\Phi)(\vartheta)) \\
&= V(\Phi)(\Pi_{o\langle o\alpha\rangle})(V(\Phi)(\vartheta)) \\
&= V(\Phi)(\mathfrak{P}(k)).
\end{aligned}$$

Accordingly $\mathfrak{M} \models \mathfrak{P}(k)$ as desired.

We thereby conclude that $\mathfrak{M} \models \mathfrak{P}(k)$.

Thus $\Gamma \models \mathfrak{P}(j)$ for all $j \in [0, n[$. In particular, since $\mathfrak{P}(n-1) = \varphi$, $\Gamma \models \varphi$. $\square$

Since every standard model is a general model, we get the following corollary.

3.6 Corollary. *Let $\Gamma \subseteq \overline{W}_o$ and $\varphi \in W_o$ be such that $\Gamma \vdash \varphi$. Then $\mathfrak{M} \models \varphi$ for every standard model $\mathfrak{M}$ of Γ.*

4. (In)Completeness

We finally arrive at the differing property of the two notions of semantics that we have considered; completeness. This is a sort of converse of soundness, stating that all truths are provable. We will in this section verify that general semantics makes the simple theory of types a complete formal system, while the opposite hold true of standard semantics. The proof of completeness is not as straightforward as that of soundness, though, and will require a nontrivial amount of machinery. The proof of incompleteness, meanwhile, is, in comparison, almost direct (and quite elementary).

The Notion of Consistency

We start off with a few lemmata aiming towards the definition, and subsequent results, of consistency.

4.1 Lemma. *Let* $\Gamma \subseteq \overline{W}_o$, $\alpha \in \mathcal{T}$, $\mathbf{x}, \mathbf{y} \in X_\alpha$ *and* $\psi \in W_\alpha$ *be such that* $\mathbf{y}$ *is not free in* ψ *and* $\mathbf{x}$ *does not occur in* ψ. *Then* $\mathfrak{a}\,\mathrm{SF}(\mathbf{x})(\mathrm{S}(\mathbf{y})(\mathbf{x})(\psi))(\varphi)\mathfrak{b} \in W_o$ *and*

$$\Gamma \vdash \mathfrak{a}\,\mathrm{SF}(\mathbf{x})(\psi)(\varphi)\mathfrak{b} \Rightarrow \Gamma \vdash \mathfrak{a}\,\mathrm{SF}(\mathbf{x})(\mathrm{S}(\mathbf{y})(\mathbf{x})(\psi))(\varphi)\mathfrak{b}$$

for all $\varphi \in W$ *and* $\mathfrak{a}, \mathfrak{b} \in \mathcal{S}^*$ *such that* $\mathfrak{a}\,\mathrm{SF}(\mathbf{x})(\psi)(\varphi)\mathfrak{b} \in W_o$.

Proof. If $\mathbf{y} = \mathbf{x}$ there is nothing to prove, hence assume $\mathbf{y} \neq \mathbf{x}$. We proceed by induction on φ.

- First consider $\mathsf{s} \in \mathcal{B}$ and let $\mathfrak{a}, \mathfrak{b} \in \mathcal{S}^*$ be such that $\mathfrak{a}\,\mathrm{SF}(\mathbf{x})(\psi)(\mathsf{s})\mathfrak{b} \in W_o$. If $\mathsf{s} = \mathbf{x}$ then $\mathfrak{a}\,\mathrm{SF}(\mathbf{x})(\psi)(\mathsf{s})\mathfrak{b} = \mathfrak{a}\psi\mathfrak{b}$ and $\mathfrak{a}\,\mathrm{SF}(\mathbf{x})(\mathrm{S}(\mathbf{y})(\mathbf{x})(\psi))(\mathsf{s})\mathfrak{b} = \mathfrak{a}\,\mathrm{S}(\mathbf{y})(\mathbf{x})(\psi)\mathfrak{b}$ by definition of SF, whence

$$\begin{aligned}\Gamma \vdash \mathfrak{a}\,\mathrm{SF}(\mathbf{x})(\psi)(\mathsf{s})\mathfrak{b} &\Leftrightarrow \Gamma \vdash \mathfrak{a}\psi\mathfrak{b}\\ &\Rightarrow \Gamma \vdash \mathfrak{a}\,\mathrm{S}(\mathbf{y})(\mathbf{x})(\psi)\mathfrak{b}\\ &\Leftrightarrow \Gamma \vdash \mathfrak{a}\,\mathrm{SF}(\mathbf{x})(\mathrm{S}(\mathbf{y})(\mathbf{x})(\psi))(\mathsf{s})\mathfrak{b}\end{aligned}$$

  by α-conversion, whereby $\mathfrak{a}\,\mathrm{SF}(\mathbf{x})(\mathrm{S}(\mathbf{y})(\mathbf{x})(\psi))(\mathsf{s})\mathfrak{b} \in W_o$ by Theorem 3.3 of [7]. Otherwise

$$\begin{aligned}\mathfrak{a}\,\mathrm{SF}(\mathbf{x})(\mathrm{S}(\mathbf{y})(\mathbf{x})(\psi))(\mathsf{s})\mathfrak{b} &= \mathfrak{a}\mathsf{s}\mathfrak{b}\\ &= \mathfrak{a}\,\mathrm{SF}(\mathbf{x})(\psi)(\mathsf{s})\mathfrak{b} \in W_o\end{aligned}$$

  by definition of SF, whereby

$$\Gamma \vdash \mathfrak{a}\,\mathrm{SF}(\mathbf{x})(\psi)(\mathsf{s})\mathfrak{b} \Leftrightarrow \Gamma \vdash \mathfrak{a}\,\mathrm{SF}(\mathbf{x})(\mathrm{S}(\mathbf{y})(\mathbf{x})(\psi))(\mathsf{s})\mathfrak{b}.$$

- Consider $[\lambda \mathbf{z}\vartheta]$, where $\beta, \gamma \in \mathcal{T}$, $\mathbf{z} \in X_\gamma$ and $\vartheta \in W_\beta$ is such that

$$\Gamma \vdash \mathfrak{c}\,\mathrm{SF}(\mathbf{x})(\psi)(\vartheta)\mathfrak{d} \Rightarrow \Gamma \vdash \mathfrak{c}\,\mathrm{SF}(\mathbf{x})(\mathrm{S}(\mathbf{y})(\mathbf{x})(\psi))(\vartheta)\mathfrak{d}$$

for all $\mathfrak{c}, \mathfrak{d} \in \mathcal{S}^*$ such that $\mathfrak{c}\,\mathrm{SF}(\boldsymbol{x})(\psi)(\vartheta)\mathfrak{d} \in W_o$. Assume $\mathfrak{a}, \mathfrak{b} \in \mathcal{S}^*$ are such that $\mathfrak{a}\,\mathrm{SF}(\boldsymbol{x})(\psi)([\lambda z\vartheta])\mathfrak{b} \in W_o$. If $\boldsymbol{z} = \boldsymbol{x}$ then

$$\begin{aligned}\mathfrak{a}\,\mathrm{SF}(\boldsymbol{x})(\mathrm{S}(\boldsymbol{y})(\boldsymbol{x})(\psi))([\lambda z\vartheta])\mathfrak{b} &= \mathfrak{a}[\lambda z\vartheta]\mathfrak{b}\\ &= \mathfrak{a}\,\mathrm{SF}(\boldsymbol{x})(\psi)([\lambda z\vartheta])\mathfrak{b} \in W_o\end{aligned}$$

by the definition of SF, so that

$$\Gamma \vdash \mathfrak{a}\,\mathrm{SF}(\boldsymbol{x})(\psi)([\lambda z\vartheta])\mathfrak{b} \Leftrightarrow \Gamma \vdash \mathfrak{a}\,\mathrm{SF}(\boldsymbol{x})(\mathrm{S}(\boldsymbol{y})(\boldsymbol{x})(\psi))([\lambda z\vartheta])\mathfrak{b}.$$

Otherwise

$$\mathfrak{a}\,\mathrm{SF}(\boldsymbol{x})(\psi)([\lambda z\vartheta])\mathfrak{b} = \mathfrak{a}[\lambda z\,\mathrm{SF}(\boldsymbol{x})(\psi)(\vartheta)]\mathfrak{b}$$

and

$$\mathfrak{a}\,\mathrm{SF}(\boldsymbol{x})(\mathrm{S}(\boldsymbol{y})(\boldsymbol{x})(\psi))([\lambda z\vartheta])\mathfrak{b} = \mathfrak{a}[\lambda z\,\mathrm{SF}(\boldsymbol{x})(\mathrm{S}(\boldsymbol{y})(\boldsymbol{x})(\psi))(\vartheta)]\mathfrak{b}$$

by the definition of SF, whence

$$\mathfrak{a}\,\mathrm{SF}(\boldsymbol{x})(\mathrm{S}(\boldsymbol{y})(\boldsymbol{x})(\psi))([\lambda z\vartheta])\mathfrak{b} \in W_o$$

and

$$\begin{aligned}\Gamma \vdash \mathfrak{a}\,\mathrm{SF}(\boldsymbol{x})(\psi)([\lambda z\vartheta])\mathfrak{b} &\Leftrightarrow \Gamma \vdash \mathfrak{a}[\lambda z\,\mathrm{SF}(\boldsymbol{x})(\psi)(\vartheta)]\mathfrak{b}\\ &\Rightarrow \Gamma \vdash \mathfrak{a}[\lambda z\,\mathrm{SF}(\boldsymbol{x})(\mathrm{S}(\boldsymbol{y})(\boldsymbol{x})(\psi))(\vartheta)]\mathfrak{b}\\ &\Leftrightarrow \Gamma \vdash \mathfrak{a}\,\mathrm{SF}(\boldsymbol{x})(\mathrm{S}(\boldsymbol{y})(\boldsymbol{x})(\psi))([\lambda z\vartheta])\mathfrak{b}\end{aligned}$$

by induction hypothesis.

- Finally let $\beta, \gamma \in \mathcal{T}$ and consider $[\eta\vartheta]$, where $\eta_{\beta\gamma}, \vartheta_\beta \in W$ are such that

$$\Gamma \vdash \mathfrak{c}\,\mathrm{SF}(\boldsymbol{x})(\psi)(\tau)\mathfrak{d} \Rightarrow \Gamma \vdash \mathfrak{c}\,\mathrm{SF}(\boldsymbol{x})(\mathrm{S}(\boldsymbol{y})(\boldsymbol{x})(\psi))(\tau)\mathfrak{d}$$

for all $\mathfrak{c}, \mathfrak{d} \in \mathcal{S}^*$ such that $\mathfrak{a}\,\mathrm{SF}(\boldsymbol{x})(\psi)(\tau)\mathfrak{b} \in W_o$, for both $\tau \in \{\eta, \vartheta\}$. Let $\mathfrak{a}, \mathfrak{b} \in \mathcal{S}^*$ be such that $\mathfrak{a}\,\mathrm{SF}(\boldsymbol{x})(\psi)([\eta\vartheta])\mathfrak{b} \in W_o$. Then

$$\mathfrak{a}\,\mathrm{SF}(\boldsymbol{x})(\psi)([\eta\vartheta])\mathfrak{b} = \mathfrak{a}[\mathrm{SF}(\boldsymbol{x})(\psi)(\eta)\,\mathrm{SF}(\boldsymbol{x})(\psi)(\vartheta)]\mathfrak{b} \in W_o$$

and

$$\mathfrak{a}\,\mathrm{SF}(\boldsymbol{x})(\mathrm{S}(\boldsymbol{y})(\boldsymbol{x})(\psi))([\eta\vartheta])\mathfrak{b} = [\mathrm{SF}(\boldsymbol{x})(\mathrm{S}(\boldsymbol{y})(\boldsymbol{x})(\psi))(\eta)\,\mathrm{SF}(\boldsymbol{x})(\mathrm{S}(\boldsymbol{y})(\boldsymbol{x})(\psi))(\vartheta)]\mathfrak{b} \in W_o$$

by definition of SF, whence

$$\begin{aligned}\Gamma \vdash \mathfrak{a}\,\mathrm{SF}(\boldsymbol{x})(\psi)([\eta\vartheta])\mathfrak{b} &\Leftrightarrow \Gamma \vdash \mathfrak{a}[\mathrm{SF}(\boldsymbol{x})(\psi)(\eta)\,\mathrm{SF}(\boldsymbol{x})(\psi)(\vartheta)]\mathfrak{b}\\ &\Rightarrow \Gamma \vdash \mathfrak{a}[\mathrm{SF}(\boldsymbol{x})(\mathrm{S}(\boldsymbol{y})(\boldsymbol{x})(\psi))(\eta)\,\mathrm{SF}(\boldsymbol{x})(\psi)(\vartheta)]\mathfrak{b}\\ &\Rightarrow \Gamma \vdash \mathfrak{a}[\mathrm{SF}(\boldsymbol{x})(\mathrm{S}(\boldsymbol{y})(\boldsymbol{x})(\psi))(\eta)\,\mathrm{SF}(\boldsymbol{x})(\mathrm{S}(\boldsymbol{y})(\boldsymbol{x})(\psi))(\vartheta)]\mathfrak{b}\\ &\Leftrightarrow \Gamma \vdash \mathfrak{a}\,\mathrm{SF}(\boldsymbol{x})(\mathrm{S}(\boldsymbol{y})(\boldsymbol{x})(\psi))([\eta\vartheta])\mathfrak{b}\end{aligned}$$

by induction hypothesis.

□

4.2 Lemma. *For all* $\Gamma \subseteq \overline{W}_o$*,* $\varphi \in W$*,* $\alpha \in \mathcal{T}$*,* $\mathfrak{x}, \mathfrak{y} \in X_\alpha$ *and* $\mathfrak{a}, \mathfrak{b} \in \mathcal{S}^*$ *the following hold:*

1. $\mathfrak{a}\varphi\mathfrak{b} \in W_o \Leftrightarrow \mathfrak{a}\,\mathrm{SB}(\mathfrak{x})(\mathfrak{y})(\varphi)\mathfrak{b} \in W_o$.
2. *If* $\mathfrak{y}$ *does not occur in* φ *then* $\Gamma \vdash \mathfrak{a}\varphi\mathfrak{b} \Leftrightarrow \Gamma \vdash \mathfrak{a}\,\mathrm{SB}(\mathfrak{x})(\mathfrak{y})(\varphi)\mathfrak{b}$.

Proof. 1. If φ is a single character string, then $\mathrm{SB}(\mathfrak{x})(\mathfrak{y})(\varphi) = \varphi$ and the conclusion is trivial. Otherwise, it is a special case of Lemma 2.10 of [7].

2. By 1 this is a meaningful statement. Otherwise the proof is the same argument as the proof of Theorem 3.8.1 of [7]; adjoining $\mathfrak{a}$ and $\mathfrak{b}$ does not have any impact.

□

4.3 Lemma. *Let* $\Gamma \subseteq \overline{W}_o$*,* $\alpha, \beta \in \mathcal{T}$*,* $\varphi_\alpha, \psi_\beta \in W$*,* $\mathfrak{x} \in X_\beta$ *and* $\mathfrak{a}, \mathfrak{b} \in \mathcal{S}^*$ *be such that* $\mathfrak{a}[[\lambda\mathfrak{x}\varphi]\psi]\mathfrak{b} \in W_o$*. Then* $\mathfrak{a}\,\mathrm{SF}(\mathfrak{x})(\psi)(\varphi)\mathfrak{b} \in W_o$ *and*

$$\Gamma \vdash \mathfrak{a}[[\lambda\mathfrak{x}\varphi]\psi]\mathfrak{b} \Rightarrow \Gamma \vdash \mathfrak{a}\,\mathrm{SF}(\mathfrak{x})(\psi)(\varphi)\mathfrak{b}$$

provided $\mathfrak{x} \notin \mathsf{VF}(\psi)$ *and no free variable of* ψ *is bound in* φ *(in particular, if* ψ *is closed).*

Proof. Since $[[\lambda\mathfrak{x}\varphi]\psi]$ is clearly not a single character, the first claim follows by Lemma 2.10 of [7]. Furthermore let $\mathfrak{y}, \mathfrak{z} \in X_\beta$ be distinct and such that neither occurs in $\mathfrak{a}[[\lambda\mathfrak{x}\varphi]\psi]\mathfrak{b}$, and let $\vartheta = \mathrm{S}(\mathfrak{x})(\mathfrak{z})(\psi)$. By Lemma 4.2 above

$$\Gamma \vdash \mathfrak{a}[[\lambda\mathfrak{x}\,\mathrm{SB}(\mathfrak{x})(\mathfrak{y})(\varphi)]\psi]\mathfrak{b}.$$

Now, by α-conversion

$$\Gamma \vdash \mathfrak{a}[[\lambda\mathfrak{x}\,\mathrm{SB}(\mathfrak{x})(\mathfrak{y})(\varphi)]\vartheta]\mathfrak{b},$$

whence

$$\Gamma \vdash \mathfrak{a}\,\mathrm{S}(\mathfrak{x})(\vartheta)(\mathrm{SB}(\mathfrak{x})(\mathfrak{y})(\varphi))\mathfrak{b}$$

by β-contraction. Since $\mathfrak{x}$ is not bound in $\mathrm{SB}(\mathfrak{x})(\mathfrak{y})(\varphi)$,

$$\mathrm{S}(\mathfrak{x})(\vartheta)(\mathrm{SB}(\mathfrak{x})(\mathfrak{y})(\varphi)) = \mathrm{SF}(\mathfrak{x})(\vartheta)(\mathrm{SB}(\mathfrak{x})(\mathfrak{y})(\varphi)).$$

Furthermore, since $\mathfrak{x}$ does not occur in ϑ

$$\mathrm{SF}(\mathfrak{x})(\vartheta)(\mathrm{SB}(\mathfrak{x})(\mathfrak{y})(\varphi)) = \mathrm{SB}(\mathfrak{x})(\mathfrak{y})(\mathrm{SF}(\mathfrak{x})(\vartheta)(\varphi))$$

by Lemma 2.24.2 of [7]. By Lemma 4.2 again

$$\Gamma \vdash \mathfrak{a}\,\mathrm{SF}(\mathfrak{x})(\vartheta)(\varphi)\mathfrak{b}$$

since $\boldsymbol{y}$ does not occur in $\mathrm{SF}(\boldsymbol{x})(\vartheta)(\varphi)$. Note that $\boldsymbol{z}$ is not free in ϑ, as $\boldsymbol{x}$ was not so in ψ; thus

$$\Gamma \vdash \mathfrak{a}\,\mathrm{SF}(\boldsymbol{x})(\mathrm{S}(\boldsymbol{z})(\boldsymbol{x})(\vartheta))(\varphi)\mathfrak{b}$$

by Lemma 4.1. Since $\mathrm{S}(\boldsymbol{z})(\boldsymbol{x})(\vartheta) = \psi$,

$$\Gamma \vdash \mathfrak{a}\,\mathrm{SF}(\boldsymbol{x})(\psi)(\varphi)\mathfrak{b}$$

as desired. □

4.4 Lemma (Formal theorems)**.** *For all* $\alpha, \beta \in \mathcal{T}$, $\boldsymbol{x} \in \mathcal{X}$ *and* $\varphi, \psi, \vartheta, \tau \in W$ *we have*

1. $\vdash ([\Pi_{o\langle o\alpha\rangle}\varphi_{o\alpha}] \to [\varphi_{o\alpha}\psi_\alpha])$.
2. $\vdash (\bot \to \varphi_o)$.
3. $\vdash ((\neg\varphi_o) \to (\varphi_o \to \bot))$.
4. $\vdash (\varphi_o \to \varphi_o)$.
5. $\vdash ((\neg\varphi_o) \vee \varphi_o)$.
6. $\vdash ((\varphi_o \to \psi_o) \to ((\psi_o \to \varphi_o) \to (\varphi_o \equiv \psi_o)))$.
7. $\vdash (\varphi_o \to (\neg(\neg\varphi_o)))$.
8. $\vdash ((\neg(\neg\varphi_o)) \to \varphi_o)$.
9. $\vdash (\varphi_o \equiv (\neg(\neg\varphi_o)))$.
10. $\vdash (((\neg\varphi_o) \to \varphi_o) \to \varphi_o)$.
11. $\vdash ((\bot \vee \varphi_o) \equiv \varphi_o)$.
12. $\vdash \top$.
13. $\vdash ((\top \vee \varphi_o) \equiv \top)$.
14. $\vdash ([\varphi_{o\alpha}\psi_\alpha] \to [\varphi_{o\alpha}[\iota_{\alpha\langle o\alpha\rangle}\varphi_{o\alpha}]])$.
15. $\vdash ((\forall \boldsymbol{x}_\alpha \varphi_o) \to \varphi_o)$.

For $\alpha, \beta \in \mathcal{T}$, $\varphi, \psi, \vartheta, \tau \in \overline{W}$ *and* $\boldsymbol{x} \in \mathcal{X}$.

16. $\vdash ((\varphi_o \to \psi_o) \to (((\neg\varphi_o) \to \psi_o) \to \psi_o))$.
17. $\vdash ((\varphi_\alpha \equiv \psi_\alpha) \to ([\vartheta_{o\alpha}\varphi_\alpha] \to [\vartheta_{o\alpha}\psi_\alpha]))$.
18. $\vdash (\varphi_\alpha \equiv \varphi_\alpha)$.
19. $\vdash ((\varphi_\alpha \equiv \psi_\alpha) \to (\psi_\alpha \equiv \varphi_\alpha))$.

20. $\vdash ((\varphi_\alpha \equiv \psi_\alpha) \to ((\psi_\alpha \equiv \vartheta_\alpha) \to (\varphi_\alpha \equiv \vartheta_\alpha)))$.

21. $\vdash (\varphi_o \to ((\varphi_o \equiv \psi_o) \to \psi_o))$.

22. $\vdash ((\neg\varphi_o) \to ((\varphi_o \equiv \psi_o) \to (\neg\psi_o)))$.

23. $\vdash (\varphi_o \to (\psi_o \to (\varphi_o \equiv \psi_o)))$.

24. $\vdash ((\neg\varphi_o) \to ((\neg\psi_o) \to (\varphi_o \equiv \psi_o)))$.

25. $\vdash ((\varphi_{\alpha\beta} \equiv \psi_{\alpha\beta}) \to ((\vartheta_\beta \equiv \tau_\beta) \to ([\varphi_{\alpha\beta}\vartheta_\beta] \equiv [\psi_{\alpha\beta}\tau_\beta])))$.

26. $\vdash ([\lambda \mathbf{x}_\beta[\varphi_{\alpha\beta}\mathbf{x}_\beta]] \equiv \varphi_{\alpha\beta})$.

27. $\vdash ((\exists \mathbf{x}_\alpha[\varphi_{o\alpha}\mathbf{x}_\alpha]) \to [\varphi_{o\alpha}(\iota \mathbf{x}_\alpha[\varphi_{o\alpha}\mathbf{x}_\alpha])])$.

28. $\vdash ([\varphi_{o\alpha}(\iota \mathbf{x}_\alpha(\neg[\varphi_{o\alpha}\mathbf{x}_\alpha]))] \to [\Pi_{o\langle o\alpha\rangle}\varphi_{o\alpha}])$.

29. $\vdash (([\varphi_{\alpha\beta}(\iota \mathbf{x}_\beta([\varphi_{\alpha\beta}\mathbf{x}_\beta] \not\equiv [\psi_{\alpha\beta}\mathbf{x}_\beta]))] \equiv [\psi_{\alpha\beta}(\iota \mathbf{x}_\beta([\varphi_{\alpha\beta}\mathbf{x}_\beta] \not\equiv [\psi_{\alpha\beta}\mathbf{x}_\beta]))]) \to$
$(\varphi_{\alpha\beta} \equiv \psi_{\alpha\beta}))$.

In order to avoid too much distortion of the grand scheme by not too enlightening details, we will defer the proofs to an appendix (A). With these at our hands, however, we are ready to give some corollaries of 4.2 and 4.3.

4.5 Corollary. *For all* $\varphi \in W$, $\alpha \in \mathcal{T}$, $\mathbf{x}, \mathbf{y} \in X_\alpha$ *and* $\mathfrak{a}, \mathfrak{b} \in \mathcal{S}^*$ *such that* $\mathfrak{a}\varphi\mathfrak{b} \in \overline{W}$ *and* $\mathbf{y}$ *does not occur in* φ *we have*

$$\vdash (\mathfrak{a}\varphi\mathfrak{b} \equiv \mathfrak{a}\,\mathrm{SB}(\mathbf{x})(\mathbf{y})(\varphi)\mathfrak{b}).$$

Proof. By 18, $\vdash (\mathfrak{a}\varphi\mathfrak{b} \equiv \mathfrak{a}\varphi\mathfrak{b})$, whence the claim follows by Lemma 4.2. □

4.6 Corollary. *For all* $\alpha, \beta \in \mathcal{T}$, $\varphi_\alpha, \psi_\beta \in W$, $\mathbf{x} \in X_\beta$ *and* $\mathfrak{a}, \mathfrak{b} \in \mathcal{S}^*$ *such that* $\mathfrak{a}[[\lambda\mathbf{x}\varphi]\psi]\mathfrak{b} \in \overline{W}$, $\mathbf{x} \notin \mathsf{VF}(\psi)$ *and no free variable of* ψ *is bound in* φ *(in particular, if* ψ *is closed) we have that*

$$\vdash (\mathfrak{a}[[\lambda\mathbf{x}\varphi]\psi]\mathfrak{b} \equiv \mathfrak{a}\,\mathrm{SF}(\mathbf{x})(\psi)(\varphi)\mathfrak{b}).$$

Proof. By 18, $\vdash (\mathfrak{a}[[\lambda\mathbf{x}\varphi]\psi]\mathfrak{b} \equiv \mathfrak{a}[[\lambda\mathbf{x}\varphi]\psi]\mathfrak{b})$, whence the claim follows by Lemma 4.3. □

We now turn to the definition of consistency and its elementary properties, where the interpretation of $\bot$ will finally play an important part. Along this line it is interesting to note that thus far the sets of closed well-formed formulae (or axiomatisations) considered has never been assumed to be free of contradictions, i.e. consistent.

4.7 Definition (Consistency)**.** Let $\Gamma \subseteq \overline{W}_o$. If $\Gamma \nvdash \bot$ then Γ is called *consistent*, and otherwise *inconsistent*. Γ is *maximal consistent* if it is consistent and maximal, with respect to the ordering $(\mathcal{P}(\overline{W}_o), \subseteq)$, with this property.

There is an alternative characterisation of inconsistency.

4.8 Lemma. *Let* $\Gamma \subseteq \overline{W}_o$*.* Γ *is inconsistent if and only if* $\Gamma \vdash \varphi$ *for all* $\varphi \in W_o$*.*

Proof. If $\Gamma \vdash \varphi$ for all $\varphi \in W_o$ then in particular $\Gamma \vdash \bot$, whence Γ is inconsistent.

If Γ is inconsistent then $\Gamma \vdash \bot$. By 2 of 4.4, $\vdash (\bot \to \varphi)$ for every $\varphi \in W_o$, whence $\Gamma \vdash \varphi$ by Modus ponens for all such φ. □

One of the central aspects of proofs are their finite character, which is encapsulated in the following proposition. This will have interesting consequences for complete semantics, to be exploited in the final part of this section.

4.9 Proposition. $\Gamma \subseteq \overline{W}_o$ *is consistent if and only if every finite* $\Delta \subseteq \Gamma$ *is consistent.*

Proof. We prove the contrapositive statement.

Assume first that $\Delta \subseteq \Gamma$ is finite and inconsistent, that is $\Delta \vdash \bot$. Then $\Gamma \vdash \bot$, i.e. Γ is inconsistent.

Conversely, assume that $\Gamma \vdash \bot$ and let $\mathfrak{P}$ be a proof thereof with length $n \in \mathbb{Z}^+$. Let $\Delta = \{\varphi \in \Gamma \,|\, \exists k \in [0, n[: \varphi = \mathfrak{P}(k)\}$. $\Delta \subseteq \Gamma$ is finite, since $|\Delta| \leq n$. Let $k \in [0, n[$. Then $\mathfrak{P}(k)$ is either a formal axiom, a formula of Γ, or can be inferred from $\{\mathfrak{P}(l) \,|\, 0 \leq l < k\}$ by some rule of inference. If $\mathfrak{P}(k) \in \Gamma$ then $\mathfrak{P}(k) \in \Delta$ by construction. Thus $\mathfrak{P}(k)$ is either a formal axiom, a formula of Δ, or can be inferred from $\{\mathfrak{P}(l) \,|\, 0 \leq l < k\}$ by some rule of inference. Hence $\mathfrak{P}$ is a proof of $\bot$ from Δ, whereby the latter is inconsistent. □

There is also a different characterisation of maximal consistency.

4.10 Lemma. *Let* $\Gamma \subseteq \overline{W}_o$*.* Γ *is maximal consistent if and only if it is consistent and, for all* $\varphi \in \overline{W}_o$*, exactly one of* $\varphi \in \Gamma$ *or* $(\neg\varphi) \in \Gamma$ *holds.*

Proof. Suppose that Γ is maximal consistent and let $\varphi \in \overline{W}_o$. If $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{(\neg\varphi)\}$ are consistent, then $\varphi \in \Gamma$ and $(\neg\varphi) \in \Gamma$ by maximality. Hence $\Gamma \vdash \bot$ by 3 of 4.4 and Modus ponens. So Γ is inconsistent, which is absurd since it is maximal consistent. Hence this case will not occur.

If $\Gamma \cup \{\varphi\}$ is consistent but $\Gamma \cup \{(\neg\varphi)\}$ is inconsistent, then $\varphi \in \Gamma$ by maximality, while $(\neg\varphi) \notin \Gamma$ since Γ is consistent. Similarly (mutatis mutandis) if $\Gamma \cup \{\varphi\}$ is inconsistent and $\Gamma \cup \{(\neg\varphi)\}$ is consistent, then $\varphi \notin \Gamma$ and $(\neg\varphi) \in \Gamma$.

Finally, if both $\Gamma \cup \{\varphi\}$ and $\Gamma \cup \{(\neg\varphi)\}$ are inconsistent, then $\Gamma \vdash (\varphi \to \bot)$ and $\Gamma \vdash ((\neg\varphi) \to \bot)$ by the Deduction Theorem. Then $\Gamma \vdash \bot$ by 16 of 4.4 and Modus ponens. But then Γ is inconsistent, which contradicts its maximal consistency. So this case cannot occur.

Conversely, suppose that Γ is consistent and that, for all $\varphi \in \overline{W}_o$, exactly one of $\varphi \in \Gamma$ or $(\neg\varphi) \in \Gamma$. Let $\Delta \subseteq \overline{W}_o$ be consistent and such that $\Gamma \subseteq \Delta$. Take $\varphi \in \Delta$ and suppose that $\varphi \notin \Gamma$, whence $(\neg\varphi) \in \Gamma$ by assumption. Then $\Delta \vdash \varphi$ and $\Delta \vdash (\neg\varphi)$, so that $\Delta \vdash \bot$ by 3 of 4.4 and Modus ponens, contradicting the consistency of Δ, whence $\Delta \subseteq \Gamma$. Thus Γ is maximal consistent. □

Remark 3. Thus $\Gamma \vdash \varphi$ if and only if $\varphi \in \Gamma$, whenever $\varphi \in \overline{W}_o$ and $\Gamma \subseteq \overline{W}_o$ is maximal consistent.

4.11 Lemma. *If $\Gamma \subseteq \overline{W}_o$ is consistent, then there is a maximal consistent $\Delta \subseteq \overline{W}_o$ containing Γ.*

Proof. Let $\kappa = |\overline{W}_o|$ and $c : \kappa \longrightarrow \overline{W}_o$ be a witnessing bijection. Define $f : S(\kappa) \longrightarrow \mathcal{P}(\overline{W}_o)$ by ordinal recursion as

$$f(\alpha) = \begin{cases} \Gamma & \text{if } \alpha = 0 \\ f(\beta) & \text{if } \alpha = S(\beta) \text{ and } f(\beta) \cup \{c(\beta)\} \text{ is inconsistent} \\ f(\beta) \cup \{c(\beta)\} & \text{if } \alpha = S(\beta) \text{ and } f(\beta) \cup \{c(\beta)\} \text{ is consistent} \\ \bigcup_{\beta \in \alpha} f(\beta) & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Clearly f is well-defined and increasing (between the standard ordering of $S(\kappa)$ and $\subseteq_{\mathcal{P}(\overline{W}_o)}$). We now verify that $f(\alpha)$ is consistent for all $\alpha \in S(\kappa)$, by ordinal induction.

- $f(0) = \Gamma$ is consistent by assumption.
- Assume that $\beta \in S(\kappa)$ is such that $S(\beta) \in S(\kappa)$ and $f(\gamma)$ is consistent for all $\gamma \in S(\beta)$. If $f(\beta) \cup \{c(\beta)\}$ is inconsistent then $f(S(\beta)) = f(\beta)$ is consistent. Otherwise $f(S(\beta)) = f(\beta) \cup \{c(\beta)\}$ is consistent.
- Consider a limit ordinal $\alpha \in S(\kappa)$ and suppose that $f(\beta)$ is consistent for all $\beta \in \alpha$. Assume towards a contradiction that $f(\alpha)$ is inconsistent, so that, by Proposition 4.9, there is a finite $\Xi \subseteq f(\alpha)$ which is inconsistent. For every $\varphi \in \Xi$ there is a $\beta \in \alpha$ such that $\varphi \in f(\beta)$; hence there is an $r : \Xi \longrightarrow \alpha$ such that $\varphi \in f(r(\varphi))$. Since $r[\Xi]$ is finite, it has a largest member γ. Then $\gamma \in \alpha$ and $f(\beta) \subseteq f(\gamma)$ for all $\beta \in r[\Xi]$. In particular $\varphi \in f(r(\varphi)) \subseteq f(\gamma)$ for all $\varphi \in \Xi$, i.e. $\Xi \subseteq f(\gamma)$. But then $f(\gamma)$ is already inconsistent, which contradicts our supposition. Hence $f(\alpha)$ is consistent.

Let $\Delta = f(\kappa)$, which is in particular consistent and contains Γ. Moreover, if $\Theta \subseteq \overline{W}_o$ is consistent and such that $\Delta \subseteq \Theta$, then given some $\varphi \in \Theta$ we must in particular have that $f(c^{-1}(\varphi)) \subseteq \Delta \subseteq \Theta$. Thus $f(c^{-1}(\varphi)) \cup \{\varphi\} \subseteq \Theta$ is consistent, whereby

$$\varphi \in f(c^{-1}(\varphi)) \cup \{\varphi\} = f(S(c^{-1}(\varphi))) \subseteq f(\kappa) = \Delta.$$

Hence $\Theta = \Delta$, which is thus maximal consistent. □

The essential part of the proof of completeness is the equivalence between consistency and having a model for a set of well-formed formulae. One direction in this equivalence is an easy consequence of the Soundness Theorem.

4.12 Proposition. *Let $\Gamma \subseteq \overline{W}_o$. If Γ has a (general) model, then it is consistent.*

Proof. Suppose that Γ is inconsistent, that is $\Gamma \vdash \bot$. By the Soundness Theorem 3.5 this entails $\Gamma \models \bot$. Thus, were $\mathfrak{M}$ any model of Γ we would have $\mathfrak{M} \models \bot$, which is impossible by Lemma 2.14. Hence Γ has no model.

Thus, if Γ has a model, it must be consistent. $\square$

The converse, the so called Model Existence Theorem, will be the goal of the second-next subsection to follow.

Formulae for Variables

Before we reach the Model Existence Theorem, however, we will need some additional machinery. The purpose is to obtain a way of, given a model and an assignment, replacing all free variables of a well-formed formula by closed formulae whose denotations matches the variables evaluations under the assignment in question. With this we would only need to consider the evaluation of closed formulae in a model. The numbering of variables will also prove useful in the last subsection, where we consider the universal closure of a formula.

4.13 Definition. Let $\mathfrak{X} : \mathbb{N} \longrightarrow \mathcal{X}$ be a bijection witnessing the countability of $\mathcal{X}$. Given $\boldsymbol{x} \in \mathcal{X}$ we will call $\mathfrak{X}^{-1}(\boldsymbol{x})$ the *index* of $\boldsymbol{x}$. Furthermore define $\mathsf{M} : W \longrightarrow \mathbb{N}$ by

$$\mathsf{M}(\varphi) = \max(\{\mathfrak{X}^{-1}(\boldsymbol{x}) + 1 \mid \boldsymbol{x} \in \mathsf{VF}(\varphi)\} \cup \{0\})$$

for all $\varphi \in W$. Moreover, let $\mathsf{P} = \{\mathsf{f} : \mathcal{X} \longrightarrow W \mid \forall \boldsymbol{x} \in \mathcal{X} : \mathsf{f}(\boldsymbol{x}) \in \{\boldsymbol{x}\} \cup \overline{W}_{\mathsf{T}(\boldsymbol{x})}\}$, and define $\mathfrak{f} : \mathbb{N} \longrightarrow (W^W)^{\mathsf{P}}$ recursively by

$$\begin{aligned} \mathfrak{f}(0)(\mathsf{f})(\varphi) &= \varphi \\ \mathfrak{f}(k+1)(\mathsf{f})(\varphi) &= \mathfrak{f}(\mathsf{f})(k)(\mathrm{SF}(\mathfrak{X}(k))(\mathsf{f}(\mathfrak{X}(k)))(\varphi)) \end{aligned}$$

for all $k \in \mathbb{N}$, $\mathsf{f} \in \mathsf{P}$ and $\varphi \in W$. Finally we define $\mathfrak{F} : \mathsf{P} \longrightarrow W^W$ by

$$\mathfrak{F}(\mathsf{f})(\varphi) = \mathfrak{f}(\mathsf{M}(\varphi))(\mathsf{f})(\varphi)$$

for all $\mathsf{f} \in \mathsf{P}$ and $\varphi \in W$.

We will extend the notation of Definition 2.5 to P, that is

$$\mathsf{f}^{\boldsymbol{x}}_{\varphi}(\boldsymbol{z}) = \begin{cases} \varphi & \text{if } \boldsymbol{z} = \boldsymbol{x} \\ \mathsf{f}(\boldsymbol{z}) & \text{otherwise} \end{cases}$$

for all $\boldsymbol{z}, \boldsymbol{x} \in \mathcal{X}$, $\varphi \in \{\boldsymbol{x}\} \cup \overline{W}_{\mathsf{T}(\boldsymbol{x})}$ and $\mathsf{f} \in \mathsf{P}$.

It might be worth to point out that, while $\mathfrak{F}$ indeed substitutes for every free variable of its second argument, it might, depending on the first argument, leave some of them unchanged, since the number of calls to $\mathfrak{f}$ is independent of whether said variables actually disappear. The possibility of leaving variables unchanged is a technical necessity to be exploited in the proof of the Model Existence Theorem.

Remark 4. For all $\alpha, \beta \in \mathcal{T}$, $\varphi_\alpha, \psi_{\beta\alpha} \in W$, $\vartheta \in \overline{W}_\alpha$ and $x \in X_\beta$

$$\begin{aligned} M([\psi\varphi]) &\geq M(\psi), M(\varphi) \\ M([\lambda x \varphi]) &\leq M(\varphi) \\ M(\vartheta) &= 0. \end{aligned}$$

4.14 Lemma. *Let* $f \in P$ *and* $\varphi \in W$*. Then*

$$\mathfrak{F}(f)(\varphi) = \mathfrak{f}(n)(f)(\varphi)$$

for all $n \geq M(\varphi)$*.*

Proof. By induction on the difference $m - M(\varphi)$, the base case (i.e. when the difference is 0) holding true by definition. Thus let $k \in \mathbb{N}$, $k \geq M(\varphi)$ be such that

$$\mathfrak{F}(f)(\varphi) = \mathfrak{f}(k)(f)(\varphi).$$

Then

$$\begin{aligned} \mathfrak{f}(k+1)(f)(\varphi) &= \mathfrak{f}(k)(f)(\mathrm{SF}(\mathcal{X}(k))(f(\mathcal{X}(k)))(\varphi)) \\ &= \mathfrak{f}(k)(f)(\varphi) \\ &= \mathfrak{F}(f)(\varphi), \end{aligned}$$

since $\mathcal{X}(k) \notin \mathsf{VF}(\varphi)$. □

Remark 5. For all $f \in P$, $n \in \mathbb{N}$ and $\varphi \in \overline{W}$, $\mathfrak{f}(n)(f)(\varphi) = \varphi$.

4.15 Lemma. *Let* $f \in P$ *and* $\alpha, \beta \in \mathcal{T}$*. Then*

$$\mathfrak{F}(f)([\varphi\psi]) = [\mathfrak{F}(f)(\varphi)\mathfrak{F}(f)(\psi)]$$

for all $\varphi_{\alpha\beta}, \psi_\beta \in W$*.*

Proof. We first verify by induction that, for every $n \in \mathbb{N}$,

$$\mathfrak{f}(n)(f)([\varphi\psi]) = [\mathfrak{f}(n)(f)(\varphi)\mathfrak{f}(n)(f)(\psi)]$$

for all $\varphi_{\alpha\beta}, \psi_\beta \in W$.

- Trivially

  $$\mathfrak{f}(0)(f)([\varphi\psi]) = [\varphi\psi] = [\mathfrak{f}(0)(f)(\varphi)\mathfrak{f}(0)(f)(\psi)]$$

  for all $\varphi_{\alpha\beta}, \psi_\beta \in W$.

- Let $k \in \mathbb{N}$ be such that

$$\mathfrak{f}(k)(f)([\varphi\psi]) = [\mathfrak{f}(k)(f)(\varphi)\mathfrak{f}(k)(f)(\psi)]$$

for all $\varphi_{\alpha\beta}, \psi_\beta \in W$. Then

$$\begin{aligned}\mathfrak{f}(k+1)(f)([\varphi\psi]) &= \mathfrak{f}(k)(f)(\mathrm{SF}(\mathcal{X}(k))(f(\mathcal{X}(k)))([\varphi\psi]))\\ &= \mathfrak{f}(k)(f)([\mathrm{SF}(\mathcal{X}(k))(f(\mathcal{X}(k)))(\varphi)\,\mathrm{SF}(\mathcal{X}(k))(f(\mathcal{X}(k)))(\psi)])\\ &= [\mathfrak{f}(k)(f)(\mathrm{SF}(\mathcal{X}(k))(f(\mathcal{X}(k)))(\varphi))\mathfrak{f}(k)(f)(\mathrm{SF}(\mathcal{X}(k))(f(\mathcal{X}(k)))(\psi))]\\ &= [\mathfrak{f}(k+1)(f)(\varphi)\mathfrak{f}(k+1)(f)(\psi)]\end{aligned}$$

for all $\varphi_{\alpha\beta}, \psi_\beta \in W$.

Now the claim follows, since $M([\varphi\psi]) \geq M(\varphi), M(\psi)$ and thus

$$\begin{aligned}\mathfrak{F}(f)([\varphi\psi]) &= \mathfrak{f}(M([\varphi\psi]))(f)([\varphi\psi])\\ &= [\mathfrak{f}(M([\varphi\psi]))(f)(\varphi)\mathfrak{f}(M([\varphi\psi]))(f)(\psi)]\\ &= [\mathfrak{F}(f)(\varphi)\mathfrak{F}(f)(\psi)]\end{aligned}$$

by Lemma 4.14. □

4.16 Lemma. *Let* $f \in P$ *and* $x \in \mathcal{X}$. *Then*

$$\mathfrak{F}(f)([\lambda x\varphi]) = [\lambda x\mathfrak{F}(f_x^x)(\varphi)]$$

for all $\varphi \in W$.

Proof. We first show that for every $n \in \mathbb{N}$ we have

$$\mathfrak{f}(n)(f)([\lambda x\varphi]) = [\lambda x\mathfrak{f}(n)(f_x^x)(\varphi)]$$

for all $\varphi \in W$. This is by induction on n.

- We have that

$$\begin{aligned}\mathfrak{f}(0)(f)([\lambda x\varphi]) &= [\lambda x\varphi]\\ &= [\lambda x\mathfrak{f}(0)(f_x^x)(\varphi)]\end{aligned}$$

for all $\varphi \in W$ by definition.

- Let $k \in \mathbb{N}$ be such that

$$\mathfrak{f}(k)(f)([\lambda x\psi]) = [\lambda x\mathfrak{f}(k)(f_x^x)(\psi)]$$

for all $\psi \in W$, and let $\varphi \in W$. We distinguish two cases. First, if $\mathcal{X}(k) \neq x$ then

$$\begin{aligned}\mathfrak{f}(k+1)(f)([\lambda x\varphi]) &= \mathfrak{f}(k)(f)(\mathrm{SF}(\mathcal{X}(k))(f(\mathcal{X}(k)))([\lambda x\varphi]))\\ &= \mathfrak{f}(k)(f)([\lambda x\,\mathrm{SF}(\mathcal{X}(k))(f(\mathcal{X}(k)))(\varphi)])\\ &= \mathfrak{f}(k)(f)([\lambda x\,\mathrm{SF}(\mathcal{X}(k))(f_x^x(\mathcal{X}(k)))(\varphi)])\\ &= [\lambda x\mathfrak{f}(k)(f_x^x)(\mathrm{SF}(\mathcal{X}(k))(f_x^x(\mathcal{X}(k)))(\varphi))]\\ &= [\lambda x\mathfrak{f}(k+1)(f_x^x)(\varphi)].\end{aligned}$$

If, on the other hand, $\mathcal{X}(k) = \mathbf{x}$, then

$$\begin{aligned}\mathfrak{f}(k+1)(f)([\lambda\mathbf{x}\varphi]) &= \mathfrak{f}(k)(f)(\mathrm{SF}(\mathcal{X}(k))(f(\mathcal{X}(k)))([\lambda\mathbf{x}\varphi]))\\ &= \mathfrak{f}(k)(f)([\lambda\mathbf{x}\varphi])\\ &= \mathfrak{f}(k)(f)([\lambda\mathbf{x}\,\mathrm{SF}(\mathbf{x})(\mathbf{x})(\varphi)])\\ &= \mathfrak{f}(k)(f)([\lambda\mathbf{x}\,\mathrm{SF}(\mathcal{X}(k))(f^{\mathbf{x}}_{\mathbf{x}}(\mathcal{X}(k)))(\varphi)])\\ &= [\lambda\mathbf{x}\mathfrak{f}(k)(f^{\mathbf{x}}_{\mathbf{x}})(\mathrm{SF}(\mathcal{X}(k))(f^{\mathbf{x}}_{\mathbf{x}}(\mathcal{X}(k)))(\varphi))]\\ &= [\lambda\mathbf{x}\mathfrak{f}(k+1)(f^{\mathbf{x}}_{\mathbf{x}})(\varphi)].\end{aligned}$$

Whichever the case

$$\mathfrak{f}(k+1)(f)([\lambda\mathbf{x}\varphi]) = [\lambda\mathbf{x}\mathfrak{f}(k+1)(f^{\mathbf{x}}_{\mathbf{x}})(\varphi)].$$

Thus

$$\mathfrak{f}(n)(f)([\lambda\mathbf{x}\varphi]) = [\lambda\mathbf{x}\mathfrak{f}(n)(f^{\mathbf{x}}_{\mathbf{x}})(\varphi)]$$

for all $n \in \mathbb{N}$ and $\varphi \in W$, as claimed. Therefore

$$\begin{aligned}\mathfrak{F}(f)([\lambda\mathbf{x}\varphi]) &= \mathfrak{f}(M(\varphi))(f)([\lambda\mathbf{x}\varphi])\\ &= [\lambda\mathbf{x}\mathfrak{f}(M(\varphi))(f^{\mathbf{x}}_{\mathbf{x}})(\varphi)]\\ &= [\lambda\mathbf{x}\mathfrak{F}(f^{\mathbf{x}}_{\mathbf{x}})(\varphi)]\end{aligned}$$

by Lemma 4.14, as $M(\varphi) \geq M([\lambda\mathbf{x}\varphi])$, for all $\varphi \in W$. □

Attentive readers will recognise the following lemma as part 1 of Lemma 2.24 from [7], generalised to the current context.

4.17 Lemma. *Let $\alpha, \beta \in \mathcal{T}$, $\mathbf{x}_\alpha, \mathbf{y}_\beta \in \mathcal{X}$ and $\varphi_\alpha, \psi_\beta \in W$ such that $\mathbf{x} \neq \mathbf{y}$, $\mathbf{x}$ is not free in ψ and $\mathbf{y}$ is not free in φ. Then*

$$\mathrm{SF}(\mathbf{x})(\varphi)(\mathrm{SF}(\mathbf{y})(\psi)(\vartheta)) = \mathrm{SF}(\mathbf{y})(\psi)(\mathrm{SF}(\mathbf{x})(\varphi)(\vartheta)).$$

Proof. By induction on ϑ.

- Let $s \in \mathcal{B}$.

  If $\mathbf{x} = s$, then $\mathbf{y} \neq s$, whence

$$\begin{aligned}\mathrm{SF}(\mathbf{x})(\varphi)(\mathrm{SF}(\mathbf{y})(\psi)(s)) &= \mathrm{SF}(\mathbf{x})(\varphi)(s)\\ &= \varphi\\ &= \mathrm{SF}(\mathbf{y})(\psi)(\varphi)\\ &= \mathrm{SF}(\mathbf{y})(\psi)(\mathrm{SF}(\mathbf{x})(\varphi)(s))\end{aligned}$$

  since $\mathbf{y}$ is not free in $\mathrm{SF}(\mathbf{x})(\varphi)(s)$.

If $\mathbf{y} = \mathrm{s}$, then symmetrically

$$\begin{aligned}
\mathrm{SF}(\mathbf{y})(\psi)(\mathrm{SF}(\mathbf{x})(\varphi)(\mathrm{s})) &= \mathrm{SF}(\mathbf{y})(\psi)(\mathrm{s}) \\
&= \psi \\
&= \mathrm{SF}(\mathbf{x})(\varphi)(\psi) \\
&= \mathrm{SF}(\mathbf{x})(\varphi)(\mathrm{SF}(\mathbf{y})(\psi)(\mathrm{s})).
\end{aligned}$$

If finally $\mathbf{x} \neq \mathrm{s} \neq \mathbf{y}$, then

$$\begin{aligned}
\mathrm{SF}(\mathbf{x})(\varphi)(\mathrm{SF}(\mathbf{y})(\psi)(\mathrm{s})) &= \mathrm{SF}(\mathbf{x})(\varphi)(\mathrm{s}) \\
&= \mathrm{s} \\
&= \mathrm{SF}(\mathbf{y})(\psi)(\mathrm{s}) \\
&= \mathrm{SF}(\mathbf{y})(\psi)(\mathrm{SF}(\mathbf{x})(\varphi)(\mathrm{s})).
\end{aligned}$$

In any case $\mathrm{SF}(\mathbf{x})(\varphi)(\mathrm{SF}(\mathbf{y})(\psi)(\mathrm{s})) = \mathrm{SF}(\mathbf{y})(\psi)(\mathrm{SF}(\mathbf{x})(\varphi)(\mathrm{s}))$.

- Let $\gamma, \delta \in \mathcal{T}$, $\mathbf{z} \in \mathrm{X}_\delta$ and $\tau \in \mathcal{W}_\gamma$ be such that $\mathrm{SF}(\mathbf{x})(\varphi)(\mathrm{SF}(\mathbf{y})(\psi)(\tau)) = \mathrm{SF}(\mathbf{y})(\psi)(\mathrm{SF}(\mathbf{x})(\varphi)(\tau))$.

  If $\mathbf{z} = \mathbf{x}$ we have that

$$\begin{aligned}
\mathrm{SF}(\mathbf{x})(\varphi)(\mathrm{SF}(\mathbf{y})(\psi)([\lambda \mathbf{z} \tau])) &= \mathrm{SF}(\mathbf{x})(\varphi)([\lambda \mathbf{z}\, \mathrm{SF}(\mathbf{y})(\psi)(\tau)]) \\
&= [\lambda \mathbf{z}\, \mathrm{SF}(\mathbf{y})(\psi)(\tau)] \\
&= \mathrm{SF}(\mathbf{y})(\psi)([\lambda \mathbf{z} \tau]) \\
&= \mathrm{SF}(\mathbf{y})(\psi)(\mathrm{SF}(\mathbf{x})(\varphi)([\lambda \mathbf{z} \tau])).
\end{aligned}$$

  If $\mathbf{z} = \mathbf{y}$ we have, symmetrically

$$\begin{aligned}
\mathrm{SF}(\mathbf{x})(\varphi)(\mathrm{SF}(\mathbf{y})(\psi)([\lambda \mathbf{z} \tau])) &= \mathrm{SF}(\mathbf{x})(\varphi)([\lambda \mathbf{z} \tau]) \\
&= [\lambda \mathbf{z}\, \mathrm{SF}(\mathbf{x})(\varphi)(\tau)] \\
&= \mathrm{SF}(\mathbf{y})(\psi)([\lambda \mathbf{z}\, \mathrm{SF}(\mathbf{x})(\varphi)(\tau)]) \\
&= \mathrm{SF}(\mathbf{y})(\psi)(\mathrm{SF}(\mathbf{x})(\psi)([\lambda \mathbf{z} \tau])).
\end{aligned}$$

  Finally, if $\mathbf{x} \neq \mathbf{z} \neq \mathbf{y}$ we have that

$$\begin{aligned}
\mathrm{SF}(\mathbf{x})(\varphi)(\mathrm{SF}(\mathbf{y})(\psi)([\lambda \mathbf{z} \tau])) &= \mathrm{SF}(\mathbf{x})(\varphi)([\lambda \mathbf{z}\, \mathrm{SF}(\mathbf{y})(\psi)(\tau)]) \\
&= [\lambda \mathbf{z}\, \mathrm{SF}(\mathbf{x})(\varphi)(\mathrm{SF}(\mathbf{y})(\psi)(\tau))] \\
&= [\lambda \mathbf{z}\, \mathrm{SF}(\mathbf{y})(\psi)(\mathrm{SF}(\mathbf{x})(\varphi)(\tau))] \\
&= \mathrm{SF}(\mathbf{y})(\psi)([\lambda \mathbf{z}\, \mathrm{SF}(\mathbf{x})(\varphi)(\tau)]) \\
&= \mathrm{SF}(\mathbf{y})(\psi)(\mathrm{SF}(\mathbf{x})(\varphi)([\lambda \mathbf{z} \tau])).
\end{aligned}$$

  In either case $\mathrm{SF}(\mathbf{x})(\varphi)(\mathrm{SF}(\mathbf{y})(\psi)([\lambda \mathbf{z} \tau])) = \mathrm{SF}(\mathbf{y})(\psi)(\mathrm{SF}(\mathbf{x})(\varphi)([\lambda \mathbf{z} \tau]))$.

- Finally consider $\gamma, \delta \in \mathcal{T}$ and $\tau_{\gamma\delta}, \eta_\delta \in W$ such that $\mathrm{SF}(\mathbf{x})(\varphi)(\mathrm{SF}(\mathbf{y})(\psi)(\nu)) = \mathrm{SF}(\mathbf{y})(\psi)(\mathrm{SF}(\mathbf{x})(\varphi)(\nu))$ for both $\nu \in \{\tau, \eta\}$. Then

$$\begin{aligned}
\mathrm{SF}(\mathbf{x})(\varphi)(\mathrm{SF}(\mathbf{y})(\psi)([\tau\eta])) &= \mathrm{SF}(\mathbf{x})(\varphi)([\mathrm{SF}(\mathbf{y})(\psi)(\tau)\,\mathrm{SF}(\mathbf{y})(\psi)(\eta)]) \\
&= [\mathrm{SF}(\mathbf{x})(\varphi)(\mathrm{SF}(\mathbf{y})(\psi)(\tau))\,\mathrm{SF}(\mathbf{x})(\varphi)(\mathrm{SF}(\mathbf{y})(\psi)(\eta))] \\
&= [\mathrm{SF}(\mathbf{y})(\psi)(\mathrm{SF}(\mathbf{x})(\varphi)(\tau))\,\mathrm{SF}(\mathbf{y})(\psi)(\mathrm{SF}(\mathbf{x})(\varphi)(\eta))] \\
&= \mathrm{SF}(\mathbf{y})(\psi)([\mathrm{SF}(\mathbf{x})(\varphi)(\tau)\,\mathrm{SF}(\mathbf{x})(\varphi)(\eta)]) \\
&= \mathrm{SF}(\mathbf{y})(\psi)(\mathrm{SF}(\mathbf{x})(\varphi)([\tau\eta])).
\end{aligned}$$

□

4.18 Lemma. *Let $\alpha \in \mathcal{T}$, $\mathbf{x} \in X_\alpha$, $\psi \in \overline{W}_\alpha$ and $f \in P$. Then*

$$\mathrm{SF}(\mathbf{x})(\psi) \circ \mathfrak{f}(m)(f) = \mathfrak{f}(m)(f) \circ \mathrm{SF}(\mathbf{x})(\psi)$$

for all $m \in \mathbb{N}$ such that $m \leq \mathcal{X}^{-1}(\mathbf{x})$.

Proof. By induction on m.

- By definition

$$\mathrm{SF}(\mathbf{x})(\psi)(\mathfrak{f}(0)(f)(\varphi)) = \mathrm{SF}(\mathbf{x})(\psi)(\varphi) = \mathfrak{f}(0)(f)(\mathrm{SF}(\mathbf{x})(\psi)(\varphi))$$

  for all $\varphi \in W$.

- Assume $k \in \mathbb{N}$ is such that if $k \leq \mathcal{X}^{-1}(\mathbf{x})$ then

$$\mathrm{SF}(\mathbf{x})(\psi) \circ \mathfrak{f}(k)(f) = \mathfrak{f}(k)(f) \circ \mathrm{SF}(\mathbf{x})(\psi),$$

  and consider $k+1$. Suppose $k+1 \leq \mathcal{X}^{-1}(\mathbf{x})$. Then $k < \mathcal{X}^{-1}(\mathbf{x})$, whereby $f(\mathcal{X}(k)) \neq \mathbf{x}$, so that $f(\mathcal{X}(k)) \in \overline{W}$ by definition of P. Hence

$$\begin{aligned}
\mathrm{SF}(\mathbf{x})(\psi)(\mathfrak{f}(k+1)(f)(\varphi)) &= \mathrm{SF}(\mathbf{x})(\psi)(\mathfrak{f}(k)(f)(\mathrm{SF}(\mathcal{X}(k))(f(\mathcal{X}(k)))(\varphi))) \\
&= \mathfrak{f}(k)(f)(\mathrm{SF}(\mathbf{x})(\psi)(\mathrm{SF}(\mathcal{X}(k))(f(\mathcal{X}(k)))(\varphi))) \\
&= \mathfrak{f}(k)(f)(\mathrm{SF}(\mathcal{X}(k))(f(\mathcal{X}(k)))(\mathrm{SF}(\mathbf{x})(\psi)(\varphi))) \\
&= \mathfrak{f}(k+1)(f)(\mathrm{SF}(\mathbf{x})(\psi)(\varphi))
\end{aligned}$$

  for all $\varphi \in W$, by Lemma 4.17.

□

4.19 Lemma. *Let $f, g \in P$. For all $n \in \mathbb{N}$ such that $f \restriction \mathcal{X}\big[[0, n[\big] = g \restriction \mathcal{X}\big[[0, n[\big]$ we have*

$$\mathfrak{f}(n)(f)(\varphi) = \mathfrak{f}(n)(g)(\varphi)$$

for every $\varphi \in W$.

Proof. By induction on n.

- By definition

$$\mathfrak{f}(0)(f)(\varphi) = \varphi = \mathfrak{f}(0)(g)(\varphi)$$

for all $\varphi \in W$.

- Assume that $k \in \mathbb{N}$ is such that if $f \restriction \mathcal{X}\big[[0,k[\big] = g \restriction \mathcal{X}\big[[0,k[\big]$ then

$$\mathfrak{f}(k)(f)(\psi) = \mathfrak{f}(k)(g)(\psi)$$

for all $\psi \in W$. If $f \restriction \mathcal{X}\big[[0,k+1[\big] = g \restriction \mathcal{X}\big[[0,k+1[\big]$ then in particular $f(\mathcal{X}(k)) = g(\mathcal{X}(k))$, whence

$$\begin{aligned}\mathfrak{f}(k+1)(f)(\varphi) &= \mathfrak{f}(k)(f)(\mathrm{SF}(\mathcal{X}(k))(f(\mathcal{X}(k)))(\varphi)) \\ &= \mathfrak{f}(k)(f)(\mathrm{SF}(\mathcal{X}(k))(g(\mathcal{X}(k)))(\varphi)) \\ &= \mathfrak{f}(k)(g)(\mathrm{SF}(\mathcal{X}(k))(g(\mathcal{X}(k)))(\varphi)) \\ &= \mathfrak{f}(k+1)(g)(\varphi)\end{aligned}$$

for all $\varphi \in W$ by induction hypothesis.

□

4.20 Lemma. *Let $\alpha \in \mathcal{T}$, $\mathbf{x} \in X_\alpha$, $\psi \in \overline{W}_\alpha$ and $f \in P$. Then*

$$\mathrm{SF}(\mathbf{x})(\psi)(\mathfrak{F}(f_x^x)(\varphi)) = \mathfrak{F}(f_\psi^x)(\varphi)$$

for all $\varphi \in W$.

Proof. Let $m = \mathcal{X}^{-1}(\mathbf{x})$. We will show that

$$\mathrm{SF}(\mathbf{x})(\psi)(\mathfrak{f}(n)(f_x^x)(\varphi)) = \mathfrak{f}(n)(f_\psi^x)(\varphi)$$

for all $\varphi \in W$, for all $n \in \mathbb{N}$ such that $n > m$, by induction on n.

- In the base case we see that

$$\begin{aligned}\mathrm{SF}(\mathbf{x})(\psi)(\mathfrak{f}(m+1)(f_x^x)(\varphi)) &= \mathrm{SF}(\mathbf{x})(\psi)(\mathfrak{f}(m)(f_x^x)(\mathrm{SF}(\mathcal{X}(m))(f_x^x(\mathcal{X}(m)))(\varphi))) \\ &= \mathrm{SF}(\mathbf{x})(\psi)(\mathfrak{f}(m)(f_x^x)(\mathrm{SF}(\mathbf{x})(\mathbf{x})(\varphi))) \\ &= \mathrm{SF}(\mathbf{x})(\psi)(\mathfrak{f}(m)(f_x^x)(\varphi)) \\ &= \mathrm{SF}(\mathbf{x})(\psi)(\mathfrak{f}(m)(f_\psi^x)(\varphi)) \\ &= \mathfrak{f}(m)(f_\psi^x)(\mathrm{SF}(\mathbf{x})(\psi)(\varphi)) \\ &= \mathfrak{f}(m)(f_\psi^x)(\mathrm{SF}(\mathbf{x})(f_\psi^x(\mathbf{x}))(\varphi)) \\ &= \mathfrak{f}(m+1)(f_\psi^x)(\varphi),\end{aligned}$$

for all $\varphi \in W$, where the third and fourth equalities follows by lemmata 4.19 and 4.18, respectively.

- Now assume that $k \in \mathbb{N}$ is such that $k > m$ and

$$\mathrm{SF}(x)(\psi)(\mathfrak{f}(k)(f^x_x)(\vartheta)) = \mathfrak{f}(k)(f^x_\psi)(\vartheta)$$

for all $\vartheta \in W$. Then

$$\begin{aligned}\mathrm{SF}(x)(\psi)(\mathfrak{f}(k+1)(f^x_x)(\varphi)) &= \mathrm{SF}(x)(\psi)(\mathfrak{f}(k)(f^x_x)(\mathrm{SF}(\mathcal{X}(k))(f^x_x(\mathcal{X}(k)))(\varphi))) \\ &= \mathfrak{f}(k)(f^x_\psi)(\mathrm{SF}(\mathcal{X}(k))(f^x_x(\mathcal{X}(k)))(\varphi)) \\ &= \mathfrak{f}(k)(f^x_\psi)(\mathrm{SF}(\mathcal{X}(k))(f^x_\psi(\mathcal{X}(k)))(\varphi)) \\ &= \mathfrak{f}(k+1)(f^x_\psi)(\varphi)\end{aligned}$$

for all $\varphi \in W$.

In particular, given $\varphi \in W$, if $K = \max(M(\varphi), m+1)$, then

$$\begin{aligned}\mathrm{SF}(x)(\psi)(\mathfrak{F}(f^x_x)(\varphi)) &= \mathrm{SF}(x)(\psi)(\mathfrak{f}(K)(f^x_x)(\varphi)) \\ &= \mathfrak{f}(K)(f^x_\psi)(\varphi) \\ &= \mathfrak{F}(f^x_\psi)(\varphi),\end{aligned}$$

since $K > m$. ☐

The Model Existence Theorem

With all of the above lemmata, we are finally ready to prove the Model Existence Theorem, which will be the last essential leap towards the Completeness Theorem (4.28).

4.21 Theorem (Model Existence Theorem)**.** *Let $\Gamma \subseteq \overline{W}_o$. If Γ is consistent, there is a general model $\mathfrak{M} = (\{D_\alpha\}_{\alpha\in\mathcal{T}}, \{k_c\}_{c\in\mathcal{L}\cup\mathcal{C}})$, where $|D_\alpha| \leq \max(|\mathcal{C}|, \aleph_0)$ for all $\alpha \in \mathcal{T}$, such that $\mathfrak{M} \models \Gamma$.*

Proof. In this proof all references given by single numerals (e.g. 4) refer to the corresponding entry in Lemma 4.4.

Let $\Delta \subseteq \overline{W}_o$ be a maximal consistent superset of Γ. Define a relation $\sim \subseteq \overline{W}$ by

$$\varphi \sim \psi \Leftrightarrow \big(T(\varphi) = T(\psi) \wedge \Delta \vdash (\varphi_{T(\varphi)} \equiv \psi_{T(\varphi)})\big)$$

for all $\varphi, \psi \in \overline{W}$.

Sublemma 4.21.1. *$\sim$ is an equivalence relation on $\overline{W}$.*

Proof. Let $\varphi, \psi, \vartheta \in \overline{W}$.

By 18 $\vdash (\varphi_{T(\varphi)} \equiv \varphi_{T(\varphi)})$, whence $\varphi \sim \varphi$. Thus $\sim$ is reflexive.

If $\varphi \sim \psi$ then $T(\varphi) = T(\psi)$ and $\Delta \vdash (\varphi_{T(\varphi)} \equiv \psi_{T(\varphi)})$. By 19 $\vdash ((\varphi_{T(\varphi)} \equiv \psi_{T(\varphi)}) \to (\psi_{T(\varphi)} \equiv \varphi_{T(\varphi)}))$, whereby $\Delta \vdash (\psi_{T(\varphi)} \equiv \varphi_{T(\varphi)})$ by Modus ponens. Thus $\sim$ is symmetric.

If $\varphi \sim \psi$ and $\psi \sim \vartheta$ then $T(\varphi) = T(\psi) = T(\vartheta)$, $\Delta \vdash (\varphi_{T(\varphi)} \equiv \psi_{T(\varphi)})$ and $\Delta \vdash (\psi_{T(\varphi)} \equiv \vartheta_{T(\varphi)})$. By 20 $\vdash ((\varphi_{T(\varphi)} \equiv \psi_{T(\varphi)}) \to ((\psi_{T(\varphi)} \equiv \vartheta_{T(\varphi)}) \to (\varphi_{T(\varphi)} \equiv \vartheta_{T(\varphi)})))$, whereby $\Delta \vdash (\psi_{T(\varphi)} \equiv \vartheta_{T(\varphi)})$ by Modus ponens. Thus $\sim$ is transitive. ☐

Let $I = \overline{W}/\sim$ and $\pi : \overline{W} \longrightarrow I$ be the canonical projection, and let $r : I \longrightarrow \overline{W}$ be some right inverse of π (so that $\pi(r(q)) = q$ and $r(\pi(a)) \sim a$). Furthermore let $I_\alpha = \pi[\overline{W}_\alpha]$,for every $\alpha \in \mathcal{T}$, so that $\bigcup_{\alpha\in\mathcal{T}} I_\alpha = I$ (and the union is disjoint).

Sublemma 4.21.2. *Let* $\varphi, \psi, \vartheta, \eta \in \overline{W}$ *be such that* $\varphi \sim \vartheta$ *and* $\psi \sim \eta$. *Then* $[\varphi\psi] \sim [\vartheta\eta]$.

Proof. By 25 and Modus ponens. □

To define the embryo of our frame to be, let $\{M_\alpha\}_{\alpha\in\mathcal{T}}$ be given by

- $M_o = \{0, 1\}$
- $M_i = I_i$
- $M_{\alpha\beta} = (M_\beta \rightharpoonup M_\alpha)$

for all $\alpha, \beta \in \mathcal{T}$ (recall that $M_\beta \rightharpoonup M_\alpha$ is the set of all *partial* functions from M_β to M_α). Define $u : \overline{W}_o \longrightarrow M_o$ by

$$u(\varphi) = \begin{cases} 1 & \text{if } \varphi \in \Delta \\ 0 & \text{if } (\neg\varphi) \in \Delta \end{cases}$$

for all $\varphi \in \overline{W_o}$. This is well-defined since Δ is maximal consistent. In fact, it is even well-defined on equivalence classes of $\sim$. To see this, suppose that $\varphi, \psi \in \overline{W}_o$ are such that $\varphi \sim \psi$. If $u(\varphi) = 1$ then $\varphi \in \Delta$, whereby $\Delta \vdash \psi$ by Modus ponens since $\vdash (\varphi_o \rightarrow ((\varphi_o \equiv \psi_o) \rightarrow \psi_o))$ by 21, and thus $\psi \in \Delta$ by Remark 3. On the other hand, if $u(\varphi) = 0$ then $(\neg\varphi) \in \Delta$, whence $\Delta \vdash (\neg\psi)$ by Modus ponens since $\vdash ((\neg\varphi_o) \rightarrow ((\varphi_o \equiv \psi_o) \rightarrow (\neg\psi_o)))$ by 22. Thus we see that

$$u(\varphi) = u(\psi)$$

for all $\varphi, \psi \in \overline{W}_o$ such that $\varphi \sim \psi$. Hence there is a unique $u' : I_o \longrightarrow M_o$ satisfying

$$u'(\pi(\varphi)) = u(\varphi)$$

for all $\varphi \in \overline{W}_o$, and u' is surjective. Now consider the class function

$$H((\alpha, f), (\beta, g)) = \begin{cases} (\alpha\beta, h) \in \mathcal{T} \times ((M_\alpha)^{g[I_\beta]})^{I_{\alpha\beta}} & \text{if } \alpha, \beta \in \mathcal{T},\ f : I_\alpha \rightarrowtail M_\alpha \\ \text{s.t. } h(q)(e) = f(\pi([r(q)r(g^{-1}(e))])) & \text{and } g : I_\beta \rightarrowtail M_\beta \\ \text{for all } q \in I_{\alpha\beta},\ e \in g[I_\beta] & \\ \varnothing & \text{otherwise} \end{cases}$$

(recall that $\rightarrowtail$ denotes an injective function) and define g, a function on $\mathcal{T}$, by recursion as follows

- $g(o) = (o, u')$

- $g(i) = (i, \mathrm{Id}_{I_i})$
- $g(\alpha\beta) = H(g(\alpha), g(\beta))$ for all $\alpha, \beta \in \mathcal{T}$.

Sublemma 4.21.3. *For every* $\alpha \in \mathcal{T}$ *there is an (injective)* $f : I_\alpha \rightarrowtail M_\alpha$ *such that* $g(\alpha) = (\alpha, f)$.

Proof. By induction on $\alpha \in \mathcal{T}$.

- $g(o) = (o, u')$ where $u' : I_o \longrightarrow M_o$, whence it suffices to verify the injectivity of u', that is that $u(\varphi) = u(\psi) \Rightarrow \varphi \sim \psi$ for all $\varphi, \psi \in \overline{W}_o$. Thus let $\varphi, \psi \in \overline{W}_o$ be such that $u(\varphi) = u(\psi)$. If $u(\varphi) = 1$ then $\varphi, \psi \in \Delta$, whence $\Delta \vdash (\varphi_o \equiv \psi_o)$ by 23 and Modus ponens. If $u(\varphi) = 0$ then $(\neg\varphi), (\neg\psi) \in \Delta$, whereby $\Delta \vdash (\varphi_o \equiv \psi_o)$ by 24 and Modus ponens. Hence u' is injective.
- $g(i) = (i, \mathrm{Id}_{I_i})$, and Id_{I_i} is clearly injective.
- Suppose $\alpha, \beta \in \mathcal{T}$ are such that there are (injective) $f_\alpha : I_\alpha \rightarrowtail M_\alpha$ and $f_\beta : I_\beta \rightarrowtail M_\beta$, such that $g(\alpha) = (\alpha, f_\alpha)$ and $g(\beta) = (\beta, f_\beta)$. Consider $g(\alpha\beta) = H(g(\alpha), g(\beta))$. Since $\alpha, \beta \in \mathcal{T}$, $f_\alpha : I_\alpha \rightarrowtail M_\alpha$ and $f_\beta : I_\beta \rightarrowtail M_\beta$, we see that $H(g(\alpha), g(\beta)) = (\alpha\beta, h)$ where $h : I_{\alpha\beta} \longrightarrow M_\alpha^{f_\beta[I_\beta]}$. Since $M_\alpha^{f_\beta[I_\beta]} \subseteq (M_\beta \rightharpoonup M_\alpha) = M_{\alpha\beta}$, it only remains to verify that h is injective. Let $q_1, q_2 \in I_{\alpha\beta}$ be such that $h(q_1) = h(q_2)$. Then, supposing $\psi \in \overline{W}_\beta$, we have that

  $$h(q_1)(f_\beta(\pi(\psi))) = h(q_2)(f_\beta(\pi(\psi))).$$

  But

  $$\begin{aligned} h(q_n)(f_\beta(\pi(\psi))) &= f_\alpha(\pi([r(q_n)r(f_\beta^{-1}(f_\beta(\pi(\psi))))])) \\ &= f_\alpha(\pi([r(q_n)r(\pi(\psi))])) \end{aligned}$$

  for $n \in \{1, 2\}$, and since f_α is injective

  $$\pi([r(q_1)r(\pi(\psi))]) = \pi([r(q_2)r(\pi(\psi))]). \tag{4}$$

  Since $r(\pi(\psi)) \sim \psi$ we see that

  $$([r(q_n)r(\pi(\psi))] \sim [r(q_n)\psi])$$

  for both $n \in \{1, 2\}$, by Sublemma 4.21.2. This together with (4) yields

  $$[r(q_1)\psi] \sim [r(q_1)r(\pi(\psi))] \sim [r(q_2)r(\pi(\psi))] \sim [r(q_2)\psi],$$

  that is

  $$\Delta \vdash ([r(q_1)\psi] \equiv [r(q_2)\psi]).$$

Thus in particular (if ψ is the formula $(\imath x_\beta([r(q_1)x_\beta] \not\equiv [r(q_2)x_\beta]))$)

$$\Delta \vdash ([r(q_1)(\imath x_\beta([r(q_1)x_\beta] \not\equiv [r(q_2)x_\beta]))] \equiv [r(q_2)(\imath x_\beta([r(q_1)x_\beta] \not\equiv [r(q_2)x_\beta]))]).$$

By 29 and Modus ponens

$$\Delta \vdash (r(q_1) \equiv r(q_2)),$$

(intuitively since the "functions" $r(q_1)$ and $r(q_2)$ should be equal on an element singled out to make them differ) whereby $q_1 = \pi(r(q_1)) = \pi(r(q_2)) = q_2$. So h is injective.

This concludes the induction. $\square$

For every $\alpha \in \mathcal{T}$ let $s_\alpha : I_\alpha \rightarrowtail M_\alpha$ be such that $g(\alpha) = (\alpha, s_\alpha)$. Furthermore let $D_\alpha = s_\alpha[I_\alpha]$ for each $\alpha \in \mathcal{T}$, and $k_c = s_{T(c)}(\pi(c))$ for every $c \in \mathcal{L} \cup \mathcal{C}$. Finally let $\mathfrak{M} = (\{D_\alpha\}_{\alpha\in\mathcal{T}}, \{k_c\}_{c\in\mathcal{L}\cup\mathcal{C}})$. Since $s_\alpha : I_\alpha \longrightarrow D_\alpha$ is bijective for every $\alpha \in \mathcal{T}$, $|D_\alpha| = |I_\alpha| \leq |\overline{W}_\alpha| \leq |W| = \max(|\mathcal{C}|, \aleph_0)$.

Sublemma 4.21.4. *For all $\alpha, \beta \in \mathcal{T}$ and $\varphi_\alpha, \psi_\beta \in \overline{W}$ it is the case that*

$$s_\alpha(\pi([\varphi\psi])) = s_{\alpha\beta}(\pi(\varphi))(s_\beta(\pi(\psi))).$$

Proof. By definition

$$\begin{aligned} s_{\alpha\beta}(\pi(\varphi))(s_\beta(\pi(\psi))) &= s_\alpha(\pi([r(\pi(\varphi))r(s_\beta^{-1}(s_\beta(\pi(\psi))))])) \\ &= s_\alpha(\pi([r(\pi(\varphi))r(\pi(\psi))])) \\ &= s_\alpha(\pi([\varphi\psi])), \end{aligned}$$

where the last equality follows by 4.21.2 (since $r(\pi(\varphi)) \sim \varphi$ and $r(\pi(\psi)) \sim \psi$). $\square$

Sublemma 4.21.5. *$\mathfrak{M}$ is a non-degenerate structured frame.*

Proof. Since $|D_\alpha| = |I_\alpha|$ for all $\alpha \in \mathcal{T}$, it is clear that $\{D_\alpha\}_{\alpha\in\mathcal{T}}$ is a non-degenerate frame. We show that $\mathfrak{M}$ fulfils the other requirements of a structured frame (see Definition 2.1).

- Since Δ is maximal consistent, $\top \in \Delta$ and $\perp \notin \Delta$ by 12, whence

$$s_o(\pi(\top)) = u'(\pi(\top)) = u(\top) = 1,$$
$$s_o(\pi(\perp)) = u'(\pi(\perp)) = u(\perp) = 0.$$

  So $D_o = \{0, 1\}$.

- Let $\alpha, \beta \in \mathcal{T}$, and consider $D_\alpha \subseteq M_\alpha, D_\beta \subseteq M_\beta$. Let $a \in D_{\alpha\beta} = s_{\alpha\beta}[I_{\alpha\beta}]$ and observe that

$$s_{\alpha\beta} : I_{\alpha\beta} \longrightarrow M_\alpha^{s_\beta[I_\beta]} = M_\alpha^{D_\beta}.$$

  Thus, if $q = s_{\alpha\beta}^{-1}(a)$ then

$$a(p) = s_\alpha(\pi([r(q)r(s_\beta^{-1}(p))])) \in D_\alpha$$

  for all $p \in D_\beta$, so $a \in D_\alpha^{D_\beta}$. Thus $D_{\alpha\beta} \subseteq D_\alpha^{D_\beta}$.

- Let $\alpha \in \mathcal{T}$ and $c_\alpha \in \mathcal{L} \cup \mathcal{C}$. Then $k_c = s_\alpha(\pi(c)) \in D_\alpha$.
- Consider $k_N \in D_{oo}$. It satisfies $k_N : \{0,1\} \longrightarrow \{0,1\}$

$$\begin{aligned} k_N(1) &= s_{oo}(\pi(N))(1) \\ &= s_{oo}(\pi(N))(s_o(\pi(\top))) \\ &= s_o(\pi([N\top])) \\ &= s_o(\pi(\bot)) \\ &= 0, \end{aligned}$$

since $(\neg\top) \sim \bot$ by 9, and

$$\begin{aligned} k_N(0) &= s_{oo}(\pi(N))(0) \\ &= s_{oo}(\pi(N))(s_o(\pi(\bot))) \\ &= s_o(\pi([N\bot])) \\ &= s_o(\pi(\top)) \\ &= 1 \end{aligned}$$

by Sublemma 4.21.4. Hence

$$k_N(n) = 1 - n$$

for all $n \in \{0,1\}$.

- Now consider $k_A \in D_{\langle oo\rangle o}$. We have that

$$\begin{aligned} k_A(0)(m) &= s_{\langle oo\rangle o}(\pi(A))(0)(m) \\ &= s_{\langle oo\rangle o}(\pi(A))(s_o(\pi(\bot)))(m) \\ &= s_{oo}(\pi([A\bot]))(m) \\ &= s_{oo}(\pi([A\bot]))(s_o(\pi(r(s_o^{-1}(m))))) \\ &= s_o(\pi([[A\bot]r(s_o^{-1}(m))])) \\ &= s_o(\pi((\bot \vee r(s_o^{-1}(m))))) \\ &= s_o(\pi(r(s_o^{-1}(m)))) \\ &= s_o(s_o^{-1}(m)) \\ &= m \end{aligned}$$

since $(\bot \vee r(s_o^{-1}(m))) \sim r(s_o^{-1}(m))$ by 11, while

$$\begin{aligned}
k_A(1)(m) &= s_{\langle oo\rangle o}(\pi(A))(1)(m) \\
&= s_{\langle oo\rangle o}(\pi(A))(s_o(\pi(\top)))(m) \\
&= s_{oo}(\pi([A\top]))(m) \\
&= s_{oo}(\pi([A\top]))(s_o(\pi(r(s_o^{-1}(m))))) \\
&= s_o(\pi([[A\top]r(s_o^{-1}(m))])) \\
&= s_o(\pi((\top \vee r(s_o^{-1}(m))))) \\
&= s_o(\pi[\top]) \\
&= 1
\end{aligned}$$

since $(\top \vee r(s_o^{-1}(m))) \sim \top$ by 13, for all $m \in \{0, 1\}$. Thus

$$k_A(n)(m) = 1 - (1 - n) \cdot (1 - m)$$

for all $n, m \in \{0, 1\}$.

- Let $\alpha \in \mathcal{T}$ and consider $k_{\Pi_{o\langle o\alpha\rangle}} \in D_{o\langle o\alpha\rangle}$. First let $f \in D_{o\alpha}$ be such that $f(d) = 1$ for all $d \in D_\alpha$. Let $\varphi \in \overline{W}_{o\alpha}$ be such that $f = s_{o\alpha}(\pi(\varphi))$, and $\psi \in \overline{W}_\alpha$. Then

$$\begin{aligned}
1 &= f(s_\alpha(\pi(\psi))) \\
&= s_{o\alpha}(\pi(\varphi))(s_\alpha(\pi(\psi))) \\
&= s_o(\pi([\varphi\psi])) \\
&= u([\varphi\psi])
\end{aligned}$$

whence $[\varphi\psi] \in \Delta$. In particular (take $\psi = (\imath x_\alpha(\neg[\varphi x_\alpha]))$) $[\varphi(\imath x_\alpha(\neg[\varphi x_\alpha]))] \in \Delta$, whence $\Delta \vdash [\Pi_{o\langle o\alpha\rangle}\varphi]$ by 28. Thus

$$\begin{aligned}
k_{\Pi_{o\langle o\alpha\rangle}}(f) &= s_{o\langle o\alpha\rangle}(\pi(\Pi_{o\langle o\alpha\rangle}))(s_{o\alpha}(\pi(\varphi))) \\
&= s_o(\pi([\Pi_{o\langle o\alpha\rangle}\varphi])) \\
&= 1.
\end{aligned}$$

Conversely, if $g \in D_{o\alpha}$ is such that $k_{\Pi_{o\langle o\alpha\rangle}}(g) = 1$, then, similarly to above, $s_o(\pi([\Pi_{o\langle o\alpha\rangle}\vartheta])) = 1$, where $\vartheta \in \overline{W}_{o\alpha}$ is such that $s_\alpha(\pi(\vartheta)) = g$. Thus $[\Pi_{o\langle o\alpha\rangle}\vartheta] \in \Delta$, whence by 1 $\Delta \vdash [\vartheta\tau]$ for all $\tau \in W_\alpha$. In particular, if $d \in D_\alpha$ and $\tau \in \overline{W}_\alpha$ is such that $s_\alpha(\pi(\tau)) = d$, then

$$\begin{aligned}
1 &= s_o(\pi([\vartheta\tau])) \\
&= s_{o\alpha}(\pi(\vartheta))(s_\alpha(\pi(\tau))) \\
&= g(d).
\end{aligned}$$

To sum up,

$$k_{\Pi_{o\langle o\alpha\rangle}}(h) = \begin{cases} 1 & \text{if } h(d) = 1 \text{ for all } d \in D_\alpha \\ 0 & \text{otherwise} \end{cases}$$

for all $h \in D_{o\alpha}$.

- Finally, let $\alpha \in \mathcal{T}$ and consider $k_{\iota_{\alpha\langle o\alpha\rangle}} \in D_{\alpha\langle o\alpha\rangle}$. Assume $f \in D_{o\alpha}$ is such that there is a $d \in D_\alpha$ with $f(d) = 1$. Let $\varphi_{o\alpha}, \psi_\alpha \in \overline{\mathcal{W}}$ be such that $f = s_{o\alpha}(\pi(\varphi))$ and $d = s_\alpha(\pi(\psi))$. Like before

  $$u([\varphi\psi]) = 1$$

  whence $[\varphi\psi] \in \Delta$. By 14, $\Delta \vdash [\varphi[\iota_{\alpha\langle o\alpha\rangle}\varphi]]$, whence

  $$\begin{aligned} 1 &= s_o(\pi([\varphi[\iota_{\alpha\langle o\alpha\rangle}\varphi]])) \\ &= s_{o\alpha}(\pi(\varphi))(s_\alpha(\pi([\iota_{\alpha\langle o\alpha\rangle}\varphi]))) \\ &= s_{o\alpha}(\pi(\varphi))(s_{\alpha\langle o\alpha\rangle}(\pi(\iota_{\alpha\langle o\alpha\rangle}))(s_{o\alpha}(\pi(\varphi)))) \\ &= f(k_{\iota_{\alpha\langle o\alpha\rangle}}(f)) \end{aligned}$$

  as desired.

Thus $\mathfrak{M}$ is a non-degenerate structured frame. □

To show that $\mathfrak{M}$ is in fact a general model, let $\alpha \in \mathcal{T}$ and consider $q = s_{\langle o\alpha\rangle\alpha}(\pi(Q_{\langle o\alpha\rangle\alpha}))$. Let $a, b \in D_\alpha$ and denote $\varphi = r(s_\alpha^{-1}(a))$, $\psi = r(s_\alpha^{-1}(b))$. Then

$$\begin{aligned} q(a)(b) &= s_{\langle o\alpha\rangle\alpha}(\pi(Q_{\langle o\alpha\rangle\alpha}))(a)(b) \\ &= s_{o\alpha}(\pi([Q_{\langle o\alpha\rangle\alpha}\varphi]))(b) \\ &= s_o(\pi([[Q_{\langle o\alpha\rangle\alpha}\varphi]\psi])) \\ &= s_o(\pi((\varphi \equiv \psi))) \\ &= u((\varphi \equiv \psi)) \\ &= \begin{cases} 1 & \text{if } (\varphi \equiv \psi) \in \Delta \\ 0 & \text{if } (\neg(\varphi \equiv \psi)) \in \Delta \end{cases} \\ &= \begin{cases} 1 & \text{if } \pi(\varphi) = \pi(\psi) \\ 0 & \text{if } \pi(\varphi) \neq \pi(\psi) \end{cases} \\ &= \begin{cases} 1 & \text{if } s_\alpha^{-1}(a) = s_\alpha^{-1}(b) \\ 0 & \text{if } s_\alpha^{-1}(a) \neq s_\alpha^{-1}(b) \end{cases} \\ &= \begin{cases} 1 & \text{if } a = b \\ 0 & \text{if } a \neq b. \end{cases} \end{aligned}$$

So q is the equality predicate for D_α, as required in Definition 2.8.

Furthermore define $\mathcal{I} : A_{\mathfrak{M}} \longrightarrow P$ (recall Definition 4.13) by

$$\mathcal{I}(\Phi)(x) = r(s^{-1}_{T(x)}(\Phi(x)))$$

for all $x \in \mathcal{X}$ and $\Phi \in A_{\mathfrak{M}}$.

Sublemma 4.21.6. *For all $\alpha \in \mathcal{T}$, $\Phi \in \mathcal{A}_{\mathfrak{M}}$ and $\varphi \in W_\alpha$ we have $\mathfrak{F}(\mathcal{I}(\Phi))(\varphi) \in \overline{W}_\alpha$.*

Proof. By induction on $|VF(\varphi)|$.

- If $\psi \in \overline{W}_\alpha$ then

$$\begin{aligned}\mathfrak{F}(\mathcal{I}(\Phi))(\psi) &= \mathfrak{f}(M(\varphi))(\mathcal{I}(\Phi))(\psi)\\ &= \mathfrak{f}(0)(\mathcal{I}(\Phi))(\psi)\\ &= \psi \in \overline{W}_\alpha.\end{aligned}$$

- Assume that $k \in \mathbb{N}$ is such that $\mathfrak{F}(\mathcal{I}(\Phi))(\vartheta) \in \overline{W}_\alpha$ for all $\vartheta \in W_\alpha$ with $|VF(\vartheta)| = k$. Let $\psi \in W$ be such that $|VF(\psi)| = k + 1$, and $m = M(\psi) - 1 \in \mathbb{N}$. Then $\mathcal{X}(m) \in VF(\psi)$,

  $$\mathrm{SF}(\mathcal{X}(m))(\mathcal{I}(\Phi)(\mathcal{X}(m)))(\psi) \in W_\alpha$$

  and

  $$VF(\mathrm{SF}(\mathcal{X}(m))(\mathcal{I}(\Phi)(\mathcal{X}(m)))(\psi)) = VF(\psi) \setminus \{\mathcal{X}(m)\}$$

  by 2.18 of [7], whereby $|VF(\mathrm{SF}(\mathcal{X}(m))(\mathcal{I}(\Phi)(\mathcal{X}(m)))(\psi))| = k$ and

  $$m \geq M(\mathrm{SF}(\mathcal{X}(m))(\mathcal{I}(\Phi)(\mathcal{X}(m)))(\psi)).$$

  Thus

  $$\begin{aligned}\mathfrak{F}(\mathcal{I}(\Phi))(\psi) &= \mathfrak{f}(M(\psi))(\mathcal{I}(\Phi))(\psi)\\ &= \mathfrak{f}(m)(\mathcal{I}(\Phi))(\mathrm{SF}(\mathcal{X}(m))(\mathcal{I}(\Phi)(\mathcal{X}(m)))(\psi))\\ &= \mathfrak{F}(\mathcal{I}(\Phi))(\mathrm{SF}(\mathcal{X}(m))(\mathcal{I}(\Phi)(\mathcal{X}(m)))(\psi)) \in \overline{W}_\alpha\end{aligned}$$

  by induction hypothesis.

□

Finally define $V : \mathcal{A}_{\mathfrak{M}} \longrightarrow (\bigcup_{\alpha \in \mathcal{T}} D_\alpha)^W$ by

$$V(\Phi)(\varphi) = s_{T(\varphi)}(\pi(\mathfrak{F}(\mathcal{I}(\Phi))(\varphi)))$$

for all $\Phi \in \mathcal{A}_{\mathfrak{M}}$ and $\varphi \in W$.

Sublemma 4.21.7. *Given $\Phi \in A_{\mathfrak{M}}$, $\alpha \in \mathcal{T}$, $x \in X_\alpha$ and $d \in D_\alpha$,*

$$\mathcal{I}(\Phi)^x_\psi = \mathcal{I}(\Phi^x_d),$$

where $\psi = r(s^{-1}_\alpha(d))$.

Proof. Clearly

$$\begin{aligned}
\mathcal{I}(\Phi)^{\mathbf{x}}_{\psi}(\mathbf{y}) &= \begin{cases} \psi & \text{if } \mathbf{y} = \mathbf{x} \\ \mathcal{I}(\Phi)(\mathbf{y}) & \text{otherwise} \end{cases} \\
&= \begin{cases} r(s_\alpha^{-1}(d)) & \text{if } \mathbf{y} = \mathbf{x} \\ r(s_\alpha^{-1}(\Phi(\mathbf{y}))) & \text{otherwise} \end{cases} \\
&= \begin{cases} r(s_\alpha^{-1}(\Phi^{\mathbf{x}}_{d}(\mathbf{y}))) & \text{if } \mathbf{y} = \mathbf{x} \\ r(s_\alpha^{-1}(\Phi^{\mathbf{x}}_{d}(\mathbf{y}))) & \text{otherwise} \end{cases} \\
&= \mathcal{I}(\Phi^{\mathbf{x}}_{d})(\mathbf{y})
\end{aligned}$$

for all $\mathbf{y} \in \mathcal{X}$. □

We now show that V is a valuation for $\mathfrak{M}$.

- By definition $V(\Phi)(\varphi) \in D_\alpha$ for all $\Phi \in \mathcal{A}_{\mathfrak{M}}$, $\alpha \in \mathcal{T}$ and $\varphi \in W_\alpha$.
- Let $\Phi \in \mathcal{A}_{\mathfrak{M}}$, $\alpha \in \mathcal{T}$ and $\mathbf{x} \in X_\alpha$. Then

$$\begin{aligned}
V(\Phi)(\mathbf{x}) &= s_\alpha(\pi(\mathfrak{F}(\mathcal{I}(\Phi))(\mathbf{x}))) \\
&= s_\alpha(\pi(\mathfrak{f}(\mathcal{X}^{-1}(\mathbf{x}) + 1)(\mathcal{I}(\Phi))(\mathbf{x}))) \\
&= s_\alpha(\pi(\mathfrak{f}(\mathcal{X}^{-1}(\mathbf{x}))(\mathcal{I}(\Phi))(\mathrm{SF}(\mathbf{x})(\mathcal{I}(\Phi)(\mathbf{x}))(\mathbf{x})))) \\
&= s_\alpha(\pi(\mathfrak{f}(\mathcal{X}^{-1}(\mathbf{x}))(\mathcal{I}(\Phi))(\mathcal{I}(\Phi)(\mathbf{x})))) \\
&= s_\alpha(\pi(\mathfrak{f}(\mathcal{X}^{-1}(\mathbf{x}))(\mathcal{I}(\Phi))(r(s_\alpha^{-1}(\Phi(\mathbf{x})))))) \\
&= s_\alpha(\pi(r(s_\alpha^{-1}(\Phi(\mathbf{x}))))) \\
&= \Phi(\mathbf{x})
\end{aligned}$$

  since $r(s_\alpha^{-1}(\Phi(\mathbf{x})))$ is closed.
- Let $\Phi \in \mathcal{A}_{\mathfrak{M}}$, $\alpha \in \mathcal{T}$ and $c_\alpha \in \mathcal{L} \cup \mathcal{C}$. Then

$$\begin{aligned}
V(\Phi)(c) &= s_\alpha(\pi(\mathfrak{F}(\mathcal{I}(\Phi))(c))) \\
&= s_\alpha(\pi(c)) \\
&= k_c
\end{aligned}$$

  as desired.
- Let $\Phi \in \mathcal{A}_{\mathfrak{M}}$, $\alpha, \beta \in \mathcal{T}$, $\varphi \in W_\alpha$, $\mathbf{x} \in X_\beta$ and $\mathbf{d} \in D_\beta$. Furthermore let $\psi \in \overline{W}_\beta$ be

such that $s_\beta(\pi(\psi)) = d$. Then

$$\begin{aligned} V(\Phi)([\lambda x\varphi])(d) &= s_{\alpha\beta}(\pi(\mathfrak{F}(\mathfrak{I}(\Phi))([\lambda x\varphi])))(s_\beta(\pi(\psi))) \\ &= s_\alpha(\pi([\mathfrak{F}(\mathfrak{I}(\Phi))([\lambda x\varphi])\psi])) \\ &= s_\alpha(\pi([[\lambda x\mathfrak{F}(\mathfrak{I}(\Phi)^x_x)(\varphi)]\psi])) \\ &= s_\alpha(\pi(\mathrm{SF}(x)(\psi)(\mathfrak{F}(\mathfrak{I}(\Phi)^x_x)(\varphi)))) \\ &= s_\alpha(\pi(\mathfrak{F}(\mathfrak{I}(\Phi)^x_\psi)(\varphi))) \\ &= s_\alpha(\pi(\mathfrak{F}(\mathfrak{I}(\Phi^x_d))(\varphi))) \\ &= V(\Phi^x_d)(\varphi) \end{aligned}$$

by, in order, definition, Sublemma 4.21.4, Lemma 4.16, Corollary 4.6, Lemma 4.20, Sublemma 4.21.7 and definition again.

- Let $\Phi \in \mathcal{A}_{\mathfrak{M}}$, $\alpha, \beta \in \mathcal{T}$ and $\varphi_{\alpha\beta}, \psi_\beta \in W$. Then

$$\begin{aligned} V(\Phi)([\varphi\psi]) &= s_\alpha(\pi(\mathfrak{F}(\mathfrak{I}(\Phi))([\varphi\psi]))) \\ &= s_\alpha(\pi([\mathfrak{F}(\mathfrak{I}(\Phi))(\varphi)\mathfrak{F}(\mathfrak{I}(\Phi))(\psi)])) \\ &= s_{\alpha\beta}(\pi(\mathfrak{F}(\mathfrak{I}(\Phi))(\varphi)))(s_\beta(\pi(\mathfrak{F}(\mathfrak{I}(\Phi))(\psi)))) \\ &= V(\Phi)(\varphi)(V(\Phi)(\psi)). \end{aligned}$$

Hence $\mathfrak{M}$ is a general model. To show that $\mathfrak{M} \models \Gamma$, let $\varphi \in \Gamma \subseteq \Delta$. Then we have, for all $\Phi \in \mathcal{A}_{\mathfrak{M}}$,

$$\begin{aligned} V(\Phi)(\varphi) &= s_o(\pi(\mathfrak{F}(\Phi)(\varphi))) \\ &= s_o(\pi(\varphi)) \\ &= u(\varphi) \\ &= 1. \end{aligned}$$

Hence $\mathfrak{M} \models \Gamma$. □

The "interesting consequences" of Proposition 4.9 we promised earlier are all hidden in the following corollary and its corollary, the proof of which will venture slightly outside the fixed setting we have used in the rest of the thesis (except for Theorem 4.30).

4.22 Corollary (Compactness Theorem). *Let $\Gamma \subseteq \overline{W}_o$. Γ has a general model if and only if every finite subset thereof has a general model.*

4.23 Corollary (Löwenheim-Skolem Theorem). *Let $\Gamma \subseteq \overline{W}_o$, $\mathfrak{M} = (\{D_\alpha\}_{\alpha\in\mathcal{T}}, \{k_c\}_{c\in\mathcal{L}\cup\mathcal{C}})$ be a general model and $\alpha \in \mathcal{T}$. Assume that $|D_\alpha| \geq \aleph_0$ and $\mathfrak{M} \models \Gamma$. Then for every cardinal $\kappa \geq |D_\alpha|$ there is a general model $\mathfrak{K} = (\{E_\alpha\}_{\alpha\in\mathcal{T}}, \{l_c\}_{c\in\mathcal{L}\cup\mathcal{C}})$ such that $\mathfrak{K} \models \Gamma$ and $|E_\alpha| \geq \kappa$.*

Sketch of proof. Assume $\mathfrak{M} \models \Gamma$ and let $\kappa \geq |D_\alpha|$ be a cardinal. Let C'_α be a set of cardinality κ disjoint from $\mathcal{S}$, and consider the theory of types with $\mathcal{C} \cup C'_\alpha$ as constant symbols, where all constants of C'_α have type α; let W' denote the set of well-formed formulae in this instantiation of the theory and let P' be the W' version of P above. Let $\Delta = \Gamma \cup \{(a \not\equiv b) \in W' \mid a, b \in C'_\alpha \wedge a \neq b\}$. We will prove that Δ has a model $\mathfrak{K}$.

Let $\Theta \subseteq \Delta$ be finite and $K \subseteq C'_\alpha$ be the set of all new constants occurring in Θ. Then K is finite as well. Since D_α is infinite there is an injective function $f : K \rightarrowtail D_\alpha$, and we define

$$j_c = \begin{cases} f(c) & c \in C'_\alpha \\ k_c & c \in \mathcal{L} \cup \mathcal{C} \end{cases}$$

for all $c \in \mathcal{L} \cup \mathcal{C} \cup C'_\alpha$. Let $\mathfrak{J} = (\{D_\alpha\}_{\alpha \in \mathcal{T}}, \{j_c\}_{c \in \mathcal{L} \cup \mathcal{C} \cup C'_\alpha})$, so that $\mathfrak{J}$ is a non-degenerate structured frame. Furthermore, for all $\varphi \in W'$ there are $n_\varphi \in \mathbb{N}$, $f_\varphi \in P'$ and $\psi_\varphi \in W$ such that

$$\varphi = \mathfrak{f}(n_\varphi)(f_\varphi)(\psi_\varphi)$$

and no $x \in VF(\psi_\varphi) \setminus VF(\varphi)$ is bound in ψ_φ. For all $\Phi \in A_{\mathfrak{J}} = A_{\mathfrak{M}}$ and $\varphi \in W'$ we thus define ${}_\varphi\Phi \in A_{\mathfrak{J}}$ by

$${}_\varphi\Phi(x) = \begin{cases} j_{f_\varphi(x)} & \text{if } x \in VF(\psi_\varphi) \setminus VF(\varphi) \\ \Phi(x) & \text{otherwise} \end{cases}$$

for all $x \in \mathcal{X}$. Then let $V' : \mathcal{A}_{\mathfrak{J}} \longrightarrow (\bigcup_{\alpha \in \mathcal{T}} D_\alpha)^{W'}$ be given by

$$V'(\Phi)(\varphi) = V({}_\varphi\Phi)(\psi_\varphi)$$

for all $\Phi \in A_{\mathfrak{J}}$ and $\varphi \in W'$. Clearly (though not necessarily readily) this makes $\mathfrak{J}$ a general model, and $\mathfrak{J} \models \Theta$.

Since every finite subset Θ of Δ has a general model, Δ itself has a general model $\mathfrak{K} = (\{E_\alpha\}_{\alpha \in \mathcal{T}}, \{l_c\}_{c \in \mathcal{L} \cup \mathcal{C}})$. Moreover, since $l_a \neq l_b$ for distinct $a, b \in C'_\alpha$, we have $|E_\alpha| \geq |C'_\alpha| = \kappa$. $\square$

Completeness and incompleteness

Having established the equivalence of consistency and existence of models, we can now give a proof of the Completeness Theorem. However, since the Model Existence Theorem and Deduction Theorem, which are needed in this proof, only treats closed well-formed formulae, we will need to define the *universal closure* of a well-formed formula. It will be shown that a well-formed formula and its universal closure are essentially equivalent.

4.24 Definition. For $\varphi \in W_o$ we define $\overline{\varphi} = \mathbb{V}(M(\varphi))(\varphi)$, where

$$\mathbb{V}(0)(\varphi) = \varphi$$

$$\mathbb{V}(n+1)(\varphi) = \begin{cases} (\forall \mathfrak{X}(n) \mathbb{V}(n)(\varphi)) & \text{if } \mathfrak{X}(n) \in VF(\varphi) \\ \mathbb{V}(n)(\varphi) & \text{otherwise} \end{cases}$$

for all $n \in \mathbb{N}$.

The universal closure is of course closed, as the next lemma shows.

4.25 Lemma. *For every $\varphi \in W_o$ we have $\overline{\varphi} \in \overline{W}_o$.*

Proof. First note that $\mathsf{VF}(\mathbb{V}(n+1)(\varphi)) \subseteq \mathsf{VF}(\mathbb{V}(n)(\varphi))$ for every $n \in \mathbb{N}$, whence in particular $\mathsf{VF}(\overline{\varphi}) \subseteq \mathsf{VF}(\mathbb{V}(n)(\varphi))$ for all $n \in \mathbb{N}$ such that $n \leq M(\varphi)$. Furthermore, for all $n \in \mathbb{N}$ we have

$$\mathcal{X}(n) \notin \mathsf{VF}(\mathbb{V}(n+1)(\varphi))$$

by definition. Hence

$$\mathcal{X}(n) \notin \mathsf{VF}(\overline{\varphi})$$

for all $n \in \mathbb{N}$, $n < M(\varphi)$. Since $\mathcal{X}^{-1}(\mathbf{x}) < M(\varphi)$ for all $\mathbf{x} \in \mathsf{VF}(\varphi)$, we have

$$\mathsf{VF}(\varphi) \cap \mathsf{VF}(\overline{\varphi}) = \varnothing.$$

Thus

$$\mathsf{VF}(\overline{\varphi}) = \varnothing$$

as claimed. □

4.26 Lemma. *Let $\mathfrak{M} = (\{D_\alpha\}_{\alpha \in \mathcal{T}}, \{k_c\}_{c \in \mathcal{L} \cup \mathcal{C}})$ be a general model. Then $\mathfrak{M} \models \varphi \Rightarrow \mathfrak{M} \models \overline{\varphi}$ for all $\varphi \in W_o$. In particular $\Gamma \models \varphi \Rightarrow \Gamma \models \overline{\varphi}$ for all $\Gamma \subseteq \overline{W}_o$ and $\varphi \in W_o$.*

Proof. Suppose that $\varphi \in W_o$ is such that $\mathfrak{M} \models \varphi$ and let $\alpha \in \mathcal{T}$, $\mathbf{x} \in X_\alpha$ and $\Phi \in A_{\mathfrak{M}}$. Then

$$V(\Phi^{\mathbf{x}}_{d})(\varphi) = 1$$

for every $d \in D_\alpha$. Hence

$$\mathfrak{M} \models (\forall \mathbf{x} \varphi)$$

as well. So $\mathfrak{M} \models \varphi \Rightarrow \mathfrak{M} \models (\forall \mathbf{x} \varphi)$ for all $\varphi \in W_o$ and $\mathbf{x} \in \mathcal{X}$.

We now prove that, for all $\varphi \in W_o$ such that $\mathfrak{M} \models \varphi$,

$$\mathfrak{M} \models \mathbb{V}(n)(\varphi)$$

for all $n \in \mathbb{N}$, by induction on n. So let $\varphi \in W$ and suppose $\mathfrak{M} \models \varphi$.

- Since $\mathbb{V}(0)(\varphi) = \varphi$ this case is trivial.
- Assume $k \in \mathbb{N}$ is such that $\mathfrak{M} \models \mathbb{V}(k)(\varphi)$. If $\mathcal{X}(k) \notin \mathsf{VF}(\varphi)$ then $\mathbb{V}(k+1)(\varphi) = \mathbb{V}(k)(\varphi)$. Otherwise $\mathbb{V}(k+1)(\varphi) = (\forall \mathcal{X}(k) \mathbb{V}(k)(\varphi))$. In either case

$$\mathfrak{M} \models \mathbb{V}(k+1)(\varphi).$$

In particular $\mathfrak{M} \models \overline{\varphi}$. □

4.27 Lemma. *For all* $\varphi \in W_o$, $\vdash (\overline{\varphi} \to \varphi)$.

Proof. Let $\varphi \in W_o$. We first prove that $\vdash (\mathbb{V}(n)(\varphi) \to \varphi)$ for all $n \in \mathbb{N}$, whence the claim follows as a special case.

- To begin with, $\vdash (\mathbb{V}(0)(\varphi) \to \varphi)$ by definition and Lemma 4.4.4.
- Suppose $k \in \mathbb{N}$ is such that $\vdash (\mathbb{V}(k)(\varphi) \to \varphi)$. If $\mathfrak{X}(k) \notin \mathsf{VF}(\varphi)$ then $\mathbb{V}(k+1)(\varphi) = \mathbb{V}(k)(\varphi)$. Otherwise $\mathbb{V}(k+1)(\varphi) = (\forall \mathfrak{X}(k)\mathbb{V}(k)(\varphi))$. In any case, since $\vdash ((\forall \boldsymbol{x}\varphi) \to \varphi)$ for all $\varphi \in W_o$ and $\boldsymbol{x} \in \mathcal{X}$ by Lemma 4.4.15,

$$\vdash (\mathbb{V}(k+1)(\varphi) \to \varphi).$$

We conclude that $\vdash (\mathbb{V}(n)(\varphi) \to \varphi)$ for all $n \in \mathbb{N}$, whence

$$\vdash (\overline{\varphi} \to \varphi)$$

as claimed. □

Now for the payoff of our strife; the Completeness Theorem of general semantics.

4.28 Theorem (Completeness Theorem)**.** *Let* $\Gamma \subseteq \overline{W_o}$ *and* $\varphi \in W_o$ *be such that* $\Gamma \models \varphi$. *Then* $\Gamma \vdash \varphi$.

Proof. By Lemma 4.26, $\Gamma \models \varphi$ only if $\Gamma \models \overline{\varphi}$. Thus $\Gamma \cup \{(\neg\overline{\varphi})\}$ has no model, whereby it is inconsistent. In particular

$$\Gamma \cup \{(\neg\overline{\varphi})\} \vdash \overline{\varphi}$$

by Lemma 4.8, whence

$$\Gamma \vdash ((\neg\overline{\varphi}) \to \overline{\varphi})$$

by the Deduction Theorem. Hence

$$\Gamma \vdash \overline{\varphi}$$

by Lemma 4.4.10. Finally Lemma 4.27 yields

$$\Gamma \vdash \varphi.$$

□

4.29 Corollary. *The following are equivalent for all* $\varphi \in \overline{W_o}$ *and* $\psi \in W_o$*:*

- $\varphi \vdash \psi$.
- $\varphi \models \psi$.

- $\vdash (\varphi \rightarrow \psi)$.
- $\models (\varphi \rightarrow \psi)$.

Having thus verified completeness of general semantics, we turn to the corresponding questions for standard semantics. As has been stated several times, however, standard semantics are in fact incomplete. In particular, the two notions of semantics are indeed different, i.e. there are general models which are not standard (in general the model constructed in Theorem 4.21 is not standard).

4.30 Theorem (Incompleteness Theorem)**.** *Standard semantics are incomplete, that is for some choice of the non-logical constants $\mathcal{C}$, there is a $\Gamma \subseteq \overline{W}_o$ and a $\varphi \in W_o$ such that φ is true in every standard model of Γ but $\Gamma \nvdash \varphi$.*

Sketch of proof. Assume that standard semantics is complete. If $\Delta \subseteq \overline{W}_o$ does not have a standard model, then in particular $\mathfrak{M} \models \bot$ for all standard models $\mathfrak{M}$ of Δ, whence $\Delta \vdash \bot$. Thus we have a Model Existence Theorem, that is if $\Delta \subseteq \overline{W}_o$ is consistent then Δ has a standard model. From this both a Compactness Theorem and a Löwenheim-Skolem Theorem (cf. 4.22 and 4.23) follows in the same way as before.

Suppose $\mathcal{C} = \{0_i, S_{ii}\}$, and let Γ consist of the following closed well-formed formulae

1. $(\forall x_i(\forall y_i(([S_{ii}x_i] = [S_{ii}y_i]) \rightarrow (x_i = y_i))))$
2. $(\forall x_i([S_{ii}x_i] \neq 0_i))$
3. $(\forall p_{oi}([p_{oi}0_i] \rightarrow ((\forall x_i([p_{oi}x_i] \rightarrow [p_{oi}[S_{ii}x_i]])) \rightarrow (\forall x_i[p_{oi}xi]))))$.

Let $\mathfrak{N} = (\{E_\alpha\}_{\alpha \in \mathcal{T}}, \{l_c\}_{c \in \mathcal{L} \cup \mathcal{C}})$ be the standard model constructed in Proposition 2.19 with the following modifications:

$$k_{0_i} = 0$$
$$k_{S_{ii}} = S \text{ (the "real" successor operator on the natural numbers).}$$

Clearly $\mathfrak{N} \models \Gamma$. In particular Γ is consistent, having a standard model with countable domain of individuals. By the Löwenheim-Skolem result above, there must be a standard model $\mathfrak{M} = (\{D_\alpha\}_{\alpha \in \mathcal{T}}, \{k_c\}_{c \in \mathcal{L} \cup \mathcal{C}})$ of Γ such that $|D_i| \geq \aleph_1$.

However, since $\mathfrak{M} = (\{D_\alpha\}_{\alpha \in \mathcal{T}}, \{k_c\}_{c \in \mathcal{L} \cup \mathcal{C}})$ is a standard model, every subset of D_i is represented by some element of D_{oi}. Hence, by 3 the only subset of D_i which contains k_{0_i} and the successors of all its elements is D_i itself, which is thereby isomorphic to $\mathbb{N}$ (see for example Theorem 5.4. of [6]). This is clearly a contradiction. □

In [4], Henkin argues for incompleteness (in the above sense) by Gödel's theorems. We have chosen the above argument, which is similar to that given in [2], since it seems more direct, if not as generally applicable (though we could in fact have done without the restriction on $\mathcal{C}$). Note, however, that we have in fact established the a priori stronger result of non-compactness of standard semantics. In particular, the model constructed in the proof of Theorem 4.21 is in general not standard.

We have thus shown that there is a systematic and reasonable way to interpret the well-formed formulae of simple type theory which corresponds *exactly* to the "natural interpretation" given by the permitted deductions (in fact, as we have shown in Theorem 4.28, this "natural interpretation" can be made precise). From another viewpoint, however, it could be argued that the "intuitive" interpretation (i.e. the one captured by the notion of standard model) should be the focus of our study, whence we should try to find a deductive system capturing it as closely as possible. While the deductive system in question does not allow too much, in that we cannot prove false statements (3.5), it is not perfect either, since there are true (in the standard sense) statements which are unprovable (4.30).

A. Formal theorems

We will here provide proofs of the following formal theorems, which are the contents of Lemma 4.4. We will let the use of some "obvious" lemmata, such as part 1 of Lemma 3.10 of [7], be implicit, to avoid too tedious proofs. Let $\alpha, \beta \in \mathcal{T}$, $\varphi, \psi, \vartheta, \tau \in W$ and $\mathbf{x} \in \mathcal{X}$.

1. $\vdash ([\Pi_{o\langle o\alpha\rangle}\varphi_{o\alpha}] \to [\varphi_{o\alpha}\psi_\alpha])$.
2. $\vdash (\bot \to \varphi_o)$.
3. $\vdash ((\neg\varphi_o) \to (\varphi_o \to \bot))$.
4. $\vdash (\varphi_o \to \varphi_o)$.
5. $\vdash ((\neg\varphi_o) \vee \varphi_o)$.
6. $\vdash ((\varphi_o \to \psi_o) \to ((\psi_o \to \varphi_o) \to (\varphi_o \equiv \psi_o)))$.
7. $\vdash (\varphi_o \to (\neg(\neg\varphi_o)))$.
8. $\vdash ((\neg(\neg\varphi_o)) \to \varphi_o)$.
9. $\vdash (\varphi_o \equiv (\neg(\neg\varphi_o)))$.
10. $\vdash (((\neg\varphi_o) \to \varphi_o) \to \varphi_o)$.
11. $\vdash ((\bot \vee \varphi_o) \equiv \varphi_o)$.
12. $\vdash \top$.
13. $\vdash ((\top \vee \varphi_o) \equiv \top)$.
14. $\vdash ([\varphi_{o\alpha}\psi_\alpha] \to [\varphi_{o\alpha}[\iota_{\alpha\langle o\alpha\rangle}\varphi_{o\alpha}]])$.
15. $\vdash ((\forall \mathbf{x}_\alpha \varphi_o) \to \varphi_o)$.

Proof. 1. By axiom 5

$$\vdash ([\Pi_{o\langle o\alpha\rangle}f_{o\alpha}] \to [f_{o\alpha}x_\alpha]),$$

which by part 2 of Theorem 3.8 of [7] implies that

$$\vdash ([\Pi_{o\langle o\alpha\rangle}\mathbf{f}_{o\alpha}] \to [\mathbf{f}_{o\alpha}\mathbf{x}_\alpha])$$

for some $\mathbf{x}_\alpha, \mathbf{f}_{o\alpha} \in \mathcal{X}$ which does not occur in φ or ψ. By Proposition 3.9 of [7] we have

$$\vdash ([\Pi_{o\langle o\alpha\rangle}\varphi_{o\alpha}] \to [\varphi_{o\alpha}\psi_\alpha]).$$

2. By 1

$$\vdash ([\Pi_{o\langle oo\rangle}[\lambda p_o p_o]] \to [[\lambda p_o p_o]\varphi_o]),$$

whence $\vdash (\bot \to \varphi_o)$ by β-contraction.

3. By axiom 2 $\vdash (p_o \to (p_o \vee q_o))$, whence

$$\vdash ((\neg\varphi_o) \to ((\neg\varphi_o) \vee \bot))$$

like before. This is the same as $\vdash ((\neg\varphi_o) \to (\varphi_o \to \bot))$.

4. By part 2 of Lemma 3.6 of [7], $\vdash (p_o \to p_o)$, whence $\vdash (\varphi_o \to \varphi_o)$ like before.

5. The above is exactly $\vdash ((\neg\varphi_o) \vee \varphi_o)$.

6. By Lemma 3.11 part 8 of [7], $\vdash (((p_o \wedge q_o) \to r_o) \to (p_o \to (q_o \to r_o)))$. Hence, like before,

$$\vdash (((p_o \wedge q_o) \to (p_o \wedge q_o)) \to (p_o \to (q_o \to (p_o \wedge q_o))))$$

whereby, since $\vdash ((p_o \wedge q_o) \to (p_o \wedge q_o))$ by 4,

$$\vdash (p_o \to (q_o \to (p_o \wedge q_o)))$$

by Modus ponens. Similarly to before (that is, by applying Proposition 3.9 of [7]) this yields

$$\vdash ((\varphi_o \to \psi_o) \to ((\psi_o \to \varphi_o) \to ((\varphi_o \to \psi_o) \wedge (\psi_o \to \varphi_o)))),$$

that is

$$\vdash ((\varphi_o \to \psi_o) \to ((\psi_o \to \varphi_o) \to (\varphi_o \leftrightarrow \psi_o))). \tag{5}$$

By axiom 2 we get, using Proposition 3.9 of [7] once more,

$$\vdash ((\neg(\psi_o \to \varphi_o)) \to ((\neg(\psi_o \to \varphi_o)) \vee (\varphi_o \equiv \psi_o))). \tag{6}$$

Similarly, by Lemma 3.6 part 3 of [7],

$$\vdash ((\varphi_o \equiv \psi_o) \to ((\neg(\psi_o \to \varphi_o)) \vee (\varphi_o \equiv \psi_o))). \tag{7}$$

Furthermore, $\vdash ((x_o \leftrightarrow y_o) \to (x_o \equiv y_o))$ by axiom 10^o. Letting $\mathbf{p}_o, \mathbf{q}_o, \mathbf{R}_{\langle oo\rangle o} \in \mathcal{X}$ be distinct and without occurrences in φ_o, ψ_o and $Q_{\langle oo\rangle o}$, we thereby see that

$$\vdash ((\mathbf{p}_o \leftrightarrow \mathbf{q}_o) \to (\mathbf{p}_o \equiv \mathbf{q}_o))$$

by Theorem 3.8.2 of [7], where

$$((\mathbf{p}_o \leftrightarrow \mathbf{q}_o) \to (\mathbf{p}_o \equiv \mathbf{q}_o)) = \mathrm{SF}(\mathbf{R}_{\langle oo\rangle o})(Q_{\langle oo\rangle o})(((\mathbf{p}_o \leftrightarrow \mathbf{q}_o) \to [[\mathbf{R}_{\langle oo\rangle o}\mathbf{p}_o]\mathbf{q}_o]))$$

and $\mathcal{VB}(((\mathbf{p}_o \leftrightarrow \mathbf{q}_o) \rightarrow [[\mathbf{R}_{\langle oo \rangle o} \mathbf{p}_o] \mathbf{q}_o])) = \varnothing$; thus

$$\vdash ((\varphi_o \leftrightarrow \mathbf{q}_o) \rightarrow (\varphi_o \equiv \mathbf{q}_o))$$

and

$$\vdash ((\varphi_o \leftrightarrow \psi_o) \rightarrow (\varphi_o \equiv \psi_o))$$

by Proposition 3.9 of [7]. With (7) this gives

$$\vdash ((\varphi_o \leftrightarrow \psi_o) \rightarrow ((\neg(\psi_o \rightarrow \varphi_o)) \vee (\varphi_o \equiv \psi_o))) \tag{8}$$

by Lemma 3.10.2 of [7]. By (6), (8) and Lemma 3.10.3 of [7] we get

$$\vdash (((\neg(\psi_o \rightarrow \varphi_o)) \vee (\varphi_o \leftrightarrow \psi_o)) \rightarrow ((\neg(\psi_o \rightarrow \varphi_o)) \vee (\varphi_o \equiv \psi_o))),$$

that is

$$\vdash (((\psi_o \rightarrow \varphi_o) \rightarrow (\varphi_o \leftrightarrow \psi_o)) \rightarrow ((\psi_o \rightarrow \varphi_o) \rightarrow (\varphi_o \equiv \psi_o))),$$

which together with (5) and Lemma 3.10.2 of [7] gives

$$\vdash ((\varphi_o \rightarrow \psi_o) \rightarrow ((\psi_o \rightarrow \varphi_o) \rightarrow (\varphi_o \equiv \psi_o))).$$

7. By Lemma 3.11.1 of [7], $\vdash (p_o \rightarrow (\neg(\neg p_o)))$, whereby $\vdash (\varphi_o \rightarrow (\neg(\neg \varphi_o)))$.

8. By Lemma 3.11.2 of [7], $\vdash ((\neg(\neg p_o)) \rightarrow p_o)$, whereby $\vdash ((\neg(\neg \varphi_o)) \rightarrow \varphi_o)$.

9. By 6

$$\vdash ((\varphi_o \rightarrow (\neg(\neg \varphi_o))) \rightarrow (((\neg(\neg \varphi_o)) \rightarrow \varphi_o) \rightarrow (\varphi_o \equiv (\neg(\neg \varphi_o))))),$$

whence

$$\vdash (\varphi_o \equiv (\neg(\neg \varphi_o)))$$

by 7, 8 and Modus ponens.

10. Since $\vdash (\varphi_o \rightarrow \varphi_o)$ and $\vdash ((\neg(\neg \varphi_o)) \rightarrow \varphi_o)$, we have $\vdash (((\neg(\neg \varphi_o)) \vee \varphi_o) \rightarrow \varphi_o)$ by Lemma 3.10.3 of [7]. This is the same as

$$\vdash (((\neg \varphi_o) \rightarrow \varphi_o) \rightarrow \varphi_o).$$

11. By Lemma 3.6.3 of [7],

$$\vdash (\varphi_o \rightarrow (\bot \vee \varphi_o)).$$

Conversely, by 2 we know that

$$\vdash (\bot \rightarrow \varphi_o),$$

while

$$\vdash (\varphi_o \to \varphi_o)$$

by 4, whence

$$\vdash ((\bot \vee \varphi_o) \to \varphi_o)$$

by Lemma 3.10.3 in [7]. By 6 and Modus ponens

$$\vdash ((\bot \vee \varphi_o) \equiv \varphi_o).$$

12. By 2, $\vdash (\bot \to \top)$. That $\vdash ((\neg\top) \to \bot)$ is an instance of 8. By Lemma 3.10.2 of [7], this gives $\vdash ((\neg\top) \to \top)$. By 10 and Modus ponens $\vdash \top$.

13. Like before, $\vdash (\top \to (\top \vee \varphi_o))$ by axiom 2. Similarly, by Lemma 3.6.3 of [7],

$$\vdash (\top \to ((\neg(\top \vee \varphi_o)) \vee \top)),$$

whence $\vdash ((\top \vee \varphi_o) \to \top)$ by above and Modus ponens. Thus, by 6 and Modus ponens,

$$\vdash ((\top \vee \varphi_o) \equiv \top).$$

14. By axiom 11 $\vdash ([f_{o\alpha}x_\alpha] \to [f_{o\alpha}[\iota_{\alpha\langle o\alpha\rangle} f_{o\alpha}]])$. Thus

$$\vdash ([\varphi_{o\alpha}\psi_\alpha] \to [\varphi_{o\alpha}[\iota_{\alpha\langle o\alpha\rangle} \varphi_{o\alpha}]])$$

like before.

15. Let $\mathbf{y}_\alpha \in \mathcal{X}\backslash\{\mathbf{x}_\alpha\}$ be such that $\mathbf{y}_\alpha$ does not occur in φ_o, and let $\eta_o = \mathrm{SB}(\mathbf{x}_\alpha)(\mathbf{y}_\alpha)(\varphi_o)$. By 1, $\vdash ([\Pi_{o\langle o\alpha\rangle}[\lambda\mathbf{x}_\alpha\eta_o]] \to [[\lambda\mathbf{x}_\alpha\eta_o]\mathbf{x}_\alpha])$. By β-contraction,

$$\vdash ([\Pi_{o\langle o\alpha\rangle}[\lambda\mathbf{x}_\alpha\eta_o]] \to \eta_o).$$

Since $\mathbf{y}_\alpha$ does not occur in φ_o, $\vdash ([\Pi_{o\langle o\alpha\rangle}[\lambda\mathbf{x}_\alpha\varphi_o]] \to \varphi_o)$ by Lemma 4.2. This is exactly

$$\vdash ((\forall\mathbf{x}_\alpha\varphi_o) \to \varphi_o).$$

□

Let $\alpha, \beta \in \mathcal{T}$, $\varphi, \psi, \vartheta, \tau \in \overline{W}$ (this is mostly to facilitate the proofs, since it allows for applications of e.g. the Deduction Theorem) and $\mathbf{x} \in \mathcal{X}$.

16. $\vdash ((\varphi_o \to \psi_o) \to (((\neg\varphi_o) \to \psi_o) \to \psi_o))$.

17. $\vdash ((\varphi_\alpha \equiv \psi_\alpha) \to ([\vartheta_{o\alpha}\varphi_\alpha] \to [\vartheta_{o\alpha}\psi_\alpha]))$.

18. $\vdash (\varphi_\alpha \equiv \varphi_\alpha)$.

19. $\vdash ((\varphi_\alpha \equiv \psi_\alpha) \to (\psi_\alpha \equiv \varphi_\alpha))$.

20. $\vdash ((\varphi_\alpha \equiv \psi_\alpha) \to ((\psi_\alpha \equiv \vartheta_\alpha) \to (\varphi_\alpha \equiv \vartheta_\alpha)))$.

21. $\vdash (\varphi_o \to ((\varphi_o \equiv \psi_o) \to \psi_o))$.

22. $\vdash ((\neg\varphi_o) \to ((\varphi_o \equiv \psi_o) \to (\neg\psi_o)))$.

23. $\vdash (\varphi_o \to (\psi_o \to (\varphi_o \equiv \psi_o)))$.

24. $\vdash ((\neg\varphi_o) \to ((\neg\psi_o) \to (\varphi_o \equiv \psi_o)))$.

25. $\vdash ((\varphi_{\alpha\beta} \equiv \psi_{\alpha\beta}) \to ((\vartheta_\beta \equiv \tau_\beta) \to ([\varphi_{\alpha\beta}\vartheta_\beta] \equiv [\psi_{\alpha\beta}\tau_\beta])))$.

26. $\vdash ([\lambda \mathbf{x}_\beta[\varphi_{\alpha\beta}\mathbf{x}_\beta]] \equiv \varphi_{\alpha\beta})$.

27. $\vdash ((\exists \mathbf{x}_\alpha[\varphi_{o\alpha}\mathbf{x}_\alpha]) \to [\varphi_{o\alpha}(\imath \mathbf{x}_\alpha[\varphi_{o\alpha}\mathbf{x}_\alpha])])$.

28. $\vdash ([\varphi_{o\alpha}(\imath \mathbf{x}_\alpha(\neg[\varphi_{o\alpha}\mathbf{x}_\alpha]))] \to [\Pi_{o\langle o\alpha\rangle}\varphi_{o\alpha}])$.

29. $\vdash (([\varphi_{\alpha\beta}(\imath \mathbf{x}_\beta([\varphi_{\alpha\beta}\mathbf{x}_\beta] \not\equiv [\psi_{\alpha\beta}\mathbf{x}_\beta]))] \equiv [\psi_{\alpha\beta}(\imath \mathbf{x}_\beta([\varphi_{\alpha\beta}\mathbf{x}_\beta] \not\equiv [\psi_{\alpha\beta}\mathbf{x}_\beta]))]) \to (\varphi_{\alpha\beta} \equiv \psi_{\alpha\beta}))$.

Proof. 16. Trivially we have that $\{(\varphi_o \to \psi_o), ((\neg\varphi_o) \to \psi_o)\} \vdash (\varphi_o \to \psi_o)$ and $\{(\varphi_o \to \psi_o), ((\neg\varphi_o) \to \psi_o)\} \vdash ((\neg\varphi_o) \to \psi_o)$. By Lemma 3.10.3 of [7],

$$\{(\varphi_o \to \psi_o), ((\neg\varphi_o) \to \psi_o)\} \vdash (((\neg\varphi_o) \vee \varphi_o) \to \psi_o),$$

whence $\{(\varphi_o \to \psi_o), ((\neg\varphi_o) \to \psi_o)\} \vdash \psi_o$ by 5 and Modus ponens. Now the Deduction Theorem assures that

$$(\varphi_o \to \psi_o) \vdash (((\neg\varphi_o) \to \psi_o) \to \psi_o)$$

and

$$\vdash ((\varphi_o \to \psi_o) \to (((\neg\varphi_o) \to \psi_o) \to \psi_o)),$$

as desired.

17. Clearly $(\varphi_\alpha \equiv \psi_\alpha) \vdash (\varphi_\alpha \equiv \psi_\alpha)$. Hence

$$(\varphi_\alpha \equiv \psi_\alpha) \vdash (\forall f_{o\alpha}([f_{o\alpha}\varphi_\alpha] \to [f_{o\alpha}\psi_\alpha]))$$

by Lemma 4.3 (twice). By 15 and Modus ponens,

$$(\varphi_\alpha \equiv \psi_\alpha) \vdash ([f_{o\alpha}\varphi_\alpha] \to [f_{o\alpha}\psi_\alpha]).$$

Hence

$$(\varphi_\alpha \equiv \psi_\alpha) \vdash ([\vartheta_{o\alpha}\varphi_\alpha] \to [\vartheta_{o\alpha}\psi_\alpha])$$

by Theorem 3.8.2 of [7], whereby

$$\vdash ((\varphi_\alpha \equiv \psi_\alpha) \to ([\vartheta_{o\alpha}\varphi_\alpha] \to [\vartheta_{o\alpha}\psi_\alpha]))$$

by the Deduction Theorem.

18. Let $\mathfrak{y} \in X_\alpha \setminus \{x_\alpha, y_\alpha\}$ be without occurrences in φ. By 4 we have

$$\vdash ([f_{o\alpha}\varphi_\alpha] \to [f_{o\alpha}\varphi_\alpha]),$$

whence

$$\vdash (\forall f_{o\alpha}([f_{o\alpha}\varphi_\alpha] \to [f_{o\alpha}\varphi_\alpha]))$$

by Theorem 3.8.3 of [7]. By β-expansion we get

$$\vdash [[\lambda \mathfrak{y}_\alpha(\forall f_{o\alpha}([f_{o\alpha}\varphi_\alpha] \to [f_{o\alpha}\mathfrak{y}_\alpha]))]\varphi_\alpha],$$

since φ is closed. Similarly

$$\vdash [[[\lambda x_\alpha[\lambda \mathfrak{y}_\alpha(\forall f_{o\alpha}([f_{o\alpha}x_\alpha] \to [f_{o\alpha}\mathfrak{y}_\alpha]))]]\varphi_\alpha]\varphi_\alpha],$$

whereby

$$\vdash [[[\lambda x_\alpha[\lambda y_\alpha(\forall f_{o\alpha}([f_{o\alpha}x_\alpha] \to [f_{o\alpha}y_\alpha]))]]\varphi_\alpha]\varphi_\alpha]$$

by α-conversion. Thus $\vdash (\varphi_\alpha \equiv \varphi_\alpha)$.

19. Let $\mathfrak{y}_\alpha \in \mathcal{X}$ be without occurrences in $Q_{\langle o\alpha\rangle\alpha}$, φ_α and ψ_α. By 17 and Modus ponens

$$(\varphi_\alpha \equiv \psi_\alpha) \vdash ([[\lambda \mathfrak{y}_\alpha[[Q_{\langle o\alpha\rangle\alpha}\mathfrak{y}_\alpha]\varphi_\alpha]]\varphi_\alpha] \to [[\lambda \mathfrak{y}_\alpha[[Q_{\langle o\alpha\rangle\alpha}\mathfrak{y}_\alpha]\varphi_\alpha]]\psi_\alpha]).$$

By β-contraction, this simplifies to

$$(\varphi_\alpha \equiv \psi_\alpha) \vdash ([[Q_{\langle o\alpha\rangle\alpha}\varphi_\alpha]\varphi_\alpha] \to [[Q_{\langle o\alpha\rangle\alpha}\psi_\alpha]\varphi_\alpha]).$$

Since $(\varphi_\alpha \equiv \psi_\alpha) \vdash [[Q_{\langle o\alpha\rangle\alpha}\varphi_\alpha]\varphi_\alpha]$ by 18, we get

$$(\varphi_\alpha \equiv \psi_\alpha) \vdash (\psi_\alpha \equiv \varphi_\alpha)$$

by Modus ponens. Hence

$$\vdash ((\varphi_\alpha \equiv \psi_\alpha) \to (\psi_\alpha \equiv \varphi_\alpha))$$

by the Deduction Theorem.

20. By 17 and Modus ponens

$$\{(\varphi_\alpha \equiv \psi_\alpha), (\psi_\alpha \equiv \vartheta_\alpha)\} \vdash ([[Q_{\langle o\alpha\rangle\alpha}\varphi_\alpha]\psi_\alpha] \to [[Q_{\langle o\alpha\rangle\alpha}\varphi_\alpha]\vartheta_\alpha]).$$

Since $\{(\varphi_\alpha \equiv \psi_\alpha), (\psi_\alpha \equiv \vartheta_\alpha)\} \vdash (\varphi_\alpha \equiv \psi_\alpha)$,

$$\{(\varphi_\alpha \equiv \psi_\alpha), (\psi_\alpha \equiv \vartheta_\alpha)\} \vdash (\varphi_\alpha \equiv \vartheta_\alpha)$$

by Modus ponens. Thus

$$\vdash ((\varphi_\alpha \equiv \psi_\alpha) \to ((\psi_\alpha \equiv \vartheta_\alpha) \to (\varphi_\alpha \equiv \vartheta_\alpha)))$$

by the Deduction Theorem.

21. By 17 and Modus ponens

$$\{\varphi_o, (\varphi_o \equiv \psi_o)\} \vdash ([[\lambda p_o p_o]\varphi_o] \to [[\lambda p_o p_o]\psi_o]),$$

whence $\{\varphi_o, (\varphi_o \equiv \psi_o)\} \vdash (\varphi_o \to \psi_o)$ by β-contraction. Since $\{\varphi_o, (\varphi_o \equiv \psi_o)\} \vdash \varphi_o$ we get

$$\{\varphi_o, (\varphi_o \equiv \psi_o)\} \vdash \psi_o$$

by Modus ponens. Hence

$$\vdash (\varphi_o \to ((\varphi_o \equiv \psi_o) \to \psi_o))$$

by the Deduction Theorem.

22. Similarly to above we have that

$$\{(\neg\varphi_o), (\varphi_o \equiv \psi_o)\} \vdash ([[\lambda p_o(\neg p_o)]\varphi_o] \to [[\lambda p_o(\neg p_o)]\psi_o])$$

whence

$$\{(\neg\varphi_o), (\varphi_o \equiv \psi_o)\} \vdash ((\neg\varphi_o) \to (\neg\psi_o)),$$
$$\{(\neg\varphi_o), (\varphi_o \equiv \psi_o)\} \vdash (\neg\psi_o)$$

and

$$\vdash ((\neg\varphi_o) \to ((\varphi_o \equiv \psi_o) \to (\neg\psi_o)))$$

by β-contraction, Modus ponens and the Deduction Theorem, respectively.

23. Since $\{\varphi_o, \psi_o\} \vdash \varphi_o$ and $\{\varphi_o, \psi_o\} \vdash \psi_o$,

$$\{\varphi_o, \psi_o\} \vdash (\psi_o \to \varphi_o)$$

and

$$\{\varphi_o, \psi_o\} \vdash (\varphi_o \to \psi_o)$$

by the Deduction Theorem. By 6 and Modus ponens

$$\{\varphi_o, \psi_o\} \vdash (\varphi_o \equiv \psi_o).$$

Therefore

$$\vdash (\varphi_o \to (\psi_o \to (\varphi_o \equiv \psi_o)))$$

by the Deduction Theorem.

24. By axiom 2 and Theorem 3.8.2 of [7], $\{(\neg\varphi_o), (\neg\psi_o)\} \vdash ((\neg\varphi_o) \to ((\neg\varphi_o) \vee \psi_o))$ and $\{(\neg\varphi_o), (\neg\psi_o)\} \vdash ((\neg\psi_o) \to ((\neg\psi_o) \vee \varphi_o))$, whence

$$\{(\neg\varphi_o), (\neg\psi_o)\} \vdash ((\neg\varphi_o) \vee \psi_o)$$
$$\{(\neg\varphi_o), (\neg\psi_o)\} \vdash ((\neg\psi_o) \vee \varphi_o)$$

by Modus ponens. Like above, 6 and Modus ponens give

$$\{(\neg\varphi_o), (\neg\psi_o)\} \vdash (\varphi_o \equiv \psi_o).$$

Thus

$$\vdash ((\neg\varphi_o) \to ((\neg\psi_o) \to (\varphi_o \equiv \psi_o)))$$

by the Deduction Theorem.

25. Let $\mathbf{f}_{\alpha\beta}, \mathbf{y}_\beta \in \mathcal{X}$ be without occurrences in $\varphi_{\alpha\beta}$, $\psi_{\alpha\beta}, \vartheta_\beta$, τ_β, $Q_{\langle o\langle\alpha\beta\rangle\rangle\langle\alpha\beta\rangle}$, $Q_{\langle o\alpha\rangle\alpha}$ and $Q_{\langle o\beta\rangle\beta}$ (they are also distinct, since their types differ) and $\Gamma = \{(\varphi_{\alpha\beta} \equiv \psi_{\alpha\beta}), (\vartheta_\beta \equiv \tau_\beta)\}$. By 17 and Modus ponens we get

$$\Gamma \vdash ([[\lambda \mathbf{y}_\beta[[Q_{\langle o\alpha\rangle\alpha}[\varphi_{\alpha\beta}\vartheta_\beta]][\varphi_{\alpha\beta}\mathbf{y}_\beta]]]\vartheta_\beta] \to [[\lambda \mathbf{y}_\beta[[Q_{\langle o\alpha\rangle\alpha}[\varphi_{\alpha\beta}\vartheta_\beta]][\varphi_{\alpha\beta}\mathbf{y}_\beta]]]\tau_\beta]),$$
$$\Gamma \vdash ([[\lambda \mathbf{f}_{\alpha\beta}[[Q_{\langle o\alpha\rangle\alpha}[\varphi_{\alpha\beta}\vartheta_\beta]][\mathbf{f}_{\alpha\beta}\tau_\beta]]]\varphi_{\alpha\beta}] \to [[\lambda \mathbf{f}_{\alpha\beta}[[Q_{\langle o\alpha\rangle\alpha}[\varphi_{\alpha\beta}\vartheta_\beta]][\mathbf{f}_{\alpha\beta}\tau_\beta]]]\psi_{\alpha\beta}]).$$

By β-contraction this simplifies to

$$\Gamma \vdash ([[Q_{\langle o\alpha\rangle\alpha}[\varphi_{\alpha\beta}\vartheta_\beta]][\varphi_{\alpha\beta}\vartheta_\beta]] \to [[Q_{\langle o\alpha\rangle\alpha}[\varphi_{\alpha\beta}\vartheta_\beta]][\varphi_{\alpha\beta}\tau_\beta]]),$$
$$\Gamma \vdash ([[Q_{\langle o\alpha\rangle\alpha}[\varphi_{\alpha\beta}\vartheta_\beta]][\varphi_{\alpha\beta}\tau_\beta]] \to [[Q_{\langle o\alpha\rangle\alpha}[\varphi_{\alpha\beta}\vartheta_\beta]][\psi_{\alpha\beta}\tau_\beta]]),$$

that is

$$\Gamma \vdash ((([\varphi_{\alpha\beta}\vartheta_\beta] \equiv [\varphi_{\alpha\beta}\vartheta_\beta]) \to ([\varphi_{\alpha\beta}\vartheta_\beta] \equiv [\varphi_{\alpha\beta}\tau_\beta])),$$
$$\Gamma \vdash ((([\varphi_{\alpha\beta}\vartheta_\beta] \equiv [\varphi_{\alpha\beta}\tau_\beta]) \to ([\varphi_{\alpha\beta}\vartheta_\beta] \equiv [\psi_{\alpha\beta}\tau_\beta])).$$

By 18 and Modus ponens, we get

$$\Gamma \vdash ([\varphi_{\alpha\beta}\vartheta_\beta] \equiv [\varphi_{\alpha\beta}\tau_\beta]),$$

whence

$$\Gamma \vdash ([\varphi_{\alpha\beta}\vartheta_\beta] \equiv [\psi_{\alpha\beta}\tau_\beta])$$

by Modus ponens again. Thus

$$\vdash ((\varphi_{\alpha\beta} \equiv \psi_{\alpha\beta}) \to ((\vartheta_\beta \equiv \tau_\beta) \to ([\varphi_{\alpha\beta}\vartheta_\beta] \equiv [\psi_{\alpha\beta}\tau_\beta])))$$

by the Deduction Theorem.

26. Let $\mathbf{y}_\beta, \mathbf{z}_\beta \in \mathcal{X} \setminus \{x_\beta, \mathbf{x}_\beta\}$ be distinct and without occurrences in $\varphi_{\alpha\beta}$ and $Q_{\langle o\alpha\rangle\alpha}$. Denote by $\eta_{\alpha\beta} = \mathrm{SB}(\mathbf{x}_\beta)(\mathbf{y}_\beta)(\varphi_{\alpha\beta})$. Since $\vdash ([\varphi_{\alpha\beta}\mathbf{z}_\beta] \equiv [\varphi_{\alpha\beta}\mathbf{z}_\beta])$ by 18, we get

$$\vdash ([\eta_{\alpha\beta}\mathbf{z}_\beta] \equiv [\varphi_{\alpha\beta}\mathbf{z}_\beta])$$

by Lemma 4.2. Now $\mathbf{x}_\beta$ does not occur in $\eta_{\alpha\beta}$, whence $\mathrm{S}(\mathbf{x}_\beta)(\mathbf{z}_\beta)(\eta_{\alpha\beta}) = \eta_{\alpha\beta}$. Thus β-expansion gives

$$\vdash ([[\lambda\mathbf{x}_\beta[\eta_{\alpha\beta}\mathbf{x}_\beta]]\mathbf{z}_\beta] \equiv [\varphi_{\alpha\beta}\mathbf{z}_\beta]).$$

By Lemma 4.2 again

$$\vdash ([[\lambda\mathbf{x}_\beta[\varphi_{\alpha\beta}\mathbf{x}_\beta]]\mathbf{z}_\beta] \equiv [\varphi_{\alpha\beta}\mathbf{z}_\beta]),$$

whence

$$\vdash (\forall\mathbf{z}_\beta([[\lambda\mathbf{x}_\beta[\varphi_{\alpha\beta}\mathbf{x}_\beta]]\mathbf{z}_\beta] \equiv [\varphi_{\alpha\beta}\mathbf{z}_\beta])) \tag{9}$$

by Theorem 3.8.3 of [7]. By axiom 10 and α-conversion

$$\vdash ((\forall\mathbf{z}_\beta([f_{\alpha\beta}\mathbf{z}_\beta] \equiv [g_{\alpha\beta}\mathbf{z}_\beta])) \to (f_{\alpha\beta} \equiv g_{\alpha\beta})),$$

whereby

$$\vdash ((\forall\mathbf{z}_\beta([[\lambda\mathbf{x}_\beta[\varphi_{\alpha\beta}\mathbf{x}_\beta]]\mathbf{z}_\beta] \equiv [\varphi_{\alpha\beta}\mathbf{z}_\beta])) \to ([\lambda\mathbf{x}_\beta[\varphi_{\alpha\beta}\mathbf{x}_\beta]] \equiv \varphi_{\alpha\beta}))$$

by Theorem 3.8.2 of [7]. This yields

$$\vdash ([\lambda\mathbf{x}_\beta[\varphi_{\alpha\beta}\mathbf{x}_\beta]] \equiv \varphi_{\alpha\beta})$$

together with (9) and Modus ponens.

27. Let $\mathbf{y}_\alpha, \mathbf{z}_\alpha \in \mathcal{X} \setminus \{x_\alpha, \mathbf{x}_\alpha\}$ be distinct and without occurrences in $\varphi_{o\alpha}$. Furthermore let $\eta_{o\alpha} = \mathrm{SB}(\mathbf{x}_\alpha)(\mathbf{z}_\alpha)(\varphi_{o\alpha})$. By axiom 11, $\vdash ([f_{o\alpha}x_\alpha] \to [f_{o\alpha}[\iota_{\alpha\langle o\alpha\rangle}f_{o\alpha}]])$. By Theorem 3.8.2 of [7] this implies

$$\vdash ([[\lambda\mathbf{y}_\alpha[\eta_{o\alpha}\mathbf{y}_\alpha]]\mathbf{y}_\alpha] \to [[\lambda\mathbf{y}_\alpha[\eta_{o\alpha}\mathbf{y}_\alpha]][\iota_{\alpha\langle o\alpha\rangle}[\lambda\mathbf{y}_\alpha[\eta_{o\alpha}\mathbf{y}_\alpha]]]]),$$

whence

$$\vdash ([\eta_{o\alpha}\mathbf{y}_\alpha] \to [\eta_{o\alpha}[\iota_{\alpha\langle o\alpha\rangle}[\lambda\mathbf{y}_\alpha[\eta_{o\alpha}\mathbf{y}_\alpha]]]])$$

by iterated β-contraction. Differently put

$$\vdash ((\neg[\eta_{o\alpha}\mathbf{y}_\alpha]) \vee [\eta_{o\alpha}(\imath\mathbf{y}_\alpha[\eta_{o\alpha}\mathbf{y}_\alpha])]).$$

By axiom 3 and Theorem 3.8.2 of [7]

$$\vdash (((\neg[\eta_{o\alpha}\mathbf{y}_\alpha]) \vee [\eta_{o\alpha}(\imath\mathbf{y}_\alpha[\eta_{o\alpha}\mathbf{y}_\alpha])]) \to ([\eta_{o\alpha}(\imath\mathbf{y}_\alpha[\eta_{o\alpha}\mathbf{y}_\alpha])] \vee (\neg[\eta_{o\alpha}\mathbf{y}_\alpha]))),$$

whence

$$\vdash ([\eta_{o\alpha}(\imath\mathbf{y}_\alpha[\eta_{o\alpha}\mathbf{y}_\alpha])] \vee (\neg[\eta_{o\alpha}\mathbf{y}_\alpha]))$$

by Modus ponens. By β-expansion

$$\vdash ([\eta_{o\alpha}(\imath\mathbf{y}_\alpha[\eta_{o\alpha}\mathbf{y}_\alpha])] \vee [[\lambda\mathbf{y}_\alpha(\neg[\eta_{o\alpha}\mathbf{y}_\alpha])]\mathbf{y}_\alpha]).$$

Thus

$$\vdash (\forall\mathbf{y}_\alpha([\eta_{o\alpha}(\imath\mathbf{y}_\alpha[\eta_{o\alpha}\mathbf{y}_\alpha])] \vee [[\lambda\mathbf{y}_\alpha(\neg[\eta_{o\alpha}\mathbf{y}_\alpha])]\mathbf{y}_\alpha])) \tag{10}$$

by Theorem 3.8.3 of [7]. By axiom 6, $\vdash ((\forall x_\alpha(p_o \vee [f_{o\alpha}x_\alpha])) \to (p_o \vee [\Pi_{o\langle o\alpha\rangle}f_{o\alpha}]))$, so that

$$\vdash ((\forall\mathbf{y}_\alpha(p_o \vee [f_{o\alpha}\mathbf{y}_\alpha])) \to (p_o \vee [\Pi_{o\langle o\alpha\rangle}f_{o\alpha}]))$$

by Theorem 3.8.1 of [7]. By part 2 of the same theorem (applied twice)

$$\begin{aligned}\vdash((\forall\mathbf{y}_\alpha([\eta_{o\alpha}(\imath\mathbf{y}_\alpha[\eta_{o\alpha}\mathbf{y}_\alpha])] \vee [[\lambda\mathbf{y}_\alpha(\neg[\eta_{o\alpha}\mathbf{y}_\alpha])]\mathbf{y}_\alpha])) \to \\ ([\eta_{o\alpha}(\imath\mathbf{y}_\alpha[\eta_{o\alpha}\mathbf{y}_\alpha])] \vee [\Pi_{o\langle o\alpha\rangle}[\lambda\mathbf{y}_\alpha(\neg[\eta_{o\alpha}\mathbf{y}_\alpha])]]))\end{aligned}$$

since η is closed. Hence, with (10),

$$\vdash ([\eta_{o\alpha}(\imath\mathbf{y}_\alpha[\eta_{o\alpha}\mathbf{y}_\alpha])] \vee [\Pi_{o\langle o\alpha\rangle}[\lambda\mathbf{y}_\alpha(\neg[\eta_{o\alpha}\mathbf{y}_\alpha])]])$$

by Modus ponens, that is

$$\vdash ([\eta_{o\alpha}(\imath\mathbf{y}_\alpha[\eta_{o\alpha}\mathbf{y}_\alpha])] \vee (\forall\mathbf{y}_\alpha(\neg[\eta_{o\alpha}\mathbf{y}_\alpha]))). \tag{11}$$

Lemma 3.6.3 and Theorem 3.8.2 of [7] gives

$$\vdash ([\eta_{o\alpha}(\imath \mathbf{y}_\alpha[\eta_{o\alpha}\mathbf{y}_\alpha])] \to ((\neg(\neg(\forall \mathbf{y}_\alpha(\neg[\eta_{o\alpha}\mathbf{y}_\alpha])))) \vee [\eta_{o\alpha}(\imath \mathbf{y}_\alpha[\eta_{o\alpha}\mathbf{y}_\alpha])])) \tag{12}$$

like before. By 7 we get

$$\vdash ((\forall \mathbf{y}_\alpha(\neg[\eta_{o\alpha}\mathbf{y}_\alpha])) \to (\neg(\neg(\forall \mathbf{y}_\alpha(\neg[\eta_{o\alpha}\mathbf{y}_\alpha]))))),$$

and since

$$\vdash ((\neg(\neg(\forall \mathbf{y}_\alpha(\neg[\eta_{o\alpha}\mathbf{y}_\alpha])))) \to ((\neg(\neg(\forall \mathbf{y}_\alpha(\neg[\eta_{o\alpha}\mathbf{y}_\alpha])))) \vee [\eta_{o\alpha}(\imath \mathbf{y}_\alpha[\eta_{o\alpha}\mathbf{y}_\alpha])]))$$

by axiom 2 and Theorem 3.8.2 of [7],

$$\vdash ((\forall \mathbf{y}_\alpha(\neg[\eta_{o\alpha}\mathbf{y}_\alpha])) \to ((\neg(\neg(\forall \mathbf{y}_\alpha(\neg[\eta_{o\alpha}\mathbf{y}_\alpha])))) \vee [\eta_{o\alpha}(\imath \mathbf{y}_\alpha[\eta_{o\alpha}\mathbf{y}_\alpha])])) \tag{13}$$

by Lemma 3.10.2 of [7]. Part 3 of the same lemma together with (12) and (13) now guarantees

$$\begin{aligned}\vdash&((([\eta_{o\alpha}(\imath \mathbf{y}_\alpha[\eta_{o\alpha}\mathbf{y}_\alpha])] \vee (\forall \mathbf{y}_\alpha(\neg[\eta_{o\alpha}\mathbf{y}_\alpha]))) \to \\ &\quad ((\neg(\neg(\forall \mathbf{y}_\alpha(\neg[\eta_{o\alpha}\mathbf{y}_\alpha])))) \vee [\eta_{o\alpha}(\imath \mathbf{y}_\alpha[\eta_{o\alpha}\mathbf{y}_\alpha])])).\end{aligned}$$

Thus, by (11) and Modus ponens,

$$\vdash ((\neg(\neg(\forall \mathbf{y}_\alpha(\neg[\eta_{o\alpha}\mathbf{y}_\alpha])))) \vee [\eta_{o\alpha}(\imath \mathbf{y}_\alpha[\eta_{o\alpha}\mathbf{y}_\alpha])]),$$

that is

$$\vdash ((\exists \mathbf{y}_\alpha[\eta_{o\alpha}\mathbf{y}_\alpha]) \to [\eta_{o\alpha}(\imath \mathbf{y}_\alpha[\eta_{o\alpha}\mathbf{y}_\alpha])]).$$

By Theorem 3.8.1 of [7]

$$\vdash ((\exists \mathbf{x}_\alpha[\eta_{o\alpha}\mathbf{x}_\alpha]) \to [\eta_{o\alpha}(\imath \mathbf{x}_\alpha[\eta_{o\alpha}\mathbf{x}_\alpha])]),$$

whereby

$$\vdash ((\exists \mathbf{x}_\alpha[\varphi_{o\alpha}\mathbf{x}_\alpha]) \to [\varphi_{o\alpha}(\imath \mathbf{x}_\alpha[\varphi_{o\alpha}\mathbf{x}_\alpha])])$$

by Lemma 4.2 (thrice).

28. Let $\mathbf{y}_\alpha \in \mathcal{X} \setminus \{\mathbf{x}_\alpha\}$ be without occurrences in $\varphi_{o\alpha}$, $\Gamma = \{[\varphi_{o\alpha}(\imath \mathbf{x}_\alpha(\neg[\varphi_{o\alpha}\mathbf{x}_\alpha]))]$, $(\exists \mathbf{x}_\alpha[[\lambda \mathbf{x}_\alpha(\neg[\eta_{o\alpha}\mathbf{x}_\alpha])]\mathbf{x}_\alpha])\}$ and $\eta_{o\alpha} = \mathrm{SB}(\mathbf{x}_\alpha)(\mathbf{y}_\alpha)(\varphi_{o\alpha})$. By 27 and Modus ponens

$$\Gamma \vdash [[\lambda \mathbf{x}_\alpha(\neg[\eta_{o\alpha}\mathbf{x}_\alpha])](\imath \mathbf{x}_\alpha[[\lambda \mathbf{x}_\alpha(\neg[\eta_{o\alpha}\mathbf{x}_\alpha])]\mathbf{x}_\alpha])],$$

whence

$$\Gamma \vdash [[\lambda \mathbf{x}_\alpha(\neg[\eta_{o\alpha}\mathbf{x}_\alpha])](\imath \mathbf{x}_\alpha(\neg[\eta_{o\alpha}\mathbf{x}_\alpha]))]$$

and

$$\Gamma \vdash (\neg[\eta_{o\alpha}(\imath\mathbf{x}_\alpha(\neg[\eta_{o\alpha}\mathbf{x}_\alpha]))])$$

by β-contraction. By two applications of Lemma 4.2 we see that

$$\Gamma \vdash (\neg[\varphi_{o\alpha}(\imath\mathbf{x}_\alpha(\neg[\varphi_{o\alpha}\mathbf{x}_\alpha]))]).$$

Furthermore,

$$\Gamma \vdash ((\neg[\varphi_{o\alpha}(\imath\mathbf{x}_\alpha(\neg[\varphi_{o\alpha}\mathbf{x}_\alpha]))]) \to ([\varphi_{o\alpha}(\imath\mathbf{x}_\alpha(\neg[\varphi_{o\alpha}\mathbf{x}_\alpha]))] \to \bot))$$

by 3, whence

$$\Gamma \vdash \bot$$

by Modus ponens. Thus

$$\Gamma \vdash (\forall\mathbf{x}_\alpha(\neg(\neg[\eta_{o\alpha}\mathbf{x}_\alpha])))$$

by 2 and Modus ponens, whereby

$$[\varphi_{o\alpha}(\imath\mathbf{x}_\alpha(\neg[\varphi_{o\alpha}\mathbf{x}_\alpha]))] \vdash ((\exists\mathbf{x}_\alpha[[\lambda\mathbf{x}_\alpha(\neg[\eta_{o\alpha}\mathbf{x}_\alpha])]\mathbf{x}_\alpha]) \to (\forall\mathbf{x}_\alpha(\neg(\neg[\eta_{o\alpha}\mathbf{x}_\alpha]))))$$

by the Deduction Theorem.
Since $(\exists\mathbf{x}_\alpha[[\lambda\mathbf{x}_\alpha(\neg[\eta_{o\alpha}\mathbf{x}_\alpha])]\mathbf{x}_\alpha]) = (\neg(\forall\mathbf{x}_\alpha(\neg[[\lambda\mathbf{x}_\alpha(\neg[\eta_{o\alpha}\mathbf{x}_\alpha])]\mathbf{x}_\alpha])))$ we get that

$$[\varphi_{o\alpha}(\imath\mathbf{x}_\alpha(\neg[\varphi_{o\alpha}\mathbf{x}_\alpha]))] \vdash ((\neg(\forall\mathbf{x}_\alpha(\neg(\neg[\eta_{o\alpha}\mathbf{x}_\alpha])))) \to (\forall\mathbf{x}_\alpha(\neg(\neg[\eta_{o\alpha}\mathbf{x}_\alpha]))))$$

by β-contraction. By 10 and Modus ponens

$$[\varphi_{o\alpha}(\imath\mathbf{x}_\alpha(\neg[\varphi_{o\alpha}\mathbf{x}_\alpha]))] \vdash (\forall\mathbf{x}_\alpha(\neg(\neg[\eta_{o\alpha}\mathbf{x}_\alpha]))),$$

and thus

$$[\varphi_{o\alpha}(\imath\mathbf{x}_\alpha(\neg[\varphi_{o\alpha}\mathbf{x}_\alpha]))] \vdash (\neg(\neg[\eta_{o\alpha}\mathbf{x}_\alpha]))$$

by 15 and Modus ponens. By 8 and Modus ponens

$$[\varphi_{o\alpha}(\imath\mathbf{x}_\alpha(\neg[\varphi_{o\alpha}\mathbf{x}_\alpha]))] \vdash [\eta_{o\alpha}\mathbf{x}_\alpha],$$

and since $\mathbf{x}_\alpha$ is not free in $\eta_{o\alpha}$

$$[\varphi_{o\alpha}(\imath\mathbf{x}_\alpha(\neg[\varphi_{o\alpha}\mathbf{x}_\alpha]))] \vdash [\Pi_{o\langle o\alpha\rangle}\eta_{o\alpha}]$$

by Generalisation. Now Lemma 4.2 applies, yielding

$$[\varphi_{o\alpha}(\imath\mathbf{x}_\alpha(\neg[\varphi_{o\alpha}\mathbf{x}_\alpha]))] \vdash [\Pi_{o\langle o\alpha\rangle}\varphi_{o\alpha}],$$

whence

$$\vdash ([\varphi_{o\alpha}(\imath\mathbf{x}_\alpha(\neg[\varphi_{o\alpha}\mathbf{x}_\alpha]))] \to [\Pi_{o\langle o\alpha\rangle}\varphi_{o\alpha}])$$

by the Deduction Theorem.

29. Let $\mathbf{y}_\beta, \mathbf{z}_\beta \in \mathcal{X}\backslash\{x_\beta, \mathbf{x}_\beta\}$ be distinct, without occurrences in $\varphi_{\alpha\beta}$, $\psi_{\alpha\beta}$, $Q_{\langle o\langle\alpha\beta\rangle\rangle\langle\alpha\beta\rangle}$, $Q_{\langle o\alpha\rangle\alpha}$ and $Q_{\langle o\beta\rangle\beta}$. Define $\eta_{\alpha\beta} = \mathrm{SB}(\mathbf{x}_\beta)(\mathbf{z}_\beta)(\varphi_{\alpha\beta})$ and $\upsilon_{\alpha\beta} = \mathrm{SB}(\mathbf{x}_\beta)(\mathbf{z}_\beta)(\psi_{\alpha\beta})$. By above

$$\vdash([[\lambda\mathbf{y}_\beta([\eta_{\alpha\beta}\mathbf{y}_\beta] \equiv [\upsilon_{\alpha\beta}\mathbf{y}_\beta])](\imath\mathbf{y}_\beta(\neg[[\lambda\mathbf{y}_\beta([\eta_{\alpha\beta}\mathbf{y}_\beta] \equiv [\upsilon_{\alpha\beta}\mathbf{y}_\beta])]\mathbf{y}_\beta]))] \rightarrow [\Pi_{o\langle o\beta\rangle}[\lambda\mathbf{y}_\beta([\eta_{\alpha\beta}\mathbf{y}_\beta] \equiv [\upsilon_{\alpha\beta}\mathbf{y}_\beta])]]]),$$

whence

$$\vdash([[\lambda\mathbf{y}_\beta([\eta_{\alpha\beta}\mathbf{y}_\beta] \equiv [\upsilon_{\alpha\beta}\mathbf{y}_\beta])](\imath\mathbf{y}_\beta(\neg([\eta_{\alpha\beta}\mathbf{y}_\beta] \equiv [\upsilon_{\alpha\beta}\mathbf{y}_\beta])))] \rightarrow [\Pi_{o\langle o\beta\rangle}[\lambda\mathbf{y}_\beta([\eta_{\alpha\beta}\mathbf{y}_\beta] \equiv [\upsilon_{\alpha\beta}\mathbf{y}_\beta])]]])$$

by β-contraction. Another application of β-contraction yields

$$\vdash(([\eta_{\alpha\beta}(\imath\mathbf{y}_\beta(\neg([\eta_{\alpha\beta}\mathbf{y}_\beta] \equiv [\upsilon_{\alpha\beta}\mathbf{y}_\beta])))] \equiv [\upsilon_{\alpha\beta}(\imath\mathbf{y}_\beta(\neg([\eta_{\alpha\beta}\mathbf{y}_\beta] \equiv [\upsilon_{\alpha\beta}\mathbf{y}_\beta])))]) \rightarrow [\Pi_{o\langle o\beta\rangle}[\lambda\mathbf{y}_\beta([\eta_{\alpha\beta}\mathbf{y}_\beta] \equiv [\upsilon_{\alpha\beta}\mathbf{y}_\beta])]]]),$$

that is

$$\vdash(([\eta_{\alpha\beta}(\imath\mathbf{y}_\beta([\eta_{\alpha\beta}\mathbf{y}_\beta] \not\equiv [\upsilon_{\alpha\beta}\mathbf{y}_\beta]))] \equiv [\upsilon_{\alpha\beta}(\imath\mathbf{y}_\beta([\eta_{\alpha\beta}\mathbf{y}_\beta] \not\equiv [\upsilon_{\alpha\beta}\mathbf{y}_\beta]))]) \rightarrow (\forall\mathbf{y}_\beta([\eta_{\alpha\beta}\mathbf{y}_\beta] \equiv [\upsilon_{\alpha\beta}\mathbf{y}_\beta]))).$$

By α-conversion

$$\vdash(([\eta_{\alpha\beta}(\imath\mathbf{x}_\beta([\eta_{\alpha\beta}\mathbf{x}_\beta] \not\equiv [\upsilon_{\alpha\beta}\mathbf{x}_\beta]))] \equiv [\upsilon_{\alpha\beta}(\imath\mathbf{x}_\beta([\eta_{\alpha\beta}\mathbf{x}_\beta] \not\equiv [\upsilon_{\alpha\beta}\mathbf{x}_\beta]))]) \rightarrow (\forall\mathbf{y}_\beta([\eta_{\alpha\beta}\mathbf{y}_\beta] \equiv [\upsilon_{\alpha\beta}\mathbf{y}_\beta]))).$$

By eight applications of Lemma 4.2,

$$\vdash(([\varphi_{\alpha\beta}(\imath\mathbf{x}_\beta([\varphi_{\alpha\beta}\mathbf{x}_\beta] \not\equiv [\psi_{\alpha\beta}\mathbf{x}_\beta]))] \equiv [\psi_{\alpha\beta}(\imath\mathbf{x}_\beta([\varphi_{\alpha\beta}\mathbf{x}_\beta] \not\equiv [\psi_{\alpha\beta}\mathbf{x}_\beta]))]) \rightarrow (\forall\mathbf{y}_\beta([\varphi_{\alpha\beta}\mathbf{y}_\beta] \equiv [\psi_{\alpha\beta}\mathbf{y}_\beta]))),$$

and since

$$\vdash ((\forall\mathbf{y}_\beta([\varphi_{\alpha\beta}\mathbf{y}_\beta] \equiv [\psi_{\alpha\beta}\mathbf{y}_\beta])) \rightarrow (\varphi_{\alpha\beta} \equiv \psi_{\alpha\beta}))$$

by axiom 10 and Theorem 3.8.2 and 1 of [7], we get

$$\vdash(([\varphi_{\alpha\beta}(\imath\mathbf{x}_\beta([\varphi_{\alpha\beta}\mathbf{x}_\beta] \not\equiv [\psi_{\alpha\beta}\mathbf{x}_\beta]))] \equiv [\psi_{\alpha\beta}(\imath\mathbf{x}_\beta([\varphi_{\alpha\beta}\mathbf{x}_\beta] \not\equiv [\psi_{\alpha\beta}\mathbf{x}_\beta]))]) \rightarrow (\varphi_{\alpha\beta} \equiv \psi_{\alpha\beta}))$$

by Lemma 3.10.2 of [7].

□

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