UPPSALA
UNIVERSITET

Matrix Integrals

Calculating Matrix Integrals Using Feynman Diagrams

Adam Friberg

2014-06-26

Supervisor: Maxim Zabzine
Subject Reader: Ulf Lindström
Examinator: Susanne Mirbt
Degree Project C in Physics, 15 Credits

Department of Physics and Astronomy
Division of Theoretical Physics
Bachelor Programme in Physics
Uppsala University

Abstract

In this project, we examine how integration over matrices is performed. We investigate and develop a method for calculating matrix integrals of the general form

$$\int \mathcal{D}M e^{-\operatorname{Tr}(V(M))},$$

over the set of real square matrices M.

Matrix integrals are used for calculations in several different areas of physics and mathematics; for example quantum field theory, string theory, quantum chromodynamics, and random matrix theory.

Our method consists of ways to apply perturbative Taylor expansions to the matrix integrals, reducing each term of the resulting Taylor series to a combinatorial problem using *Wick's theorem*, and representing the terms of the Wick sum graphically with the help of Feynman diagrams and fat graphs. We use the method in a few examples that aim to clearly demonstrate how to calculate the matrix integrals.

Sammanfattning

I detta projekt undersöker vi hur integration över matriser genomförs. Vi undersöker och utvecklar en metod för beräkning av matrisintegraler på den allmänna formen

$$\int \mathcal{D}M e^{-\operatorname{Tr}(V(M))},$$

över mängden av alla reell-värda kvadratiska matriser M.

Matrisintegraler används för beräkningar i ett flertal olika områden inom fysik och matematik, till exempel kvantfältteori, strängteori, kvantkromodynamik och slumpmatristeori.

Vår metod består av sätt att applicera perturbativa Taylorutvecklingar på matrisintegralerna, reducera varje term i den resulterande Taylorserien till ett kombinatoriellt problem med hjälp av *Wicks sats*, och att representera termerna i Wicksumman grafiskt med hjälp av Feynmandiagram. Vi använder metoden i några exempel som syftar till att klart demonstrera hur beräkningen av matrisintegraler går till.

Contents

Chapter 1

Introduction

1.1 Background

The calculation of matrix integrals is something that is commonly utilized in several areas of modern physics and mathematics. Notable mentions are, among others:

- Electromagnetic response and transport properties in disordered or irregular quantum systems. [6]
- Counting of maps, triangulations [6] and quadrangulations [2] in quantum field theory.
- Path integrals in quantum field theory. [6]
- Studying the spectrum of energy levels in large nuclei. [6]
- Two-dimensional quantum gravity. [6]
- Planar diagrams in quantum chromodynamics. [6]
- The Kardar–Parisi–Zhang equation. [6]
- Random growth models with random matrices. [6]
- Supersymmetric gauge theories in string theory. [6]
- Counting of maps, foldings, and knots in combinatorics. [6]

1.2 Problem Formulation

The goal of this project is to specify *how* integration over matrices A is performed, and to define and calculate different examples of matrix integrals. We will define *what it means* to integrate over matrices, and we will develop and investigate a method to *calculate* matrix integrals of various forms.

We seek to, starting from [2] and [6], independently and thoroughly develop and derive this method, and to verify that the resulting method is the same as, or least equivalent to, the methods described in [2] and [6].

We will present the results in the form of derivations, expressions and methods for the calculation of the matrix integrals, as well as some simple examples that demonstrate the use of the developed methods. A general matrix integral can be written on the form

$$\int \mathcal{D}M e^{-\operatorname{Tr}(V(M))}, \tag{1.1}$$

with the integral taken over some set of matrices M. Here, we will focus on the most basic set: the set of real n-by-n square matrices, that is

$$M \in \mathbb{R}^{n\times n}. \tag{1.2}$$

1.3 Theory

1.3.1 Diagonalization of Matrices

In one of the first few steps towards deriving a method for calculating matrix integrals, we will need to make use of a simple fact that follows from the finite-dimensional *spectral theorem*.

Proposition 1.1 (Diagonalization of real symmetric matrices).

For a real and symmetric matrix

$$A = A^T \in \mathbb{R}^{n\times n}, \tag{1.3}$$

it follows from the spectral theorem that A has the (real) eigenvalues $\lambda_i \in \mathbb{R}$ and can be diagonalized into a matrix

$$D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \tag{1.4}$$

by an orthogonal matrix O, that is,

$$OAO^T = D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \tag{1.5}$$

$$\Leftrightarrow A = O^T DO. \tag{1.6}$$

1.3.2 Wick's Theorem

We will arrive at a point where we need to make use of a theorem for relating long and somewhat chaotic calculations of many derivatives to an, in comparison, trivial combinatorial problem. This theorem is called Wick's theorem. [4, 5]

Theorem 1.2 (Wick's Theorem).

For the expectation value

$$\langle x_{i_1} x_{i_2} \cdots x_{i_m} \rangle , \tag{1.7}$$

we have

$$\langle x_{i_1} x_{i_2} \cdots x_{i_m} \rangle = \sum \Delta_{i_{p_1} i_{p_2}} \Delta_{i_{p_3} i_{p_4}} \cdots \Delta_{i_{p_{m-1}} i_{p_m}}, \tag{1.8}$$

where $\Delta_{ij} := \langle x_i x_j \rangle$ *denotes the* propagator *between the space points* x_i *and* x_j*, and the sum is taken over all pairings*

$$\left(i_{p_1}, i_{p_2}\right), \left(i_{p_3}, i_{p_4}\right), \ldots, \left(i_{p_{m-1}}, i_{p_m}\right) \tag{1.9}$$

of $i_1, i_2, \ldots, i_m$.

Chapter 2

Method

In this project, we develop a method to calculate matrix integrals. Starting from [2] and [6], we develop the method independently, and we carefully prove that all assumptions that we make are correct and can be applied in our calculations.

We verify that the resulting method yields the same results from calculations as the methods described in [2] and [6] do. Since we do not regard the contents of our reference literature as facts, but rather as guidelines for our choice of work flow, we are able to verify the validity of the methods in [2] and [6].

2.1 Outline

Our first step is to calculate the one-dimensional Gaussian integral

$$\int_{-\infty}^{+\infty} dx e^{-\frac{\alpha}{2}x^2} \tag{2.1}$$

(see section 3.1), and the multi-dimensional Gaussian integral

$$\int_{\mathbb{R}^n} d^n x e^{-\frac{1}{2}x^T A x}, \tag{2.2}$$

where x is a column vector and A is a real symmetric matrix (see section 3.2). These simple integrals will evaluate to constants, and knowledge of the constants will slightly simplify some subsequent calculations.

Next, we calculate the integral

$$\int_{-\infty}^{+\infty} dx x^m e^{-\frac{\alpha}{2}x^2}, \tag{2.3}$$

by introducing the so called *generating function*

$$Z[j] := \int_{-\infty}^{+\infty} dx e^{-\frac{\alpha}{2}x^2 + jx} \tag{2.4}$$

(see section 3.3). We relate the wanted integral to the generating function, whose value we can easily calculate. We also calculate the similar but multi-dimensional integrals

$$\int_{\mathbb{R}^n} d^n x x_i x_j e^{-\frac{1}{2}x^T A x} \tag{2.5}$$

and

$$\int_{\mathbb{R}^n} d^n x x_i x_j x_k x_l e^{-\frac{1}{2}x^T A x}, \tag{2.6}$$

and try to calculate the integral

$$\int_{\mathbb{R}^n} d^n x x_{i_1} x_{i_2} \cdots x_{i_m} e^{-\frac{1}{2}x^T A x}, \tag{2.7}$$

also by defining a generating function, which we relate to the wanted integral (see section 3.4, section 3.5, and section 3.6). Since the latter two of these three integrals leads to a significantly more complex analytical calculation than the first of these three integrals, we introduce a different and much simpler way to calculate the involved expressions, namely, Wick's theorem (see Theorem 1.2).

After applying Wick's theorem to the integrals, we end up with an elementary combinatorial problem. Even more convenient is to represent this diagrammatically, using *Feynman diagrams* [2, 4, 6]. We introduce the appropriate nomenclature and notations for the diagrams, and establish a way to assign the diagrams values (see section 3.7).

We then calculate the integral

$$\int_{\mathbb{R}^n} d^n x e^{-\frac{1}{2}x^T A x + \alpha x^j}, \tag{2.8}$$

with an additional term introduced in the exponent, by rewriting the integral using Taylor expansion, and exchanging the order of the summation and the integration (see section 3.8). The resulting expression is a sum, where each term is an integral of a form that we are already familiar with.

Our next step is to calculate the *matrix integral*

$$\int_{\mathbb{R}^{n\times n}} \mathcal{D}M e^{-\frac{1}{2}\operatorname{Tr}\left(M^2\right)}. \tag{2.9}$$

By rewriting the integral, we can make it assume a form that resembles the integrals that we have already calculated (see section 3.9).

We then calculate the matrix integral

$$\int_{\mathbb{R}^{n\times n}} \mathcal{D}M M_{ij} M_{kl} e^{-\frac{1}{2}\operatorname{Tr}\left(M^2\right)}, \tag{2.10}$$

and begin to calculate the matrix integral

$$\int_{\mathbb{R}^{n\times n}} \mathcal{D}M M_{i_1 j_1} M_{i_2 j_2} \cdots M_{i_m j_m} e^{-\frac{1}{2}\operatorname{Tr}\left(M^2\right)}. \tag{2.11}$$

Like when calculating previous integrals, we rewrite the integral on a form we are familiar with (see section 3.10 and section 3.11). We apply Wick's theorem to the latter of the two integrals to reduce the otherwise complicated calculation to a more convenient combinatorial problem.

Just like in the case with integrals over vectors, we can also here represent the resulting combinatorial sum graphically using Feynman diagrams [2, 6]. We introduce the appropriate nomenclature and the notations necessary to extend the Feynman diagrams to usage with matrix elements (see section 3.12).

As the last step, we apply all of the tools that we have presented to calculate the integral

$$\int_{\mathbb{R}^{n\times n}} \mathcal{D}M e^{-\frac{1}{2}\operatorname{Tr}\left(M^2\right)+g\operatorname{Tr}\left(M^3\right)} \tag{2.12}$$

(see section 3.13).

Chapter 3

Results

3.1 One-dimensional Gaussian Integral

We begin by calculating the value of the one-dimensional Gaussian integral,

$$Z = \int_{-\infty}^{+\infty} dx e^{-\frac{\alpha}{2}x^2} \tag{3.1}$$

$$\Rightarrow Z^2 = \left(\int_{-\infty}^{+\infty} dx e^{-\frac{\alpha}{2}x^2}\right)\left(\int_{-\infty}^{+\infty} dy e^{-\frac{\alpha}{2}y^2}\right) \tag{3.2}$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx dy e^{-\frac{\alpha}{2}\left(x^2+y^2\right)}. \tag{3.3}$$

We substitute the variables

$$x =: r\cos(\theta), \tag{3.4}$$

$$y =: r\sin(\theta) \tag{3.5}$$

$$\Rightarrow Z^2 = \int_0^{+\infty} r dr \int_0^{2\pi} d\theta e^{-\frac{\alpha}{2}ar^2} \tag{3.6}$$

$$= 2\pi \int_0^{+\infty} dr r e^{-\frac{\alpha}{2}r^2}. \tag{3.7}$$

We perform another substitution,

$$z := r^2 \tag{3.8}$$

$$\Rightarrow Z^2 = \pi \int_0^{+\infty} dz e^{-\frac{\alpha}{2}z} \tag{3.9}$$

$$= \pi\left[-\frac{2e^{-\frac{\alpha}{2}z}}{\alpha}\right]_0^{+\infty} \tag{3.10}$$

$$= \frac{2\pi}{\alpha} \tag{3.11}$$

$$\Rightarrow Z^2 = \frac{2\pi}{\alpha} \tag{3.12}$$

$$\Rightarrow Z = \sqrt{\frac{2\pi}{\alpha}}. \tag{3.13}$$

The value of this integral is a constant, and knowledge of this constant will allow us to simplify the subsequent calculations.

3.2 Multi-dimensional Gaussian Integral

Next, we calculate the value of the multi-dimensional Gaussian integral, that has the form

$$Z = \int_{\mathbb{R}^n} d^n x e^{-\frac{1}{2}x^T A x}, \tag{3.14}$$

where x is the column vector

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \tag{3.15}$$

and A is a real symmetric matrix, that is

$$A = A^T \in \mathbb{R}^{n \times n} \tag{3.16}$$

$$\Leftrightarrow A_{ij} = A_{ji}. \tag{3.17}$$

Since the matrix A is real and symmetric, we can make use of a result that follows from the spectral theorem, see Proposition 1.1. The spectral theorem implies that A has the eigenvalues $\lambda_i \in \mathbb{R}$ and can be diagonalized into a matrix $D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ by an orthogonal matrix O (see Proposition 1.1), that is

$$D = OAO^T = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \tag{3.18}$$

$$\Leftrightarrow A = O^T D O \tag{3.19}$$

$$\Rightarrow Z = \int_{\mathbb{R}^n} d^n x e^{-\frac{1}{2}x^T A x} \tag{3.20}$$

$$= \int_{\mathbb{R}^n} d^n x e^{-\frac{1}{2}x^T O^T D O x} \tag{3.21}$$

$$= \int_{\mathbb{R}^n} d^n x e^{-\frac{1}{2}(Ox)^T D (Ox)}. \tag{3.22}$$

We perform the variable substitution

$$y := Ox \tag{3.23}$$

$$\Rightarrow d^n y = \det\left(\frac{dy}{dx}\right) d^n x \tag{3.24}$$

$$\Rightarrow d^n x = \frac{1}{\det(O)} d^n y \tag{3.25}$$

$$\Rightarrow Z = \int_{\mathbb{R}^n} d^n y \frac{1}{\det(O)} e^{-\frac{1}{2}y^T D y} \tag{3.26}$$

$$= \int_{\mathbb{R}^n} d^n y \frac{1}{\det(O)} e^{-\frac{1}{2}\sum_{i=1}^{n} \lambda_i y_i^2} \tag{3.27}$$

$$= \frac{1}{\det(O)} \prod_{i=1}^{n} \int_{-\infty}^{+\infty} dy_i e^{-\frac{1}{2}\lambda_i y_i^2}. \tag{3.28}$$

Using Equation 3.13, we get

$$Z = \frac{1}{\det(O)} \prod_{i=1}^{n} \sqrt{\frac{2\pi}{\lambda_i}} \tag{3.29}$$

$$= \frac{1}{\det(O)} \frac{(2\pi)^{n/2}}{\sqrt{\det(D)}} \tag{3.30}$$

$$= \frac{(2\pi)^{n/2}}{\sqrt{\det(O^T D O)}} \tag{3.31}$$

$$= \frac{(2\pi)^{n/2}}{\sqrt{\det(A)}} \tag{3.32}$$

$$\Rightarrow Z = \frac{(2\pi)^{n/2}}{\sqrt{\det(A)}}. \tag{3.33}$$

Also here, the result is a constant that we will use in subsequent calculations.

3.3 Generating Function

We then calculate the integral

$$I = \int_{-\infty}^{+\infty} dx x^m e^{-\frac{\alpha}{2}x^2}, \tag{3.34}$$

by introducing and defining the *generating function* $Z[j]$. With the help of derivatives, we relate the wanted integral to the generating function, whose value we can easily calculate.

$$Z[j] := \int_{-\infty}^{+\infty} dx e^{-\frac{\alpha}{2}x^2 + jx} \tag{3.35}$$

$$\Rightarrow I = \left. \frac{d^m Z[j]}{dj^m} \right|_{j=0}. \tag{3.36}$$

We evaluate the generating function

$$Z[j] = \int_{-\infty}^{+\infty} dx e^{-\frac{\alpha}{2}x^2 + jx} \tag{3.37}$$

$$= \int_{-\infty}^{+\infty} dx e^{-\frac{\alpha}{2}\left(x - \frac{j}{\alpha}\right)^2 + \frac{j^2}{2\alpha}}. \tag{3.38}$$

We substitute the variable

$$y := x - \frac{j}{\alpha} \tag{3.39}$$

$$\Rightarrow Z[j] = \int_{-\infty}^{+\infty} dy e^{-\frac{\alpha}{2}y^2 + \frac{j^2}{2\alpha}} \tag{3.40}$$

$$= e^{\frac{j^2}{2\alpha}} \int_{-\infty}^{+\infty} dy e^{-\frac{\alpha}{2}y^2}. \tag{3.41}$$

Using Equation 3.13, we see

$$Z[j] = e^{\frac{j^2}{2\alpha}} \int_{-\infty}^{+\infty} dy e^{-\frac{\alpha}{2}y^2} \tag{3.42}$$

$$= \sqrt{\frac{2\pi}{\alpha}} e^{\frac{j^2}{2\alpha}} \tag{3.43}$$

$$\Rightarrow I = \left. \frac{d^m Z[j]}{dj^m} \right|_{j=0} \tag{3.44}$$

$$= \sqrt{\frac{2\pi}{\alpha}} \frac{d^m}{dj^m} \left. \left(e^{\frac{j^2}{2\alpha}} \right) \right|_{j=0} \tag{3.45}$$

$$\Rightarrow I = \sqrt{\frac{2\pi}{\alpha}} \frac{d^m}{dj^m} \left. \left(e^{\frac{j^2}{2\alpha}} \right) \right|_{j=0}. \tag{3.46}$$

3.4 Two-point Function

Similarly, we calculate the multi-dimensional integral

$$I = \int_{\mathbb{R}^n} d^n x x_i x_j e^{-\frac{1}{2}x^T A x}. \tag{3.47}$$

To do this, we define the *generating function* $Z[J]$. Again, we relate the wanted integral, with the help of partial derivatives, to the generating function, whose value we can easily calculate.

$$Z[J] := \int_{\mathbb{R}^n} d^n x e^{-\frac{1}{2}x^T A x + x^T J} \tag{3.48}$$

$$\Rightarrow I = \left.\frac{\partial^2 Z[J]}{\partial J_i \partial J_j}\right|_{J=0}, \tag{3.49}$$

where

$$J = \begin{pmatrix} J_1 \\ J_2 \\ \vdots \\ J_n \end{pmatrix}. \tag{3.50}$$

We see that

$$Z[J] = \int_{\mathbb{R}^n} d^n x e^{-\frac{1}{2}x^T A x + x^T J} \tag{3.51}$$

$$= \int_{\mathbb{R}^n} d^n x e^{-\frac{1}{2}\left(x - A^{-1}J\right)^T A \left(x - A^{-1}J\right) + \frac{1}{2}J^T A^{-1} J}. \tag{3.52}$$

We substitute the variable

$$y := x - A^{-1}J \tag{3.53}$$

$$\Rightarrow Z[J] = \int_{\mathbb{R}^n} d^n y e^{-\frac{1}{2}y^T A y + \frac{1}{2}J^T A^{-1} J} \tag{3.54}$$

$$= e^{\frac{1}{2}J^T A^{-1} J} \int_{\mathbb{R}^n} d^n y e^{-\frac{1}{2}y^T A y}. \tag{3.55}$$

Using Equation 3.33, we see that

$$Z[J] = e^{\frac{1}{2}J^T A^{-1} J} \int_{\mathbb{R}^n} d^n y e^{-\frac{1}{2}y^T A y} \tag{3.56}$$

$$= \frac{(2\pi)^{n/2}}{\sqrt{\det(A)}} e^{\frac{1}{2}J^T A^{-1} J} \tag{3.57}$$

$$\Rightarrow I = \left.\frac{\partial^2 Z[J]}{\partial J_i \partial J_j}\right|_{J=0} \tag{3.58}$$

$$= \frac{(2\pi)^{n/2}}{\sqrt{\det(A)}} \frac{\partial^2}{\partial J_i \partial J_j} \left(e^{\frac{1}{2} J^T A^{-1} J} \right) \Bigg|_{J=0} . \tag{3.59}$$

Using

$$J^T A^{-1} J = \sum_{k=1}^{n} \sum_{l=1}^{n} \left(A^{-1}\right)_{kl} J_k J_l, \tag{3.60}$$

$$Z_0 := Z[0] \tag{3.61}$$

$$= \frac{(2\pi)^{n/2}}{\sqrt{\det(A)}}, \tag{3.62}$$

we see

$$I = Z_0 \frac{\partial^2}{\partial J_i \partial J_j} \left(e^{\frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \left(A^{-1}\right)_{kl} J_k J_l} \right) \Bigg|_{J=0} \tag{3.63}$$

$$= Z_0 \frac{\partial}{\partial J_j} \left(\left(\frac{1}{2} \sum_{a=1}^{n} \left(A^{-1}\right)_{ai} J_a + \frac{1}{2} \sum_{b=1}^{n} \left(A^{-1}\right)_{ib} J_b \right) \cdot e^{\frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \left(A^{-1}\right)_{kl} J_k J_l} \right) \Bigg|_{J=0} \tag{3.64}$$

$$= Z_0 \frac{\partial}{\partial J_j} \left(\sum_{a=1}^{n} \left(A^{-1}\right)_{ia} J_a e^{\frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \left(A^{-1}\right)_{kl} J_k J_l} \right) \Bigg|_{J=0} \tag{3.65}$$

$$= Z_0 \left(\left(A^{-1}\right)_{ij} + \sum_{a=1}^{n} \left(A^{-1}\right)_{ia} J_a \left(\frac{1}{2} \sum_{b=1}^{n} \left(A^{-1}\right)_{bj} J_b + \frac{1}{2} \sum_{c=1}^{n} \left(A^{-1}\right)_{jc} J_c \right) \right) e^{\frac{1}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \left(A^{-1}\right)_{kl} J_k J_l} \Bigg|_{J=0} \tag{3.66}$$

$$= Z_0 \left(\left(A^{-1}\right)_{ij} + \sum_{a=1}^{n} \left(A^{-1}\right)_{ia} J_a \cdot \sum_{b=1}^{n} \left(A^{-1}\right)_{jb} J_b \right) e^{\frac{1}{2} J^T A^{-1} J} \Bigg|_{J=0} \tag{3.67}$$

$$= Z_0 \left(A^{-1}\right)_{ij} \tag{3.68}$$

$$\Rightarrow I = Z_0 \left(A^{-1}\right)_{ij}, \tag{3.69}$$

where $Z_0 = Z[0]$ is a constant that by itself is of low importance, but will follow us throughout this chapter.

3.5 Four-point Function

We also calculate the multi-dimensional integral

$$I = \int_{\mathbb{R}^n} d^n x x_i x_j x_k x_l e^{-\frac{1}{2}x^T A x}. \tag{3.70}$$

We define the *generating function*

$$Z[J] := \int_{\mathbb{R}^n} d^n x e^{-\frac{1}{2}x^T A x + x^T J} \tag{3.71}$$

$$\Rightarrow I = \left.\frac{\partial^4 Z[J]}{\partial J_i \partial J_j \partial J_k \partial J_l}\right|_{J=0}. \tag{3.72}$$

Using Equation 3.57 we see that

$$I = \left.\frac{\partial^4 Z[J]}{\partial J_i \partial J_j \partial J_k \partial J_l}\right|_{J=0} \tag{3.73}$$

$$= Z_0 \frac{\partial^4}{\partial J_i \partial J_j \partial J_k \partial J_l} \left.\left(e^{\frac{1}{2}J^T A^{-1} J}\right)\right|_{J=0} \tag{3.74}$$

$$= Z_0 \frac{\partial^4}{\partial J_i \partial J_j \partial J_k \partial J_l} \left.\left(e^{\frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}\left(A^{-1}\right)_{pq} J_p J_q}\right)\right|_{J=0} \tag{3.75}$$

$$= Z_0 \frac{\partial^3}{\partial J_j \partial J_k \partial J_l} \left(\left(\frac{1}{2}\sum_{a=1}^{n}\left(A^{-1}\right)_{ai} J_a + \frac{1}{2}\sum_{b=1}^{n}\left(A^{-1}\right)_{ib} J_b\right) \right.$$
$$\left.\left. \cdot e^{\frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}\left(A^{-1}\right)_{pq} J_p J_q}\right)\right|_{J=0} \tag{3.76}$$

$$= Z_0 \frac{\partial^3}{\partial J_j \partial J_k \partial J_l} \left.\left(\sum_{a=1}^{n}\left(A^{-1}\right)_{ia} J_a e^{\frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}\left(A^{-1}\right)_{pq} J_p J_q}\right)\right|_{J=0} \tag{3.77}$$

$$= Z_0 \frac{\partial^2}{\partial J_k \partial J_l} \left(\left(A^{-1}\right)_{ij} + \sum_{a=1}^{n}\left(A^{-1}\right)_{ia} J_a \left(\frac{1}{2}\sum_{b=1}^{n}\left(A^{-1}\right)_{bj} J_b \right.\right.$$
$$\left.\left.\left. + \frac{1}{2}\sum_{c=1}^{n}\left(A^{-1}\right)_{jc} J_c\right) e^{\frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}\left(A^{-1}\right)_{pq} J_p J_q}\right)\right|_{J=0} \tag{3.78}$$

$$= Z_0 \frac{\partial^2}{\partial J_k \partial J_l} \left(\left(\left(A^{-1}\right)_{ij}\right.\right.$$
$$\left.\left.\left. + \sum_{a=1}^{n}\left(A^{-1}\right)_{ia} J_a \sum_{b=1}^{n}\left(A^{-1}\right)_{jb} J_b\right) e^{\frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}\left(A^{-1}\right)_{pq} J_p J_q}\right)\right|_{J=0} \tag{3.79}$$

$$= Z_0 \frac{\partial}{\partial J_l} \Bigg(\Bigg(\Big(\left(A^{-1}\right)_{jk} \sum_{a=1}^{n} \left(A^{-1}\right)_{ia} J_a + \left(A^{-1}\right)_{ik} \sum_{b=1}^{n} \left(A^{-1}\right)_{jb} J_b$$
$$+ \Big(\left(A^{-1}\right)_{ij} + \sum_{a=1}^{n} \left(A^{-1}\right)_{ia} J_a \sum_{b=1}^{n} \left(A^{-1}\right)_{jb} J_b \Big)$$
$$\cdot \Big(\frac{1}{2} \sum_{c=1}^{n} \left(A^{-1}\right)_{ck} J_c + \frac{1}{2} \sum_{d=1}^{n} \left(A^{-1}\right)_{kd} J_d \Big) \Big)$$
$$\cdot e^{\frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} \left(A^{-1}\right)_{pq} J_p J_q} \Bigg) \Bigg|_{J=0} \tag{3.80}$$

$$= Z_0 \frac{\partial}{\partial J_l} \Bigg(\Bigg(\Big(\left(A^{-1}\right)_{jk} \sum_{a=1}^{n} \left(A^{-1}\right)_{ia} J_a + \left(A^{-1}\right)_{ik} \sum_{b=1}^{n} \left(A^{-1}\right)_{jb} J_b$$
$$+ \left(A^{-1}\right)_{ij} \sum_{c=1}^{n} \left(A^{-1}\right)_{kc} J_c + \sum_{a=1}^{n} \left(A^{-1}\right)_{ia} J_a \sum_{b=1}^{n} \left(A^{-1}\right)_{jb} J_b \sum_{c=1}^{n} \left(A^{-1}\right)_{kc} J_c \Big)$$
$$\cdot e^{\frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} \left(A^{-1}\right)_{pq} J_p J_q} \Bigg) \Bigg|_{J=0} \tag{3.81}$$

$$= Z_0 \Bigg(\Bigg(\Big(\left(A^{-1}\right)_{ij} \left(A^{-1}\right)_{kl} + \left(A^{-1}\right)_{ik} \left(A^{-1}\right)_{jl} + \left(A^{-1}\right)_{il} \left(A^{-1}\right)_{jk}$$
$$+ \left(A^{-1}\right)_{il} \sum_{b=1}^{n} \left(A^{-1}\right)_{jb} J_b \sum_{b=1}^{n} \left(A^{-1}\right)_{kc} J_c$$
$$+ \left(A^{-1}\right)_{jl} \sum_{b=1}^{n} \left(A^{-1}\right)_{ia} J_a \sum_{b=1}^{n} \left(A^{-1}\right)_{kc} J_c$$
$$+ \left(A^{-1}\right)_{kl} \sum_{b=1}^{n} \left(A^{-1}\right)_{ia} J_a \sum_{b=1}^{n} \left(A^{-1}\right)_{jb} J_b$$
$$+ \Big(\left(A^{-1}\right)_{jk} \sum_{a=1}^{n} \left(A^{-1}\right)_{ia} J_a + \left(A^{-1}\right)_{ik} \sum_{b=1}^{n} \left(A^{-1}\right)_{jb} J_b$$
$$+ \left(A^{-1}\right)_{ij} \sum_{c=1}^{n} \left(A^{-1}\right)_{kc} J_c + \sum_{a=1}^{n} \left(A^{-1}\right)_{ia} J_a \sum_{b=1}^{n} \left(A^{-1}\right)_{jb} J_b \sum_{c=1}^{n} \left(A^{-1}\right)_{kc} J_c \Big)$$
$$\cdot \Big(\frac{1}{2} \sum_{d=1}^{n} \left(A^{-1}\right)_{dk} J_d + \frac{1}{2} \sum_{e=1}^{n} \left(A^{-1}\right)_{ke} J_e \Big) \Big)$$
$$\cdot e^{\frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} \left(A^{-1}\right)_{pq} J_p J_q} \Bigg) \Bigg|_{J=0} \tag{3.82}$$

$$= Z_0 \Big(\left(A^{-1}\right)_{ij} \left(A^{-1}\right)_{kl} + \left(A^{-1}\right)_{ik} \left(A^{-1}\right)_{jl}$$

$$+\left(A^{-1}\right)_{il}\left(A^{-1}\right)_{jk}\Big) \tag{3.83}$$

$$\Rightarrow I = Z_0\Big(\left(A^{-1}\right)_{ij}\left(A^{-1}\right)_{kl}$$
$$+\left(A^{-1}\right)_{ik}\left(A^{-1}\right)_{jl}+\left(A^{-1}\right)_{il}\left(A^{-1}\right)_{jk}\Big). \tag{3.84}$$

This portrays the level of complexity of calculating these integrals analytically, using derivatives.

3.6 m-point Function and Wick's Theorem

We then begin to calculate the integral

$$I = \int_{\mathbb{R}^n} d^n x x_{i_1} x_{i_2} \cdots x_{i_m} e^{-\frac{1}{2}x^T Ax}. \tag{3.85}$$

We once again define the same *generating function* $Z[J]$, and relate the wanted integral to partial derivatives of the generating function.

$$Z[J] := \int_{\mathbb{R}^n} d^n x e^{-\frac{1}{2}x^T Ax + x^T J}, \tag{3.86}$$

$$\Rightarrow I = \left.\frac{\partial^m Z[J]}{\partial J_{i_1}\partial J_{i_2}\cdots\partial J_{i_m}}\right|_{J=0}. \tag{3.87}$$

Using Equation 3.57 we see

$$I = \left.\frac{\partial^m Z[J]}{\partial J_{i_1}\partial J_{i_2}\cdots\partial J_{i_m}}\right|_{J=0} \tag{3.88}$$

$$= Z_0 \left.\frac{\partial^m}{\partial J_{i_1}\partial J_{i_2}\cdots\partial J_{i_m}}\left(e^{\frac{1}{2}J^T A^{-1} J}\right)\right|_{J=0}. \tag{3.89}$$

Note that, for odd m, every term that comes from the derivatives contains some J_i, and the value of the integral is thus zero.

Since calculating this seemingly elementary expression analytically would require several pages of paper (an assumption based on the rapidly increasing complexity of the calculations in section 3.4 and section 3.5), we introduce an alternative way to calculate these integrals.

We introduce and define the m-point function

$$\langle x_{i_1} x_{i_2} \cdots x_{i_m}\rangle := \frac{1}{Z_0}\int_{\mathbb{R}^n} d^n x x_{i_1} x_{i_2} \cdots x_{i_m} e^{-\frac{1}{2}x^T Ax}, \tag{3.90}$$

with the same multi-dimensional *generating function* $Z[J]$ as before. We then relate the m-point function to the generating function using partial derivatives, and make use of Theorem 1.2 to turn the very untidy calculation of the derivatives into a combinatorial problem.

$$Z[J] := \int_{\mathbb{R}^n} d^n x e^{-x^T A x + x^T J}, \tag{3.91}$$

which implies that

$$\langle x_{i_1} x_{i_2} \cdots x_{i_m} \rangle = \frac{1}{Z_0} \frac{\partial^m Z[J]}{\partial J_{i_1} \partial J_{i_2} \cdots \partial J_{i_m}} \bigg|_{J=0} \tag{3.92}$$

$$= \frac{\partial^m}{\partial J_{i_1} \partial J_{i_2} \cdots \partial J_{i_m}} \left(e^{\frac{1}{2} J^T A^{-1} J} \right) \bigg|_{J=0}. \tag{3.93}$$

We reformulate Wick's theorem (Theorem 1.2) to match the notation used here, and we can then use it to calculate this integral.

Theorem 3.1 (Wick's Theorem).

$$\frac{\partial^m}{\partial J_{i_1} \partial J_{i_2} \cdots \partial J_{i_m}} \left(e^{\frac{1}{2} J^T A^{-1} J} \right) \bigg|_{J=0} = \sum \left(A^{-1} \right)_{i_{p_1} i_{p_2}} \left(A^{-1} \right)_{i_{p_3} i_{p_4}} \cdots \left(A^{-1} \right)_{i_{p_{m-1}} i_{p_m}}, \tag{3.94}$$

where the sum is taken over all pairings

$$\left(i_{p_1}, i_{p_2} \right), \left(i_{p_3}, i_{p_4} \right), \ldots, \left(i_{p_{m-1}}, i_{p_m} \right) \tag{3.95}$$

of $i_1, i_2, \ldots, i_m$.

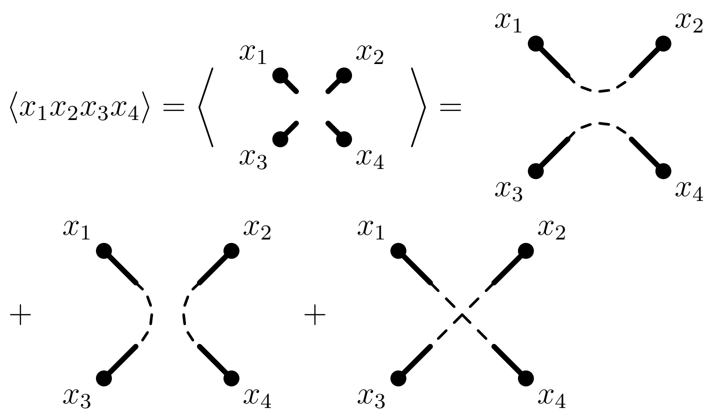
3.7 Feynman Diagrams

This combinatorial problem can be solved with even less effort using a visual representation of the combinatorial sum [2, 6]. We now introduce a way to represent each term of the sum with a drawing of a so called *Feynman diagram* [2, 4, 6], and use this method to calculate a few examples of integrals that would be very inconvenient to calculate using partial derivatives.

We assign to each factor x_i^p a *vertex* with p *half-edges*. Then we connect the different half-edges in pairs with a *propagator*. To the propagator belonging to the vertices x_i, x_j we assign a *weight* $(A^{-1})_{ij}$. We connect the half-edges in every possible way, and sum together the permutations, after calculating the product of the weights of the propagators for each permutation (see Example 3.2 and Example 3.3). [2, 6]

The resulting diagrams that we draw here are then called *Feynman diagrams* [2, 4, 6].

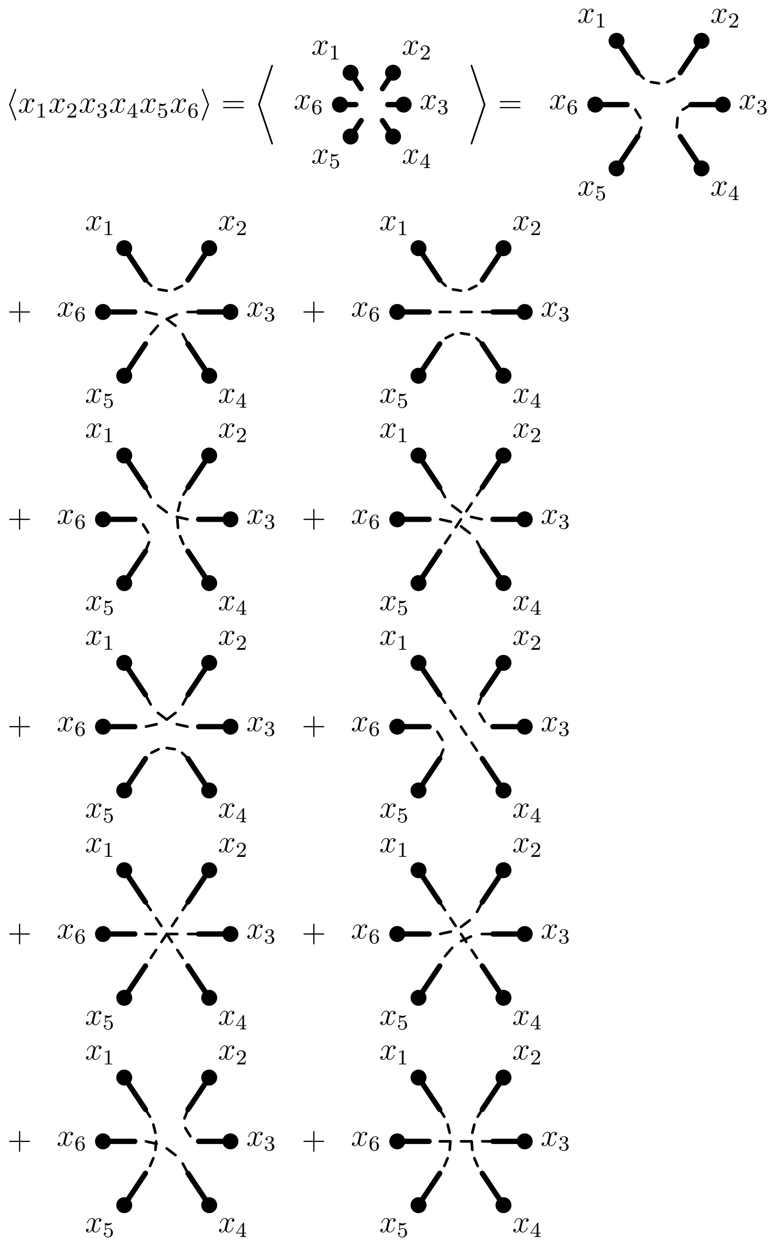
Example 3.2 (Four-point function).

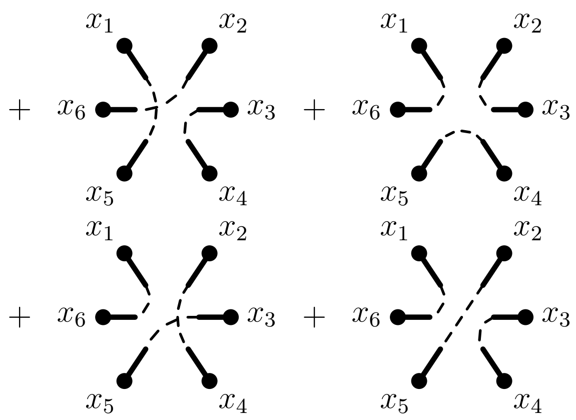


(3.96)

$$= \left(A^{-1}\right)_{12}\left(A^{-1}\right)_{34} + \left(A^{-1}\right)_{13}\left(A^{-1}\right)_{24} + \left(A^{-1}\right)_{14}\left(A^{-1}\right)_{23}. \tag{3.97}$$

Example 3.3 (Six-point function).





(3.98)

$$
\begin{aligned}
&= \left(A^{-1}\right)_{12}\left(A^{-1}\right)_{34}\left(A^{-1}\right)_{56} + \left(A^{-1}\right)_{12}\left(A^{-1}\right)_{35}\left(A^{-1}\right)_{46} \\
&+ \left(A^{-1}\right)_{12}\left(A^{-1}\right)_{36}\left(A^{-1}\right)_{45} + \left(A^{-1}\right)_{13}\left(A^{-1}\right)_{24}\left(A^{-1}\right)_{56} \\
&+ \left(A^{-1}\right)_{13}\left(A^{-1}\right)_{25}\left(A^{-1}\right)_{46} + \left(A^{-1}\right)_{13}\left(A^{-1}\right)_{26}\left(A^{-1}\right)_{45} \\
&+ \left(A^{-1}\right)_{14}\left(A^{-1}\right)_{23}\left(A^{-1}\right)_{56} + \left(A^{-1}\right)_{14}\left(A^{-1}\right)_{25}\left(A^{-1}\right)_{36} \\
&+ \left(A^{-1}\right)_{14}\left(A^{-1}\right)_{26}\left(A^{-1}\right)_{35} + \left(A^{-1}\right)_{15}\left(A^{-1}\right)_{23}\left(A^{-1}\right)_{46} \\
&+ \left(A^{-1}\right)_{15}\left(A^{-1}\right)_{24}\left(A^{-1}\right)_{36} + \left(A^{-1}\right)_{15}\left(A^{-1}\right)_{26}\left(A^{-1}\right)_{34} \\
&+ \left(A^{-1}\right)_{16}\left(A^{-1}\right)_{23}\left(A^{-1}\right)_{45} + \left(A^{-1}\right)_{16}\left(A^{-1}\right)_{24}\left(A^{-1}\right)_{35} \\
&+ \left(A^{-1}\right)_{16}\left(A^{-1}\right)_{25}\left(A^{-1}\right)_{34}.
\end{aligned}
\tag{3.99}
$$

3.8 Perturbative Expansion

We now calculate the integral

$$
I = \int_{\mathbb{R}^n} d^n x e^{-\frac{1}{2}x^T A x + \alpha x^j}, \tag{3.100}
$$

with a new term introduced in the exponent.

Rewriting the integral using Taylor expansion, and exchanging the order of the summation and the integration, gives us

$$
\begin{aligned}
I &= \int_{\mathbb{R}^n} d^n x e^{-\frac{1}{2}x^T A x + \alpha x^j} && (3.101) \\
&= \int_{\mathbb{R}^n} d^n x e^{\alpha x^j} e^{-\frac{1}{2}x^T A x} && (3.102) \\
&= \int_{\mathbb{R}^n} d^n x \sum_{k=0}^{\infty} \frac{\left(\alpha x^j\right)^k}{k!} e^{-\frac{1}{2}x^T A x} && (3.103)
\end{aligned}
$$

$$= \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \int_{\mathbb{R}^n} d^n x x^{jk} e^{-\frac{1}{2}x^T A x}. \tag{3.104}$$

Each term in this sum bears strong resemblance to previously performed integrals (see section 3.6), and can thus be solved in the same way. For α small enough, the first few terms in the sum are dominating, and we can be satisfied with calculating only the first few terms.

Example 3.4 (Perturbative expansion for $j = 4$).

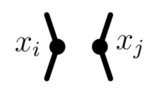
$$I = \int_{\mathbb{R}^n} d^n x e^{-\frac{1}{2}x^T A x + \alpha x^4} \tag{3.105}$$

$$= \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} \int_{\mathbb{R}^n} d^n x x^{4k} e^{-\frac{1}{2}x^T A x} \tag{3.106}$$

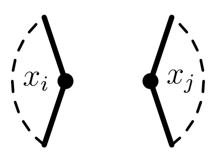
$$= Z_0 \sum_{k=0}^{\infty} \left\langle x^{4k} \right\rangle \tag{3.107}$$

$$= Z_0 \left(\langle 1 \rangle + \sum_{i=1}^{n} \sum_{j=1}^{n} \langle x_i x_i x_j x_j \rangle + \cdots \right) \tag{3.108}$$

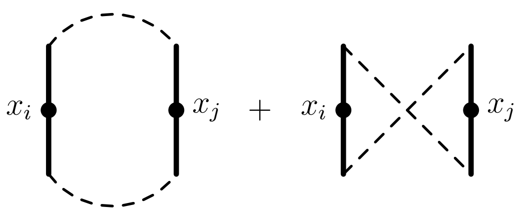
$$= Z_0 \left(1 + \sum_{i=1}^{n} \sum_{j=1}^{n} \left\langle \quad \right\rangle + \cdots \right) \tag{3.109}$$



$$= Z_0 \left(1 + \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\right. \right.$$



$$+ \quad + \quad \left. \right) + \cdots \left. \right) \tag{3.110}$$



$$= Z_0 \left(1 + \sum_{i=1}^{n} \sum_{j=1}^{n} \left(\left(A^{-1}\right)_{ii} \left(A^{-1}\right)_{jj} + 2 \left(A^{-1}\right)_{ij}^2 \right) + \cdots \right) \tag{3.111}$$

$$\approx Z_0 \left(1 + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\left(A^{-1}\right)_{ii}\left(A^{-1}\right)_{jj} + 2\left(A^{-1}\right)_{ij}^{2}\right)\right). \quad (3.112)$$

3.9 Basic Matrix Integral

We calculate the *matrix integral*

$$I = \int_{\mathbb{R}^{n\times n}} \mathcal{D}M e^{-\frac{1}{2}\operatorname{Tr}\left(M^2\right)}. \quad (3.113)$$

For the matrix trace, we see that

$$\operatorname{Tr}\left(M^2\right) = \sum_{i=1}^{n}\left(M^2\right)_{ii}, \quad (3.114)$$

$$\left(M^2\right)_{ii} = \sum_{j=1}^{n} M_{ij}M_{ji} \quad (3.115)$$

$$\Rightarrow \operatorname{Tr}\left(M^2\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}M_{ji} \quad (3.116)$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} M_{ij}M_{kl}\delta_{il}\delta_{jk}. \quad (3.117)$$

To solve the integral, we identify the matrix M with the previously used vector x, and the Kronecker delta functions δ_{il}, δ_{jk} with the previously used matrix A. We define

$$x := \begin{pmatrix} M_{11} \\ M_{12} \\ \vdots \\ M_{1n} \\ M_{21} \\ M_{22} \\ \vdots \\ M_{nn} \end{pmatrix} \in \mathbb{R}^{n^2} \quad (3.118)$$

$$\Leftrightarrow M_{ij} =: x_{n(i-1)+j} \quad \forall i,j \in \mathbb{N}\cap[1,n], \quad (3.119)$$

$$\delta_{il}\delta_{jk} =: A_{n(i-1)+j,n(k-1)+l} \quad \forall i,j,k,l \in \mathbb{N}\cap[1,n], \quad (3.120)$$

$$\Rightarrow x^T A x = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} M_{ij}M_{kl}\delta_{il}\delta_{jk} \quad (3.121)$$

$$= \operatorname{Tr}\left(M^2\right). \quad (3.122)$$

Note that each element in M corresponds to exactly one element in x, and vice versa. Additionally, our matrix A is still real and symmetric.

We can now also define the measure

$$\mathcal{D}M := \prod_{i,j} dM_{ij} \tag{3.123}$$
$$= \prod_{i,j} dx_{n(i-1)+j} \tag{3.124}$$
$$= d^{n^2}x \tag{3.125}$$
$$\Rightarrow I = \int_{\mathbb{R}^{n^2}} d^{n^2}x e^{-\frac{1}{2}x^T Ax}. \tag{3.126}$$

This integral now assumes a form that is nearly identical to integrals we have already solved. The value of this integral is a constant which we will use in subsequent calculations. From Equation 3.33 we see that

$$I = \frac{(2\pi)^{n^2/2}}{\sqrt{\det(A)}}. \tag{3.127}$$

3.10 Two-point Function of Matrix Elements

We calculate the matrix integral

$$I = \int_{\mathbb{R}^{n\times n}} \mathcal{D}M M_{ij} M_{kl} e^{-\frac{1}{2}\operatorname{Tr}\left(M^2\right)}. \tag{3.128}$$

Like in the previous section, we rewrite the integral on a form we are familiar with. We identify

$$M_{ij} =: x_{n(i-1)+j} \quad \forall i,j \in \mathbb{N}\cap[1,n], \tag{3.129}$$
$$\delta_{il}\delta_{jk} =: A_{n(i-1)+j,n(k-1)+l} \quad \forall i,j,k,l \in \mathbb{N}\cap[1,n], \tag{3.130}$$
$$\Rightarrow I = \int_{\mathbb{R}^{n^2}} d^{n^2}x x_{n(i-1)+j} x_{n(k-1)+l} e^{-\frac{1}{2}x^T Ax}. \tag{3.131}$$

From Equation 3.69 we then see that

$$I = \int_{\mathbb{R}^{n^2}} d^{n^2}x x_{n(i-1)+j} x_{n(k-1)+l} e^{-\frac{1}{2}x^T Ax} \tag{3.132}$$
$$= Z_0 \left(A^{-1}\right)_{n(i-1)+j,n(k-1)+l} \tag{3.133}$$
$$= Z_0 A_{n(i-1)+j,n(k-1)+l} \tag{3.134}$$

$$= Z_0 \delta_{il} \delta_{jk}, \tag{3.135}$$

where (from Equation 3.127)

$$Z_0 := \int_{\mathbb{R}^{n\times n}} \mathcal{D}M e^{-\frac{1}{2}\operatorname{Tr}\left(M^2\right)} \tag{3.136}$$

$$= \frac{(2\pi)^{n^2/2}}{\sqrt{\det(A)}}. \tag{3.137}$$

3.11 m-point Function of Matrix Elements and Wick's Theorem

Suppose we want to calculate the matrix integral

$$I = \int_{\mathbb{R}^{n\times n}} \mathcal{D}M M_{i_1 j_1} M_{i_2 j_2} \cdots M_{i_m j_m} e^{-\frac{1}{2}\operatorname{Tr}\left(M^2\right)}. \tag{3.138}$$

We introduce the m-point function

$$\left\langle \prod_{k=1}^{m} M_{i_k j_k} \right\rangle := \frac{1}{Z_0} \int_{\mathbb{R}^{n\times n}} \mathcal{D}M M_{i_1 j_1} M_{i_2 j_2} \cdots M_{i_m j_m} e^{-\frac{1}{2}\operatorname{Tr}\left(M^2\right)} \tag{3.139}$$

$$= \frac{1}{Z_0} \int_{\mathbb{R}^{n^2}} d^{n^2}x \prod_{k=1}^{m} x_{n(i_k-1)+j_k} e^{-\frac{1}{2}x^T A x}, \tag{3.140}$$

with the same definitions of A and x as previously, and begin to calculate this integral.

From Equation 3.89 we see

$$\left\langle \prod_{k=1}^{m} M_{i_k j_k} \right\rangle = \frac{1}{Z_0} \int_{\mathbb{R}^{n^2}} d^{n^2}x \prod_{k=1}^{m} x_{n(i_k-1)+j_k} e^{-\frac{1}{2}x^T A x} \tag{3.141}$$

$$= \left(\prod_{k=1}^{m} \frac{\partial}{\partial J_{n(i_k-1)+j_k}} \right) \left(e^{\frac{1}{2}J^T A^{-1} J} \right) \Bigg|_{J=0} \tag{3.142}$$

Again, we can use Wick's theorem to reduce this long calculation of the partial derivatives to a much more elegant combinatorial problem. We reformulate Wick's theorem (Theorem 1.2) using the same notations as here.

Theorem 3.5 (Wick's Theorem).

$$\left(\prod_{k=1}^{m}\frac{\partial}{\partial J_{n(i_k-1)+j_k}}\right)\left(e^{\frac{1}{2}J^T A^{-1} J}\right)\Bigg|_{J=0}$$
$$= \sum \left(A^{-1}\right)_{i_{p_1} i_{p_2}} \left(A^{-1}\right)_{i_{p_3} i_{p_4}} \cdots \left(A^{-1}\right)_{i_{p_{m-1}} i_{p_m}}, \quad (3.143)$$

where the sum is taken over all pairings

$$\left(i_{p_1}, i_{p_2}\right), \left(i_{p_3}, i_{p_4}\right), \ldots, \left(i_{p_{m-1}}, i_{p_m}\right) \quad (3.144)$$

of $i_1, i_2, \ldots, i_m$.

Note that our definition of A leads to

$$A^{-1} = A. \quad (3.145)$$

Proof.

$$\left(A^2\right)_{ab} = \sum_{p=1}^{n^2} A_{ap} A_{pb}. \quad (3.146)$$

Define

$$a := n\,(i-1) + j, \quad (3.147)$$
$$b := n\,(k-1) + l, \quad (3.148)$$
$$p =: n\,(q-1) + r. \quad (3.149)$$

Note that, for each $a \in \mathbb{N} \cap [1, n^2]$, there exists a unique combination of $i, j \in \mathbb{N} \cap [1, n]$ such that this equality holds. The same is true for $b \in \mathbb{N} \cap [1, n^2]$ with $k, l \in \mathbb{N} \cap [1, n]$, and $p \in \mathbb{N} \cap [1, n^2]$ with $q, r \in \mathbb{N} \cap [1, n]$.

$$\left(A^2\right)_{ab} = \sum_{p=1}^{n^2} A_{ap} A_{pb} \quad (3.150)$$
$$= \sum_{q=1}^{n} \sum_{r=1}^{n} A_{n(i-1)+j, n(q-1)+r} A_{n(q-1)+r, n(k-1)+l} \quad (3.151)$$
$$= \sum_{q=1}^{n} \sum_{r=1}^{n} \delta_{ir} \delta_{jq} \delta_{lq} \delta_{kr} \quad (3.152)$$

$$= \delta_{jl}\delta_{ik} \tag{3.153}$$
$$= \delta_{n(i-1)+j,n(k-1)+l} \tag{3.154}$$
$$= \delta_{ab} \tag{3.155}$$
$$= (\mathbb{1})_{ab} \tag{3.156}$$
$$\Rightarrow A^2 = \mathbb{1} \tag{3.157}$$
$$\Rightarrow A^{-1} = A. \tag{3.158}$$

□

This implies that

$$\left(\prod_{k=1}^{m} \frac{\partial}{\partial J_{n(i_k-1)+j_k}}\right)\left(e^{\frac{1}{2}J^T A^{-1} J}\right)\Bigg|_{J=0}$$
$$= \sum \left(A^{-1}\right)_{i_{p_1} i_{p_2}} \left(A^{-1}\right)_{i_{p_3} i_{p_4}} \cdots \left(A^{-1}\right)_{i_{p_{m-1}} i_{p_m}}, \tag{3.159}$$
$$= \sum A_{i_{p_1} i_{p_2}} A_{i_{p_3} i_{p_4}} \cdots A_{i_{p_{m-1}} i_{p_m}}, \tag{3.160}$$
$$= \sum \sum A_{n\left(j_{p_1}-1\right)+k_{p_1}, n\left(j_{p_2}-1\right)+k_{p_2}} A_{n\left(j_{p_3}-1\right)+k_{p_3}, n\left(j_{p_4}-1\right)+k_{p_4}} \cdots A_{n\left(j_{p_{m-1}}-1\right)+k_{p_{m-1}}, n(j_{p_m}-1)+k_{p_m}} \tag{3.161}$$
$$= \sum \sum \delta_{j_{p_1} k_{p_2}} \delta_{j_{p_2} k_{p_1}} \delta_{j_{p_3} k_{p_4}} \delta_{j_{p_4} k_{p_3}} \cdots \delta_{j_{p_{m-1}} k_{p_m}} \delta_{j_{p_m} k_{p_{m-1}}}, \tag{3.162}$$

where the single sum is taken over all pairings

$$\left(i_{p_1}, i_{p_2}\right), \left(i_{p_3}, i_{p_4}\right), \ldots, \left(i_{p_{m-1}}, i_{p_m}\right) \tag{3.163}$$

of $i_1, i_2, \ldots, i_m$ for $i \in \mathbb{N} \cap [1, n^2]$, and the double sum is taken over all pairings

$$\left(j_{p_1}, j_{p_2}\right), \left(j_{p_3}, j_{p_4}\right), \ldots, \left(j_{p_{m-1}}, j_{p_m}\right) \tag{3.164}$$

of $j_1, j_2, \ldots, j_m$, and

$$\left(k_{p_1}, k_{p_2}\right), \left(k_{p_3}, k_{p_4}\right), \ldots, \left(k_{p_{m-1}}, k_{p_m}\right) \tag{3.165}$$

of $k_1, k_2, \ldots, k_m$, for $j, k \in \mathbb{N} \cap [1, n]$, respectively. Once again, note that, for each $i \in \mathbb{N} \cap [1, n^2]$, there exists a unique combination of $j, k \in \mathbb{N} \cap [1, n]$ such that $i = n\,(j-1) + k$.

3.12 Feynman Diagrams and Fat Graphs

Like in the case with a multi-dimensional integral over a vector (see section 3.7), this combinatorial problem can be solved with less effort by representing each term of the combinatorial sum graphically with a Feynman diagram. [2, 4, 6]

We therefore assign, to each factor with cyclically contracted indices $M_{i_1 i_2} M_{i_2 i_3} \cdots M_{i_m i_1}$, a *vertex* with m double-line *half-edges*, each with a pair of indices corresponding to a matrix element; see for example Figure 3.1, in which $m = 4$. The single lines are oriented, and the indices are constant along these. [2, 6]

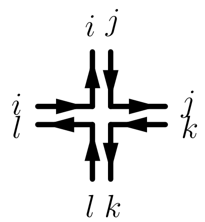


Figure 3.1: The vertex corresponding to $\langle M_{ij} M_{jk} M_{kl} M_{li} \rangle$.

Then we connect the different half-edges in pairs with an oriented double-line edge *propagator* with the same indices; see Figure 3.2. [2, 6]

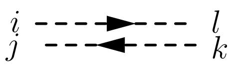


Figure 3.2: The Feynman propagator.

To each propagator with the indices i, j and k, l we assign a *weight* $\delta_{il}\delta_{jk}$, corresponding to the value of the two-point function (similar to how we defined the propagator in section 3.7). We connect the half-edges in every possible way while respecting the orientation of the single lines by making sure it is preserved, and sum together the permutations after calculating the product of the weights of the propagators for each permutation. [2, 6]

The diagrams drawn here are also a form of *Feynman diagrams* [2, 4, 6]. Since they consist of double-lines, they are sometimes called *fat graphs* as well [2, 6].

Example 3.6 (Fat graph).

We calculate

$$\left\langle \mathrm{Tr}\left(M^4\right)\right\rangle . \tag{3.166}$$

We note that

$$\mathrm{Tr}\left(M^4\right) = \sum_{i=1}^{n}\left(M^4\right)_{ii} \tag{3.167}$$

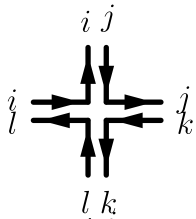
$$= \sum_{i=1}^{n}\sum_{k=1}^{n}\left(M^2\right)_{ik}\left(M^2\right)_{ki} \tag{3.168}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} M_{ij}M_{jk}M_{kl}M_{li} \tag{3.169}$$

$$\Rightarrow \left\langle \mathrm{Tr}\left(M^4\right)\right\rangle = \left\langle \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} M_{ij}M_{jk}M_{kl}M_{li}\right\rangle \tag{3.170}$$

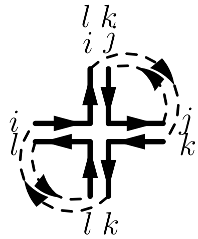
$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} \langle M_{ij}M_{jk}M_{kl}M_{li}\rangle \tag{3.171}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} \Bigg\langle$$

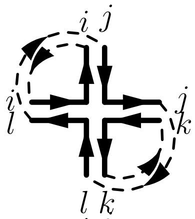


$$\Bigg\rangle \tag{3.172}$$

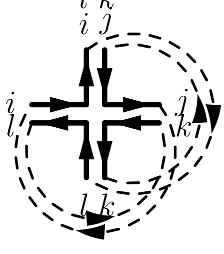
$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} \Bigg($$



$$+$$



$$+$$



$$\Bigg) \tag{3.173}$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} \left(\delta_{ik}\delta_{jj}\delta_{ik}\delta_{ll}\right.$$

$$\left. +\delta_{jl}\delta_{ii}\delta_{jl}\delta_{kk} + \delta_{ij}\delta_{jk}\delta_{kl}\delta_{li}\right) \tag{3.174}$$

$$= 2n^3 + n. \tag{3.175}$$

3.13 Perturbative Expansion of a Matrix Integral

We now calculate one final example matrix integral, using perturbative expansion in addition to the previously demonstrated techniques, putting all the tools that we have developed to use.

Example 3.7 (Perturbative expansion of a matrix integral).

Calculate the matrix integral

$$I = \frac{1}{Z_0} \int_{\mathbb{R}^{n\times n}} \mathcal{D}M e^{-\frac{1}{2}\operatorname{Tr}\left(M^2\right) + g\operatorname{Tr}\left(M^3\right)}. \tag{3.176}$$

We express the matrix trace as a sum of matrix elements. We see that

$$\operatorname{Tr}\left(M^3\right) = \sum_{j=1}^{n} \left(M^3\right)_{jj} \tag{3.177}$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} M_{jk} \left(M^2\right)_{kj} \tag{3.178}$$

$$= \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} M_{jk} M_{kl} M_{lj}. \tag{3.179}$$

Like in section 3.8, we then rewrite the integral using Taylor expansion, and exchange the order of summation and integration. This implies that

$$I = \frac{1}{Z_0} \int_{\mathbb{R}^{n\times n}} \mathcal{D}M e^{-\frac{1}{2}\operatorname{Tr}\left(M^2\right) + g\operatorname{Tr}\left(M^3\right)} \tag{3.180}$$

$$= \frac{1}{Z_0} \int_{\mathbb{R}^{n\times n}} \mathcal{D}M e^{g\operatorname{Tr}\left(M^3\right)} e^{-\frac{1}{2}\operatorname{Tr}\left(M^2\right)} \tag{3.181}$$

$$= \frac{1}{Z_0} \int_{\mathbb{R}^{n\times n}} \mathcal{D}M \sum_{i=0}^{\infty} \frac{1}{i!} \left(g\operatorname{Tr}\left(M^3\right)\right)^i e^{-\frac{1}{2}\operatorname{Tr}\left(M^2\right)} \tag{3.182}$$

$$= \frac{1}{Z_0} \sum_{i=0}^{\infty} \frac{g^i}{i!} \int_{\mathbb{R}^{n\times n}} \mathcal{D}M \left(\sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} M_{jk} M_{kl} M_{lj} \right)^i e^{-\frac{1}{2}\operatorname{Tr}\left(M^2\right)} \tag{3.183}$$

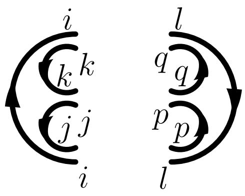
$$= \sum_{i=0}^{\infty} \frac{g^i}{i!} \left\langle \left(\sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} M_{jk} M_{kl} M_{lj} \right)^i \right\rangle \tag{3.184}$$

$$= \langle 1 \rangle + g \left\langle \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} M_{jk} M_{kl} M_{lj} \right\rangle + \frac{g^2}{2} \left\langle \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{p=1}^{n} \sum_{q=1}^{n} M_{ij} M_{jk} M_{ki} M_{lp} M_{pq} M_{ql} \right\rangle + \cdots . \tag{3.185}$$

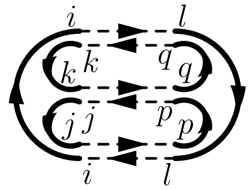
We calculate each term of this expression by representing it with a Feynman diagram. We first note that the second term contains an odd number of matrix elements. Similar to what we saw in section 3.6, this term thus evaluates to zero. We also see, that for small g, the first few terms are dominant. Therefore, it is enough to calculate a finite number of terms of the Taylor series; otherwise calculating this expression would simply not be feasible. In this example, we are satisfied with calculating the integral up to and including and order of g^2. Note that, when we draw the diagrams in the following calculation, we sometimes leave out arrows and labels, purely to reduce the clutter in the figures.

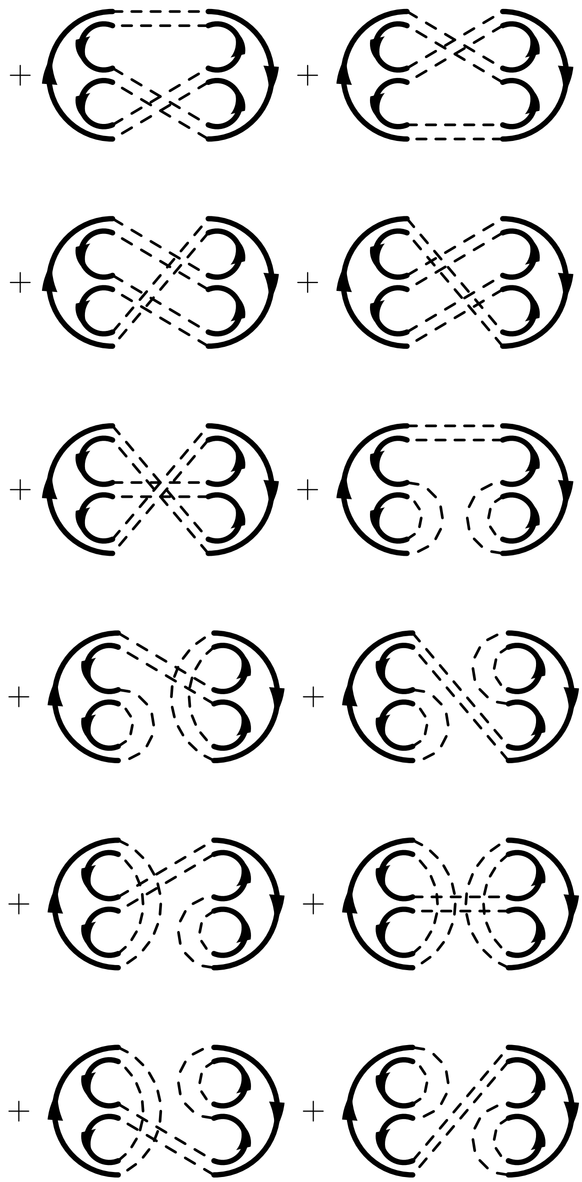
$$\Rightarrow I = 1 + 0 + \frac{g^2}{2} \left\langle \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{p=1}^{n} \sum_{q=1}^{n} M_{ij} M_{jk} M_{ki} M_{lp} M_{pq} M_{ql} \right\rangle + \cdots \tag{3.186}$$

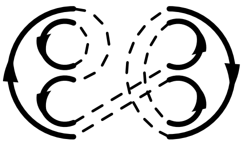
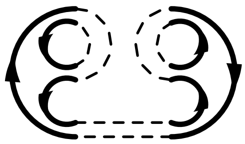
$$\approx 1 + \frac{g^2}{2} \left\langle \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{p=1}^{n} \sum_{q=1}^{n} M_{ij} M_{jk} M_{ki} M_{lp} M_{pq} M_{ql} \right\rangle \tag{3.187}$$

$$= 1 + \frac{g^2}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{p=1}^{n} \sum_{q=1}^{n} \Big\langle$$  $$\Big\rangle \tag{3.188}$$

$$= 1 + \frac{g^2}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{p=1}^{n} \sum_{q=1}^{n} \Bigg($$





$+$  $+$  $\Bigg)$ (3.189)

$$= 1 + \frac{g^2}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{n} \sum_{p=1}^{n} \sum_{q=1}^{n} (\delta_{il}\delta_{kq}\delta_{kq}\delta_{jp}\delta_{jp}\delta_{il}$$
$$+\delta_{il}\delta_{kq}\delta_{kp}\delta_{jl}\delta_{jq}\delta_{ip} + \delta_{iq}\delta_{kp}\delta_{kl}\delta_{jq}\delta_{jp}\delta_{il} + \delta_{iq}\delta_{kp}\delta_{kp}\delta_{jl}\delta_{jl}\delta_{iq}$$
$$+\delta_{ip}\delta_{kl}\delta_{kl}\delta_{jq}\delta_{jq}\delta_{ip} + \delta_{ip}\delta_{kl}\delta_{kq}\delta_{jp}\delta_{jl}\delta_{iq} + \delta_{il}\delta_{kq}\delta_{ik}\delta_{jj}\delta_{lq}\delta_{pp}$$
$$+\delta_{iq}\delta_{kp}\delta_{ik}\delta_{jj}\delta_{ll}\delta_{pq} + \delta_{ip}\delta_{kl}\delta_{ik}\delta_{jj}\delta_{lp}\delta_{qq} + \delta_{ii}\delta_{jk}\delta_{kl}\delta_{jq}\delta_{lq}\delta_{pp}$$
$$+\delta_{ii}\delta_{jk}\delta_{kq}\delta_{jp}\delta_{ll}\delta_{pq} + \delta_{ii}\delta_{jk}\delta_{kp}\delta_{jl}\delta_{lp}\delta_{qq} + \delta_{ij}\delta_{kk}\delta_{jl}\delta_{iq}\delta_{lq}\delta_{pp}$$
$$+\delta_{ij}\delta_{kk}\delta_{jq}\delta_{ip}\delta_{ll}\delta_{pq} + \delta_{ij}\delta_{kk}\delta_{jp}\delta_{il}\delta_{lp}\delta_{qq}) \tag{3.190}$$

$$= 1 + \frac{g^2}{2}\Big(n^3 + n + n + n^3 + n^3 + n + n^3 + n^3 + n^3$$
$$+ n^3 + n^3 + n^3 + n^3 + n^3 + n^3\Big) \tag{3.191}$$

$$= 1 + \frac{g^2}{2}\Big(12n^3 + 3n\Big) \tag{3.192}$$

We now have the tools required to calculate an arbitrarily complicated matrix integral on the form

$$\int_{\mathbb{R}^{n\times n}} \mathcal{D}M M_{i_1 j_1} M_{i_2 j_2} \cdots M_{i_m j_m} e^{-\frac{1}{2}\operatorname{Tr}\left(M^2\right) + \sum_{k=1}^{p} \alpha_k \operatorname{Tr}\left(M^k\right)}. \tag{3.193}$$

Chapter 4

Discussion & Recommendation

4.1 Discussion

4.1.1 Limitations

There are apparent limitations to the theory of matrix integrals investigated in this project.

What might be most limiting is the fact that our results are valid only for integration over the set of real-valued square matrices. Of course, since the matrix trace is defined only for square matrices, the actual limitation of this project does not lie in that the matrices must be square; rather, the limitation consists of the fact that we allow only real-valued matrices.

Something else that can seem like a limitation is the fact that we several times, without further discussion, exchange the order of integration of summation. In general, however, this is not a problem. As long as the functions we are summing over converge uniformly, the value of the integral expression remains unchanged.

4.1.2 Evaluation

Taking these limitations into account, the obtained results and methods fulfill the goal of this project. We are now able to calculate a general matrix integral of the form shown in Equation 1.1. (In fact, what we can calculate is actually slightly more complicated, see Equation 3.193.) The method which allows us to calculate matrix integrals using Feynman diagrams greatly simplifies the, potentially extremely complicated, analytical expressions we obtain while calculating derivatives of the generating functions.

Our developed method agrees well with the methods presented in [2] and [6]. While it is not identically the same, it is equivalent to those methods, and it yields the same results.

4.2 Recommendation

Following up this project, there are two obvious extensions to this theory of matrix integrals. Our recommendations are to

1. Evaluate the matrix integrals over different sets of matrices. The most evident and notable sets are

   (a) The set of complex square matrices, that is

   $$M \in \mathbb{C}^{n\times n}. \tag{4.1}$$

   (b) The set of unitary matrices, that is

   $$M^\dagger = M^{-1} \tag{4.2}$$
   $$\Rightarrow MM^\dagger = \mathbb{1}. \tag{4.3}$$

   (c) The set of Hermitian matrices, that is

   $$M = M^\dagger. \tag{4.4}$$

2. Evaluate the matrix integrals for matrices whose size approaches infinity. This is called the *large-N limit* [1], and can be performed with a method called *saddle-point approximation* [3].

Chapter 5

Conclusions

In this project, we have examined and performed integration over matrices. We have developed a method for calculating matrix integrals of the general form

$$\int \mathcal{D}M e^{-\operatorname{Tr}(V(M))}, \tag{5.1}$$

over the set of real square matrices M.

We calculated the matrix integrals by applying a perturbative Taylor expansion to the matrix integrals. Using *Wick's theorem*, we simplified the calculations of the integrals further, to a combinatorial problem. By representing the terms of the Wick sum graphically with Feynman diagrams and fat graphs, we calculated the final results of the integrals. Compared to calculating the integrals strictly analytically using partial derivatives of a generating function, our method has the obvious advantage of making the calculations less complicated, and being intuitively simple to use. We demonstrated the usage of this developed method in a few important examples that are representative for the method.

Bibliography

[1] E. Brezin, C. Itzykson, G. Parisi, and J. B. Zuber. Planar Diagrams. *Communications in Mathematical Physics*, 59(1):35–51, 1978.

[2] B. Eynard. *Counting Surfaces: Combinatorics, Matrix Models and Algebraic Geometry (Progress in Mathematical Physics)*, volume 114 of *Progress in Mathematical Physics*. Springer Birkhäuser, first edition, December 2013.

[3] M. Mariño. Les Houches Lectures on Matrix Models and Topological Strings. July 2005.

[4] M. Srednicki. *Quantum Field Theory*. Progress in Mathematical Physics. Cambridge University Press, first edition, 2007.

[5] G.-C. Wick. The Evaluation of the Collision Matrix. *Physical Review*, 80:268–272, October 1950.

[6] J.-B. Zuber. Introduction to Random Matrices. December 2012.