UPPSALA
UNIVERSITET

U.U.D.M. Project Report 2014:34

Quiver Algebras

Max Morin

Examensarbete i matematik, 15 hp
Handledare och examinator: Qimh Xantcha
December 2014

Department of Mathematics
Uppsala University

Quiver Algebras

Max Morin

December 8, 2014

Abstract

We define algebras and explore a certain class of algebras called quiver algebras. We also introduce the concept of a module of an algebra, and prove many useful theorems regarding modules and algebras. Finally, we give a simple characterisation of all algebras that are isomorphic to quiver algebras.

Acknowledgements

I want to thank my advisor Qimh Xantcha for suggesting this topic and giving me much help along the way.

Contents

In this work, the symbol $\subset$ always denotes a *proper* subset relation and $\subseteq$ is used when the relation is not necessarily proper, analogously with the common usage of $<$ and $\leqslant$. This presentation mainly follows [1].

Ring Theory

Definition 1. A **ring** is a triple $(R, +, \cdot)$ where R is a set and $+$ and $\cdot$ are binary operations $R \times R \to R$ such that $(R, +)$ is an abelian group and for all $a, b, c \in R$ we have

$$(ab)c = a(bc) \tag{1}$$
$$a(b+c) = ab + ac \tag{2}$$
$$(a+b)c = ac + bc. \tag{3}$$

A ring is called **commutative** if $ab = ba$ for all a and b in R and **unital** if there is an element 1 such that $1a = a1 = a$ for all $a \in R$. All rings in this text shall be assumed unital, and in this section the letter R shall always denote a ring.

A nonempty subset $I \subseteq R$ is called a **right ideal** if

$$\forall a, b \in I : a + b \in I \wedge -a \in I \tag{4}$$
$$\forall a \in I, \forall r \in R : ar \in I. \tag{5}$$

If (5) is replaced by

$$\forall a \in I, \forall r \in R : ra \in I \tag{6}$$

we instead have a **left ideal**. If I is both a left and a right ideal we call it a **two-sided ideal**. A **proper ideal** is an ideal I that is a proper subset of R, and it is said to be **maximal** if there is no ideal $I' \subset R$ such that $I \subset I'$. If I is a two-sided ideal, we define I^n to be the ideal given by all linear combinations of products of n elements of I, with $I^0 = R$. The ideal I is **nilpotent** if $I^n = \{0\}$ for some $n \in \mathbf{N}$.

A **ring homomorphism** between two rings R and S is a map $\varphi : R \to S$ such that

$$\varphi(a+b) = \varphi(a) + \varphi(b) \tag{7}$$
$$\varphi(ab) = \varphi(a)\varphi(b) \tag{8}$$
$$\varphi(1) = 1. \tag{9}$$

If it is bijective, it is a **ring isomorphism** and we say that R is **isomorphic** to S and write $R \cong S$.

Definition 2. An element $e \in R$ is called **idempotent** if $e^2 = e$. A ring always have the idempotents 0 and 1, which are **trivial**. An idempotent e is **central** if $er = re$ for all $r \in R$.

Two idempotents e_1 and e_2 are **orthogonal** if $e_1e_2 = e_2e_1 = 0$. Finally, e is **primitive** if it cannot be written as $e_1 + e_2$ for any nonzero orthogonal idempotents e_1 and e_2. This is equivalent to that 0 and e are the only idempotents of eRe, since if we have some e' different from 0 and e, then $e = e' + (e - e')$ and conversely if e is not primitive, $e = e_1 + e_2$ where both e_1 and e_2 are idempotents other than 0 and e in eRe.

A ring R is **connected** or **indecomposable** if it cannot be written as $R \cong R_1 \times R_2$ for some unital nontrivial rings R_1 and R_2. Equivalently, R is connected if it has no nontrivial central idempotents. To see this equivalence, note that given a central idempotent $e \neq 0, 1$, we have $R \cong Re \times R(1 - e)$ and conversely, if $R \cong R_1 \times R_2$ then the units of R_1 and R_2 are both central idempotents in $R_1 \times R_2$.

A set of primitive pairwise orthogonal idempotents $\{e_1, \dots, e_n\}$ that sums to 1 is called a **complete set of primitive orthogonal idempotents**. Such a set is usually not unique.

Proposition 1. *Let R be a ring with a complete set $\{e_i\}_{i=1}^{n}$ of primitive orthogonal idempotents. R is connected if and only if there does not exist a nontrivial partition $I \cup J$ of $\{1, \dots, n\}$ such that for any $i \in I$ and $j \in J$ we have $e_iRe_j = e_jRe_i = 0$. In such a case, $\sum_{i\in I} e_i$ and $\sum_{j\in J} e_j$ are central idempotents of R.*

Proof. If: Assume that there is such a partition and let $c = \sum_{j\in J} e_j$. Since the e_j are orthogonal idempotents, $c^2 = \sum_{j\in J} e_j^2 = \sum_{j\in J} e_j = c$, and for each $i \in I$, we have $ce_i = e_ic = 0$ and for each $j \in J$ we have $ce_j = e_jc = e_j$. Now take any $r \in R$. We assumed $e_ire_j = e_jre_i = 0$ for any $i \in I$ and $j \in J$. Thus

$$\begin{aligned} cr &= \left(\sum_{j\in J} e_j\right) r = \left(\sum_{j\in J} e_j r\right) \cdot 1 = \left(\sum_{j\in J} e_j r\right) \left(\sum_{i\in I} e_i + \sum_{j'\in J} e'_j\right) \\ &= \left(\sum_{j\in J, i\in I} e_j r e_i\right) + \left(\sum_{j,j'\in J} e_j r e_{j'}\right) = \left(\sum_{j,j'\in J} e_j r e_{j'}\right) \\ &= \left(\sum_{i\in I} e_i + \sum_{j\in J} e_j\right) r \left(\sum_{j'\in J} e_{j'}\right) = rc \end{aligned}$$

and c is a central idempotent so $R = Rc \times R(1 - c)$ is a nontrivial decomposition of R.

Only if: Assume R is not connected, so it contains a central idempotent $c \neq 0, 1$. Then

$$c = 1 \cdot c \cdot 1 = \left(\sum_{l=1}^{n} e_l\right) c \left(\sum_{m=1}^{n} e_m\right) = \left(\sum_{m=1}^{n} e_m c e_m\right). \tag{10}$$

Let $c_m = e_m c e_m$. Then $c_m^2 = c_m$ so c_m is idempotent and since e_m is primitive, $c_m = e_m$ or $c_m = 0$. Now $I = \{i | c_i = 0\}$ and $J = \{j | c_j = e_j\}$ is a partition as the one needed. Since $c \neq 0, 1$, neither I nor J is empty, and for $i \in I$ we have $e_ic = ce_i = 0$ and for $j \in J$

we have $e_j c = c e_j = e_j$, so for any $i \in I, j \in J$ we have $e_i R e_j = e_i R c e_j = e_i c R e_j = 0$ and similarly for $e_j R e_i$.

□

The Jacobson Radical

Definition 3. The **Jacobson Radical** of the ring R, denoted by $\operatorname{rad} R$, is the intersection of all maximal right ideals in R.

Proposition 2. *The following conditions are equivalent:*

1. $r \in \operatorname{rad} R$.
2. r *is in the intersection of all maximal left ideals of* R.
3. *For every* $r' \in R$, $1 - rr'$ *has a right inverse.*
4. *For every* $r' \in R$, $1 - r'r$ *has a left inverse.*
5. *For every* $r' \in R$, $1 - rr'$ *has a two-sided inverse.*
6. *For every* $r' \in R$, $1 - r'r$ *has a two-sided inverse.*

Thus $\operatorname{rad} R$ *is the intersection of all maximal left ideals of* R*, so* $\operatorname{rad} R$ *is a two-sided ideal.*

Proof. Obviously, $5 \Rightarrow 3$ and $6 \Rightarrow 4$. We show $1 \Rightarrow 3$ by proving the contrapositive. Assume that $r \in R$ and there is an $r' \in R$ such that $1 - rr'$ does not have a right inverse. Then $1 - rr'$ generates a proper right ideal, which is then a subset of some maximal right ideal I. Now r cannot be in I, since then $rr' \in I$ and $1 \in I$ so $I = R$ is not maximal. Thus r is not in $\operatorname{rad} R$.

To show $3 \Rightarrow 1$, assume that $r \notin \operatorname{rad} R$. This means that we can find a maximal right ideal I such that $r \notin I$, and then $I + rR$ is a new right ideal that contains I as a proper subset, so $I + rR = R$. This means there are $i \in I$ and $r' \in R$ such that $i + rx = 1$, so $i = 1 - rr' \in I$ and thus does not have a right inverse. That $2 \iff 4$ is shown similarly.

Assume x is a two-sided inverse of $1 - rr'$. Then

$$(1 - r'r)(1 + r'xr) = 1 - r'r + r'xr - r'rr'xr = 1 - r'r + r'(1 - rr')xr = 1 \tag{11}$$

and

$$(1 + r'xr)(1 - r'r) = 1 - r'r + r'xr - r'xrr'r = 1 - r'r + r'x(1 - rr')r = 1 \tag{12}$$

so $1 + r'xr$ is a two-sided inverse of $1 - r'r$, which shows that $5 \iff 6$.

Now, assume 3 and let $r \in A$ be such that $1 - rr'$ has a right inverse for every r' and x be a right inverse of $1 - rr'$. Then $(1 - rr')x = 1 \Rightarrow x = 1 - r(-r'x)$, where the right-hand side must have a right inverse y. Then $1 = xy = y + rr'xy = y + rr'$ so $y = 1 - rr'$ and x is also a left inverse of $1 - rr'$, so $3 \Rightarrow 5$. Similarly, $4 \Rightarrow 6$. □

Corollary 1. *For any ring* R*, we have that* $\operatorname{rad}(R/\operatorname{rad} R) = \{0\}$*. Furthermore, if* I *is a two-sided nilpotent ideal of* R*, then* $I \subseteq \operatorname{rad} R$*.*

Proof. Each maximal ideal of R is also a maximal ideal of $R/\operatorname{rad} R$ under the canonical homomorphism from R to $R/\operatorname{rad} R$. The intersection of these is $\{0 + \operatorname{rad} R\}$, so $\operatorname{rad}(R/\operatorname{rad} R) = \{0\}$. For the second statement, let $I^n = \{0\}$ and $r \in I$. For every $r' \in R$

$$(1 + r'r + (r'r)^2 + \cdots + (r'r)^{n-1})(1 - r'r) = 1 - (r'r)^n = 1 \tag{13}$$

so $1 - r'r$ has a left inverse and $r \in \operatorname{rad} R$. Thus $I \subseteq \operatorname{rad} R$. □

Proposition 3. *If* $f : R \to S$ *is a surjective homomorphism of rings,* $f(\operatorname{rad} R) \subseteq \operatorname{rad} S$*.*

Proof. We use that $r \in \operatorname{rad} R \iff \forall r' \in R : 1 - rr'$ has a right inverse. Let's call this inverse $x_{r,r'}$. Now, for every $s \in S$, we know that we have a $s' \in R$ such that $f(s') = s$, so for every $r \in \operatorname{rad} R$

$$(1 - f(r)s)f(x_{r,s'}) = f(1 - rs')f(x_{r,s'}) = f((1 - rs')x_{r,s'}) = f(1) = 1. \tag{14}$$

This means that $f(r) \in \operatorname{rad} S$ for every $r \in \operatorname{rad} R$, so $f(\operatorname{rad} R) \subseteq \operatorname{rad} S$. □

Definition 4. A ring R with a complete set of primitive orthogonal idempotents $\{e_1, \ldots, e_n\}$ is **basic** if $e_i R \not\cong e_j R$ whenever $i \neq j$.

K-algebras

On vector spaces, we can sometimes define a multiplication such that they also become rings with this multiplication. Intuitively, a vector space over some field $\mathbf{K}$ that is also a ring is what is called a $\mathbf{K}$-algebra. The exact definition follows after the examples.

Example 1. The following are examples of $\mathbf{C}$-algebras.

1. $\mathbf{M}_n(\mathbf{C})$, the vector space of $n \times n$ matrices over $\mathbf{C}$, with the familiar matrix multiplication.

2. $\mathbf{C}[x]$, the vector space of polynomials over $\mathbf{C}$ in a variable x, with the usual polynomial multiplication.

3. If A is a $\mathbf{K}$-algebra, we can define a new $\mathbf{K}$-algebra A^{op} by using the same vector space structure but with the multiplication $*$ of A^{op} defined by $a * b = ba$.

4. Given any vector space V over some field $\mathbf{K}$, we can form the **endomorphism algebra** $\mathrm{End}(V)$, consisting of all linear maps from V to itself (called **endomorphisms**), where addition and scalar multiplication is given pointwise and multiplication is given by function composition. This is a generalisation of $\mathbf{M}_n(\mathbf{C})$ in the sense that $\mathbf{M}_n(\mathbf{C})$ is $\mathrm{End}(\mathbf{C}^n)$, given a basis.

Definition 5. A unital ring $(A, +, \cdot, 1)$ that is also a vector space over some field $\mathbf{K}$ is called a **K-algebra** if

$$\forall a, b \in A, \alpha \in \mathbf{K} : \alpha(ab) = (\alpha a)b = a(\alpha b). \tag{15}$$

In this work, we assume that $\mathbf{K}$ is algebraically closed.

Given two $\mathbf{K}$-algebras A and B, a **K-algebra homomorphism** is a ring homomorphism $\varphi : A \to B$ such that

$$\varphi(\alpha a) = \alpha\varphi(a) \tag{16}$$

for all $a \in A, \alpha \in \mathbf{K}$. If φ is bijective we call it an **isomorphism** and say that A and B are **isomorphic**. The set of all homomorphisms from A to B form a $\mathbf{K}$-vector space with pointwise addition and scalar multiplication and is denoted by $\mathrm{Hom}(A, B)$. We say that a vector subspace B of a $\mathbf{K}$-algebra A is a **K-subalgebra** if B contains the multiplicative identity of A and B is closed under multiplication.

Note 1. While we refer to the above as a $\mathbf{K}$-algebra, it is in many contexts worth studying similar structures that must not necessarily have associative multiplication nor a multiplicative unit. In such contexts the above definition is that of a *unital associative* $\mathbf{K}$*-algebra*

Of course, the definitions we did for rings, for example the definition of an ideal and of the Jacobson radical, also apply to $\mathbf{K}$-algebras.

Proposition 4. *If I is a two-sided nilpotent ideal of A such that A/I is isomorphic to $\mathbf{K}^n$ for some n, then $I = \mathrm{rad}\, A$.*

Proof. By Corollary 1, $I \subseteq \mathrm{rad}\, A$. To show $\mathrm{rad}\, A \subseteq I$, note that $\mathrm{rad}(A/I) = 0$ and let $\pi : A \to A/I$ be the canonical homomorphism. By Proposition 3, $\pi(\mathrm{rad}\, A) = 0$, so $\mathrm{rad}\, A \subseteq \ker \pi = I$ and thus $\mathrm{rad}\, A = I$. □

Modules of K-algebras

Given a $\mathbf{K}$-algebra A, we can define a **(left)** A**-module** as a $\mathbf{K}$-vector space M together with a $\mathbf{K}$-algebra homomorphism $\rho : A \to \mathrm{End}(M)$. The module shall be written simply as M and given $a \in A$ and $m \in M$, we shall usually write $\rho(a)(m)$ simply as am. The module M is **finite-dimensional** if M is finite-dimensional as a vector space. A subspace $M' \subseteq M$ is a **submodule** of M if $am' \in M'$ for all $a \in A$ and $m' \in M'$.

An A-**module homomorphism** between A-modules M and N is a linear map $\varphi : M \to N$ such that $\varphi(am) = a\varphi(m)$ for all $a \in A$ and $m \in M$. If φ is bijective we call it an **isomorphism**, and say that M and N are **isomorphic**. For every module M, there is a special homomorphism $\mathrm{id} : M \to M$ defined by $\mathrm{id}(m) = m$. We denote the vector space of module homomorphisms from M to N by $\mathrm{Hom}_A(M, N)$, and write $\mathrm{End}_A(M) = \mathrm{Hom}_A(M, M)$. This can be made into an algebra by defining multiplication as composition of maps. Note that $\mathrm{End}_A(M)$ is not at all the same as $\mathrm{End}(M)$! The maps in $\mathrm{End}_A(M)$ must respect multiplication by elements in A.

A module M is said to be **generated** by the set $X \subseteq M$ if every $m \in M$ can be written as $m = a_1x_1 + \cdots + a_nx_n$ for some $a_1, \ldots, a_n \in A$ and $x_1, \ldots, x_n \in X$. If there is a finite set $X \subseteq M$ such that M is generated by X, M is **finitely generated**.

Given two or more A-modules $M_1, \ldots, M_n$, we can define the **direct sum** of $M_1, \ldots, M_n$

$$\bigoplus_{i=1}^{n} M_i = M_1 \oplus \cdots \oplus M_n \tag{17}$$

as the direct sum of the vector spaces $M_1, \ldots M_n$ with the action of $a \in A$ given by

$$a(m_1, \ldots, m_n) = (am_1, \ldots, am_n). \tag{18}$$

An A-module M is said to be **indecomposable** if there is no decomposition $M = M_1 \oplus M_2$ where both M_1 and M_2 are non-zero.

Note 2. This above definition of a module is from [2]. Another equivalent definition is that a left A-module is a $\mathbf{K}$-vector space M together with a map $\cdot : A \times M \to M$ such that for $\alpha \in K$, $a, b \in A$, $m, n \in M$

$$(a + b)(m + n) = am + an + bm + bn \tag{19}$$

$$(ab)m = a(bm) \tag{20}$$

$$1m = m \tag{21}$$

$$\alpha(am) = (\alpha a)m = a(\alpha m) \tag{22}$$

Modules are also referred to as *representations* and are the prime objects of study in *representation theory*.

Every left ideal I can be viewed as a module over A where the elements of A acts by the ordinary left multiplication on I. In particular, A can be regarded as a module over itself, denoted ${}_AA$.

Example 2. We investigate $\mathrm{End}_A({}_AA)$. For each $a \in A$, there is a corresponding map $\varphi_a \in \mathrm{End}_A({}_AA)$, acting by $x \mapsto xa$. This is an endomorphism of ${}_AA$, since for any $b \in A$, we have $\varphi_a(bx) = (bx)a = b(xa) = b\varphi_a(x)$. These are also all the endomorphisms, since an endomorphism is uniquely defined by its value on 1:

$$\varphi(x) = \varphi(x1) = x\varphi(1).$$

Moreover, $\varphi_a \circ \varphi_{a'} = \varphi_{a'a}$. Thus, we have an isomorphism from the *opposite* algebra A^{op} to $\mathrm{End}_A({}_AA)$ given by $a \mapsto \varphi_a$.

If I is a left ideal of A and M is an A-module, the set of all linear combinations of the form $\sum_{n=1}^{N} i_n m_n$ with $i_n \in I$ and $m_n \in M$, denoted IM, is a submodule of M since for any $a \in A$ we have $a \sum_{n=1}^{N}(i_n m_n) = \sum_{n=1}^{N} a(i_n m_n) = \sum_{n=1}^{N}(a i_n)m_n \in IM$.

We can define the radical of a module similarly to the radical of a ring.

Definition 6. The **Jacobson Radical** of a module M of A is the intersection of all maximal submodules of M.

We note that $\mathrm{rad}\, A = \mathrm{rad}\, {}_AA$, so the radical of a module is a generalisation of the radical of a ring.

We will need the following basic fact about homomorphisms of modules:

Proposition 5. *Let* M *and* N *be* A*-modules and let* $\varphi : M \to N$ *be a homomorphism. Then the kernel of* φ *is a submodule of* M *and the image of* φ *is a submodule of* N*. Furthermore,* $M/\ker(\varphi) \cong N$.

Proof. Since $\varphi(0) = 0$, both $\ker\varphi$ and $\mathrm{im}\,\varphi$ are nonempty. If $m, m' \in \ker\varphi$, then $\varphi(am) = a\varphi(m) = 0$ and $\varphi(m+m') = \varphi(m)+\varphi(m') = 0$, so $\ker\varphi$ is a submodule of M. If $n, n' \in \mathrm{im}\,\varphi$, then $n = \varphi(m)$ and $n' = \varphi(m')$ for some $m, m' \in M$, so $an = a\varphi(m) = \varphi(am) \in \mathrm{im}\,\varphi$ and $n + n' = \varphi(n + n') \in \mathrm{im}\,\varphi$, so $\mathrm{im}\,\varphi$ is a submodule of N.

The isomorphism $M/\ker(\varphi) \to \mathrm{im}(\varphi)$ is given by $m + \ker(\varphi) \mapsto \varphi(m)$. It is obviously a homomorphism and injective, and since $\varphi(m) = \varphi(m') \implies \varphi(m - m') = 0 \implies m + \ker(\varphi) = m' + \ker(\varphi)$ it is also surjective, thus an isomorphism. ☐

The Unique Decomposition of ${}_AA$

Lemma 1 (Nakayama's Lemma). *Let* A *be an algebra and* M *a finitely generated module of* A*, and* $I \subseteq \mathrm{rad}\, A$ *be a two-sided ideal of* A*. If* $IM = M$*, then* $M = \{0\}$.

Proof. Assume that $M = IM$ and that at least $n > 0$ elements are needed to generate M. Let $\{m_1, \dots, m_n\}$ be such a set. That $M = IM$ implies that there are $a_1, \dots, a_n \in I$ such that

$$m_1 = \sum_{i=1}^{n} a_i m_i \Rightarrow (1 - a_i)m_1 = \sum_{i=2}^{n} a_i m_i \tag{23}$$

and since a_i is in rad A, we have that $(1 - a_i)$ is invertible and so m_1 is a linear combination of $m_2, \dots, m_n$ and M is generated by $m_2, \dots, m_n$, contradiction. Thus M does not require $n > 0$ elements to be generated, so $M = \{0\}$. □

Corollary 2. *For a finite-dimensional* **K***-algebra* A*, the radical* rad A *is nilpotent.*

Proof. Since rad A is finite-dimensional, $(\operatorname{rad} A)^n = (\operatorname{rad} A)^{n+1}$ for some n and by Nakayama's Lemma $(\operatorname{rad} A)^n = \{0\}$. □

Corollary 3. *If* $f \in \operatorname{Hom}_A(M, N)$*, then* $f(\operatorname{rad} M) \subseteq \operatorname{rad} N$.

Proof. Let S be a simple submodule of N and let $\varphi \in \operatorname{Hom}_A(N, S)$. Then $\varphi \circ f \in \operatorname{Hom}_A(M, S)$, so $\varphi \circ f(\operatorname{rad} M) = 0$, thus $f(\operatorname{rad} M) \subseteq \operatorname{rad} N$. □

Proposition 6. *Let* A *be a* **K***-algebra with some idempotent* e*, and let* M *be an* A*-module. The map*

$$\theta_M : \operatorname{Hom}_A(Ae, M) \to eM \tag{24}$$

defined by $\theta_M(\varphi) = e\varphi(e)$ *is an isomorphism of vector spaces.*

Proof. It is clear that θ_M is a linear map. Its inverse is given by $\theta_M^{-1}(em)(x) = xem$, since $\theta_M(\theta_M^{-1}(em)) = e(\theta_M^{-1}(em)(e)) = eeem = em$ and $\theta_M^{-1}(\theta_M(\varphi))(x) = \theta_M^{-1}(e\varphi(e))(x) = xe\varphi(e) = \varphi(x)$. □

The lemma and theorem below, as well as the proofs, are from [2].

Definition 7. Let T be an operator on some vector space V and let λ be an eigenvalue of T. The **generalised eigenspace** of T corresponding to λ consists of the vectors such that $(T - \lambda \operatorname{id})^k v = 0$ for some k. This space is the kernel of $(T - \lambda \operatorname{id})^k$ for some large enough k, and we can decompose V into a direct sum of generalised eigenspaces of T.

Lemma 2. *Let* A *be a finite-dimensional* **K***-algebra and* M *an indecomposable* A*-module. Then any homomorphism* θ *from* M *to itself is either an isomorphism or nilpotent. Furthermore, the sum of any set of nilpotent homomorphisms from* M *to itself is nilpotent.*

Proof. The generalised eigenspaces of θ are all submodules of M. Thus, since M is indecomposable, θ can only have a single eigenvalue λ. If $\lambda = 0$, then θ is nilpotent, else it is an isomorphism.

To show that a sum of nilpotent homomorphisms is nilpotent, it is enough to show that a sum of two nilpotent homomorphisms are nilpotent. Let θ_1 and θ_2 be nilpotent homomorphisms. Now, if $\theta = \theta_1 + \theta_2$ is not nilpotent, it is an isomorphism. In such a case, $\theta^{-1}\theta = \theta^{-1}\theta_1 + \theta^{-1}\theta_2 = \operatorname{id}$, and since $\theta^{-1}\theta_1$ is not an isomorphism $\operatorname{id} - \theta^{-1}\theta_2 = \theta^{-1}\theta_1$ is nilpotent.

Let m be a vector such that $\theta_2(m) = 0$. Then $(\mathrm{id} - \theta^{-1}\theta_2)(m) = m$, so $(\mathrm{id} - \theta^{-1}\theta_2)^k(m) = m$ for all $k > 0$, a contradiction.

Thus $\theta = \theta_1 + \theta_2$ is nilpotent. □

Theorem 1. *Let A be a $\mathbf{K}$-algebra. Every finite-dimensional A-module M has a unique decomposition into indecomposable modules, meaning that $M \cong \bigoplus_{i=1}^{n} M_i$ for some set of indecomposable modules $\{M_i\}_{i=1}^{n}$. If there is some other decomposition $M \cong \bigoplus_{i=1}^{n'} M'_i$ into indecomposable modules, then $n = n'$ and there is some permutation π of $\{1, \dots, n\}$ such that $M_i \cong M'_{\pi(i)}$ for all i.*

Proof. That there is a decomposition of M is clear by induction. Any module is either indecomposable, or it's possible to write it as a sum of two modules of smaller dimension.

Now let $i_s : M_s \to M$ and $i'_s : M'_s \to M$ be the natural injections from M_s and M'_s to M, and let $p_s : M \to M_s$ and $p'_s : M \to M'_s$ be the natural projections. To see that two decompositions must be equivalent, we proceed by induction on n.

Let $\theta_s = p_1 i'_s p'_s i_1 : M_1 \to M_1$. Then $\sum_{s=1}^{n} \theta_s = \mathrm{id}$. By the above lemma, θ_s is an isomorphism for some s.

We find that $M'_s \cong \mathrm{im}(p'_s i_1) \oplus \ker(p_1 i'_s)$, where an isomorphism is given by

$$\varphi(m) = (p'_s i_1 \theta_s^{-1} p_1 i'_s(m), m - p'_s i_1 \theta_s^{-1} p_1 i'_s(m)). \tag{25}$$

It is clear that the first component is in $\mathrm{im}(p'_s i_1)$, and the second component is in $\ker(p_1 i'_s)$ since $p_1 i'_s(m - p'_s i_1 \theta_s^{-1} p_1 i'_s(m)) = p_1 i'_s(m) - \theta_s \theta_s^{-1} p_1 i'_s(m) = 0$. An inverse of φ is given by $\varphi^{-1}(m_1, m_2) = m_1 + m_2$.

Since M_1 is indecomposable and $p_1 i'_s p'_s i_1$ is an isomorphism, $\ker(p_1 i'_s) = \{0\}$. Thus, $p'_s i_1 : M_1 \to M'_s$ and $p_1 i'_s : M'_s \to M_1$ are isomorphisms.

Now let $B = \bigoplus_{i=2}^{n} M_i$ and $B' = \bigoplus_{i=1, i \neq s}^{n'} M'_i$ so that $M \cong M_1 \oplus B \cong M'_s \oplus B'$ and let h be the composition of the injection $B \to M$ and the projection $M \to B'$. Since B and B' are of the same dimension, h is an isomorphism if $\ker h = \{0\}$. Let ν be in B and $h(\nu) = 0$. Then $\nu \in M'_s$, so $p_1 i'_s(\nu) = 0$. Since $p_1 i'_s$ is an isomorphism, it follows that $\nu = 0$. Thus $B \cong B'$ and by the induction assumption, the two decompositions are the same up to isomorphism of the summands and permutation of the indices. □

Simple and Semisimple Modules

This section is mainly based on [2].

Definition 8. A module without proper submodules is called **simple**. A module that is a direct sum of simple modules is **semisimple**. Similarly, an algebra such that any module of it is semisimple is a semisimple algebra.

Example 3. Let A be an algebra. Given a simple module M of dimension n, a semisimple module of A is given by $\mathrm{End}(M)$, with the action defined by left multiplication of A.

$\mathrm{End}(M)$ is isomorphic to nM, which is defined as $M \oplus M \oplus \cdots \oplus M$ (n times). Let $m_1, \ldots, m_n$ be a basis of M. Then an isomorphism $\mathrm{End}(M) \to nM$ is given by $x \mapsto (xm_1, xm_2, \ldots, xm_n)$.

Lemma 3 (Schur's Lemma)**.** *Let M and N be modules of some algebra A, and let $\varphi : M \to N$ be a homomorphism that is not identically zero. If M is simple, then φ is injective. If N is simple, then φ is surjective. Thus, if they are both simple, φ is an isomorphism. In that case, $\varphi = \lambda\,\mathrm{id}$ for some $\lambda \in \mathbf{K}$.*

Proof. $\ker\varphi$ is a submodule of M, so if M is simple and φ not identically zero, $\ker\varphi = \{0\}$ so φ is injective. In the same way, $\mathrm{im}\,\varphi$ is a submodule of N so it must be N. In the case that φ is an isomorphism, let λ be an eigenvalue, which exists since $\mathbf{K}$ is algebraically closed. Then $\varphi - \lambda\,\mathrm{id}$ is a homomorphism $M \to N$ that is not an isomorphism. Thus it is zero, so $\varphi = \lambda\,\mathrm{id}$. □

Proposition 7. *Let M be a module of some algebra A. Then $\mathrm{rad}\,M$ is the set of all $m \in M$ such that for any simple module S of A and any $\varphi \in \mathrm{Hom}_A(M, S)$, we have $\varphi(m) = 0$.*

Proof. The maximal submodules of M are precisely those T such that M/T is simple. Let the set of m such as above be denoted I. Any simple submodule of M can be obtained as M/T for some maximal T, so if $m \in \mathrm{rad}\,M \subseteq T$, then $m \equiv 0 \mod T$, meaning $m \in I$. Thus $\mathrm{rad}\,M \subseteq I$. Now, if $m \in I$, then $m \in \mathrm{rad}\,M$ since if it was not, we would have $m \notin N$ for some maximal submodule N of M. Then $m \not\equiv 0 \mod N$, so m is not in the kernel of the natural surjection $M \to M/N$ and so $m \notin I$, meaning $I \subseteq \mathrm{rad}\,M$. □

For the special case of the radical of A itself, this means that $a \in \mathrm{rad}(A)$ if and only if a acts by 0 on every simple module of A.

Now we completely determine the finite-dimensional modules of matrix algebras (that is, the algebras $\mathbf{M}_n(\mathbf{K})$), and sums of matrix algebras. First note that if $A = \bigoplus_{i=1}^{n} A_i$ is an algebra where the units of each summand is $1_1, 1_2, \ldots, i_n$, and if we have some module M of A, then

$$M = 1_1M \oplus 1_2M \oplus \ldots \oplus 1_nM$$

where 1_iM is a module of A_i, and the action of $a = (a_1, a_2, \dots, a_n) \in A$ on $m_1, m_2, \dots, m_n$ is given by $(a_1m_1, a_2m_2, \dots, a_nm_n)$. Note also that M is simple (or irreducible) if and only if M_i is zero for all i except for one, which is simple (or irreducible) in A_i.

Proposition 8. *The simple modules of* $A = \bigoplus_{i=1}^{r} \mathbf{M}_{n_i}(\mathbf{K})$ *are* $\mathbf{K}^{n_1}, \mathbf{K}^{n_2}, \dots, \mathbf{K}^{n_r}$. *Moreover,* $\operatorname{rad}(A) = \{0\}$.

Proof. By the above comment, we may consider each component separately. Clearly, $\mathbf{K}^n$ is a simple module of $\mathbf{M}_n(\mathbf{K})$, since the map $\rho : \mathbf{M}_n(\mathbf{K}) \to \operatorname{End}(\mathbf{K}^n)$ defining the action on $\mathbf{K}^n$ is surjective. Since the map is even bijective, $\operatorname{rad}(A) = 0$ by Proposition 7. □

Lemma 4. *Let* $\{M_i\}_{i=1}^{n}$ *be pairwise non-isomorphic finite-dimensional simple modules of* A, *and let* N *be a submodule of* $M = \bigoplus_{i=1}^{n} k_iM_i$. *Then* $N \cong \bigoplus_{i=1}^{n} l_iM_i$ *where* $l_i \leqslant k_i$. *The inclusion* $\varphi : N \to M$ *is a direct sum of inclusions* $\varphi_i : l_iM_i \to k_iM_i$ *given by* $\varphi_i(m_1, \dots, m_{l_i}) = X_i(m_1, \dots, m_{l_i})^T$ *where* X_i *is some* k_i*-by-*l_i *matrix with linearly independent columns.*

Proof. All submodules of N must be submodules of M, so the statement that $N \cong \bigoplus_{i=1}^{n} l_iM_i$ follows from Theorem 1. To find the matrix corresponding to the inclusion, we consider the inclusion φ of l_iM_i into k_iM_i. Now, consider $\varphi(n_1, \dots, n_{l_i}) = (m_1, \dots, m_{k_i}) \in M_i$. The element m_1 will be given by the sum

$$\varphi_{1,1}(n_1) + \varphi_{1,2}(n_2) + \cdots + \varphi_{1,l_i}(n_{l_i})$$

where each φ is an automorphism of M_i, and is thus given by $\lambda\,\mathrm{id}$ for some $\lambda \in \mathbf{K}$ by Schur's Lemma. We thus find a matrix such that $\varphi(n_1, \dots, n_{l_i}) = X(m_1, \dots, m_{k_i})^T$. This matrix must have linearly independent columns since $(\dim M_i) \operatorname{rank} X = \dim \operatorname{im} \varphi = \dim l_iM_i$. □

Lemma 5. *Let* M *be a finite-dimensional simple* A*-module and let* $m_1, \dots, m_n \in M$ *be any set of linearly independent vectors. For any set* $m'_1, \dots, m'_n \in M$, *there is an* $a \in A$ *such that* $am_i = m'_i$ *for all* $1 \leqslant i \leqslant n$.

Proof. Assume there is a vector $(m'_1, \dots, m'_n \in M)$ such that there is no a fulfilling the theorem. Then the image of the map $A \to nM$ given by $a \mapsto (am_1, \dots, am_n)$ is a proper submodule of M, corresponding to some n-by-r matrix X with $r < n$ by Lemma 4. Thus, for $a = 1$, there are vectors $u_1, \dots, u_n \in M$ such that $X(u_1, \dots, u_n)^T = (m_1, \dots, m_n)^T$. Since $r < n$, there are nonzero vectors $q_1, \dots, q_n$ such that $(q_1, \dots, q_n)X = 0$. Then $\sum q_im_i = (q_1, \dots, q_n)X(u_1, \dots, u_n)^T = 0$, which contradicts the linear independence of $m_1, \dots, m_n$. □

Theorem 2. *1. Let* M *be a finite-dimensional simple module of* A *given by the map* $\rho : A \to \operatorname{End}(M)$. *Then* ρ *is surjective.*

2. *Let* M *be a semisimple module with decomposition* $M = \bigoplus_{i=1}^{r} M_i$ *where* $\{M_i\}$ *are pairwise non-isomorphic simple modules given by maps* $\{\rho_i\}$*. Then the map*

$$\rho = \bigoplus_{i=1}^{r} \rho_i : A \to \bigoplus_{i=1}^{r} \mathrm{End}(M_i) \tag{26}$$

is surjective.

Proof. 1. Let $B = \operatorname{im} \rho$. The claim is then that $B = \mathrm{End}(M)$. Take $c \in \mathrm{End}(M)$ and let $m_1, \ldots, m_n$ be a basis of M. By Lemma 5, there is an $a \in A$ such that $av_i = cv_i$ for $1 \leqslant i \leqslant n$. Then $\rho(a) = c$, so c is in B.

2. By Example 3, $\bigoplus_{i=1}^{r} \mathrm{End}(M_i)$ is semisimple. Let $B = \operatorname{im} \rho$. By Lemma 4, $B = \bigoplus_{i=1}^{r} B_i$, where B_i is a submodule of $\mathrm{End}(M_i)$. Now the first part of the theorem tells us that $B_i = \mathrm{End}(M_i)$, so $B = \mathrm{End}(M)$.

□

Proposition 9. *A finite-dimensional algebra* A *has only finitely many simple modules* M_i *up to isomorphism, all those are finite-dimensional, and*

$$A/\operatorname{rad} A \cong \bigoplus_{i} \mathrm{End}(M_i). \tag{27}$$

Proof. Any simple module M is generated by any nonzero $m \in M$, so M is finite-dimensional. To show that there are finitely many non-isomorphic simple modules of A, assume that $M_1, \ldots, M_n$ are non-isomorphic simple modules of A. The map

$$\bigoplus_{i=1}^{n} \rho_i : A \to \bigoplus_{i=1}^{n} \mathrm{End}(M_i) \tag{28}$$

is surjective, so $\dim A \geqslant \sum_{i=1}^{n} \dim \mathrm{End}(M_i) \geqslant n$, thus there are only finitely many non-isomorphic simple modules of A. Lastly, The kernel of $\bigoplus_{i=1}^{n} \rho_i$ is $\operatorname{rad} A$ by Proposition 7, so the theorem follows from Proposition 5. □

Note that $\dim \bigoplus_{i=1}^{n} \mathrm{End}(M_i) = \bigoplus_{i=1}^{n} (\dim M_i)^2$, so $\bigoplus_{i=1}^{n} (\dim M_i)^2 \leqslant \dim A$, with equality if and only if $\operatorname{rad} A = \{0\}$.

We summarise our findings in a theorem.

Theorem 3 (The Artin-Wedderburn Theorem)**.** *Let* A *be a finite-dimensional algebra. The following conditions are equivalent:*

1. A *is semisimple.*

2. ${}_AA$ *is semisimple.*

3. $\operatorname{rad} A = \{0\}$.

4. $\sum_M (\dim M)^2 = \dim A$, *where sum is over the simple modules of* A.

5. $A \cong \bigoplus_i \mathbf{M}_{n_i}(\mathbf{K})$ *from some finite set of positive integers* n_i.

Proof. By Proposition 9, 3 $\iff$ 4, and if $\operatorname{rad} A = \{0\}$ then the map is an isomorphism so 3 $\implies$ 5.

By Proposition 8, 5 $\implies$ 1 and 5 $\implies$ 3.

By definition, 1 $\implies$ 2. We need to show that 2 $\implies$ 5 to conclude the theorem. Note that ${}_AA = \bigoplus_{i=1}^{k} n_i V_i$ where V_i are simple non-isomorphic A-modules. We consider $\operatorname{End}_A({}_AA)$. We know that $\operatorname{End}_A(V_i) = \mathbf{K}$ by Schur's Lemma, so $\operatorname{End}_A(n_i V_i) = \mathbf{M}_{n_i}(\mathbf{K})$. It also follows from Schur's Lemma that an endomorphism cannot map a simple module to another simple module, so and $\operatorname{End}_A({}_AA) \cong \bigoplus_{i=1}^{k} \mathbf{M}_{n_i}(\mathbf{K})$. By Exercise 2, $A^{op} \cong \bigoplus_{i=1}^{k} \mathbf{M}_{n_i}(\mathbf{K})$, and by taking the opposite of both sides,

$$(A^{op})^{op} \cong \left(\bigoplus_{i=1}^{k} \mathbf{M}_{n_i}(\mathbf{K}) \right)^{op}.$$

Since $\mathbf{M}_{n_i}(\mathbf{K})^{op} \cong \mathbf{M}_{n_i}(\mathbf{K})$, with an isomorphism given by transposition, and $(A^{op})^{op} \cong A$, we have that

$$A \cong \bigoplus_{i=1}^{k} \mathbf{M}_{n_i}(\mathbf{K}).$$

□

Corollary 4. *For any module* M *of* A, *we have* $\operatorname{rad} M = (\operatorname{rad} A)M$. *This implies that*

$$\operatorname{rad}^n M = (\operatorname{rad} A)^n M. \tag{29}$$

Proof. For any $m \in M$, the map $a \mapsto am$ is an A-module homomorphism ${}_AA \to M$, showing that $(\operatorname{rad} A)m \subseteq \operatorname{rad} M$, so $(\operatorname{rad} A)M \subseteq \operatorname{rad} M$.

Now consider $M/(\operatorname{rad}(A)M)$. This is an $A/\operatorname{rad} A$-module and is thus semisimple, since $A/\operatorname{rad} A$ is semisimple. The map $\pi : M \to M/(\operatorname{rad}(A)M)$ must then map $\operatorname{rad} M$ to 0 by Proposition 7, and so $\operatorname{rad} M \subseteq \ker \pi = \operatorname{rad}(A)M$. □

Lemma 6. *A finite-dimensional algebra* A *is basic if and only if* $A/\operatorname{rad} A \cong \mathbf{K} \times \mathbf{K} \times \cdots \times \mathbf{K}$. *By the Artin-Wedderburn Theorem, this is equivalent to saying that all simple modules are of dimension* 1.

Proof. This follows from the fact that $Ae_i \cong Ae_j \iff A/\operatorname{rad} Ae_i \cong A/\operatorname{rad} Ae_j$, which is proven in [1]. □

Theorem 4 (The Wedderburn-Malcev Theorem)**.** *Let* A *be a finite-dimensional* $\mathbf{K}$*-algebra. There exists a* $\mathbf{K}$*-subalgebra* B *of* A *such that* A *as a vector space is equal to* $B \oplus \operatorname{rad} A$ *and the restriction of the canonical homomorphism* $\pi : A \to A/\operatorname{rad} A$ *to* B *is a* $\mathbf{K}$*-algebra isomorphism.*

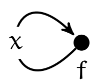
Proof. This is shown in [3]. □

Quivers

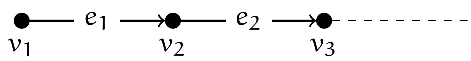
Definition 9. A **quiver** is a directed graph. Formally, it is a quadruple $Q = (V, E, s, t)$ where V and E are sets and s and t are maps from E to V. We call the elements of V the **vertices** of the quiver and the elements of E the **arrows** or **edges**. Q is finite if both V and E are finite sets. We call $s(e)$ the **source** and $t(e)$ the **target** of the arrow e.

Example 4. The following are some examples of quivers

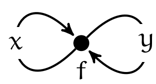
1. The quiver $Q = (\{f\}, \{x\}, x \mapsto f, x \mapsto f)$, illustrated as



2. The quiver $Q' = (\{v_i\}_{i=1}^{\infty}, \{e_i\}_{i=1}^{\infty}, e_i \mapsto v_i, e_i \mapsto v_{i+1})$, drawn as



3. We often give quivers only by their graphical representation. Let Q'' be given as



The Path Algebra

Definition 10. A **path** of **length** $l \geqslant 1$ from a to b in a quiver Q is a sequence

$$a|\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_l|b \tag{30}$$

such that $s(\alpha_1) = a, t(\alpha_i) = s(\alpha_{i+1}), t(\alpha_l) = b$ for $1 \leqslant i < l$. We call a the **source** and b the **target** of the path. For each $a \in V$, we also have the **stationary path**, denoted

$$\varepsilon_a = a|a \tag{31}$$

which is said to have length 0. A path of length $l \geqslant 1$ is called a cycle if its source and target coincide and a cycle of length 1 is called a **loop**. An **acyclic** quiver is one without cycles.

We can concatenate two paths $a|\alpha_1, \dots, \alpha_n|b$ and $c|\beta_1, \dots, \beta_m|d$ if $b = c$; we take the path with the source a, target d and the concatenation of the arrow sequences as its arrow sequence. Symbolically:

$$(a|\alpha_1, \dots, \alpha_n|b)(c|\beta_1, \dots, \beta_m|d) = a|\alpha_1, \dots, \alpha_n\beta_1, \dots, \beta_m|d. \tag{32}$$

This is again a path, since $t(\alpha_n) = b = c = s(\beta_1)$. We use this operation form a new algebra from a quiver.

Definition 11. The **path algebra** $\mathbf{K}Q$ of a quiver Q is the $\mathbf{K}$-algebra with the set of all paths of Q as its vector space basis and multiplication defined by concatenation of paths extended by linearity, meaning that

$$\left(\sum_{i=1}^{n} \alpha_i a_i\right)\left(\sum_{j=1}^{n} \beta_i b_i\right) = \sum_{1 \leqslant i,j \leqslant n} \alpha_i \beta_j (a_i b_j) \tag{33}$$

where $a_i b_j$ is the concatenation of the path a_i with the path b_j if possible, else 0. We often denote paths by their arrow sequences.

Example 5. In $\mathbf{C}Q$ with Q defined as above, we add and multiply elements as in the polynomial algebra $\mathbf{C}[x]$, with the unit polynomial given by $x^0 = \varepsilon_f$. Similarly, $\mathbf{C}Q'$ is isomorphic to $\mathbf{C}\langle x, y\rangle$, the polynomials over $\mathbf{C}$ in two non-commuting indeterminates. This means that $xy \neq yx$ in $\mathbf{C}Q''$ and so $\mathbf{C}Q''$ is an example of an algebra that is not commutative.

Let $\mathbf{K}Q_i$ be the subspace generated by all paths of length i in Q. Then

$$\mathbf{K}Q = \bigoplus_{i=0}^{\infty} \mathbf{K}Q_i \tag{34}$$

and $(\mathbf{K}Q_m) \cdot (\mathbf{K}Q_n) \subseteq \mathbf{K}Q_{m+n}$ for all $m, n \geqslant 0$.

Proposition 10. 1. $\mathbf{K}Q$ *has an identity element if and only if the set of vertices* V *is finite. This identity is then given by* $\sum_{v \in V} \varepsilon_v$.

2. $\mathbf{K}Q$ *is finite-dimensional if and only if* V *is finite and* Q *is acyclic.*

3. *If* Q *is finite the set* $\{\varepsilon_a : a \in V\}$ *is a complete set of orthogonal idempotents of* $\mathbf{K}Q$.

Proof. 1. If V has finitely many elements $v_1, \dots, v_n$ the element $1 = \sum_{i=1}^{n} \varepsilon_i$ is an identity. If V is infinite, there cannot be an identity 1 since it can only be a linear combination of finitely many paths that can only have one target each, so there is some $a \in V$ such that $1\varepsilon_a = 0$, contradiction.

2. If Q is finite and acyclic, each path has length at most $|V| - 1$ and there are at most $|E|$ choices for each edge in a path, so the total number of paths is finite.

   Conversely, if Q is infinite, either V is infinite, in which case we have an infinite number of stationary paths, or E is infinite, in which case we have an infinite number of paths of length 1. In either case, $\mathbf{K}Q$ is infinite-dimensional. If Q has a cycle c, there are infinitely many paths of the form c^k for natural number k so $\mathbf{K}Q$ is again infinite-dimensional.

3. All the stationary paths are obviously orthogonal idempotents. Since they sum to 1, we only need to show that they are primitive, meaning that 0 and ε_a are the only idempotents of $\varepsilon_a \mathbf{K}Q \varepsilon_a$. The elements of $\varepsilon_a \mathbf{K}Q \varepsilon_a$ must be of the form $x = \alpha\varepsilon_a + w$ where $\alpha \in \mathbf{K}$ and w is a linear combination of cycles through a. If x is an idempotent, $0 = x^2 - x = (\alpha^2 - \alpha)\varepsilon_a + (2\alpha - 1)w + w^2$ so $w = 0$ and $\alpha^2 - \alpha = 0$, so α is 0 or 1 and x is ε_a or 0.

□

Theorem 5. *Let* $Q = (V, E, s, t)$ *be a finite connected quiver and* A *an associative unital* $\mathbf{K}$*-algebra. Let* $\varphi_V : V \to A$ *and* $\varphi_E : E \to A$ *be such that:*

1. $\sum_{v \in V} \varphi_V(v) = 1$.

2. *for* $v, v' \in V$*:*
$$\varphi_V(v)\varphi_V(v') = \begin{cases} \varphi_V(v) & v = v' \\ 0 & v \neq v' \end{cases}. \tag{35}$$

3. *for every* $e \in E : \varphi_E(e) = \varphi_V(s(e))\varphi_E(e)\varphi_V(t(e))$.

Then there is a unique $\mathbf{K}$*-algebra homomorphism* $\varphi : \mathbf{K}Q \to A$ *such that* $\varphi(\varepsilon_v) = \varphi_V(v)$ *for* $v \in V$ *and* $\varphi(e) = \varphi_E(e)$ *for* $e \in E$.

Proof. To show uniqueness, note that for $v \in V$, the value of $\varphi(\varepsilon_v)$ must be $\varphi_V(v)$, and similarly for any path $e_1, \ldots, e_n$ we must have $\varphi(e_1, \ldots, e_n) = \varphi_E(e_1) \cdots \varphi_E(e_n)$. Thus we can have at most one such homomorphism φ. On the other hand, extending this φ linearly, we see that it preserves products and unit, so it is clearly a homomorphism. $\square$

Definition 12. Let Q be a finite connected quiver. The two-sided ideal generated by all arrows of $\mathbf{K}Q$, denoted R_Q or simply R if Q is clear from context, is called the **arrow ideal** of $\mathbf{K}Q$.

R_Q is thus spanned by the set all paths of length at least one, and we have a vector space decomposition

$$R_Q = \bigoplus_{i \geqslant 1} \mathbf{K}Q_i. \tag{36}$$

It follows that $R_Q^l = \bigoplus_{i>l} \mathbf{K}Q_i$ and $R_Q^l / R_Q^{l+1} \cong \mathbf{K}Q_l$ as a vector space.

Proposition 11. *Let* Q *be an acyclic finite quiver. Then* $R = \operatorname{rad} \mathbf{K}Q$.

Proof. Since $\mathbf{K}Q/R \cong \mathbf{K}Q_0 \cong \mathbf{K}^{|V|}$ where $\mathbf{K}Q_0$ is is the subalgebra of $\mathbf{K}Q$ generated by the stationary paths and V is the vertices of Q, it follows from Proposition 4 that since R is nilpotent, $R = \operatorname{rad} \mathbf{K}Q$. $\square$

Quiver algebras

Definition 13. Let Q be a finite quiver and R be the arrow ideal of $\mathbf{K}Q$. A two-sided ideal $I \subseteq \mathbf{K}Q$ is **admissible** if

$$R^m \subseteq I \subseteq R^2 \tag{37}$$

for some $m \geqslant 2$. If I is an admissible ideal of $\mathbf{K}Q$, we say that (Q, I) is a **bound quiver**, and than $\mathbf{K}Q/I$ is a **quiver algebra**.

Example 6. The ideal R^m is admissible for every $m \geqslant 2$. If Q is acyclic, any ideal contained in R^2 is admissible since R is nilpotent.

Let Q'' be as in Example 4. Then $I = \langle xy - yx \rangle$ is an admissible ideal of $\mathbf{K}Q''$ and $\mathbf{K}Q''/I \cong \mathbf{K}[x, y]$, the algebra of polynomials in two commuting indeterminates.

We usually define admissible ideals in term of its generators, which we call *relations*.

Definition 14. A **relation** in Q with coefficients in $\mathbf{K}$ is a $\mathbf{K}$-linear combination of paths of length at least two having the same source and target. We write it as

$$\rho = \sum_{i=1}^{m} \lambda_i w_i \tag{38}$$

where $\lambda_i \in \mathbf{K}$ are all non-zero and w_i are paths, all with the same source and target.

If $m = 1$, we call our relation a **monomial relation**, if $m = 2$, it is a **commutativity relation**.

Proposition 12. *Let* Q *be a finite quiver bound by* I. *Then* $\mathbf{K}Q/I$ *is finite-dimensional.*

Proof. Let R be the arrow ideal of $\mathbf{K}Q$ and let $m \geqslant 2$ be an integer such that $R^m \subseteq I$. Since we can have a basis of $\mathbf{K}Q/I$ of paths of length at most m and there are only finitely many edges of Q, there are only finitely many paths in the basis, so $\mathbf{K}Q/I$ is finite-dimensional. □

Proposition 13. *Given a finite quiver* Q *and an admissible ideal* I *of* Q, *there is a set of relations generating* I.

Proof. Let $V = \{v_1, v_2, \dots, v_n\}$ be the vertex set of Q. I is generated by some set $\{a_j\}_{j \in J}$ for some index set J. We can decompose each a_j into relations by $a_j = \sum_{k=1}^{n} \varepsilon_{v_k} a_j \varepsilon_{v_k}$. Each of the summands is then either a relation or 0, and $I = \langle \varepsilon_{v_k} a_j \varepsilon_{v_k} | 1 \leqslant k \leqslant n, j \in J \rangle$. □

Lemma 7. *Let* $\mathbf{K}Q/I$ *be a bound quiver algebra. The set* $\{e_a = \varepsilon_a + I\}$ *is a complete set of primitive orthogonal idempotents of* $\mathbf{K}Q/I$.

Proof. By Proposition 10, $\{\varepsilon_a\}$ is a complete set of orthogonal idempotents of $\mathbf{K}Q$. Thus $e_a e_b = (\varepsilon_a \varepsilon_b) + I$, which is $0 + I$ when $a \neq b$. and the set $\{e_a\}$ sums to $1 + I$. We must check that they are primitive, meaning that e_a and $0 + I$ are the only idempotents of $e_a(\mathbf{K}Q/I)e_a$. Let e be an idempotent of $e_a(\mathbf{K}Q/I)e_a$, which must then be of the form $\lambda\varepsilon_a + w + I$ for some $\lambda \in \mathbf{K}$ and linear combination w of cycles through a. Since $e^2 - e = 0$,

$$(\lambda^2 - \lambda)\varepsilon_a + (2\lambda - 1)w + w^2 + I = 0 + I, \tag{39}$$

and since all paths in I has length at least 2, $\lambda^2 = \lambda$ and so λ is 0 or 1. If $\lambda = 0$, then $e = w$. Since I is admissible, $e^m = 0 + I$ for some m, and since e is idempotent, $e = e^m = 0 + I$. If $\lambda = 1$, then $e - e_a = w + I$ is an idempotent, since $(e - e_a)^2 = e^2 - 2ee_a + e_a^2 = e - e_a$, and as above, w must be $0 + I$. Thus $e - e_a = 0 + I$, so $e = e_a$. □

Lemma 8. *Let* $Q = (V, E, s, t)$ *be a finite quiver bound by* I. *The following conditions are equivalent:*

1. Q *is connected.*
2. $\mathbf{K}Q$ *is connected.*
3. $\mathbf{K}Q/I$ *is connected.*

Proof. If $Q = (V, E, s, t)$ is not connected, it has a connected component with some vertex set $V' \neq V$. Then $\sum_{v \in V'} \varepsilon_v$ is a central idempotent in KQ different from 0 and 1, so KQ is not connected.

If KQ is not connected, KQ has some central idempotent $c \neq 0, 1$ that is a sum of stationary paths by Proposition 1. Then $c + I$ a central idempotent in KQ/I, and it is not $0 + I$ (since $c \notin I$), and not $1 + I$, so KQ/I is not connected.

By proposition 1, if KQ/I is not connected there is be a partition of $I \cup J$ of V with I and J nonempty such that for every $i \in I$ and $j \in J$, we have that $\varepsilon_i(KQ/I)\varepsilon_j = \varepsilon_j(KQ/I)\varepsilon_i = 0$, so there cannot be any arrows between a vertex in I to a vertex in J, so Q is not connected. □

Lemma 9. *Let Q be a finite quiver bound by I and let R be the arrow ideal of KQ. Then $\operatorname{rad}(KQ/I) = R/I$ and KQ/I is basic.*

Proof. Let m be an integer such that $R^m \subseteq I$. Then $(R/I)^m = 0$, so R/I is nilpotent. Since $(KQ/I)/(R/I) \cong KQ/R$, it follows from Proposition 4 that $\operatorname{rad} KQ/I = R/I$, and by Lemma 6, KQ is basic. □

Corollary 5. *We have that $\operatorname{rad}^l(KQ/I) = (R/I)^l$ for positive integers l.*

From the above lemmata, we can conclude that $\left\{w + \operatorname{rad}^2(KQ/I) | w \in E\right\}$ is a basis of the vector space $\operatorname{rad}(KQ/I)/\operatorname{rad}^2(KQ/I)$.

Historical Digression: Gabriel's Theorem

The term *quiver* were first introduced by Peter Gabriel in [4]. He chose to use a new term rather than *directed graph* to emphasise the study of *representations of quivers*, rather than any of the many other aspects of directed graphs. This section is an informal discussion of his paper.

In his paper, he determined exactly which quivers have finitely many nonequivalent representations, a result known as *Gabriel's Theorem*, given below without proof. First, we shall need some definitions.

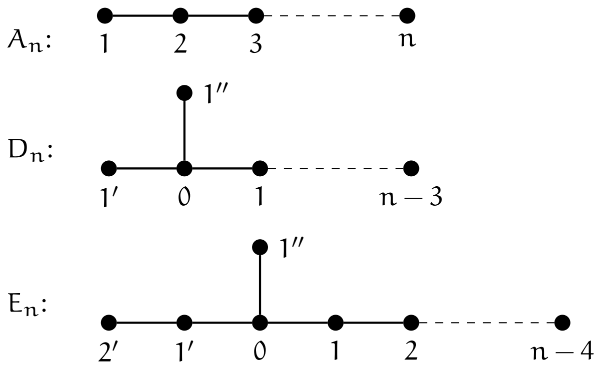
Definition 15. Let K be some field. A **representation of the quiver** $Q = (V, E, s, t)$ assigns to each vertex $v \in V$ a K-vector space W_v and to each arrow $e \in E$ a linear map φ_e from $W_{s(e)}$ to $W_{t(e)}$. A **subrepresentation** of a representation $(\{W_v\}_{v \in V}, \{\varphi_e\}_{e \in E})$ is a representation $(\{W'_v\}_{v \in V}, \{\varphi'_e\}_{e \in E})$ such that W'_v is a subspace of W_v for all v, and φ'_e is the restriction of φ_e to $W'_{s(e)}$. It is thus necessary that $W'_{t(e)}$ contains $\operatorname{im} \varphi'_e$.

A representation is **indecomposable** if it is not the direct sum of two nontrivial subrepresentations, where the direct sum of $(\{U_v\}, \{\varphi_e\})$ and $(\{W_v\}, \{\psi_e\})$ is given by taking the direct sum $U_v \oplus W_v$ as the vector space corresponding to v. For the map corresponding to $e \in E$, we take the "direct sum"

$$\varphi_e \oplus \psi_e : (U_{s(e)} \oplus W_{s(e)}) \to (U_{t(e)} \oplus W_{t(e)}),$$

defined as mapping (u, w) to $(\varphi_e(u), \psi_e(w))$.

The following graphs are vital for Gabriel's Theorem:



Theorem 6. *A connected quiver* Q *has finitely many isomorphism classes of indecomposable representations exactly when its underlying graph (the graph with directions of arrows forgotten) is of one of* A_n *for positive* n, D_n *for* $n \geqslant 4$ *or* E_n *for* $6 \leqslant n \leqslant 8$.

Representations of Q are interesting because they can be shown to be in correspondence with modules of $\mathbf{K}Q$. One can show that not only are modules of $\mathbf{K}Q$ in bijection with representations of Q, but they are so in a very natural way. In fact, one can give an intuitive definition of homomorphisms of representations Q such that they are in correspondence with the homomorphisms of modules of $\mathbf{K}Q$.

Every module M of $\mathbf{K}Q$ give rise to a representation of $Q = (V, E, s, t)$ by letting $W_v = \varepsilon_v M$ for $v \in V$ and φ_e be the function from $\varepsilon_{s(e)}M$ to $\varepsilon_{t(e)}M$ given by restriction of e to $\varepsilon_{s(e)}M$, for all $e \in E$.

Conversely, given a representation $(W_v)_{v \in V}$ of Q, a module over $\mathbf{K}Q$ is given by by the direct sum W of all W_v. Let $f_v : W \to W_v$ be projection on W_v, and $g_v : W_v \to W$ be the natural injection from W_v into W. The action of ε_v is then given by $g_v \circ f_v$ and the action of $e \in E$ by $g_{t(e)} \circ \varphi_e \circ f_{s(e)}$.

The above processes are each others inverses, so we have a bijection between representations of Q and modules of $\mathbf{K}Q$.

Example 7. In the case of A_1, an indecomposable representation is just an indecomposable vector space, so the only indecomposable representation is $\mathbf{K}$.

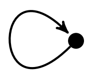
Example 8. We seek all the indecomposable representations of the quiver A_2.

Any representation is given by two vector spaces V_1 and V_2 and a map $\varphi : V_1 \to V_2$. First of all, if $\ker \varphi$ is nontrivial, it will be a direct summand of the representation, and the same with the complement of the image in V_2. These can in turn be decomposed into one-dimensional vector spaces. What is left is

where φ is now an isomorphism. This can be decomposed into two-dimensional subrepresentations (meaning that each vector space is one-dimensional) by simply decomposing $V_1 \setminus \ker \varphi$ into one-dimensional vector spaces and taking these, together with their images under φ, to be subrepresentations.

Thus, the indecomposable representations of A_2 (up to isomorphism) are given by $(V_1 = \mathbf{K}, V_2 = \{0\}, \varphi = 0)$, $(V_1 = \mathbf{K}, V_2 = \{0\}, \varphi = 0)$, and $(V_1 = \mathbf{K}, V_2 = \mathbf{K}, \varphi = \mathrm{id})$.

Example 9. The quiver



should have infinitely many indecomposable representations by Gabriel's Theorem. A representation here is a vector space V and an automorphism φ of V. We can take V to be $\mathbf{K}^n$ and φ to be a matrix with ones on the diagonal and superdiagonal. Then, φ is a single Jordan block and thus indecomposable.

Quivers of Finite-Dimensional Algebras

Definition 16. Let A be a basic and connected finite dimensional $\mathbf{K}$-algebra with a complete set of primitive orthogonal idempotents $\{e_1, \dots, e_n\}$. The quiver of A, written Q_A, is defined as having the vertex set $\{1, \dots, n\}$, with as many arrows from a to b as the dimension of the vector space $e_a(\operatorname{rad} A / \operatorname{rad}^2 A)e_b$.

Since A is finite-dimensional, $e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b$ is finite-dimensional so Q_A is finite.

Note that we do not define what labels the arrows have, so the quiver of A is defined up to isomorphism. Perhaps more worrying, its not evident from the construction that Q_A does not depend on the choice of idempotents.

Theorem 7. *Q_A does not depend on the choice of idempotents of A. Furthermore, for any pair of primitive orthogonal idempotents e_a and e_b, an isomorphism from $e_a(\operatorname{rad} A)e_b/e_a(\operatorname{rad}^2 A)e_b$ to $e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b$ is given by*

$$\varphi(e_a x e_b + \operatorname{rad}^2 A) = e_a(x + \operatorname{rad}^2 A)e_b. \tag{40}$$

Proof. Let $\{e_1, e_2, \dots, e_n\}$ and $\{e'_1, e'_2, \dots, e'_m\}$ be two sets of primitive orthogonal idempotents of A. Each set of primitive orthogonal idempotents of A gives rise to a decomposition $\bigoplus_{i=1}^{n} Ae_i$ of ${}_AA$ into indecomposable submodules, and by Theorem 1, $n = m$ and for some renumbering of the set of e'_is, $Ae_i \cong Ae'_i$ for all $1 \leqslant i \leqslant n$. We must show that

$$\dim(e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b) = \dim(e'_a(\operatorname{rad} A/\operatorname{rad}^2 A)e'_b). \tag{41}$$

The kernel of the A-module homomorphism from $\operatorname{rad}(A)e_b$ to $(\operatorname{rad} A/\operatorname{rad}^2 A)e_b$ given by $xe_b \mapsto (x + \operatorname{rad}^2 A)e_b$ is $\operatorname{rad}^2(A)e_b$, so by Proposition 5,

$$(\operatorname{rad} A/\operatorname{rad}^2 A)e_b \cong (\operatorname{rad}(A)e_b)/(\operatorname{rad}^2(A)e_b). \tag{42}$$

By Proposition 6,

$$\begin{aligned} e_a \operatorname{rad}(Ae_b)/\operatorname{rad}^2(Ae_b) &\cong \operatorname{Hom}_A(Ae_a, \operatorname{rad}(Ae_b)/\operatorname{rad}^2(Ae_b)) && (43)\\ &\cong \operatorname{Hom}_A(Ae'_a, \operatorname{rad}(Ae'_b)/\operatorname{rad}^2(Ae'_b)) && (44)\\ &\cong e'_a(\operatorname{rad} A/\operatorname{rad}^2 A)e'_b && (45) \end{aligned}$$

so

$$\dim(e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b) = \dim(e'_a(\operatorname{rad} A/\operatorname{rad}^2 A)e'_b). \tag{46}$$

To see that

$$\begin{aligned} \varphi : e_a(\operatorname{rad} A)e_b/e_a(\operatorname{rad}^2 A)e_b &\to e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b && (47)\\ e_a x e_b + \operatorname{rad}^2 A &\mapsto e_a(x + \operatorname{rad}^2 A)e_b && (48) \end{aligned}$$

is an isomorphism, note that the kernel of the map from $e_a(\operatorname{rad} A)e_b$ to $e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b$ given by the same formula is $e_a(\operatorname{rad}^2 A)e_b$ so by Proposition 5, φ is an isomorphism. $\square$

Now we want to show that if A is connected, then Q_A is connected, which is true in case A is basic.

Lemma 10. *Let A be a finite-dimensional algebra and let $Q_A = (V, E, s, t)$ be the quiver of A. For each $\alpha \in E$, let $x_\alpha \in e_a(\operatorname{rad} A)e_b$ be such that $\{x_\alpha + \operatorname{rad}^2 A | \alpha : a \to b\}$ is a basis of $e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b$.*

1. *For any two points $a, b \in V$, any element $x \in e_a(\operatorname{rad} A)e_b$ can be written as a linear combination of elements of the form $x_{\alpha_1} x_{\alpha_2} \dots x_{\alpha_k}$ where $\alpha_1 \alpha_2 \dots \alpha_k$ is a path from a to b.*

2. *Each x_α uniquely determines a nonzero non-isomorphism $\tilde{x}_\alpha : Ae_a \to Ae_b$ such that $\tilde{x}_\alpha(e_a) = x_\alpha$, $\operatorname{im} \tilde{x}_\alpha \subseteq (\operatorname{rad} A)e_b$ and $\operatorname{im} \tilde{x}_\alpha \nsubseteq (\operatorname{rad}^2 A)e_b$.*

Proof. 1. As a vector space, we can decompose $\operatorname{rad} A$ as $(\operatorname{rad} A/\operatorname{rad}^2 A) \oplus \operatorname{rad}^2 A$, and so $e_a(\operatorname{rad} A)e_b \cong e_a(\operatorname{rad} A/\operatorname{rad}^2 A)e_b \oplus e_a(\operatorname{rad}^2 A)e_b$. Thus x can be written as

$$x = \sum_{\alpha : a \to b} x_\alpha \lambda_\alpha \mod e_a(\operatorname{rad}^2 A)e_b, \tag{49}$$

where all λ_α are scalars. Since $\operatorname{rad} A = \bigoplus_{a,b \in V} e_a(\operatorname{rad} A)e_b$, we can write $e_a(\operatorname{rad}^2 A)e_b = \sum_{c \in V}(e_a(\operatorname{rad} A)e_c)(e_c(\operatorname{rad} A)e_b)$, so

$$x = \sum_{\alpha : a \to b} x_\alpha \lambda_\alpha + \sum_{\beta : a \to c} \sum_{\gamma : c \to b} x_\beta x_\gamma \lambda_{\beta,\gamma} \mod e_a(\operatorname{rad}^3 A)e_b. \tag{50}$$

$e_a(\operatorname{rad}^3 A)e_b$, can in turn be decomposed into longer sums $\mod e_a(\operatorname{rad}^4 A)e_b$, and so on, eventually giving x as a linear combination of paths mod 0, since $\operatorname{rad} A$ is nilpotent.

2. By Proposition 6, x_α is mapped to $\tilde{x}_\alpha$ by the isomorphism $\theta^{-1}_{(\operatorname{rad} A)e_b} : e_a(\operatorname{rad} A)e_b \to \operatorname{Hom}_A(Ae_a, \operatorname{rad}(A)e_b)$, mapping x to $\tilde{x}$ where $\tilde{x}(y) = yx$. This $\tilde{x}_\alpha$ satisfies all conditions stated in the theorem and is unique since $\theta^{-1}_{(\operatorname{rad} A)e_b}$ is invertible.

□

Corollary 6. *If A is a basic, connected and finite-dimensional, the quiver $Q_A = (V, E, s, t)$ of A is connected.*

Proof. Assume Q_A is not connected. We have some partition $V' \cup V''$ of V such that there are no edges between any two points in different parts of V. By Proposition 6, we have the vector space isomorphism $e_a A e_b \cong \operatorname{Hom}_A(Ae_a, Ae_b)$. Since A is basic, $Ae_a \ncong Ae_b$, so $\operatorname{Hom}_A(Ae_a, Ae_b)$ does not contain any isomorphisms. For $\varphi \in \operatorname{Hom}_A(Ae_a, Ae_b)$, $\varphi(e_a)$ is in Ae_b so it may be written as $(\lambda 1 + x)e_b$ where $\lambda \in \mathbf{K}$ and $x \in \operatorname{rad} A$. If $\lambda \neq 0$, then $1 - (-\lambda^{-1})x$ has a two-sided inverse y by Proposition 2, so $\varphi(a\lambda^{-1} y e_a) = ae_b$, but φ is not an isomorphism, so $\lambda = 0$ and $\varphi(e_a) = x \in \operatorname{rad}(Ae_b)$. Thus the image of φ is contained in

$\text{rad}(Ae_b)$, so $\text{Hom}_A(Ae_a, Ae_b) \cong \text{Hom}_A(Ae_a, \text{rad}(Ae_b)) \cong e_a \,\text{rad}(A)e_b$, which is $\{0\}$ whenever $a \in V', b \in V''$ or $a \in V'', b \in V'$ by assumption. Thus $e_a A e_b = \{0\}$ for a and b in different part of V and so A is not connected. □

Lemma 11. *Let* (Q, I) *be a bound quiver and* $A = \mathbf{K}Q/I$. *Then* $Q_A = Q$.

Proof. The set $\{e_a = \varepsilon_a + I | a \in V\}$ is a complete set of orthogonal idempotents of A, giving a correspondence between points of Q and those of Q_A, and by Corollary 5, the arrows of Q from a to b are in bijective correspondence with $e_a \,\text{rad}(\mathbf{K}Q/I)e_b$. □

Theorem 8. *Let* A *be a basic and connected finite-dimensional* $\mathbf{K}$*-algebra. There is an admissible ideal* I *of* $\mathbf{K}Q_A$ *such that* $A \cong \mathbf{K}Q_A/I$.

Proof. The idea is to find a homomorphism $\varphi : \mathbf{K}Q_A \to A$ such that $\ker\varphi$ is an admissible ideal. Then, by Proposition 5, $\mathbf{K}Q_A/\ker\varphi \cong A$.

As usual, let $Q_A = (V, E, s, t)$ and let $\{e_1, e_2, \dots, e_n\}$ be the set of primitive orthogonal idempotents of A used to construct Q_A. For each $\alpha \in E$, let x_α be defined such that $x_\alpha + \text{rad}^2 A | \alpha : a \to b$ forms a basis of $e_a(\text{rad}\, A/\text{rad}^2 A)e_b$. By Theorem 5, there is a unique homomorphism from $\mathbf{K}Q_A$ to A mapping ε_a to e_a for each $a \in V$ and $\alpha \in E$ to x_α, which we call φ.

Now we must show that φ is surjective. By Theorem 4, $A = B \oplus \text{rad}(A)$ with B as in the theorem. All the elements of $\text{rad}(A)$ is in the image of φ by Lemma 10, and B is spanned by $\{e_1, e_2, \dots, e_n\} = \{\varphi(1), \varphi(2), \dots, \varphi(n)\}$.

To show that $\ker\varphi$ is admissible, we must show that $R^m \subseteq \ker\varphi \subseteq R^2$ for some m. Note that $\varphi(R_Q) \subseteq \text{rad}(A)$ by the definition of φ, so $\varphi(R_Q^m) \subseteq \text{rad}(A)^m$, which is $\{0\}$ for some m since the radical is nilpotent, so $R_Q^m \subseteq \ker\varphi$. To see that $\ker\varphi \subseteq R^2$, note that any $x \in \mathbf{K}Q$ can be written as

$$\sum_{a\in V} \lambda_a \varepsilon_a + \sum_{\alpha\in E} \mu_\alpha \alpha + y$$

where $y \in R^2$. If $\varphi(x) = 0$, we have

$$\sum_{a\in V} \lambda_a e_a = -\sum_{\alpha\in E} \mu_\alpha x_\alpha + \varphi(y)$$

which is in R and thus nilpotent, so $\lambda_a = 0$ for all a. Similarly, $\sum_{\alpha\in E} \mu_\alpha x_\alpha = -\varphi(y)$ which is in $\text{rad}^2 A$, and since x_α is a basis of $\text{rad}\, A/\text{rad}^2 A$, $\mu_\alpha = 0$ for all α. Thus $x = y \in \text{rad}^2 A$, so $\ker\varphi \subseteq R^2$. □

In the case where A is not connected, we can apply this theorem to each connected subalgebra of A, meaning that if $A = A_1 \oplus A_2 \oplus \dots \oplus A_n$, we get a quiver $Q_A = Q_{A_1} + Q_{A_2} + \dots + Q_{A_n}$, where the sum of graphs is defined as follows: if $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, the vertex set of $G_1 + G_2$ is $V_1 \cup V_2$ and the edge set is $E_1 \cup E_2$. The vertex sets are assumed to be disjoint (otherwise, we relabel the vertices of one of the graphs). We have ideals $I_1, I_2, \ldots, I_n$ such that $\mathbf{K}Q_{A_i}/I_i \cong A_i$ for all $0 < i \leqslant n$, and thus $\mathbf{K}Q_A/(I_1 + I_2 + \cdots + I_n) \cong A$.

References

[1] Assem, I., Skowronski, A. & Simson, D. *Elements of the Representation Theory of Associative Algebras* (Cambridge University Press (CUP), 2006). http://dx.doi.org/10.1017/CBO9780511614309.

[2] Etingof, P. *et al.* Introduction to representation theory (2011). http://www-math.mit.edu/~etingof/replect.pdf. Lecture notes.

[3] Pierce, R. S. *Associative Algebras* (Springer, 1982). http://link.springer.com/book/10.1007/978-1-4757-0163-0.

[4] Gabriel, P. Unzerlegbare darstellungen i. *Manuscripta Mathematica* **6**, 71–103 (1972). http://dx.doi.org/10.1007/BF01298413.