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Jump-Diffusion Models and Implied Volatility

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Contents

Chapter 1

Abstract

The origin of this thesis came from a statement found in the book Financial Modelling With Jump Processes by Rama Cont and Peter Tankov. In the context of option pricing and volatilities, the introduction states:

”Models with jumps, by contrast, not only lead to a variety of smile/ skew patterns but also propose a simple explanation in terms of market anticipations: the presence of a skew is attributed to the fear of large negative jumps by market participants.”

This statement is written without any reference or proof and the object of this thesis was to examine whether it is true or not. The results from this thesis does confirm the statement.

The statement has been tested by using a Monte-Carlo method in a theoretical pricing model.

Chapter 2

Introduction

Whenever options are mentioned in this thesis, the reader may interpret this as european call options, as put options and american options are not covered in this thesis.

2.1 Incomplete Markets

When using this option pricing model, we assume an incomplete market. A complete market is defined as every contingent claim can be replicated by a portfollio consisting of existing assets on the market. Therefore, in a complete market, all contingent claims are redundant since one can gain the same result with the existing products on the market. Although this property does make thing easier in pricing theoretical derivatives, it is not something which we can assume as a property of our existing market.

When one allows jumps in the model of the underlying's price, market completeness is destroyed. Contingent claims are then not redundant but actually become an important product on the market (which it evidentally is, based on the amount of derivatives on the market today). Prices of contingent claims in incomplete markets are not unique and we will not find a perfect hedge, as one would in a complete market. A perfect hedge is defined as a trading strategy which eliminates all risk of a position.

2.2 Implied Volatility

When one is speaking about the volatility, it may not be clear of what one is speaking about. Statistically speaking the volatility is defined as the standard deviation. When one is talking about the financial market there are three main volatilities that one may mean. There is the **historical volatility** which is the standard deviation of a previous time period of the underlying. There is the **actual volatility** which is the volatility that the underlying will actually have. Since we can't see the future, this volatility is not observable in any way. Then there is the **implied volatility**, which is defined as the volatility that one receives when using the Black and Scholes formula backwards. Since all of the parameters of this formula are observable from the market, one simply puts these in the formula and solves for the volatility. It is this volatility which will be studied in this thesis.

2.3 Volatility Smile and Skew

There are several common patterns when one is plotting the implied volatility to the strike prices. The most common ones are the smiles and the skews. The smile U-shaped (as a smile) and the implied volatility is higher for out-of-the-money options and in-the-money options than when the strike price is at the money.

The skew which is the pattern which is relevant for this thesis, is defined as a decreasing curve. So as the strike price increases, the implied volatility decreases.

Chapter 3

Method

The formula used to simulate the price of the underlying was the following:

$$X_t = X_0 e^{-\frac{\sigma^2}{2}t + \sigma W_t}(1-\gamma)^{N_t} e^{\lambda\gamma t}$$

where

- X_0 is the initial stock price
- σ is the volatility of the underlying
- t is the time of the simulation in years
- $W_t \sim \mathcal{N}(\mu, \sigma^2 t)$
- γ is the amount that the price of the underlying drops when a jump occurs
- $N_t \sim Pois(\lambda t)$ where $Pois$ is the Poisson distribution
- λ is the intensity of the Poisson process

The constant interest rate is for simplicity assumed to be 0 and is therefore not taken into consideration. This is without loss of generality since we can use the bank account as a numeraire.

As we can see from the formula, the stock price depends both on a normally distributed variable as well as a variable which is Poisson distributed. It is the latter variable which makes the jumps of the formula. In this formula, as one can see, only negative jumps are allowed. The objective for that comes from the statement that we wish to examine, if the "fear of large negative jumps by market participants" leads to a skew pattern.

3.1 Valuing the Option

To value the option, the expected value of these options was calculated using the following formula:

$$\mathbb{E}\big(X_t - k\big)^+ = \frac{\sum_{i=1}^{n}(x_i - k)^+}{n}$$

where x_i, $i = 1, ..., n$ are the simulated stock prices. More about how the calculations are made can be found in the end of this thesis, under the chapter Appendix where the code can be found and is also explained.

3.2 Adding a Jump Process

As we will see in the plots, the implied volatilites will always be higher than the σ which is defined. The reason is the jump function which is added and will increase the volatility. The following proof shows that this is the case:

$$\mathbb{E}(g(YZ)) = \int \mathbb{E}\Big[g(yZ)\Big] f(y)dy \geq \int g(y)f(y)dy = \mathbb{E}g(Y)$$

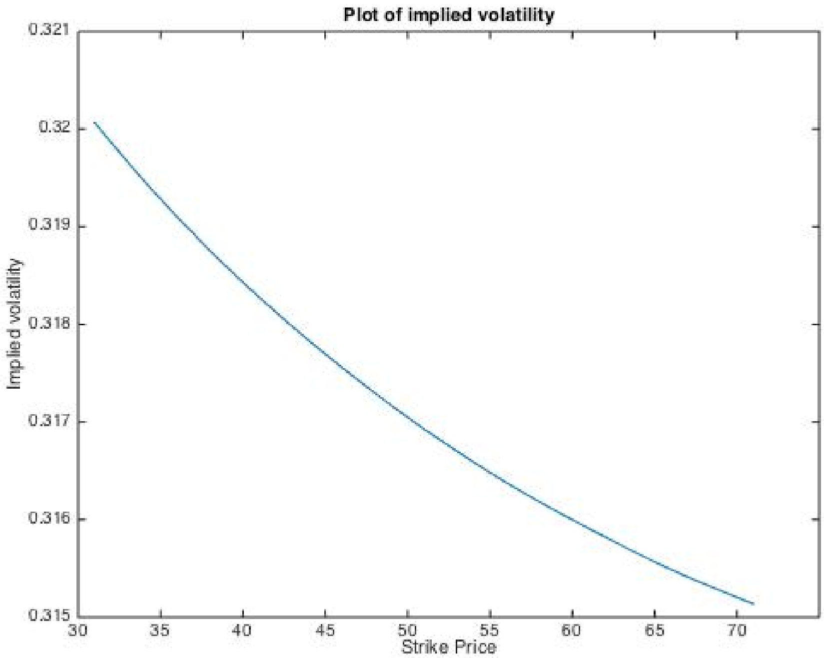
where $Y = X_0 e^{-\frac{\sigma^2}{2} + \sigma W_t}$, $Z = (1-\gamma)^{N_t} e^{\lambda\gamma t}$ and f is the density function of Y. The inequality follows from the Jensen's inequality with g being the implied volatility function which is convex.

Chapter 4

Results

The stock prices were simulated 20 million times for each set of parameters. The following plots for the implied volatilities was given.

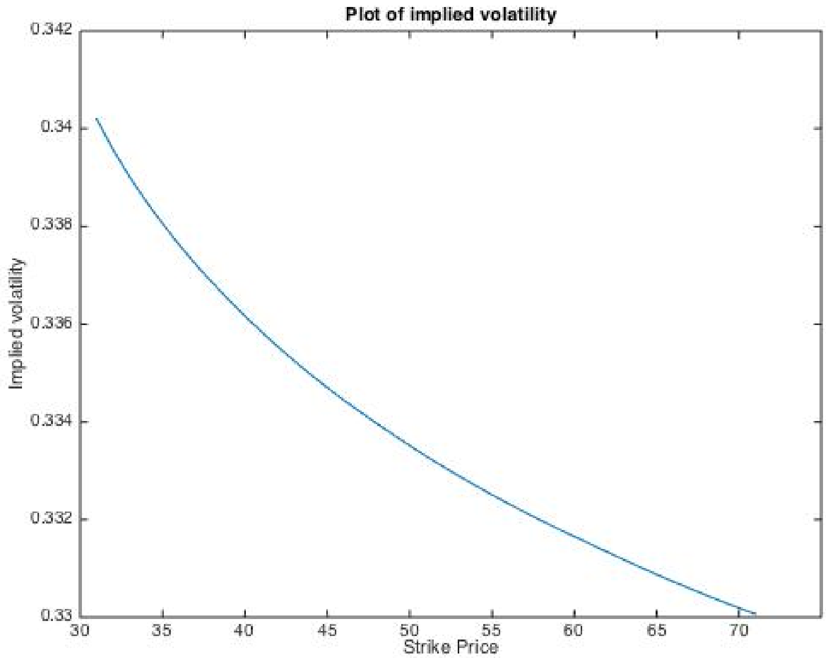
Figure 4.1:



The following parameters were used for the above plot:

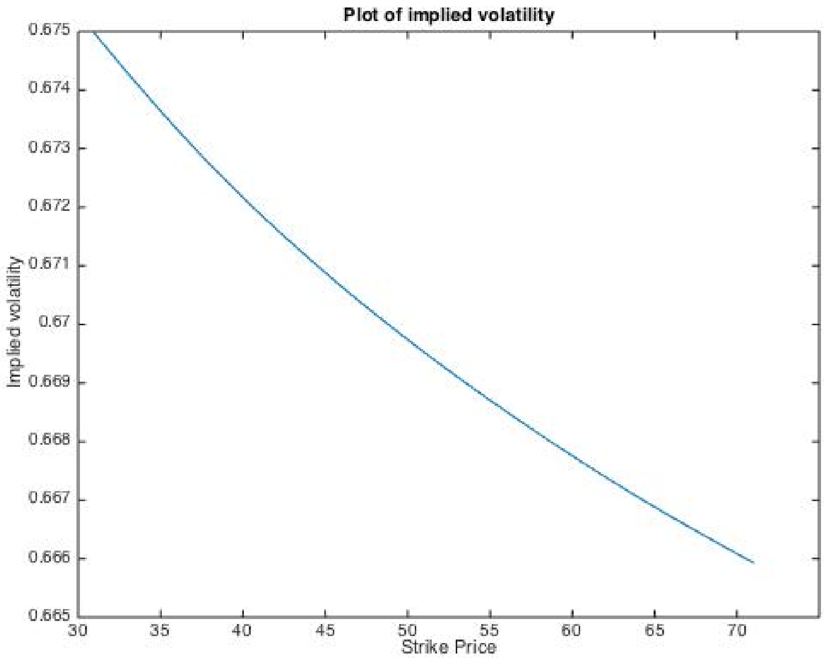
- $X_0 = 50$
- $\sigma = 0.3$
- $t = 1$
- $\gamma = 0.1$
- $\lambda = 1$

Figure 4.2:



The parameters in this simulation are the same as the one above except with more frequent jumps ($\lambda = 2$)

Figure 4.3:



- $X_0 = 50$
- $\sigma = 0.6$
- $t = 1$
- $\gamma = 0.2$
- $\lambda = 2$

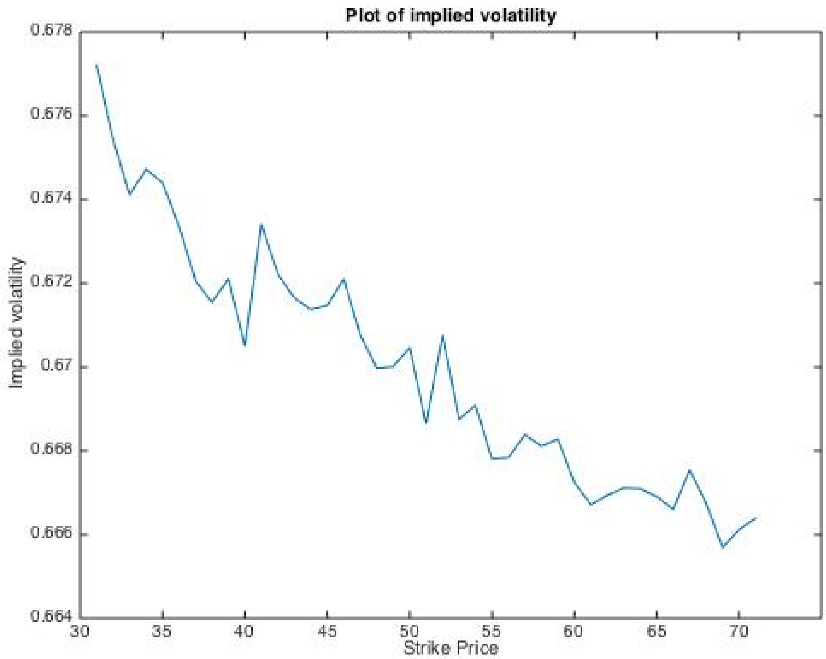
As we can see from the plots, the implied volatilities are strictly decreasing when the strike prices are increasing.

In all of the above graphs, the same stock price simulations are used for every strike price.

4.1 Different stock price simulations for different K

Here there are different simulations of the stock prices for the different strike prices. For each strike price, there are five million stock prices simulated. The parameters for these simulations are the same as the one on the previous page.

Figure 4.4:



The plot is more volatile than the plots which are using the same stock prices for each strike price but we can still see a negative trend as the strike price increases. If enough simulations were used, the curve should become as smooth as the previous ones. When, as in this simulation, a limited number of simulations are used, the unsystematical noise makes a large difference. If the statement that higher strike price reduces the implied volatility is true, the above curve would become as smooth as the ones before when enough simulations are used. Then it is systematical that a higher strike price decreases the implied volatility.

4.2 Numerical Problems

In these simulations, one may encounter numerical problems when looking at the low and the high strike prices.
Low strike prices: When calculating the implied volatility, the price of the option must be larger than $x - k$, the stock price minus the strike price. If it is not, the implied volatility can not be defined as there is arbitrage on the market and one could buy the option and short the stock. One would then have made an riskless profit.
Examples of implied volatilities which are not defined exist further down.
High strike prices: When none of the simulations reach over the strike price and $x - k$ is zero, the value of the option will become 0. The implied volatility of an option with zero value will be zero as well, which comes as no surprise. For an option to not have any value there has to be a deterministic price of the underlying. At least to the point that the underlying does not reach over the strike price. Since this is generally not the case, this is a problem of the numerics.

4.3 Letting T $\to 0$

The aim of this section is to test how well the simulation program work when the parameter for time will decrease. How short time period can we use to not receive the numerical problems described above?

The following parameters are used for each of the plots.

- $X_0 = 50$
- $\sigma = 0.5$
- $\gamma = 0.2$
- $\lambda = 2$

Figure 4.5: t = 0.5

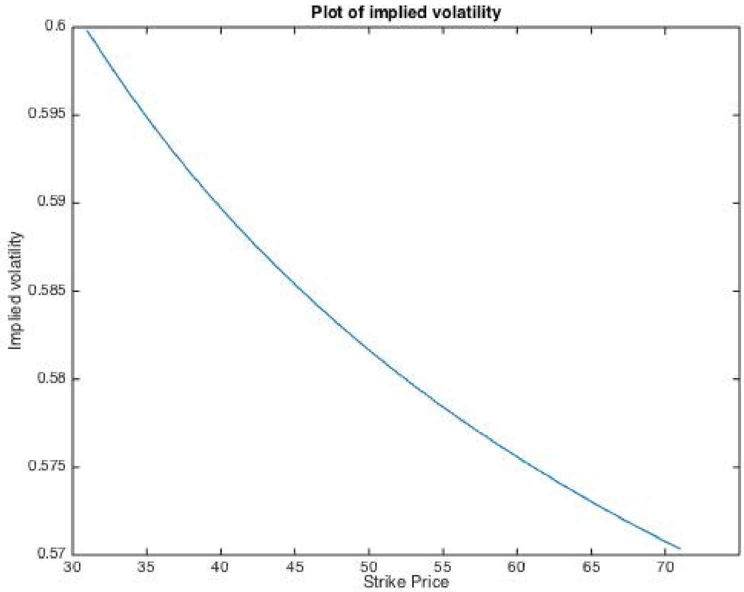


Figure 4.6: t = 0.1

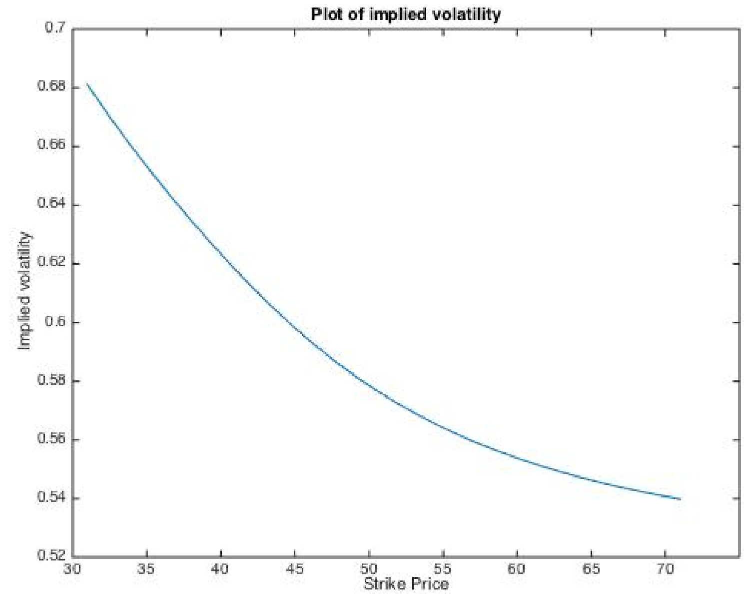


Figure 4.7: t = 0.05

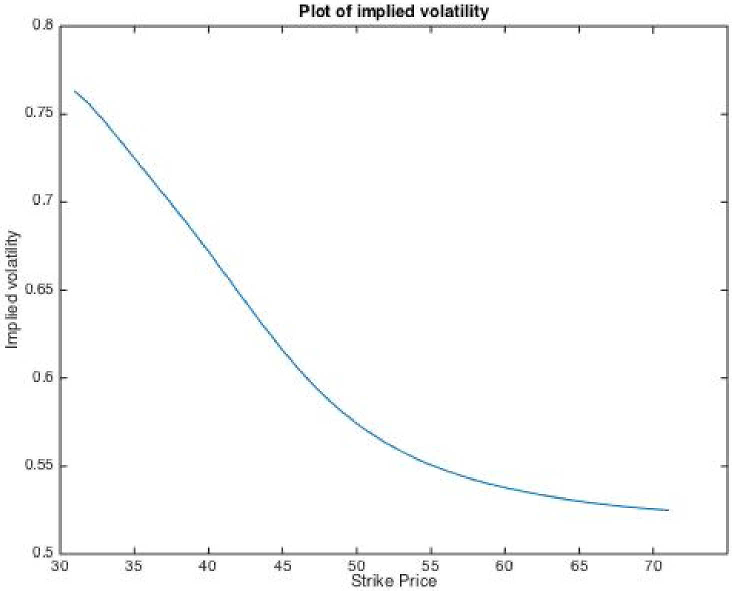


Figure 4.8: t = 0.01

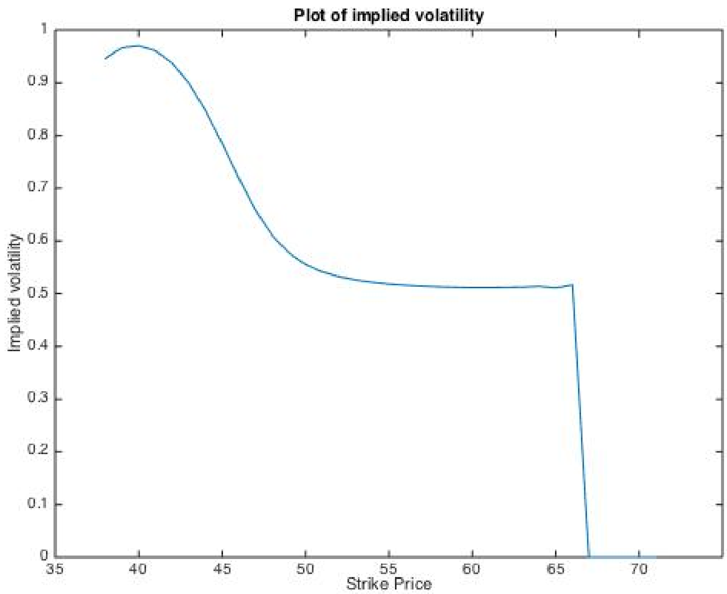


Figure 4.9: t = 0.005

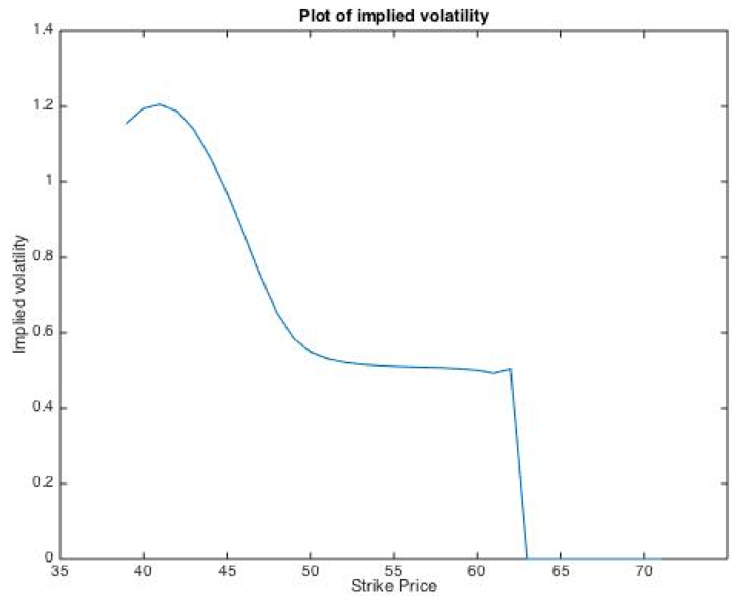
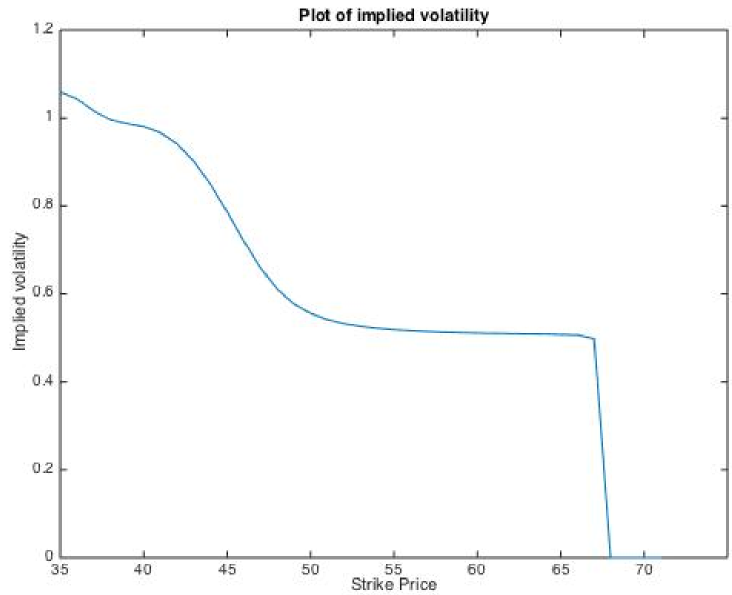


Figure 4.10: t = 0.001



The six different plots have t values of 0.5, 0.1, 0.05, 0.01, 0.005 and 0.001. Judging from the plots, the numerical problems appear to start when T is less than 0.05. In the third to last plot with t=0.01, the implied volatilities are not defined for the lowest strike prices and are 0 for strike prices higher than 66. For strike prices which are in the money the volatilities are significantly larger. As the option becomes at the money and out of the money, the jump process does not seem to matter much (the sigma of the normally distributed variable is 0.5).

The second to last plot with t=0.005 is similiar to the previous one, but more extreme. 100 million simulations are used for this plot, for a more reliable result. Compared to the previous plot it has higher implied volatility for the options which are in the money and there are more of the options which has an implied volatility of 0 for the high strike prices. Both of the plots (and the ones before, just not as much) have an implied volatility close to the volatility of the normally distributed variable for when the option is at the money or above. For both the plots the implied volatility seem to peak for strike prices around 40. Since initial stock price is 50 and the jump process makes the price drop by 20 %, there was probably not a single simulation which jumped twice.

For the plot with the smallest t, t=0.001, we see something similiar as with the previous two plots. There is still something happening when the strike price is around 40, the difference is that the peak is not at that point. It looks more like a saddle point. Nonetheless it is worth to mention that when t becomes small, the jump size becomes more important when observing the plots. 200 million simulations was used for this plot.

For the lowest strike prices in the last three plots, the implied volatilties are not defined. This is, as previously stated, because the option price was lower than $x - k$.

Chapter 5

Conclusion

As all the plots in this thesis show, there appears to be a skew in the implied volatility when using a pricing model for the underlying which allows only negative jumps. From all of the various simulations with different parameters, the implied volatilities decrease as the strike price increases. This is even true for when the time period is approaching zero, even though the graphs looks different. When t decreases we could see that the size of the jumps were becoming more important for the value of the implied volatility.

Chapter 6

References

- Arbitrage Theory in Continuous Time
  Tomas Björk
  Third Edition 2009

- Financial Modelling With Jump Processes
  Rama Cont and Peter Tankov
  2004

Chapter 7

Appendix

7.1 Underlying Function

```
function x = underlying(lambda, t, x_0, sigma, gamma)
mu = 0 ; % The drift of the underlying
x = x_0 * exp(- ((sigma^2) /2)*t + sqrt(t)*sigma * normrnd(mu, t))*(1-gamma)^poissrnd(lam
end
```

This program simply simulates our stock prices, with the five input variables seen in the first line. There are two random variables here, the normally distributed (normrnd) which is used in the black scholes model and the poisson distributed (poissrnd) which simulates the jumps.

7.2 Simulation Program

```
clc
clear all
tic
sigma = 0.6; % Standard deviation of the underlying
gamma = 0.2; % Jump size
lambda = 2; % Intensity of the jumps
t = 1; % Time
x_0 =50; % Initial stock price
n = 5000000; % Number of stock simulations
m = zeros(n,1);
ca = zeros(n,41);
```

```
for i=1:n
m(i,1) = underlying(lambda, t, 50, sigma, gamma);
end
for i=30:70 % Different strike price
   for j=1:n
       ca(j,i-29) = m(j)-i;
       if ca(j,i-29) < 0
          ca(j,i-29) = 0;
       end
   end
end
c = sum(ca)/n;
vol = zeros(41,1);
for i=1:41
    vol(i) = blsimpv(50,i+29,0,t,c(i));
end
vol
toc
x = linspace(1,41,41);
plot(x,vol)
title('Plot of implied volatility')
xlabel('Strike Price')
ylabel('Implied volatility')
h = gca;
h.XTick = [0,5,10,15,20,25,30,35,40];
h.XTickLabel = {'30','35','40','45','50','55','60','65','70'};
```

In the beginning of this program, the variables that are used in the underlying program are defined. These values are changed for the different plots in this thesis. The number of times that the underlying will be simulated is then defined. A vector which will contain these values are defined, as well as the the matrix which will contain the prices of every option for the different strike prices. The stock prices are then simulated and the option prices are calculated. An average of the option prices are then calculated for every strike price. The vector of the implied volatilities is created and then calculated with the for-loop. To receive the implied volatilites, we use the function **blsimpv** which gives us the implied volatilities by putting in the inputs **Original price, Strike price, interest rate, time to expiration** and **option call price**.After this the main part of the program is finished, and all that is left is to create a graphical visualisation of the results, which is done be creating a simple graph.

7.3 Simulation Program with different stock prices for different K

```
clc
clear all
tic
n = 2000000; % Number of stock simulations
sigma = 0.6; % Standard deviation of the underlying
gamma = 0.2; % Jump size
lambda = 2; % Intensity of the jumps
t = 1; % Time
m = zeros(n,41);
ca = zeros(n,41);
for j=1:41
    for i=1:n
        m(i,j) = underlying(lambda, t, 50, sigma, gamma);
    end
end
for i=1:41 % Different strike price
   for j=1:n
       ca(j,i) = m(j,i)-(i+29);
       if ca(j,i) < 0
          ca(j,i) = 0;
       end
   end
end
c = sum(ca)/n;
vol = zeros(41,1);
for i=1:41
    vol(i) = blsimpv(50,i+29,0,t,c(i));
end
vol
toc
x = linspace(1,41,41);
plot(x,vol)
title('Plot of implied volatility')
xlabel('Strike Price')
ylabel('Implied volatility')
h = gca;
h.XTick = [0,5,10,15,20,25,30,35,40];
h.XTickLabel = {'30','35','40','45','50','55','60','65','70'};
```

The difference between this program and the previous one is first seen when the matrix m is created. In the previous program, m is simply a vector. In this program, m is a 41 times n matrix where every vector contains the stock prices for the 41 different strike prices used. All of these stock prices are then used to calculate the option prices in the following loops, and the rest is the same.