

On a Probability Distribution Convolution Approach to Clear-Sky Index and a Generalized Ångström Equation

Joakim Munkhammar^a, Joakim Widén^b

^a (Corresponding author) Built Environment Energy Systems Group, Department of Engineering Sciences, Ångström Laboratory, Uppsala University, SE- 751 21 Uppsala, Sweden. e-mail: joakim.munkhammar@angstrom.uu.se, phone: +46704464271

^b Built Environment Energy Systems Group, Department of Engineering Sciences, Ångström Laboratory, Uppsala University, SE- 751 21 Uppsala, Sweden. e-mail: joakim.widen@angstrom.uu.se

Abstract

We show that by modeling solar beam irradiance approximately as a simple Bernoulli distribution and diffuse irradiance as a Gamma distribution, a generalized Ångström equation relating solar irradiation to sunshine hours follows directly as a consequence of the convolution of beam and diffuse irradiance distributions into a distribution for the clear-sky index.

Keywords: Ångström equation, Clear-sky index.

1. Introduction

Estimations of solar irradiation on Earth's surface is important for many purposes, including solar energy potential studies and simulations of solar energy systems. It has proven possible to estimate solar irradiation from duration of sunshine hours relative to clear-sky irradiation, as was first investigated in [1] and [2, 3]. This can be expressed as the Ångström equation relating average solar irradiation to duration of bright sunshine [3, 4]:

$$\frac{\bar{H}}{\bar{H}_c} = \alpha + (1 - \alpha)S \quad (1)$$

where \bar{H} is the monthly average of daily horizontal surface radiation, \bar{H}_c is the average daily clear sky solar radiation, $S \in [0, 1]$ is the fraction of bright sunshine and α is a parameter for each location. This equation has been mostly empirical throughout the literature, but attempts have been made to give a physical description, see for example [5, 4, 6]. There is a deep connection between different aspects of the clear-sky index and the Ångström equation which is not completely understood [6].

When modeling the instantaneous clear-sky index a fit-to-distribution model is typically adopted, ranging from a single Gamma distribution [7] to mixture

models including triple Normal distributions [8]. In this paper we propose fit-to-distribution models for clearness indices corresponding to the beam and diffuse irradiance components. These can be combined to yield the global clearness index with a convolution approach. We show that by doing so, a generalized Ångström equation is a direct consequence.

2. Convolution approach

The instantaneous Global Horizontal Irradiance (GHI) G is composed of Beam Horizontal Irradiance (BHI) G_b and Diffuse Horizontal Irradiance (DHI) G_d over time t according to:

$$G(t) = G_d(t) + G_b(t). \quad (2)$$

Normalizing by the Global Horizontal Irradiance for clear sky conditions $G_c(t)$ gives the so-called clear-sky index κ :

$$\kappa(t) \equiv \frac{G(t)}{G_c(t)} = \kappa_d(t) + \kappa_b(t), \quad (3)$$

where $\kappa_d(t) \equiv G_d(t)/G_c(t)$ and $\kappa_b(t) \equiv G_b(t)/G_c(t)$. It should be emphasized here that there is a complex dependency between κ_d and κ_b [4]. Modeling such a dependency requires a more detailed statistical analysis. That is outside the scope of this paper, which aims to provide a connection between a convolution approach and a generalized Ångström equation. Let's assume that κ_d is represented by a stochastic variable X_d and κ_b is represented by a stochastic variable X_b , then an analogue for κ can be setup as stochastic variable X :

$$X = X_d + X_b \quad (4)$$

The diffuse irradiance index κ_d mainly represents scattered solar irradiance from all over the sky dome, whereas the beam irradiance index κ_b mainly represents direct solar irradiance. In the literature the distribution for κ_d is typically assumed to be unimodal whereas κ_b is assumed bimodal [9]. It is here assumed that the bimodal peaks of κ_b correspond to two states: bright sunshine, no bright sunshine. This is motivated by the fact that beam irradiance essentially only reaches the Earth's surface when no clouds obscure the sun [4]. That is, κ_b is approximately a two-state distribution with one peak at zero corresponding to cloud coverage blocking the sun and one peak between zero and one corresponding to bright sunshine. The bright sunshine peak is dependent on the interpretation of the definition "bright sunshine" in this particular probability distribution approach. We introduce a scaling parameter β that relates the beam irradiance distribution X_b to a two-state Bernoulli distribution $X'_b = \beta X'_b$ where:

$$X'_b \sim \text{Bernoulli}(p). \quad (5)$$

The scaling factor β is assumed to be the average beam irradiance index above a threshold σ , since to some extent the beam irradiance index resembles the near

binary properties of beam radiation as bright sunshine or not [4]. This also defines p as the proportion of beam irradiance index fractions greater than σ . In total this represents a fit-to-distribution for the beam irradiance index by the Bernoulli distribution. The probability mass function (PMF) f of a Bernoulli distributed stochastic variable $X \sim \text{Bernoulli}(p)$ is defined as:

$$f(k, p) = \begin{cases} p & \text{if } k = 1, \\ 1 - p & \text{if } k = 0. \end{cases} \quad (6)$$

The diffuse radiation was modeled by a Gamma probability density distribution as a typical positive unimodal distribution [10, p.60]:

$$G(x; \gamma, \delta) = \frac{1}{\delta^\gamma \Gamma(\gamma)} x^{\gamma-1} e^{-x/\delta}. \quad (7)$$

In this study parameters γ and δ were set by the fit-to-distribution routine *gamfit* in Matlab. The data set used in the simulations was one year of minute-resolution data on DHI and BHI for the location of Norrköping, Sweden (59°35'31" N 17°11'8" E) [11]. Global clear sky irradiance was calculated with the Ineichen-Perez model [12]. In order to avoid infinities in the data set the clearness indices were only evaluated for solar angles above 20 degrees. This data is shown in Figure 1 along with a the fit-to-distribution of the Gamma distribution for DHI and the Bernoulli distribution for BHI with scaling parameter $\beta = 0.69$, which is a consequence of using threshold $\sigma = 0.01$.

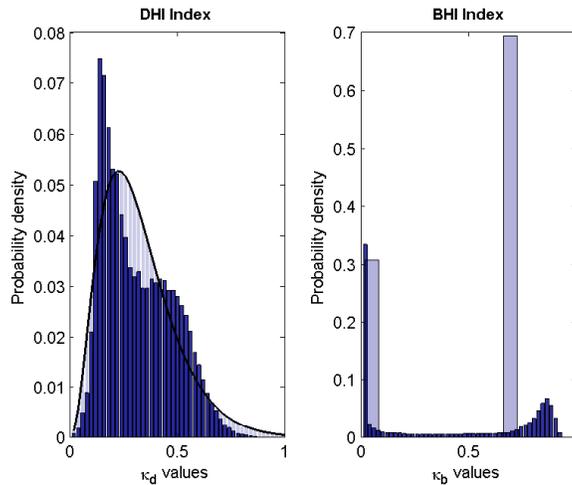


Figure 1: This plot shows histograms and fit-to-distributions for the DHI and BHI clear-sky indices.

3. The generalized Ångström equation

GHI is, by definition, the clear-sky radiation weighted by the clear sky index according to:

$$G = (X_d + X_b) G_c, \quad (8)$$

where G and G_c are now considered stochastic variables as well. If we take the expectation value we get:

$$E[G] = E[(X_d + X_b)G_c] = E[X_d + \beta X'_b]E[G_c], \quad (9)$$

where we can identify the ratio of monthly average daily horizontal irradiation \bar{H} and the monthly average of clear-sky irradiation \bar{H}_c via the ratio of expectation values for G and G_c :

$$\frac{\bar{H}}{\bar{H}_c} \equiv \frac{E[G]}{E[G_c]}. \quad (10)$$

Despite X_d and X_b being correlated, the expectation value of the sum of X_d and X_b is equal to the sum of the expectation values, which brings the following expression:

$$\frac{\bar{H}}{\bar{H}_c} = E[X_d] + \beta E[X'_b]. \quad (11)$$

We have that $E[X_d] = \gamma\delta$ is the mean value of the Gamma distribution (7) representing average solar irradiation from cloudiness. $E[X'_b]$ is the expectation value of the Bernoulli distribution

$$E[X'_b] = p \quad (12)$$

This brings a generalization of the Ångström equation for estimates of solar irradiation:

$$\frac{\bar{H}}{\bar{H}_c} = \gamma\delta + \beta p \quad (13)$$

where we can identify $\gamma\delta \sim \alpha$ and $p \equiv S$ in the Ångström equation (1), but β is not necessarily equivalent to $1 - \alpha$. This constitutes the generalization of the Ångström equation (1) in this paper. It should be pointed out that (13) is similar to the Ångström-Prescott equation with the difference that the Ångström-Prescott equation is based on extraterrestrial radiation instead of clear-sky irradiance [4, 13]. If using the meteorological data [11] and the fit-to-distribution routines, one arrives at $\gamma\delta = 0.32$ and $\beta = 0.64$ coupled with the previously calculated $\beta = 0.64$ for a threshold of $\sigma \sim 0.01$. these estimates can be compared with Ångström's original results for Stockholm, Sweden in 1924: $\alpha = 0.25$, $1 - \alpha = 0.75$ [3]. A numerical calculation of β and p from different σ levels based on the data set is shown in Figure 2. For this particular set of data p is estimated at 0.69 based on $\sigma = 0.01$.

A complete statistical investigation regarding the modeling of the interdependency of κ_b and κ_d could benefit this understanding as well.

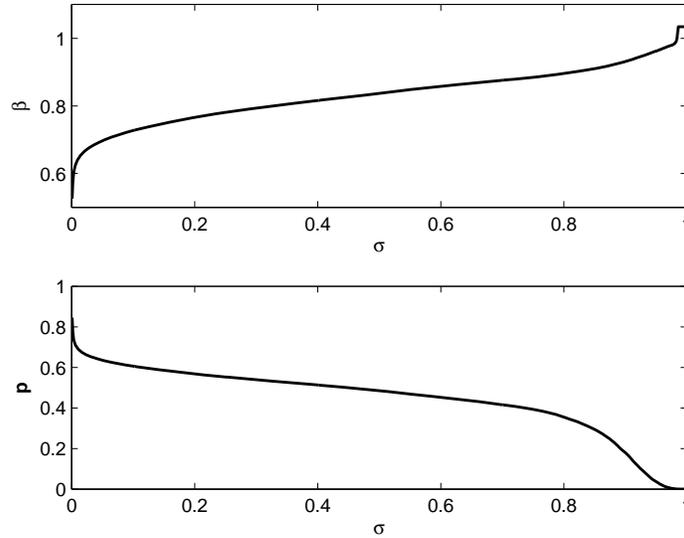


Figure 2: This figure shows the relationship between the scaling parameter β and the probability p in the Bernoulli distribution for different values of the threshold parameter σ .

4. Conclusions

In this paper we propose a fit-to-distribution approach for modeling the BHI and DHI clear-sky indices as separate probability distributions. Using a convolution approach and a statistical analysis based on expectation values we derive a generalized Ångström equation.

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