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High-fidelity numerical simulation of the dynamic beam equation

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Abstract

A high-fidelity finite difference approximation of the the dynamic beam equation is derived. Different types of well-posed boundary conditions are analysed. The boundary closures are based on the summation-by-parts (SBP) framework and the boundary conditions are imposed using a penalty (SAT) technique, to guarantee linear stability. The resulting SBP-SAT approximation leads to fully explicit time integration. The accuracy and stability properties of the newly derived SBP-SAT approximations are demonstrated for both 1-D and 2-D problems.

Key words: finite difference methods, high-order derivative, high-order accuracy, stability, boundary treatment, dynamic beam equation

1 Introduction

The dynamic beam equation (DBE), also known as the Euler-Bernoulli beam equation, is a standard model of flexible body dynamics and is thus of interest in many engineering applications where beams are used as the basis of supporting structure or as axles, e.g when studying vibrations of buildings or railway structures. The DBE is derived from Euler-Bernoulli beam theory, one of the simplest beam theories dating back to the 18th century. The model includes potential energy arising due to strain forces from the bending of the beam and kinetic energy due to the lateral displacement of the beam.

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The governing equation of the DBE is a linear partial differential equation (PDE) that is second order in time and fourth order in space.

In the literature, numerical analysis of the DBE in time-domain is very scarce, in particular concerning higher-order methods. Most numerical methods target the analysis of natural frequencies of beams (see for example [4]), by Laplace transform the DBE and study a single frequency at a time. FEM is a well-proven technique for these types of time-independent problems, especially if non-cartesian geometries are considered. A drawback with this approach is that time-dependent phenomena cannot easily be approximated, as well broad-band sources. In the present study we analyse and solve the DBE in time-domain using an explicit finite difference method. This approach is computationally much more demanding (than solving the DBE in frequency domain) since the dispersive nature of the DBE requires very small time-steps to resolve the high-frequency part of the solution. A well-proven approach to reduce the computational cost (or rather the degrees of freedom) is to employ a high-order accurate finite-difference method. (See the pioneering paper by Kreiss and Oliger [18].) The major difficulty with higher-order finite difference methods is to obtain a stable boundary treatment, that has received considerable past attention. (For examples, see [20, 39, 37, 1, 5, 13].)

A robust and well-proven high-order finite difference methodology that ensures stability of time-dependent partial differential equations (PDEs) is the summation-by-parts-simultaneous approximation term (SBP-SAT) method. The SBP-SAT method combines semi-discrete operators that satisfy a summation-by-parts (SBP) formula [17], with physical boundary conditions implemented using the simultaneous approximation term (SAT) method [6]. (The SAT method was originally developed for pseudo-spectral approximations [8, 9]. Examples of the pseudo-spectral penalty approach can be found in [13, 43].) Examples of the SBP-SAT approach can be found in [35, 26, 29, 30, 36, 24, 19, 7, 15, 14, 16, 22, 2].

The SBP-SAT approach has so far been developed for problems involving first and second derivatives in space. However, there are many problems where higher order derivatives are present. Some examples include the Korteweg-de Vries and the Boussinesq equations (describing nonlinear water waves), soliton models in neuroscience [38], the Euler-Lagrange equation for beams [42, 10], and the Cahn-Hilliard equation which describes the process of phase separation. Recently, high-order accurate SBP operators for third- and fourth derivatives were derived [32].

In contrast to hyperbolic problems [24, 25, 41, 3] solutions to the DBE are dispersive, meaning that the group velocity of a wave governed by the DBE is frequency dependent. Hence, the DBE is a dispersive wave equation. To capture the dispersive nature of the equation efficiently it is even more essential

that high-order (i.e. higher than second order) spatially accurate numerical methods are used to capture the high-frequency parts of the solution. The time-dependent Schrödinger equation is also a dispersive wave equation that have been successfully solved using the SBP-SAT method [2, 33, 34]. However, the numerical treatment of the DBE is much more challenging due to the high order derivative. Another distinction is that the DBE has a second derivative in time while the Schrödinger equation has a first derivative in time. (By splitting the Schrödinger equation into real and imaginary components it is possible to obtain an equation similar to the DBE for the real component.)

Since the wave speed scales as the frequency, the physics require that we take much smaller time-step (k) compared to the spatial grid-size (h), in order to resolve the fastest going waves (i.e., the high-frequency part of the solution). Hence, the physics require $k \simeq h^2$. In the present study we will derive an explicit time-integration with a CFL condition similar to the physical requirement. This physical restriction on the time-step is another motivation to employ spatially high-order accurate methods to allow for larger h . As the DBE involves a fourth derivative in space, the numerical boundary treatment is challenging. In particular the treatment of *clamped* boundary conditions, requires novel treatment when employing the SBP-SAT technique.

The main focus in the present study is to construct high-order accurate explicit (i.e., do not require solving any equation system to obtain the difference approximation) SBP-SAT approximations of the DBE, for quite general type of boundary conditions.

In Section 2 the DBE model is introduced, including the most common types of boundary conditions. In Section 3 the SBP-SAT method is introduced. Stability analysis of the various SBP-SAT approximations are presented in Section 4. Time integration is analysed and discussed in Section 5. In Section 6 the accuracy and stability properties of the newly developed SBP-SAT approximations are verified by performing numerical simulations. The extension to 2-D applications is addressed in Section 7. Section 8 summarizes the work. The SBP operator for the fourth-order accurate case is presented in the Appendix.

2 The dynamic beam equation

For a beam of length L with its axis along the x -direction, denote the deflection of the beam from its axis as $u(x, t)$. The governing equation of the one

dimensional DBE is then given by,

$$\begin{aligned} \mu \frac{\partial^2 u(x,t)}{\partial t^2} &= -\frac{\partial^2}{\partial x^2} \left(E(x)I(x) \frac{\partial^2 u(x,t)}{\partial x^2} \right) + F(x,t), \quad 0 \leq x \leq L, \quad t \geq 0 \\ u(x,0) &= f_1(x), \quad \frac{d}{dt}u(x,0) = f_2(x), \quad 0 \leq x \leq L, \end{aligned} \quad (1)$$

where $E(x)$ is the elastic modulus of the beam, $I(x)$ the second moment of area of the cross section of the beam and μ the mass per unit length. Here $f_{1,2}(x)$ are initial data and $F(x,t)$ a forcing function. For a homogeneous beam, E and I are independent of x .

Remark The stability analysis in the present study allows for variable coefficients, but would require SBP operators that can approximate $(a(x)u_{xx})_{xx}$. Such operators do not yet exist. In the present study we assume constant coefficients since SBP operators for constant coefficient fourth derivatives were recently derived in [32]. There are many applications that have piecewise constant coefficients. Treatment of piecewise constant coefficients have been done earlier for the acoustic wave equation using constant coefficient SBP operators by introducing internal interfaces (see for example [27, 24]). This extension is something we intend to analyse in a coming study.

Using the notation u_{xxxx} , u_t and u_{tt} for the fourth, first and second partial derivatives of $u(x,t)$ in space and time respectively, the DBE is reduced to,

$$\begin{aligned} \mu u(x,t)_{tt} &= -EIu(x,t)_{xxxx} + F(x,t), \quad 0 \leq x \leq L, \quad t \geq 0 \\ u(x,0) &= f_1(x), \quad u(x,0)_t = f_2(x), \quad 0 \leq x \leq L. \end{aligned} \quad (2)$$

As previously mentioned the DBE is a dispersive wave equation and it is therefore possible to formulate a dispersion relation. The dispersion relation $\omega = \omega(\kappa)$ determines how the temporal oscillations are linked to the spatial oscillations, i.e., it is the function for which the plane waves $e^{i\kappa x}e^{i\omega(\kappa)t}$ solves the PDE. Here, $\omega(\kappa)$ is the frequency and κ is the wave-number. Plugging $e^{i\kappa x}e^{i\omega(\kappa)t}$ into (2) and solving for $\omega(\kappa)$ results in

$$\omega(\kappa)^2 = \frac{EI}{\mu} \kappa^4 \implies \omega(\kappa) = \pm \sqrt{\frac{EI}{\mu}} |\kappa|^2.$$

The group velocity is then given by

$$\frac{\partial}{\partial \kappa} \omega(\kappa) = \pm 2 \sqrt{\frac{EI}{\mu}} \kappa, \quad (3)$$

showing that the group velocity (or wave speed) depends on the wave-number of the wave. In addition, the expression for the group velocity indicates that wave propagation will occur in two opposite directions.

Remark The dispersion relation (3) shows that the wave speed scales as κ . This means that we will have to take much smaller time-step (k) compared to the spatial grid-size (h), in order to resolve the fastest going waves (i.e., the high-frequency part of the solution). Hence, the physics require $k \simeq h^2$. This is clearly seen in Figure 4, where we initiate the simulation with a broadband Gaussian profile.

Dividing (2) through by EI , setting $c = \frac{\mu}{EI}$ and denoting the scaled forcing function $\tilde{F}$ results in,

$$\begin{aligned} cu(x, t)_{tt} &= -u(x, t)_{xxxx} + \tilde{F}(x, t), \quad 0 \leq x \leq L, \quad t \geq 0 \\ u(x, 0) &= f_1(x), \quad u(x, 0)_t = f_2(x), \quad 0 \leq x \leq L. \end{aligned} \tag{4}$$

This is the form of the DBE that will be analysed throughout the present study.

2.1 Wellposedness

Let the inner product for real-valued functions $u, v \in L^2[0, L]$ be defined by $(u, v) = \int_0^L u v dx$, and let the corresponding norm be $\|u\|^2 = (u, u)$. In the present study we focus on the Initial-Boundary value problem (IBVP) of (4). We will employ the energy method to analyse wellposedness of the most common type of boundary conditions. (In the present study we assume that the solution to the IBVP of Eq. 4 is unique, which can be trivially shown since the problem is linear.)

Remark The analysis of wellposedness is done for the homogeneous case, i.e., by setting the forcing function $\tilde{F}$ and boundary data to zero. This will be referred to as *the homogeneous version*. For certain boundary conditions a stronger energy estimate can be derived (such as strong wellposedness [12]) but this will not be analysed in the present study. Wellposedness for the non-homogeneous version follows directly from the energy analysis of the homogeneous version [12].

Multiplying the homogeneous version of (4) by u_t and integrating by parts (two times) leads to,

$$c(u_t, u_{tt}) = -u_t u_{xxx}|_0^L + (u_{tx}, u_{xxx}) = -u_t u_{xxx}|_0^L + u_{tx} u_{xx}|_0^L - (u_{txx}, u_{xx}).$$

By adding the transpose we obtain,

$$c(u_t, u_{tt}) + c(u_{tt}, u_t) = -2u_t u_{xxx}|_0^L + 2u_{tx} u_{xx}|_0^L - (u_{txx}, u_{xx}) - (u_{xx}, u_{txx}) .$$

This can be written,

$$\frac{d}{dt}E^{(c)} = -2u_t u_{xxx}|_0^L + 2u_{tx} u_{xx}|_0^L , \quad (5)$$

where the continuous energy $E^{(c)}$ is given by,

$$E^{(c)} = c\|u_t\|^2 + \|u_{xx}\|^2. \quad (6)$$

For $E^{(c)}$ to be an energy it is required that $E^{(c)} \geq 0$, which implies that both μ and EI have to be greater than zero, all in accordance with the underlying physics as both the mass per unit length μ and EI , which can be thought of as the 'stiffness' of the beam, are non-negative quantities. Furthermore, for the problem to be well-posed it is required that $\frac{d}{dt}E^{(c)} \leq 0$ and therefore there are restrictions on the boundary conditions.

The number of boundary conditions (BC) required to obtain a well-posed model is given by analysing the boundary terms in (5), rewritten like

$$\frac{d}{dt}E^{(c)} = BT|_0^1 = w^T G w|_0^L ,$$

where

$$w = \begin{bmatrix} u_t \\ u_{xt} \\ u_{xx} \\ u_{xxx} \end{bmatrix} \quad G = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} .$$

The number of BC at $x = L$ equals the number of positive eigenvalues to G , and the number of negative eigenvalues to G yields the correct number of BC at $x = 0$. We now diagonalise G ,

$$G = S \Lambda S^T ,$$

where $\Lambda = \text{diag}(1, 1, -1, -1)$ hold the eigenvalues to G . We have two positive and two negative eigenvalues, and hence 2 boundary conditions are to be prescribed at each boundary. The diagonalizer S holds the eigenvectors to G (the first column belongs to the first eigenvalue) and is given by (here we have normalised the eigenvectors),

$$S = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \end{bmatrix} .$$

We introduce the characteristic variables $\tilde{w} = S^T w$ and the boundary terms can be rewritten $w^T G w = \tilde{w}^T \Lambda \tilde{w} = \sum_{j=1}^4 \lambda_j \tilde{w}_j^2$, where λ_j is the j th eigenvalue and $\tilde{w}_j$ the corresponding characteristic variable. By specifying the characteristic variables corresponding to the two positive eigenvalues at $x = L$, i.e. $\tilde{w}_1$ and $\tilde{w}_2$, and the two characteristic variables corresponding to the two negative eigenvalues at $x = 0$, i.e. $\tilde{w}_3$ and $\tilde{w}_4$, we obtain the most dissipative set of well-posed boundary conditions given by,

$$\begin{aligned} u_{xt} - u_{xx} &= g_0^{(1)}(t), \quad u_t + u_{xxx} = g_0^{(2)}(t), \quad x = 0, \\ u_{xt} + u_{xx} &= g_L^{(1)}(t), \quad u_t - u_{xxx} = g_L^{(2)}(t), \quad x = L. \end{aligned} \quad (7)$$

where $g_0^{(1)}(t)$, $g_0^{(2)}(t)$, $g_L^{(1)}(t)$, $g_L^{(2)}(t)$ are boundary data. In the present study (7) will be referred to as *characteristic boundary conditions*, and introduce damping through the boundaries.

Inserting the characteristic boundary conditions (7) with zero boundary data ($g_{0,L}^{(1,2)}(t) = 0$) into (5) leads to

$$\frac{d}{dt} E^{(c)} = -2(u_t)^2|_L - 2(u_t)^2|_0 - 2(u_{tx})^2|_L - 2(u_{tx})^2|_0. \quad (8)$$

Hence, the solution will be dampened through the boundaries.

2.2 Boundary conditions

A less rigorous treatment of the boundary conditions can be made by simply studying the right hand side of (5) and choosing a set of boundary conditions (2 at each side) that fulfils the condition $\frac{d}{dt} E^{(c)} \leq 0$ (here assuming homogeneous boundary data and zero forcing).

A few of the most commonly used and well-posed boundary conditions (found in literature) for DBE are listed below,

$$\begin{aligned} u &= 0, \quad u_x = 0, & \text{'Clamped'}, \\ u_{xx} &= 0, \quad u_{xxx} = 0, & \text{'Free'}, \\ u &= 0, \quad u_{xx} = 0, & \text{'Hinged'}, \\ u_x &= 0, \quad u_{xxx} = 0, & \text{'Sliding'}. \end{aligned} \quad (9)$$

Next to each of the boundary conditions in (9) the physical interpretation of the boundary conditions imposed on a homogeneous beam is stated. All of these boundary conditions yield energy conservation, i.e., $\frac{d}{dt} E^{(c)} = 0$, assuming zero boundary data and zero forcing function.

Remark Any combination of the four different types of boundary conditions given by (9) yield energy conservation and thus wellposedness. Hence, we can for example specify *clamped* on one side and *sliding* on the other.

In the present study the first, third and fourth set of boundary conditions in (9), (here referred to as *clamped*, *hinged* and *sliding*) are of special interest as they are in particular difficult to impose using the SBP-SAT method. For wellposedness it is not necessary to impose homogeneous boundary conditions and thus a generalisation of the clamped boundary conditions in (9) is

$$\begin{aligned} u &= g_0^{(1)}(t), \quad u_x = g_0^{(2)}(t), \quad x = 0, \\ u &= g_L^{(1)}(t), \quad u_x = g_L^{(2)}(t), \quad x = L. \end{aligned} \tag{10}$$

3 The finite difference method

SBP operators are essentially central finite difference stencils closed at the boundaries with a careful choice of one-sided difference stencils, to mimic the underlying integration-by-parts formula in a discrete norm. In the present paper we address the SBP operators by the accuracy of the central scheme and the type of norm which they are based on.

3.1 Definitions

The following definitions are needed later in the present study. The domain $(0 \leq x \leq L)$ is discretized using the following m equidistant grid points:

$$x_i = (i - 1) h, \quad i = 1, 2, \dots, m, \quad h = \frac{1}{m-1}.$$

The approximate solution at grid point x_i is denoted v_i , and the discrete solution vector is $v^T = [v_1, v_2, \dots, v_m]$. Similarly, we define an inner product for discrete real-valued vector functions $u, v \in \mathbf{R}^m$ by $(u, v)_H = u^T H v$, where H is positive definite. The corresponding norm is $\|v\|_H^2 = v^T H v$.

Definition 3.1 *To simplify notation we will write v_t for the time-derivative of the discrete solution vector, i.e., $\frac{dv}{dt}$.*

The following vectors (and matrix) will be frequently used:

$$\begin{aligned} e_1 &= [1, 0, \dots, 0]^T, \quad e_m = [0, \dots, 0, 1]^T, \\ B &= e_m e_m^T - e_1 e_1^T \end{aligned} \tag{11}$$

The following definition (first stated in [30, 24]) is central to the present study:

Definition 3.2 *An explicit 2pth-order accurate finite difference scheme with minimal stencil width for the Cauchy problem is denoted a 2pth-order accurate narrow-stencil.*

3.2 SBP operator

The definitions for first- and second-derivative SBP operators can be found in earlier papers (see for example [6, 26, 30, 24, 23, 28, 31]). In a recent paper [32] third- and fourth-derivative SBP operators were derived. Finite difference SBP operators may be further categorized by the structure of their norm: a) diagonal, b) diagonal interior with block boundary closures, c) fully banded. In the present study we focus on diagonal-norm SBP operators.

For completeness we restate (first introduced in [32]) the definition of a fourth-derivative SBP operator:

Definition 3.3 *A difference operator*

$D_4 = H^{-1} (N - e_1 d_{3;1} + e_m d_{3;m} + d_{1;1}^T d_{2;1} - d_{1;m}^T d_{2;m})$ approximating $\partial^4/\partial x^4$, using a 2pth-order accurate narrow-stencil in the interior, is said to be a 2pth-order diagonal-norm fourth-derivative SBP operator if $H = H^T > 0$ is diagonal, $N = N^T \geq 0$, $d_{1;1}v \simeq u_x|_l$, $d_{1;m}v \simeq u_x|_r$, $d_{2;1}v \simeq u_{xx}|_l$, $d_{2;m}v \simeq u_{xx}|_r$, and $d_{3;1}v \simeq u_{xxx}|_l$, $d_{3;m}v \simeq u_{xxx}|_r$ are finite difference approximations of the first, second and third derivatives at the left and right boundary points.

The SBP operator for the fourth-order accurate case is presented in the Appendix. For further details concerning the various SBP operators we refer to [32], where the second- and sixth-order accurate SBP operators are presented.

Remark In [40] it is shown that a stable approximation of a PDE involving derivatives up to order s yields a convergence rate of order $r+s$, where r is the order of accuracy at the boundaries. If the highest spatial derivative order is 4 in the PDE (such as the DBE model), it is enough to employ zeroth-order accurate boundary stencils (in the case of fourth-order accurate SBP operators), and still obtain fourth order convergence, assuming stability. The second-order accurate fourth-derivative SBP operator is inconsistent with $r = -2$, and still this is enough to obtain second order convergence. The fourth-order accurate SBP operator have $r = 0$ (and we expect fourth order convergence). The sixth-order accurate SBP operator have $r = 1$ and we expect fifth order convergence, assuming that we employ a time discretisation that is accurate enough. The expected convergence behaviours are verified in tables 3, 4 and 5.

The following two novel lemmas are central to the present study:

Lemma 3.4 *The dissipative part N of a diagonal-norm fourth-derivative SBP operator has the following property:*

$$v^T N v = h \alpha_2 \left((d_{2;1}v)^2 + (d_{3;m}v)^2 \right) + v^T \tilde{N}_2 v , \quad (12)$$

where $\tilde{N}_2$ is symmetric and positive semi-definite, and α_2 a positive constant, independent of h .

Lemma 3.5 *The dissipative part N of a diagonal-norm fourth-derivative SBP operator has the following property:*

$$v^T N v = h^3 \alpha_3 \left((d_{3;1}v)^2 + (d_{3;m}v)^2 \right) + v^T \tilde{N}_3 v , \quad (13)$$

where $\tilde{N}_3$ is symmetric and positive semi-definite, and α_3 a positive constant, independent of h .

The values of α_2 and α_3 , were derived numerically for the second-, fourth- and sixth-order accurate finite difference SBP discretizations. The results are presented in Table 1.

$\alpha_2^{(2nd)}$	$\alpha_2^{(4th)}$	$\alpha_2^{(6th)}$	$\alpha_3^{(2nd)}$	$\alpha_3^{(4th)}$	$\alpha_3^{(6th)}$
1.250	0.548	0.322	0.4	1.088	0.156

Table 1: $\alpha_{2,3}$ in Eqs. 12 and 13, for the second-, fourth- and sixth-order accurate narrow-diagonal fourth-derivative SBP operators.

3.3 Semi-discrete boundary conditions

The semi-discrete approximations of the boundary conditions given by (9) can be written,

$$\begin{aligned} e_i v &= 0, & d_{1;i} v &= 0, & (i = 1, m) & \text{'Clamped'}, \\ d_{2;i} v &= 0, & d_{3;i} v &= 0, & (i = 1, m) & \text{'Free'}, \\ e_i v &= 0, & d_{1;i} v &= 0, & (i = 1, m) & \text{'Hinged'}, \\ d_{1;i} v &= 0, & d_{3;i} v &= 0, & (i = 1, m) & \text{'Sliding'}. \end{aligned} \quad (14)$$

The semi-discrete approximation of the characteristic boundary conditions (7) can be written,

$$\begin{aligned} d_{1;1} v_t - d_{2;1} v &= g_0^{(1)}(t), & e_1 v_t + d_{3;1} v &= g_0^{(2)}(t), & x = 0, \\ d_{1;m} v_t + d_{2;m} v &= g_L^{(1)}(t), & e_m v_t - d_{3;m} v &= g_L^{(2)}(t), & x = L. \end{aligned} \quad (15)$$

4 Stability analysis

In this section we present the SBP-SAT approximations of (2) combined with the various types of boundary conditions presented in Section 2.2. In the following we assume the same initial conditions $v = f_1$ and $v_t = f_2$ as in the continuous case.

Remark The stability analysis is done for the homogeneous case, i.e., by setting the forcing function $\tilde{F}$ and boundary data to zero. This will be referred to as *the homogeneous version*. For certain boundary conditions a stronger stability estimate can be derived (such as strong stability [12]) but this will not be done in the present study. Stability for the non-homogeneous problem follows directly from the stability analysis of the homogeneous problem [12].

4.1 Characteristic boundary conditions

A semi-discrete SBP-SAT approximation of the problem as formulated in (2) with the characteristic boundary conditions (7) is given by,

$$\begin{aligned} cv_{tt} = & -D_4v + \tau_0^{(1)}H^{-1}d_{1;1}^T \left(d_{1;1}v_t - d_{2;1}v - g_0^{(1)} \right) \\ & + \tau_0^{(2)}H^{-1}e_1 \left(e_1^T v_t + d_{3;1}v - g_0^{(2)} \right) \\ & + \tau_L^{(1)}H^{-1}S_m^T \left(d_{1;m}v_t + d_{2;m}v - g_L^{(1)} \right) \\ & + \tau_L^{(2)}H^{-1}e_m \left(e_m^T v_t - d_{3;m}v - g_L^{(2)} \right) + \tilde{F}. \end{aligned} \quad (16)$$

The following lemma is one of the main results in the present study:

Lemma 4.1 *Eq. 16 is stable if D_4 is a fourth-derivative SBP operator and $\tau_0^{(1)} = \tau_0^{(2)} = \tau_L^{(1)} = \tau_L^{(2)} = -1$ hold.*

Proof Multiplying the homogeneous version of (16) by $v_t^T H$, and adding the transpose leads to,

$$\begin{aligned} \frac{d}{dt}E_H = & -2(1 + \tau_0^{(1)})(d_{1;1}v_t)d_{2;1}v + 2(1 + \tau_0^{(2)})(v_t)_1d_{3;1}v \\ & + 2(1 + \tau_L^{(1)})(d_{1;m}v_t)d_{2;m}v - 2(1 + \tau_L^{(2)})(v_t)_md_{3;m}v \\ & + 2\tau_0^{(1)}(d_{1;1}v_t)^2 + 2\tau_0^{(2)}(v_t)_1^2 + 2\tau_L^{(1)}(d_{1;m}v_t)^2 + 2\tau_L^{(2)}(v_t)_m^2, \end{aligned}$$

where $E_H = c\|v_t\|_H^2 + v^T N v$ is the semi-discrete counterpart to the continuous energy $E^{(c)}$ given by (6). By choosing $\tau_0^{(1)} = \tau_0^{(2)} = \tau_L^{(1)} = \tau_L^{(2)} = -1$,

$$\frac{d}{dt}E_H = -2(S_1v_t)^2 - 2(v_t)_1^2 - 2(S_mv_t)^2 - 2(v_t)_m^2,$$

which exactly mimics the corresponding continuous energy estimate (8). $\square$

4.2 Free boundary conditions

A semi-discrete SBP-SAT approximation of the problem as formulated in (2) with the *free* boundary conditions is given by,

$$\begin{aligned}
cv_{tt} = & -D_4v - \tau_0^{(1)} H^{-1} d_{1;1}^T \left(d_{2;1}v - g_0^{(1)} \right) \\
& + \tau_0^{(2)} H^{-1} e_1 \left(d_{3;1}v - g_0^{(2)} \right) \\
& + \tau_L^{(1)} H^{-1} S_m^T \left(d_{2;m}v - g_L^{(1)} \right) \\
& - \tau_L^{(2)} H^{-1} e_m \left(d_{3;m}v - g_L^{(2)} \right) + \tilde{F}.
\end{aligned} \tag{17}$$

The following lemma is one of the main results in the present study:

Lemma 4.2 *Eq. 17 is stable if D_4 is a fourth-derivative SBP operator and $\tau_0^{(1)} = \tau_0^{(2)} = \tau_L^{(1)} = \tau_L^{(2)} = -1$ hold.*

Proof Multiplying the homogeneous version of (16) by $v_t^T H$, and adding the transpose leads to,

$$\begin{aligned}
\frac{d}{dt} E_H = & -2(1 + \tau_0^{(1)})(d_{1;1}v_t)d_{2;1}v + 2(1 + \tau_0^{(2)})(v_t)_1 d_{3;1}v \\
& + 2(1 + \tau_L^{(1)})(d_{1;m}v_t)d_{2;m}v - 2(1 + \tau_L^{(2)})(v_t)_m d_{3;m}v,
\end{aligned}$$

where $E_H = c\|v_t\|_H^2 + v^T N v$ is the semi-discrete counterpart to the continuous energy $E^{(c)}$ given by (6). By choosing $\tau_0^{(1)} = \tau_0^{(2)} = \tau_L^{(1)} = \tau_L^{(2)} = -1$,

$$\frac{d}{dt} E_H = 0,$$

which mimic the corresponding continuous energy estimate. $\square$

4.3 Clamped boundary conditions

A semi-discrete approximation of (4) with the *clamped* boundary conditions using the SBP-SAT method is then given by

$$\begin{aligned}
cv_{tt} = & -D_4v + H^{-1}(d_{3;1} - \tau_0^{(1)} e_1^T)^T \left(e_1^T v - g_0^{(1)} \right) \\
& - H^{-1}(d_{2;1} + \tau_0^{(2)} d_{1;1})^T \left(d_{1;1}v - g_0^{(2)} \right) \\
& - H^{-1}(d_{3;m} + \tau_L^{(1)} e_m^T)^T \left(e_m^T v - g_L^{(1)} \right) \\
& + H^{-1}(d_{2;m} - \tau_L^{(2)} d_{1;m})^T \left(d_{1;m}v - g_L^{(2)} \right) + \tilde{F}.
\end{aligned} \tag{18}$$

The following lemma is one of the main results in the present study:

Lemma 4.3 *Eq. 18 is stable if D_4 is a fourth-derivative SBP operator, and $\tau_{0,L}^{(1)} \geq \frac{2}{h^3\alpha_3}$, $\tau_{0,L}^{(2)} \geq \frac{2}{h\alpha_2}$ hold.*

Proof Multiplying the homogeneous version of (18) by $v_t^T H$, and adding the transpose leads to,

$$\begin{aligned} \frac{d}{dt} (\|v_t\|_H^2 + v^T N v) &= \frac{d}{dt} \left(2(v_1)d_{3;1}v - 2(d_{1;1}v)d_{2;1}v - \tau_0^{(1)}v_1^2 - \tau_0^{(2)}(d_{1;1}v)^2 \right) \\ &\quad - \frac{d}{dt} \left(2(v_m)d_{3;m}v - 2(d_{1;m}v)d_{2;m}v + \tau_L^{(1)}v_m^2 + \tau_L^{(2)}(d_{1;m}v)^2 \right) \end{aligned}$$

By using Lemmas 3.4, 3.5, and the fact that D_4 is a diagonal-norm fourth-derivative SBP operator we obtain,

$$\frac{d}{dt} \left(\tilde{E}_H + w_1^T A_1 w_1 + w_m^T A_m w_m \right) = 0,$$

where $\tilde{E}_H = \|v_t\|_H^2 + \frac{1}{2}v^T \tilde{N}_2 v + \frac{1}{2}v^T \tilde{N}_3 v$, and

$$\begin{aligned} w_1 &= \begin{bmatrix} v_1 \\ d_{1;1}v \\ d_{2;1}v \\ d_{3;1}v \end{bmatrix}, \quad A_1 = \begin{bmatrix} \tau_0^{(1)} & 0 & 0 & -1 \\ 0 & \tau_0^{(2)} & 1 & 0 \\ 0 & 1 & \frac{h}{2}\alpha_2 & 0 \\ -1 & 0 & 0 & \frac{h^3}{2}\alpha_3 \end{bmatrix}, \\ w_m &= \begin{bmatrix} v_m \\ d_{1;m}v \\ d_{2;m}v \\ d_{3;m}v \end{bmatrix}, \quad A_m = \begin{bmatrix} \tau_L^{(1)} & 0 & 0 & 1 \\ 0 & \tau_L^{(2)} & -1 & 0 \\ 0 & -1 & \frac{h}{2}\alpha_2 & 0 \\ 1 & 0 & 0 & \frac{h^3}{2}\alpha_3 \end{bmatrix}. \end{aligned}$$

By choosing $\tau_{0,L}^{(1)} \geq \frac{2}{h^3\alpha_3}$, $\tau_{0,L}^{(2)} \geq \frac{2}{h\alpha_2}$, $A_{1,m}$ are positive semi-definite and $\tilde{E}_H + w_1^T A_1 w_1 + w_m^T A_m w_m$ defines a semi-norm. $\square$

4.4 Sliding boundary conditions

A semi-discrete approximation of (4) with the *sliding* boundary conditions using the SBP-SAT method is then given by

$$\begin{aligned} cv_{tt} &= -D_4 v - H^{-1}(d_{2;1} + \tau_0 d_{1;1})^T \left(d_{1;1}v - g_0^{(1)} \right) \\ &\quad - H^{-1}e_1 \left(d_{3;1}v - g_0^{(2)} \right) \\ &\quad + H^{-1}(d_{2;m} - \tau_L d_{1;m})^T \left(d_{1;m}v - g_L^{(1)} \right) \\ &\quad + H^{-1}e_m \left(d_{3;m}v - g_L^{(2)} \right) + \tilde{F}. \end{aligned} \tag{19}$$

The following lemma is one of the main results in the present study:

Lemma 4.4 *Eq. 19 is stable if D_4 is a fourth-derivative SBP operator, and $\tau_{0,L} \geq \frac{1}{h\alpha_2}$ hold.*

Proof Multiplying the homogeneous version of (19) by $v_t^T H$, and adding the transpose leads to,

$$\begin{aligned} \frac{d}{dt} (\|v_t\|_H^2 + v^T N v) &= \frac{d}{dt} (-2(d_{1;1}v)d_{2;1}v - \tau_0(d_{1;1}v)^2) \\ &\quad - \frac{d}{dt} (-2(d_{1;m}v)d_{2;m}v + \tau_L(d_{1;m}v)^2) \end{aligned}$$

By using Lemma 3.4 and the fact that D_4 is a diagonal-norm fourth-derivative SBP operator we obtain,

$$\frac{d}{dt} \left(\tilde{E}_H + w_1^T A_1 w_1 + w_m^T A_m w_m \right) = 0,$$

where $\tilde{E}_H = \|v_t\|_H^2 + v^T \tilde{N}_2 v$, and

$$\begin{aligned} w_1 &= \begin{bmatrix} d_{1;1}v \\ d_{2;1}v \end{bmatrix}, & A_1 &= \begin{bmatrix} \tau_0 & 1 \\ 1 & h\alpha_2 \end{bmatrix}, \\ w_m &= \begin{bmatrix} d_{1;m}v \\ d_{2;m}v \end{bmatrix}, & A_m &= \begin{bmatrix} \tau_L & -1 \\ -1 & h\alpha_2 \end{bmatrix}. \end{aligned}$$

By choosing $\tau_{0,L} \geq \frac{1}{h\alpha_2}$, $A_{1,m}$ are positive semi-definite and $\tilde{E}_H + w_1^T A_1 w_1 + w_m^T A_m w_m$ defines a semi-norm. $\square$

To elucidate the importance of Lemma 4.4, we present the numerical solution of (19) with two different choices of penalty parameters $\tau_{0,L}$ in Figure 1. Choosing the stability limit, i.e., $\tau_{0,L} = \frac{1}{h\alpha_2}$ leads to inaccurate results, which is clearly visible close to the boundaries. By making the penalty parameters twice as big the results improved dramatically. If Lemma 4.4 is violated, setting $\tau_{0,L} \leq \frac{1}{h\alpha_2}$ the solution diverges quickly. This behaviour is not limited to this particular boundary condition, we also see it for the other types of boundary conditions. Hence, as a rule of thumb, avoid choosing the penalty parameters exactly as the stability limit.

Figure 1: The numerical solution of (19) with two different stable choices of penalty parameters $\tau_{0,L}$. The initial data is presented in Figure 3(a). In (a) on the stability limit (see Lemma 4.4). Solutions presented at $t=0.01$, and $m = 201$. Notice the large deviation close to the boundaries.

4.5 Hinged boundary conditions

A semi-discrete approximation of (4) with the *hinged* boundary conditions using the SBP-SAT method is then given by

$$\begin{aligned}
cv_{tt} = & -D_4v + H^{-1}(d_{3;1} - \tau_0 e_1^T)^T \left(e_1^T v - g_0^{(1)} \right) \\
& + H^{-1}d_{1;1}^T \left(d_{2;1}v - g_0^{(2)} \right) \\
& - H^{-1}(d_{3;m} + \tau_L e_m^T)^T \left(e_m^T v - g_L^{(1)} \right) \\
& - H^{-1}d_{1;m}^T \left(d_{2;m}v - g_L^{(2)} \right) + \tilde{F}.
\end{aligned} \tag{20}$$

The following lemma is one of the main results in the present study:

Lemma 4.5 *Eq. 20 is stable if D_4 is a fourth-derivative SBP operator, and $\tau_{0,L} \geq \frac{1}{h^3\alpha_3}$ hold.*

Proof Multiplying the homogeneous version of (20) by $v_t^T H$, and adding the transpose leads to,

$$\begin{aligned}
\frac{d}{dt} (\|v_t\|_H^2 + v^T N v) = & \frac{d}{dt} (2v_1 d_{3;1} v - \tau_0 v_1^2) \\
& - \frac{d}{dt} (2v_m d_{3;m} v + \tau_L v_m^2)
\end{aligned}$$

By using Lemma 3.5 and the fact that D_4 is a diagonal-norm fourth-derivative SBP operator we obtain,

$$\frac{d}{dt} \left(\tilde{E}_H + w_1^T A_1 w_1 + w_m^T A_m w_m \right) = 0,$$

where $\tilde{E}_H = \|v_t\|_H^2 + v^T \tilde{N}_3 v$, and

$$w_1 = \begin{bmatrix} v_1 \\ d_{3;1}v \end{bmatrix}, \quad A_1 = \begin{bmatrix} \tau_0 & -1 \\ -1 & h^3 \alpha_3 \end{bmatrix},$$

$$w_m = \begin{bmatrix} v_m \\ d_{3;m}v \end{bmatrix}, \quad A_m = \begin{bmatrix} \tau_L & 1 \\ 1 & h^3 \alpha_3 \end{bmatrix}.$$

By choosing $\tau_{0,L} \geq \frac{1}{h^3 \alpha_3}$, $A_{1,m}$ are positive semi-definite and $\tilde{E}_H + w_1^T A_1 w_1 + w_m^T A_m w_m$ defines a semi-norm. $\square$

5 Time integration

The time-discretisation of the semi-discrete SBP-SAT models presented in Section 4 will be analysed. The first model (16) have both a first and second derivative in time while the rest of the problems have only a second derivative in time. For hyperbolic problems involving both first and second derivatives in time one often (see for example [11, 28]) rewrite the system on first order form by introducing an auxiliary variable, and time-integrate using an explicit Runge-Kutta method. However, for dispersive wave equations such as DBE this approach can lead to a very restrictive CFL condition, if first derivative terms are present.

Consider the following ODE system (with m unknowns),

$$\begin{aligned} v_{tt} &= -H^{-1}Cv_t - H^{-1}Av + H^{-1}G(t), & t \geq 0, \\ v(0) &= f_1, v_t(0) = f_2, & t = 0, \end{aligned} \tag{21}$$

where f_1 and f_2 are initial data, $G(t)$ a known time-dependent vector function. A and C are $m \times m$ matrices.

Definition 5.1 *If $A = A^T$ is positive semi-definite and $H = H^T$ is positive definite,*

$$E_H = \|v_t\|_H^2 + v^T A v,$$

is non-negative and $E_H = 0$ imply that $v_t = 0$. Hence, E_H defines a semi-norm.

Lemma 5.2 *If $A = A^T \geq 0$, $C + C^T \geq 0$ and $H = H^T > 0$, the ODE system (21) is stable.*

Proof Multiply the homogeneous version of (21) by $v_t^T H$, and adding the transpose leads to,

$$\begin{aligned} v_t^T H v_{tt} + v_{tt}^T H^T v_t &= -v_t^T A v - v^T A^T v_t - v_t^T C v_t - v_t^T C^T v_t \\ &\Downarrow \\ \frac{d}{dt} E_H &= -v_t^T (C + C^T) v_t . \end{aligned}$$

In the last step we use the fact that $A = A^T$ is positive semi-definite and $H = H^T$ is positive definite such that $E_H = \|v_t\|_H^2 + v^T A v$ defines a semi-norm (see Definition 5.1). If $C + C^T \geq 0$ the time-growth of E_H is non-positive. $\square$

The semi-discrete SBP-SAT models presented in Section 4 fulfil (21) such that the conditions in Lemma 5.2 hold.

Remark The second order ODE system (21) allows for linear time-growth if A is positive semi-definite (even with homogeneous data, i.e., $G(t) = 0$). This however requires two things: 1) The zero eigenvalue is a double root, and 2) $f_2 \neq 0$. This is consistent with the characteristics of the underlying IBVP, that also allows linear time-growth, when there are zero double roots (see [32]). If A is negative definite, there can be no time-growth. Nevertheless, the ODE system is stable if A is negative semi-definite. (See [12] for details regarding the various types of stability definitions.)

5.1 Explicit Runge-Kutta methods

When first derivative terms are present in the DBE model, an explicit Runge-Kutta method have severe CFL restrictions (see (23)). This has been verified in numerical computations of (16) using the fourth order Runge-Kutta method.

To explain this behaviour we will study a scalar ODE system involving both first and second derivative terms,

$$\begin{aligned} y_{tt} &= -\alpha y_t - \beta y + g(t) , \\ y(0) &= f_1 , y_t(0) = f_2 , \end{aligned} \tag{22}$$

where f_1 and f_2 are initial data, and $g(t)$ known time-dependent data.

Remark Starting from (21) and assuming periodic boundary conditions (here referred to as the *Cauchy* problem), we can simultaneously diagonalise $H^{-1}C$ and $H^{-1}A$, and for each Fourier mode derive a scalar ODE of the form given by (22). The Cauchy problem do share much of the characteristics with the corresponding IBVP, when it comes to expected the CFL condition. This motivates the study of a scalar ODE system, concerning the expected CFL condition for a similar ODE system.

The solution to the homogeneous version of (22) can be written $y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$, where $c_{1,2}$ are give by the initial data and $r_{1,2}$ by solving the characteristic equation $r^2 - \alpha r - \beta = 0$, i.e., $r_{1,2} = -\frac{\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}$.

To time-integrate (22) using the standard fourth order Runge-Kutta method, an auxiliary variable $p = y_t$ is introduced, to obtain an ODE-system with only first derivative terms. The CFL restriction is given by $k \cdot \max |r_{1,2}| \leq 2.8$, where k denotes the time-step.

We now consider the case $\alpha = a \cdot h^{-3}$ and $\beta = b \cdot h^{-4}$, in (22), where a and b are constants independent of h .

Remark This particular choice of h -scaling is motivated by the ODE system (16). The h -scaling (where h is the spatial grid-size) of $H^{-1}A$ and $H^{-1}C$ are given by h^{-4} and h^{-3} respectively.

Hence, for sufficiently small grid-spacing h , $\max |r_{1,2}| \simeq a \cdot h^{-3}$, and the CFL condition is given by

$$k \leq \frac{2.8 h^3}{a}. \quad (23)$$

Hence, we expect a very severe time-step restriction when using an explicit Runge-Kutta method to solve (16). In fact, this is exactly what we found by extensive numerical testing. For this reason we will use a finite difference method to time-integrate (16). This will be analysed in the coming section.

Remark The dispersion relation (3) shows that the wave speed scales as κ . Hence, the physics require $k \simeq h^2$, to capture the high-frequency part of the solution. An efficient time-integrator should therefore allow a time-step similar to what is required from the physical consideration. This is why we refer to (23) as restrictive. Ideally we want something similar to (24).

For the other boundary conditions in Section 4, the first derivative component $H^{-1}Cv_t$ is not present. This will lead to the CFL condition given by

$$k \leq \frac{2.8 h^2}{b}, \quad (24)$$

which agree with the physical requirement. In the present study we will therefore employ a compact finite difference method that yield a CFL similar to (24) also when first derivatives are present in the ODE model.

5.2 First and second derivatives in time

An alternative time-discretisation technique (compared to Runge-Kutta methods) is to employ a finite difference method. For the case with both first and second derivative terms present in (21), higher than second-order accurate approximations are far from trivial and will not be investigated in the present study. However, we will see that the CFL condition with the finite difference method will yield an estimate similar to (24). Hence, a second-order accurate finite difference approximation (in time) will lead to fourth order convergence due to the CFL condition (assuming that the problem is stable).

Let $t_n = nk$, $n = 0, 1, \dots$ denote the discrete time-levels, where k is the time-step, and introduce the notation $v^n = v(t_n)$. We introduce the following second-order accurate finite difference approximations, $D_0 v^n = \frac{v^{n+1} - v^{n-1}}{2k}$ and $D_+ D_- v^n = \frac{v^{n+1} - 2v^n + v^{n-1}}{k^2}$, of the first and second derivatives in (21). The first derivative in the initial step can be approximated with $D_+ v^0 = \frac{v^1 - v^0}{k}$. By replacing the first derivative in the initial step by $D_+ v^0$ and inserting appropriate Taylor expansions we obtain the following second-order accurate approximation of (21),

$$\begin{aligned} H \frac{v^{n+1} - 2v^n + v^{n-1}}{k^2} &= -Av^n - C \frac{v^{n+1} - v^{n-1}}{2k} + G^n, n = 1, \dots \\ v^0 &= f_1, \quad \frac{v^1 - f_1}{k} = f_2 - \frac{k}{2} H^{-1} (A f_1 + C f_2 - H G^0). \end{aligned} \quad (25)$$

Definition 5.3 *If $A = A^T$ is positive semi definite and $B = B^T$ is positive definite,*

$$E^n = (D_+ v^n, B D_+ v^n) + \left(\frac{v^{n+1} + v^n}{2}, A \frac{v^{n+1} + v^n}{2} \right),$$

is non-negative for all v^n, v^{n+1} and $E^n = 0$ imply $v^{n+1} = v^n$. Hence, E^n defines a semi-norm. Here we define $(D_+ v^n, B D_+ v^n) = (D_+ v^n)^T B (D_+ v^n)$.

The following Lemma is central to the present study,

Lemma 5.4 *The matrix $(H - \frac{k^2}{4} A)$ is positive definite if the time-step is chosen as $k^2 < 4 \frac{\min(h_i)}{\max(a_i)}$, where h_i and a_i are the eigenvalues to H and A , respectively.*

Proof The eigenvalue problem $(H - \frac{k^2}{4}A)X = \lambda X$ can be written $(I - \frac{k^2}{4}H^{-1}A)X = \lambda H^{-1}X$. Since A and H are symmetric and H is positive definite, the eigenvalues to $H^{-1}A$ are given by $\frac{a_i}{h_i}$, where a_i and h_i are the eigenvalues to A and H respectively (since this is a generalized symmetric-definite eigenvalue problem). Hence, the eigenvalues to $I - \frac{k^2}{4}H^{-1}A$ are given by $1 - \frac{k^2}{4}\frac{a_i}{h_i}$. To guarantee that λ_i are positive, it is enough to require that $1 - \frac{k^2}{4}\frac{a_i}{h_i} > 0$ (since $h_i > 0$). By choosing $k^2 < 4\frac{\min(h_i)}{\max(a_i)}$ we guarantee that $\lambda_i > 0$. $\square$

We will make use of the following two relations,

$$\begin{aligned} & (v^{n+1} - v^{n-1}, H(v^{n+1} + v^{n-1})) + (v^{n+1} + v^{n-1}, H(v^{n+1} - v^{n-1})) \\ &= 2(v^{n+1} - v^n, H(v^{n+1} - v^n)) + 4(v^{n+1}, Hv^n) \\ &- 2(v^n - v^{n-1}, H(v^n - v^{n-1})) - 4(v^n, Hv^{n-1}) \end{aligned} \quad (26)$$

$$\begin{aligned} (v^{n+1}, Av^n) + (v^n, Av^{n+1}) &= \frac{1}{2}(v^{n+1} + v^n, A(v^{n+1} + v^n)) \\ &- \frac{1}{2}(v^{n+1} - v^n, A(v^{n+1} - v^n)) \end{aligned} \quad (27)$$

The following Theorem is one of the main results of the study.

Theorem 5.5 *Eq. 25 is a stable approximation of (21) if Lemmas 5.2 and 5.4 hold.*

Proof Start by setting $G = 0$ (since G does not affect stability), and rewriting (25) to,

$$H \frac{v^{n+1} + v^{n-1}}{k^2} = - \left(A - 2\frac{H}{k^2} \right) v^n - C \frac{v^{n+1} - v^{n-1}}{2k}.$$

Multiplying by $\left(\frac{v^{n+1} - v^{n-1}}{2k} \right)^T$ leads to,

$$\begin{aligned} \left(\frac{v^{n+1} - v^{n-1}}{2k}, H \left(\frac{v^{n+1} + v^{n-1}}{k^2} \right) \right) &= - \left(\frac{v^{n+1} - v^{n-1}}{2k}, \left(A - 2\frac{H}{k^2} \right) v^n \right) \\ &- \left(\frac{v^{n+1} - v^{n-1}}{2k}, C \left(\frac{v^{n+1} - v^{n-1}}{2k} \right) \right). \end{aligned}$$

By adding the transpose, assuming $H = H^T$ and $A = A^T$, we obtain

$$\begin{aligned} & \left(\frac{v^{n+1} - v^{n-1}}{2k}, H \left(\frac{v^{n+1} + v^{n-1}}{k^2} \right) \right) + \left(\frac{v^{n+1} + v^{n-1}}{k^2}, H \left(\frac{v^{n+1} - v^{n-1}}{2k} \right) \right) \\ &= - \left(\frac{v^{n+1} - v^{n-1}}{2k}, \left(A - 2\frac{H}{k^2} \right) v^n \right) - \left(v^n, \left(A - 2\frac{H}{k^2} \right) \frac{v^{n+1} - v^{n-1}}{2k} \right) \\ & - \left(\frac{v^{n+1} - v^{n-1}}{2k}, (C + C^T) \left(\frac{v^{n+1} - v^{n-1}}{2k} \right) \right) . \end{aligned}$$

By using the two relations (26) and (27) we obtain,

$$\begin{aligned} & \left(D_+ v^n, \left(H - \frac{k^2}{4} A \right) D_+ v^n \right) + \left(\frac{v^{n+1} + v^n}{2}, A \frac{v^{n+1} + v^n}{2} \right) = \\ & \left(D_+ v^{n-1}, \left(H - \frac{k^2}{4} A \right) D_+ v^{n-1} \right) + \left(\frac{v^{n-1} + v^n}{2}, A \frac{v^{n-1} + v^n}{2} \right) \\ & - k (D_0 v^n, (C + C^T) D_0 v^n) . \end{aligned}$$

If Lemma 5.4 holds this can be written

$$\frac{E^{n+1} - E^n}{k} = - (D_0 v^n, (C + C^T) D_0 v^n) ,$$

where the semi-norm E^n is given by Definition 5.3, where $B = \left(H - \frac{k^2}{4} A \right)$ is positive definite. If $(C + C^T) \geq 0$ we conclude that the time-growth of the semi-norm E^n is bounded. $\square$

The fully discrete energy estimate in Theorem 5.5 is completely analogous to the semi-discrete energy estimate in Lemma 5.2, assuming that we fulfil the CFL condition, $k^2 < 4 \frac{\min(h_i)}{\max(a_i)}$. For the semi-discrete SBP-SAT models presented in Section 4, h_j is proportional to h and a_j is proportional to h^{-3} , implying that the CFL conditions is given by (24) for some constant $b > 0$. (The value of b depends on the type of boundary condition and the specific SBP operator employed.) In Table 2 we present sharp estimates of the CFL, i.e., $\alpha = \frac{k}{h^2}$ for the different types BC and accuracy of the SBP operator.

5.3 Second derivative in time

Now we consider the case where there is no first derivative present in the ODE system (21), i.e., $C = 0$. A compact and explicit high-order accurate time marching method can be achieved by following the technique employed in [27].

By replacing the first and second derivatives in (21) by D_+v^0 and $D_+D_-v^n$, and inserting appropriate Taylor expansions we obtain,

$$\begin{aligned}\frac{v^{n+1}-2v^n+v^{n-1}}{k^2} &= v_{tt}^n + \frac{k^2}{12}v_{tttt}^n + \mathcal{O}(k^4), n = 1, \dots \\ \frac{v^1-v^0}{k} &= v_t^0 + \frac{k}{2}v_{tt}^0 + \frac{k^2}{6}v_{ttt}^0 + \mathcal{O}(k^3) .\end{aligned}$$

By using (21) we can replace the time derivatives (sometimes referred to as the *modified equation approach*) to obtain the following compact fourth-order accurate explicit time discretisation (although the initial step is third-order accurate),

$$\begin{aligned}\frac{v^{n+1}-2v^n+v^{n-1}}{k^2} &= -H^{-1}Av^n + H^{-1}G(t_n) \\ &\quad + \frac{k^2}{12}(H^{-1}A(H^{-1}Av^n - H^{-1}G(t_n)) + H^{-1}G_{tt}(t_n)), n = 1, \dots \\ \frac{v^1-f_1}{k} &= f_2 - \frac{k}{2}(H^{-1}Af_1 - H^{-1}G(0)) - \frac{k^2}{6}(H^{-1}Af_2 - H^{-1}G_t(0)) .\end{aligned}$$

This can be written in the more convenient form (from a computational point of view),

$$\begin{aligned}v^0 &= f_1 , \\ v^1 &= (I_m - \frac{k^2}{2}H^{-1}A)f_1 + k(I_m - \frac{k^2}{6}H^{-1}A)f_2 + \frac{k^2}{2}H^{-1}G(0) + \frac{k^3}{6}H^{-1}G_t(0) , \\ v^{n+1} &= (2I_m - k^2H^{-1}A + \frac{k^4}{12}(H^{-1}A)^2)v^n - v^{n-1} \\ &\quad + k^2(I_m - \frac{k^2}{12}H^{-1}A)H^{-1}G(t_n) + \frac{k^4}{12}H^{-1}G_{tt}(t_n), n = 1, \dots\end{aligned}\tag{28}$$

where I_m is the unit $m \times m$ matrix. Stability of (28) follows if

$$\rho(2I - k^2H^{-1}A + \frac{k^4}{12}(H^{-1}A)^2) \leq 2 ,$$

where $\rho(M)$ represents the spectral radius of M . (For further details see [27].)

Remark The above stability requirement can only be satisfied if $H^{-1}A$ is positive semi-definite, which is guaranteed if $A = A^T$ is positive semi-definite and $H = H^T$ is positive definite.

Let $\alpha = \frac{k}{h^2}$ and introduce $\tilde{H}^{-1}\tilde{A} = h^4H^{-1}A$. The CFL condition can be written,

$$\alpha \leq \frac{\sqrt{12}}{\sqrt{\rho(\tilde{H}^{-1}\tilde{A})}} .$$

Hence, the CFL condition is equivalent to (24) for some constant $b > 0$.

Accuracy / BC	<i>Characteristic</i>	<i>Clamped</i>	<i>Free</i>	<i>Hinged</i>	<i>Sliding</i>
<i>2nd</i>	0.500	0.730	0.867	0.774	0.867
<i>4th</i>	0.377	0.491	0.650	0.569	0.569
<i>6th</i>	0.218	0.318	0.650	0.517	0.372

Table 2: Sharp estimates of $\alpha = \frac{k}{h^2}$ for the different types of BC and accuracy of the SBP operator.

Remark The spectral radius of $\tilde{H}^{-1}\tilde{A}$ is independent of h for sufficiently large m (typically $m > 50$) but problem dependent. Hence, it is sufficient to compute $\rho(\tilde{H}^{-1}\tilde{A})$ for a small m to obtain a fairly sharp estimate of the CFL condition.

In Table 2 we present sharp estimates of the CFL, i.e., $\alpha = \frac{k}{h^2}$ for different types of BC and accuracy of the SBP operator.

Remark The particular multi-step method (that we refer to as the finite difference method in time) have the attractive property of being time-symmetric (since it is a central finite difference approximation). Explicit Runge-Kutta methods or non-central multi-step methods do not share this property. For a non-dissipative semi-discrete ODE system, time-symmetric integration imply discrete energy-conservation.

6 Computations

In this section the accuracy and efficiency properties of the SBP-SAT approximations analysed in Section 4 will be verified. We will focus in particular on the cases with characteristic and clamped BC. The SBP-SAT approximation with characteristic BC given by (16) is time-integrated with the second-order accurate finite difference approximation (25). The SBP-SAT approximation with clamped BC given by (18) is time-integrated with the fourth-order accurate finite difference approximation (28).

6.1 Convergence study

The convergence rate is calculated as

$$q = \log_{10} \left(\frac{\|u - v^{(m_1)}\|_h}{\|u - v^{(m_2)}\|_h} \right) / \log_{10} \left(\frac{m_1}{m_2} \right)^{1/d}, \quad (29)$$

where d is the dimension ($d = 1$ in the 1-D case), u is the analytic solution, and $v^{(m_1)}$ the corresponding numerical solution with m_1 unknowns. $\|u - v^{(m_1)}\|_h$ is the discrete l^2 norm of the error.

Remark For the case of characteristic boundary conditions we employ a second-order accurate time discretisation and we expect fourth order convergence (since we have a CFL, $k < \alpha h^2$ for stability) for both the fourth- and sixth-order accurate SBP-SAT approximations. For the other types of BC we employ a fourth-order accurate time discretisation and the convergence is not affected. (Hence, the sixth-order accurate SBP-SAT approximations should give fifth order convergence.) The expected convergence behaviours are verified in tables 3 and 4.

The function,

$$u(x, t) = \sin(\kappa x) \cos\left(\frac{\kappa}{\sqrt{c}} t^2\right), \quad (30)$$

is an analytic solution to the DBE equation (4). In the following we will use this solution as initial and boundary data, in order to verify the accuracy and convergence behaviour of the SBP-SAT approximations. In the present study we set $\kappa = 7\pi$ and $c = 1$. The numerical approximations are integrated in time to $t = 1$ with a time step $k = 0.1h^2$. (In Table 2 sharp CFL estimates are presented.)

The SBP-SAT approximation with characteristic BC is given by (16). In Table 3 we present the convergence results using the second- fourth- and sixth-order accurate SBP operators. The time-integrator is second-order accurate, but due to the CFL condition (see Table 2) we expect fourth order (temporal) convergence for the fourth- and sixth-order accurate SBP-SAT approximations. To obtain the optimal fifth order (spatial) convergence a higher order version of (25) is required. That is however out of scope in the present study, but is something we hope to address in a coming study.

m	$\log l^{2(2nd)}$	$q^{(2nd)}$	$\log l^{2(4th)}$	$q^{(4th)}$	$\log l^{2(6th)}$	$q^{(6th)}$
101	-1.72		-3.20		-3.53	
201	-2.32	1.99	-4.40	3.97	-4.71	3.93
301	-2.68	2.00	-5.10	3.99	-5.41	3.98
401	-2.93	2.00	-5.60	3.99	-5.91	4.00

Table 3: $\log(l^2 - errors)$ and convergence rates comparing the second-, fourth- and sixth-order accurate SBP-SAT approximations (16) at $t = 1$.

The SBP-SAT approximation with clamped BC is given by (18). In Table 4 we present the convergence results using the second- fourth- and sixth-order accurate SBP operators. The time-integrator is fourth-order accurate, and due to the CFL condition (see Table 2) we expect full order (spatial) convergence. The penalty parameters were set to $\tau_{0,L}^{(1)} = \frac{2}{h^3 \alpha_3}$, $\tau_{0,L}^{(2)} = \frac{2}{h \alpha_2}$.

m	$\log l^2(2^{nd})$	$q^{(2nd)}$	$\log l^2(4^{th})$	$q^{(4th)}$	$\log l^2(6^{th})$	$q^{(6th)}$
101	-1.52		-3.64		-4.47	
201	-2.32	2.67	-4.60	3.19	-6.29	6.05
301	-2.47	0.87	-5.28	3.81	-7.21	5.18
401	-2.68	1.67	-5.75	3.81	-7.89	5.47

Table 4: $\log(l^2 - errors)$ and convergence rates comparing the second-, fourth- and sixth-order accurate SBP-SAT approximations (18) at $t = 1$.

6.2 Efficiency study

In Figure 2 we compare the efficiency for different orders of accuracy. We plot the $l^2 - error$ as a function of runtime. The efficiency study is performed for both clamped and characteristic BC. We integrate to $t = 1$ with the optimal CFL presented in Table 2. This can be compared to the convergence analysis found in Tables 3 and 4. The results clearly support the claim that an efficient solution of the DBE model requires a stable and higher-order accurate numerical method.

Figure 2: The l^2 -error as a function of runtime for the second-, fourth- and sixth-order accurate SBP-SAT approximations. The solution is integrated to $t = 1$. We present the results for both characteristic and clamped boundary conditions.

In Figure 3 we present a direct comparison of three different BC: Clamped, Hinged and Free, to elucidate the different boundary behaviours. As initial data we use a Gaussian centered at $x = 0.6$. The solutions are presented at $t = 0.01$ using $m = 201$ grid-points.

Figure 3: Numerical solution of the DBE model with different boundary conditions (Free, Clamped and Hinged). For (b)-(d) the solution is presented at $t=0.01$, and $m = 201$.

7 Two-dimensional simulation

A two-dimensional extension of the DBE model with clamped BC is given by,

$$\begin{aligned}
 cu_{tt} &= -(u_{xxxx} + u_{yyyy}), & 0 \leq (x, y) \leq 1, \quad t \geq 0 \\
 u &= u_x = u_y = 0, & (x, y) \in \Omega, \quad t \geq 0, \\
 u &= e^{-1000((x-0.5)^2 + (y-0.5)^2)}, \quad u_t = 0, & 0 \leq (x, y) \leq 1, \quad t = 0,
 \end{aligned} \tag{31}$$

where Ω is the boundary of the unit square. The construction of the corresponding SBP-SAT approximation is performed by treating the different spatial dimensions separately. It is therefore useful to derive a corresponding

1-D SBP-SAT approximation first. For more details on how to construct multi-D SBP-SAT approximations from the corresponding 1-D problem, see for example [31, 21, 41].

The SBP-SAT approximation of the corresponding 1-D problem is given by (18), which can be written, $v_{tt} = -Mv$, assuming homogeneous data. M is a positive semi-definite $m \times m$ -matrix if Lemma 4.3 holds. The 2-D domain is here discretized using m points in both the x - and y -direction (to simplify notation). Let I_m denote the unit matrix of size $m \times m$. A discrete approximation of the 2-D problem (31) can be written $v_{tt} = -((M \otimes I_m) + (I_m \otimes M))v$, where we make use of the Kronecker product:

$$C \otimes D = \begin{bmatrix} c_{0,0} D & \cdots & c_{0,q-1} D \\ \vdots & & \vdots \\ c_{p-1,0} D & \cdots & c_{p-1,q-1} D \end{bmatrix},$$

where C is a $p \times q$ matrix and D is an $m \times n$ matrix. The discrete solution is now defined as a "vector of vectors" of the form

$$v^T = [v_{1,1}, v_{1,2}, \dots, v_{1,m}, v_{2,1}, v_{2,2}, \dots, v_{2,m}, \dots, v_{x_i, y_j}, \dots, v_{m,1}, v_{m,2}, \dots, v_{m,m}]$$

Stability of the 2-D problem is guaranteed if the corresponding 1-D problem is stable (i.e., that M is positive semi-definite).

Figure 4: A sixth-order accurate simulation of (31) using the SBP-SAT method with 401 unknowns (i.e., $h = 0.0025$) at 6 different times. The initial data (a) is a Gaussian pulse. Notice the dispersive wave propagation with the fast high-frequency part.

The initial data (see Figure 4) is a Gaussian pulse, and thus contains of

a spectrum of frequencies. Thus, due to the dispersive nature of the DBE, if the SBP-SAT method is implemented correctly the wave packet should break apart into many wave-fronts moving at different speed depending on their frequencies. A simulation of the 2-D model using 401×401 grid-points is presented in Figure 4 for 6 different times. Here we employ a sixth-order accurate SBP-SAT approximation with time-step $k = 0.1h^2$.

To test the accuracy properties of the 2-D problem we performed a convergence study, shown in Table 5, comparing the second-, fourth- and sixth-order SBP-SAT discretisations. The initial data is a Gaussian pulse. The solution is then advanced and recorded at $t = 0.003$, when the dispersive waves have reached the boundaries (see Figure 5). For the present case we have no analytic solution. Instead a highly resolved simulation with the sixth-order accurate SBP-SAT discretisation using 801×801 grid-points is employed as a reference solution. To obtain a confident measure on the rate of convergence requires (i) a fairly grid-converged reference solution, ii) that the numerical method is proven stable and accurate, and (iii) that tested solutions are far from grid-convergence. All of these assumptions are fulfilled in the present convergence study for the 2-D problem.

The point wise relative errors are presented in Figure 5 for the second- and sixth-order accurate cases with 401×401 grid-points. The solutions on the coarsest grid (51×51) are poorly resolved for all discretisations, which explains the low convergence rate in the second row (in Table 5). An efficiency study is presented in Figure 6. We plot the $l^2 - error$ as a function of runtime. We integrate to $t = 0.003$ with the optimal CFL presented in Table 2. This can be compared to the convergence analysis found in Table 5. The results clearly support the claim that an efficient solution of the DBE model requires a stable and higher-order accurate numerical method.

m	$\log l^{2(2nd)}$	$q^{(2nd)}$	$\log l^{2(4th)}$	$q^{(4th)}$	$\log l^{2(6th)}$	$q^{(6th)}$
51×51	-1.75		-2.16		-2.37	
101×101	-2.22	1.58	-2.64	1.58	-3.21	2.79
201×201	-2.58	1.19	-4.00	4.54	-4.80	5.30
401×401	-3.16	1.91	-5.15	3.80	-6.59	5.95

Table 5: $\log(l^2 - errors)$ and convergence rates comparing the second-, fourth- and sixth-order accurate SBP-SAT approximations of (31) at $t = 0.003$. We compare against a reference solution with 801×801 grid-points using the sixth-order accurate discretisation.

Figure 5: Comparing the point wise relative errors for the second- and sixth-order accurate simulations of (31) at $t = 0.003$, using 401×401 grid-points . The initial data and the solution at $t = 0.003$ are shown in Figure 4.

Figure 6: The l^2 -error as a function of runtime for the second-, fourth- and sixth-order accurate SBP-SAT approximations of (31). The solution is integrated to $t = 0.003$. Compare with Table 5.

8 Conclusions and future work

The main focus has been to construct explicit, stable and high-order accurate SBP-SAT approximations of the DBE model with various boundary conditions. To guarantee stability we employ novel SBP operators (derived in [32]) to approximate the fourth-derivative terms, combined with the SAT technique to impose the boundary conditions weakly. One of the main results in the present study is the proof that it is necessary to "borrow" from the fourth-derivative SBP operator in order to derive a stable boundary treatment.

Numerical computations in 1-D and 2-D corroborate the stability- and accuracy-properties and show that the higher-order accurate approximations are superior to the corresponding second-order accurate SBP-SAT schemes.

Another main result concerns the CFL conditions for the DBE models involving both first and second derivatives in time. It is shown that explicit Runge-Kutta methods have severe CFL restrictions and a better choice is to use a finite difference method.

In a coming study we will derive a higher-order accurate finite difference approximation of the temporal derivatives when both first and second derivatives are present. We will also extend the SBP-SAT method to DBE models with non-constant coefficients.

APPENDIX

Below we list the fourth-order accurate fourth-derivative SBP operator (for the second- and sixth-order accurate cases see [32]). The fourth-derivative SBP operator is given by,

$$D_4 = H^{-1} (N - e_1 d_{3;1} + e_m d_{3;m} + d_{1;1}^T d_{2;1} - d_{1;m}^T d_{2;m}) .$$

The boundary closures (the coefficients) are presented for the H and N matrices. Notice that the coefficients in the norm should be multiplied by the grid-step h , and the coefficients in N should be multiplied by $\frac{1}{h^3}$.

The boundary derivative operators are given by

$$\begin{aligned} d_{1;1}v &= \frac{-11v_1+18v_2-9v_3+2v_4}{6h}, & d_{1;m}v &= \frac{+11v_m-18v_{m-1}+9v_{m-2}-2v_{m-3}}{6h} \\ d_{2;1}v &= \frac{2v_1-5v_2+4v_3-v_4}{h^2}, & d_{2;m}v &= \frac{2v_m-5v_{m-1}+4v_{m-2}-v_{m-3}}{h^2} \\ d_{3;1}v &= \frac{-v_1+3v_2-3v_3+v_4}{h^3}, & d_{3;m}v &= \frac{+v_m-3v_{m-1}+3v_{m-2}-v_{m-3}}{h^3} \end{aligned}$$

The internal schemes of D_4 is given by

$$(D_4v)_j = \frac{-v_{j-3} + 12v_{j-2} - 39v_{j-1} + 56v_j - 39v_{j+1} + 12v_{j+2} - v_{j+3}}{6h^4}$$

The upper part of the norm H is given by,

$$\begin{aligned} H_{1,1} &= \frac{35809}{100800} & H_{3,3} &= \frac{5701}{5600} & H_{5,5} &= \frac{35191}{33600} \\ H_{2,2} &= \frac{13297}{11200} & H_{4,4} &= \frac{45109}{50400} & H_{6,6} &= \frac{33503}{33600} \end{aligned}$$

The upper part of N is given by,

$$\begin{aligned} N_{1,1} &= \frac{4596181}{1814400} & N_{2,3} &= -\frac{9558943}{907200} & N_{3,6} &= -\frac{120619}{907200} \\ N_{1,2} &= -\frac{10307743}{1814400} & N_{2,4} &= \frac{2177057}{907200} & N_{4,4} &= \frac{3146581}{453600} \\ N_{1,3} &= \frac{160961}{43200} & N_{2,5} &= -\frac{11351}{86400} & N_{4,5} &= -\frac{4614143}{907200} \\ N_{1,4} &= -\frac{535019}{907200} & N_{2,6} &= \frac{204257}{1814400} & N_{4,6} &= \frac{24587}{14400} \\ N_{1,5} &= \frac{109057}{1814400} & N_{3,3} &= \frac{4938581}{453600} & N_{5,5} &= \frac{185709}{22400} \\ N_{1,6} &= -\frac{29273}{604800} & N_{3,4} &= -\frac{786473}{151200} & N_{5,6} &= -\frac{11293343}{1814400} \\ N_{2,2} &= \frac{8368543}{604800} & N_{3,5} &= \frac{1141057}{907200} & N_{6,6} &= \frac{16787381}{1814400} \end{aligned}$$

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