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Stability analysis of an adaptively sampled controller for SISO systems with nonlinear feedback*

Johannes Nygren and Kristiaan Pelckmans

Abstract—In this paper, a stability criterion is derived for variable sampled SISO systems where the feedback loop is subject to a sector bounded non-linearity. The new stability criterion is closely related to the well-known circle stability criterion. The main motivation for variable sampling is to reduce communication cost when the feedback is supported by battery driven, energy limited wireless communication links. An energy aware variable sampling control synthesis for heat regulation in a room is also given.

I. INTRODUCTION

Wireless sensor networks (WSNs) are becoming a more and more fundamental part of control systems for maintaining closed loop feedback. The reasons are that WSN nodes are relatively cheap and easy to deploy and maintain since no cabling is required. Communicating through a wireless channel, the network forwards data packets from sensors to actuators, subject to packet losses, channel fading, variable time delays and other typical features of a wireless network. Previous work has been done to mitigate the impact of these network specific drawbacks on closed loop performance, see e.g. [3], [4] and [12]. However, for short range WSNs with relatively few nodes, these network features play a much less important role and the sole challenge that remains is that of WSN energy conservation.

Since WSN nodes mostly utilize batteries as the sole energy source, the WSN has a limited energy budget which must be used wisely in order to avoid costly and cumbersome battery replacements too frequently. Given that the radio chip is the primary energy consumer in a wireless node [10], the most natural way to conserve energy is to minimize the amount of transmissions while still keeping the closed loop performance on an acceptable level, i.e. to try to "sample only when necessary".

The problem of communication energy constrained control is still rather open, and people currently attack the problem from different angles. Two main approaches are event-based or event-triggered control [16], [17], [11] where the process state is measured continuously and a data transmission executes when the state reaches a certain threshold, and self-triggered control [15], where the control algorithm decides when it needs to be executed again based on current state

measurements. Self-triggered control conditions for stability have been developed for homogeneous and polynomial nonlinear systems [1], in order to decrease unnecessary use of CPU resources in embedded control systems. An energy-aware model predictive control (MPC) scheme for short-range WSNs has been developed which incorporates communication constraints [2].

The main contribution of this paper is the development of a stability condition for a class of linear systems subject to variable sampling and nonlinear sector-bounded feedback. If the sampling period is assumed to be fixed, the new stability condition coincides exactly with the well-known circle stability criterion.

The open loop system is assumed to incorporate a single integrator, and the open loop impulse response is assumed to be finite and monotonically non-decreasing. Such open loop behaviour is common e.g. in process industry [6], where systems with monotonic step responses are regulated by PI controllers. The sector-bounded feedback nonlinearity may be time-varying and is often representing data quantization.

The link between sensor and actuator is assumed to be supported by a single wireless sensor node. The act of sensing is assumed to consume a negligible amount of energy compared to the act of transmission, and therefore the node can measure the output without sending which allows for a combined event-based and self-triggered control strategy. Furthermore, the sensor node is assumed to know the reference signal and therefore sends control error measurements to the actuator, subject to quantization in order to decrease the data packet size.

To avoid misconceptions, it is important to note that the term "sampling time" refers to the time instances when the sensor node transmits and the actuator receives data (which is assumed to happen at the same time instance), and "sampling period" refers to the time difference between two consecutive sampling time instances.

A combined self-triggered and event-based control algorithm is applied to control the temperature in a room in a simulation study, which is a typical application of wireless sensor networks [13], [7]. Stability of the algorithm is supported by the theory. The simulations suggest a very practical design procedure; i.e. design the controller first, assuming a fixed sampling period, and then design the self-triggered/event-based sampling rule as an add-on to decrease unnecessary communication. Another paper dealing with combined self-triggered/event-based sampling algorithms is [9].

The paper is organized as follows. Section II gives the

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formal framework and states the main stability result of the paper. Section III illustrates the theory with a simulation study of a heat control system. Section IV concludes the paper. The proof of the main result in Section II is given in Appendix.

The following notation is used. Bold face is used to denote matrices and vectors. Scalars are either upper- or lower case. $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}^+$ is the set of all non-negative real numbers including zero, $\mathbb{N}$ is the set of all strictly positive integers, and $\mathbb{N}_0$ is the set of all non-negative integers including zero. The differentiation operator is denoted as p and the forward shift operator is denoted as q . The vector with all ones is denoted $\mathbf{1}$. The function $\text{floor}(x, y) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ returns the value ay , where $a = \arg \max_{b \in \mathbb{N}_0} : by \leq x$.

II. GENERAL SETTING AND MAIN RESULTS

Assume that there is a continuous time, SISO open loop system

$$z(t) = G(p)\sigma(t), \quad (1)$$

where $z(t) \in \mathbb{R}$ is the output signal and $\sigma(t) \in \mathbb{R}$ is the open loop input signal at time $t \in \mathbb{R}^+$. $G(p)$ is a transfer operator with impulse response $g : \mathbb{R}^+ \rightarrow \mathbb{R}$, and therefore, (1) can equivalently be described as

$$z(t) = \int_0^\infty g(\tau)\sigma(t-\tau)d\tau. \quad (2)$$

Let $r(t) \in \mathbb{R}$ be the reference signal at time $t \in \mathbb{R}^+$, and denote the control error as

$$\varepsilon(t) = r(t) - z(t). \quad (3)$$

Assume that there is a sensor which measures $z(t)$, and that the sensor is aware of the reference signal $r(t)$. Assume also that this sensor forwards quantized output error measurements at time instances $\{t_k\}_{k=0}^\infty$, which satisfies

$$t_0 = 0, \quad (4)$$

$$t_{k+1} = t_k + \Delta_k, \quad (5)$$

$$\infty > d_{\max} \geq \Delta_k \geq d_{\min} > 0. \quad (6)$$

Hence, the following sensor model is implemented

$$\bar{\varepsilon}(t) = f(t, \varepsilon(t)), \quad (7)$$

$$\hat{\varepsilon}(t) = \bar{\varepsilon}(t_k) \quad \forall t \in [t_k, t_{k+1}), \quad (8)$$

where $f : \mathbb{R}^+ \times \mathbb{R} \rightarrow \mathbb{R}$ satisfies the following sector bound

$$K_{\min}x^2 \leq xf(t, x) \leq K_{\max}x^2, \quad \forall t, x \in \mathbb{R}, \quad (9)$$

for some $0 < K_{\min} \leq K_{\max}$.

The actuator scales the quantized error measurement with a time-varying gain

$$\sigma(t) = \gamma(t)\hat{\varepsilon}(t). \quad (10)$$

The gain $\gamma(t) \in \mathbb{R}$ is assumed to be strictly positive and piecewise constant in the same way as $\hat{\varepsilon}(t)$, i.e.

$$\gamma(t) > 0 \quad \forall t \in \mathbb{R}^+, \quad (11)$$

$$\gamma(t) = \gamma(t_k) \quad \forall t \in [t_k, t_{k+1}). \quad (12)$$

Before proceeding to the main stability result, two standard definitions of bounded function sets are needed.

Definition 1: Define $\mathcal{A}$ as the set of stable and causal impulse responses in continuous time, i.e. impulse responses $g : \mathbb{R}^+ \rightarrow \mathbb{R}$ of the form

$$g(t) = \sum_{k=0}^\infty g_k \delta(t - t_k) + \tilde{g}(t), \quad (13)$$

where $\delta(t)$ is the Dirac delta function, $0 \leq t_0 < t_1 < \dots$, $\tilde{g} : \mathbb{R}^+ \rightarrow \mathbb{R}$ is measurable, $\sum_{k=0}^\infty |g_k| < \infty$, and $\int_0^\infty |\tilde{g}(\tau)|d\tau < \infty$.

For all $g \in \mathcal{A}$, the corresponding Laplace transforms $G(s)$ are well-defined and converge for all $\text{Re } s \geq 0$. For a proof of the fact that $\mathcal{A}$ is actually the set of all continuous time, causal and stable impulse responses, see [14], Theorem 6.4.30.

It should also be noted that systems with impulse responses in $\mathcal{A}$ are quite general since they might be infinite dimensional such as, for example, systems with time delays.

Definition 2: The set L_2 consists of all measurable functions $y : \mathbb{R}^+ \rightarrow \mathbb{R}$ which are bounded in the following sense

$$\int_0^\infty y(\tau)^2 d\tau < \infty. \quad (14)$$

Theorem 1: Consider the closed loop system (1)-(12). Assume that the open loop transfer operator $G(p)$ is stable except for a single integrator, i.e. that the open loop impulse response is given by $g(t) = g_\infty + \bar{g}(t)$, where $0 < g_\infty < \infty$, $\int_0^\infty |\bar{g}(\tau)|d\tau < \infty$, and $\bar{g}(t) \in \mathcal{A}$. Define $H(q) = \sum_{k=1}^\infty h(k)q^{-k}$, $h(k) = \int_{(k-1)d_{\min}}^{kd_{\min}} g(\tau)d\tau$ as the discrete time transfer operator obtained when sampling $G(p)$ using zero order hold and sampling period $d_{\min}$. Assume that at least one of the following conditions hold:

- 1) The open loop impulse response is monotonically non-decreasing, i.e. $(g(t) - g(\tau))(t - \tau) \geq 0$ for all $t, \tau \in \mathbb{R}$.
- 2) $(h(k) - h(j))(k - j) \geq 0$ for all $k, j \in \mathbb{N}_0$ and $\Delta_k \in \{ad_{\min} : a \in \mathbb{N}\}$.

Also assume that the reference signal satisfies $r(t) = \tilde{r}(t) + r_\infty$ for some $r_\infty \in \mathbb{R}$ and $\tilde{r} \in L_2$. Under these assumptions, the closed loop system is L_2 -stable in the sense that $\varepsilon \in L_2$ if the time-varying actuator gain satisfies

$$\alpha < \gamma(t_k) < \frac{d_{\min}}{\Delta_k K_{\max} \lim_{\omega \rightarrow 0} [-\text{Re } H(e^{i\omega})]} - \alpha, \quad (15)$$

for some $\alpha > 0$.

Proof. See appendix. $\square$

Remark 1: If the sampling period and the actuator gain are constant, i.e. if $\Delta_k = d_{\min}$ and $\gamma(t) = \gamma$, then the stability bound (15) of Theorem 1 coincides exactly with the circle stability criterion. From the assumptions on the open loop impulse response it follows that $\lim_{\omega \rightarrow 0} [-\text{Re } H(e^{i\omega})] = \sup_{\omega \in \mathbb{R}} [-\text{Re } H(e^{i\omega})]$ (see the derivations (53) and (54) in Appendix) and therefore, the stability criterion becomes

$$\gamma < \frac{1}{K_{\max} \sup_{\omega \in \mathbb{R}} [-\text{Re } H(e^{i\omega})]}. \quad (16)$$

This is equivalent to the circle criterion which states that the closed loop system is stable if $1/K_{\max} + \gamma H(q)$ is positive real. Hence, Theorem 1 can be seen as a step towards a generalization of the circle criterion for systems with non-uniform sampling.

III. SIMULATION

Theorem 1 might be used to design self-triggered control algorithms, or possibly combined event- and self-triggered algorithms. In this section, a combined event- and self-triggered algorithm is applied in a simulation study to control the temperature in a room. The system is given by

$$z(t) = \frac{0.5}{(0.03p + 1)(0.7p + 1)}u(t) + \frac{0.01p + 0.5}{(0.03p + 1)(0.7p + 1)}w(t), \quad (17)$$

where $z(t) + 20^\circ\text{C}$ is the indoor temperature (process variable), $w(t) - 10^\circ\text{C}$ is the outdoor temperature (disturbance), and $u(t) + 50.6^\circ\text{C}$ is the water temperature in the radiator (manipulated variable). Also, the time scale is in hours. The time constant of the radiator temperature is 0.03 hours and the time constant of the indoor temperature is 0.7 hours. The system and problem setting is taken from [5], example 7.4.

The goal of the controller is to keep the indoor temperature within $20 \pm 1^\circ\text{C}$ as the outdoor temperature behaves in the following way

$$w(t) = \frac{1}{p + 1}d(t), \quad (18)$$

where $d(t)$ is any signal in the interval $\pm 10^\circ\text{C}$. Figure 1 shows $w(t)$ during the simulation time, which is 24 hours.

First, a PI controller is designed assuming a fixed sampling period of 0.21 hours. The sampling period is 30% of the time constant in the room, and the PI controller is tuned to achieve $20 \pm 1^\circ\text{C}$ indoor temperature at this sampling period. The PI controller is given by

$$u(t) = \frac{\gamma(0.7\bar{p} + 1)}{\bar{p}}(-z(t)), \quad (19)$$

where $\gamma = 8.95$. The sampling period is at the high end of the rule of thumb given in [18] (Chapter 6) for PI controllers, i.e. a sampling period between 10 – 30% of the integration time, which is 0.7 hours.

The controller is sampled by the Tustin formula, i.e.

$$\bar{p} = \frac{2}{0.21} \frac{e^{-0.21p} - 1}{e^{-0.21p} + 1}. \quad (20)$$

Note that the reference signal is zero, i.e. $\varepsilon(t) = r(t) - z(t) = -z(t)$. Also, condition 2 and the assumptions on the open loop system $G(p)$ given in Theorem 1 holds. Therefore if (15) is satisfied, then closed loop L_2 -stability is guaranteed.

Next, a combined event- and self-triggered algorithm is supplied as an add-on to the PI controller, with $d_{\min} = 0.21$ hours. It is assumed that a sensor node measures the indoor temperature each $t = ad_{\min}$ time instant, for all $a \in \mathbb{N}$. At each measuring instant, the node decides whether or not to transmit the measurement to the actuator at the radiator site. Each transmitted data packet at each transmission time t_k contains the measurement $\hat{\varepsilon}(t_k) = \varepsilon(t_k)$.

Fig. 1. Outdoor temperature = $w(t) - 10^\circ\text{C}$.

After transmission and receiving of a data packet at time t_k , the sensor and actuator computes $\bar{t}_{k+1}$ which is an upper bound to the next transmission time t_{k+1} . In this way the sensor and actuator does not have to communicate $\bar{t}_{k+1}$ between each other. The following self-trigger rule is used to determine $\bar{t}_{k+1}$

$$\bar{t}_{k+1} = t_k + \bar{\Delta}_k, \quad (21)$$

$$\bar{\Delta}_k = \text{floor} \left(\frac{4.2 - d_{\min}}{1 + e^{100(|\varepsilon(t_k)| - 0.1)}} + d_{\min}, d_{\min} \right). \quad (22)$$

It turns out that $d_{\max} = \text{floor}((4.2 - d_{\min})/(1 + e^{-10}) + d_{\min}, d_{\min}) = 19d_{\min}$. If one wants to avoid energy consuming calculations at the sensor site, $\bar{\Delta}_k$ can be calculated using a lookup-table. Based on $\bar{t}_{k+1}$, the actuator implements the following gain

$$\gamma(t_k) = d_{\min} / \bar{\Delta}_k. \quad (23)$$

Assuming that the previous transmission was at t_k , the sensor node decides to transmit again at time instant $t = ad_{\min}$ if either $t = \bar{t}_{k+1}$ or if

$$|\varepsilon(ad_{\min})| > |\varepsilon(ad_{\min} - d_{\min})| + 0.05. \quad (24)$$

Hence, the control law becomes a combined event based and self-triggered algorithm. It follows from Theorem 1 that the closed loop system is L_2 -stable since (15) holds.

Figures 2 and 3 show the simulation results for two control laws. The solid line (A1) is the combined self-triggered and event-based algorithm, and the dash-dotted line (A2) is the PI controller with a fixed transmission time equal to $d_{\min}$. The upper graph in Figure 2 mainly indicates two things. First, it is seen that the control specification of keeping $z(t)$ inside ± 1 is satisfied everywhere for both control laws. Secondly, A1 behaves approximately the same as A2 when $z(t)$ is close to ± 1 , which occurs when there are large fluctuations in the outdoor temperature. When the outdoor temperature disturbances are smaller, A2 makes the indoor temperature fluctuate less than A1, but that is not important

Fig. 2. The upper graph shows indoor temperature = $z(t) + 20^\circ\text{C}$, and the lower graph shows the cumulative number of sensor-to-actuator transmissions.

Fig. 3. Water temperature in radiator = $u(t) + 50.6^\circ\text{C}$.

since the goal is to keep $z(t)$ inside ± 1 . This indicates that a reasonable design procedure is to design a fixed controller with fixed sampling period $d_{\min}$, and then design the self-triggered/event-based transmission algorithm in order to decrease unnecessary communication. To summarize, A1 is able to maintain the same control performance as A2 while reducing the number of transmissions by about 45% in this particular simulation.

IV. CONCLUSIONS

In this paper, a stability criterion was derived for a class of linear systems with monotonically non-decreasing impulse responses, subject to variable sampling periods and sector-bounded nonlinear feedback. A simulation study of climate control in a room was included to illustrate the effectiveness of the variable sampling approach. The variable sampling approach was able to decrease the number of samples by

45% compared to the corresponding fixed sampling controller. Furthermore, the simulations suggest that a reasonable synthesis approach is to first design the controller for a fixed sampling period, and then to design the variable sampling strategy as an add-on to decrease unnecessary sampling.

This paper opens up some interesting future research directions. In future publications we (the authors) will extend results to non-monotone systems, systems without integrators, and MIMO systems.

APPENDIX - PROOF OF THEOREM 1

The proof of Theorem 1 relies on the following generic Lemma. The lemma defines a set of gain sequences $\{\gamma(t_k)\}_{k=0}^\infty$ for which the closed loop system is L_2 -stable.

Lemma 1: Consider the closed loop system (1)-(12). Assume that the open loop transfer operator $G(p)$ is stable except for a single integrator, i.e. that the open loop impulse response is given by $g(t) = g_\infty + \bar{g}(t)$, where $0 < g_\infty < \infty$, $\int_0^\infty |\bar{g}(\tau)| d\tau < \infty$, and $\dot{\bar{g}}(t) \in \mathcal{A}$. Also assume that the reference signal satisfies $r(t) = \tilde{r}(t) + r_\infty$ for some $r_\infty \in \mathbb{R}$ and $\tilde{r} \in L_2$. Under these assumptions, the closed loop system is L_2 -stable in the sense that $\varepsilon \in L_2$ if the gain sequence $\{\gamma(t_k)\}_{k=0}^\infty$ satisfies

$$\beta \leq \gamma(t_k) \Delta_k \leq \frac{2}{K_{\max} x_k} - \beta, \quad (25)$$

for some $\beta > 0$, where

$$x_k = g_\infty + \sum_{j=0}^{k-1} \frac{1}{\Delta_j} \left| \int_{t_k - t_{j+1}}^{t_k - t_j} g_\infty - g(\tau) d\tau \right| + \sum_{j=k+1}^\infty \frac{1}{\Delta_k} \left| \int_{t_j - t_{k+1}}^{t_j - t_k} g_\infty - g(\tau) d\tau \right|. \quad (26)$$

Proof. First, define $\sigma^* : \mathbb{R}^+ \rightarrow \mathbb{R}$ as any piecewise stationary function which satisfies

$$\sigma^*(t) = \sigma^*(t_k), \quad t \in [t_k, t_{k+1}). \quad (27)$$

A specific choice of σ^* will be made later. Also define $\varepsilon^*(t) = r(t) - G(p)\sigma^*(t)$ for all $t \geq 0$. Introduce the following vectors

$$\varepsilon = [\varepsilon(t_0), \varepsilon(t_1), \dots, \varepsilon(t_N)]^T \in \mathbb{R}^{N+1}, \quad (28)$$

$$\varepsilon_* = [\varepsilon^*(t_0), \varepsilon^*(t_1), \dots, \varepsilon^*(t_N)]^T \in \mathbb{R}^{N+1}, \quad (29)$$

$$\sigma = [\sigma(t_0)\Delta_0, \dots, \sigma(t_N)\Delta_N]^T \in \mathbb{R}^{N+1}, \quad (30)$$

$$\sigma_* = [\sigma^*(t_0)\Delta_0, \dots, \sigma^*(t_N)\Delta_N]^T \in \mathbb{R}^{N+1}. \quad (31)$$

Then define $\mathbf{X} = \text{diag}(\{x_k\}_{k=0}^N) \in \mathbb{R}^{(N+1) \times (N+1)}$ and

$$\mathbf{H} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ h_{1,1} & 0 & 0 & \dots & 0 \\ h_{1,2} & h_{2,2} & 0 & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \\ h_{1,N} & & h_{N-1,N} & h_{N,N} & 0 \end{bmatrix}, \quad (32)$$

where

$$h_{k,j} = \frac{1}{\Delta_{k-1}} \int_{t_j - t_k}^{t_j - t_{k-1}} g(\tau) d\tau. \quad (33)$$

Assuming that the system initializes from rest at $t = t_0 = 0$, it follows from the definitions (1), (3) and (10), as well

as the piecewise constant properties (8),(12) and (27), that $\mathbf{H}(\boldsymbol{\sigma}_* - \boldsymbol{\sigma}) = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_*$. Therefore,

$$\begin{aligned}
& (\boldsymbol{\sigma}_* - \boldsymbol{\sigma})^T [\mathbf{H} + \mathbf{H}^T + \mathbf{X}] (\boldsymbol{\sigma}_* - \boldsymbol{\sigma}) \\
&= 2(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_*)^T (\boldsymbol{\sigma}_* - \boldsymbol{\sigma}) + (\boldsymbol{\sigma}_* - \boldsymbol{\sigma})^T \mathbf{X} (\boldsymbol{\sigma}_* - \boldsymbol{\sigma}) \\
&= \boldsymbol{\sigma}^T (\mathbf{X}\boldsymbol{\sigma} - 2\boldsymbol{\varepsilon}) + 2\boldsymbol{\varepsilon}_*^T (\boldsymbol{\sigma} - \boldsymbol{\sigma}_*) + 2\boldsymbol{\sigma}_*^T (\boldsymbol{\varepsilon} - \mathbf{X}\boldsymbol{\sigma}) + \boldsymbol{\sigma}_*^T \mathbf{X}\boldsymbol{\sigma}_* \\
&= \sum_{k=0}^N \Delta_k \sigma(t_k) \{x_k \Delta_k \sigma(t_k) - 2\varepsilon(t_k)\} \\
&+ 2 \sum_{k=0}^N \Delta_k \varepsilon^*(t_k) \{\sigma(t_k) - \sigma^*(t_k)\} \\
&+ 2 \sum_{k=0}^N \Delta_k \sigma^*(t_k) \{\varepsilon(t_k) - x_k \Delta_k \sigma(t_k)\} + \sum_{k=0}^N x_k \Delta_k^2 \sigma^*(t_k)^2.
\end{aligned} \tag{34}$$

Now choose

$$\sigma^*(t) = \begin{cases} r_\infty/(g_\infty \Delta_0), & t \in [0, t_1], \\ 0, & t \geq t_1, \end{cases} \tag{35}$$

which implies that

$$\begin{aligned}
\varepsilon^*(t) &= \tilde{r}(t) - \int_0^t \bar{g}(t-\tau) \sigma^*(\tau) d\tau + r_\infty - g_\infty \int_0^t \sigma^*(\tau) d\tau \\
&=_{(t \geq t_1)} \tilde{r}(t) - \frac{r_\infty}{g_\infty \Delta_0} \int_0^{t_1} \bar{g}(t-\tau) d\tau.
\end{aligned} \tag{36}$$

The triangle inequality implies that

$$\varepsilon_*^T \varepsilon_* \leq \sum_{k=0}^N \tilde{r}(t_k)^2 + \frac{r_\infty^2}{g_\infty^2 \Delta_0^2} \sum_{k=1}^N \left(\int_0^{t_1} \bar{g}(t_k - \tau) d\tau \right)^2. \tag{37}$$

The first term of (37) is bounded for all N by assumption. Schwarz' inequality on the second term gives

$$\left(\int_0^{t_1} \bar{g}(t_k - \tau) d\tau \right)^2 \leq t_1 \int_0^{t_1} \bar{g}(t_k - \tau)^2 d\tau = t_1 \int_{t_k - t_1}^{t_k} \bar{g}(\tau)^2 d\tau. \tag{38}$$

Since $\dot{\bar{g}} \in \mathcal{A}$, $\bar{g}(t)$ is well-defined for all $t \in \mathbb{R}^+$. This fact, together with the boundedness assumption $\int_0^\infty |\bar{g}(\tau)| d\tau < \infty$ implies that there is a non-increasing function $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\mu(t) \geq |\bar{g}(t)|$ for all $t \in \mathbb{R}^+$ and $\int_0^\infty \mu(\tau) d\tau < \infty$. Let $n \in \mathbb{N}$ be any integer such that $n d_{\min} \geq t_1$. Hence,

$$\begin{aligned}
\sum_{k=1}^N \int_{t_k - t_1}^{t_k} \bar{g}(\tau)^2 d\tau &\leq \sum_{k=0}^{N-1} \int_{k d_{\min}}^{k d_{\min} + t_1} \mu(\tau)^2 d\tau \\
&\leq \sum_{k=0}^{N-1} \int_{k d_{\min}}^{(k+n) d_{\min}} \mu(\tau)^2 d\tau \leq n \sum_{k=0}^{N-1} \int_{k d_{\min}}^{(k+1) d_{\min}} \mu(\tau)^2 d\tau \\
&= n \int_0^N \mu(\tau)^2 d\tau \leq n \int_0^\infty \mu(\tau)^2 d\tau < \infty,
\end{aligned} \tag{39}$$

implying that also the second term of (37) is bounded for all N . This means that $\varepsilon_*^T \varepsilon_*$ is bounded.

Since x_k is given by (26), it follows that $\mathbf{H} + \mathbf{H}^T + \mathbf{X} - g(\infty) \mathbf{1}^T \mathbf{1}$ is positive semi-definite for all N , by applying the Gerschgorin disc theorem [8]. Therefore $\mathbf{H} + \mathbf{H}^T + \mathbf{X}$ is also positive semi-definite for all N since $g(\infty) > 0$. Equation

(34) now gives the following inequality

$$\begin{aligned}
& \sum_{k=0}^N \Delta_k \sigma(t_k) \{2\varepsilon(t_k) - x_k \Delta_k \sigma(t_k)\} \leq \\
& + 2 \sum_{k=0}^N \Delta_k \varepsilon^*(t_k) \sigma(t_k) + 2\varepsilon^*(0) \Delta_0 \sigma^*(0) \\
& + 2\Delta_0 \sigma^*(0) \{\varepsilon(0) - x_0 \Delta_0 \sigma(0)\} + x_0 \Delta_0^2 \sigma^*(0)^2.
\end{aligned} \tag{40}$$

Since $\varepsilon_*^T \varepsilon_*$ is bounded, Cauchy's inequality implies that $\varepsilon_*^T \boldsymbol{\sigma} \leq \sqrt{\varepsilon_*^T \varepsilon_*} \sqrt{\boldsymbol{\sigma}^T \boldsymbol{\sigma}} \leq c \sqrt{\boldsymbol{\sigma}^T \boldsymbol{\sigma}}$ for some $c \in \mathbb{R}^+$. Inserting $\sigma(t_k) = \gamma(t_k) \hat{\varepsilon}(t_k)$, using the sector bound (9), and exploiting the fact that $\gamma(t_k) \leq 2/(K_{\max} g_\infty) - \beta$, (40) turns into

$$\begin{aligned}
& \sum_{k=0}^N \Delta_k \gamma(t_k) \{2/K_{\max} - x_k \Delta_k \gamma(t_k)\} \hat{\varepsilon}(t_k)^2 \leq \\
& \sum_{k=0}^N \Delta_k \gamma(t_k) \hat{\varepsilon}(t_k) \{2\varepsilon(t_k) - x_k \Delta_k \gamma(t_k) \hat{\varepsilon}(t_k)\} \leq \\
& c \left(\frac{2}{K_{\max} g_\infty} - \beta \right) \sqrt{\sum_{k=0}^N \hat{\varepsilon}(t_k)^2} + C,
\end{aligned} \tag{41}$$

for some constant $C \in \mathbb{R}$.

Equation (25) implies that the left hand side factor satisfies $\gamma(t_k) \Delta_k \{2/K_{\max} - x_k \Delta_k \gamma(t_k)\} \geq x_k \beta^2 > 0$ for all $k \in \mathbb{N}_0$. Using the sector bound (9), it is then concluded that

$$K_{\min}^2 \sum_{k=0}^\infty \varepsilon(t_k)^2 \leq \sum_{k=0}^\infty \hat{\varepsilon}(t_k)^2 < \infty. \tag{42}$$

Note specifically that $\hat{\varepsilon} \in L_2$ because of (42) and the piecewise constant property (8). Furthermore, $\varepsilon(t)$ can be written as

$$\begin{aligned}
\varepsilon(t) &= \varepsilon(t_k) + r(t) - r(t_k) - (z(t) - z(t_k)) \\
&= \varepsilon(t_k) + \tilde{r}(t) - \tilde{r}(t_k) - (z(t) - z(t_k)),
\end{aligned} \tag{43}$$

where

$$\begin{aligned}
z(t) - z(t_k) &= \int_0^\infty g(\tau) [\hat{\varepsilon}(t - \tau) - \hat{\varepsilon}(t_k - \tau)] d\tau \\
&=_{(t \geq t_k)} \int_0^\infty [1 - e^{-p(t-t_k)}] g(\tau) \hat{\varepsilon}(t - \tau) d\tau,
\end{aligned} \tag{44}$$

and

$$[1 - e^{-p(t-t_k)}] g(\tau) = \begin{cases} \bar{g}(\tau) + g_\infty, & 0 \leq \tau < t - t_k \\ (1 - e^{-p(t-t_k)}) \bar{g}(\tau), & \tau \geq t - t_k. \end{cases} \tag{45}$$

From the assumption on the impulse response it holds that $[1 - e^{-p(t-t_k)}] g : \mathbb{R}^+ \rightarrow \mathbb{R}$ is well-defined everywhere for all $t \geq t_k$ and $\int_0^\infty |[1 - e^{-p(t-t_k)}] g(\tau)| d\tau < \infty$. Therefore, there is a measurable function $\eta : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $\int_0^\infty \eta(\tau) d\tau < \infty$ and $\eta(\tau) \geq |[1 - e^{-p(t-t_k)}] g(\tau)|$ for all $\tau \in \mathbb{R}^+$ and all $t - t_k \in [0, d_{\max}]$. Hence,

$$\zeta(t) \triangleq \int_0^\infty \eta(\tau) |\hat{\varepsilon}(t - \tau)| d\tau, \tag{46}$$

$$\int_{t_k}^{t_{k+1}} (z(\tau) - z(t_k))^2 d\tau \leq \int_{t_k}^{t_{k+1}} \zeta(\tau)^2 d\tau. \tag{47}$$

Since $|\hat{\varepsilon}| \in L_2$ due to (42) and $\eta \in \mathcal{A}$, it holds that $\zeta \in L_2$. It is now seen that

$$\begin{aligned} \int_0^\infty \varepsilon(\tau)^2 d\tau &= \sum_{k=0}^\infty \int_{t_k}^{t_{k+1}} \varepsilon(\tau)^2 d\tau \leq \sum_{k=0}^\infty \Delta_k \varepsilon(t_k)^2 \\ &+ \sum_{k=0}^\infty \int_{t_k}^{t_{k+1}} (\tilde{r}(\tau) - \tilde{r}(t_k))^2 d\tau + \int_0^\infty \zeta(\tau)^2 d\tau < \infty. \end{aligned} \quad (48)$$

Thus, it is concluded that $\varepsilon \in L_2$. $\square$

The bound (25) can not be used as an update rule for t_{k+1} as it is since it depends on future transmission times $\{t_j\}_{j=k+1}^\infty$ due to the third term in (26). First it is noted that the absolute values in (26) can be removed since at least one of conditions 1 and 2 holds. Since $t_{k+1} - t_k \geq d_{\min}$ for all $k \in \mathbb{N}_0$, it follows that the third term of (26) can be bounded as

$$\begin{aligned} &\sum_{j=k+1}^\infty \frac{1}{\Delta_k} \int_{t_j - t_{k+1}}^{t_j - t_k} (g_\infty - g(\tau)) d\tau \\ &\leq \sum_{i=0}^\infty \frac{1}{d_{\min}} \int_{id_{\min}}^{(i+1)d_{\min}} (g_\infty - g(\tau)) d\tau \\ &= \frac{1}{d_{\min}} \int_0^\infty (g_\infty - g(\tau)) d\tau. \end{aligned} \quad (49)$$

The inequality of (49) follows from condition 1 or condition 2. If condition 1 holds, $g_\infty - g(\tau)$ is non-increasing. Alternatively if condition 2 holds, $\int_{kd_{\min}}^{(k+1)d_{\min}} (g_\infty - g(\tau)) d\tau$ is non-increasing over $k \in \mathbb{N}_0$.

The second term of (26) depends on past transmission instances $\{t_j\}_{j=0}^{k-1}$ and the current transmission time t_k and can therefore be calculated in practice. In order to avoid the computational burden of doing so, the second term is bounded as follows

$$\begin{aligned} &\sum_{j=0}^{k-1} \frac{1}{\Delta_j} \int_{t_k - t_{j+1}}^{t_k - t_j} (g_\infty - g(\tau)) d\tau \\ &\leq \sum_{j=0}^{k-1} \frac{1}{d_{\min}} \int_{t_k - t_{j+1}}^{t_k - t_j} (g_\infty - g(\tau)) d\tau \\ &= \frac{1}{d_{\min}} \int_0^{t_k - t_0} (g_\infty - g(\tau)) d\tau \\ &\leq \frac{1}{d_{\min}} \int_0^\infty (g_\infty - g(\tau)) d\tau. \end{aligned} \quad (50)$$

Hence, Lemma 1 implies that the closed loop system is L_2 -stable if

$$\beta \leq \gamma(t_k) \Delta_k \leq \frac{2}{K_{\max} \bar{x}} - \beta, \quad (51)$$

where

$$\bar{x} = g(\infty) + \frac{2}{d_{\min}} \int_0^\infty (g_\infty - g(\tau)) d\tau. \quad (52)$$

Now consider the following derivation for the sampled system $H(q)$.

$$\begin{aligned} &2 \lim_{\omega \rightarrow 0} [-\operatorname{Re} H(e^{i\omega})] \\ &= \sup_{\omega \in \mathbb{R}} \operatorname{Re} \left[\frac{h(\infty)(e^{i\omega} + 1)}{e^{i\omega} - 1} - 2H(e^{i\omega}) \right] \\ &= h(\infty) + 2 \sup_{\omega \in \mathbb{R}} \operatorname{Re} \left[\frac{h(\infty)}{e^{i\omega} - 1} - H(e^{i\omega}) \right] \end{aligned} \quad (53)$$

$$\begin{aligned} &= h(\infty) + 2 \sup_{\omega \in \mathbb{R}} \sum_{j=0}^\infty (h(\infty) - h(j)) \cos(j\omega) \\ &= h(\infty) + 2 \sum_{j=0}^\infty (h(\infty) - h(j)) \\ &= d_{\min} g_\infty + 2 \int_0^\infty (g_\infty - g(\tau)) d\tau = d_{\min} \bar{x}. \end{aligned}$$

The first equality derives from the fact that the real part of $(e^{i\omega} + 1)/(e^{i\omega} - 1)$ is zero. The fourth equality derives from the fact that $h(\infty) - h(j) \geq 0$ for all $j \in \mathbb{N}_0$. The supremum over ω is thus obtained for $\omega = 0$ and therefore,

$$\sup_{\omega \in \mathbb{R}} [-\operatorname{Re} H(e^{i\omega})] = \lim_{\omega \rightarrow 0} [-\operatorname{Re} H(e^{i\omega})]. \quad (54)$$

Now, (51) can be rewritten as

$$\beta \leq \gamma(t_k) \Delta_k \leq \frac{d_{\min}}{K_{\max} \lim_{\omega \rightarrow 0} [-\operatorname{Re} H(e^{i\omega})]} - \beta. \quad (55)$$

Then, Theorem 1 follows directly by letting $\alpha = \beta/d_{\max}$. $\square$

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