Pricing exotic power options

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Abstract

A two folded thesis concerning the pricing of an exotic option with Nord Pool spot price as underlying, namely an Asian option. The project is made in collaboration with the business unit Portfolio Management, which belongs to the business division Asset Optimization and Trading, at Vattenfall AB. In the first part of the thesis existing ideas regarding the dynamics of the Nord Pool spot price are extended. A three factor mean reverting jump process is assumed to reflect the spot price. The model incorporates a seasonal price trend estimated by wavelets and non-stationary jump processes estimated by a moving average approach. In the second part we use the model to price an arithmetic average Asian option. We use the Monte Carlo method and simulate paths of the underlying and apply the contract function. The model is evaluated by comparing the characteristics of the simulated trajectories versus the historical spot prices.
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1 Introduction

This paper is a degree project for the business unit Portfolio Management at Vattenfall AB. The scope of the thesis lies on the development of an application for pricing Asian option with the Nord Pool system price as underlying instrument. It ranges from identifying a suitable process reflecting the spot price to implementing the model as a Matlab application. In this first section a brief background of the market is presented. Thereafter a section on the characteristics of the spot price follows. Next there are a short overview of the Asian option. Then a more formal framework follows. In the following chapter the framework is set. The preceding section is on the estimation of the parameters of the model where after a section on the Monte Carlo simulation follows. Finally the result is presented and in the last section some concluding remarks are given.

1.1 Background

Vattenfall is one of the leading energy companies in Europe and one of the largest generators of electricity and the largest producer of heat. Concerning these assets, Vattenfall participate in all parts of the value chain, that is, generation, distribution and sales. In 2013 Vattenfall had approximately 32000 employees. Vattenfall produce energy from a variety of sources, in 2013, 1% of the produced energy came from biomass, 2% from wind power, 20% from hydropower, 29% from nuclear power and 48% from fossil based power. In the Nordic area, however, 35% of the total production is based on hydro power and 56% of the produced energy comes from nuclear power plants. The business division Asset Optimization and Trading, AOT, is the center of wholesale and trading activities. The mission for AOT is to optimize and hedge production and also manage Vattenfall’s external customer portfolios. Also, AOT proprietary trading adds additional value to the Vattenfall.

This degree project was suggested by the business unit Portfolio Management, PM, at AOT. Even though Asian options are not continuously traded at Nord pool, incentives are found pointing at promising business cases. In fact, this thesis is an extension to a proposed trading case in 2014. Moreover, Asian options makes a good hedging instrument in risk management.

1.2 Nord Pool

Today Nord Pool is the leading power market in Europe. Nord Pool consists partly of the Elspot market which is a day-ahead market in the Nordic and
Baltic regions. Further, Nord pool also incorporates the intra-day market, Elbas, which is a continuously traded market place with hourly quotations. Included in this market is Sweden, Finland, Norway, Denmark, Estonia, Latvia, and Lithuania. In turn some of these areas are divided into smaller regions, known as price areas. This is a natural implication of limitations in the transmission system. For instance, Sweden is divided into four different price areas, Norway in five and Denmark are divided in two areas.

Recent years Nord Pool has expanded and now also the N2EX market in the UK is associated with it. The role they play is summarized in four points,

- provider of liquid secure markets for power,
- provider of accurate information of the whole market, ensuring transparency,
- provider of market access for power traders,
- being the counter party for all trades and guarantor of settlement and delivery.

They fill their mission by having 361 companies from 20 different countries trading on Nord Pool. The turnover of the power trading was aggregated to 493 TWh in 2013. On their homepage a comparison is made and the turnover is put in perspective, it approximately corresponds to a 61 year accumulated consumption of Oslo.

Companies connected to Nord Pool have different approaches to the market. The four main types of members are, producers, distributors, suppliers and traders. The power traded at Nord pool increases yearly. Of course, with this comes a demand for risk management which in turn makes the financial contracts traded increasing as well. The financial contracts, which are used for hedging and risk management are traded through Nasdaq OMX commodities. The different time horizons for the contracts makes the hedging flexible, it stretches from daily contracts to annual contracts up to six years ahead.

2 Spot price process

The spot price process is characterized by a large number of complex features affecting the price. As in this paper, the pricing of Asian options is considered, a quantitative approach rather than a fundamental is used. This
implies the main objective is to identify and thereby also enable the generation of a process with proper distributional properties. Also, the simpler model the better and, naturally, the simpler model the more difficult it is to get the model to incorporate all the properties of the process. Hence, a challenge is to find the right trade off between incorporating properties characterizing the distribution of the spot price in a model and the complicity of the model.

2.1 Trends and seasonality

A common assumption regarding the power market is the presence of deterministic trends. Many different approaches are found in the literature regarding this matter. From a modelling perspective there are mainly two aspects that make this feature especially important. First, if the spot price is assumed to invoke some seasonal pattern or trend then the estimation of the stochastics of the process is affected by this seasonality. By detrending or deseasonalizing the data properly the errors of the estimation decreases. A thorough study regarding this matter is found in [7]. Second, from a forecasting point of view, knowing the underlying deterministic effect is of course desirable.

Depending on the time resolution of the data, different levels of seasonality is exposed, that is intra-day, intra-week, monthly or yearly seasonality [1]. As we here model the daily spot price, considering intra day seasonality is not of interest. However, both weekly and longer, monthly to yearly, seasonal effects are found. As this thesis concerns option pricing and the desirable result is to find the correct distributional properties of the spot price the observed data is detrended in order to properly estimate the parameters of the stochastic process.

The long term seasonal component, hereafter referred to LTSC, using the same notations and terms as in [13] and [7], reflects the trends over time horizons corresponding to yearly, quarterly and monthly behaviour. In the Nordic countries these kinds of seasonal effects occur mainly due to climate related effects and may be deduced there by. It is easy to convince oneself that the cold winter and, naturally also the relatively warm summer, results in fluctuations in the load and in turn the price changes will reflect the changes in consumption behaviour, the demand, over the year. The hydro balance is also an effect of the climate. The precipitation during the winter months melts in the spring which raises the water levels both in the ground as well as in the reservoirs and in turn affects the price level. These are some of the main factors which give raise to the long term seasonal effects. Also, the market psychology might affect the price. For instance if a producer misjudges the load quantities systematically the price is of course
also systematically affected.

There are several different suggestions on how to model this feature. The most frequently stated approaches in the literature are piecewise linear functions, what also is known as dummies, combinations of sinusoidal functions and wavelets. Some of the more recent papers using piecewise constant functions are [3] and [5], recent examples of the sinusoidal combination approach is found in [1] and [12] and the wavelet approach is recently used in for instance [13] and [7]. Each approach holds their own pros and cons.

As it is impossible to exactly predict the loads, the human behaviouristics will push the seasonal trends from year to year in a stochastic manner. Also it is clear that, for instance, the cold periods, the warm periods and the spring flood, that is, examples of weather conditions affecting the price, do not occur exactly at the same time each year, which in turn result in fluctuations in the seasonal patterns. Due to this lack of static behaviour wavelets are assumed to model the long term behaviour. Foremost, the choice of the wavelet approach relies on [7] in which it is found that with the parameter estimation as the objective, wavelets are preferable. The approach is explained in section 11. In addition, it should be acknowledged that [13] gives a extensive study on estimating and forecasting the LTSC.

Moreover, the *the short term seasonal component*, hereafter referred to as STSC, again using the same notations and linguistic terms as in [13] and [7], reflects the intra-week seasonality. It is known that the load fluctuates day to day as a natural effect due to holidays and "low activity days". On non-working days the load decreases which is reflected in the price level. There are mainly two approaches used to process the intra-week seasonality, both which relies on piecewise dummies, namely using mean and median estimates over the week days. [7] accordingly the median is more robust to outliers but the differences are not substantial. Nevertheless, here the median estimate is used. A more formal description is found in section 11.

**2.2 Spikes**

A profound feature of the spot price is the price spikes. By a visual contemplation of the observed spot price process, see figure (1) one realizes that indeed extreme observations occur. This phenomenon, which also could be referred to as outliers, are here referred to as *spikes*. The main reason causing the behaviour is the non-storability condition of electricity [18].

A variety of procedures concerning the identification of spikes are suggested in the literature. In [7] a broad study comprising the more common ap-
Observed spot prices 2003-2013

Figure 1: Observed daily spot prices, $S(t)$, from 2003 - 2013

Two approaches are found. Based on the same paper, a brief summary is given over some of the methods. There are fixed price thresholds where price levels exceeding that particular threshold are considered as spikes. In addition, also variable price thresholds are suggested, where prices exceeding a certain level defined as some percentile of the observed prices are considered spikes. These two approaches, in turn, also have their analogies using price differences instead of the price levels. From a mathematical point of view, a more sophisticated, or at least complicated method is the use of wavelets to identify extreme values in the price series (compare “signal” in signal processing). The data is decomposed and rebuilt to a certain level and what is not reflected in this rebuilt level is considered as spikes. Again, this summary is brief and does not claim to include even the majority of all approaches and details of them, therefore we refer to [7] for an extensive review.

Here, a spike is associated with some specific properties. First, when the spike occur, a sudden check is found in the price level. The size of the check is measured as the amplitude of the spike. The following days, the price level should decrease, corresponding to a quick reduction of that certain price level. This pattern should at most allocate a couple of days or up to a week to be classified as a spike. Also, note there are two types of spikes, illustrated in figure (1), both negative and positive. Regarding the negative spikes the
symmetric properties holds. A formal definition will follow in section 6

As Asian options lies in the scope of this paper the importance of modelling spikes properly becomes obvious. Considering the contract function, dependent of the average over the spot prices, one realizes that it will be distinctly affected by price spikes. In turn, this directly affects the distribution of which the expectation is derived. Hence, the influence of spikes on the price of a path dependent option, as the Asian option, is worth pointing out specially. This may be compared to a European option, for instance treated in [10], where it is assumed that only a spike occurring right before the expiry date gives influence on the price of the option. This is due to the assumption of quick reduction. A final remark on how to define and identify spikes is that it in some ways always will, to some extent, be a subjective matter relying for instance partly on experiences.

2.3 Mean reversion

It is known that the stochastics of commodity markets also holds a mean reversion effect on the price process. A straightforward explanation of this property is that in an equilibrium frame work, a natural and expected effect of a high price level, would be that higher marginal cost producers will supply the market with the asset and hence due to the higher quantities available the price level in turn decreases. Considering a relatively low price level, the supply is expected to decrease as the producers with higher costs will not produce the same quantities, pushing the price upwards, back towards the equilibrium level [14].

This feature implies that a the model properly reflecting the behaviour of the spot price also, of course, should carry this property. One of the most common models holding this property is the Ornstein-Uhlenbeck process. Here we use a three factor stochastic model where the Ornstein-Uhlenbeck is assumed to reflect the price movements under normal circumstances, that is, periods when the market does not experience any extraordinary abnormalities like power outage or extreme weather conditions etc.

In addition, the mean reversion of the spikes clearly is greater in their nature. As a result thereof, the part of the model reflecting the spikes, is also mean reverting, however, at another rate, reflecting this extreme phenomenon.
2.4 No-arbitrage assumption on the power market

A trivial example on the stock market illustrating the no-arbitrage assumption is that the value of a stock must at least increase with a rate equivalent to the risk free interest rate $r$. Informally, we say that the price of the stock without respecting this aspect is the risky price of the stock and the dynamics of the stock is said to be associated with the a risky probability measure. Respecting this feature, informally, we say that we have derived the risk-neutral price of the stock and the stochastic dynamics are associated with a risk-neutral probability measure, some times referred to as the equivalent martingale measure. In order for the market to be arbitrage free, the right price must be the risk free price. Otherwise, depending on the assumptions of the dynamics, one could lend money from the bank, buy the stock and pay the interest rate to the bank or sell the stock short and invest the money in the bank.

Concerning commodities, a standard approach pricing a derivative is to create a portfolio perfectly replicating the pay out of the contract and often this implies storing the right amount of the underlying asset under the contract duration. Then, what is meant by the arbitrage free market is that the price of the market force the price of derivative and the replicating portfolio to be equal [17]. If we consider the electricity market the reasoning fails due to the reasonable assumption of electricity as an asset being non-storable. However, there are forwards and futures on the market which gives raise to questions on how to model the arbitrage free link between the spot price and the future/forward contracts. In [17] a suggested approach is to consider the market price of risk as the "missing" link on the arbitrage free market. Moreover, a framework for this kind of questions are presented and evaluated. On a shorter time horizon they show a linear dependence between the spot price and the future price. This result, however, is model-dependent and hence the linear relationship is only associated with their specific stochastic model of the spot price. What is interesting is that the implied market risk obtained from the Asian options, that where traded Nord Pool and the future contract, match the result obtained by simulations of the model and in comparisons of the the result with the future price. Despite that the result refer years back, it could be an option for calibrating the model. On the contrary, the staff at Vattenfall portfolio management claims the opposite, any clear relationship can not be found.

As a result thereby, here we simply, aware of the pros and cons, use the future price as an extension of the seasonal trend around which the price fluctuates when performing the Monte Carlo simulations. Due to the no-arbitrage assumption the "market-guess" should be the best guess. In addition, [10] states that a pragmatic way of approaching this problem is to estimate the
parameters of the model on historical data but adjust the seasonality or the deterministic trend to the observable forward curve. This is exactly what is done here and the monthly forward prices are used for this purpose.

3 Model selection

As the Nordic power market still is relatively immature compared to the stock market and other commodity markets, there is not any clear consensus on which model captures the most of the spot price behaviour. In addition, what should also be taken in consideration is the calibration, the more complex the model the more parameters to calibrate. As Nord pool does not serve a liquidity near the magnitude of today’s stock markets, the uncertainty this matter addresses should be considered. What could be said, though, is that an evolution of the models indeed is observable and the progress is ongoing. One of the simpler models is the one factor model found in [11],

\[ P(t) = f(t) + X_t, \]
\[ dX_t = -\kappa X_t dt + \sigma dZ_t, \]

where \( P(t) \) is the spot price, \( X_0 = x_0, \kappa > 0 \) and \( dZ \) follows a standard Brownian motion. What also plays an important role is \( f(t) \) which corresponds to the seasonal behaviour of the price. Moreover, \( X_t \) is assumed to follow a Ornstein-Uhlenbeck process, mean reverting to a long run mean which is zero. The process is further developed in the same paper and extended to a two factor model accounting for the oil price as well. Also, in the paper they claim a seasonal volatility pattern, possibly including mean reversion as well, might be present and the statement is in addition visually supported. They conclude that the extension is promising but they also admits that considering the non-storability of the electricity, an important feature which needs to be included is jumps. That is, because the short run in-elasticity of the supply. Indeed, models which include jumps have also been suggested, for instance in [18],

\[ P(t) = s(t) + S(t) + e^{J_{dt}} + X_t, \]
\[ dX_t = \beta(L - X_t)dt + \sigma dB_t, \]

where \( s(t) \) and \( S(t) \) is the seasonal deterministic trend, weekly and yearly, and the sum corresponding to \( f(t) \) in 1. Further, \( \log(J) \sim N(\mu, \sigma^2), \) \( dq_t \) is a Poisson process and \( B_t \) is standard Brownian motion. The model admits jumps but is open for criticism from other perspectives. In [12] a sound
objection is that the mean reversion of the spikes is more accentuated than the mean reversion of prices observed under what could be called normal circumstances. A solution to this matter suggested in [10] is,

\[ S_t = \exp(f(t) + X_t + Y_t), \]
\[ dX_t = -\alpha X_t dt + \sigma dW_t, \]
\[ dY_t = -\beta Y_t - dt + J_t dN_t, \]

where \( S(t) \) is the spot price, \( f(t) \) reflects the seasonality and \( \alpha \) and \( \beta \) the mean reversion of the process under normal circumstances and the spikes respectively. Further \( J \) is normally or exponential distributed, both cases are studied. The spikes are assumed to occur according to the stationary Poisson process, \( N_t \). A final assumption made, is that \( N_t, W_t \) and \( J_t \) are mutually independent. Moreover, in the paper indications of stochastic volatilities is mentioned. Also the stationary condition of the Poisson process is remarked as vague due to the tendencies of spikes occurring more frequently certain periods over the year. A similar approach is suggested in [12], in contrast to [10] regime switching approach is suggested where the process is dependent on the reservoir levels.

In this paper we try to obey some characteristics not captured, to our knowledge, so far in any model of the spot price. We include the non-stationary of the Poisson process. By a visually examination we approximate the distribution of the amplitudes of the spikes. In addition, we add a mixture of the qualities of the above models. A final requirement from Vattenfall was that the model would not be exhibiting the threshold where it becomes non-user friendly due to complexity. This criteria is of course reasonable and considered in the selection process of the model. In this section we satisfy with presenting the above approaches and some pros and cons and leave a formal set up of our model to the section 6.

4 Asian options

Asian options are exotic options belonging to the category of path dependent options. That is, the value of the option is a function of the realized path. The two common versions of the Asian option are based on the arithmetic average or geometric average of the path of the underlying asset. Here we consider arithmetic average options. More specifically, we have a combined monthly look back of a quarter and the price is weighted according to the hours in the respective month. Today Asian options are not traded at Nord Pool, however, one and a half decade ago they were. The Asian options on Nord Pool were connected to the simultaneous traded future block and they
also shared the same settlement period. Even though the Asian option in
time followed the Eltermin, the underlying asset was the spot price. The day
after the last day in the delivery period was the settlement day.

From the perspective of risk management the Asian option serve to a great
extent to protect the owner/holder of the option against the very volatile
spot market with the extreme price spikes included. Moreover, from
the perspective of trading, Asian options may be viewed as bets on the volatility
of the underlying spot price. The one with the best estimate of the distribu-
tion will be rewarded. A more formal definition of the Asian option is found
in section 6.

5 Data

The data is provided in-house by Vattenfall. A reasonable consideration
is how large the data set should be due to calibration purposes. In [13] a
maximum of four years are suggested for instance. The simulation will be
made on a daily time grid and therefore the data used for the calibration
also is obtained with the same granularity. That is, the data consists of the
daily average spot prices at Nord Pool in EUR/MWh.

6 Framework

This section present to the reader the framework and set up of the thesis.
The chapter will take on the mathematical framework and present on which
assumptions the model relies. However, it is worthwhile noting that the
focus in this paper rather lies on a practical approach and therefore some
details is abbreviated or only referred to. We begin with an overview of the
selected model assumed to reflect the spot price.

6.1 The Model

We start by presenting the model, aiming to get the reader an initial overview
and ease the reading. Thus, be aware of that the needed explanations of the
details it holds will be carried out in the coming sections. The model selected
and assumed to reflect the spot price process, $S_t$, is in its simplest form stated
as,

$$ S_t = \exp(\Gamma_t + f(t)) $$
The seasonal fluctuations are respected by \( f(t) = l(t) + w(t) \), where \( l \) and \( w \) corresponds to the LTSC and the STSC respectively. The dynamics of the stochastic part is described by \( \Gamma_t = X_t + Y_t^+ + Y_t^- \) where \( X_t, Y_t^+ \) and \( Y_t^- \) are associated with the movements of the price process under normal circumstances, when positive spikes occur and when negative spikes occur, respectively. That is, the core stochastics of the spot price is modelled by a sum of three independent processes, namely,

\[
\begin{align*}
    dX_t &= \alpha X_t dt + \sigma dW_t \\
    dY_t^+ &= \alpha^+ Y_t^+ dt + \gamma^+ dq(\lambda_t^+) \\
    dY_t^- &= \alpha^- Y_t^- dt + \gamma^- dq(\lambda_t^-)
\end{align*}
\]

where \( \alpha, \alpha^+ < 0 \) and \( \alpha^- < 0 \). Worthwhile noting further, is that \( X_t \) is a Ornstein-Uhlenbeck process, reverting to a mean which here is assumed to be zero. This will on the other hand not be the case for \( S_t \) due to the included seasonal trend \( f(t) \) to which it will revert. Also the process for the spikes are mean reverting, at a relatively higher rate compared to \( X_t \). The rate of mean reversion is denoted \( \alpha, \alpha^+ \) and \( \alpha^- \) for the normal process, the positive prices spikes and the negative price spikes respectively. The notations \( ^- \) and \( ^+ \) addresses the connection to the positive and negative spikes respectively. As mentioned, the spikes are described by a Poisson process, more precisely a non-stationary compound Poisson process. The amplitudes of the spikes are assumed to be random and the variables corresponding to those are denoted \( \gamma^+ \) and \( \gamma^- \). Moreover, we assume that they are exponentially distributed with different parameters, \( \Lambda^+ \) and \( \Lambda^- \), that is, \( \gamma^+ \sim \text{Exp}(\Lambda^+) \) and \( \gamma^- \sim \text{Exp}(\Lambda^-) \). The intensity of the Poisson process associated with the occurrences of the spikes are \( \lambda_t^+ \) and \( \lambda_t^- \). These two assumptions will be treated in more detail in section 11. For now we satisfy with a brief visual inspection of spikes identified in the sample. Of course to this point it might be considered meaningless to investigate the spikes without having any formal definition. What we can say though, is that to some extent defining the spikes is a subjective matter. Nevertheless, let us consider figure 2, to get an intuitive feeling, where spikes using a subjective definition is plotted. These, what could be spikes, are projected on single year and is visualized in figure 3. As we see the assumption of a time dependent intensity can not likely be rejected, rather supported though. The positive spikes seems to occur in winter or spring time and the negative during the summer months. Further, by looking, again only to get an intuitive feeling for the model, at figure 4 and 5 the assumption that amplitudes are exponentially distributed, at least at this point, with a quick ocular survey, seems reasonable.
Figure 2: Days with large daily differences marked green or red corresponding to what could be considered positive or negative spikes.

6.2 Seasonal components

We assume two seasonal components being present in the spot price process. These are described an estimated in two different ways. The LTSC is estimated using a wavelet approach and in addition the STCS is estimated using the median over the specific weekdays in the sample. It is put forward in 11 that the seasonal pattern should be approximately the same year to year, however it is stressed that due to, for instance weather conditions, the seasonal pattern might be disturbed. Approaching this matter we use a wavelets. We denote the long term seasonal component $l(t)$. Looking at a shorter periods ranging over a week, a weekly pattern is also found. This is due to to the behavioural patterns of the society. As to illustrate, on weekdays the load is generally greater than in the weekends when the industry are down. Let us denote the short term seasonal component $w(t)$. Further we also introduce the deseasonalized price series, which naturally is $\Gamma_t = log(S_t) - f(t)$, which is the stochastic part of the model. Finally, what also is needed to be defined is the daily difference,

$$\Delta\Gamma_{t_i} = \Gamma_{t_i} - \Gamma_{t_{i-1}}$$
where the $t_i$ is associated with day $i$.

### 6.3 The spike process

Spikes are one of the profound features of the spot price and several ways of defining them have been proposed. In [7] it is found that today there is not any consensus on how to define this property, however, they carefully carry out an extensive study on some of the different definitions suggested and how the definition affects the calibration. Here, we define a spike as special case of subsequent prices having certain properties, namely a **chock**, **amplitude** and **duration**. Regarding the estimation, an observed spike is a sub sample of the observed price process, having these properties. Let us first consider the **chock criteria**.
6.3.1 Chock criteria and amplitude

First, a feature that characterize a spike is the sudden large chock in the spot price which quickly is reduced. For the positive spikes, a positive chock and for the negative spikes a negative shock. A natural question is what to considered being a "large" chock. This matter is a section of its own and holds many aspects. For instance, if the chosen definition results only in very few spikes, then it is hard to draw any statistical conclusions regarding the features of the spikes. On the other, of course the definition can not be a function of wanted statistical properties. At the end, what to consider a large chock, is a subjective matter and lies in the eye of the viewer. A common condition is to view changes larger than some, more or less subjective, threshold as spikes. Here we define a large chock as a change in the deseasonalized log-price differences, $\Delta \Gamma_t$, larger than $H_1$ sample standard deviations of the deseasonalized daily log-prices differences a positive chock, and if it is smaller than the negative counterpart, a negative one. More formally, the sample standard deviation of the deseasonalized log-prices, $\hat{\sigma}_{\Delta \Gamma}$, is defined
as usual,

$$\hat{\sigma}_{\Delta \Gamma} = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (\Delta \Gamma_{t_i} - \bar{\Delta \Gamma}_t)^2}.$$ (3)

If the chock is sufficiently large, that is if

$$\Delta \Gamma_{t_i} > H_1 \hat{\sigma}_{\Delta \Gamma},$$
$$\Delta \Gamma_{t_i} < -H_1 \hat{\sigma}_{\Delta \Gamma}$$

for a positive or a negative chock, then consider the chock criteria fulfilled for the same type of spike. Hence, to this point $S_{t_i}$ could be a positive or a negative spike.

What is closely associated with this threshold is the amplitude of the spike, denoted $J$, for which one quickly realizes a natural measure, actually is $\Delta \Gamma_{t_i}$. However, in addition the subsequent prices following the chock, must also fulfil the reduction criteria, described in the next section, in order to be classified as a spike. In figure 2, actually, what is plotted is all chocks in the historical sample exceeding this threshold using $H_1 = 2.5$. As the classification is partly a function of $H_1$, the number of sufficiently large price

Figure 5: Histogram representing the amplitudes of negative spikes.
movements, marked by green and red stars, of course would change if we change \( H_1 \). To sum up so far, a necessary but not sufficient condition for a de-seasonalized daily log-price difference to be classified as a spike is a sufficiently large price movement from the previous day. A positive movement results in a positive spike and a negative in a negative spike. An additional assumption is that the amplitudes are exponentially distributed. More precisely they are assumed to follow a shifted exponential distribution. The shifting follows naturally by our definition of a spike. Simply there can not be any spikes with amplitudes less than the chock criteria. That is, the amplitudes for the positive spikes, \((J^+ - H_1\hat{\sigma}_{\Delta\Gamma}) \sim \text{Exp}(\Lambda^+)\) and the negative amplitudes for the negative spikes \((-J^- - H_1\hat{\sigma}_{\Delta\Gamma}) \sim \text{Exp}(\Lambda^-)\).

That is we have the density function for the positive spikes,

\[
f(j; \Lambda^+, \phi) = \Lambda^+ e^{-\Lambda^+(j-\phi)} \quad \text{for} \quad j \geq \phi
\]  

(4)

and for the negative, negative spikes,

\[
f(j; \Lambda^-, \phi) = \Lambda^- e^{-\Lambda^-(j-\phi)} \quad \text{for} \quad j \geq \phi
\]  

(5)

where, in this case \( \phi = H_1\hat{\sigma}_{\Delta\Gamma} \). The next step is to process the reduction criteria.

6.3.2 Reduction criteria and duration

When a large price movement take place, sooner or later, a price reduction follows the mean reversion property accordingly. If the price slowly decays it is due to the mean reversion of the process under normal circumstances. That is, it is assumed to follow \( X_t \). On the other hand, if the price level rapidly reduces it is a key feature of a spike, hence the mean reversion is associated with the spike. A natural question is to decide what actually is a sufficiently rapid reduction. Again, there are a number of suggestions found in the literature. As the chock at time \( t \) is measured by the de-seasonalized daily log-price difference \( \Delta\Gamma_t \), the reduction in turn is determined as changes in \( \Gamma \) counting for a rapid reduction in the following days. An implication is that there are two key properties of reduction that needs to be measured and defined, the relative price reduction and the time, that is the number of days, it takes for the price to be reduced. Let the threshold \( H_2 \) denote the maximum number of days in which the amplitude of the spike has been reduced by a factor \( H_3 \in [0, 1] \). In [12] a rapid reduction is defined as at least a halving time of the chock in, at most, the following 5 days, that is \( H_2 = 5 \) and \( H_3 = 0.5 \). Discussing this matter at Vattenfall, we agreed that the natural number of days should be \( H_2 = 6 \) which corresponds to a week and might exclude weekly extraordinaries which means any weekly periodic
behaviour is excluded. Formally, if
\[ \Gamma_{t_{i+j}} < H_3 \Delta \Gamma_{t_i} + \Gamma_{t_i} \text{ for some } j = 1, 2, ..., H_2. \] (6)
then \((S_{t_i}, ..., S_{t_{i+j}'})\) where \(j' = \min(j : \Gamma_{t_{i+j}} < H_3 \Delta \Gamma_{t_i} + \Gamma_{t_i})\), fulfills the reduction criteria for a positive spike. In addition, if
\[ \Gamma_{t_{i+j}} > -H_3 \Delta \Gamma_{t_i} + \Gamma_{t_i} \text{ for some } j = 1, 2, ..., H_2. \] (7)
then \((S_{t_i}, ..., S_{t_{i+j}'})\) where \(j' = \min(j : \Gamma_{t_{i+j}} > -H_3 \Delta \Gamma_{t_i} + \Gamma_{t_i})\), fulfills the reduction criteria for a negative spike. Closely associated to this, we have the duration of a spike, which we define as, \(D = j' + 1\). Now we are ready to define the spikes.

**Definition 6.1 (Positive spike).** If, for \(S_{t_i}, S_{t_{i+1}}, ..., S_{t_{i+H_2}}\), \(\Gamma_{t_i} > H_1 \hat{\sigma} \Delta \Gamma\) and \(\Gamma_{t_{i+j}} < H_3 \Delta \Gamma_{t_i} + \Gamma_{t_i} \text{ for some } j = 1, 2, ..., H_2\) holds, then \((S_{t_i}, ..., S_{t_{i+j}'})\) where \(j' = \min(j : \Gamma_{t_{i+j}} < H_3 \Delta \Gamma_{t_i} + \Gamma_{t_i})\) is a positive spike.

Moreover, a positive spike is denoted \(S^+(t_i, D, J^+)\) where \(t_i\), \(D\) and \(J^+\) corresponds to the initial time of the positive spike, the duration and the amplitude, respectively. Regarding the negative spikes, we define,

**Definition 6.2 (Negative spike).** If, for \(S_{t_i}, S_{t_{i+1}}, ..., S_{t_{i+H_2}}\), \(\Gamma_{t_i} < -H_1 \hat{\sigma} \Delta \Gamma\) and \(\Gamma_{t_{i+j}} > -H_3 \Delta \Gamma_{t_i} + \Gamma_{t_i} \text{ for some } j = 1, 2, ..., H_2\) holds, then \((S_{t_i}, ..., S_{t_{i+j}'})\) where \(j' = \min(j : \Gamma_{t_{i+j}} < H_3 \Delta \Gamma_{t_i} + \Gamma_{t_i})\) is a negative spike.

In the same way as for the positive spikes we denote a negative spike, \(S^-(t_i, D, J^-)\) where \(t_i\), \(D\) and \(J^-\) corresponds to the initial time of the negative spike, the duration and the amplitude, respectively. Furthermore, the occurrences of the spikes, that is when the check take place, following a non stationary Poisson process. So, for all positive spikes, \(S^+(t_i, D, J^+)\), \(t_i \sim Po(\lambda_i^+)\) and in the same way for all the negative spikes, \(S^-(t_i, D, J^-)\), \(t_i \sim Po(\lambda_i^-)\).

### 6.4 The Ohrstein-Uhlenbeck process

As in for instance [10] we assume a mean reverting Ohrstein-Uhlenbeck process, in general defined by \(dX_t = \alpha(\mu - X_t)dt + \sigma dW_t\), \(\alpha > 0\), to reflect the spot price under normal circumstances. Here it is slightly modified and we use
\[ dX_t = \alpha X_t dt + \sigma dW_t \text{ where } \alpha < 0 \]

The mean reversion of the Ohrstein-Uhlenbeck is assumed to reflect the effect of change in supply and demand described in earlier section.
6.5 Asian options

Let us use the following notation,

\begin{align*}
    K &= \text{strike}, \\
    t &= \text{time}, \\
    T_j &= (t_1^{(j)}, t_2^{(j)}, ..., t_{n_j}^{(j)}) \text{corresponding to month } j \text{ in the settlement period} \\
    &\text{which have } n_j \text{ days} \\
    h_j &= 24n_j, \text{hours in month } j, \\
    h &= 24 \sum_{j=1}^{3} n_j, \text{hours in settlement period}, \\
    T^0 &= (t_1^{(0)}, t_2^{(0)}, ..., t_{n_0}^{(0)}) \text{corresponding to period until settlement period starts} \\
    T &= (T_1, T_2, T_3) \text{ which is the settlement period} \\
    S_t &= \text{price of underlying at time } t, \\
    V(t, S) &= \text{price of option at time } t, \\
    \Psi() &= \text{"contract"/"pay off" - function.}
\end{align*}

Then the price of an Asian option at \( t = 0 \) given \( S(t = 0) \),

\[
    V = \frac{1}{h} e^{-rT} E[\Psi(S_{t \in T}, K)] \\
    = \frac{1}{h} e^{-rT} E [(S_{t \in T_1}, K)^+ h_1 + (S_{t \in T_2}, K)^+ h_2 + (S_{t \in T_3}, K)^+ h_3]
\]

where we have

\[
    S_{t \in T_j} = \frac{1}{n_j} \sum_{t \in T_j} S_t
\]

Note that \((x)^+ = \max(x, 0)\).

7 Estimation

Good estimation procedures are of course of great importance. For the estimations we dispose a sample of observed spot log-prices, \( S = (s_{t_0}, s_{t_1}, ..., s_{t_N}) \), where \( s_t \) is the logarithm of the observed spot price at time \( t \). Further \( N = 2^{11} \) and the sample ends at 2013-12-31. The number of observations are assumed to be sufficiently large and hence the errors should be reasonable small.
7.1 Trends and seasonality

In the introductory section we mentioned some of the ideas used for estimating the LTSC. The results indicating a robust procedure found in [7]. The idea therein relies on a filtering approach. The idea is to filter out extreme values in \( S \) to get a better estimation of the seasonality. A stepwise description of the procedure follows,

1. Approximate a long term trend, \( \hat{l}^*(t) \), from \( S \), where * indicates that this is only first estimation of a two times repeated procedure.

2. Remove \( \hat{l}^*(t) \) from \( S \), that is, let
\[
S_t - \hat{l}^*(t) = w(t) + \Gamma_t + \epsilon_t^l
\]
where \( \epsilon \) is a error term due to the estimation

3. Approximate the intra-week pattern \( \hat{w}^*(t) \) and remove it, that is,
\[
S_t - \hat{l}^*(t) - \hat{w}^*(t) = \Gamma_t + \epsilon_t^l + \epsilon_t^w = \hat{\Gamma}^*_t
\]
where \( \epsilon \) is a error term due to the estimation

4. Filter \( \hat{\Gamma}^*_t \) by removing and replacing outliers, obtain \( \hat{\Gamma}'_t \) where ' denotes that the sample is filtrated.

5. Set \( \hat{\Gamma}'_t + \hat{l}^*(t) + \hat{w}^*(t) \) which implies we obtain a filtered sample of the spot prices in accordance to
\[
S'_t = \hat{l}^*(t) + \hat{w}^*(t) + \Gamma'_t + \epsilon_t^l + \epsilon_t^w
\]
which is equivalent to,
\[
S'_t = l(t) + w(t) + \Gamma'_t
\]

6. Now, when we have removed the outliers, a better estimate of \( l(t) \) and \( w(t) \) can be found and in turn a better estimate of the parameters associated with \( \Gamma_t \). We repeat the same procedure again but on the filtered sample. Estimate \( \hat{l}'(t) \), from \( S' \) and let
\[
S'_t - \hat{l}'(t) = w(t) + \Gamma'_t + \epsilon_t^l
\]

7. Approximate the intra-week pattern \( \hat{w}(t) \) from \( S'_t - \hat{l}(t) \).

8. Remove \( \hat{l}(t) \) and \( \hat{w}(t) \) from \( S_t \),
\[
S_t - \hat{l}(t) - \hat{w}(t) = \Gamma_t + \epsilon_t^l + \epsilon_t^w = \hat{\Gamma}_t
\]
We have now obtained the estimation of $\Gamma_t, \hat{\Gamma}_t$

The estimation of the LTSC relies on wavelets which is a less periodic alternative to Fourier methods. Also, it is found to be more robust to outliers. There are a lot of families of wavelets, which briefly may be compared to short "wave"-ish patterns. What also is interesting about wavelets in contrast to Fourier methods is that they are localized in both time and space. The original unstretched high pass filter wavelet, is often referred to as the mother wavelet and the stretched one as the father wavelet. Moreover, the families have their own properties regarding their compactness and their smoothness. We follow [7] and use a Daubechies wavelet of order 24. The signal, here the spot log-price, may be decomposed into a sum of mother wavelets where the maximum levels of decomposition is $j$ where $2^j = n$. By summing the decompositions a estimate of the original signal is obtained. The more levels included the better the estimate. Recall that by summing all levels of the decompositions the original signal is, in general, perfectly restored. A detailed presentation of wavelets is not in the scope of this paper and hence, for a more detailed description of how to apply them as a tool for deseasoning we refer to [18] and a interesting introduction with a practical approach is found in [4] and regarding more specific properties of the Daubechies wavelet see for instance [2]. First, aiming on a proper filtering, we approximate $\hat{l}(t)$ by a Daubechies wavelet of order 24 with $j = 6$ corresponding roughly to a bimonthly behaviour, $2^6 = 64$.

The STCS $\hat{\omega}^*(t)$ is estimated from $S_t - \hat{\Gamma}_t(t)$ as the median over the weekdays. Thereafter, the filtering takes place. In [7] as well as [18] it is concluded that any of the tested methods therein out perform the no filtering approach. They even state that due to this it satisfy to use either one of the methods. Hence they filter out the upper and lower 2.5% of the observations in $\hat{\Gamma}^{(1)}(t)$. This is also what we do here. The values are replaced by the mean of $\hat{\Gamma}^*(t)$. One may argue that the mean is a normal value of the market when the prices are deseasonalized. When the removal and replacement is performed, we have $S'_t = \hat{l}(t) + \hat{\omega}(t) + \Gamma'_t$, where ' indicates that the sample is filtered and the outliers replaced by more normal values. We repeat the procedure again on $S'_t$ and obtain $\hat{l}(t)$ and $\hat{\omega}(t)$ which are better approximations than $\hat{l}^{(1)}(t)$ and $\hat{\omega}^{(1)}(t)$ of the seasonal trends. Finally we extract $\hat{l}(t)$ and $\hat{\omega}(t)$ from $S_t$ and this results in our deseasonalized log-price sample, $\hat{\Gamma}_t$.

The result of this procedure is visualized in figure 6. The estimated LTSC, $\hat{l}(t)$ seems to follow the yearly trends reasonable well. In figure 7 a close up is plotted and here one clearly realize that the yearly trends changes year to year and a sinusoidal combination would probably had missed the "deeper" dip in 2012 compared to 2011. Further, consider figure 8, indeed the most
of the seasonality is eliminated. Taking a more detail look at figure 9 we observe that the log price is more stable fluctuating round zero.

7.2 Spikes

In line with the rest of the questions related to electricity price modelling, any industry standard way of modelling and estimating spikes does not exist. According a suggestion in [10] a reasonable way of estimating some of the properties of the spikes is by using the experience gained by market participants, like traders and risk managers. The approach could be address to as a "hybrid approach" as we partly define a spike upon input from experiences market participants and in addition we use a parametric estimation procedure when estimating the stochastics of the spikes. That is, a maximum likelihood estimation for the amplitudes of the spikes, a regression approach for the mean reversion and a moving average method is used to estimate the intensities of the spikes. Finally note that the number of identified spikes depends on how we chose our parameters $H_1, H_2$ and $H_3$. 

Figure 6: The estimated LTSC, $\hat{l}(t)$ and $S_t$, from 2003 - 2013
Figure 7: The estimated LTSC, \( \hat{l}(t) \) and \( S_t \), from 2011 - 2012

7.2.1 The check and the amplitude

The check criteria

Different values of \( H_1 \) is found in the literature, here a brief empiric examination showed that letting \( H_1 = 2.5 \) seems to identify most of the largest movements. In figure 10, the \( \Delta\Gamma_t > H_1\hat{\sigma}_{\Delta\Gamma_t} \) and \( \Delta\Gamma_t < -H_1\hat{\sigma}_{\Delta\Gamma_t} \) are marked out, all the obvious large differences identified. Further, from a comparison between the figures 10 with 11, it comes clear that a sufficiently large jump is not necessarily what we would considered a spike. There are some jumps that do not results in a spike, atleast not in the sense a spike is addressed here. The next step is to identify those of the possible spikes which fulfils the reduction criteria.
The reduction criteria

Discussing the matter of reasonable values of $H_2$ and $H_3$ and taking suggestions in [7] into account, the values where determined to be $H_2 = 6$ and $H_3 = 0.5$ for the plots in this section. Note that we later on refine these values of the parameters by applying a simple optimization procedure. For know, let us satisfy with the values above when explaining the estimation procedure. In a first step, applied to an identification procedure on our sample $S_t$, we identify possible spikes using the threshold $H_1 = 2.5$. Then we add on the requirements $H_2 = 6$ and $H_3 = 0.5$. Hence from our sample $S$ two sub samples are obtained $\hat{\Gamma}^+ = (s^+(t_i, D, J), s^+(t_i, D, J), ..., s^+(t_i, D, J))$ where $s^+$ is an observed positive spike. And another sample for the negative spikes, with analogue notation, that is $S^- = (s^-(t_i, D, J), s^-(t_i, D, J), ..., s^-(t_i, D, J))$ where $s^-$ is a observed negative spike. Now, let us consider the identified spikes in the sub-sample $S^+$ with their amplitudes as our objective. From $S^+$ we look at a sample of the amplitudes of the positive spikes, say $J^+ = (j_1, j_2, ..., j_n)$. In figure 13 it indeed seems like the amplitudes could be exponentially distributed. However, one might object due to the slightly heavier tail than expected. Follow the same procedure for the negative spikes and obtain the analogue sample $J^-$. In figure 14 these are visualized and it
The positive amplitudes

We assume $J^+ - \phi \sim \text{Exp}(\Lambda^+)$, where $\phi$ is the smallest amplitude per definition. The density function is $f(j; \Lambda^+) = \Lambda^+ e^{-\Lambda^+(j-\phi)}$, $j \geq \phi$. The log-likelihood function is obtained by,

$$L(\Lambda^+) = \prod_{i=1}^{n} f(j_i; \Lambda^+) = \prod_{i=1}^{n} (\Lambda^+ e^{-\Lambda^+(j_i-\phi)}) = (\Lambda^+)^n e^{-\Lambda^+ \sum (j_i - \phi)}$$

and in turn, take the logarithm, differentiate and set equal to zero to
Figure 10: The deseasonalized log-spot prices changes, $\Delta \Gamma, \ 2003$-2013. $\Delta \Gamma_i > (\prec)H_1\sigma_{\Delta \Gamma}$ marked out.

find maximum, results in

$$
\log(L(\Lambda^+)) = l'(\Lambda^+) = \frac{n}{\Lambda^+} - \sum_{i=1}^{n} j_i - \phi = 0
$$

solve with respect to $\Lambda^+$ gives

$$
\Lambda^+ = \frac{n}{\sum(j_i - \phi)}
$$

and finally, realize that $\hat{\Lambda}^+ = \frac{n}{\sum(j_i - \phi)}$ is the maximum likelihood estimation of $\Lambda^+$. The analogue reasoning holds for the negative spikes. The estimated densities of the positive and negative spikes are illustrated in figure 15. Of course the value depends on how we define a spike, here we just want illustrate and uses the values mentioned in the beginning of the section.

Figure 11: The log-spot prices, $S_t$, 2003-2013. $\Delta \Gamma_t > (> -) H \sigma_{\Delta \Gamma}$ marked out.

7.2.2 The reduction and duration

The reduction corresponds to the mean reversion of the spike processes. Here we use the same approach as in [12], that is a regression, however it is a bit modified. Recall the spike processes,

\[
\begin{align*}
    dY^+_t &= \alpha^+ Y^+_t dt + J^+ dq(\lambda^+_t) \\
    dY^-_t &= \alpha^- Y^-_t dt + J^- dq(\lambda^-_t),
\end{align*}
\]

We use the positive case to illustrate the procedure and it is straightforward to apply it to the negative spike process. For a observed sample of spikes, the discrete counterpart is written,

\[
Y^+_{t_i} - Y^+_{t_{i-1}} = \alpha^+ Y^+_{t_{i-1}} \Delta t, \tag{8}
\]

where $\Delta t$ corresponds to the time increment from day $t_{i-1}$ to $t_i$, which is the same for every $i$ due to the equidistant daily time points. We use the regression

\[
Y^+_{t_i} - Y^+_{t_{i-1}} = \alpha^+ Y^+_{t_{i-1}} \Delta t + \varepsilon_{t_i},
\]
Jumps $H_1 = 2.5$ and spikes $H_1 = 2.3$, $H_2 = 6$ and $H_3 = 0.5$.

**Figure 12:** The log-spot prices, $S_t$, 2003-2013. Black * corresponds only to $H_1 = 2.5$, green and red * satisfy positive and negative spike with additional requirements $H_2 = 6$ and $H_3 = 0.5$.

In order to estimate $\alpha^+$. The estimate is obtain from $S^+$ and $S^-$. First, for all spikes in $S^+$, construct a vector, say $Y^+ = (s_{1,1}, s_{1,2}, ..., s_{m,n-1})$ where $i, j$ in $s_{i,j}$ corresponds to the log price of the $i$’th spike on its $j$’th day. Then let $Y^+ = (s_{1,2}, s_{1,3}, ..., s_{m,n})$. Using vector notation for the regression, it implies

$$Y^+ - Y^+ = \alpha^+ Y^+ \Delta t.$$ 

Rewrite and the result is

$$\left(\frac{Y^+ - Y^+}{Y^+ \Delta t}\right) \left(\frac{Y^+}{Y^+ \Delta t}\right)^* = \hat{\alpha}^+$$

where * indicate the transpose. A remark is that, in this estimate there is some mean reversion that corresponds to the process $X_t$. This is such a small part of the more extreme mean reversion of the spikes that it considered negligible.
7.2.3 Intensity and occurrence

The time dependent intensity of the non-stationary Poisson process are estimated as a moving average estimate of the found spikes. We do not consider leap-years at all in the paper, it is considered negligible. Due that leap years are simply shortened by a day, every year here consist of 365 days. The intensity is estimated from the latest $n = 11$ years, from the beginning of 2003 to the end of 2013. We have chosen a straight forward way and use a moving average approach. First, derive the average year, $h_t$. For all days $i$ in all years $j$, $i^{(j)}_i = 1, ..., 365$ and $j = 1, ..., 11$ let

$$h_{t_i} = \frac{1}{11} \sum_{j=1}^{11} I(\text{spike on day } i),$$

and $h_t$ corresponds to the average number of spikes each day. It is intuitive to address $h$ as a "typical year". Now, let, $H = (h, h, h)$ be a vector with dimension $1 \times 1095$, from which we derive the final estimate. For $2k + 1$
Amplitude size
Number of amplitudes
Histogram of negative spike amplitudes, $A^-$

Figure 14: Histogram of the observed amplitudes $A^-$ from the sample $S^-$

consecutive days, $t_{i-k}, ..., t_{i-1}, t_i, t_{i+1}, ..., t_{i+k}$ let

$$\tilde{\lambda}_{i}^{+} = \frac{1}{2k+1} \sum_{j=i-2k}^{i+2k} h(t_j)$$

where we keep the mid 365 elements in $\tilde{\lambda}_{i}^{+}$, which is the moving average estimate of $\lambda_{i}^{+}$, $\tilde{\lambda}_{i}^{+}$ of width $2k$. Note that this is only reasonable for $k \leq 365$. This is done both for the positive and the negative spikes. It is assumed that the intensity is time dependent and in 16 the estimate of time dependent is plotted.

7.3 The Ornstein-Uhlenbeck process

Remove the identified spikes from $\hat{\Gamma}_i$. After the spikes are removed, we consider the sample cleared form outliers. What is left is sub samples of the process $X_t$, between the removed spikes, so to say. Denote the sample $X = (x_{i_1}, ..., x_{i_n})$. This is the sample we approximate the parameters associated with $X_t$. As we know that when a positive spike occur the price level quickly
reduces to approximately the same level as before the spike. Therefore, when the spikes are removed from our original sample, instead of replacing the values in the sample we rearrange so that the places containing spikes instead represents the following value of the $X_t$ process. As we calibrate the model on relatively large sample the errors rising from this rearrangement will be small. In figure 17 what is left after the removal is visualized. Indeed most of the spikes are removed and it is reasonable to calibrate process on the observations. We estimate the parameters of the process by the maximum likelihood method on $X$.

One can find the distribution of $X_r$ given $X_s$ where $s < r$ of the Ornstein-Uhlenbeck process $X_t$ with long term mean $b = 0$, $dX(t) = \alpha(b - X(t))dt + \sigma dW_t$, namely,

$$X_r | x_s \sim N \left( x_s e^{-\alpha(r-s)} \frac{\sigma^2}{2\alpha} \left( 1 - e^{-2\alpha(r-s)} \right) \right)$$

which comes naturally from applying Ito formulae. Due that we know, for a normal random variable with mean $\mu$ and variance $\sigma^2$, say $Z \sim N(\mu, \sigma^2)$,
Figure 16: Estimated intensities of the positive and negative spikes $\lambda^+_t$ and $\lambda^-_t$ for an arbitrary length $k$ of the moving average estimate

the density function is

$$f_Z(z) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(z-\mu)^2}{2\sigma^2}},$$

it is implied that the transition density for our process $X$ is

$$f_{X_{t+1}|X_t=x_t}(x) = \sqrt{\frac{\alpha}{\pi \sigma^2(1-e^{-2\alpha \Delta_t})}} \exp \left( -\frac{\alpha(x-x_t)e^{-\alpha \Delta_t})^2}{\sigma^2(1-e^{-2\alpha \Delta_t})} \right).$$

Moreover, we know that if the process holds the Markov property, the joint density is the product of the transitional densities. With these assumptions we know may derive the log-likelihood function as a function of the logarithm
Deseasonal spot price \( \hat{\Gamma}_t \) and \( X \)

\[
L(\sigma, \alpha) = \frac{1}{2} \sum_{i=1}^{n} \log \left( \frac{\alpha}{\pi \sigma^2 (1 - e^{-2\alpha \Delta t})} \right) - \frac{\alpha}{\sigma^2} \sum_{i=1}^{n} \left( \frac{(x_{t_i} - x_{t_{i-1}} e^{-\alpha \Delta t})^2}{(1 - e^{-2\alpha \Delta t})} \right)
\]

(9)

Figure 17: Comparison of \( \hat{\Gamma}_t \) and \( X \)

over the joint density, namely

\[
\hat{\sigma}^2 = \frac{2\alpha}{n} \sum_{i=1}^{n} \frac{(x_{t_i} - x_{t_{i-1}} e^{-\alpha \Delta t})^2}{(1 - e^{-2\alpha \Delta t})}
\]

is obtained. Further, the ML estimation of \( \alpha \) is needed. It is indeed cumbersome if not impossible to obtain a analytical solution of the derivative. In [10] however, some simplifications are made, and an estimation of \( \alpha \) is
found as,
\[ \hat{\alpha} = \frac{1}{\Delta t} \ln \left( \frac{\sum_{i=1}^{n} x_t x_{t-1}}{\sum_{i=1}^{n} x_t^2} \right). \]

What also is remarked in [10] is that other methods yields the same result. We satisfy with this result. Note that in our case, as we have defined the process \( \alpha \) is replaced by \(-\alpha\).

## 8 Simulation

Denote the simulated daily spot log-prices by \( \tilde{S}(t_i) \) and let tilde, \( \tilde{\cdot} \), henceforth denote a simulated value. Then let
\[ \tilde{S}_t = \tilde{f}(t) + \tilde{\Gamma}_t, \]
where \( \tilde{f}(t) \) is the forwards price at time \( t \) and \( \tilde{\Gamma}_t = \tilde{X}_t + \tilde{Y}_t^+ + \tilde{Y}_t^- \). A straightforward way to simulate values for \( X_t, Y_t^+ \) and \( Y_t^- \) is to use a Euler discretization and then generate the path according to that. However, it entails some discretization error [6]. Simulations of \( X_t \) could also be done by using that we know the solution to a general Ornstein-Uhlenbeck process
\[ dX(t) = \alpha(b - X(t))dt + \sigma dW_t, \]
where \( \alpha, \beta \) and \( \sigma \) positive constants, namely,
\[ X(T) = b + e^{-\alpha T}(X_0 - b) + \sigma e^{-\alpha T} \int_0^T e^{-\alpha s} dW_s. \]
Again, note that in our case, as we have defined the process \( \alpha \) is replaced by \(-\alpha\). The result is straightforward to obtain with Ito. In this case though, recall stationarity and the special case with \( b = 0 \) considered here, and we obtain an algorithm that produces exactly the \( X_{t_i} \)'s of our process, that is,
\[ \tilde{X}_{t_{i+1}} = e^{-\alpha \Delta t} X_{t_i} + \sigma \sqrt{\frac{1}{2\alpha} \left( 1 - e^{-2\alpha \Delta t} \right)} \tilde{Z}_{t_{i+1}}, \]
where \( Z \sim N(0, 1) \). For
\[ dY_t^+ = \alpha^+ Y_t^+ dt + J^+ dq(\lambda_t^+) \]
\[ dY_t^- = \alpha^- Y_t^- dt + J^- dq(\lambda_t^-), \]
we choose a simple discretization of the processes, namely

\[ \tilde{Y}_{t_i}^+ = Y_{t_{i-1}}^+ \alpha^+ Y_{t_{i-1}}^+ \Delta t + \tilde{N}_t \]

where \( N_t \) is the generated compound Poisson process. Moreover, for \( N_t \) we use the thinning approach found in [6] combined with generated amplitudes \( \tilde{J} \).

There is an additional problem which we need to take into account. The day when the option is priced does not, in general, coincide with the last day before the start of the delivery period. Say that we price the option today, and there is time left until the delivery period starts. This means that there is an uncertainty regarding the forward price, due to that there is time left. As we do not know the distribution of the forward price, and hence cannot conditionally simulate from that, we need to approach this uncertainty in some other way. A solution is to use stratification and assume we start in the stationary distribution of \( \Gamma_t \). However, we do not know the stationary distribution and moreover the time from the pricing day to the delivery period start might not be sufficiently long to make the assumption that we start in the stationary distribution reasonable. We solve this by starting the simulation from the the day we price the option. This means we simulate paths from this day in order to get a distribution of the spot price with mean value equal to the forward price when the delivery period starts.

A final practical problem, concerning the arbitrage free price is that the average value of the simulated paths must be equal to the forward price. Otherwise there is arbitrage. Hence, if there is any ”arbitrage bias” in the simulations, we pragmatically adjust this by shifting the distribution in to the right place suggested by [9].

8.1 The Monte Carlo method

As in previous section stated, the price of an option is,

\[ V = \mathbb{E}[e^{-rT} \Psi(S(T))], \]

(10)

and the Monte Carlo Method is the ”tool” used in order to derive the expected value, namely by deriving the estimate,

\[ \hat{V} = \frac{1}{n} \sum_{i=1}^{n} V_i, \]

(11)

where \( V_i \) comes from a, sufficiently large, generated sample. This is known as the Monte Carlo estimate of the expected value. The estimator \( \hat{V} \) is
unbiased, that is, $E(\hat{V}) = V$ [6]. It relies on the strong law of large numbers which "ensures" that the Monte Carlo method indeed gives a "good" estimate of the expected value. The coding is made in Matlab and the built in pseudo random number generators \texttt{rand()} and \texttt{randn()} are used. It relies on the Marsienne twister \textit{MT19937} and is considered to have a sufficiently long period.

9 Results

This section consists of two parts. In the first part we visually compare some distributional properties. In the second part we present a case where we price an Asian option on the first quarter 2014 and see how the price changes due to variation in input parameters. A problematic concern is that there is not any data to back test the model on. From that point of view it is not straightforward to evaluate the model. However, we know that the price of an Asian option in general should be lower than its European counter part. Therefore in the second part, we also compare the result with the price on European options.

9.1 Distributional properties

Of course the result will differ from time to time if we use change the input parameters. So here we use what we have used through the entire report, $H_1 = 2.5$, $H_2 = 6$ and $H_3 = 0.5$.

\textit{QQ-plot, deseasonalized daily differences, sample vs. simulated}

A way to examine whether the sample and the simulated values are from the same distribution is to compare quantile-wise. This is made in 18 where we see that the quantile match pretty good. A straight line would represent a perfect match.

\textit{Deseasonalized daily differences, sample vs. simulated}

Another way to get an insight is to compare the daily differences straight away. In figure 19 what seems to hold the greatest difference is the variance. The simulated sample is much more homogeneous than the historical sample. This might be due to stochastic or seasonal volatility in the spot price.
9.2 Input sensitivity

Here we test for two values on the different input parameters. We test two different values of $H_1, H_2$ and $H_3$ to see how the price of the option changes. The input to the runs used to price the Asian call option, with delivery Q1 2014, is the following, $n_0 = 20 + 31$, which means we price the option November 9 2013. $n_1 = 31, n_2 = 28, n_3 = 31$ $\hat{f} = 39.9$ for $t$ in the delivery period, $K = 45, N = 10000$ where $K$ is the strike and $N$ is the number of generated paths. $H_1 = 2.5, H_2 = 6, H_3 = 0.5$, if not explicitly other values are phrased (as the case is in some examples) $r = 0.015$ where $r$ is the interest rate, $k = 15, c = 11$, $k$ corresponding to the moving average estimate of the estimation of the intensities $c$ is the number of years on which the model is calibrated.

First, as we see in the table 9.2 the value of the option changes quite dramatically when the definition, with subject to $H_1, H_2$ and $H_3$, is changed. In order to get a clearer picture of this, see figure 20. All parameters are kept fixed except for $H_1$ that varies. One possible explanation to the drastic price change is that there are so few spikes, that if just a small number extra

![QQ plot matching $S'$ and $S$](image-url)
spikes are identified, then it will give great influence to the estimated intensity. The same reasoning holds for changes in $H_3$. In figure 21 it is visualized that there seems to be a threshold at 0.3 where the price rapidly increases. Intuitively one might thing that a case where $H_3 = 0.2$ should result in a price higher than when $H_3 = 0.3$ due that it should allow more spikes, and hence more volatility. However, it might be the case that it includes a lot of spikes with amplitudes that are much lower, which in turn means that the spikes in the simulation wont have the same effect on the price of the option. Also the relationship between the amplitudes of the negative and positive spikes are particularly influential in some cases.

As mentioned earlier, there are suggestions for how to choose the values of the spikes. This is of course not to be disregarded, however, an additional way of choosing the input parameters is suggested here. What we have used is an easy, but time consuming, procedure. We simulate a great number

Figure 19: Comparison of the distribution of the historical sample and equal many simulated values of the daily deseasonalized logprice differences.
of paths, of the same length as the historical sample. In every generated
sample we slightly change the values of $H_1$, $H_2$ and $H_3$. Every time we
compare the distributional properties by a two sample Kolmogorov-Smirnov
test. This indicates tests the hypothesis that the historical sample and the
simulated one comes from the same distribution. We optimize by choosing
the inparameters giving the best $p-value$. Of course this procedure could be
discussed and there are objections. The result is that $H_1 = 2.15$, $H_2 = 6$ and
$H_3 = 0.50$. So this is also what we use in this last case where we compare
the results of some traded European options to the simulated values of the
Asian options. As we see the price of the Asian option is lower than the
price of the European option until the time to delivery gets small, that is
in December. This might be due to several reasons. One proposed reason
is that the model used in the market for pricing the European options do
not incorporate mean reversion which means that the distribution is much
wider when there are a "long" time until delivery while the model used in
this paper does not get the same range and stays more compact.

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<th>$H_3$</th>
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</table>

10 Conclusion

Due to the time constrain and the magnitude of the project, a natural re-
flexion is that indeed more time could be spent on each of the steps in the
project. Regarding the estimation of the process one could think about how
the model would behave if jumps were included. In the plots we see that
there indeed is some jumps that not are defined as spikes. So an extension
of the model could be to include a jump term in the Ornstein-Uhlenbeck
process. Stochastic or price dependent volatility could be another reason-
able feature to include. As seen in figure 19 the generated path is much
more homogeneous in this aspect. Whether this is an effect of stochastic
Figure 20: *Plot of how the price of an Asian option is affected for $H_1 = 2 : 0.01 : 3$*

Volatility, seasonal volatility or price level dependent volatility or something else is a question to investigate further. Even though, the model here investigated might be plausible for pricing Asian options, the conclusion is that the objections and concerns above needs to be taken into consideration.
Figure 21: Plot of how the price of an Asian option is affected for $H_3 = 0.2$: 0.01 : 30.8
Figure 22: Price of a Asian option compared with European option on the forward contract. Different strikes, prices of underlying and different times to maturity. "Asianj" corresponds to the adjustment to get a arbitrage free price. The plots have the same time granularity and hence they correponds to each others. For every begining of every month, two contracts are compared. One with strike far away from the price of the underlying and one with the strike close to the price of the underlying. So each two observations in each month have the same date.
References


