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Financial Modeling Under Incomplete Information

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I would like to dedicate this thesis to my parents

Athanasios and Paraskevi
and my brother
Stilianos
for their endless support during my master studies.

"There is nothing impossible to him who will try."
Alexander the Great

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Abstract

In this project our main assumption is that the price of a stock is modeled as a Geometric Brownian Motion with unknown drift and known constant volatility. In particular, we examine two cases for the drift. In the first one, we assume that the drift is modeled as a random variable with known distribution which is not directly observable by an agent in the market and it can take two values. In the second case, we assume that the drift is modeled as an Ornstein-Uhlenbeck process with known initial distribution which again is not directly observable. In view of the above assumptions, we address the following two problems. In the first one, we assume that an agent has a short position on a stock and we identify when is the optimal time to buy it back. In the second one, we assume that an agent has a self-financing portfolio consisting of one stock and a bank account and we present the optimal wealth allocation between the risky asset and the risk-free one.

Contents

Chapter 1

Introduction

In this project we address the problems of Optimal Closing of a Short-Position as well as The Portfolio Problem (Chapters 3 and 4 respectively) for 2 different stock price models of the form:

$$dX_t = \theta(t, X_t)X_t dt + \sigma X_t dW_t \quad (1)$$

where the drift $\theta(t, X_t)$ is an **unobserved** quantity while the volatility σ is considered as a known positive constant.

In Chapter 2, we introduce the stock price models and the main assumptions under which we work through out this project. Specifically, in paragraph 2.2 we assume that the drift is modeled as a random variable with known distribution and 2 possible outcomes, the Digital Drift case. Moreover, we assume that in the beginning of time ($t = 0$) a hypothetical coin toss occurred but an agent in the market is not able to observe the outcome and the only available information to her is given by the price process of the stock X_t. In view of the general theory of the **Filtering Problem** (see [8]), the **Girsanov** theorem (see [7], [8]) and the **Bayes** formula (see [8], [10]), we find the best estimate for the drift. Then, we transform (1) in such a way where all the parameters are observable, that is, equations **2.2.2.13** and **2.2.2.14**. In paragraph 2.3, we assume that the drift is modeled as an Ornstein-Uhlenbeck process and same as above, it cannot be observed. In view of the general theory of the Filtering problem and the **Kalman-Bucy** theorem (see [8]), we transform (1) in a Stochastic Differential Equation where all the parameters are observable quantities, that is, equations **2.3.2.12** and **2.3.2.13** (similar work can be found in [1], [2], [3], [11]).

In Chapter 3, we address the **optimal stopping** problem (see [7], [8]) of closing a short position for both cases described above. The main mathematical machinery that we use is the Girsanov theorem as well as the **Feynmann-Kac** theorem (see [7], [8]). The **optimal execution boundary** for both models is given by theorems **3.2.1** and **3.3.1** respectively (similar work can be found in [1], [2]).

In Chapter 4, we address the **portfolio optimization** problem (see [7]) under the assumption that an agent in the market possesses a portfolio consisting of a stock and a bank account growing with interest rate $r > 0$. Since this is a **stochastic optimal control** problem, the main mathematical machinery that we use is the **Hamilton-Jacobi-Bellman** equation (see [7], [8]) as well as the Feynmann-Kac theorem. We derive the portfolio's optimal wealth allocation as well as the agent's optimal value function for both market models described in Chapter 2. In addition, we solve the problem for the standard case where the drift is a known constant and we introduce theorem **4.2.2.2** which compares the value functions between the standard case and the Digital Drift case. The main results of this chapter are given by theorems **4.2.1**, **4.2.2.1**, **4.2.2.2** and **4.2.3.1** (similar work can be found in [4], [5], [6]).

Chapter 2
Modeling the Market

2.1 Introduction

In this chapter we formulate the main models that we will use through out this thesis. Our main assumption is that an agent in the market can only observe the price process of an asset without having any information about its drift or its driving process. Moreover, we assume that the asset's volatility is an estimated known positive constant (for example historical volatility).

2.2 The Digital Drift Case

2.2.1 Market Model

Suppose that the price of a stock is modeled by the following stochastic differential equation:

$$dX_t = \theta X_t dt + \sigma X_t dW_t, \; X_0 = x > 0$$

where, $\sigma > 0$ is constant and W_t is a Wiener process. Let $\mathscr{M}_t$ be the information generated by X_s, $s \leq t$. Moreover, we assume that θ is a random variable which can only take the values θ_1 and θ_2 ($\theta_1 < 0 < \theta_2$) and it is not directly observable. Furthermore, we assume that at time $t = 0$ an agent in the market has information about the probabilities of the events $\{\theta = \theta_1\}$ and $\{\theta = \theta_2\}$, that is, $P[\{\theta = \theta_2\}] = \rho$ and $P[\{\theta = \theta_1\}] = 1 - \rho$. Notice that only X_t is observable.

2.2.2 The Filtering Problem

As we can only observe X_t, we would like to find an estimate $\hat{\theta}_t$ of θ given $\mathscr{M}_t$. Let $\hat{\theta}_t = E[\theta|\mathscr{M}_t]$. Now, since our observations satisfy

$$dX_t = \theta X_t dt + \sigma X_t dW_t \quad (2.2.2.1)$$

we have that,

$$\frac{1}{X_t} dX_t = \theta dt + \sigma dW_t.$$

If we set $Z_t = \int\limits_0^t \frac{1}{X_s} dX_s$, we have that our observations get the form:

$$dZ_t = \theta dt + \sigma dW_t \ \ (2.2.2.2).$$

According to the general theory of the filtering problem we can define our innovation process as: $N_t = Z_t - \int\limits_0^t \hat{\theta}_s ds$. Thus,

$$N_t = \int\limits_0^t \theta ds + \sigma \int\limits_0^t dW_s - \int\limits_0^t \hat{\theta}_s \ ds \Leftrightarrow$$

$$N_t = \sigma W_t + \int\limits_0^t \theta - \hat{\theta}_s \ ds \Leftrightarrow$$

$$\frac{1}{\sigma} N_t = W_t + \int\limits_0^t \frac{1}{\sigma}(\theta - \hat{\theta}_s) \ ds \ \ (2.2.2.3).$$

Let $\hat{W}_t = W_t + \int\limits_0^t \frac{1}{\sigma}(\theta - \hat{\theta}_s) ds$, which is a Wiener process. Then we have that,

$$d\hat{W}_t = dW_t + \frac{1}{\sigma}(\theta - \hat{\theta}_t)dt \Leftrightarrow$$

$$d\hat{W}_t = \frac{1}{\sigma}(\theta dt + \sigma dW_t - \hat{\theta}_t dt) \Leftrightarrow$$

$$d\hat{W}_t = \frac{1}{\sigma}(\frac{1}{X_t} dX_t - \hat{\theta}_t dt) \Leftrightarrow$$

$$dX_t = \hat{\theta}_t X_t dt + \sigma X_t d\hat{W}_t \ \ (2.2.2.4).$$

Therefore, our observations will be of the form:

$$dZ_t = \hat{\theta}_t dt + \sigma d\hat{W}_t \ \ (2.2.2.5).$$

Now, we would like to find the Stochastic Differential Equation that $\hat{\theta}_t$ satisfies. Let $\mathfrak{X}_t$ be the information generated by the process $(Z_s)_{s \leq t}$ as appearing in (2.2.2.2). In view of the Girsanov theorem if we let:

$$\frac{dQ}{dP} = \mathrm{M}_T = \exp\left(-\frac{1}{2}\int\limits_0^T \frac{\theta^2}{\sigma^2} ds - \int\limits_0^T \frac{\theta}{\sigma} dW_s\right) =$$

$$\exp\left(\frac{\theta^2}{2\sigma^2} T - \frac{\theta}{\sigma} Z_T\right), \text{ on } \mathfrak{X}_t \ \ (2.2.2.6)$$

we have that Z_t is a Q- Wiener process independent of θ. Moreover, notice that $\mathscr{M}_t$=$\mathscr{X}_t$ so $\hat{\theta}_t = E[\theta|\mathscr{M}_t] = E[\theta|\mathscr{X}_t]$. Let $(\Omega, \mathscr{M}, Q)$ be a probability space and ϕ be the distribution of θ, i.e., $\phi(B) = Q(\theta^{-1}(B)),\ B \subset \mathscr{R}$ Borel. By the Bayes formula we have that:

$$\hat{\theta}_t = E[\theta|\mathscr{M}_t] = \frac{E^Q[\theta \, \frac{dP}{dQ}|\mathscr{M}_t]}{E^Q[\frac{dP}{dQ}|\mathscr{M}_t]} = \mathrm{f}(t, Z_t) \quad (2.2.2.7)$$

where,

$$E^Q[\theta \, \frac{dP}{dQ}|\mathscr{M}_t] = \int\limits_{\mathscr{R}} x \exp(-\frac{x^2}{2\sigma^2}t + \frac{x}{\sigma^2}Z_t)d\phi(x) \quad (2.2.2.8)$$

and

$$E^Q[\frac{dP}{dQ}|\mathscr{M}_t] = \int\limits_{\mathscr{R}} \exp(-\frac{x^2}{2\sigma^2}t + \frac{x}{\sigma^2}Z_t)d\phi(x) \quad (2.2.2.9)$$

Now, by applying Ito's formula (with differentiation under the integral sign) on $\hat{\theta}_t = \mathrm{f}(t, Z_t)$ and using (2.2.2.5) as the dynamics of Z_t as well as Fubini's theorem combined with the independence of θ and Z we end up to:

$$d\hat{\theta}_t = \sigma \, \mathrm{f}_z(t, Z_t)d\hat{W}_t \quad (2.2.2.10)$$

By the Bayes formula we have that:

$$\mathrm{f}_z(t, Z_t) = \frac{1}{\sigma^2}\left[\frac{\int\limits_{\mathscr{R}} x^2 \exp(-\frac{x^2}{2\sigma^2}t + \frac{x}{\sigma^2}Z_t)d\phi(x)}{\int\limits_{\mathscr{R}} \exp(-\frac{x^2}{2\sigma^2}t + \frac{x}{\sigma^2}Z_t)d\phi(x)} - \left(\frac{\int\limits_{\mathscr{R}} x \exp(-\frac{x^2}{2\sigma^2}t + \frac{x}{\sigma^2}Z_t)d\phi(x)}{\int\limits_{\mathscr{R}} \exp(-\frac{x^2}{2\sigma^2}t + \frac{x}{\sigma^2}Z_t)d\phi(x)}\right)^2\right]$$

$$\Leftrightarrow$$

$$\mathrm{f}_z(t, Z_t) = \frac{1}{\sigma^2}\left(E[\theta^2|\mathscr{M}_t] - E[\theta|\mathscr{M}_t]^2\right) = \frac{1}{\sigma^2}\mathrm{var}(\theta|\mathscr{M}_t) \quad (2.2.2.11)$$

Thus, (2.2.2.10) becomes

$$d\hat{\theta}_t = \frac{1}{\sigma}\mathrm{var}(\theta|\mathscr{M}_t)d\hat{W}_t \quad (2.2.2.12)$$

In addition, if we let $\rho_t = P[\{\theta = \theta_2\}|\mathscr{M}_t]$ for $t > 0$, that is that the estimate for the probability of the drift changes over time, then (2.2.2.4) becomes:

$$dX_t = [\rho_t(\theta_2 - \theta_1) + \theta_1]X_t dt + \sigma X_t d\hat{W}_t \quad (2.2.2.13)$$

where,

$$d\rho_t = \frac{(\theta_2 - \theta_1)}{\sigma}\rho_t(1 - \rho_t)d\hat{W}_t \quad (2.2.2.14)$$

Regarding equation (2.2.2.14) see [2], equation (2.5). Finally, notice that under this representation of X_t all the coefficients are observable quantities.

2.3 The Ornstein-Uhlenbeck Drift Case

2.3.1 Market Model

Suppose that the price of a stock is modeled by the following stochastic differential equation:

$$dX_t = \theta_t X_t dt + \sigma X_t dW_t,\ X_0 = x > 0$$

where, $\sigma > 0$ is constant and W_t is a Wiener process. Let $\mathscr{M}_t$ be the information generated by $X_s,\ s \leq t$. Moreover, we assume that the process θ_t satisfies the Ornstein-Uhlenbeck stochastic differential equation:

$$d\theta_t = (k - \theta_t)dt + udV_t \quad (2.3.1.1)$$

where, V_t is a Wiener process, $u > 0$, $k < 0$ constants. Furthermore, we assume that the initial value of the drift process has normal distribution, $\theta_0 \sim \mathcal{N}[\mu_0, \sigma_0^2]$ which is independent of W_t and V_t and that only X_t is observable.

2.3.2 The Filtering Problem

Since the drift process θ_t satisfies the Ornstein-Uhlenbeck stochastic differential equation we have that:

$$d\theta_t = (k - \theta_t)dt + udV_t \Leftrightarrow$$

$$d\theta_t - \theta_t dt = kdt + udV_t \Leftrightarrow$$

$$e^t d\theta_t - e^t \theta_t dt = ke^t dt + ue^t dV_t \Leftrightarrow$$

$$d(e^t \theta_t) = ke^t dt + ue^t dV_t \Leftrightarrow$$

$$\theta_t = \theta_0 e^{-t} + k(1 - e^{-t}) + u \int_0^t e^{s-t} dV_s$$

with $E[\theta_t | \mathcal{N}_t] = \theta_0 e^{-t} + k(1 - e^{-t})$, where $\mathcal{N}_t$ is the information generated by $\theta_s,\ s \leq t$. As we can only observe X_t, we would like to find an estimate $\hat{\theta}_t$ of θ_t given $\mathscr{M}_t$. Let $\hat{\theta}_t = E[\theta_t | \mathscr{M}_t]$. Now, since our observations satisfy

$$dX_t = \theta_t X_t dt + \sigma X_t dW_t \quad (2.3.2.1)$$

we have that,

$$\frac{1}{X_t} dX_t = \theta_t dt + \sigma dW_t.$$

If we set $Z_t = \int\limits_0^t \frac{1}{X_s} dX_s$, we have that our observations get the form:

$$dZ_t = \theta_t dt + \sigma dW_t \quad (2.3.2.2).$$

According to the general theory of the filtering problem we can define our innovation process as: $N_t = Z_t - \int\limits_0^t \hat{\theta}_s ds$. Thus,

$$N_t = \int_0^t \theta_s ds + \sigma \int_0^t dW_s - \int_0^t \hat{\theta}_s ds \Leftrightarrow$$

$$N_t = \sigma W_t + \int_0^t \theta_s - \hat{\theta}_s ds \Leftrightarrow$$

$$\frac{1}{\sigma} N_t = W_t + \int_0^t \frac{1}{\sigma}(\theta_s - \hat{\theta}_s) ds \quad (2.3.2.3).$$

Let $\hat{W}_t = W_t + \int\limits_0^t \frac{1}{\sigma}(\theta_s - \hat{\theta}_s) ds$, which is a Wiener process. Then we have that,

$$d\hat{W}_t = dW_t + \frac{1}{\sigma}(\theta_t - \hat{\theta}_t) dt \Leftrightarrow$$

$$d\hat{W}_t = \frac{1}{\sigma}(\theta_t dt + \sigma dW_t - \hat{\theta}_t dt) \Leftrightarrow$$

$$d\hat{W}_t = \frac{1}{\sigma}(\frac{1}{X_t} dX_t - \hat{\theta}_t dt) \Leftrightarrow$$

$$dX_t = \hat{\theta}_t X_t dt + \sigma X_t d\hat{W}_t \quad (2.3.2.4).$$

Therefore, our observations will be of the form:

$$dZ_t = \hat{\theta}_t dt + \sigma d\hat{W}_t \quad (2.3.2.5).$$

Now, we would like to find the Stochastic Differential Equation that $\hat{\theta}_t$ satisfies. The system equation is given by (2.3.1.1), while the observations satisfy (2.3.2.2). We note that $E[\theta_t - k|\mathcal{M}_t] = E[\theta_t|\mathcal{M}_t] - k = \hat{\theta}_t - k$ and that $d(\theta_t - k) = d\theta_t$. Hence, since the process θ_t has normal distribution in view of the Kalman-Bucy filter theorem we have that:

$$d\hat{\theta}_t = \frac{\sigma^2 + S(t)}{\sigma^2}(k - \hat{\theta}_t) dt + \frac{S(t)}{\sigma^2} dZ_t \Leftrightarrow$$

$$d\hat{\theta}_t = [k(1 + \frac{S(t)}{\sigma^2}) - \hat{\theta}_t] dt + \frac{S(t)}{\sigma} d\hat{W}_t \quad (2.3.2.6)$$

where, the Mean Square Error $S(t) = E[(\theta_t - \hat{\theta}_t)^2]$ satisfies the deterministic Riccati equation:

$$\frac{dS(t)}{dt} = -\frac{1}{\sigma^2}S^2(t) - 2S(t) + u^2 \quad (2.3.2.7)$$

where,

$$S(0) = E[(\theta_0 - \hat{\theta}_0)^2] = \mathrm{Var}(\theta_0) = \sigma_0^2 \quad (2.3.2.8).$$

The solution of the above equation is:

$$S(t) = (\sqrt{\sigma^4 + \sigma^2 u^2})\tanh(A + Bt) - \sigma^2 \quad (2.3.2.9)$$

where,

$$A = \tanh^{-1}\left(\frac{\sigma^2 + \sigma_0^2}{\sqrt{\sigma^4 + \sigma^2 u^2}}\right)$$

$$B = \sqrt{1 + \frac{u^2}{\sigma^2}}$$

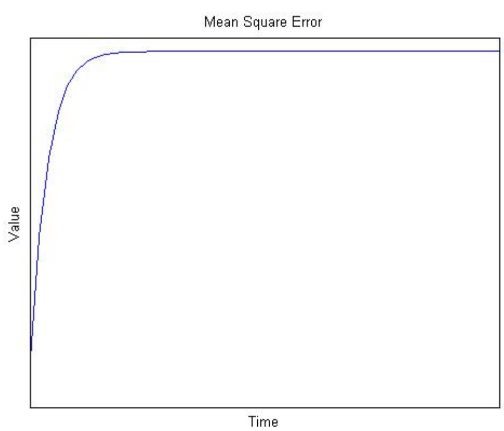
If we let $t \to +\infty$ we have that $S(t) \to \sqrt{\sigma^4 + \sigma^2 u^2} - \sigma^2 = s^* > 0$, the steady state of the Mean Square Error.

According to the above result, it can be seen that in the long run the Mean Square Error is a positive constant. Therefore, more information will not have crucial effect on variance reduction of the estimator but it will only renew $\hat{\theta}_t$. In addition, the monotonicity of the solution $S(t)$ depends on the choice of the parameters σ, u and σ_0.

Namely, if $\sqrt{\sigma^4 + \sigma^2 u^2} - (\sigma^2 + \sigma_0^2) > 0$, we have that:

$$S(t) \nearrow \text{ and } 0 < \sigma_0 \leq S(t) \leq s^* \quad (2.3.2.10)$$

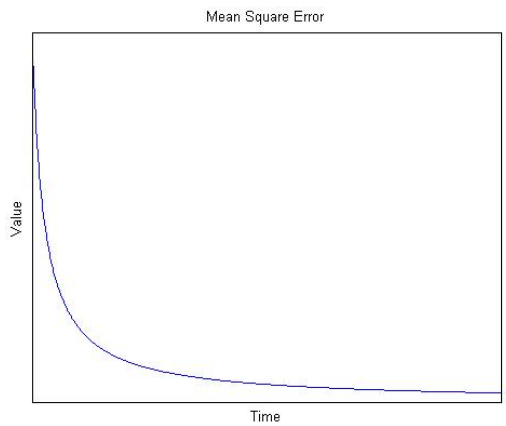
(The unrealistic scenario, since even if the set of our observations is infinite we will never be able to determine the initial distribution of θ_t)



while if $\sqrt{\sigma^4+\sigma^2u^2}-(\sigma^2+\sigma_0^2)\leq 0$, we have that:

$$S(t)\searrow \text{ and } 0<s^*\leq S(t)\leq \sigma_0 \quad (2.3.2.11)$$

(The realistic scenario, since we know the initial distribution of θ_t)



Now, under (2.3.2.11) since the Mean Square Error in the long-run will be a positive constant, our market model it is natural to take the form:

$$dX_t = \hat{\theta}_t X_t dt + \sigma X_t d\hat{W}_t \quad (2.3.2.12)$$
$$d\hat{\theta}_t = [k(1+\frac{s^*}{\sigma^2}) - \hat{\theta}_t]dt + \frac{s^*}{\sigma} d\hat{W}_t \quad (2.3.2.13)$$

Chapter 3

Optimal Closing of a Short-Position

3.1 Introduction

In this chapter we assume that an agent has a short-position on the stock X_t and that there are no dividends and no transaction costs. The problem that we will try to address is to identify when is the best time for the agent to buy back the stock. Let us suppose that she does not execute the order if the price is too high for profit, that is, the agent believes in a small drift. Therefore, there exists a threshold $h(t) > 0$ such that when the stock price hits $h(t)$, she immediately closes her position. The mathematical interpretation of the above problem is the solution of:

$$u(x) = \inf_{\tau \geq 0} E_{0,x}[X_\tau] = E_{0,x}[X_{\tau^*}] \quad (3.1.1)$$

where, u is called the value function and τ^* is the first exit time of the process X_t from the "stop-later" region $x \in (0, h)$. In the "stop-now" region where $x \geq h$ it is $u(x) = x$. As a result, the solution of (3.1.1) coincides with the determination of the free boundary $h(t)$ between the two regions.

3.2 The Digital Drift Case

Let us assume that our model is governed by (2.2.2.13) and (2.2.2.14). Let $\{\mathcal{F}\}_{t \geq 0}$ be the information generated by the process ρ_t, $(\Omega, \mathcal{F}, P)$ be a filtered probability space and T$\leq +\infty$ be a given constant.

Since $X_t = x \exp(\int_0^t \rho_s(\theta_2 - \theta_1) + \theta_1 ds - \frac{1}{2}\sigma^2 t + \sigma \hat{W}_t)$, we have that under P equation (3.1.1) becomes:

$$u(x) = \inf_{\tau \geq 0} E^P_{0,x}[X_\tau] = x \inf_{\tau \geq 0} E^P_{0,x}[\exp(\int_0^\tau \rho_s(\theta_2 - \theta_1) + \theta_1 ds) \exp(-\frac{1}{2}\sigma^2 \tau + \sigma \hat{W}_\tau)]$$

$$(3.2.1)$$

In view of the Girsanov theorem, if we put:

$$M_t = \exp(-\frac{1}{2}\sigma^2 t + \sigma \hat{W}_t) = \exp(-\frac{1}{2}\int_0^t \sigma^2 ds + \int_0^t \sigma d\hat{W}_t),\ t \leq T$$

and

$$\frac{dQ}{dP} = M_T, \text{ on } \mathscr{F}_T$$

then, Q is a probability measure on $\mathscr{F}_T$ and the process:

$$\hat{W}_t^Q = -\int_0^t \sigma ds + \hat{W}_t \Leftrightarrow$$

$$d\hat{W}_t^Q = -\sigma dt + d\hat{W}_t$$

is a Wiener process with respect to Q. Moreover, the stochastic integral representation of ρ_t under Q is of the form:

$$d\rho_t = \frac{(\theta_2 - \theta_1)}{\sigma}\rho_t(1-\rho_t)d\hat{W}_t \Leftrightarrow$$

$$d\rho_t = \frac{(\theta_2 - \theta_1)}{\sigma}\rho_t(1-\rho_t)(d\hat{W}_t^Q + \sigma dt) \Leftrightarrow$$

$$d\rho_t = (\theta_2 - \theta_1)\rho_t(1-\rho_t)dt + \frac{(\theta_2 - \theta_1)}{\sigma}\rho_t(1-\rho_t)d\hat{W}_t^Q \quad (3.2.2).$$

Hence, under the probability measure Q we have that (3.2.1) becomes:

$$u(x) = x \ \inf_{\tau \geq 0} E_{0,\rho}^Q[\exp(\int_0^\tau \rho_s(\theta_2 - \theta_1) + \theta_1 ds)]. \quad (3.2.3)$$

Now, according to equation (3.2.3) it can be seen that the solution of the agent's original problem can be reduced to the solution of the problem:

$$V(\rho) = \inf_{\tau \geq 0} E_{0,\rho}^Q[\exp(\int_0^\tau \rho_s(\theta_2 - \theta_1) + \theta_1 ds)]. \quad (3.2.4)$$

Notice that the smaller the probability for the drift to be θ_2, the bigger the agent's profit. Thus, as above we postulate that there exists a threshold $h \in (0,1)$ with the corresponding "stop-later" region of the process ρ_t to be $A = (0,h)$. Hence, the agent closes her position immediately when $\rho_t = h$. Under the postulated strategy and in view of the Feynman-Kac theorem we have the following:

Theorem 3.2.1
The value function V satisfies the boundary value problem:

$$\frac{1}{2}\frac{(\theta_2-\theta_1)^2}{\sigma^2}\rho^2(1-\rho)^2V^h_{\rho\rho}+(\theta_2-\theta_1)\rho(1-\rho)V^h_\rho+(\rho(\theta_2-\theta_1)+\theta_1)V^h=0,\ \rho\in A$$

$$V^h(\rho)=1,\ h\leq\rho<1$$

where the free boundary h coincides with the positive, with respect to γ, solution of:

$$\frac{1}{2}\frac{(\theta_2-\theta_1)^2}{\sigma^2}\gamma(\gamma-1)+(\theta_2-\theta_1)\gamma+\theta_1=0$$

Proof

If we make the change of variables $\rho=\dfrac{\phi}{1+\phi}$ and set $V^h(\rho)=\dfrac{1}{1+\phi}u(\phi)$ we have that:

$$V^h_\rho=u^h_\phi(1+\phi)-u^h,\ V^h_{\rho\rho}=u^h_{\phi\phi}(1+\phi)^3.$$

Thus, the partial differential equation takes the form:

$$\frac{1}{2}\frac{(\theta_2-\theta_1)^2}{\sigma^2}\phi^2u^h_{\phi\phi}+(\theta_2-\theta_1)\phi u^h_\phi+\theta_1u^h=0\ \ (3.2.5).$$

Now, if we make the ansantz $u(\phi)=\phi^\gamma$ we have that (3.2.5) becomes:

$$\frac{1}{2}\frac{(\theta_2-\theta_1)^2}{\sigma^2}\gamma(\gamma-1)+(\theta_2-\theta_1)\gamma+\theta_1=0$$

which has two distinct solutions of opposite signs which we denote them by $\{\gamma_-,\ \gamma_+\}$. Notice that for $\gamma=1$, it is:

$$\frac{1}{2}\frac{(\theta_2-\theta_1)^2}{\sigma^2}\gamma(\gamma-1)+(\theta_2-\theta_1)\gamma+\theta_1=\theta_2>0$$

so $\gamma_+\in(0,1)$. Therefore, (3.2.5) has the following solution:

$$u^h(\phi)=C_1u^h_1+C_2u^h_2=C_1\phi^{\gamma_-}+C_2\phi^{\gamma_+}$$

where, u^h_1 and u^h_2 are independent solutions of (3.2.5). Since we want the solution to decline due to the fact that the agent will not have incentives to buy back the stock, it suffices to consider:

$$u^h(\phi)=C\phi^{\gamma_+}.$$

As a result, the solution of the agent's initial problem is:

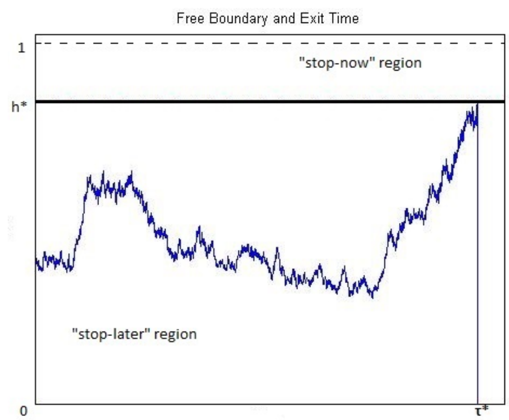
$$V^h(\rho)=C\rho^{\gamma_+}(1-\rho)^{1-\gamma_+},\ \ \rho\in A$$

$$V^h(\rho) = 1, \text{ for } h \leq \rho < 1$$

Since $V^h(h) = 1$, we have that $C = h^{-\gamma_+}(1-h)^{\gamma_+ - 1}$. In addition by the "Higher-Order-Contact" condition we have that:

$$\frac{d^{(\text{right})}V^h}{d\rho}|_h = \frac{d^{(\text{left})}V^h}{d\rho}|_h$$

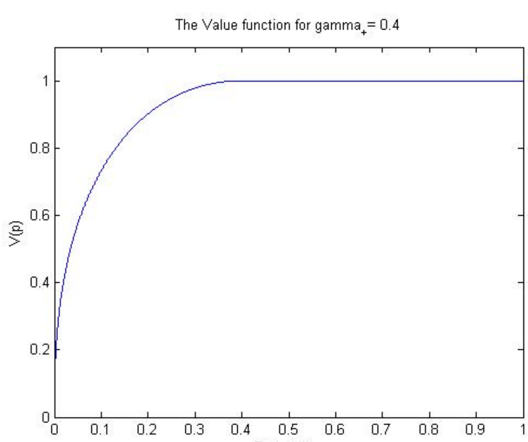
from which we derive that the optimal level h^* is given by: $h^* = \gamma_+$, constant.■



Remarks

1) $V^h(\rho)$ is concave in ρ. For $\rho \in (0, \gamma_+)$ we have that $V^h(\rho) \nearrow$ while for $\rho \in [\gamma_+, 1)$ it is $V^h(\rho) = 1$.

2) For fixed θ_1, θ_2 we have that the smaller the volatility σ, the smaller the γ_+. Thus, the margin for profit for the agent is bigger according to our analysis above. Exactly the same effect occurs when σ is fixed, while the difference between θ_2 and θ_1 increases.



3.3 The Ornstein-Uhlenbeck Drift Case

Let us assume that our model is governed by (2.3.2.12) and (2.3.2.13). Let $\{\mathcal{N}\}_{t\geq 0}$ be the information generated by the process $\hat{\theta}_t$, $(\Omega, \mathcal{N}, P)$ be a filtered probability space and T$\leq +\infty$ be a given constant.

Since $X_t = x \exp(\int_0^t \hat{\theta}_s ds - \frac{1}{2}\sigma^2 t + \sigma \hat{W}_t)$, we have that under P equation (3.1.1) becomes:

$$u(x) = \inf_{\tau\geq 0} E^P_{0,x}[X_\tau] = x \inf_{\tau\geq 0} E^P_{0,x}[\exp(\int_0^\tau \hat{\theta}_s ds) \exp(-\frac{1}{2}\sigma^2 \tau + \sigma \hat{W}_\tau)] \quad (3.3.1)$$

In view of the Girsanov theorem, if we put:

$$M_t = \exp(-\frac{1}{2}\sigma^2 t + \sigma \hat{W}_t) = \exp(-\frac{1}{2}\int_0^t \sigma^2 ds + \int_0^t \sigma d\hat{W}_t),\ t \leq T$$

and

$$\frac{dQ}{dP} = M_T,\ \text{on } \mathcal{N}_T$$

then, Q is a probability measure on $\mathcal{N}_T$ and the process:

$$\hat{W}^Q_t = -\int_0^t \sigma ds + \hat{W}_t \Leftrightarrow$$

$$d\hat{W}^Q_t = -\sigma dt + d\hat{W}_t$$

is a Wiener process with respect to Q. Moreover, the stochastic integral representation of $\hat{\theta}_t$ under Q is of the form:

$$d\hat{\theta}_t = [k(1 + \frac{s^*}{\sigma^2}) - \hat{\theta}_t]dt + \frac{s^*}{\sigma} d\hat{W}_t \Leftrightarrow$$

$$d\hat{\theta}_t = [k(1 + \frac{s^*}{\sigma^2}) - \hat{\theta}_t]dt + \frac{s^*}{\sigma}(d\hat{W}^Q_t + \sigma dt) \Leftrightarrow$$

$$d\hat{\theta}_t = [k(1 + \frac{s^*}{\sigma^2}) + s^* - \hat{\theta}_t]dt + \frac{s^*}{\sigma} d\hat{W}^Q_t \Leftrightarrow$$

$$d\hat{\theta}_t = (\alpha - \hat{\theta}_t)dt + \beta d\hat{W}^Q_t \quad (3.3.2)$$

where,

$$\alpha = k(1 + \frac{s^*}{\sigma^2}) + s^* \text{ and } \beta = \frac{s^*}{\sigma}.$$

Hence, under the probability measure Q we have that (3.3.1) becomes:

$$u(x) = x\ \mathrm{inf}_{\tau\geq 0} E^{Q}_{0,\hat{\theta}}[\exp(\int_0^\tau \hat{\theta}_s ds)]. \quad (3.3.3)$$

Now, according to equation (3.3.3) it can be seen that the solution of the agent's original problem can be reduced to the solution of the problem:

$$V(\hat{\theta}) = \mathrm{inf}_{\tau\geq 0} E^{Q}_{0,\hat{\theta}}[\exp(\int_0^\tau \hat{\theta}_s ds)]. \quad (3.3.4)$$

Notice that the smaller the drift, the bigger the agent's profit. Thus, as above we postulate that there exists a threshold $\lambda \in \mathscr{R}$ with the corresponding "stop-later" region of the process $\hat{\theta}_t$ to be $A = (-\infty, \lambda)$. Hence, the agent closes her position immediately when $\hat{\theta}_t = \lambda$. Under the postulated strategy and in view of the Feynman-Kac theorem we have the following:

Theorem 3.3.1
The value function V satisfies the boundary value problem:

$$\frac{1}{2}\beta^2 V^{\lambda}_{\hat{\theta}\hat{\theta}} + (\alpha - \hat{\theta})V^{\lambda}_{\hat{\theta}} + \hat{\theta}V^{\lambda} = 0, \text{ for } \hat{\theta} \in A$$

$$V^{\lambda}(\hat{\theta}) = 1, \text{ for } \hat{\theta} \geq \lambda \ \square$$

Remark
The general solution of the above problem is of the form:

$$V^{\lambda}(\hat{\theta}) = \exp\left(\hat{\theta} - 2\sqrt{\frac{1}{2}\beta^2 + \alpha}\right)\mathscr{G}\left(-\frac{1}{4}\beta^2 + \frac{1}{2}\alpha,\ \frac{1}{2}\ ;\ 2(\sqrt{\frac{1}{2}\beta^2 + \alpha})(\hat{\theta} - (\beta^2 + \alpha)^2)\right)$$

where,

$$\mathscr{G}\left(-\frac{1}{4}\beta^2 + \frac{1}{2}\alpha,\ \frac{1}{2}\ ;\ 2(\sqrt{\frac{1}{2}\beta^2 + \alpha})(\hat{\theta} - (\beta^2 + \alpha)^2)\right)$$

is an arbitrary solution of the degenerate hypergeometric ordinary differential equation:

$$\hat{\theta}V^{\lambda}_{\hat{\theta}\hat{\theta}} + (\frac{1}{2} - \hat{\theta})V^{\lambda}_{\hat{\theta}} + (\frac{1}{4}\beta^2 + \frac{1}{2}\alpha)V^{\lambda} = 0$$

Chapter 4

The Portfolio Problem

4.1 Introduction

In this chapter we assume that the agent possesses a self-financing portfolio consisting of a risky asset X_t (no dividends, no transaction costs) and a bank account with interest rate $r > 0$ such that $dB_t = rB_t dt$. Under the market models described in Chapter 2 we address the portfolio's optimal rebalancing problem. In addition, we introduce a comparison theorem for the portfolio value functions between the complete information model where the drift is known constant and equal to the expected value of the random variable θ and the Digital Drift model.

4.2 Portfolio Rebalance

Since the portfolio is self-financing that means that at time t and once the new price of the risky asset X_t has been quoted the agent readjusts her position without bringing or removing any wealth. Finally, we introduce the terms of lending and borrowing at a riskless rate. The former can be interpreted as savings in the bank account (by selling the risky asset X_t), while the latter can be considered as withdrawals from the bank account (for investing in the risky asset X_t). Now let $T > 0$ be the time horizon and $\Phi(x) = \dfrac{x^{\gamma-1}}{\gamma-1}$, $\gamma \in (1,2)$, be the Constant Relative Risk Aversion utility function of the agent's final time wealth. Φ expresses the agent's risk aversion, i.e., the higher the γ, the more risk averse is the agent. Let us define P_t as the value of the portfolio at time $t \geq 0$ and $u^0(t),\ u^1(t)$ the proportions invested in B_t and X_t respectively. Then, we have that the dynamics of P_t are given by:

$$dP_t = P_t[u^0(t,\cdot)\frac{dB_t}{B_t} + u^1(t,\cdot)\frac{dX_t}{X_t}] \quad (4.1.1)$$

where,

$$u^0(t,\cdot) + u^1(t,\cdot) = 1,\ t \geq 0 \quad (4.1.2).$$

Notice that equation (4.1.2) means that the agent is fully invested and that unlimited short sales are allowed. Moreover, we demand u^0 and u^1 to be admissible control laws, that is, to be adapted to the filtration generated by the risky asset X_t.

4.2.1 The Market Under Complete Information

Let us assume that we are under the following model:

$$dX_t = E[\theta]X_t dt + \sigma X_t dW_t \quad (4.2.1.1)$$

where,

$$E[\theta] = \theta_1 + \rho(\theta_2 - \theta_1) \quad (4.2.1.2)$$

Theorem 4.2.1
Let $V(t,p) = \max_{u^1} E_{t,p}[\Phi(P_T)],\ (t,p) \in [0,T] \times \mathcal{R}_+$, be the optimal value function of the agent's portfolio wealth. Then,

1) $V(t,p) = \dfrac{p^{\gamma-1}}{\gamma-1}\exp\left\{\left[r + \dfrac{(E[\theta]-r)^2}{2\sigma^2(2-\gamma)}\right](\gamma-1)(T-t)\right\}$

2) The optimal portfolio wealth allocation is given by:

$$u_*^1 = \frac{E[\theta]-r}{\sigma^2(2-\gamma)},\ u_*^1 = 1 - u_*^0$$

Proof

Under the model given by (4.2.1.1) we have that (4.1.1) becomes:

$$dP_t = P_t\{[r + u^1(t,p)(E[\theta]-r)]dt + u^1(t,p)\sigma dW\} \quad (4.2.1.3)$$

Therefore, we have that the optimal value function V satisfies the Hamilton-Jacobi-Bellman equation:

$$V_t + \max_{u^1}\{[r + u^1(E[\theta]-r)]pV_p + \frac{1}{2}(u^1)^2\sigma^2 p^2 V_{pp}\} = 0 \quad (4.2.1.5)$$

$$V(T,\rho,p) = \Phi \quad (4.2.1.6)$$

Thus, if we let

$$f(u^1) = [r + u^1(E[\theta]-r)]pV_p + \frac{1}{2}(u^1)^2\sigma^2 p^2 V_{pp}$$

and maximize it with respect to u^1 we have that the optimal portfolio allocation is given by:

$$u_*^1(t) = -\frac{(E[\theta]-r)V_p}{\sigma^2 p V_{pp}},\ u_*^0(t) = 1 - u_*^1(t) \quad (4.2.1.7)$$

By (4.2.1.7) we have that (4.2.1.5) takes the form:

$$V_t + [r + u_*^1(E[\theta]-r)]pV_p + \frac{1}{2}(u_*^1)^2\sigma^2 p^2 V_{pp} = 0 \quad (4.2.1.8)$$

Hence, if we try:

$$V(t,p) = f(t)\frac{p^{\gamma-1}}{\gamma-1} \quad (4.2.1.9)$$

where $f(T) = 1$, we have that:

$$f(t) = \exp\left(A(T-t)\right) \quad (4.2.1.10)$$

where,

$$A = [r + u_*^1(E[\theta] - r)](\gamma - 1) + \frac{1}{2}(u_*^1)^2\sigma^2(\gamma-1)(\gamma-2).$$

As a result by (4.2.1.7), (4.2.1.9) and (4.2.1.10) we have that,

$$u_*^1 = \frac{E[\theta] - r}{\sigma^2(2-\gamma)},\ u_*^1 = 1 - u_*^0 \quad (4.2.1.11)$$

and

$$V(t,p) = \frac{p^{\gamma-1}}{\gamma-1}\exp\left\{\left[r + \frac{(E[\theta]-r)^2}{2\sigma^2(2-\gamma)}\right](\gamma-1)(T-t)\right\} \quad (4.2.1.12) \blacksquare$$

Remark
Notice, that in this case the optimal portfolio allocation does not depend on time but only on the agent's risk aversion level γ and remains constant in time. Also, notice that the optimal value function is convex in $E[\theta]$.

4.2.2 The Digital Drift Case

Suppose that the price of a stock is modeled by (2.2.2.13) and (2.2.2.14). Then, (4.1.1) takes the form:

$$dP_t = P_t[u^0(t,p,\rho)\frac{dB_t}{B_t} + u^1(t,p,\rho)\frac{dX_t}{X_t}] \ \Leftrightarrow$$

$$dP_t = P_t\{u^0(t,p,\rho)rdt + u^1(t,p,\rho)\{[\ \rho_t(\theta_2-\theta_1)+\theta_1]dt + \sigma d\hat{W}_t\}\} \Leftrightarrow$$

$$dP_t = P_t\{\ \{\ r + u^1(t,p,\rho)[\rho_t(\theta_2-\theta_1)+\theta_1-r]\ \}dt + u^1(t,p,\rho)\sigma d\hat{W}_t\ \} \quad (4.2.2.1)\ .$$

Moreover, let

$$V(t,p,\rho) = \ \max_{u^1}\ E_{t,p,\rho}[\Phi(P_T)] \quad (4.2.2.2)$$

be the optimal value function, where V is sufficiently smooth. In order to find the optimal portfolio allocation $u^1_*(t,p,\rho)$ we will make use of the dynamic programming argument. Therefore, in order to derive the Hamilton-Jacobi-Bellman equation we apply Ito's formula on $V(t,p,\rho)$. Notice that the processes P_t and ρ_t are driven by the same Wiener process. Thus, we have that:

$$dV = \frac{dV}{dt}dt + \frac{dV}{d\rho}d\rho_t + \frac{dV}{dp}dP_t + \frac{1}{2}\left(\frac{d^2V}{d\rho^2}(d\rho_t)^2 + 2\frac{d^2V}{d\rho dp}(d\rho_t)(dP_t) + \frac{d^2V}{dp^2}(dP_t)^2\right)$$

$$\Leftrightarrow$$

$$dV = Adt + Bd\hat{W}_t$$

where,

$$A = V_t + \frac{1}{2}\frac{(\theta_2-\theta_1)^2}{\sigma^2}\rho_t^2(1-\rho_t)^2V_{\rho\rho} + (\theta_2-\theta_1)\rho_t(1-\rho_t)u^1(t,p,\rho)P_tV_{\rho p} +$$

$$\frac{1}{2}u^1(t,p,\rho)^2\sigma^2P_t^2V_{pp} + P_t\{\ r + u^1(t,p,\rho)[\rho_t(\theta_2-\theta_1)+\theta_1-r]\ \}V_p \quad (4.2.2.3)$$

and

$$B = \frac{(\theta_2-\theta_1)}{\sigma}\rho_t(1-\rho_t)V_\rho + P_tu^1(t,p,\rho)\sigma V_p \quad (4.2.2.4)$$

As a result, we have that the Hamilton-Jacobi-Bellman equation is of the form:

$$V_t + \max_{u^1}[\mathscr{L}^{u^1}V] = 0 \ \Leftrightarrow$$

$$V_t + \frac{1}{2}\frac{(\theta_2-\theta_1)^2}{\sigma^2}\rho^2(1-\rho)^2V_{\rho\rho} + rpV_p + \max_{u^1}\{C\} = 0 \quad (4.2.2.5)$$

where,

$$C = (\theta_2-\theta_1)\rho(1-\rho)u^1pV_{\rho p} + pu^1[\rho(\theta_2-\theta_1)+\theta_1-r]V_p + \frac{1}{2}(u^1)^2\sigma^2p^2V_{pp}$$

Hence, by maximizing C with respect to u^1 we have that the optimal portfolio allocation will be given by:

$$u_*^1(t,p,\rho) = -\frac{(\theta_2-\theta_1)\rho(1-\rho)pV_{\rho p} + [\rho(\theta_2-\theta_1)+\theta_1-r]pV_p}{\sigma^2 p^2 V_{pp}} \quad (4.2.2.6)$$

and

$$u_*^0(t,p,\rho) = 1 - u_*^1(t,p,\rho) \quad (4.2.2.7)$$

Thus, by substituting (4.2.2.6) into (4.2.2.5) we have that $V(t,p,\rho)$ satisfies:

$$V_t + \frac{1}{2}\frac{(\theta_2-\theta_1)^2}{\sigma^2}\rho^2(1-\rho)^2 V_{\rho\rho} + rpV_p + C_* = 0 \quad (4.2.2.8)$$

$$V(T,\rho,p) = \Phi \quad (4.2.2.9)$$

where,

$$C_* = (\theta_2-\theta_1)\rho(1-\rho)u_*^1 pV_{\rho p} + pu_*^1[\rho(\theta_2-\theta_1)+\theta_1-r]V_p + \frac{1}{2}(u_*^1)^2\sigma^2 p^2 V_{pp} \quad (4.2.2.10).$$

Let us try a solution of the form:

$$V(t,p,\rho) = \left(\frac{p^{\gamma-1}}{\gamma-1}\right) H(t,\rho)^{2-\gamma}, \text{ such that } H(T,\rho)=1,\ H \geq 0$$

$$\text{where } (t,p,\rho) \in [0,T] \times \mathscr{R}_+ \times (0,1)$$

$$(4.2.2.11)$$

Then, we have that:

$$V_t = \left(\frac{p^{\gamma-1}}{\gamma-1}\right)(2-\gamma)H^{1-\gamma}H_t,\ V_\rho = \left(\frac{p^{\gamma-1}}{\gamma-1}\right)(2-\gamma)H^{1-\gamma}H_\rho$$

$$V_{\rho\rho} = \left(\frac{p^{\gamma-1}}{\gamma-1}\right)(2-\gamma)\left[(1-\gamma)H^{-\gamma}H_\rho^2 + H_{\rho\rho}^{1-\gamma}\right],\ V_p = p^{\gamma-2}H^{2-\gamma}$$

$$V_{pp} = (\gamma-2)p^{\gamma-3}H^{2-\gamma},\ V_{\rho p} = p^{\gamma-2}(2-\gamma)H^{1-\gamma}H_\rho$$

Thus, we have that (4.2.2.6) becomes:

$$u_*^1(t,\rho) = \frac{(\theta_2-\theta_1)\rho(1-\rho)H_\rho}{\sigma^2 H} + \frac{\rho(\theta_2-\theta_1)+\theta_1-r}{\sigma^2(2-\gamma)} \quad (4.2.2.12)$$

$$u_*^0(t,\rho) = 1 - u_*^1(t,\rho) \quad (4.2.2.13)$$

Now, by (4.2.2.12) we have that (4.2.2.10) takes the form:

$$C_* = \frac{1}{2}\frac{(\theta_2-\theta_1)^2\rho^2(1-\rho)^2(2-\gamma)p^{\gamma-1}H_\rho^2}{\sigma^2 H^\gamma}$$
$$+\frac{(\theta_2-\theta_1)\rho(1-\rho)[\rho(\theta_2-\theta_1)+\theta_1-r]p^{\gamma-1}H^{1-\gamma}H_\rho}{\sigma^2}$$
$$+\frac{1}{2}\frac{[\rho(\theta_2-\theta_1)+\theta_1-r]^2p^{\gamma-1}H^{2-\gamma}}{\sigma^2(2-\gamma)}$$
$$(4.2.2.14)$$

Therefore, by (4.2.2.8) in view of (4.2.2.14) and all the above we derive the following:

Theorem 4.2.2.1
The value function of the agent's problem is given by (4.2.2.11), where H satisfies the following linear partial differential equation:

$$H_t + \frac{1}{2\sigma^2}(\theta_2-\theta_1)^2\rho^2(1-\rho)^2 H_{\rho\rho}$$
$$+\left(\frac{\gamma-1}{2-\gamma}\right)\left(\frac{\theta_2-\theta_1}{\sigma^2}\right)\{\rho(1-\rho)[\rho(\theta_2-\theta_1)+\theta_1-r]\}H_\rho$$
$$+\left(\frac{\gamma-1}{2-\gamma}\right)\left\{r+\frac{1}{2\sigma^2(2-\gamma)}[\rho(\theta_2-\theta_1)+\theta_1-r]^2\right\}H=0$$

such that: $H(T,\rho)=1$.
The stochastic representation of H is given by:

$$H(t,\rho)=$$
$$\exp\left(r\frac{(\gamma-1)(T-t)}{2-\gamma}\right)E_{t,\rho}\left[\exp\left(\frac{\gamma-1}{2\sigma^2(2-\gamma)^2}\int_t^T[Y_s(\theta_2-\theta_1)+\theta_1-r]^2ds\right)\right]$$

where the dynamics of Y_t are given by:

$$dY_t=\mu(t,Y_t)dt+\sigma(t,Y_t)d\Pi_t$$

where,

$$\mu(t,Y_t)=\left(\frac{(\gamma-1)(\theta_2-\theta_1)}{(2-\gamma)\sigma^2}\right)\{Y_t(1-Y_t)[Y_t(\theta_2-\theta_1)+\theta_1-r]\}$$

and

$$\sigma(t,Y_t)=\frac{(\theta_2-\theta_1)Y_t(1-Y_t)}{\sigma}$$

Moreover, the optimal portfolio allocation is given by (4.2.2.12) and (4.2.2.13)
□

Remarks

1) The stochastic representation of $H(t,\rho)$ can be derived by a straightforward application of the Feynman-Kac theorem. Namely, the solution of the general boundary value problem on $[0,T]\times\mathscr{R}$

$$H_t + \mu(t,\rho)H_\rho + \frac{1}{2}\sigma(t,\rho)^2 H_{\rho\rho} + r(t,\rho)H = 0$$

$$H(T,\rho) = \Phi(\rho)$$

by considering the process $G_s = H(s,\rho_s)\exp\left(\int_t^s r(u,\rho_u)du\right)$ is given by:

$$H(t,\rho) = E_{t,\rho}\left[\Phi(\rho_T)\exp\left(\int_t^T r(s,\rho_s)ds\right)\right]$$

2) Notice that in paragraph **3.2** we have identified when is optimal the agent to buy back the stock when she is in a short position.

Now, let as assume that in the beginning of time our agent is fully informed about the distribution of the drift as described in paragraph **(2.2.1)**. Then, by (4.2.2.6) and (4.2.2.8) we derive that:

If $P[\theta=\theta_2]=1=\rho$, then:

$$V(t,p,1) = \mathscr{U}^1(t,p) = \frac{p^{\gamma-1}}{\gamma-1}\exp\left\{\left[r + \frac{(\theta_2-r)^2}{2\sigma^2(2-\gamma)}\right](\gamma-1)(T-t)\right\} \quad (4.2.2.15)$$

which is the optimal value function when the drift is known, constant and equal to θ_2, with the optimal wealth allocation to be constant and given by:

$$u_*^{1,1} = \frac{\theta_2-r}{\sigma^2(2-\gamma)} > 0 \quad (4.2.2.16)$$

If $P[\theta=\theta_2]=0=\rho$, then:

$$V(t,p,0) = \mathscr{U}^0(t,p) = \frac{p^{\gamma-1}}{\gamma-1}\exp\left\{\left[r + \frac{(\theta_1-r)^2}{2\sigma^2(2-\gamma)}\right](\gamma-1)(T-t)\right\} \quad (4.2.2.17)$$

which is the optimal value function when the drift is known, constant and equal to θ_1, with the optimal wealth allocation to be constant and given by:

$$u_*^{1,0} = \frac{\theta_1-r}{\sigma^2(2-\gamma)} < 0 \quad (4.2.2.18)$$

Moreover, if we do not adopt the filtering technique as we did in paragraph **2.2.2** and work solely under the model described in paragraph **2.1**, we have that our agent's portfolio process will be given by:

$$dP_t = P_t\{[r + u_\theta^1(t,p)(\theta - r)]dt + u_\theta^1(t,p)\sigma dW_t\} \quad (4.2.2.19)$$

Let $\{\mathscr{D}\}_{t\geq 0}$ be the information generated by the process P_t and notice that u_θ is an admissible portfolio strategy if and only if it is $\mathscr{D}_t$ adapted. Then the optimal value function is given by:

$$V^\theta(t,p) = \max_{u_\theta^1} E[\Phi(P_T)] \quad (4.2.2.20)$$

Now, let $Z_t = P_t^{\gamma-1}$. By Ito's formula we have that:

$$Z_T = p^{\gamma-1}\exp\left\{\left[(\gamma-1)[r + u_\theta^1(\theta - r)] + \frac{1}{2}(\gamma-1)(\gamma-2)(u_\theta^1)^2\sigma^2\right]T\right\} \times$$

$$\exp\left\{-\frac{1}{2}\int_0^T(\gamma-1)^2(u_\theta^1)^2\sigma^2 ds + \int_0^T(\gamma-1)u_\theta^1\sigma dW_s\right\}$$

Now, if we define

$$M_t = \exp\left\{-\frac{1}{2}\int_0^t(\gamma-1)^2(u_\theta^1)^2\sigma^2 ds + \int_0^t(\gamma-1)u_\theta^1\sigma dW_s\right\},\ t \leq T$$

and a measure L such that:

$$\frac{dL}{dP} = M_T, \text{on } \mathscr{D}_T$$

then in view of Girsanov's theorem we have that (4.2.2.20) becomes:

$$V^\theta(t,p) =$$

$$\max_{u_\theta^1} E^L\left[\frac{1}{\gamma-1}p^{\gamma-1}\exp\left\{\left[(\gamma-1)[r + u_\theta^1(\theta - r)] + \frac{1}{2}(\gamma-1)(\gamma-2)(u_\theta^1)^2\sigma^2\right](T-t)\right\}\right]$$

(4.2.2.21)

Lemma 4.2.2.1
For the optimal value functions given by (4.2.2.11) and (4.2.2.21) we have that:

$$V^\theta(t,p) = V(t,p,\rho),\ \forall (t,p) \in [0,T] \times \mathscr{R}_+$$

Proof

The result comes immediately by (2.2.2.2), (2.2.2.3), (2.2.2.5), (2.2.2.13), (4.2.2.1) and (4.2.2.19) ■

Now, let us assume that our agent decides to follow the admissible portfolio strategy:

$$u^1_\theta = \frac{E[\theta]-r}{\sigma^2(\gamma-2)} = \alpha, \text{ constant}$$

Then, for our agent's value function, according to (4.2.2.21), we have that:

$$V^\theta_\alpha(t,p) \leq V^\theta(t,p) \quad (4.2.2.22)$$

where,

$$V^\theta_\alpha(t,p) =$$
$$E^L\left[\frac{1}{\gamma-1}p^{\gamma-1}\exp\left\{\left[(\gamma-1)[r+\alpha(\theta-r)]+\frac{1}{2}(\gamma-1)(\gamma-2)\alpha^2\sigma^2\right](T-t)\right\}\right]$$

Furthermore, notice that the function $V^\theta_\alpha(t,p)$ is convex in θ. As a result, in view of Jensen's inequality we have that:

$$\phi(E^L[\theta]) \leq E^L[\phi(\theta)] \Leftrightarrow$$
$$\frac{p^{\gamma-1}}{\gamma-1}\left[\exp\left\{\left[r+\frac{(E^L[\theta]-r)^2}{2\sigma^2(2-\gamma)}\right](\gamma-1)(T-t)\right\}\right] \leq V^\theta_\alpha(t,p) \Leftrightarrow$$
$$\mathscr{K}(t,p,\rho) \leq V^\theta_\alpha(t,p) \leq V(t,p,\rho) \quad (4.2.2.23)$$

where,

$$\mathscr{K}(t,p,\rho) = \frac{p^{\gamma-1}}{\gamma-1}\left[\exp\left\{\left[r+\frac{(\rho(\theta_2-\theta_1)+\theta_1-r)^2}{2\sigma^2(2-\gamma)}\right](\gamma-1)(T-t)\right\}\right]$$

(4.2.2.24)

Remark
Notice that the function $\mathscr{K}(t,p,\rho)$ is the optimal value function derived given by **Theorem 4.2.1** where the drift is known constant and equal to the expected value of the random variable θ with the optimal portfolio wealth allocation to be given by α.

Now, in view of (4.2.2.21) we have that:

$$V^{\theta}(t,p) =$$

$$\max_{u_\theta^1} E^L \left[\frac{1}{\gamma-1} p^{\gamma-1} \exp\left\{ \left[(\gamma-1)[r + u_\theta^1(\theta - r)] + \frac{1}{2}(\gamma-1)(\gamma-2)(u_\theta^1)^2\sigma^2 \right] (T-t) \right\} \right]$$

$$\leq$$

$$\frac{\rho}{\gamma-1} p^{\gamma-1} \max_{u_\theta^1} \left\{ \exp\left\{ \left[(\gamma-1)[r + u_\theta^1(\theta_2 - r)] + \frac{1}{2}(\gamma-1)(\gamma-2)(u_\theta^1)^2\sigma^2 \right] (T-t) \right\} \right\}$$

$$+$$

$$\frac{(1-\rho)}{\gamma-1} p^{\gamma-1} \max_{u_\theta^1} \left\{ \exp\left\{ \left[(\gamma-1)[r + u_\theta^1(\theta_1 - r)] + \frac{1}{2}(\gamma-1)(\gamma-2)(u_\theta^1)^2\sigma^2 \right] (T-t) \right\} \right\}$$

$$=$$

$$\rho \mathscr{U}^1(t,p) + (1-\rho)\mathscr{U}^0(t,p)$$

$$(4.2.2.25)$$

where the last equality comes from (4.2.2.15) and (4.2.2.17).

Let, $\mathscr{X}(t,p,\rho) = \rho\mathscr{U}^1(t,p) + (1-\rho)\mathscr{U}^0(t,p)$. Notice that this represents the optimal value function under the assumption that the agent is not only aware about the distribution of the drift ρ, but also she is fully informed about the outcome of the "coin toss" occurred in the beginning of time. Therefore, at time $t = 0$ her decision will naturally be to take position in both possible scenarios with respect to the distribution of the random variable θ and at time $t > 0$ to follow the strategy indicated by $\mathscr{U}^1(t,p)$ or $\mathscr{U}^0(t,p)$.

From all the above we derive the following:

Theorem 4.2.2.2
Let $\gamma \in (0,1)$ and the agent's constant relative risk aversion to be given by $0 < \gamma - 1 < 1$. If $V(t,p,\rho)$ is the optimal value function as described in (4.2.2.11) then $\forall(t,p,\rho) \in [0,T] \times \mathscr{R}_+ \times (0,1)$ we have that:

$$\mathscr{K}(t,p,\rho) \leq V(t,p,\rho) \leq \mathscr{X}(t,p,\rho) \;\; \square$$

4.2.3 The Ornstein-Uhlenbeck Drift case

Similarly as in 4.2.2, now we suppose that the market is governed by equations (2.3.2.12) and (2.3.2.13). Therefore, (4.1.1) becomes:

$$dP_t = P_t\{[r + u^1(t,p,\hat{\theta})(\hat{\theta}_t - r)]dt + u^1(t,p,\hat{\theta})\sigma d\hat{W}_t\} \quad (4.2.3.1).$$

By defining the optimal value function of the agent's wealth as:

$$V(t,p,\hat{\theta}) = \max_{u^1} E_{t,p,\hat{\theta}}[\Phi(P_T)] \quad (4.2.3.2)$$

where again V is sufficiently smooth, we have that by Ito's formula (again $\hat{\theta}_t$ and P_t are driven by the same Wiener process $\hat{W}_t$) the Hamilton-Jacobi-Bellman equation is of the form:

$$V_t + \max_{u^1}[\mathscr{L}^{u^1}V] = 0 \iff$$

$$V_t + [k(1 + \frac{s^*}{\sigma^2}) - \hat{\theta}]V_{\hat{\theta}} + rpV_p + \frac{1}{2}(\frac{s^*}{\sigma})^2 V_{\hat{\theta}\hat{\theta}} + \max_{u^1}[A] = 0 \quad (4.2.3.3)$$

where,

$$A = [p(\hat{\theta} - r)V_p + s^* pV_{\hat{\theta}p}]u^1 + \frac{1}{2}\sigma^2(u^1)^2 p^2 V_{pp}$$

Thus, if we maximize A with respect to u^1 we have that:

$$u^1_*(t,p,\hat{\theta}) = -\frac{s^* pV_{\hat{\theta}p} + (\hat{\theta} - r)pV_p}{\sigma^2 p^2 V_{pp}}$$

(4.2.3.4)

and

$$u^0_*(t,p,\hat{\theta}) = 1 - u^1_*(t,p,\hat{\theta}) \quad (4.2.3.5)$$

As a result, (4.2.3.3) under (4.2.3.4) becomes:

$$V_t + [k(1 + \frac{s^*}{\sigma^2}) - \hat{\theta}]V_{\hat{\theta}} + rpV_p + \frac{1}{2}(\frac{s^*}{\sigma})^2 V_{\hat{\theta}\hat{\theta}} + A_* = 0 \quad (4.2.3.6)$$

$$V(T,p,\hat{\theta}) = \Phi \quad (4.2.3.7)$$

where,

$$A_* = [p(\hat{\theta} - r)V_p + s^* pV_{\hat{\theta}p}]u^1_* + \frac{1}{2}\sigma^2(u^1_*)^2 p^2 V_{pp} \quad (4.2.3.8)$$

Similarly as in 4.2.2, let us apply a solution of the form:

$$V(t,p,\hat{\theta}) = \frac{p^{\gamma-1}}{\gamma-1} L^{2-\gamma}(t,\hat{\theta}) \text{ , such that } L(T,\hat{\theta}) = 1, \ L \geq 0$$

$$\text{where } (t,p,\hat{\theta}) \in [0,T] \times \mathscr{R}_+ \times \mathscr{R}$$

$$(4.2.3.9)$$

where $L(t,\hat{\theta})$ to be determined. Then, (4.2.3.4) becomes:

$$u_*^1(t,\hat{\theta}) = \frac{s^* L_{\hat{\theta}}}{\sigma^2 L} + \frac{\hat{\theta}-r}{\sigma^2(2-\gamma)} \quad (4.2.3.10)$$

$$u_*^0(t,\hat{\theta}) = 1 - u_*^1(t,\hat{\theta}) \quad (4.2.3.11)$$

Now, if we substitute (4.2.3.10) into (4.2.3.8) we derive that:

$$A_* = \frac{1}{2}\frac{(s^*)^2(2-\gamma)p^{\gamma-1}L_{\hat{\theta}}^2}{\sigma^2 L^{\gamma}}$$

$$+\frac{s^*(\hat{\theta}-r)p^{\gamma-1}L^{1-\gamma}L_{\hat{\theta}}}{\sigma^2}$$

$$+\frac{1}{2}\frac{(\hat{\theta}-r)^2}{\sigma^2(2-\gamma)}p^{\gamma-1}L^{2-\gamma}$$

$$(4.2.3.12)$$

Thus, by combining all the above we derive the following:

Theorem 4.2.3.1
The value function of the agent's problem is given by (4.2.3.9), where L satisfies the following linear partial differential equation:

$$L_t + \frac{1}{2}\frac{(s^*)^2}{\sigma^2}L_{\hat{\theta}\hat{\theta}} + \left[k(1+\frac{s^*}{\sigma^2}) - \hat{\theta} + \frac{s^*(\gamma-1)(\hat{\theta}-r)}{\sigma^2(2-\gamma)}\right]L_{\hat{\theta}}$$

$$+\left(\frac{\gamma-1}{2-\gamma}\right)\left[r + \frac{1}{2}\frac{(\hat{\theta}-r)^2}{\sigma^2(2-\gamma)}\right]L = 0$$

The stochastic representation of L is given by:

$$L(t,\hat{\theta}) =$$

$$\exp\left(r\frac{(\gamma-1)(T-t)}{2-\gamma}\right) \ E_{t,\hat{\theta}}\left[\exp\left(\frac{\gamma-1}{2\sigma^2(2-\gamma)^2}\int_t^T (Y_s-r)^2 ds\right)\right]$$

where the dynamics of Y_t are given by:

$$dY_t = \mu(t, Y_t)dt + \sigma(t, Y_t)d\Pi_t$$

where Π is a Wiener process and

$$\mu(t, Y_t) = \left[\left(k(1 + \frac{s^*}{\sigma^2}) - \frac{s^*(\gamma - 1)r}{\sigma^2(2 - \gamma)} \right) - \left(1 - \frac{s^*(\gamma - 1)}{\sigma^2(2 - \gamma)} \right) Y_t \right]$$

and

$$\sigma(t, Y_t) = \frac{(s^*)}{\sigma}, \text{ constant.}$$

Moreover, the optimal portfolio allocation is given by (4.2.3.10) and (4.2.3.11) $\square$

Remarks

1) The stochastic representation of $L(t, \hat{\theta})$ can be derived exactly as described in the case of theorem (4.2.2.1)

2) If we assume that we are under complete information about the drift, that is $Y_t = \mu_0$ constant, then by theorem (4.2.3.1) we derive instantly (4.2.1.13) and (4.2.1.14).

3) Notice that in paragraph **3.2** we have identified when is optimal the agent to buy back the stock when she is in a short position.

Chapter 5

Conclusions

In the unobserved world the main task is to find the best estimate of the drift and to identify the stochastic differential equation that satisfies. Hence, equations **2.2.2.13** and **2.2.2.14** for the Digital Drift case and **2.3.2.12** and **2.3.2.13** for the Ornstein -Uhlenbeck one, are the building blocks of this project.

In Chapter 3, we derived the optimal execution boundary for the optimal stopping problem of closing a short position. Namely, for the Digital Drift case in theorem **3.2.1** we presented an explicit solution under an infinite time horizon (for finite time horizon see [2]) while theorem **3.3.1** provides a method of how to derive it when the drift is modeled as an Ornstein-Uhlenbeck process.

In Chapter 4, theorem **4.2.1** gives us the optimal portfolio value function in view of the standard model (where both the drift and the volatility are known constants), while theorems **4.2.2.1** and **4.3.2.1** describe how to identify it for our main models. Finally, theorem **4.2.2.2** gives us a comparison measure between the complete information model and the Digital Drift one. This specific result says that one can take advantage of the randomness in the drift and achieve higher value function. Notice, that this is intuitively in contrast with the proposed strategies described in Chapter 3. This is due to the fact that an agent in the market would rather be informed of the drift instead of not knowing it, as the decision about whether or not to close her position would be instant.

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