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Option Pricing in Discrete Time and Connections between the Binomial Model and Black-Scholes Model

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Contents

1 Abstract

This thesis concerns option pricing in discrete time and connections between the binomial model and the continuous Black-Scholes model. It contains an introduction to the binomial model, and furthermore introduces some basic concepts in arbitrage theory. It will be shown how a financial derivative priced with the binomial model satisfies Black-Scholes equation, and how the price of the underlying stock in the binomial model converge in distribution to the stock price in Black-Scholes model. In conclusion there are two different proofs showing that the price for an European call option in the binomial model converges to the price for the option in the Black-Scholes model.

2 Introduction

2.1 The Binomial Model

There are two financial instruments in this discrete time model, a bond and a stock. Let the process B_t be the price of the bond at time t and let S_t denote the price of one share of the stock at time t. The running time t goes from $t = 0$ to $t = T$, where T is fixed. The constant R is a deterministic short rate of interest. Thus the bond price dynamics are defined by

$$\begin{aligned} B_{n+1} &= (1+R)B_n, \\ B_0 &= 1. \end{aligned}$$

The random variables $X_0, X_1, ..., X_{T-1}$ take the values u and d. Suppose they are independent and identically distributed with the probability mass function

$$P(X_n = k) = \begin{cases} p_u, & \text{if } k = u \\ p_d, & \text{if } k = d \end{cases} \quad \text{where } p_u + p_d = 1.$$

Then define the dynamics of the stock price by

$$\begin{aligned} S_{n+1} &= S_n * X_n, \\ S_0 &= s. \end{aligned}$$

The stock dynamics may be illustrated with a lattice. Clearly going up and then down gives the same stock price as first going down and then up, and is in that meaning **recombining** [1].

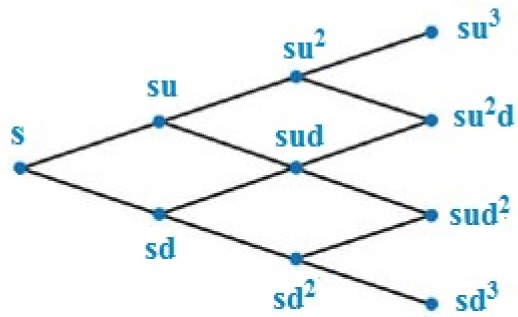

Figure 1

Define a **portfolio strategy** h such that

$$\{h_t = (x_t, y_t);\ t = 1, ..., T\}$$

where h_t is a function of $S_0, S_1, ..., S_{t-1}$, and for any portfolio strategy h the property $h_0 = h_1$ holds. The portfolio h has a corresponding **value process** given by

$$V_t{}^h = x_t(1 + R) + y_t S_t.$$

Clearly the market value of the portfolio $h_t = (x_t, y_t)$ at time t is given by the value process V_t^h. Consider x_t to be the amount of money that we are placing in the bank at time $t-1$ and it is being held until time t. The entity y_t represents the number of shares that we are purchasing at time $t-1$ and they are being held until time t [1].

We are especially interested in the self-financing portfolios which are mathematically defined in the following way:

Definition 2.1.1 A given portfolio strategy h is **self financing** if

$$x_t(1 + R) + y_t S_t = x_{t+1} + y_{t+1} S_t$$

holds for $t = 0, ..., T - 1$.

The interpretation is that the market value of $h_t = (x_t, y_t)$, established at $t-1$, is equal to the purchase value of the new portfolio $h_{t+1} = (x_{t+1}, y_{t+1})$, which is constructed at t and kept until $t + 1$. We are now able to define an arbitrage possibility.

Definition 2.1.2 An **arbitrage** possibility is any self-financing portfolio h satisfying the conditions

$$\begin{aligned} V_0{}^h &= 0, \\ P(V_T{}^h \geq 0) &= 1, \\ P(V_T{}^h > 0) &> 0. \end{aligned}$$

Arbitrage portfolios occur as a result of mispricing on the market. The following condition is necessary and sufficient for absence of arbitrage

$$d < (1 + R) < u.$$

Definition 2.1.3 Assume $d < (1 + R) < u$ holds. Let Q be a probability measure with the property $Q(X_n = d) = q_d$ and $Q(X_n = u) = q_u$. The probabilities q_d and q_u which satisfy

$$s = \frac{1}{1+R} E^Q[S_{t+1} \mid S_t = s]$$

are called the martingale probabilities [1].

Definition 2.1.3 gives us the following

$$\begin{aligned}
s &= \frac{1}{1+R} E^Q[S_{t+1} \mid S_t = s] \\
\Leftrightarrow s &= \frac{1}{1+R} \sum_x xQ(S_t X_t = x \mid S_t = s) \\
\Leftrightarrow s &= \frac{1}{1+R} \sum_x \frac{xQ(sX_t = x)Q(S_t = s)}{Q(S_t = s)} \\
\Leftrightarrow s &= \frac{1}{1+R}(suQ(sX_t = su) + sdQ(sX_t = sd)) \\
\Leftrightarrow 1 &= \frac{1}{1+R}(uQ(X_t = u) + dQ(X_t = d)) \\
\Leftrightarrow 1 &= \frac{1}{1+R}(uq_u + dq_d)).
\end{aligned}$$

Thus the martingale probabilities are given by

$$\begin{cases} q_u = \dfrac{(1+R) - d}{u - d} \\ q_d = \dfrac{u - (1+R)}{u - d}. \end{cases}$$

Definition 2.1.4 The financial derivative **contingent claim** is a random variable X given by

$$X = \Phi(S_T)$$

where Φ is the contract function and $\Phi : \mathbb{R} \to \mathbb{R}$.

We consider the claim X to be a contract that pays the holder of the contract the amount X at time T. The European call option on the stock is an example of a contingent claim. This option has a strike price K and therefore we have $X = \max[0, S_T - K]$. Note that the price of the claims which we are studying only depends on the value of the stock price at time T, and not on the total path of the price process. It would complicate the theory and the binomial lattice would not be recombining [1].

Naturally the question 'what is a reasonable price for the option?' occurs. In this problem of pricing a given claim X we consider the price process

$$\{\Pi(t; X);\ t = 0, ..., T\}.$$

Definition 2.1.5 The contingent claim X is **reachable** if a self-financing portfolio h satisfy

$$V_T^h = X, \quad \text{with probability 1.}$$

The self-financing portfolio h is then said to be **replicating** and the market is **complete** if every contingent claim can be replicating.

We are now able to define the only reasonable price process for the claim X

$$\Pi(t; X) = V_t^h, \quad t = 0, ..., T.$$

Proposition 2.1.6 Assume that the claim X is reachable and portfolio h is replicating. If there is a possibility at any t to purchase X for a price less than V_t^h or to sell X for a higher price than V_t^h, then there is a possibility to make an arbitrage profit.

Proposition 2.1.7 Consider a claim $X = \Phi(S_T)$, then the theoretical price at $t = 0$ of the given claim excluding arbitrage possibilities is given by

$$\Pi(0; X) = \frac{1}{(1+R)^T} \mathrm{E}^Q[X],$$

where Q is the martingale measure, or more precise

$$\Pi(0; X) = \frac{1}{(1+R)^T} * \sum_{k=0}^{T} \binom{T}{k} q_u^k q_d^{T-k} \Phi\big(su^k d^{T-k}\big)$$

[1].

Illustration 2.1.8 (Martingale Probabilities)
Consider an European call option where $T = 1$ i.e. there is only one period. Furthermore let $S_0 = 120$, $u = 1.5$, $d = 0.5$, $p_u = 0.7$, $p_d = 0.3$ and $R = 0$, hence the price process is given by

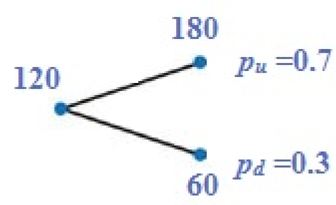

Figure 2

Condition (1) is satisfied which implies an arbitrage free market. The call option has strike price $K = 140$ and since the claim X for a call option is of the form $X = \max[0, S_T - K]$ we have

$$X = \begin{cases} 40, & \text{if } S_1 = 180 \\ 0, & \text{if } S_1 = 60 \end{cases}$$

Let $x =$ units in the bank account and let $y =$ shares of the stock, then the replicating portfolio satisfies the following system

$$\begin{cases} x + 180y = 40 \\ x + 60y = 0 \end{cases} \Leftrightarrow \begin{cases} x = -20 \\ y = \dfrac{1}{3} \end{cases}$$

The replicating portfolio is constructed by borrowing 20 units from the bank, and then invest the money in one third of a share in the stock. The cost of the portfolio at time 0 is therefore given by

$$-20 + \frac{1}{3} * 120 = 20.$$

The conclusion is that the price of the option at time 0 has to be 20 to exclude arbitrage possibilities, but the probabilities p_u and p_d imply the price

$$\begin{aligned} \mathrm{E}^P[(S_1 - K)^+] &= 40 * 0.7 + 0 * 0.3 \\ &= 28. \end{aligned}$$

Moreover if anyone buys the option from us for 28, we are able to make a risk-free profit. By applying the theory from above we get that the martingale probabilities are both equal to 0.5. Hence this gives us the theoretical price 20 as we expected since

$$\begin{aligned} \mathrm{E}^Q[(S_1 - K)^+] &= 40 * 0.5 + 0 * 0.5 \\ &= 20. \end{aligned}$$

Illustration 2.1.9 (The Binomial Model with Two Periods)
Now we consider an European call option with two periods i.e. where $T = 2$. Let $S_0 = 100$, $u = 1.5$, $d = 0.5$, and $R = 0$, thus the price process is constructed in the following way

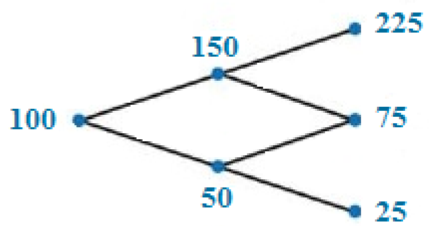

Figure 3

Once again property (1) is fulfilled and implying an arbitrage free market, in other words we are able to calculate the price of a replicating portfolio by risk neutral valuation. The option has strike price $K = 100$, and to compute the price for the option at time 0 the thought is to use induction on the time variable by moving backwards in the tree from $t = T$ to $t = 0$. Thus we begin by considering the following part of the stock dynamics

Figure 4

This part of the dynamics can be considered as a one period model and we are able to apply the one period theory directly. For any claim X the relation

$$\Pi(T; X) = X$$

holds. Hence by letting x = units in the bank account and letting y = shares of the stock, the replicating portfolio satisfies the system

$$\begin{cases} x + 225y = 125 \\ x + 75y = 0 \end{cases} \Leftrightarrow \begin{cases} x = -62.5 \\ y = \dfrac{5}{6} \end{cases}$$

The cost of the portfolio is therefore given by

$$-62.5 + \frac{5}{6} * 150 = 62.5.$$

By considering the other node at $t = 1$ we are able to compute the corresponding replicating portfolio in the same way. The result is illustrated in *Figure 5*.

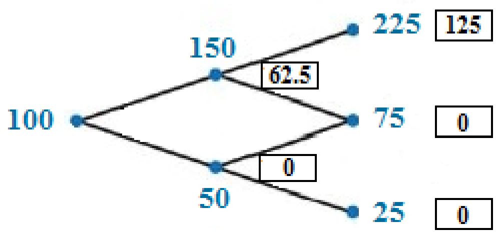

Figure 5

The computations for the cost of the replicating portfolio at $t = 0$ can now be easily preformed using the following system

$$\begin{cases} x + 150y = 62.5 \\ x + 50y = 0 \end{cases} \Leftrightarrow \begin{cases} x = -31,25 \\ y = 0.625 \end{cases}$$

Hence,

$$\Pi(0, X) = -31.25 + 100 * 0.625 = 31.25.$$

The conclusion is that the price of the option at time 0 has to be 31.25 to exclude arbitrage opportunities.

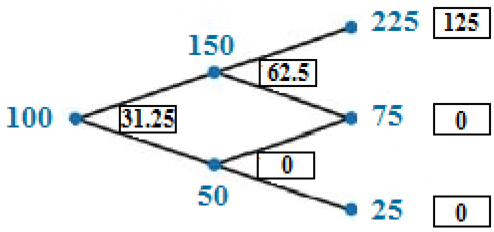

Figure 6

Our result coincides as anticipated with the martingale theory. From the theory above the martingale probabilities are both equal to 0.5. Then by proposition 2.1.7

$$\Pi(0; (S_2 - 100)^+) = 0.5^2 * \Phi(225) + 0 + 0 = 31.25.$$

2.2 Suitable Parameters for the Binomial Model

The binomial model may appear to be simple but if the period length is chosen to be very small, the number of possible values increase fast after several steps. Notice that the price of the stock will always be positive, and therefore we are able to consider the logarithm of the price. Define v as the expected yearly growth rate, moreover

$$v = \mathrm{E}[\ln(S_T/s)]$$

where S_T is the stock price after one year and s is the initial value of the stock. Furthermore define σ^2 to be the annual variance. In particular,

$$\sigma^2 = \mathrm{Var}[\ln(S_T/s)].$$

Assume that the period length is given and small in comparison with 1, then the parameters p, u and d can be chosen in the following way

$$\begin{aligned} p &= \frac{1}{2} + \frac{1}{2}\left(\frac{v}{\sigma}\right)\sqrt{\Delta t} \\ u &= e^{\sigma\sqrt{\Delta t}} \\ d &= e^{-\sigma\sqrt{\Delta t}}. \end{aligned}$$

These parameters contribute the binomial model to closely match the values v and σ. The match improves when Δt gets smaller, and becomes accurate when Δt goes to 0 [5].

Illustration 2.2.1
Consider a stock with the expected yearly growth rate $v = 14\%$ and the volatility of that growth rate is $\sigma = 30\%$. Assume that $\Delta t = 0.25$ then by calculations according to the formulas stated above

$$p = 0.616667, \quad u = 1.161834, \quad d = 0.860708.$$

Thereafter by using that

$$\begin{aligned} \mathrm{E}[\ln(S_1)] &= p\ln(u) + (1-p)\ln(d) = v\Delta t \\ \mathrm{Var}[\ln(S_1)] &= p(1-p)(\ln(u) - \ln(d))^2 = \sigma^2\Delta t \end{aligned}$$

it follows that

$$v = 0.13999, \quad \sigma = 0.29172.$$

3 Derivation of Black-Scholes Equation

The binomial model is as mentioned a discrete time model. The question arises if there exists a relation to a continuous time model, more precise the Black-Scholes model. We will begin by studying if a financial derivative priced with the binomial model satisfies Black-Scholes equation.

Consider a local region as illustrated in *Figure 3* with period length Δt and let f be a function of time and stock price. Furthermore let f be the price of a financial derivative with expiration date T and assume that f is of class C^3. Suppose that the interest rate is equal to zero. Let the function g be the pay-off function and if g is a call option, then the boundary condition is given by

$$f(T, s) = g(s) = (S_T - K)^+.$$

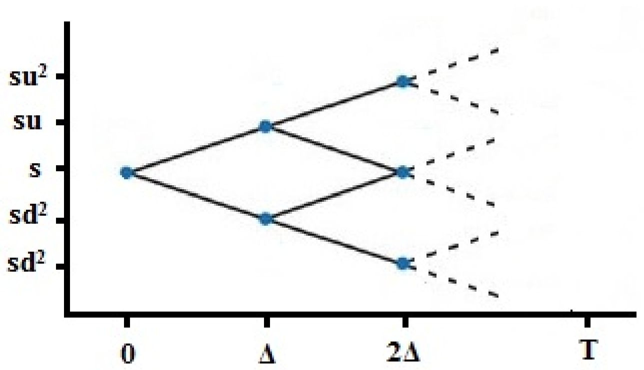

Figure 5

Let the short rate of interest be equal to zero and u and d are as above given by

$$u = e^{\sigma\sqrt{\Delta t}},$$
$$d = e^{-\sigma\sqrt{\Delta t}}.$$

Furthermore let q be the martingale probability and also the probability of an upward movement. Note that

$$\lim_{\Delta t \to 0} su^2 = \lim_{\Delta t \to 0} se^{2\sigma\sqrt{\Delta t}} = s,$$

$$\lim_{\Delta t \to 0} sd^2 = \lim_{\Delta t \to 0} se^{-2\sigma\sqrt{\Delta t}} = s.$$

When the period length Δt goes to zero we are therefore able to consider the points $(2\Delta t, su^2)$, $(2\Delta t, s)$ and $(2\Delta t, sd^2)$ as one single point. Hence we can make the following approximation.

$$\frac{\partial f}{\partial t} \approx \frac{f(2\Delta t, s) - f(0, s)}{2\Delta t}$$

$$\approx \frac{f(2\Delta t, s) - \left[(1-q)^2 f(2\Delta t, sd^2) + 2q(1-q) f(2\Delta t, s) + q^2 f(2\Delta t, su^2)\right]}{2\Delta t}$$

$$= \frac{(1 - 2q(1-q)) f(2\Delta t, s) - q^2 f(2\Delta t, su^2) - (1-q)^2 f(2\Delta t, sd^2)}{2\Delta t}$$

Theorem 3.1.1 (Taylor's Theorem for functions $\mathbb{R}^2 \to \mathbb{R}$) If the function $f(x, y)$ has continuous derivatives of at least order $n+1$ in a neighbourhood of the point (x, y), then

$$\begin{aligned}
f(x+h, y+k) = f(x, y) &+ \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x, y) \\
&+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x, y) + \ldots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(x, y) \\
&+ \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(x + \tau h, y + \tau k),
\end{aligned}$$

where $0 < \tau < 1$ [6].

Taylor's theorem can be used to rewrite $f(2\Delta t, su^2)$ and $f(2\Delta t, sd^2)$. We are able to apply the theorem because of the assumption that f is of class C^3.

$$\begin{aligned}
f(2\Delta t, su^2) &= f(2\Delta t, s) + \frac{\partial f}{\partial S}(2\Delta t, s)(su^2 - s) + \frac{\partial^2 f}{\partial S^2}(2\Delta t, s)\frac{(su^2 - s)^2}{2!} + \ldots \\
&= f(2\Delta t, s) + \frac{\partial f}{\partial S}(2\Delta t, s)s(u^2 - 1) + \frac{\partial^2 f}{\partial S^2}(2\Delta t, s)\frac{s^2(u^2 - 1)^2}{2!} + \ldots
\end{aligned}$$

$$\begin{aligned}
f(2\Delta t, sd^2) &= f(2\Delta t, s) + \frac{\partial f}{\partial S}(2\Delta t, s)(sd^2 - s) + \frac{\partial^2 f}{\partial S^2}(2\Delta t, s)\frac{(sd^2 - s)^2}{2!} + \ldots \\
&= f(2\Delta t, s) + \frac{\partial f}{\partial S}(2\Delta t, s)s(d^2 - 1) + \frac{\partial^2 f}{\partial S^2}(2\Delta t, s)\frac{s^2(d^2 - 1)^2}{2!} + \ldots
\end{aligned}$$

Now use the Taylor expansions to rewrite the expression for the partial derivative.

$$\frac{\partial f}{\partial t} \approx \frac{(1-2q(1-q))f(2\Delta t, s) - q^2 f(2\Delta t, su^2) - (1-q)^2 f(2\Delta t, sd^2)}{2\Delta t}$$

$$= \frac{\big[1-2q(1-q)\big] f(2\Delta t, s)}{2\Delta t}$$

$$- \frac{q^2\left[f(2\Delta t, s) + \frac{\partial f}{\partial S}(2\Delta t, s)(su^2 - s) + \frac{\partial^2 f}{\partial S^2}(2\Delta t, s)\frac{(su^2-s)^2}{2!} + \dots\right]}{2\Delta t}$$

$$- \frac{(1-q)^2\left[f(2\Delta t, s) + \frac{\partial f}{\partial S}(2\Delta t, s)(sd^2 - s) + \frac{\partial^2 f}{\partial S^2}(2\Delta t, s)\frac{(sd^2-s)^2}{2!} + \dots\right]}{2\Delta t}$$

$$= \frac{f(2\Delta t, s)\big[(1-2q+2q^2) - q^2 - (1-q)^2\big]}{2\Delta t}$$

$$- \frac{s\frac{\partial f}{\partial S}(2\Delta t, s)\big[(1-q)^2(d^2-1) + q^2(u^2-1)\big]}{2\Delta t}$$

$$- \frac{s^2\frac{\partial^2 f}{\partial S^2}(2\Delta t, s)\big[(1-q)^2(d^2-1)^2 + q^2(u^2-1)^2\big] + \mathcal{O}(\Delta t^{5/2})}{4\Delta t}$$

As previously mentioned when the short rate of interest is equal to zero the martingale probabilities satisfy

$$\begin{gathered} qu - (1-q)d = 1 \\ \iff q^2u^2 - 2q(1-q) + (1-q)^2d^2 = 1. \end{gathered}$$

Hence,

$$\frac{\partial f}{\partial t} \approx -\frac{s^2\frac{\partial^2 f}{\partial S^2}(2\Delta t, s)\big[(u-1)^2(d^2-1)^2 + (1-d)^2(u^2-1)^2\big] + \mathcal{O}(\Delta t^{5/2})}{4\Delta t(u-d)^2}$$

$$= -\frac{s^2\frac{\partial^2 f}{\partial S^2}(2\Delta t, s)\big[u^4 + d^4 - 2(u^3+d^3) + 2(u+d) - 2\big] + \mathcal{O}(\Delta t^{5/2})}{4\Delta t(u^2 - 2 + d^2)}$$

$$= -\frac{s^2 \frac{\partial^2 f}{\partial S^2}(2\Delta t, s)\big[8\sigma^4 \Delta t^2 + \mathcal{O}(\Delta t^3)\big] + \mathcal{O}(\Delta t^{5/2})}{4\Delta t\big(4\sigma^2 \Delta t + \mathcal{O}(\Delta t^2)\big)}$$

$$= -\frac{s^2\sigma^2 \frac{\partial^2 f}{\partial S^2}(2\Delta t, s) + \mathcal{O}(\sqrt{\Delta t})}{2 + \mathcal{O}(\Delta t)}$$

The approximation improves when Δt gets smaller, and becomes exact when Δt goes to 0.

It has now been shown that

$$\frac{\partial f}{\partial t} \approx -\frac{s^2\sigma^2 \frac{\partial^2 f}{\partial S^2}(2\Delta t, s) + \mathcal{O}(\sqrt{\Delta t})}{2 + \mathcal{O}(\Delta t)}$$

Taking the limit gives us that

$$\lim_{\Delta t \to 0} \quad -\frac{s^2\sigma^2 \frac{\partial^2 f}{\partial S^2}(2\Delta t, s) + \mathcal{O}(\sqrt{\Delta t})}{2 + \mathcal{O}(\Delta t)} \quad = -\frac{1}{2}s^2\sigma^2 \frac{\partial^2 f}{\partial S^2}$$

$$\implies \frac{\partial f}{\partial t} + \frac{1}{2}s^2\sigma^2 \frac{\partial^2 f}{\partial S^2} = 0$$

This is the well-known Black-Scholes equation when the interest rate is zero, and the desired result.

4 Convergence of Stock Price

The previous result for the binomial model and Black-Scholes equation makes us wonder if there exist further connections between the binomial model and the Black-Scholes model. We will continue by studying the stock price for the two models.

The stock price $S(T)$ in Black-Scholes model with rate of interest equal to zero, is given by

$$S(T) = se^{-\frac{\sigma^2}{2}T+\sigma\sqrt{T}N(0,1)}$$

[1]. We aim to show that the stock price in the binomial model converge in distribution to the stock price in the Black-Scholes model. To prove that $S^n(T) \xrightarrow{d} S(T)$, theorem 4.1.2 stated below with proof, is required.

4.1 Normal Approximation of the Binomial Distribution

The proof of theorem 4.1.2 is built on the continuity theorem which we shall therefore state.

Theorem 4.1.1 (Continuity theorem) Assume that a sequence of distribution functions $F_1, F_2, F_3, \ldots$ has corresponding characteristic functions $\phi_1, \phi_2, \phi_3, \ldots$.

i) If the following limit exists

$$\lim_{n\to\infty} \phi_n(t) = \phi(t)$$

and is continuous at 0, then F is the distribution function corresponding to the characteristic function ϕ, and $F_n \to F$.

ii) Conversely, if a distribution function F with corresponding characteristic function ϕ satisfies $F_n(t) \to F$, then $\phi_n(t) \to \phi(t)$ for each t [3].

Theorem 4.1.2 Let $X_n \sim \mathrm{Bin}(n, p_n)$ and assume that $np_n(1-p_n) \to \infty$ as $n \to \infty$, then

$$\frac{X_n - np_n}{\sqrt{np_n(1-p_n)}} \xrightarrow{d} \mathrm{N}(0,1) \quad \text{as } n \to \infty \qquad [4].$$

Proof. Let $X_n \sim \mathrm{Bin}(n, p_n)$ and assume that $np_n(1-p_n) \to \infty$ as $n \to \infty$. The characteristic function $\phi_n(t)$ is given by

$$\phi_n(t) = \mathrm{E}\left[e^{itX_n}\right].$$

The interest of the characteristic functions arises because of the continuity theorem. Thus we will start by calculating the characteristic function for the random variable $\frac{X_n-np_n}{\sqrt{np_n(1-p_n)}}$, with X_n defined as above .

$$\mathrm{E}\left[\exp\left(it\ \frac{X_n - np_n}{\sqrt{np_n(1-p_n)}}\right)\right] = \sum_{k=0}^{n}\exp\left(it\ \frac{k-np_n}{\sqrt{np_n(1-p_n)}}\right)\mathrm{P}(X_n = k)$$

$$= \exp\left(\frac{-itnp_n}{\sqrt{np_n(1-p_n)}}\right)\sum_{k=0}^{n}\exp\left(\frac{itk}{\sqrt{np_n(1-p_n)}}\right)\binom{n}{k}p_n^k(1-p_n)^{n-k}$$

$$= \exp\left(\frac{-itnp_n}{\sqrt{np_n(1-p_n)}}\right)\sum_{k=0}^{n}\binom{n}{k}\left(p_n\exp\left(\frac{it}{\sqrt{np_n(1-p_n)}}\right)\right)^k(1-p_n)^{n-k}$$

$$= \exp\left(\frac{-itnp_n}{\sqrt{np_n(1-p_n)}}\right)\left(p_n\exp\left(\frac{it}{\sqrt{np_n(1-p_n)}}\right)+1-p_n\right)^n$$

$$= \left(p_n\exp\left(\frac{it(1-p_n)}{\sqrt{np_n(1-p_n)}}\right)+(1-p_n)\exp\left(\frac{-itp_n}{\sqrt{np_n(1-p_n)}}\right)\right)^n$$

The fourth equality follows from the binomial theorem. Rewriting the expression by using Taylor expansions centred at zero gives us the following,

$$\left(p_n\exp\left(\frac{it(1-p_n)}{\sqrt{np_n(1-p_n)}}\right)+(1-p_n)\exp\left(\frac{-itp_n}{\sqrt{np_n(1-p_n)}}\right)\right)^n$$

$$= \Bigg(p_n\left(1+\frac{(1-p_n)it}{\sqrt{np_n(1-p_n)}}+\frac{(1-p_n)^2(it)^2}{2!np_n(1-p_n)}+\frac{(1-p_n)^3(it)^3}{3!(np_n(1-p_n))^{3/2}}+\dots\right)$$

$$+\ (1-p_n)\left(1-\frac{p_nit}{\sqrt{np_n(1-p_n)}}+\frac{p_n^2(it)^2}{2!np_n(1-p_n)}-\frac{p_n^3(it)^3}{3!(np_n(1-p_n))^{3/2}}+\dots\right)\Bigg)^n$$

$$= \Bigg(p_n+(1-p_n)+\frac{p_n(1-p_n)it-p_n(1-p_n)it}{\sqrt{np_n(1-p_n)}}$$

$$+\frac{p_n(1-p_n)^2(it)^2+(1-p_n)p_n^2(it)^2}{2!np_n(1-p_n)}+\frac{p_n(1-p_n)^3(it)^3-(1-p_n)p_n^3(it)^3}{3!(np_n(1-p_n))^{3/2}}+\dots\Bigg)^n$$

$$= \left(1 + \frac{p_n(it)^2 - p_n^2(it)^2}{2!np_n(1-p_n)} + \frac{p_n(it)^3 - 3p_n^2(it)^3 + 2p_n^3(it)^3}{3!(np_n(1-p_n))^{3/2}} + \dots\right)^n$$

$$= \left(1 + \frac{-t^2}{2n} + \frac{p_n t^3 - 3p_n^2 t^3 + 2p_n^3 t^3}{3!(np_n(1-p_n))^{3/2}} + \dots\right)^n$$

Now let $Z \sim N(\mu, \sigma^2)$ and the characteristic function for the random variable Z is then given by

$$\mathrm{E}[e^{itZ}] = \int_{-\infty}^{\infty} e^{itz} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(z-\mu)^2}{2\sigma^2}} dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(z^2 - 2z(\mu+\sigma^2 it)+\mu^2)}{2\sigma^2}} dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(z-(\mu+\sigma^2 it))^2}{2\sigma^2}} e^{\frac{2\mu\sigma^2 it - \sigma^4 t^2}{2\sigma^2}} dz$$

$$= e^{\mu it - \frac{1}{2}\sigma^2 t^2}$$

We shall now take the limit of the characteristic function $\phi_n(t)$. From the assumption that $np_n(1-p_n) \to \infty$ as $n \to \infty$ it follows that

$$\lim_{n \to \infty} \phi_n(t) = \lim_{n \to \infty} \left(1 + \frac{-t^2}{2n} + \frac{p_n(it)^3 - 3p_n^2(it)^3 + 2p_n^3(it)^3}{3!(np_n(1-p_n))^{3/2}} + \dots\right)^n = e^{-\frac{1}{2}t^2}.$$

The function $e^{-\frac{1}{2}t^2}$ is the characteristic function for a normal distributed random variable with expected value 0 and variance 1. The characteristic function is continuous at 0 and therefore by the Continuity theorem it follows that

$$\frac{X_n - np_n}{\sqrt{np_n(1-p_n)}} \xrightarrow{d} \mathrm{N}(0,1).$$

□

4.2 Weak Convergence

We shall now return to the assignment of proving that $S^n(T) \xrightarrow{d} S(T)$. Let q be the martingale probability of an upward movement, and let $u = e^{\sigma\sqrt{\Delta t}}$ and $d = e^{-\sigma\sqrt{\Delta t}}$ as above. The stock price $S^n(T)$ in the binomial model with n steps at time T is then given by

$$S^n(T) = \begin{cases} su^n, & \binom{n}{n} q^n \\ su^{n-1}d, & \binom{n}{n-1} q^{n-1}(1-q) \\ \vdots & \vdots \\ sud^{n-1}, & \binom{n}{1} q(1-q)^{n-1} \\ sd^n, & \binom{n}{0}(1-q)^n \end{cases}$$

which is equivalent to

$$S^n(T) = su^j d^{n-j}, \quad \text{with probability } \binom{n}{j} q^j (1-q)^{n-j}.$$

Thus we are able to write $S^n(T)$ as

$$S^n(T) = su^{X_n} d^{n-X_n}, \text{ where } X_n \sim \text{Bin}(n, q).$$

We can rewrite the expression for the stock price in the following way

$$\begin{aligned}
S^n(T) &= s\exp\big(X_n \ln u + (n - X_n)\ln d\big) \\
&= s\exp\big(X_n \ln(u/d) + n\ln d\big) \\
&= s\exp\big(X_n 2\sigma\sqrt{T/n} - \sigma\sqrt{Tn}\big) \\
&= s\exp\left(2\sigma\sqrt{Tq(1-q)}\left(\frac{X_n - nq}{\sqrt{nq(1-q)}}\right) + 2\sigma q\sqrt{Tn} - \sigma\sqrt{Tn}\right) \\
&= s\exp\left(2\sigma\sqrt{T}\sqrt{\frac{u-2+d}{u^2-2+d^2}}\left(\frac{X_n - nq}{\sqrt{nq(1-q)}}\right) + \sigma\sqrt{Tn}\left(\frac{2-(u+d)}{u-d}\right)\right) \\
&= s\exp\left(2\sigma\sqrt{T}\sqrt{\frac{\frac{\sigma^2 T}{n} + \frac{\sigma^4 T^2}{12n^2} + \dots}{\frac{4\sigma^2 T}{n} + \frac{4\sigma^4 T^2}{3n^2} + \dots}}\left(\frac{X_n - nq}{\sqrt{nq(1-q)}}\right) \; + \; \sigma\sqrt{Tn}\left(\frac{-\frac{\sigma^2 T}{n} - \frac{\sigma^4 T^2}{12n^2} + \dots}{\frac{2\sigma\sqrt{T}}{\sqrt{n}} + \frac{\sigma^3 T^{3/2}}{3n^{3/2}} + \dots}\right)\right) \\
&= s\exp\left(2\sigma\sqrt{T}\sqrt{\frac{1 + \frac{\sigma^2 T}{12n} + \dots}{4 + \frac{4\sigma^2}{3n} + \dots}}\left(\frac{X_n - nq}{\sqrt{nq(1-q)}}\right) \; + \; \frac{-\sigma^2 T - \frac{\sigma^4 T^2}{12n^{3/2}} + \dots}{2 + \frac{\sigma^2 T^2}{3n} + \dots}\right)
\end{aligned}$$

Note that

$$nq(1-q) = n\left(\frac{u-2+d}{u^2-2+d^2}\right) = n\left(\frac{1+\frac{\sigma^2 T}{12n}+\dots}{4+\frac{4\sigma^2}{3n}+\dots}\right) \longrightarrow \infty \quad \text{as } n \to \infty$$

Hence we can apply theorem 4.1.1. which gives us

$$\lim_{n\to\infty} s\exp\left(2\sigma\sqrt{T}\sqrt{\frac{1+\frac{\sigma^2 T}{12n}+\dots}{4+\frac{4\sigma^2}{3n}+\dots}}\left(\frac{X_n - nq}{\sqrt{nq(1-q)}}\right) + \frac{-\sigma^2 T - \frac{\sigma^4 T^2}{12n^{3/2}}+\dots}{2+\frac{\sigma^2 T^2}{3n}+\dots}\right)$$
$$= s\exp\left(\sigma\sqrt{T}N(0,1) - \frac{\sigma^2}{2}T\right).$$

It has now been shown that the stock price $S^n(T)$ converge in distribution to the stock price in the Black-Scholes model,

$$S^n(T) \xrightarrow{d} S(T) = s\exp\left(\sigma\sqrt{T}N(0,1) - \frac{\sigma^2}{2}T\right).$$

5 Convergence of the Price of the European Call Option

The convergence of the stock price makes us wonder whether the price of the European Call option also converges. We will show below in two different ways that it is in fact true.

Assume that the rate of interest is equal to zero. The price of the European call option for the binomial model converge to the theoretical price of the European call option in Black-Scholes model,

$$\mathrm{E}[(S^n(T) - K)^+] \longrightarrow \mathrm{E}[(S(T) - K)^+].$$

5.1 Proof 1

The property

$$\mathrm{E}[g(X)] = \sum_j g(X)\mathrm{P}(X = j)$$

gives us the following

$$\begin{aligned}
\mathrm{E}[(S^n(T) - K)^+] &= \sum_{j=0}^{n} \max[S(T) - K, 0] \binom{n}{j} q^j (1-q)^{n-j} \\
&= \sum_{su^j d^{n-j} - K > 0} (su^j d^{n-j} - K) \binom{n}{j} q^j (1-q)^{n-j}.
\end{aligned}$$

To see for which values on j the relation $su^j d^{n-j} - K > 0$ is fulfilled, rewrite the inequality in the following way

$$\begin{aligned}
su^j d^{n-j} - K > 0 &\Longleftrightarrow j \ln u + (n-j) \ln d > -\ln\left(\frac{s}{K}\right) \\
&\Longleftrightarrow j > \frac{-n(-\sigma\sqrt{T/n})}{2\sigma\sqrt{T/n}} - \frac{\ln\left(\frac{s}{K}\right)}{2\sigma\sqrt{T/n}} \\
&\Longleftrightarrow j > \frac{n}{2} - \frac{\ln\left(\frac{s}{K}\right)}{2\sigma\sqrt{T/n}}.
\end{aligned}$$

For simplicity, let $\alpha = \dfrac{n}{2} - \dfrac{\ln\left(\frac{s}{K}\right)}{2\sigma\sqrt{T/n}}$.

The expected value can then be written as follows

$$\mathrm{E}[(S^n(T)-K)^+] = \sum_{j>\alpha} s\binom{n}{j}(qu)^j(d(1-q))^{n-j} - \sum_{j>\alpha} K\binom{n}{j}q^j(1-q)^{n-j}$$

$$= s\mathrm{P}(X_n > \alpha) - K\mathrm{P}(Y_n > \alpha)$$

where $X_n \sim \mathrm{Bin}(n, qu)$ and $Y_n \sim \mathrm{Bin}(n, q)$. Please note that $qu+(1-q)d = 1$ and X_n is therefore binomial distributed with parameters (n, qu).

$s\mathrm{P}(X_n > \alpha) - K\mathrm{P}(Y_n > \alpha)$

$$= s\mathrm{P}\left(\frac{X_n - nqu}{\sqrt{nqud(1-q)}} > \frac{\alpha - nqu}{\sqrt{nqud(1-q)}}\right) - K\mathrm{P}\left(\frac{Y_n - nq}{\sqrt{nq(1-q)}} > \frac{\alpha - nq}{\sqrt{nq(1-q)}}\right)$$

$$= s\mathrm{P}\left(\frac{X_n - nqu}{\sqrt{nq(1-q)}} > \frac{\frac{n}{2} - \frac{\ln\left(\frac{s}{K}\right)}{2\sigma\sqrt{T/n}} - nqu}{\sqrt{nq(1-q)}}\right) - K\mathrm{P}\left(\frac{Y_n - nq}{\sqrt{nq(1-q)}} > \frac{\alpha - nq}{\sqrt{nq(1-q)}}\right)$$

$$= s\mathrm{P}\left(\frac{X_n - nqu}{\sqrt{nq(1-q)}} > \frac{\left(\frac{\sqrt{n}}{2} - \frac{\ln\left(\frac{s}{K}\right)}{2\sigma\sqrt{T}} - \sqrt{n}qu\right)(u-d)}{\sqrt{(1-d)(u-1)}}\right)$$

$$- K\mathrm{P}\left(\frac{Y_n - nq}{\sqrt{nq(1-q)}} > \frac{\left(\frac{\sqrt{n}}{2} - \frac{\ln\left(\frac{s}{K}\right)}{2\sigma\sqrt{T}} - \sqrt{n}q\right)(u-d)}{\sqrt{(1-d)(u-1)}}\right)$$

$$= s\mathrm{P}\left(\frac{X_n - nqu}{\sqrt{nq(1-q)}} > \frac{\frac{\sqrt{n}}{2}(u-d) - \frac{\ln\left(\frac{s}{K}\right)}{2\sigma\sqrt{T}}(u-d)}{\sqrt{(u-2+d)}}\right)$$

$$- K\mathrm{P}\left(\frac{Y_n - nq}{\sqrt{nq(1-q)}} > \frac{\frac{\sqrt{n}}{2}(u-d) - \sqrt{n}(1-d) - \frac{\ln\left(\frac{s}{K}\right)}{2\sigma\sqrt{T}}(u-d)}{\sqrt{(1-d)(u-1)}}\right)$$

$$= s\mathrm{P}\left(\frac{X_n - nqu}{\sqrt{nq(1-q)}} > -\frac{1}{2}\sqrt{n(u-2+d)} - \frac{\ln\left(\frac{s}{K}\right)}{2\sigma\sqrt{T}}\sqrt{\frac{u^2-2+d^2}{u-2+d}}\right)$$

$$- K\mathrm{P}\left(\frac{Y_n - nq}{\sqrt{nq(1-q)}} > \frac{1}{2}\sqrt{n(u-2+d)} - \frac{\ln\left(\frac{s}{K}\right)}{2\sigma\sqrt{T}}\sqrt{\frac{u^2-2+d^2}{u-2+d}}\right)$$

$$= s\mathrm{P}\left(\frac{X_n - nqu}{\sqrt{nq(1-q)}} > -\frac{1}{2}\sqrt{n\left(\frac{\sigma^2 T}{n} + \frac{\sigma^4 T^2}{12n^2} + \dots\right)} - \frac{\ln\left(\frac{s}{K}\right)}{2\sigma\sqrt{T}}\sqrt{\frac{\frac{4\sigma^2 T}{n} + \frac{4\sigma^4 T^2}{3n^2} + \dots}{\frac{\sigma^2 T}{n} + \frac{\sigma^4 T^2}{12n^2} + \dots}}\right)$$

$$- K\mathrm{P}\left(\frac{Y_n - nq}{\sqrt{nq(1-q)}} > \frac{1}{2}\sqrt{n\left(\frac{\sigma^2 T}{n} + \frac{\sigma^4 T^2}{12n^2} + \dots\right)} - \frac{\ln\left(\frac{s}{K}\right)}{2\sigma\sqrt{T}}\sqrt{\frac{\frac{4\sigma^2 T}{n} + \frac{4\sigma^4 T^2}{3n^2} + \dots}{\frac{\sigma^2 T}{n} + \frac{\sigma^4 T^2}{12n^2} + \dots}}\right)$$

$$= s\mathrm{P}\left(\frac{X_n - nqu}{\sqrt{nq(1-q)}} > -\frac{1}{2}\sqrt{\sigma^2 T + \frac{\sigma^4 T^2}{12n} + \dots} \quad - \frac{\ln\left(\frac{s}{K}\right)}{2\sigma\sqrt{T}}\sqrt{\frac{4 + \frac{4\sigma^2 T}{3n} + \dots}{1 + \frac{\sigma^2 T}{12n} + \dots}}\right)$$

$$- K\mathrm{P}\left(\frac{Y_n - nq}{\sqrt{nq(1-q)}} > \frac{1}{2}\sqrt{\sigma^2 T + \frac{\sigma^4 T^2}{12n} + \dots} \quad - \frac{\ln\left(\frac{s}{K}\right)}{2\sigma\sqrt{T}}\sqrt{\frac{4 + \frac{4\sigma^2 T}{3n} + \dots}{1 + \frac{\sigma^2 T}{12n} + \dots}}\right)$$

Now let $n \to \infty$. The equality below follows by theorem 4.1.2 (Normal approximation of the binomial distribution) stated above.

$$\lim_{n\to\infty} s\mathrm{P}\left(\frac{X_n - nqu}{\sqrt{nq(1-q)}} > -\frac{1}{2}\sqrt{\sigma^2 T + \frac{\sigma^4 T^2}{12n} + \dots} \quad - \frac{\ln\left(\frac{s}{K}\right)}{2\sigma\sqrt{T}}\sqrt{\frac{4 + \frac{4\sigma^2 T}{3n} + \dots}{1 + \frac{\sigma^2 T}{12n} + \dots}}\right)$$

$$- K\mathrm{P}\left(\frac{Y_n - nq}{\sqrt{nq(1-q)}} > \frac{1}{2}\sqrt{\sigma^2 T + \frac{\sigma^4 T^2}{12n} + \dots} \quad - \frac{\ln\left(\frac{s}{K}\right)}{2\sigma\sqrt{T}}\sqrt{\frac{4 + \frac{4\sigma^2 T}{3n} + \dots}{1 + \frac{\sigma^2 T}{12n} + \dots}}\right)$$

$$= s\mathrm{P}\left(Z > -\frac{1}{2}\sigma^2 T - \frac{\ln\left(\frac{s}{K}\right)}{\sigma\sqrt{T}}\right) - K\mathrm{P}\left(U > \frac{1}{2}\sigma^2 T - \frac{\ln\left(\frac{s}{K}\right)}{\sigma\sqrt{T}}\right)$$

where $Z \sim$ N(0,1) and $U \sim$ N(0,1). Then the first equality follows by symmetry of the standard normal density function,

$$sP\left(Z > -\frac{1}{2}\sigma^2 T - \frac{\ln\left(\frac{s}{K}\right)}{\sigma\sqrt{T}}\right) - KP\left(U > \frac{1}{2}\sigma^2 T - \frac{\ln\left(\frac{s}{K}\right)}{\sigma\sqrt{T}}\right)$$

$$= sP\left(Z < \frac{\ln\left(\frac{s}{K}\right)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma^2 T\right) - KP\left(U < \frac{\ln\left(\frac{s}{K}\right)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma^2 T\right)$$

$$= sN\left(\frac{\ln\left(\frac{s}{K}\right)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma^2 T\right) - KN\left(\frac{\ln\left(\frac{s}{K}\right)}{\sigma\sqrt{T}} - \frac{1}{2}\sigma^2 T\right)$$

where N(.) is the cumulative distribution function of a stochastic variable with standard normal distribution. This is the theoretical price of the European call option for the Black-Scholes model. □

The price is consistent with Black-Scholes equation.

5.2 Proof 2

Another way of showing that $\mathrm{E}[(S^n(T) - K)^+] \to \mathrm{E}[(S(T) - K)^+]$ is by using the following theorem.

Theorem 5.2.1 The two properties that follows are equivalent.

i) $X_n \xrightarrow{d} X$.

ii) $\mathrm{E}[g(X_n)] \longrightarrow \mathrm{E}[g(X)]$ for every continuous and bounded function g [3].

The function $g(X) = \max[X - K, 0]$ for a European call option is however unbounded. Thus we need the put-call parity theorem to be able to prove the convergence.

Theorem 5.2.2 (Put-Call Parity) Let $p(t, S(t), K, T)$ and $c(t, S(t), K, T)$ denote the price of a European put and call option for the asset S. Suppose that both options have termination date T and strike price K. If $t < T$ and $\tau = T - t$, then

$$c(t, S(t), K, T) = S(t) - Ke^{-r\tau} + p(t, S(t), K, T) \qquad [2].$$

Before stating the proof let $X \sim N(0,1)$ and note that

$$\begin{aligned}
\mathrm{E}[S(T)] =& \mathrm{E}\left[se^{\sigma\sqrt{T}X-\frac{\sigma^2}{2}T}\right] \\
=& se^{-\frac{\sigma^2}{2}T}\int_{-\infty}^{\infty} e^{\sigma\sqrt{T}x}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \\
=& se^{-\frac{\sigma^2}{2}T}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\sigma\sqrt{T})^2}{2}} * e^{\frac{\sigma^2}{2}T}dx \\
=& s.
\end{aligned}$$

Now we are able to prove that

$$\mathrm{E}[(S^n(T)-K)^+] \to \mathrm{E}[(S(T)-K)^+] \quad \text{as} \quad n \to \infty.$$

Proof. Assume that s is the initial stock price. From definition 2.1.3 it then follows that $\mathrm{E}[S^n(T)] = s$. Furthermore by Put-call parity theorem we are able to do the following rewriting

$$\begin{aligned}
\mathrm{E}[(S^n(T)-K)^+] =& \mathrm{E}[S^n(T)] - \mathrm{E}[K] + \mathrm{E}[(K-S^n(T))^+] \\
=& s - K + \mathrm{E}[(K-S^n(T))^+]
\end{aligned}$$

The function $(K-S^n(T))^+$ is bounded and continuous. Hence we can apply theorem 5.2.1 and the convergence follows instantly since it has already been shown that $S^n(T) \xrightarrow{d} S(T)$.

$$\mathrm{E}[(S^n(T)-K)^+] = \mathrm{E}[S^n(T)] - \mathrm{E}[K] + \mathrm{E}[(K-S^n(T))^+]$$

$$= s - K + \mathrm{E}[(K-S^n(T))^+] \longrightarrow s - K + \mathrm{E}[(K-S(T))^+]$$

$$= \mathrm{E}[S(T)] - \mathrm{E}[K] + \mathrm{E}[(K-S(T))^+] = \mathrm{E}[(S(T)-K)^+]$$

Thus the option price in the binomial model converges to the price for the option in Black-Scholes model. □

6 References

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