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A Risk Surplus Model using Hawkes Point Processes

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Abstract

In this thesis we define a Hawkes process with exponential decay that later on is used in an application to insurance. We have also applied a simulation algorithm for the Hawkes process that are able to model cluster arrival of claims. In the classical risk model one uses a homogeneous Poisson process to model the arrival of claims which is not realistic. What is most crucial for an insurance company is to price their premiums such that the probability of bankruptcy is small. Hence, we discuss the effect of modelling claim arrivals with a Hawkes process and different choices of premium principles with respect to the probability of ruin.

Contents

1 Introduction

Insurance companies offer their customers insurance against a risk in exchange for a premium. The main challenge for actuaries is to find an optimal premium, meaning that the probability of bankruptcy for the company should be small and the price of the premium not too high. If the premium is too high, the company becomes less attractive in the market while a small premium means that the company is more likely to be bankrupt.

The classical risk model generate claim arrivals using a homogeneous Poisson process. This is not realistic since claims will not always be reported due to a constant intensity. For instance, a natural disaster will cause the arrival rate of claims to be high right after the catastrophe has occurred but several claims will be delayed and reported later, thus the arrival rate of claims will decline. An event like this will affect the insurance company for a long period of time. Models that can capture this type of contagion and clustering arrival of events are therefore very useful.

Alan G. Hawkes introduced in 1971 a general type of Hawkes process, see [4] and [5], that nowadays has become a vigorous mathematical tool when modelling contagion risk and clustering arrival of events in insurance and finance. For examples of applications see [2], [6], [7] and [8]. This paper focus on a Hawkes process with exponential decay that is defined in section 2. Section 3 applies a simulation algorithm for the Hawkes process that later on is used in section 4 where the effect of modelling claim arrivals with a Hawkes process in insurance is discussed.

2 Hawkes Process

A Hawkes point process is sometimes called a self-exciting process since the presence of past points will cause future points more likely to appear. This property is what makes it capable to model contagion and clustering arrival of events. We begin with a cluster-based definition of a Hawkes process with exponential decay.

Definition 2.1 A **Hawkes process** is a point process where the points are of one of two types, an *immigrant* or an *offspring*. The immigrants can be seen as primary shocks and its associated offspring as aftershocks. The process has the following structure:

(a) The immigrants $I = \{T_m\}_{m=1,2,\ldots}$ are distributed on $\mathbb{R}_+$ as an inhomogeneous Poisson process with rate $a + (\lambda_0 - a)e^{-\delta t}$, $t \geq 0$.

(b) The marks $\{Y_m\}_{m=1,2,\ldots}$ that are connected to the immigrants I are *i.i.d* with $Y_m \sim G$. Moreover the marks are independent of the immigrants.

(c) Every immigrant T_m generates a cluster C_m, where all clusters are independent of each other.

(d) Each cluster C_m consists of a random set of marked points, where each point belongs to a generation of order $n = 0, 1, ...$, with the following branching structure:

- Generation 0 consists of the immigrant and its corresponding mark (T_m, Y_m).
- Each immigrant generate offspring $(T_j, Y_j) \in C_m$, of generation $n = 1, 2, ...$ on (T_m, ∞), distributed according to a Poisson process with intensity $Y_m e^{-\delta(t-T_m)}$, $t > T_m$. Recursively, these points generate new offspring, of generation $n+1$ on (T_j, ∞), distributed according to a Poisson process with intensity $Y_j e^{-\delta(t-T_j)}$, where $t > T_j$. Moreover, the marks $Y_j \sim G$ are independent of the generation n. Hence, offspring generated by an immigrant is of generation 1. Further on, offspring generated by points in generation 1 is of generation 2 and so on.

(e) C represents the union of all clusters C_m, i.e. $C = \cup_{m=1,2,...} C_m$.

Therefore, a Hawkes process can be seen as a marked Poisson cluster process with marks Y_i at times $T_i \in \mathbb{R}_+$. Consequently the counting process

$$N_t = \#\text{ points in } (0, t]$$

is driven by the stochastic intensity process

$$\lambda_t = a + (\lambda_0 - a)e^{-\delta t} + \sum_{0 \le T_k < t} Y_k e^{-\delta(t-T_k)}, \quad t \ge 0.$$

The random jump-size Y_k measures the rate of contagion (of the k^{th} event) but is exponentially decaying with time and δ, where δ captures the perseverance of contagion. Note that a is the constant reversion level, which means that the intensity process $\lambda_t > a$ for all $t \in \mathbb{R}_+$.

Using the result in [2] (Theorem 2.3.1 and Remark 2.3.3), take

$$Y_k \sim \text{Exp}(\beta)$$

such that $\delta > \mu_1$, where $\mu_1 = \dfrac{1}{\beta}$ and $\mu_2 = \dfrac{2}{\beta^2} = 2\mu_1^2$. Thereafter choose

$$\lambda_0 \sim a + \Gamma,$$

where

$$\Gamma \sim \text{Gamma}(\frac{a}{\delta}, \frac{\delta\beta - 1}{\delta})$$

and the expectation of Γ is given by $\text{E}[\Gamma] = \dfrac{a\mu_1}{\delta - \mu_1}$. Then, as $N_0 = 0$

$$\lambda_t \sim \lambda_0 \text{ for all } t \ge 0.$$

Thus,

$$\mathrm{E}[\lambda_t] = \frac{a\delta}{\delta - \mu_1} \tag{2.1}$$

$$\mathrm{Var}[\lambda_t] = \frac{\mu_1^2 a\delta}{2(\delta - \mu_1)^2} \tag{2.2}$$

We will use this result throughout this thesis. Henceforth, we can calculate the expectation of N_t by

$$\mathrm{E}[N_t - \int_0^t \lambda_s ds] = 0$$
$$\Rightarrow \mathrm{E}[N_t] = \int_0^t \mathrm{E}[\lambda_s]ds = \int_0^t \frac{a\delta}{\delta - \mu_1} ds = \frac{a\delta}{\delta - \mu_1} t.$$

The variance of N_t is given by

$$\mathrm{Var}[N_t] = \mathrm{E}[N_t^2] - \mathrm{E}[N_t]^2,$$

where, again using the results in [2],

$$\mathrm{E}[N_t^2] = 2\int_0^t \mathrm{E}[\lambda_s N_s]ds + \int_0^t \mathrm{E}[\lambda_s]ds$$

and

$$\mathrm{E}[\lambda_s N_s] = \bar{k}(1 - e^{-(\delta-\mu_1)s}) + \left(\frac{a\delta}{\delta - \mu_1}\right)^2 s,$$
$$\text{where } \bar{k} = \frac{\mu_1 a\delta}{(\delta - \mu_1)^2} + \frac{\mu_1^2 a\delta}{(\delta - \mu_1)^3}.$$

Therefore,

$$\Rightarrow \mathrm{Var}[N_t] = 2\int_0^t \bar{k}(1 - e^{-(\delta-\mu_1)s}) + \left(\frac{a\delta}{\delta - \mu_1}\right)^2 sds + \int_0^t \frac{a\delta}{\delta - \mu_1} ds - \left(\frac{a\delta}{\delta - \mu_1} t\right)^2 =$$
$$= ... = \frac{a\delta}{\delta - \mu_1} t + 2\bar{k}\left(t - \frac{1 - e^{-(\delta-\mu_1)t}}{\delta - \mu_1}\right).$$

Thus,

$$\mathrm{E}[N_t] = \frac{a\delta}{\delta - \mu_1} t \tag{2.3}$$

$$\mathrm{Var}[N_t] = \frac{a\delta}{\delta - \mu_1} t + 2\bar{k}\left(t - \frac{1 - e^{-(\delta-\mu_1)t}}{\delta - \mu_1}\right). \tag{2.4}$$

3 Simulation Algorithm

Algorithm 3.1 The following algorithm, found in [1], simulates one sample path of a Hawkes process $\{(N_t, \lambda_t)\}_{t\geq 0}$ with exponentially decaying intensity conditional on λ_0 and $N_0 = 0$. The self-excited jumps have random sizes with distribution $Y \sim G$ and jump times $\{T_1, T_2, ..., T_K\}$:

Step 1. Set the initial conditions $T_0 = 0$, $\lambda_{T_0^{\mp}} = \lambda_0 > a$, $N_0 = 0$ and $k \in \{0, 1, 2, ..., K-1\}$.

Step 2. Simulate the $(k+1)^{th}$ interarrival-time S_{k+1} by

$$S_{k+1} = \begin{cases} S_{k+1}^{(1)} \wedge S_{k+1}^{(2)}, & D_{k+1} > 0 \\ S_{k+1}^{(2)}, & D_{k+1} < 0 \end{cases}$$

where

$$D_{k+1} = 1 + \frac{\delta \ln U_1}{\lambda_{T_k^+} - a}, \quad U_1 \sim U[0,1],$$

and

$$S_{k+1}^{(1)} = -\frac{1}{\delta} \ln D_{k+1}, \quad S_{k+1}^{(2)} = -\frac{1}{a} \ln U_2, \quad U_2 \sim U[0,1].$$

Step 3. Record the $(k+1)^{th}$ jump-time T_{k+1} in the intensity process λ_t by

$$T_{k+1} = T_k + S_{k+1}.$$

Step 4. Record the change at the jump-time T_{k+1} in the intensity process λ_t by

$$\lambda_{T_{k+1}+} = \lambda_{T_{k+1}-} + Y_{k+1}, \quad Y_{k+1} \sim G,$$

where

$$\lambda_{T_{k+1}^-} = (\lambda_{T_k^+} - a)e^{-\delta(T_{k+1} - T_k)} + a.$$

Step 5. Record the change at the jump-time T_{k+1} in the point process N_t by

$$N_{T_{k+1}^+} = N_{T_{k+1}^-} + 1.$$

This is a general algorithm for simulating a Hawkes process. In our case we put $\lambda_0 \sim a + \Gamma$ and $Y_k \sim \text{Exp}(\beta)$.

3.1 An Illustration of a Hawkes Process

Figure 1 shows one sample path of a Hawkes process on a small time interval to illustrate how the process behaves.

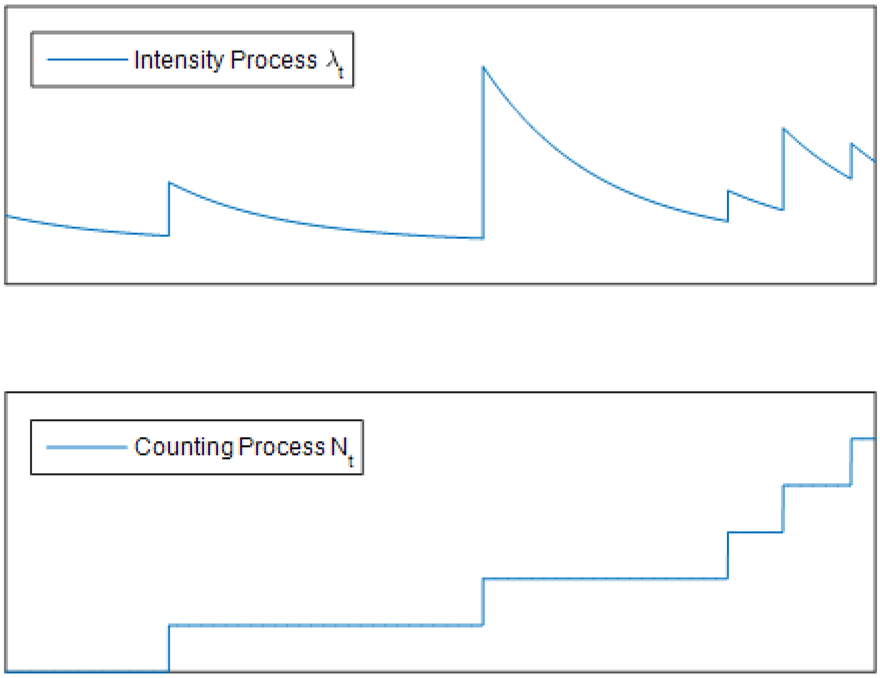

Figure 1: An illustration of a Hawkes process $\{N_t, \lambda_t\}_{t\geq 0}$.

Note that we define the Hawkes process to be *right-continuous*, i.e. T^- is the time right before the jump has occurred and T^+ is the time right after the jump has occurred. The same goes for the point process N_t where N_{T^-} is the number of events that has happened up to time T^- and N_{T^+} is the number of events that has happened up to time T^+.

4 Application of a Hawkes Process in Insurance

4.1 The Cramér-Lundberg Model

The Swedish actuary, Filip Lundberg, introduced in 1903 the **classical risk model** also known as the **Cramér-Lundberg model**. This model is the oldest in actuarial mathematics and has proved to be a very important tool. It is a collective risk model that models the surplus of an insurance company by a stochastic process. Furthermore the appearance of the model became the foundation of ruin theory.

Definition 4.1.1 (Cramér-Lundberg Process) The surplus process X_t is a stochastic process that models the surplus of an insurance company with two cash flows: incoming premiums and outgoing claims.

$$X_t = x + ct - \sum_{i=1}^{N_t} Z_i, \quad t \geq 0, \tag{4.1}$$

where

- $X_0 = x \geq 0$ is the initial surplus at $t = 0$;

- $c > 0$ is the premium income per time unit;

- N_t is a homogeneous Poisson point process with intensity λ, that counts the aggregated amount of claims up to time t, where $N_0 = 0$;

- $\{Z_i\}_{i=1,2,\ldots}$ is a sequence of $i.i.d$ positive random variables, independent of N_t, that corresponds to the claim sizes, where $Z_i \sim H$.

From definition 4.1.1 we can derive the expectation and variance of the surplus process in the following way

$$\begin{aligned}
\mathrm{E}[X_t] &= x + ct - \mathrm{E}[\sum_{i=1}^{N_t} Z_i] = \\
&= x + ct - \mathrm{E}[N_t]m = \\
&= x + (c - \lambda m)t
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Var}[X_t] &= \mathrm{Var}[\sum_{i=1}^{N_t} Z_i] = \\
&= \mathrm{Var}[\mathrm{E}[\sum_{i=1}^{N_t} Z_i | N_t]] + \mathrm{E}[\mathrm{Var}[\sum_{i=1}^{N_t} Z_i | N_t]] = \\
&= \mathrm{Var}[mN_t] + \mathrm{E}[\sigma^2 N_t] = \\
&= m^2 \mathrm{Var}[N_t] + \sigma^2 \mathrm{E}[N_t] = \\
&= m^2 \mathrm{Var}[N_t] + \sigma^2 \lambda t
\end{aligned}$$

where $m = \int_0^\infty z dH(z)$ and $\sigma^2 = \int_0^\infty (z - m)^2 dH(z)$.

Thus,

$$\mathrm{E}[X_t] = x + (c - \lambda m)t \tag{4.2}$$

$$\mathrm{Var}[X_t] = m^2 \mathrm{Var}[N_t] + \sigma^2 \lambda t \tag{4.3}$$

Figure 2 shows an illustration of a surplus process X_t, defined in 4.1. An income of a claim at time $T_{k=1,2,\ldots}$ cause the insurance company to pay out money to the policy holder and the surplus drops with Z_k, where Z_k is the random size of the k^{th} claim. The state where the surplus process falls below zero for the first time is called *ruin*, this is the case at time T_3 in figure 2. If no claims occur, the slope of the surplus process will be c.

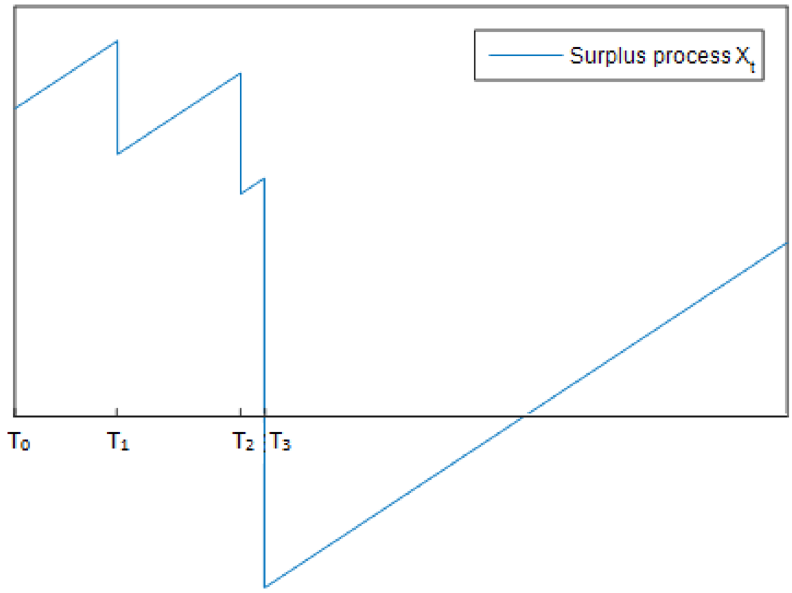

Figure 2: Realization of a Surplus process

Definition 4.1.2 (Ruin time) The ruin time τ is defined by

$$\tau = \inf\{t > 0 \mid X_t \leq 0\}.$$

If $X_t > 0$ for all t ruin does not occur and $\tau = \infty$.

When claims arrive according to a homogeneous Poisson process, as in the classical risk model, and the claims are exponentially distributed with parameter α, the probability of *ultimate ruin* equals

$$\psi(x) = P[\inf_{t>0} X_t \leq 0 \mid X_0 = x] = \frac{\lambda}{\alpha c} e^{-(\alpha - \lambda/c)x}, \tag{4.4}$$

where x is the initial surplus. *Ultimate ruin* means that we look at the ruin probability for an infinitely long time interval and not when t is fixed. See derivation of 4.4 in [3].

4.2 Net Profit Condition

What the insurance companies wants to achieve is to price c such that it covers the risk of ending up in ruin, but not for an excessive price. If the expected income in premiums is greater than the expected outflow of money, this implies that there is a positive probability for the company of not getting bankrupt. This condition is what is referred to as the **net profit condition**, i.e.

$$\mathrm{E}[ct] > \mathrm{E}[\sum_{i=1}^{N_t} Z_i]$$
$$\Rightarrow ct > \mathrm{E}[N_t]m,$$

which is equivalent to

$$\mathrm{E}[X_t] > x.$$

In the classical risk model 4.1 the net profit condition is

$$ct > \lambda mt$$
$$\Leftrightarrow \quad c > \lambda m$$

since the expected value of a Poisson process with constant intensity is λ per time unit. In case N_t is driven by a Hawkes process, we get

$$ct > \frac{a\delta m}{\delta - \mu_1} t$$
$$\Leftrightarrow c > \frac{a\delta m}{\delta - \mu_1} m.$$

4.3 The Expected Value Principle

A *premium principle* is a formula for how to price a premium against an insurance risk. Let S_t denote the accumulated expenses in the time interval $(0, t]$, $t \geq 0$,

$$S_t = \sum_{i=1}^{N_t} Z_i,$$

then the expected expenses per time unit, that we denote by $\mathrm{E}[S]$, equals

$$\mathrm{E}[S] = \frac{E[S_t]}{t}.$$

What is most essential for an insurance company is to have a positive probability of not getting ruined. By the net profit condition the price of the premium must be higher than the expected value of expenses for that to be possible. The most popular method to price a premium is according to the **expected value principle**

$$c = (1 + \theta)\mathrm{E}[S], \quad \theta > 0 \tag{4.5}$$

where the parameter θ is called **safety loading**.

4.4 Hawkes Point Processes in a Risk Model

The classical risk model 4.1 is using a homogeneous Poisson process to model claim arrivals. Hence, it would be of interest to see whether there is any difference when claims arrive according to a Hawkes process with respect to the probability of ruin. As mentioned in the introduction it is not always realistic to model claim arrivals due to a constant intensity. The Hawkes process captures the risk of contagion and clustering arrivals that can be very useful when an event like a natural disaster occur, or any other accident that may have impact on the insurance company for a long time.

Let us consider the random claim sizes Z_i exponentially distributed with parameter α, thus $m = \frac{1}{\alpha}$ and $\sigma^2 = \frac{1}{\alpha^2}$. By letting the expected number of claims, when N_t is driven by a Poisson process, equal the expected number of claims when N_t is driven by a Hawkes process, we obtain

$$\lambda t = \frac{a\delta}{\delta - \mu_1} t \quad \Leftrightarrow \quad \lambda = \frac{a\delta}{\delta - \mu_1}.$$

and therefore, rearranging

$$\frac{\mu_1}{\delta} = 1 - \frac{a}{\lambda}.$$

This leads to the subsequent steps:

1. Choose λ and a, such that $\lambda > a$.

2. Choose α and price c according to the expected value principle

$$c = (1+\theta)\frac{\lambda}{\alpha}.$$

3. Put $\gamma = 1 - \frac{a}{\lambda}$ (fixed), thus

$$\frac{\mu_1}{\delta} = \gamma.$$

This imply that the expected amount of claims will be held constant for any value of μ_1 when

$$\delta = \frac{\mu_1}{\gamma}. \tag{4.6}$$

Since μ_1 is the mean size of the self-excited jumps we can force the intensity of the Hawkes process to be constant by putting μ_1 close to zero. In that case the self-excited jumps will be so small that they are negligible and the intensity can be seen as constant. Therefore, for small μ_1, we consider the Hawkes process equivalent to a homogeneous Poisson process. If we further on increase μ_1 in steps and for each new value choose δ according to 4.6 the Hawkes process will become more and more apparent at the same time as $\mathrm{E}[N_t]$ will be constant. By this computation we are able to see how the surplus process is affected when claims arrive according to a Hawkes process compared to the Poisson case. We will get a sequence of $X_t^{(\delta)}$, $\delta > 0$, with

$$\mathrm{E}[X_t^{(\delta)}] = \underbrace{x + ct - \frac{\lambda}{\alpha}}_{= \, constant}$$

for all δ and a fixed time interval $(0, t]$. Moreover,

$$Var[X_t^{(\delta)}] = \frac{1}{\alpha^2}(\mathrm{Var}[N_t] + \lambda t). \tag{4.7}$$

What we now want to answer is:

- *Will the variance of the surplus process grow as a function of δ?*
- *Will the ruin probability grow in δ?*

We will now proceed with a numerical example where we choose $\lambda = 4$ and $a = 0.8$. Thereafter we choose $\alpha = 1$ and $\theta = \frac{1}{3}$, thus $c = (1 + \frac{1}{3})4 = \frac{16}{3}$. Furthermore

$$\gamma = 1 - \frac{a}{\lambda} = 1 - \frac{0.8}{4} = 0.8$$

By applying algorithm 3.1 we will simulate one sample path of a Hawkes process together with its associated surplus process for different values of μ_1 and δ, see illustrations below.

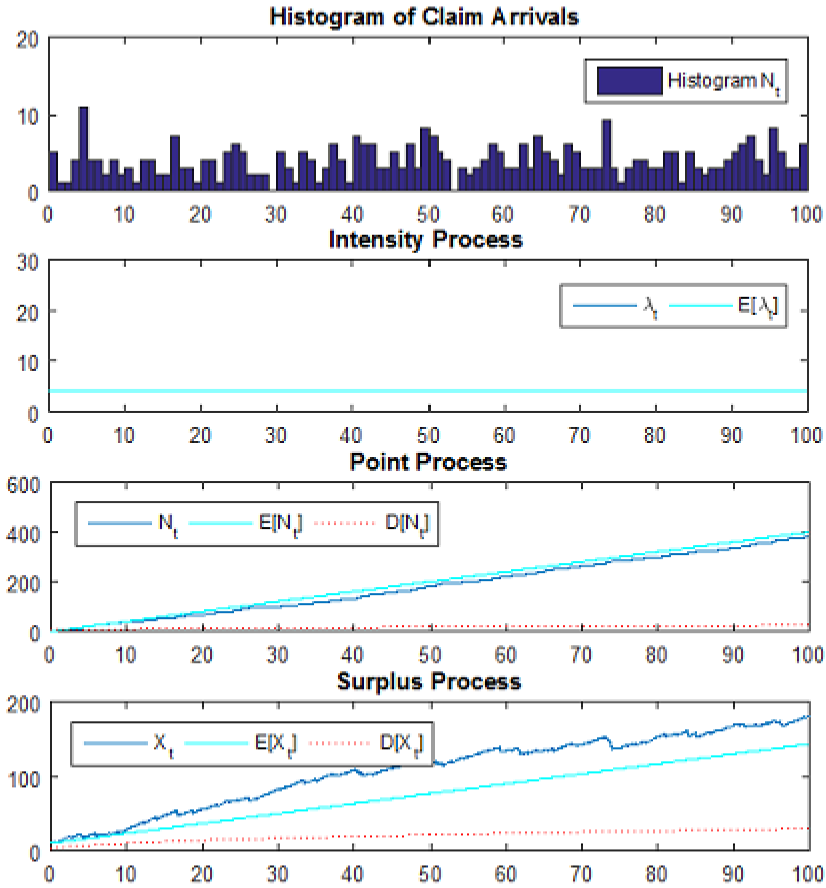

Figure 3: $\mu_1 = 0.0001$ and $\delta = 0.000125$

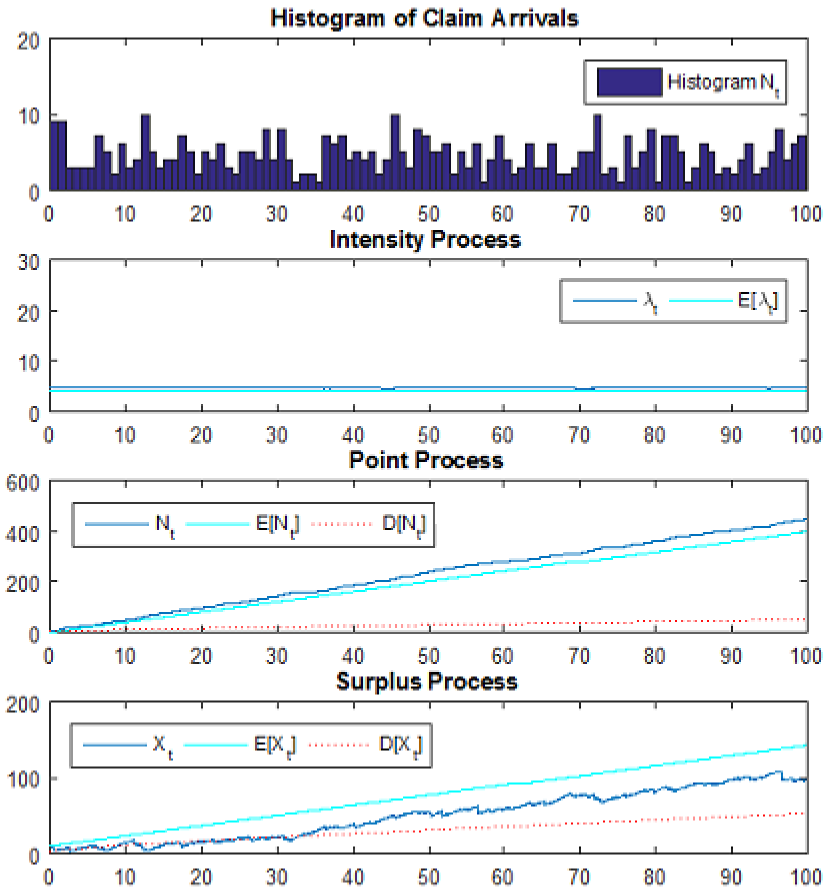

Figure 4: $\mu_1 = 0.01$ and $\delta = 0.0125$

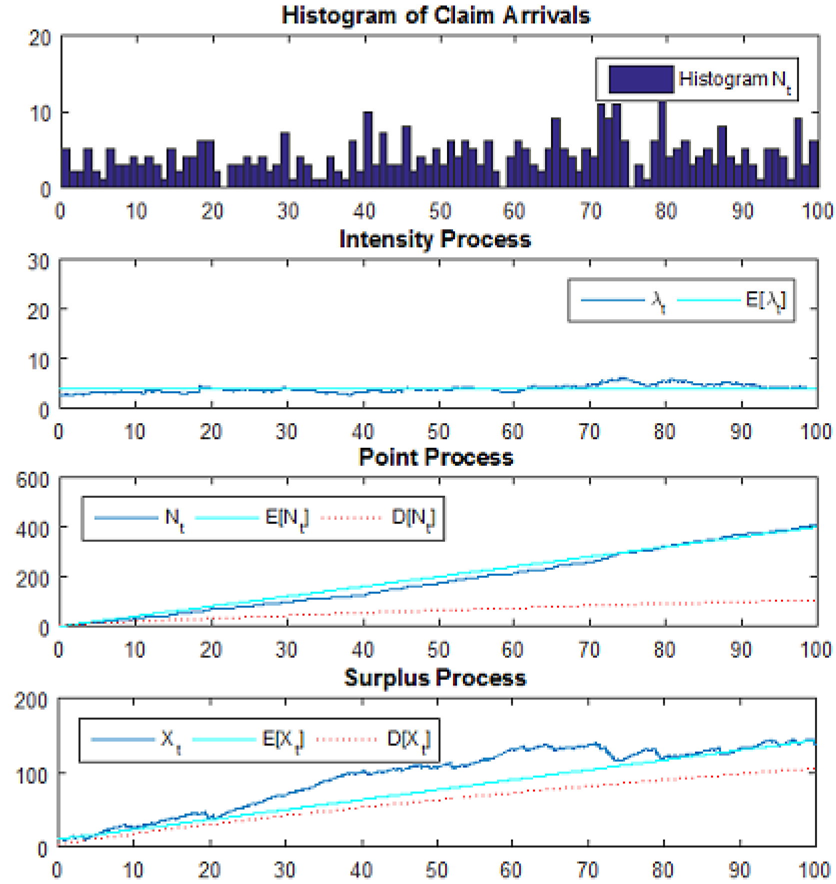

Figure 5: $\mu_1 = 0.1$ and $\delta = 0.125$

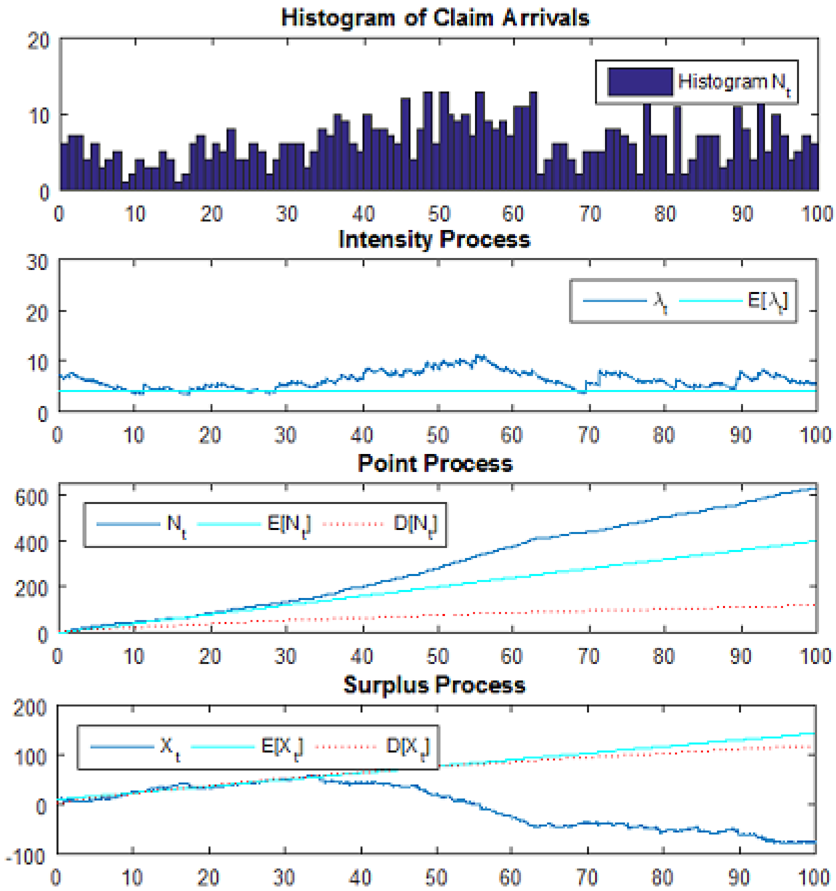

Figure 6: $\mu_1 = 0.2$ and $\delta = 0.25$

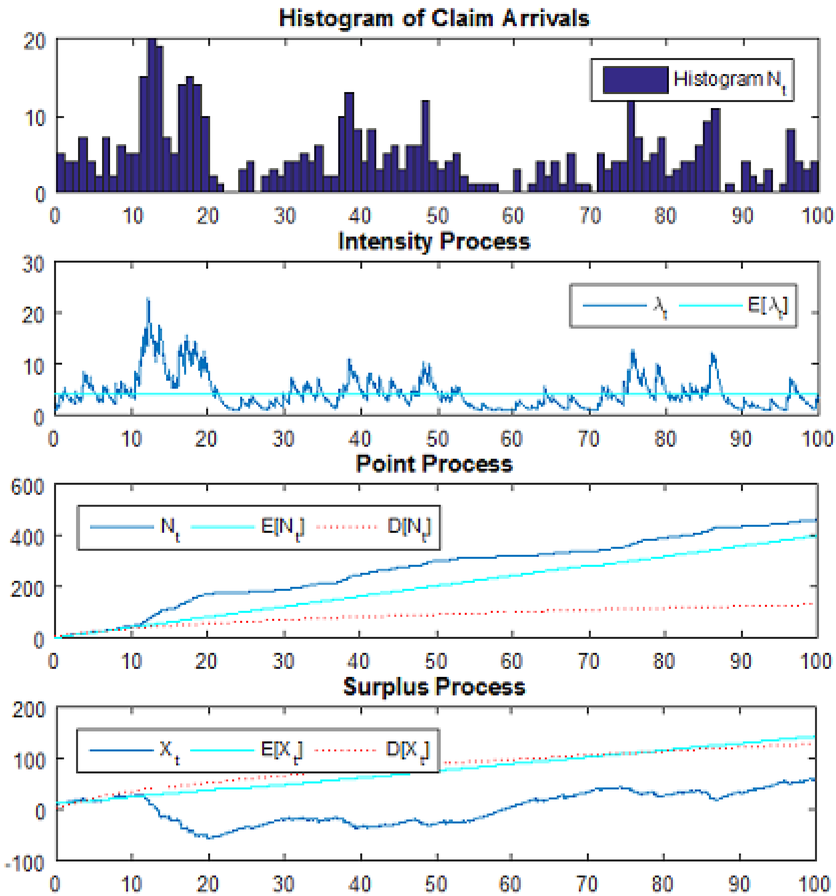

Figure 7: $\mu_1 = 1$ and $\delta = 1.25$

By studying the figures 3 - 7 we can see that as δ grows:

- Claims arrive more irregularly and more in clusters;
- The number of claims that arrive can deviate more from its expected value;
- The surplus process $X_t^{(\delta)}$ starts to fluctuate.

We will now try to answer if $\mathrm{Var}[X_t^{(\delta)}]$ grows with δ. This can be confirmed visually in figure 3 - 7 by studying the changes of the standard deviation $\mathrm{D}[X_t]$, but we will also try to prove this mathematically. We can do this by looking at the behaviour of $\mathrm{Var}[N_t]$ since $\mathrm{Var}[X_t^{(\delta)}]$ only depends on that term, see 4.7. Recall that a, λ, t now are fixed positive constants.

$$\Rightarrow \mathrm{Var}[N_t] = \underbrace{\frac{a\delta}{\delta - \mu_1}}_{=\lambda} t + 2\bar{k}\left(t - \frac{1 - e^{-(\delta-\mu_1)t}}{\delta - \mu_1}\right),$$

where

$$\begin{aligned}\bar{k} &= \frac{\mu_1 a\delta}{(\delta-\mu_1)^2} + \frac{\mu_2 a\delta}{2(\delta-\mu_1)^3} = \\ &= \underbrace{\frac{a\delta}{\delta-\mu_1}}_{=\lambda} \underbrace{(\frac{1}{\beta\delta-1} + (\frac{1}{\beta\delta-1})^2)}_{\text{const. since } \beta\delta=1/\gamma \text{ (fixed)}} \\ \Rightarrow \bar{k} &= \text{constant.}\end{aligned}$$

Thus, $\mathrm{Var}[N_t]$ only depends on

$$\frac{1-e^{-(\delta-\mu_1)t}}{\delta-\mu_1}.$$

Since $\delta - \mu_1 = \dfrac{a\delta}{\lambda}$, we get that

$$\frac{1-e^{-(\delta-\mu_1)t}}{\delta-\mu_1} = \frac{1-e^{-at\delta/\lambda}}{\frac{a\delta}{\lambda}} = \frac{\lambda}{a} \underbrace{\left(\frac{1-e^{-at\delta/\lambda}}{\delta}\right)}_{(*)}$$

If $\mathrm{Var}[N_t]$ is growing with δ, $(*)$ has to be a monotone decreasing function. Therefore the derivative of $(*)$ with respect to δ has to be negative. Let $u = \dfrac{at}{\lambda}$.

$$\begin{aligned}&\Rightarrow \frac{\mathrm{d}}{\mathrm{d}\delta}\left(\frac{1-e^{-u\delta}}{\delta}\right) < 0 \\ &\Rightarrow \frac{u\delta e^{-u\delta} + e^{-u\delta} - 1}{\delta^2} < 0.\end{aligned}$$

Since $\delta^2 > 0$, it is enough to show that the numerator is negative, i.e. that

$$\underbrace{u\delta e^{-u\delta} + e^{-u\delta} - 1}_{=\ \mathrm{g}(\delta)} < 0.$$

If $\mathrm{g}'(\delta) < 0$ it implies that $\mathrm{g}(\delta)$ is negative. Let us check if that is the case.

$$\begin{aligned}\mathrm{g}'(\delta) = \frac{\mathrm{dg}}{\mathrm{d}\delta}\left(u\delta e^{-u\delta} + e^{-u\delta} - 1\right) &< 0 \\ \Rightarrow \underbrace{-u^2\delta e^{-u\delta}}_{<\ 0} &< 0\end{aligned}$$

Thus we have proved that $(*)$ is monotone decreasing and therefore $\mathrm{Var}[X_t]$ is increasing in δ.

To answer the question if $\psi^{(\delta)}(x)$ will be growing in δ we simulate 1000 sample paths of a surplus process for each value of δ. In each simulation we store τ if the surplus process falls below zero, thereafter we can approximate the ruin probability by seeing how big proportion that ended up in ruin. Moreover we can also check whether our first case, when μ_1 is close to zero, corresponds to the ruin probability for a homogeneous Poisson process given in 4.4. The ultimate ruin probability that is given in 4.4 corresponds to the probability when $t \longrightarrow \infty$. However, since it is impossible to simulate an infinitely long time interval we will set $t = 400$ and see it as an approximation to ∞. By plotting a histogram of τ we are able to see both if the proportion of ruin times is increasing with δ and how they are distributed in time.

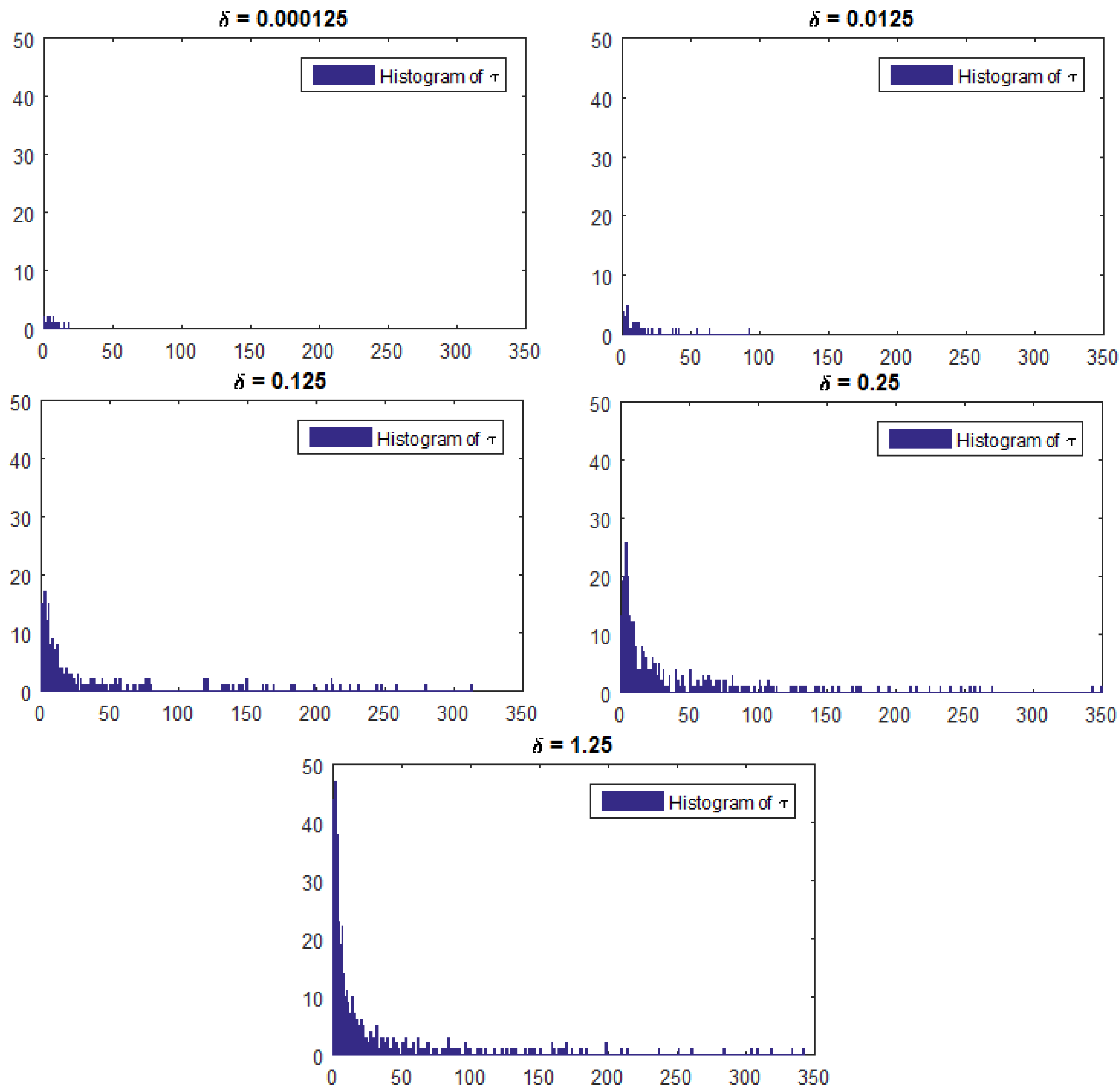

Figure 8: Histogram of τ for the surplus process X_t.

From what we can see in figure 8 it is obvious that the proportion of surplus processes that ended up in the state of ruin increased considerably when the Hawkes process became more apparent. By the simulations we can approximate the probability of ruin to 0.063 for $\delta = 0.000125$, 0.076 for $\delta = 0.0125$, 0.28 for $\delta = 0.125$, 0.38 for $\delta = 0.25$ and 0.51 for $\delta = 1$. The only expression for ruin probability that we have to compare with though is 4.4 that corresponds to the Poisson case. Studying the first case when we let the Hawkes process equal a Poisson process it seems to be consistent with the true value that is,

$$\begin{aligned}\psi(x) &= \frac{3}{4}e^{-(1-3/4)10} \approx \\ &\approx 0.0616.\end{aligned}$$

where $\psi^{(\delta=0.000125)}(x) = 0.063$. In this case, when the Hawkes process can be seen as a homogeneous Poisson process, ruin occur close to $t = 0$. Henceforth, when the Hawkes process becomes more apparent, ruin also occurs further in time. This does not arise when the intensity for the Hawkes process is almost constant.

What we can see in this application is that claims can arrive with very varying intensity even if the expected amount of claims are the same in each case. This imply that the expected expenses $\mathrm{E}[S_t]$ also are held constant for each δ. If we price premiums only with respect to the expected expenses, but not with respect to how big fluctuations we can have, it could have major consequences in the most extreme cases. Therefore the knowledge of $\mathrm{Var}[X_t^{(\delta)}]$ could be essential to know since it is a measure of how much the surplus process can deviate from its expected value. If the variance implies that the surplus process can fluctuate a lot the insurance company is exposed to a higher risk, it could therefore be reasonable to set a higher price on c to reduce the probability of bankruptcy.

4.5 The Variance Principle

Another premium principle that exists is the **variance principle**

$$c = \mathrm{E}[S] + \Theta \mathrm{Var}[S], \quad \Theta > 0, \qquad (4.8)$$
$$\text{where } \mathrm{Var}[S] = \frac{\mathrm{Var}[S_t]}{t}.$$

It depends not only on the expected expenses but also on the variance of S_t, in that way it is more sensible to higher risks, where the variance is used to determine the risk loading. Note that $\mathrm{Var}[S_t] = \mathrm{Var}[X_t]$.

A premium principle that is similar to 4.8 is the **standard deviation principle**,

$$c = \mathrm{E}[S] + \Theta\sqrt{\mathrm{Var}[S]}, \quad \Theta > 0. \qquad (4.9)$$

Let us now try to price c by using one of the premium principles 4.8 and 4.9, to see whether we can find a $\Theta > 0$ and get the same ruin probability as in the Poisson case, calculated before. We will do it for the last and most extreme case when $\mu_1 = 1$ and $\delta = 1.25$, furthermore we choose the standard deviation principle since it is expressed in the same unit as the expected value.

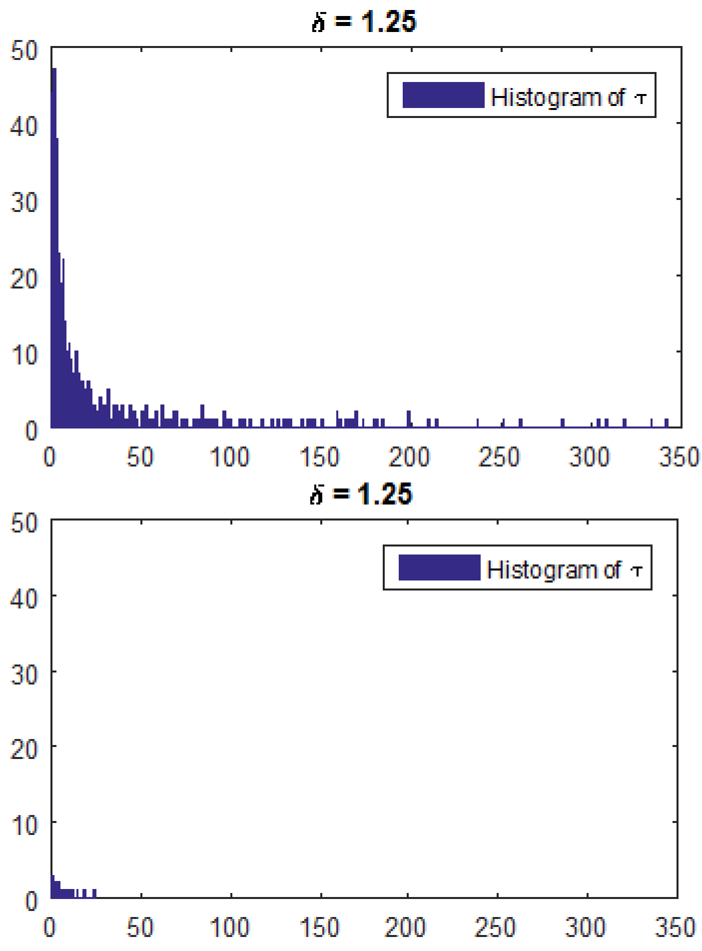

Figure 9: The upper figure is a histogram of τ when c is priced according to the expected value principle. The lower figure is a histogram of τ when c is priced according to the standard deviation principle with $\Theta = 1.25$.

In figure 9 we can see that the proportion of surplus processes that ended up in the state of ruin reduced remarkably when we priced c according to the standard deviation principle. The approximated ruin probability when c was priced according to the expected value principle was 0.51 now it is only 0.066.

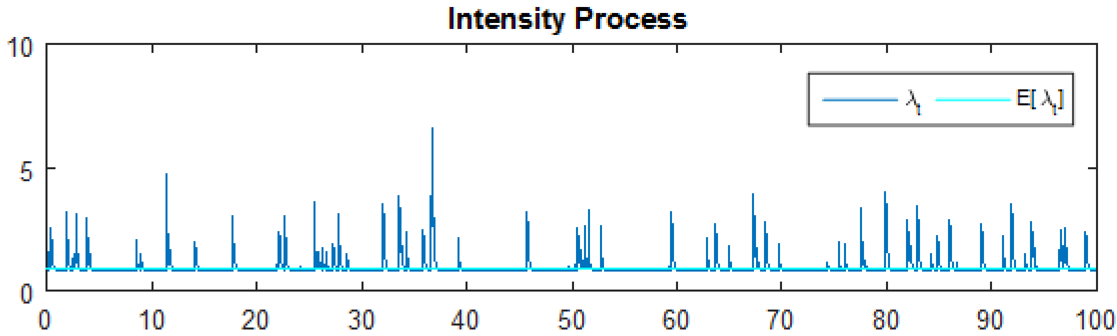

Figure 10: Intensity process λ_t for large δ.

Note that when δ is much larger than μ_1 the intensity process almost immediately return to its constant reversion level a after a claim is generated. This means that the self-exciting effect, which is characteristic for a Hawkes process, almost not exists. Therefore we do not consider this case. For an example how the intensity process can look like see figure 10.

5 Summary and Conclusion

For summary, we have defined a Hawkes process with exponential decay, and applied a simulation algorithm, that we have used in an application to insurance. We also defined the Cramér-Lundberg model that is an important tool for actuaries when pricing premiums. What is crucial is to price the premium such that the probability of bankruptcy is small, but not to an excessive price. The most popular formula that is used is the expected value principle, where the price is determined by the expected expenses. In the classical risk model one uses a homogeneous Poisson process for modelling claim arrivals, which is not optimal in reality. The idea in this thesis was to see how the surplus process is effected, when claims arrive according to a Hawkes process since it is able to capture clustering arrival of claims. This may happen after, for example, a natural disaster, like a tsunami, earthquake or flood has occurred.

By letting $\mathrm{E}[N_t]$ for the Hawkes process equal $\mathrm{E}[N_t]$ for the Poisson process we could change the values of μ_1 and δ, and at the same time hold the expected number of claims fixed. We started with the case when the Hawkes process could be seen as a Poisson process, then proceeded by increasing the values of μ_1 such that the Hawkes process became more apparent. In that way we could also see the effect of pricing c entirely by the expected expenses, since it was held fixed. Thereafter we wanted to examine whether the variance of the surplus process and the probability of ruin was growing in δ. We proved mathematically that the variance was growing in delta, which also was confirmed in the graphs. By simulations we also showed that the probability of ruin was increasing in δ. After this ascertainment we applied the standard deviation principle that price c, not only with respect to the expected expenses, but also with respect to how much the expenses can fluctuate. We did it in the last and most extreme case where we wanted to reduce the probability of ruin, such that it was approximately equal to the ruin probability in the Poisson case, which was successful.

The conclusion of this study is that if claims have tendency to arrive in clusters one should consider to use another model than the classical risk model, where claims arrive with constant intensity. The Hawkes process can be appropriate to use in this context since it capture this cluster arrival of events that may occur. It is also of note to take the variance of the expected expenses into consideration when pricing premiums, since the expected value principle not always is optimal. An appropriate choice of premium principle is of importance to keep the probability of bankruptcy small.

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