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U.U.D.M. Project Report 2015:19

d-cluster tilting modules over quotients of path algebras of type A_n

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Examensarbete i matematik, 30 hp
Handledare och examinator: Martin Herschend
Juni 2015

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Master Degree Project
Uppsala University
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June 9, 2015

Acknowledgements

I am grateful to my supervisor Martin Herschend, for his advice and suggestions concerning this thesis as well as his attention to detail which taught me much more than what is included here. I would also like to thank my professors Volodymyr Mazorchuk, Takis Konstantopoulos and Wulf Staubach for their enlightening and highly inspiring lectures.

Moreover my colleague's George Bliatsios helpful comments for my thesis and the mathematical conversations with my friend Emmanouil Androulakis are greatly appreciated. Last but not least, I would like to express my gratitude to my parents, Evgenia and Charalambos and my sister Christianna for their support throughout my studies.

Contents

1 Abstract

Let Λ be a quotient of the path algebra of a quiver of type A_n with linear orientation by an admissible ideal. We develop the representation theory of Λ. We find a necessary and sufficient condition for Λ to have a d-cluster tilting module in the case of d being equal to the global dimension of Λ.

2 Introduction

Let K be a field and Λ be a finite-dimensional K-algebra. Let $\text{mod}\Lambda$ be the category of finitely generated right Λ-modules. Representation theory aims to understand Λ by considering all indecomposable modules and the morphisms between them. In the case of a representation-finite finite-dimensional algebra Λ, Auslander-Reiten theory gives a complete picture of $\text{mod}\Lambda$. Osamu Iyama's higher dimensional Auslander-Reiten theory [1] aims to replace $\text{mod}\Lambda$ by a subcategory $\mathcal{C} \subseteq \text{mod}\Lambda$ with suitable homological properties. Such a subcategory is called an m-cluster tilting subcategory and if it is realized as $\mathcal{C} = \text{add}M$ for some $M \in \text{mod}\Lambda$, M is called an m-cluster tilting module. If such M exists, higher dimensional Auslander-Reiten theory gives analogous results to those of classical Auslander-Reiten theory for representation finite algebras. An important question is when does $\text{mod}\Lambda$ contain an m-cluster tilting module. The most elementary algebras in this theory are those for which m is equal to the global dimension d of Λ. These algebras are called *d-representation finite*.

The aim of this thesis to classify all quotients of path algebras of type A_n with linear orientation by admissible ideals that are d-representation finite. If $\Lambda_{n,I}$ is such a quotient by an admissible ideal I, we have the following result. The case $I = 0$ is equivalent to $d = 1$. Since $\text{mod}\Lambda_{n,I}$ is the unique 1-cluster tilting subcategory of $\text{mod}\Lambda_{n,I}$ and $\Lambda_{n,I}$ is representation finite, we get that $\Lambda_{n,I}$ is 1-representation finite in this case. Now assume that $I \neq 0$ and let I_l be the ideal containing all paths of length at least l. Then $\Lambda_{n,I}$ is d-representation finite if and only if $I = I_2$ or $I = I_l$ for some $l < n$ such that $n \equiv 1 \text{mod} l$ (Theorem 5.15). The cases $l = 2$ and $l = n-1$ were previously known ([1], Example 2.4(a) and [3], Theorem 3.12 respectively) but all other cases were, as far as we can tell, unkown.

3 Preliminaries

In the following, we assume that Λ is a finite-dimensional K-algebra where K is an algebraically closed field, all Λ-modules are right modules and $\text{mod}\Lambda$ denotes the category of finitely generated Λ-modules. We begin with a basic result about indecomposable modules.

Lemma 3.1. Suppose that X is a complete set of representatives of the isomorphism classes of indecomposable Λ-modules and that I is an ideal of Λ. Then the set

$$X_{\Lambda/I} = \{M \in X \mid MI = 0\}$$

is a complete set of representatives of the isomorphism classes of the indecomposable Λ/I-modules.

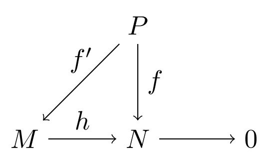
Proof. Let $M \in X_{\Lambda/I}$ and suppose towards a contradiction that M is not indecomposable as a Λ/I module. Then, there exist Λ/I-modules $M_1 \neq 0$ and $M_2 \neq 0$ such that $M = M_1 \oplus M_2$. Seeing M_1 and M_2 as Λ-modules, we have that $M = M_1 \oplus M_2$ and since M is indecomposable as a Λ module, either M_1 or M_2 is 0, a contradiction. Therefore, all modules in $X_{\Lambda/I}$ are indecomposable.

Let now N be an indecomposable non-zero Λ/I-module. If N as a Λ-module is decomposable, then it must be decomposable as a Λ/I-module too, a contradiction. So N must be indecomposable as a Λ-module. Therefore $N \cong N'$ for some $N' \in X$. Since N is a well defined Λ/I-module, we have that $NI = 0$ and therefore $N'I = 0$ which means that $N' \in X_{\Lambda/I}$. □

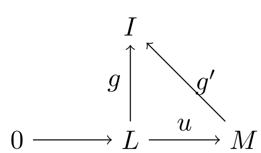
We recall some basic definitions from [4] and [6]:

Definition 3.2.

(a) A Λ-module P is *projective* if, for any epimorphism $h : M \to N$ and any $f \in \mathrm{Hom}_\Lambda(P, N)$ there exists an $f' \in \mathrm{Hom}_\Lambda(P, M)$ such that the following diagram is commutative



(b) A Λ-module I is *injective* if, for any monomorphism $u : L \to M$ and any $g \in \mathrm{Hom}_\Lambda(L, I)$ there exists a $g' \in \mathrm{Hom}_\Lambda(M, I)$ such that the following diagram is commutative



Definition 3.3. A sequence

$$\cdots \to X_{n-1} \overset{h_{n-1}}{\to} X_n \overset{h_n}{\to} X_{n+1} \overset{h_{n+1}}{\to} X_{n+2} \to \cdots$$

of right Λ-modules connected by Λ-module morphisms is called a *complex* if $h_n \circ h_{n-1} = 0$ for any n and *exact* if $\mathrm{Ker} h_n = \mathrm{Im} h_{n-1}$ for any n. In particular

$$0 \to L \overset{u}{\to} M \overset{r}{\to} N \to 0$$

is called a *short exact sequence* if u is a monomorphism, r is an epimorphism and $\mathrm{Ker} r = \mathrm{Im} u$.

Definition 3.4.

(a) A *projective resolution* of a Λ-module M is a complex

$$P_\bullet : \quad \cdots \longrightarrow P_m \xrightarrow{h_m} P_{m-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{h_1} P_0 \longrightarrow 0$$

of projective Λ-modules together with an epimorphism $h_0 : P_0 \xrightarrow{h_0} M$ of Λ-modules such that the sequence

$$\cdots \longrightarrow P_m \xrightarrow{h_m} P_{m-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{h_1} P_0 \xrightarrow{h_0} M \longrightarrow 0$$

is exact. For simplicity, we call the above sequence a projective resolution of M.

(b) An *injective resolution* of a Λ-module M is a complex

$$I^\bullet : \quad 0 \longrightarrow I^0 \xrightarrow{d^1} I^1 \longrightarrow \cdots \longrightarrow I^m \xrightarrow{d^{m+1}} I^{m+1} \longrightarrow \cdots$$

of injective Λ-modules together with a monomorphism $d^0 : M \longrightarrow I^0$ of Λ-modules such that the sequence

$$0 \longrightarrow M \xrightarrow{d^0} I^0 \xrightarrow{d^1} I^1 \longrightarrow \cdots \longrightarrow I^m \xrightarrow{d^{m+1}} I^{m+1} \longrightarrow \cdots$$

is exact. For simplicity, we call the above sequence an injective resolution of M.

Definition 3.5.

(a) Given a projective resolution in mod$A$,

$$P_\bullet : \cdots \to P_n \xrightarrow{h_n} h_{n-1} \to \cdots \to P_1 \xrightarrow{h_1} P_0 \xrightarrow{h_0} M \to 0,$$

define $K_n = \ker h_{n-1}$ for $n \geq 1$. We call K_n the *n-th syzygy* of $P_\bullet$.

(b) Given an injective resolution in mod$A$,

$$I^\bullet = 0 \to M \xrightarrow{d^0} I^0 \xrightarrow{d^1} I^1 \to \cdots \to I^n \xrightarrow{d^{n+1}} I^{n+1} \to \cdots ,$$

define $V^n = \text{coker} d^{n-1}$ for $n \geq 1$. We call V^n the *n-th cosyzygy* of $I^\bullet$.

Definition 3.6.

(a) Two modules M and N are *projectively equivalent* if there exist projective modules P and P' with $M \oplus P \cong N \oplus P'$.

(b) Two modules M and N are *injectively equivalent* if there exist injective modules I and I' with $M \oplus I \cong N \oplus I'$.

The following proposition is a basic result.

Proposition 3.7.

(a) Let $(K_n)_{n\geq 1}$ and $(K'_n)_{n\geq 1}$ be syzygies of a Λ-module M defined by two projective resolutions of M. Then, for each $n \geq 1$, K_n and K'_n are projectively equivalent.

(b) Let $(V^n)_{n\geq 1}$ and $(V^{n'})_{n\geq 1}$ be cosyzygies of a Λ-module N defined by two injective resolutions of N. Then, for each $n \geq 1$, V^n and $V^{n'}$ are injectively equivalent.

Proof.

(a) See [6], page 455.

(b) See [6], page 458.

□

This equivalence between the syzygies (respectively cosyzygies) of different projective (respecively injective) resolutions of modules allows us to speak of *the n-th syzygy* (respectively the n-th cosyzygy) of a module M, even though it is defined only up to projective (respectively injective) equivalence. We will denote the n-th syzygy of a Λ-module M by $\Omega^n M$ and we will denote the n-th cosyzygy of M by $\Omega^{-n} M$.

Definition 3.8.

(a) The *projective dimension* of a Λ-module M is the nonnegative integer $\text{pd}M = m$ such that there exists a projective resolution

$$0 \longrightarrow P_m \xrightarrow{h_m} P_{m-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{h_1} P_0 \xrightarrow{h_0} M \longrightarrow 0$$

of M of length m and M has no projective resolution of length $m-1$, if such a number m exists. If M admits no projective resolution of finite length, we define the projective dimension $\text{pd}M$ of M to be infinity.

(b) The *injective dimension* of a Λ-module N is the nonnegative integer $\text{id}N = m$ such that there exists an injective resolution

$$0 \longrightarrow N \xrightarrow{h^0} I^0 \xrightarrow{h^1} I^1 \longrightarrow \cdots \longrightarrow I^{m-1} \xrightarrow{h^m} I^m \longrightarrow 0$$

of N of length m and N has no injective resolution of length $m-1$, if such a number m exists. If N admits no injective resolution of finite length, we define the injective dimension $\text{id}N$ of N to be infinity.

Remark 3.9. If $M \in \text{mod}\Lambda$ is projective, then $\text{pd}M = 0$. Similarly, if N is injective, $\text{id}N = 0$.

Proposition 3.10. Let $M \in \text{mod}\Lambda$ and suppose that M has finite projective dimension. Then

$$\text{pd}(\Omega^k M) = \begin{cases} \text{pd}M - k & \text{if } k \leq \text{pd}M \\ 0 & \text{else} \end{cases}$$

Specifically, for $k = \text{pd}M - 1$ we have that $\text{pd}\left(\Omega^{\text{pd}M-1}M\right) = 1$.

Proof. Let $m := \text{pd}M$. It is enough to prove that $\Omega^k M$ has a projective resolution of length $m - k$ and no shorter one. But this is immediate since if

$$0 \longrightarrow P_m \xrightarrow{h_m} P_{m-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{h_1} P_0 \xrightarrow{h_0} M \longrightarrow 0$$

is a projective resolution of M, then

$$0 \longrightarrow P_m \xrightarrow{h_m} P_{m-1} \longrightarrow \cdots \longrightarrow P_{k+2} \xrightarrow{h_{k+1}} P_{k+1} \xrightarrow{h_k} \Omega^k M \longrightarrow 0$$

is a projective resolution of $\Omega^k M$ of length $m - k$. Moreover, if there exists a projective resolution of $\Omega^k M$ of length $r < m - k$

$$0 \longrightarrow P'_r \xrightarrow{h'_r} P'_{r-1} \longrightarrow \cdots \longrightarrow P'_1 \xrightarrow{h'_1} P'_0 \xrightarrow{h'_0} \Omega^k M \longrightarrow 0$$

then we have a projective resolution of length $r + k < m$ of M, namely

$$0 \longrightarrow P'_r \xrightarrow{h'_r} P'_{r-1} \longrightarrow \cdots \longrightarrow P'_1 \xrightarrow{h'_1} P'_0 \xrightarrow{h'_0} P_{k-1} \xrightarrow{h_{k-1}} \cdots P_1 \xrightarrow{h_1} P_0 \xrightarrow{h_0} M \longrightarrow 0$$

since $\Omega^k M \subset P_{k-1}$, which contradicts the fact that $\text{pd}M = m$. $\square$

Definition 3.11. The *global dimension* of Λ is defined to be

$$\text{gl.dim}\Lambda = \sup\{\text{pd}M : M \text{ is a right } \Lambda\text{-module}\}$$

where infinity is considered larger than any nonnegative integer.

In the following, we will denote the global dimension of Λ as d_Λ or simply d if Λ is clear from the context. We now define the Ext^i_Λ functor, following [5]:

Definition 3.12. For each Λ-module X, the functor $F(B) = \text{Hom}_\Lambda(M, N)$ is left exact. Its right derived functors are called the *Ext groups*:

$$\text{Ext}^i_\Lambda(M, N) = R^i\text{Hom}_\Lambda(M, -)(N).$$

In particular, $\text{Ext}^0_\Lambda(M, N)$ is $\text{Hom}_\Lambda(M, N)$.

Remark 3.13. From now on we may write $\text{Hom}(M, N)$ instead of $\text{Hom}_\Lambda(M, N)$ when Λ is clear from context. Similarly, we will write $\text{Ext}^i(M, N)$ instead of $\text{Ext}^i_\Lambda(M, N)$. To calculate $\text{Ext}^i(M, N)$ we have two ways:

- Given that

$$\cdots \longrightarrow P_m \xrightarrow{h_m} P_{m-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{h_1} P_0 \xrightarrow{h_0} M \longrightarrow 0$$

is a projective resolution of M we compute

$$0 \longrightarrow \mathrm{Hom}(P_0, N) \xrightarrow{\_\circ h_1} \mathrm{Hom}(P_1, N) \longrightarrow \cdots \longrightarrow \mathrm{Hom}(P_{m-1}, N)$$

$$\xrightarrow{\_\circ h_m} \mathrm{Hom}(P_m, N) \longrightarrow \ldots .$$

Then, $\mathrm{Ext}^0(M,N) = \mathrm{Hom}(M,N)$ and $\mathrm{Ext}^i(M,N)$ for $i \geq 1$ is the homology of this complex, that is

$$\mathrm{Ext}^i(M,N) = \mathrm{Ker}(\_\circ h_{i+1})/\mathrm{Im}(\_\circ h_i)$$

- Given that

$$0 \longrightarrow N \xrightarrow{h^0} I^0 \xrightarrow{h^1} I^1 \longrightarrow \cdots \longrightarrow I^m \xrightarrow{h^{m+1}} I^{m+1} \longrightarrow \cdots$$

is an injective resolution of N we compute

$$0 \longrightarrow \mathrm{Hom}(M, I^0) \xrightarrow{h^1\circ\_} \cdots \longrightarrow \mathrm{Hom}(M, I^m) \xrightarrow{h^{m+1}\circ\_} \mathrm{Hom}(M, I^{m+1}) \longrightarrow \ldots .$$

Then, $\mathrm{Ext}^0(M,N) = \mathrm{Hom}(M,N)$ and $\mathrm{Ext}^i(M,N)$ for $i \geq 1$ is the homology of this complex, that is

$$\mathrm{Ext}^i(M,N) = \mathrm{Ker}(h^{i+1}\circ\_)/\mathrm{Im}(h^i\circ\_)$$

With this, we can easily see that if P is a projective Λ-module, then $\mathrm{Ext}^i(P,N) = 0$ for all Λ-modules N and for all $i \geq 1$. Similarly, if I is an injective Λ-module, then $\mathrm{Ext}^i(M,I) = 0$ for all Λ-modules M and for all $i \geq 1$.

By the above and the definition of projective and injective dimension we have the following result.

Lemma 3.14.

(a) Let $M \in \mathrm{mod}\Lambda$ and suppose that $\mathrm{pd}M = n$. Then there exists a projective module $P \in \mathrm{mod}\Lambda$ such that $\mathrm{Ext}^n(M,P) \neq 0$.

(b) Let $M \in \mathrm{mod}\Lambda$ and suppose that $\mathrm{id}M = n$. Then there exists an injective module $I \in \mathrm{mod}\Lambda$ such that $\mathrm{Ext}^n(I,M) \neq 0$.

Proof.

(b) Suppose that $M \in \mathrm{mod}\Lambda$ and $\mathrm{id}M = n$. Then there exists an injective resolution

$$0 \longrightarrow M \xrightarrow{h^0} I^0 \xrightarrow{h^1} I^1 \longrightarrow \cdots \longrightarrow I^{n-1} \xrightarrow{h^n} I^n \longrightarrow 0.$$

We claim that $\mathrm{Ext}^n(I^n, M) \neq 0$. After applying $\mathrm{Hom}(I^n, -)$ we have

$$0 \longrightarrow \mathrm{Hom}(I^n, I^0) \xrightarrow{h^1 \circ \_} \cdots \longrightarrow \mathrm{Hom}(I^n, I^{n-1}) \xrightarrow{h^n \circ \_} \mathrm{Hom}(I^n, I^n) \longrightarrow 0$$

and $\mathrm{Ext}^n(I^n, M) = \mathrm{Hom}(I^n, I^n)/\mathrm{Im}(h^n \circ \_)$. Then we have

$$\mathrm{Ext}^n(I^n, M) = 0 \Leftrightarrow \mathrm{Im}(h^n \circ \_) = \mathrm{Hom}(I^n, I^n) \Leftrightarrow h^n \circ \_ \text{ is surjective.}$$

Suppose towards a contradiction that $h^n \circ \_$ is indeed surjective. Then there exists $f \in \mathrm{Hom}(I^n, I^{n-1})$ such that $h^n \circ f = \mathrm{id}_{I^n}$. Moreover, $f \circ h^n \in \mathrm{End}(I^{n-1})$ and therefore, if we set $e = f \circ h^n$, we have

$$I^{n-1} = \mathrm{Im}e \oplus \mathrm{Ker}e$$

and since I^{n-1} is injective, $\mathrm{Ker}e$ is injective too. Moreover

$$e \circ h^{n-1} = (f \circ h^n) \circ h^{n-1} = f \circ (h^n \circ h^{n-1}) = f \circ 0 = 0$$

since $h^n \circ h^{n-1} = 0$ because the projective resolution is exact. This implies that $\mathrm{Im}(h^{n-1}) \subseteq \mathrm{Ker}e$ and therefore

$$0 \longrightarrow M \xrightarrow{h^0} I^0 \xrightarrow{h^1} I^1 \longrightarrow \cdots \longrightarrow I^{n-2} \xrightarrow{h^{n-1}} \mathrm{Ker}e \longrightarrow 0 \longrightarrow 0$$

is an injective resolution of M of length $n-1$ which contradicts the assumption that $\mathrm{id}M = n$. Thus, $h^n \circ \_$ is not surjective, which means that $\mathrm{Ext}^n(I^n, M) \neq 0$ as claimed.

(a) The proof is similar to (b).

□

Definition 3.15. Let $L, M \in \mathrm{mod}\Lambda$. We call a Λ-module morphism $f : L \to M$.

(a) a *section* if there exists a Λ-module morphism $r : M \to L$ such that $r \circ f = 1_L$.

(b) *left almost split* if

  (i) f is not a section and

  (ii) for every Λ-module morphism $u : L \to U$ that is not a section there exists $u' : M \to U$ such that $u' \circ f = u$, that is u' makes the following triangle commutative

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ {\scriptstyle u}\downarrow & \swarrow{\scriptstyle u'} & \\ U & & \end{array}$$

(c) *left minimal* if every $h \in \mathrm{End}M$ such that $h \circ f = f$ is an automorphism.

(d) *left minimal almost split* if it is both left minimal and left almost split.

Dually, we define the following notions.

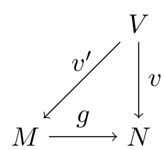
Definition 3.16. Let $M, N \in \text{mod}\Lambda$. We call a Λ-module morphism $g : M \to N$.

(a) a *retraction* if there exists a Λ-module morphism $r : N \to M$ such that $g \circ r = 1_N$.

(b) *right almost split* if

  (i) g is not a retraction and

  (ii) for every Λ-module morphism $v : V \to N$ that is not a retraction, there exists $v' : V \to M$ such that $g \circ v' = v$, that is v' makes the following triangle commutative



(c) *right minimal* if every $k \in \text{End}M$ such that $g \circ k = g$ is an automorphism.

(d) *right minimal almost split* if it is both right minimal and right almost split.

Definition 3.17. A homomorphism $f : M \to N$ in $\text{mod}\Lambda$ is said to be *irreducible* provided:

(a) f is neither a section nor a retraction and

(b) if $f = f_1 \circ f_2$, either f_1 is a retraction or f_2 is a section.

Lemma 3.18. If f is irreducible, then f is either a proper monomorphism or a proper epimorphism.

Proof. See [4], page 100. □

Definition 3.19. Let M, N be indecomposable Λ-modules. Then $\text{rad}_\Lambda(M, N)$ is the K-vector space of all $f \in \text{Hom}(M, N)$ such that f is not an isomorphism. Moreover, $\text{rad}^2_\Lambda(M, N)$ is spanned by all Λ-module morphisms $gf : M \to N$ such that $f \in \text{rad}_\Lambda(M, L)$ and $g \in \text{rad}_\Lambda(L, N)$ for some indecomposable $L \in \text{mod}\Lambda$.

Lemma 3.20. Let M, N be indecomposable Λ-modules. Then $f : M \to N$ is irreducible if and only if $f \in \text{rad}_\Lambda(M, N) \setminus \text{rad}^2_\Lambda(M, N)$.

Proof. See [4], page 101. □

Lemma 3.20 motivates the following definition.

Definition 3.21. Let M, N be indecomposable Λ-modules. We call the quotient of K-vector spaces $\mathrm{rad}_\Lambda(M, N)/\mathrm{rad}^2_\Lambda(M, N)$ *the space of irreducible morphisms between* M *and* N and we denote it by $\mathrm{Irr}(M, N)$.

Definition 3.22. A short exact sequence in modΛ

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$$

is called an *almost split sequence* provided:

(a) f is left minimal almost split and

(b) g is right minimal almost split.

Almost split sequences have several equivalent characterisations. We only mention one that we will need.

Theorem 3.23. Let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be a short exact sequence in modΛ. Then the given sequence is almost split if and only if L and N are indecomposable and f and g are irreducible.

Proof. See [4], page 105. □

Definition 3.24. Let Λ be a K-algebra. The *opposite algebra* Λ^{op} of Λ is the K-algebra with the same vector space structure as Λ and multiplication defined by $a *_{\Lambda^{\mathrm{op}}} b = b *_\Lambda a$ for all $a, b \in \Lambda^{\mathrm{op}}$.

Definition 3.25. We define the functor

$$D : \mathrm{mod}\Lambda \to \mathrm{mod}\Lambda^{\mathrm{op}}$$

as follows:

- We assign to each right Λ-module $M \in \mathrm{mod}\Lambda$ the left Λ-module defined by $D(M) = \mathrm{Hom}_K(M, K)$ as a vector space and endowed with the the left Λ-module structure given by $(a\phi)(m) = \phi(ma)$ for $\phi \in \mathrm{Hom}_K(M, K)$, $a \in \Lambda$ and $m, \in M$.
- We assign to each morphism of right Λ-modules $h : M \to N$ the morphism $D(h) : D(M) \to D(N)$ of left Λ-modules defined by $\phi \mapsto \phi \circ h \in D(M) = \mathrm{Hom}_K(M, K)$ for all $\phi \in D(N) = \mathrm{Hom}_K(N, K)$.

One can show that D is a duality of categories and we call D the *standard* K*-duality.*

For further details on D we refer to [4], page 12.

Definition 3.26. Let P be a projective Λ-module and $M, N \in \mathrm{mod}\Lambda$. Let $h : P \to M$ a Λ-module epimorphism. If for any Λ-module morphism $g : N \to P$ we have that when hg is surjective, g is also surjective, then we say that h is a *projective cover of* M.

Consider now the Λ-dual functor

$$F = \mathrm{Hom}_\Lambda(-, \Lambda) : \mathrm{mod}\Lambda \to \mathrm{mod}\Lambda^{\mathrm{op}}$$

and suppose that we have an exact sequence

$$P_1 \stackrel{p_1}{\to} P_0 \stackrel{p_0}{\to} M \to 0$$

such that $p_0 : P_0 \to M$ and $p_1 : P_1 \to \mathrm{Ker}p_0$ are projective covers. By applying the functor F we obtain an exact sequence of left Λ-modules

$$0 \to F(M) \stackrel{F(p_0)}{\to} F(P_0) \stackrel{F(p_1)}{\to} F(P_1) \to \mathrm{Coker}F(p_1) \to 0.$$

We denote $\mathrm{Coker}F(p_1)$ by $\mathrm{Tr}M$ and call it the *transpose of M*. For further details on Tr we refer to [4], page 107.

Definition 3.27. The *Auslander-Reiten translations* are defined to be the compositions of D with Tr, namely we set $\tau = D\mathrm{Tr}$ and $\tau^{-1} = \mathrm{Tr}D$.

Lemma 3.28. Let $0 \to L \to M \to N \to 0$ and $0 \to L' \to M' \to N' \to 0$ be two almost split sequences in $\mathrm{mod}\Lambda$. Then the following are equivalent:

(a) The two sequences are isomorphic.

(b) There is an isomorphism $L \cong L'$ of Λ-modules.

(c) There is an isomorphism $N \cong N'$ of Λ-modules.

Proof. See [4], page 105. □

The existence of almost split sequences and their connection to the Auslander-Reiten translations is evident by the following theorem.

Theorem 3.29.

(a) For any indecomposable nonprojective Λ-module M, there exists an almost split sequence $0 \to \tau M \to E \to M \to 0$ in $\mathrm{mod}\Lambda$.

(b) For any indecomposable noninjective Λ-module N, there exists an almost split sequence $0 \to N \to F \to \tau^{-1}N \to 0$ in $\mathrm{mod}\Lambda$.

Proof. See [4], page 120. □

Proposition 3.30. Let $M \in \mathrm{mod}\Lambda$. Then

1. $\tau M = 0$ if and only if M is projective.

2. $\tau^{-1}M = 0$ if and only if M is injective.

Proof. See [4], page 116. □

Moreover, we have the following lemma which links almost split sequences with Ext^1.

Lemma 3.31. Suppose that $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is an almost split sequence in modΛ. Then $\mathrm{Ext}^1(N, L) \neq 0$.

Proof. See [4], pages 434 and 435. □

For $M \in \mathrm{mod}\Lambda$, we denote by add$M$ the subcategory of modΛ consisting of all modules isomorphic to direct summands of finite direct sums of copies of M. With this we can define the notions of m-cluster tilting subcategory and m-cluster tilting module.

Definition 3.32. We call a module $M \in \mathrm{mod}\Lambda$ an *m-cluster tilting module* if for the subcategory $\mathrm{add}M = \mathcal{C}$ of modΛ we have

$$\mathcal{C} = \{X \in \mathrm{mod}\Lambda \mid \mathrm{Ext}^i_\Lambda(\mathcal{C}, X) = 0 \text{ for } 0 < i < m\}$$

$$= \{X \in \mathrm{mod}\Lambda \mid \mathrm{Ext}^i_\Lambda(X, \mathcal{C}) = 0 \text{ for } 0 < i < m\}.$$

In this case, we call $\mathcal{C}$ an *m-cluster tilting subcategory.*

We will denote

$$\mathcal{C}^{\perp_m} := \{X \in \mathrm{mod}\Lambda \mid \mathrm{Ext}^i_\Lambda(\mathcal{C}, X) = 0 \text{ for } 0 < i < m\}$$

$$^{\perp_m}\mathcal{C} := \{X \in \mathrm{mod}\Lambda \mid \mathrm{Ext}^i_\Lambda(X, \mathcal{C}) = 0 \text{ for } 0 < i < m\}.$$

Then, M is an m-cluster tilting module if and only if $\mathrm{add}M = \mathrm{add}M^{\perp_m} = {}^{\perp_m}\mathrm{add}M$. A basic necessary condition for a module to be an m-cluster tilting module is the following.

Proposition 3.33. Let $M \in \mathrm{mod}\Lambda$ be an m-cluster tilting module. Then add$M$ contains all projective and all injective Λ-modules.

Proof. Denote $\mathcal{C} = \mathrm{add}M$ and let P be a projective Λ-module. Then $\mathrm{Ext}^i(P, X) = 0$ for all $X \in \mathrm{mod}\Lambda$ and all $0 < i < m$. Therefore, $\mathrm{Ext}^i(P, \mathcal{C}) = 0$ for all $0 < i < m$ which implies that $P \in {}^{\perp_m}\mathcal{C}$. Similarly, if I is injective, $\mathrm{Ext}^i(\mathcal{C}, I) = 0$ for all $0 < i < m$, and therefore $I \in \mathcal{C}^{\perp_m}$. Since $\mathcal{C}^{\perp_m} = {}^{\perp_m}\mathcal{C} = \mathcal{C}$ we have that $P, I \in \mathcal{C} = \mathrm{add}M$. Since P and I were arbitrary, we have the required result. □

The existence of an m-cluster tilting module is especially important in the case of $m = d := \mathrm{gl.dim}\Lambda$. In particular, in such a case, if a d-cluster tilting module exists, it is unique (see [2], Theorem 1.6).

Definition 3.34. Λ is called *d-representation finite* if $\mathrm{gl.dim}\Lambda = d$ and there exists a d-cluster tilting Λ-module.

A necessary condition for Λ to be d-representation finite is given by the following result.

Lemma 3.35.

(a) If there exists an injective module $I \in \mathrm{mod}\Lambda$ such that $0 < \mathrm{pd}I < \mathrm{gl.dim}\Lambda$, then there exists no d-cluster tilting module in modΛ.

(b) If there exists a projective module $P \in \text{mod}\Lambda$ such that $0 < \text{id}P < \text{gl.dim}\Lambda$, then there exists no d-cluster tilting module in $\text{mod}\Lambda$.

Proof.

(a) Suppose that I is an injective Λ-module such that $0 < \text{pd}I < \text{gl.dim}\Lambda$ and there exists a d-cluster tilting module $M \in \text{mod}\Lambda$. Then, since I is injective, by Proposition 3.33 we have that $I \in \text{add}M$. Let $m = \text{pd}I$. Then, by Lemma 3.14, we have that there exists a projective Λ-module P such that $\text{Ext}^m(I, P) \neq 0$. Again by Proposition 3.33 we have that $P \in \text{add}M$, but since $\text{Ext}^m(I, P) \neq 0$ and $m < d$, we have

$$I \notin {}^{\perp_m}\{P\} \overset{P\in\text{add}M}{\supseteq} {}^{\perp_m}\text{add}M \overset{M\, d\text{-CT}}{=} \text{add}M \Rightarrow I \notin \text{add}M$$

which contradicts $I \in \text{add}M$.

(b) Similar with (a).

□

The last tool which we will use is the m-Auslander-Reiten translations, defined as in [1].

Definition 3.36. The *m-Auslander-Reiten translations* τ_m and τ_m^- are defined by $\tau_m X := \tau\Omega^{m-1}X$ and $\tau_m^- X = \tau^-\Omega^{-(m-1)}X$ for all $X \in \text{mod}\Lambda$.

We note that τ_m and τ_m^- are well defined, since if K_{m-1} and K'_{m-1} are two $(m-1)$-th syzygies of X, then by Proposition 3.7 there exist P and P' projective such that $K_{m-1} \oplus P \cong K'_{m-1} \oplus P'$, and by applying τ we have

$$\tau(K_{m-1} \oplus P) \cong \tau(K'_{m-1} \oplus P') \Rightarrow \tau K_{m-1} \oplus \tau P \cong \tau K'_{m-1} \oplus \tau P'$$

$$\overset{\tau P = \tau P' = 0 \text{ by Prop. 3.30}}{\Rightarrow} \tau K_{m-1} \cong \tau K'_{m-1}$$

and therefore, τ_m is independent of the choice of the $(m-1)$-th syzygy (and similarly for τ_m^-).

For the following result we refer to [1] (Theorem 2.8).

Proposition 3.37. Let $\mathcal{C}$ be an m-cluster tilting subcategory of $\text{mod}A$. Let $\mathcal{C}_P$ and $\mathcal{C}_I$ be the sets of isomorphism classes of indecomposable non-projective respectively non-injective Λ-modules in $\mathcal{C}$. Then τ_m and τ_m^- induce mutually inverse bijections

$$\mathcal{C}_P \underset{\tau_m^-}{\overset{\tau_m}{\rightleftarrows}} \mathcal{C}_I \quad .$$

4 Representation theory of KQ_n/I

A general question is when does modΛ have an m-cluster tilting subcategory. In this thesis we restrict ourselves in the special case of $\Lambda = KQ_n/I$, where KQ_n is the path algebra of the quiver

$$Q_n : \quad 1 \xrightarrow{a_1} 2 \xrightarrow{a_2} 3 \xrightarrow{a_3} \cdots \xrightarrow{a_{n-2}} n-1 \xrightarrow{a_{n-1}} n$$

and I an admissible ideal of KQ_n. Moreover, we further restrict to the case $m = d$, where d is the global dimension of Λ. For basic notions of quivers and their representations we refer to [4]. Since there is at most one path in Q_n from i to j for all $1 \leq i, j \leq n$, the ideal I is generated by a set of paths. Moreover, since I is admissible, each path has length at least 2. Choosing an admissible ideal I thus corresponds to choosing such paths in Q_n. We consider the algebra $A_{n,I} = KQ_n/I$. To simplify notation, we will denote $A_{n,I} = A$ and only include the dependence on n and I when needed. In this chapter, we develop the representation theory of A.

4.1 Indecomposable A-modules

We introduce some notation to distinguish the indecomposables, projective and injective A-modules. Let $M_{m,k}$ denote the representation of Q_n defined as follows

$$M_{m,k} : \underset{1}{0} \to \underset{2}{0} \to \cdots \to \underset{n-m+1-k}{0} \to \underset{n-m+2-k}{K} \to \underset{n-m+3-k}{K} \to \cdots \to \underset{n-m+1}{K} \to \underset{n-m+2}{0} \to \cdots \to \underset{n}{0}$$

where in the above:

1. We have exactly n vertices and $n-1$ arrows.
2. We have an appearance of exactly k K's.
3. We have that the last K appears in the position $n-m+1$.
4. Every arrow is the identity map, if it is possible. Otherwise, it is the zero map.

Note that from the above we also get

5. The first K appears in the position $n-m+(2-k)$.

Gabriel's theorem (see [4], page 291) asserts that KQ_n is representation finite. Moreover, it gives us a complete list of isomorphism classes of the indecomposable KQ_n-modules: a KQ_n-module M is indecomposable if and only if it is isomorphic to $M_{m,k}$ for some m, k. The following Lemma classifies all the indecomposable A-modules, using this fact.

Lemma 4.1. Let $A = KQ_n/I$. The set

$$S_A = \{M_{m,k} | 1 \leq m, k \leq n, m + k \leq n + 1, a_{n-(m+k-1)+1} \cdots a_{n-m} \notin I\}$$

is a complete set of representatives of the isomorphism classes of the indecomposable A-modules.

Proof. The result is immediate by using the above classification of indecomposable KQ_n-modules and applying Lemma 3.1. □

In the following we may write interchangibly $M_{m,k} = (m, k)$. Moreover we add the following notation:

- $\mathcal{P}_A$ for the class of all projective A-modules $M_{m,k}$ (we write $\mathcal{P}$ if A is clear from context).
- $\mathcal{I}_A$ for the class of all injective A-modules $M_{m,k}$ (we write $\mathcal{I}$ if A is clear from context).
- $\mathcal{D}_k$ for the set of all modules $M_{i,j}$ such that $i + j = k$.

4.2 The Auslander-Reiten quiver of KQ_n/I

An important tool on the investigation of the representations of A-modules is the Auslander-Reiten quiver $\Gamma(\text{mod}A)$. First we define the Auslander-Reiten quiver, as in [4].

Definition 4.2. Let A be a finite dimensional K-algebra. The quiver $\Gamma(\text{mod}A)$ is defined as follows:

(a) The vertices of $\Gamma(\text{mod}A)$ are the isomorphism classes $[X]$ of indecomposable modules X in mod$A$.

(b) Let $[M], [N]$ be the vertices in $\Gamma(\text{mod}A)$ corresponding to the indecomposable modules M, N in mod$A$. The arrows $[M] \to [N]$ are in bijective correspondence with the vectors of a basis of the K-vector space $\text{Irr}(M, N)$.

The quiver $\Gamma(\text{mod}A)$ of the module category mod$A$ is called the *Auslander-Reiten quiver* of A.

We will now construct $\Gamma(\text{mod}KQ_n)$. We know that the points of $\Gamma(\text{mod}KQ_n)$, being the isomorphism classes of the indecomposable KQ_n-modules, can be represented by the set

$$S_{KQ_n} = \{M_{m,k} | 1 \leq m, k \leq n, m + k \leq n + 1\}$$

so we have

$$\Gamma(\text{mod}KQ_n)_0 = S_{KQ_n}.$$

Since with our notation we can write $M_{m,k} = (m, k)$, this means that the vertices of $\Gamma(\text{mod}KQ_n)$ can be represented by the set of all $(m, k) \in \mathbb{Z}^2_{>0}$ with $m + k \leq n + 1$. In total, we have $1 + 2 + ... + (n + 1) = \frac{(n+1)(n+2)}{2}$ points.

The arrows of $\Gamma(\mathrm{mod}KQ_n)$ between $(m_1, k_2) := M^1$ and $(m_2, k_2) := M^2$ correspond to the vectors in the basis of $\mathrm{Irr}(M^1, M^2)$. We claim that

$$\dim\left(\mathrm{Irr}(M_{m_1,k_1}, M_{m_2,k_2})\right) = \begin{cases} 1 & \text{if } (m_1 = m_2, k_1 = k_2 - 1) \text{ or } (m_1 = m_2 - 1, k_1 = k_2 + 1) \\ 0 & \text{otherwise} \end{cases}.$$

To prove this, note first that for $\mathrm{Hom}(M^1, M^2)$ to be non-zero, it must be that the first and the last K in the representation of M^2 appear before or at the same position as the first K and the last K in the representation of M^1. This translates to $n - m_1 + (2 - k_1) \geq n - m_2 + (2 - k_2)$ and $n - m_1 + 1 \geq n - m_2 + 1$ or $m_2 + k_2 \geq m_1 + k_1$ and $m_2 \geq m_1$. Indeed, in the other cases we have one of the following diagrams

$$\begin{array}{ccccccccccccccccc}
0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & K & \longrightarrow & \cdots & \longrightarrow & K & \longrightarrow & K & \longrightarrow & \cdots & \longrightarrow & K & \longrightarrow & \cdots \\
\downarrow & & & & \downarrow & & \downarrow & & & & \downarrow & \Delta & \downarrow \phi & & & & \downarrow & & \\
0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & K & \longrightarrow & \cdots & \longrightarrow & K & \longrightarrow & \cdots
\end{array}$$

or

$$\begin{array}{ccccccccccccccccc}
\cdots & \longrightarrow & K & \longrightarrow & \cdots & \longrightarrow & K & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 \\
& & \downarrow & & & & \phi \downarrow & \Delta & \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow \\
\cdots & \longrightarrow & K & \longrightarrow & \cdots & \longrightarrow & K & \longrightarrow & K & \longrightarrow & \cdots & \longrightarrow & K & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0
\end{array}$$

and in both cases, for the squares at Δ to be commutative, ϕ has to be zero, which forces all other maps between the representations to be zero. So in this case $\mathrm{Hom}(M^1, M^2) = 0$. Therefore, the only case where a non-zero homomorphism between M^1 and M^2 appears, is when we are in the case

$$\begin{array}{ccccccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & K & \longrightarrow & \cdots & \longrightarrow & K & \longrightarrow & K & \longrightarrow & \cdots \\
& & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & K & \longrightarrow & \cdots & \longrightarrow & K & \longrightarrow & K & \longrightarrow & \cdots & \longrightarrow & K & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

where for every square

$$\begin{array}{ccc}
K & \xrightarrow{1} & K \\
\phi_1 \downarrow & & \downarrow \phi_2 \\
K & \xrightarrow[1]{} & K
\end{array}$$

we must have $\phi_1 = \phi_2$, for it to be commutative. Moreover, another obvious restriction is that the last K of the second representation must appear at least at the same position as the first K of the first representation, otherwise the only map between them is the 0 map. Therefore, we must also have $n - m_2 + 1 \geq n - m_1 + (2 - k_1)$ or $m_1 + k_1 \geq m_2 + 1$.

Given that the homomorphisms $K \to K$ are all multiples of the identity, this means that a basis of the space $\mathrm{Hom}(M^1, M^2)$ is the map $\mathcal{F}$, where $\mathcal{F} : K \to K$

is the identity and $\mathcal{F} = 0$ otherwise. Therefore, $\dim(\mathrm{Irr}(M^1, M^2)) = 1$ if and only if $\mathcal{F} : M^1 \to M^2$ is irreducible and otherwise, $\mathrm{Irr}(M^1, M^2) = 0$. We consider the three cases, $(m_1 = m_2 - 1, k_1 = k_2 + 1)$, $(m_1 = m_2, k_1 = k_2 - 1)$ and otherwise separately.

1. Case $(m_1 = m_2 - 1, k_1 = k_2 + 1)$: suppose $\phi : M_{m_1,k_1} \to M_{m_1+1,k_1-1}$ is a non-zero homomorphism and $\phi = gf$ where $f \in \mathrm{Hom}(M_{m_1,k_1}, X_s)$ and $g \in \mathrm{Hom}(X_s, M_{m_1+2,k_1-1})$ and X_s is indecomposable. Since f is non-zero we have the following commutative diagram

$$\begin{array}{rcccccccccccccccccccccc}
M_{m_1,k_1}: & 0 & \longrightarrow\cdots\longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow\cdots\longrightarrow & 0 & \longrightarrow & K & \longrightarrow\cdots\longrightarrow & K & \longrightarrow & K & \longrightarrow\cdots\longrightarrow & K & \longrightarrow & K & \longrightarrow & 0 & \longrightarrow\cdots\longrightarrow & 0 \\
f\downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f_i} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle f_j} & & \downarrow & & \downarrow & & \downarrow \\
X_s: & 0 & \longrightarrow\cdots\longrightarrow & 0 & \longrightarrow & K & \longrightarrow\cdots\longrightarrow & K & \longrightarrow & K & \longrightarrow\cdots\longrightarrow & K & \longrightarrow & 0 & \longrightarrow\cdots\longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow\cdots\longrightarrow & 0 \\
g\downarrow & \downarrow & & \downarrow & & \downarrow & & {\scriptstyle g_{i-1}}\downarrow & \Delta_1 & \downarrow{\scriptstyle g_i} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle g_j} & & \downarrow & & \downarrow & & \downarrow \\
M_{m_1+1,k_1-1} & 0 & \longrightarrow\cdots\longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow\cdots\longrightarrow & 0 & \longrightarrow & K & \longrightarrow\cdots\longrightarrow & K & \longrightarrow & K & \longrightarrow\cdots\longrightarrow & K & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow\cdots\longrightarrow & 0
\end{array}$$

   Since $g_i f_i : K \to K$ must be non-zero, we have that g_i can't be zero. Since the square Δ_1 commutes and $g_{i-1} = 0$, this means that the first K on the representation of X_s should appear exactly at the position where g_i is, i.e. at the position $n - m_1 + (2 - k_1)$. Similarly, since $g_j f_j : K \to K$ must be non-zero, we have that there must be a K at the representation of X at the position where g_j is, i.e. at the position $n - m_1$. That means that $X_s = M_{m_1,k_1}$ or $X_s = M_{m_1+1,k_1-1}$. Consider now a non-zero morphism $\Phi : M_{m_1,k_1} \to M_{m_1+1,k_1+1}$ such that $\Phi = gf$ where

$$M_{m_1,k_1} \xrightarrow{f=\begin{bmatrix} f_1 \\ \vdots \\ f_r \end{bmatrix}} X = \bigoplus_{s=1}^{r} X_s \xrightarrow{g=[\, g_1 \ \cdots \ g_r \,]} M_{m_1+1,k_1+1}$$

   and each X_s is indecomposable. Since there is i such that $g_i f_i \neq 0$, we have $f_i \neq 0$ and $g_i \neq 0$ which by the above implies $X_i \cong M_{m_1,k_1}$ or $X_i \cong M_{m_1+1,k_1-1}$. Hence f_i or g_i is an isomorphism and f is a section or g is a retraction. This proves that $\mathcal{F} : M_{m_1,k_1} \to M_{m_1+1,k_1-1}$ is irreducible.

2. Case $(m_1 = m_2, k_1 = k_2 - 1)$: similar to the previous case.

3. If we are not in the above case, then from the previous examination, we have that for $\mathcal{F} : M_{m_1,k_1} \to M_{m_2,k_2}$ to be non-zero, the following conditions must hold:

   - $m_2 \geq m_1$
   - $m_2 + k_2 \geq m_1 + k_1$
   - $m_1 + k_1 \geq m_2 + 1$

   Then we have the subcases

3a. If $m_1 = m_2$, then $m_2 + k_2 \geq m_1 + k_1$ implies $k_2 > k_1$ (otherwise we have $m_1 = m_2$ and $k_1 = k_2$) and $k_2 > k_1 + 1$ (otherwise we are in the previous case) so there exists k' with $k_2 > k' > k_1$ and $\mathcal{F} = fg$ where

$$\begin{array}{rcccccccccccccccccccccccc}
M_{m_1,k_1}: & 0 & \rightarrow \cdots \rightarrow & 0 & \rightarrow & 0 & \rightarrow \cdots \rightarrow & 0 & \rightarrow & 0 & \rightarrow \cdots \rightarrow & 0 & \rightarrow & K & \rightarrow \cdots \rightarrow & K & \rightarrow & K & \rightarrow & 0 & \rightarrow \cdots \rightarrow & 0 \\
f\downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
M_{m_1,k'}: & 0 & \rightarrow \cdots \rightarrow & 0 & \rightarrow & 0 & \rightarrow \cdots \rightarrow & 0 & \rightarrow & K & \rightarrow \cdots \rightarrow & K & \rightarrow & K & \rightarrow \cdots \rightarrow & K & \rightarrow & K & \rightarrow & 0 & \rightarrow \cdots \rightarrow & 0 \\
g\downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
M_{m_2,k_2}: & 0 & \rightarrow \cdots \rightarrow & 0 & \rightarrow & K & \rightarrow \cdots \rightarrow & K & \rightarrow & K & \rightarrow \cdots \rightarrow & K & \rightarrow & K & \rightarrow \cdots \rightarrow & K & \rightarrow & K & \rightarrow & 0 & \rightarrow \cdots \rightarrow & 0
\end{array}$$

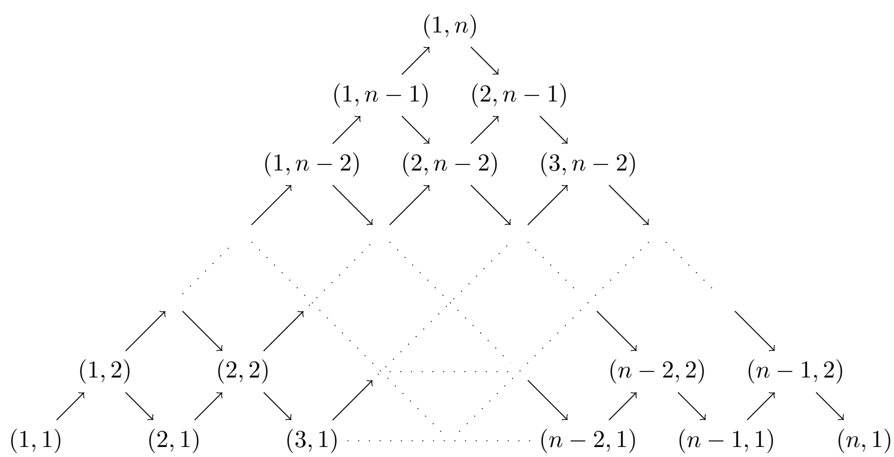
3b. If $m_1 + k_1 = m_2 + k_2$ then we have $m_2 > m_1$ (otherwise we again have $m_1 = m_2, k_1 = k_2$) and $m_2 + 1 > m_1$ (otherwise we are in the first case). So there exists m' with $m_1 < m' < m_2$ and $\mathcal{F} = fg$ where

$$\begin{array}{rcccccccccccccccccccccccc}
M_{m_2,k_2}: & 0 & \rightarrow \cdots \rightarrow & 0 & \rightarrow & K & \rightarrow \cdots \rightarrow & K & \rightarrow & K & \rightarrow \cdots \rightarrow & K & \rightarrow & K & \rightarrow \cdots \rightarrow & K & \rightarrow & K & \rightarrow & 0 & \rightarrow \cdots \rightarrow & 0 \\
f\downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
M_{m_1,k'}: & 0 & \rightarrow \cdots \rightarrow & 0 & \rightarrow & 0 & \rightarrow \cdots \rightarrow & 0 & \rightarrow & K & \rightarrow \cdots \rightarrow & K & \rightarrow & K & \rightarrow \cdots \rightarrow & K & \rightarrow & K & \rightarrow & 0 & \rightarrow \cdots \rightarrow & 0 \\
g\downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
M_{m_1,k_1}: & 0 & \rightarrow \cdots \rightarrow & 0 & \rightarrow & 0 & \rightarrow \cdots \rightarrow & 0 & \rightarrow & 0 & \rightarrow \cdots \rightarrow & 0 & \rightarrow & K & \rightarrow \cdots \rightarrow & K & \rightarrow & K & \rightarrow & 0 & \rightarrow \cdots \rightarrow & 0
\end{array}$$

3c. In this case, if $m_2 > m_1$ and $m_2 + k_2 > m_1 + k_1$, we have that $\mathcal{F} : M_{m_1,k_1} \rightarrow M_{m_2,k_2}$ is not a proper epimorphism or a proper monomorphism. Therefore, by Lemma 3.18 it is not irreducible.

With the above, we have proven the following.

Proposition 4.3. The Auslander-Reiten quiver of KQ_n is the quiver



If we want to calculate the Auslander-Reiten quiver of $A = KQ_n/I$, we know by Lemma 4.1 that $\Gamma(\text{mod}A)_0 = S_A$. That is, we know that the vertices of $\Gamma(\text{mod}A)$ are exactly the vertices of $\Gamma(\text{mod}KQ_n)$ that are not annihilated by the ideal I. The following proposition describes the Auslander-Reiten quiver of A.

Proposition 4.4. If $A = KQ_n/I$, then $\Gamma(\text{mod}A)$ is the full subquiver of $\Gamma(\text{mod}KQ_n)$ such that $\Gamma(\text{mod}A)_0 = S_A$.

Proof. If $M, N \in \text{mod}KQ_n$ and $h \in \text{Hom}(M,N)$ is irreducible in $\text{mod}KQ_n$, then if $M, N \in \text{mod}A$, h is irreducible in $\text{mod}A$. So all arrows $\alpha \in \Gamma(\text{mod}KQ_n)_1$ such that the target and the source are in $\Gamma(\text{mod}A)$, belong also in $\Gamma(\text{mod}A)_1$. So the only thing that we need to prove is that given $M, N \in \text{mod}A$, there is no $f \in \text{Hom}(M,N)$ which is irreducible in $\text{mod}A$ but not in $\text{mod}KQ_n$. Note that in the proof of the previous proposition, when proving that $\mathcal{F}$ from M_{m_1,k_1} to M_{m_2,k_2} factors we only used, as a middle indecomposable module, a module X such that if it is annihilated by I, then either M_{m_1,k_1} or M_{m_2,k_2} is annihilated too. In other words, any morphism $M \to N$ factors through X except if M and N are as described before. So $\Gamma(\text{mod}A)$ has no more arrows and the proposition is proved. $\square$

Remark 4.5. The quiver $\Gamma(\text{mod}A)$ is uniquely determined by the vertices $\Gamma(\text{mod}A)_0$. Now, given $(m,k) \in \Gamma(\text{mod}A)_0$ we have that $(m,j) \in \Gamma(\text{mod}A)_0$ for $1 \leq j \leq k$ and $(m+i, k-i) \in \Gamma(\text{mod}A)_0$ for $0 \leq i \leq k-1$. That is so because since $(m,k) \in \Gamma(\text{mod}A)_0$, (m,k) is not annihilated by I. That means that all paths of length up to $k-1$ between the points $n-m+(2-k)$ and $n-m+1$ are not in I. Thus, (m,j) and $(m+i,k-i)$ are not annihilated by I. Similarly, if (m,k) is in $\Gamma(\text{mod}A)_0$, then all points (i,j) such that $m \leq i \leq m+k-1$ and $i+j \leq m+k$ are in $\Gamma(\text{mod}A)_0$.

Now that we have the Auslander-Reiten quiver of A, there are some almost split sequences that arise naturally.

Proposition 4.6. Suppose $(m,k+1) \in \Gamma(\text{mod}A)_0$. Then the sequence

$$0 \to (m,k) \overset{\left[\begin{smallmatrix} i \\ p \end{smallmatrix}\right]}{\to} (m,k+1) \oplus (m+1,k-1) \overset{[\,-j \ q\,]}{\to} (m+1,k) \to 0$$

is almost split, where i, j are the natural inclusions, p, q the natural projections, and by convention $(m,0) = 0$.

Proof. The above maps are

$$\begin{array}{rccccccccccccccccccccccc}
(m,k): & 0 & \xrightarrow{0} & \cdots & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & K & \xrightarrow{1} & K & \xrightarrow{1} & \cdots & \xrightarrow{1} & K & \xrightarrow{1} & K & \xrightarrow{0} & 0 & \xrightarrow{0} & \cdots & \xrightarrow{0} & 0 \\
\left[\begin{smallmatrix} i \\ p \end{smallmatrix}\right]\downarrow & \downarrow 0 & & & & \downarrow 0 & & \downarrow 0 & & \downarrow \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] & & \downarrow \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] & & & & \downarrow \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] & & \downarrow \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] & & \downarrow 0 & & & & \downarrow 0 \\
(m,k+1)\oplus(m+1,k-1): & 0\oplus 0 & \xrightarrow{0} & \cdots & \xrightarrow{0} & 0\oplus 0 & \xrightarrow{0} & K\oplus 0 & \xrightarrow{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]} & K\oplus K & \xrightarrow{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]} & K\oplus K & \xrightarrow{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]} & \cdots & \xrightarrow{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]} & K\oplus K & \xrightarrow{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]} & K\oplus 0 & \xrightarrow{0} & 0\oplus 0 & \xrightarrow{0} & \cdots & \xrightarrow{0} & 0\oplus 0 \\
[\,-j \ q\,]\downarrow & \downarrow 0 & & & & \downarrow 0 & & \downarrow [\,-1 \ 0\,] & & \downarrow [\,-1 \ 1\,] & & \downarrow [\,-1 \ 1\,] & & & & \downarrow [\,-1 \ 1\,] & & \downarrow 0 & & \downarrow 0 & & & & \downarrow 0 \\
(m+1,k): & 0 & \xrightarrow{0} & \cdots & \xrightarrow{0} & 0 & \xrightarrow{0} & K & \xrightarrow{1} & K & \xrightarrow{1} & K & \xrightarrow{1} & \cdots & \xrightarrow{1} & K & \xrightarrow{0} & 0 & \xrightarrow{0} & 0 & \xrightarrow{0} & \cdots & \xrightarrow{0} & 0
\end{array}$$

from which it is easy to see that $\left[\begin{smallmatrix} i \\ p \end{smallmatrix}\right]$ is a monomorphism, $[\,-j \ q\,]$ is an epimorphism and $\text{Ker}\left[\begin{smallmatrix} i \\ p \end{smallmatrix}\right] = \text{Im}\,[\,-j \ q\,]$. Therefore the above sequence is exact. Moreover, by our previous investigation for the Auslander-Reiten quiver we know that the modules (m,k) and $(m+1,k)$ are indecomposable, while i,j and p,q are all irreducible morphisms leaving (m,k) and entering $(m+1,k)$ respectively. Therefore, by Corollary 4.4 in [4], the above sequence is almost split. □

Finally, we can calculate the Auslander-Reiten translations τ and τ^{-1} using the above result.

Lemma 4.7. For $(m,k) \in \Gamma(\text{mod}A)_0$ we have

(1) $\tau(m,k) = (m-1,k)$ if (m,k) is not projective.

(2) $\tau^{-1}(m,k) = (m+1,k)$ if (m,k) is not injective.

Proof.

(1) By Proposition 4.6 we have that the sequence

$$0 \to (m-1,k) \overset{[i\ p]}{\to} (m,k+1) \oplus (m+1,k-1) \overset{\left[\begin{smallmatrix} -j \\ q \end{smallmatrix}\right]}{\to} (m,k) \to 0$$

is almost split. By Theorem 3.29, there is an almost split sequence $0 \to \tau(m,k) \to E \to (m,k) \to 0$. By Lemma 3.28, we get $\tau(m,k) \cong (m-1,k)$.

(2) Similar to (1).

□

4.3 Projective and injective resolutions

In this section we prove a formula for calculating the projective (and, because of symmetry, injective) dimension for all indecomposable A-modules $M_{m,k} \in \Gamma(\text{mod}A)_0$.

Remark 4.8. We now want to determine the indecomposable projective and injective modules of A. A well-known result (see [4], page 79) is that the indecomposable projective modules of KQ_n/I are in bijection with the vertices of Q_n, so that for each vertex k of Q_n, the submodule of KQ_n/I spanned by all $\overline{w} = w + I$ where w is a path leaving k, is indecomposable projective. The paths w not in I form a basis of A, since they are linearly independent and any element of A can be written as a linear combination of them. Therefore, by taking all paths with a fixed starting point k, we can find the corresponding indecomposable projective module. This means we need the to check for which representation $M_{i,j}$ we have the following:

1. The first K appears in the position k (which corresponds to the paths having the same starting point k).
2. There exists no module $M_{i',j'}$ such that the first K appears in the position k and it has more K's in its representation than $M_{i,j}$ (which corresponds to being spanned by all paths not in I with starting point k).

The first condition, in our relation, translates to $k = n - i - j + 2$. The second, translates to $j = \max\{y | M_{x,y} \in \Gamma(\text{mod}A)_0 \text{ and } x + y = k\}$. Therefore, we have that

$$j = \max\{y | M_{x,y} \in \Gamma(\text{mod}A)_0 \text{ and } x + y = i + j\} \Rightarrow M_{i,j} \in \mathcal{P}. \tag{1}$$

A similar property of the indecomposable injective modules is that they are in bijection with the vertices of Q_n, so that for each vertex a of Q_n, the module with basis all paths $\overline{w} = w + I$ going into a is indecomposable injective. A similar calculation shows that

$$j = \max\{y | M_{i,y} \in \Gamma(\text{mod}A)_0\} \Rightarrow M_{i,j} \in \mathcal{I}. \tag{2}$$

Using these facts, we can explicitly calculate the first syzygy of any $M_{x,y} \in \Gamma(\text{mod}A)$. Inductively, we can calculate any syzygy which will prove useful later.

Lemma 4.9. Let $M_{x,y}$ be an indecomposable non-projective A-module. For $r \in \{2, ..., n+1\}$ define

$$y_r = \max\{j \in \{1, ..., n\} | M_{i,j} \in \Gamma(\text{mod}A)_0 \text{ and } i + j = r\}.$$

Then $(x + y - y_{x+y}, y_{x+y} - y)$ is a first syzygy of $M_{x,y}$.

Proof. Set $r = x + y$ and consider the following commutative diagram

$$\begin{array}{rccccccccccccccc}
 & & & & & & & y_r & & & & & & & & \\
 & & & & & & & & & & y_r - y & & & & & \\
 & & & & & n-(x+y-y_r)+(2-y_r) & & n-x+1 & & n-x+2 & & n-(x+y-y_r)+1 & & & & \\
 & & & & & \bullet & & \bullet & & \bullet & & \bullet & & & & \\
(x+y-y_r, y_r-y): & 0 & \to\cdots\to & 0 & \to & 0 & \to\cdots\to & 0 & \to & K & \to\cdots\to & K & \to & 0 & \to\cdots\to & 0 \\
u\downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
(x+y-y_r, y_r): & 0 & \to\cdots\to & 0 & \to & K & \to\cdots\to & K & \to & K & \to\cdots\to & K & \to & 0 & \to\cdots\to & 0 \\
s\downarrow & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
(x,y): & 0 & \to\cdots\to & 0 & \to & K & \to\cdots\to & K & \to & 0 & \to\cdots\to & 0 & \to & 0 & \to\cdots\to & 0 \\
 & & & & & \bullet & & \bullet & & & & & & & & \\
 & & & & & n-x+(2-y) & & n-x+1 & & & & & & & & \\
 & & & & & & y & & & & & & & & &
\end{array}$$

where the arrows $K \to K$ are the identity and all other arrows are the zero map. Then

$$0 \to (x + y - y_r, y_r - y) \overset{u}{\to} (x + y - y_r, y_r) \overset{s}{\to} (x, y) \to 0$$

is a short exact sequence since u is monomorphism (because the identity map is a injective), s is an epimorphism (because the identity map is surjective) and $\text{Ker}s = \text{Im}u$ as can be seen from the above diagram. Since $(x + y - y_r, y_r)$ is projective by (1), we have $\Omega(x, y) = (x + y - y_r, y_r - y)$. $\square$

And as an immediate corollary, we get the following result.

Corollary 4.10. Let $M_{x,y}$ be an indecomposable A-module. For $r \in \{2, ..., n+1\}$ define

$$y_r = \max\{j \in \{1, ..., n\} | M_{i,j} \in \Gamma(\text{mod}A)_0 \text{ and } i + j = r\}.$$

Then

$$\text{pd}(x, y) = \begin{cases} 0 & \text{if } y = y_{x+y} \\ \text{pd}(x + y - y_{x+y}, y_{x+y} - y) + 1 & \text{otherwise} \end{cases}$$

Proof. From the definition of projective dimension and syzygy we have

$$\text{pd}M = \begin{cases} 0 & \text{if } M \in \mathcal{P} \\ \text{pd}K + 1 & \text{if } M \notin \mathcal{P}, K = \Omega M \end{cases}.$$

From the Lemma 4.9, if $M \notin \mathcal{P}$ we have $\Omega M = (x + y - y_{x+y}, y_{x+y} - y)$ and the proof is complete. □

Alogrithm

We explain the algorithm by which we can calculate the projective dimensions of the vertices of $\Gamma(\text{mod}A)$ in a practical way. In this way we can assign at the point (m, k) its projective dimension. The algorithm works in this way:

We begin by calculating $\text{pd}(1, 1)$. We know that this is a projective module, so its projective dimension is 0.

Next we go to all modules (m, k) such that $m + k = 3$, i.e. $(1, 2)$ and $(2, 1)$. We know that $(1, 2)$ is projective so $\text{pd}(1, 2) = 0$. Moreover, the surjection $(1, 2) \twoheadrightarrow (2, 1)$ has $(1, 2)$ as a kernel, so a minimal projective resolution of $(2, 1)$ is

$$0 \to (2, 1) \hookrightarrow (1, 2) \twoheadrightarrow (2, 1)$$

so the projective dimension of $(2, 1)$ is 1.

Given that we have calculated now the projective dimension of all (m, k) such that $m + k \leq n$ for some n (which corresponds to the first $n - 1$ downward diagonals of the Auslander-Reiten quiver of A, starting from the left), we show how to calculate the dimension of all (x, y) such that $x + y = n + 1$. We start by finding $y_1 = \max\{y | x + y = n+1\}$. Then, for this y_1, $(y_1 - (n+1), y_1) := (x_1, y_1)$ is projective so it has projective dimension 0. Now all other (x, y) such that $x + y = n+1$ are given as $(x_1 + k, y_1 - k)$ for $0 \leq k \leq y_1 - 1$. To find the projective dimension of $(x_1 + k, y_1 - k)$, observe that the surjection $(x_1, y_1) \twoheadrightarrow (x_1 + k, y_1 - k)$ has kernel (x_1, k) and $x_1 + k \leq x_1 + y_1 - 1 \leq n$ and we know, by induction hypothesis, the projective dimension of (x_1, k). This shows that the projective dimension of $(x_1 + k, y_1 - k)$ is one more than the projective dimension of (x_1, k).

A symmetrical algorithm (going from (x, y) such that $x + y = n + 1$ to (x, y) such that $x + y = 2$, that is going through the upward slopes, from right to left) calculates the injective dimensions. This gives us a dual statement to Corollary 4.10.

Corollary 4.11. Let $M_{x,y}$ be an indecomposable A-module. For $r \in \{1,...,n\}$ define

$$\mathfrak{y}_r = \max\{j \in \{1,...,n\} | M_{r,j} \in \Gamma(\text{mod}A)_0\}.$$

Then

$$\text{id}(x,y) = \begin{cases} 0 & \text{if } y = \mathfrak{y}_x \\ 1 + \text{id}(x+y, \mathfrak{y}_x - y) & \text{otherwise} \end{cases}.$$

5 Existence of d-cluster tilting modules

In this section we will search for d-cluster tilting modules in mod$A$. We will find exactly in which cases a d-cluster tilting module exists. First we will prove that when $\{p_i | 1 \leq k\}$ is a set of generators for I such that there exist two paths with different lengths, there exists no d-cluster tilting module. Next we will show under what additional condition to the equality of all paths in the set of generators of I there actually exists a d-cluster tilting module. To do this, we will do a case by case analysis where we will represent different classes of algebras A by words in letters $\{U, S, D\}$. We begin by a definition.

Definition 5.1. The *contour* of $\Gamma(\text{mod}A)$ is defined to be the set

$$\gamma_A = \{M_{m,k} | M_{m,k}I = 0 \text{ and } (M_{m,k+1}I \neq 0 \text{ or } M_{m-1,k+1}I \neq 0)\}$$

with the convention that if $m-1 = 0$ then $M_{m-1,k+1}I \neq 0$ and if $m+k+1 = n+2$ then $M_{m,k+1}I \neq 0$.

The set γ_A consists of all modules in the "boundary" of the Auslander-Reiten quiver of A. That is, it contains all projective and injective A-modules. It is obvious that $\gamma_A = \gamma_B$ if and only if $A = B$. Moreover we have that $\gamma_A \subseteq \Gamma(\text{mod}A)_0$. We can use γ_A to uniquely define $\Gamma(\text{mod}A)_0$ and thus $\Gamma(\text{mod}A)$ as seen in the following Lemma.

Lemma 5.2. Let γ_A be defined as above. Then

$$\Gamma(\text{mod}A)_0 = \overline{\gamma_A}$$

where

$$\overline{\gamma_A} = \{M_{i,j} | \exists M_{m,k} \in \gamma_A \text{ such that } m \leq i \leq m+k-1 \text{ and } i+j \leq m+k\}.$$

Proof. Let $M_{a,b} \in \Gamma(\text{mod}A)_0$. Then, $M_{a,b}I = 0$. Define $y_0 = \max\{y | M_{a,y} \in \Gamma(\text{mod}A)_0\}$. Then, $M_{a,y_0} \in \gamma_A$ and $a = a$, $a+b \leq a+y_0$. Therefore, $M_{a,b} \in \overline{\gamma_A}$ and $\Gamma(\text{mod}A)_0 \subseteq \overline{\gamma_A}$.

Let now $M_{a,b} \in \overline{\gamma_A}$. Since there exists $M_{m,k} \in \gamma_A$ such that $m \leq a \leq m+k-1$ and $a+b \leq m+k$ we have that $M_{a,b}$ is annihilated by I and therefore $M_{a,b} \in \Gamma(\text{mod}A)_0$. Therefore $\overline{\gamma_A} \subseteq \Gamma(\text{mod}A)_0$ which completes the proof. □

So we have seen that $\Gamma(\text{mod}A)$ is uniquely defined by γ_A. We want to describe the points of γ_A as a sequence of letters.

Remark 5.3. Note that given a point $(i,j) \in \gamma_A$, we have the following distinct possibilities:

1. $(i,j+1)I = 0$. Then, we have $(i-1,j+1)I \neq 0$ (since $(i,j) \in \gamma_A$) which implies $(i-1,j+2)I \neq 0$. Therefore, $(i,j+1) \in \gamma_A$
2. $(i,j+1)I \neq 0$. Then we have two subcases:
   a. $(i+1,j)I = 0$ and therefore $(i+1,j) \in \gamma_A$.
   b. $(i+1,j)I \neq 0$ and therefore $(i+1,j-1) \in \gamma_A$.

So if $(i,j) \in \gamma_A$, at least one of $(i,j+1)$, $(i+1,j)$ and $(i+1,j-1)$ will be in γ_A.

We introduce notation to describe the different classes of γ_A. To do this, let W be a word in the alphabet $\{U,S,D\}$ (standing for "up", "straight" and "down"). Denote by W_k the k-th letter of W. Moreover we denote by L^+ the existence of 1 or more letters L and L^* the existence of 0 or more L.

Definition 5.4. Let W be a word in $\{U,S,D\}$. Then the sequence $(i_0,j_0)_W, ..., (i_m,j_m)_W$ defined by

$$(i_0,j_0)_W = (1,1)$$

$$(i_k,j_k)_W = \begin{cases} (i_{k-1}, j_{k-1}+1) & \text{if } W_k = U \\ (i_{k-1}+1, j_{k-1}) & \text{if } W_k = S \\ (i_{k-1}+1, j_{k-1}-1) & \text{if } W_k = D \end{cases}$$

is called the *contour* of W.

The contour of W defines a set of points in the following way: given a point $(i_k,j_k)_W$, we check the letter W_{k+1}. Depending on whether it is U, S or D, we go "up", "straight" or "down" respectively and include the next point. This can be used to describe the contour of A. We can construct the corresponding word W in the following way: We always have that $(1,1) \in \gamma_A$ and that $W_0 = (1,1)$ is in the contour of W. Moreover, except for the case $n = 1$, we have that $(1,2) \in \gamma_A$, since we only consider admissible ideals. We can denote this by choosing $W_1 = U$. Then $(i_1,j_1)_W = (i_0,j_0+1) = (1,1+1) = (1,2)$. Now by the previous remark we know that at least one of $(1,3)$, $(2,2)$ and $(2,1)$ will be in γ_A. If it is $(1,3)$, we define $W_2 = U$. If not, if it is $(2,2)$ we define $W_2 = S$ and if it is $(2,1)$ we define $W_2 = D$. We continue this way. Supposing we have defined W_k, we know that $(i_k,j_k)_W \in \gamma_A$. Then define

$$W_{k+1} = \begin{cases} U & \text{if } (i_k, j_k+1)_W \in \gamma_A \\ S & \text{if } (i_k, j_k+1)_W \notin \gamma_A \text{ and } (i_k+1, j_k)_W \in \gamma_A \\ D & \text{if } (i_k, j_k+1)_W \notin \gamma_A \text{ and } (i_k+1, j_k-1) \in \gamma_A \end{cases}.$$

This way the contour of W is exactly γ_A. Note that the other direction is not true: not every word W has as a contour which the contour of A for some A.

For example, if W starts with D, then $(2,0)$ is in the contour of W. However, we can describe γ_A as words in $\{U,S,D\}$. By checking all classes of these words that describe a contour γ_A, we can classify γ_A, and therefore all path algebras A. Abusing our notation slightly, we write $\gamma_A = W$ for the word W such that the contour of W is γ_A.

Definition 5.5. Let $A = KQ_n/I$. Then A is is called of *type* W if W is a word in the alphabet $\{U,S,D\}$ such that the contour of W is the same as the contour of A.

We note particularly the following two important necessary conditions for such words:

Lemma 5.6. Suppose that A is of type W. Then

(a) $W_1 = U$.

(b) There is no sequence of the form DU in W.

Proof.

(a) Since I is admissible, $(1,2) \in \gamma_A$. Since the i_k and j_k in $(i_k, j_k)_W$ are non-decreasing in k, this means that $(1,2) = (i_1, j_1)_W$ which implies by Definition 5.4 that $W_1 = U$.

(b) Suppose that there is a sequence of the form DU in W. Then (x,y), $(x+1, y-1)$ and $(x+1, y)$ are in the contour of W for some x, y. But then they are in γ_A too for some A, which contradicts directly the definition of the contour of A.

□

5.1 Cases with no d-cluster tilting module

In this section we will prove that for the majority of algebras A, the algebra A has no d-cluster tilting module, in the sense that if A is of type W and W has one of the forms

- $U^+D...$
- $U^+S^+U...$
- $U^+S^+D^+S...$

then there is no d-cluster tilting module. Note that by Lemma 5.6 the only case not included in the above list is the case $U^+S^+D^+$.

5.1.1 Case $U^+D...$

If $W = U^kD^k$ for some $k \in \mathbb{Z}_{>0}$, then it is easy to see by using Proposition 4.3 and Corollaries 4.11 and 4.10 that $I = 0$ and that the global dimension is 1. Therefore, mod$A$ is a 1-cluster tilting subcategory in this case. Instead of proving that a word $U^+D^+S...$ gives rise to an algebra with no d-cluster tilting module, we prove a more general result.

Proposition 5.7. Suppose that in the word W there exists a sequence UD and that $d_A > 1$. Then, if A is of type W, A has no d-cluster tilting module.

Proof. Suppose that U and D appear in position k and $k+1$ respectively on the word W. By Lemma 5.6, $W_{k-1} \neq D$ and $W_{k+2} \neq U$. Let $(x, y) = (i_{k-1}, j_{k-1})_W$. Then $(x, y) \in \gamma_A$ and moreover $(x, y+1) = (i_k, j_k)_W \in \gamma_A$, $(x+1, y) = (i_{k+1}, j_{k+1})_W \in \gamma_A$. In this setting, by Remark 4.8, (x, y) is projective (since $W_{k-1} \neq D$ implies $(x-1, y+1) \notin \gamma_A$) and $(x+1, y)$ is injective (since $W_{k+2} \neq U$ implies $(x+1, y+1) \notin \gamma_A$). But

$$0 \to (x, y) \to (x, y+1) \oplus (x+1, y-1) \to (x+1, y) \to 0$$

is an almost split sequence by Proposition 4.6, which implies that $\mathrm{Ext}^1(M_{x+1,y}, M_{x,y}) \neq 0$ by Lemma 3.31. So we have $\mathrm{Ext}^1(I, P) \neq 0$ where $I = (x+1, y)$ an injective module and $P = (x, y)$ a projective module. Suppose that M is a d-cluster tilting A-module, then $\mathcal{C} = \mathrm{add}M$ is a d-cluster tilting subcategory. Then, by Proposition 3.33, $P, I \in \mathcal{C}$ and $\mathcal{C} = {}^{\perp_d}\mathcal{C}$. But since $\mathrm{Ext}^1(I, P) \neq 0$, we have $P \notin {}^{\perp_d}\mathcal{C}$, a contradiction. Therefore there exists no d-cluster tilting A-module. □

5.1.2 Case $U^+S^+U...$

We begin by giving a necessary condition for the existence of a d-cluster tilting module.

Lemma 5.8. Let $M = \bigoplus_{i=1}^m X_i$ be a d-cluster tilting A-module where X_i are all indecomposable.

(a) For each X_i such that X_i is injective and non-projective there exists a unique $l_i \in \mathbb{Z}_{>0}$ such that $\tau_d^{l_i} X_i$ is projective and non-injective.

(b) For each X_i such that X_i is projective and non-injective there exists a unique $m_i \in \mathbb{Z}_{>0}$ such that $(\tau_d^-)^{m_i} X_i$ is injective and non-projective.

Proof.

(a) From Proposition 3.37, we have that τ_d is a bijection from the indecomposable non-projective modules in add$M$ to the indecomposable non-injective modules in add$M$. Let $I_i \in \mathrm{add}M$ be an injective and non-projective indecomposable module, then $\tau_d I_i \in \mathrm{add}M$. Since M is a d-cluster tilting, we must have by Lemma 3.14 that $\mathrm{pd}(\tau_d I_i) \geq d$ and since d is the global dimension of A we have $\mathrm{pd}(\tau_d I_i) = d$. Moreover we know that

$$\tau_d I_i = \tau\left(\Omega^{d-1} I_i\right)$$

and if $I_i = (x, y)$, then $\Omega^{d-1} I_i = (x', y')$ where $x' < x$ by Lemma 4.9. Since $\tau(x', y') = (x' - 1, y')$ by Lemma 4.7, we have that $\tau_d I_i = (x' - 1, y')$ and $x' - 1 < x$. So we have

$$\tau_d(x, y) = \begin{cases} 0 & \text{if } (x, y) \in \mathcal{P} \\ (\tilde{x}, \tilde{y}) & \text{otherwise} \end{cases}$$

where $\tilde{x} < x$. So $\tau_d^{\kappa}(x, y) = \tau_d^{\lambda}(x, y) \Rightarrow \tau_d^{\kappa}(x, y) = \tau_d^{\lambda}(x, y) = 0$ for $\kappa \neq \lambda \in \mathbb{Z}_{>0}$. Since A is representation-finite, the sequence

$$\tau_d(x, y), \tau_d^2(x, y), ..., \tau_d^m(x, y), ...$$

must have finitely many non-zero elements and by the definition of τ_d they must all appear before the first 0. By setting $l_i = \max\{m \in \mathbb{Z}_{>0} \mid \tau_d^{l_i} I_i \neq 0\}$ we have that $\tau_d^{l_i+1} I_i = 0$ so $\tau_d^{l_i} I_i$ is projective and l_i is the unique number in $\mathbb{Z}_{>0}$ with this property.

(b) Similar to (a).

□

Therefore, if there is a d-cluster tilting module M in A, the mapping $I_i \mapsto \tau_d^{l_i} I_i$ gives a bijection between $\mathcal{I} \setminus \mathcal{P}$ and $\mathcal{P} \setminus \mathcal{I}$. Lemma 5.8 and Lemma 4.7 imply that in this case, for every projective and non-injective module $P = (i, j)$ we must have that $\mathrm{pd}(i + 1, j) = 1$. Otherwise, there is no injective module I with $\tau_d^l I = P$ for some l and this is so because τ_d acts on X as τ on the $(d - 1)$-th syzygy of X, which always has projective dimension 1. So if we prove that for A there exists a projective non-injective module (i, j) such that $\mathrm{pd}(i + 1, j) > 1$, then there is no d-cluster tilting module for A. We claim that when the Auslander-Reiten quiver of A has a shape of the form $U^+S^+U...$ we are exactly in this case and therefore there does not exist a d-cluster tilting module.

Theorem 5.9. Let $A = KQ_n/I$ so that γ_A is the contour of

$$W = U^{k_1} S^{k_2} U...,$$

for $k_1, k_2 \geq 1$. Let d be the global dimension of A. Then A has no d-cluster tilting module

Proof. We have

$$(i_{k_1+k_2+1}, j_{k_1+k_2+1}) = (1 + k_2, 1 + k_1 + 1)$$

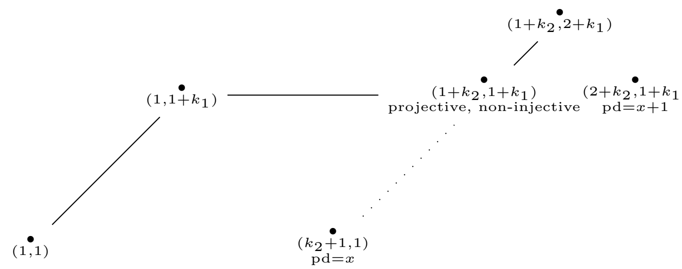
so $(1 + k_2, 2 + k_1) \in \gamma_W$. Since $(1 + k_2, 2 + k_1) \in \Gamma(\mathrm{mod} A)_0$, we have that $(2 + k_2, 1 + k_1) \in \Gamma(\mathrm{mod} A)_0$. Moreover, $(2 + k_2, 1 + k_1)$ is non-projective since $(1 + k_2, 2 + k_1) \in \Gamma(\mathrm{mod} A)_0$. We have $\mathfrak{y}_{2+k_2+1+k_1} = 2 + k_1$ and thus by Corollary 4.10 we have

$$\mathrm{pd}(2+k_2, 1+k_1) = \mathrm{pd}(2+k_2+1+k_1-(k_1+2), k_1+2-(k_1+1))+1 = \mathrm{pd}(k_2+1, 1)+1$$

so

$$\mathrm{pd}(2+k_2, 1+k_1) = \mathrm{pd}(k_2+1, 1) + 1 \geq 1 + 1 = 2$$

where we have $\mathrm{pd}(k_2+1, 1) \geq 1$ because $(k_2+1, 1)$ is not projective. The following picture shows the aforementioned modules



So in this case we have that $(1+k_2, 1+k_1)$ is projective but $\mathrm{pd}(2+k_2, 1+k_1) \geq 2$ which means that there exists no d-cluster tilting module.

□

5.1.3 Case $U^+S^+D^+S...$

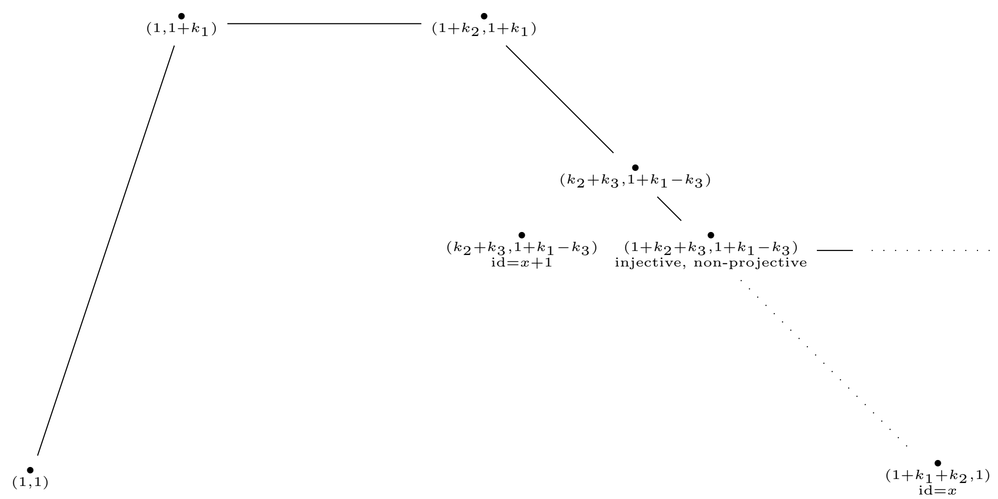
By using part (b) of Lemma 5.8, we can see similarly to the previous case that if a d-cluster tilting A-module exists and if (x, y) is an injective non-projective indecomposable A-module, then $(x-1, y)$ must have injective dimension equal to 1. The following theorem proves that this does not happen in this case.

Theorem 5.10. Let $A = KQ_n/I$ so that γ_A is the contour of

$$W = U^{k_1}S^{k_2}D^{k_3}S...$$

for $k_1, k_2, k_3 \geq 1$. Let d be the global dimension of A. Then A has no d-cluster tilting module

Proof. We give a sketch of the proof, since the proof is symmetric to the previous case, where instead of using the bijection given by $\tau_d^{l_i}$ we use the bijection given by $(\tau_d^-)^{m_i}$. In this case, the shape will be



In this case, $(1+k_2+k_3, 1+k_1-k_3)$ is injective and non-projective, and $(k_2+k_3, 1+k_1-k_3)$ has injective dimension 1 plus the injective dimension of $(1+k_1+k_2, 1)$ which has injective dimension at least 1 since it is not injective. Therefore, there exists no d-cluster tilting A-module in this case. □

5.2 Case $U^+S^+D^+$

In this section we find when $A = KQ_n/I_l$ has a d-cluster tilting module, for d equal to the global dimension, where I_l consists of all paths of length l and above. Let $A = KQ_n/I_l$ for $l \geq 2$. For $l = 2$ the answer is known, and there always exists a d-cluster tilting module.

Proposition 5.11. Let $A = KQ_n/I_2$. Then

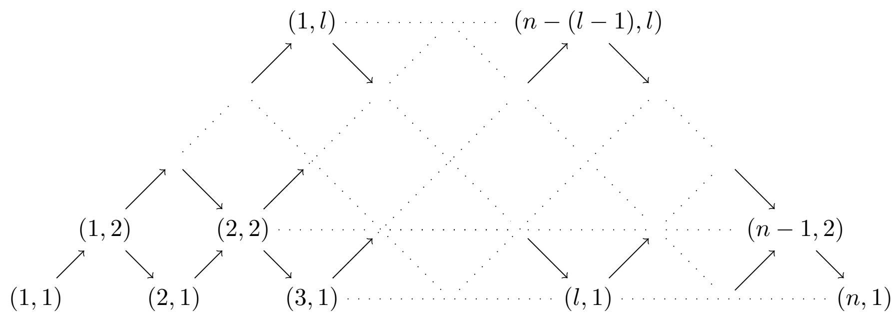
$$M = \left(\bigoplus_{P \in \mathcal{P}} P\right) \oplus \left(\bigoplus_{I \in \mathcal{I}} I\right)$$

is a d-cluster tilting module.

Proof. See [1], Example 2.4(a). □

The aim of this section is to prove that for $l \geq 3$, a d-cluster tilting module exists if and only if $n \equiv 1 \mod l$.

In this case, the Auslander-Reiten quiver of A will be:



So we can write $\gamma_A = U^{l-1}S^{n-l}D^{l-1}$. We can describe the indecomposable projective, injective and projective and injective modules on $\Gamma(\text{mod}A)_0$ with the following Lemma.

Lemma 5.12.

a) $(x, y) \in \Gamma(\text{mod}A)_0$ is projective if and only if $(x, y) = (1, j), 1 \leq j \leq l$ or $(x, y) = (i, l), 1 \leq i \leq n - j + 1$.

b) $(x, y) \in \Gamma(\text{mod}A)_0$ is injective if and only if $(x, y) = (n - j + 1, j), 1 \leq j \leq l$ or $(x, y) = (i, l), 1 \leq i \leq n - j + 1$.

c) $(x, y) \in \Gamma(\text{mod}A)_0$ is projective and injective if and only if $(x, y) = (i, l), 1 \leq i \leq n - j + 1$.

Proof.

a) By Remark 4.8, we know that (x, y) is projective if and only if $y = \max\{j|(i, j) \in \Gamma(\text{mod}A), i + j = x + y\}$. For $2 \leq i + j \leq 1 + l$ we have $\max\{j|(i, j) \in \Gamma(\text{mod}A), i + j = x + y\} = x + y - 1$, since $(1, x + y - 1) \in \Gamma(\text{mod}A)_0$. Similarly, for $l + 1 \leq x + y \leq n + 1$, $\max\{j|(i, j) \in \Gamma(\text{mod}A)i + j = x + y\} = l$ since $(x + y - l, l) \in \Gamma(\text{mod}A)_0$. Therefore, (x, y) is projective if and only if $x = 1$ and $1 \leq y \leq l$ or $y = l$ and $1 \leq x \leq n - l + 1$.

b) Similar to a).

c) Follows directly by a) and b).

□

In this case, we can also calculate the projective dimension of all indecomposable modules using the following lemma:

Lemma 5.13. Let $A = KQ_n/I_l$ and let $M_{i,j} \in \Gamma(\text{mod}A)_0$. There exist unique q, r such that

$$i - 2 = ql + r, \ \ 0 \leq r \leq l - 1.$$

Then,

$$\text{pd}(i,j) = \begin{cases} 0 & \text{if } i = 1 \text{ or } j = l \\ 2q+1 & \text{if } r < l-j \\ 2q+2 & \text{if } r \geq l-j \end{cases}.$$

Proof. We will use induction on $\nu = i+j$. For $\nu = 2$ we have $i = j = 1$ so by Corollary 4.10, $\text{pd}(1,1) = 0$ which agrees with Lemma 5.13.

Assume that Lemma 5.13 holds for $\nu \leq k$. We will prove it for $\nu = k+1$. Let $M_{i,j} \in \Gamma(\text{mod}A)_0$ be such that $i+j = k+1$. If $i = 1$ or $j = l$, we need to prove that $\text{pd}(i,j) = 0$. If $i = 1$ and $i+j = k+1$ we have $M_{i,j} = M_{1,k}$ and $k = y_{k+1}$ (since if there exists $k' > k$ such that $M_{k+1-k',k'} \in \Gamma(\text{mod}A)_0$ we need to have $k+1-k' \geq 1 \Rightarrow k \geq k'$, a contradiction). If $j = l$, we have $j = y_{k+1}$ since there does not exist $j' > l$ such that $M_{k+1-j',j'} \in \Gamma(\text{mod}A)_0$. Therefore in any case $j = y_{k+1}$ and by Lemma 5.12 $\text{pd}(1,k) = 0$ as needed.

Suppose now that $i > 1$ and $j < l$. Then $j \neq y_{k+1}$ since $M_{k+1-l,l} \in \Gamma(\text{mod}A)_0$ and $y_{k+1} = l$. Therefore, by Corollary 4.10 we have

$$\text{pd}(i,j) = 1 + \text{pd}(i+j-l, l-j)$$

and $i+j-l+l-j = i < i+j = k+1$ so by induction hypothesis we know that Lemma 5.13 holds for $(i+j-l, l-j)$. Let $i-2 = ql+r$. We consider two cases:

Case 1: If $r < l-j$, then $i+j-l-2 = ql+r+j-l = (q-1)l+(r+j)$ and $r+j < l-j+j = l \Rightarrow 0 \leq r+j \leq l-1$. So we have $i+j-l-2 = q'l+r'$ with $q' = q-1$ and $r' = r+j$. Since $l-(l+j) = j \leq r+j = r'$ we have, by induction hypothesis, that

$$\text{pd}(i+j-l, l-j) = 2q'+2 = 2(q-1)+2.$$

Therefore, we have

$$\text{pd}(i,j) = 1 + \text{pd}(i+j-l, l-j) = 2q+1$$

which agrees with the value of Lemma 5.13.

Case 2: If $r \geq l-j$, then $r+j = sl+t$ with $0 \leq t \leq l-1$. Moreover, since $l \leq r+j < l-l+l = 2l-1$, we have that $s = 1$. Then

$$i+j-l-2 = ql+r+j-l = (q-1)l+l+t = ql+t$$

so $i+j-l-2 = q'l+r'$ with $q' = q$ and $r' = t$. Additionally,

$$r' = t = r+j-l < l+j-l = j = l-(l-j)$$

so $r' < l-(l-j)$ and, by induction hypothesis, we have that

$$\text{pd}(i+j-l, l-j) = 2q'+1 = 2q+1.$$

Therefore, we have

$$\text{pd}(i,j) = 1 + \text{pd}(i+j-l, l-j) = 1 + (2q+1) = 2q+2$$

which agrees with the value of Lemma 5.13.

So in any case we have proved that the projective dimension of (i,j) is the same as Lemma 5.13 claims and the proof is complete. □

The following image is the Auslander-Reiten quiver of KQ_{25}/I_5 where we replace the indecomposable modules with their projective dimension (and we omit the arrows). Note that the shape is very symmetrical. Specifically, there are triangles and inverse triangles of indecomposable modules of the same projective dimension.

```
        0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0   0
      0   1   2   2   2   2   3   4   4   4   4   5   6   6   6   6   7   8   8   8   8   9
    0   1   1   2   2   2   3   3   4   4   4   5   5   6   6   6   7   7   8   8   8   9   9
  0   1   1   1   2   2   3   3   3   4   4   5   5   5   6   6   7   7   7   8   8   9   9   9
0   1   1   1   1   2   3   3   3   3   4   5   5   5   5   6   7   7   7   7   8   9   9   9   9
```

Of course there is a dual to Lemma 5.13 for the injective dimensions. It can be proven directly by Lemma 5.13, if we note that in this particular case, because of symmetry we have

$$\mathrm{id}(i,j) = \mathrm{pd}(n-(i+j-2),j)$$

Lemma 5.14. Let $A = KQ_n/I_l$ and let $M_{i,j} \in \Gamma(\mathrm{mod}A)_0$. There exist unique q, r such that

$$n-(i+j) = ql + r, \;\; 0 \le r \le l-1.$$

Then,

$$\mathrm{id}(i,j) = \begin{cases} 0 & \text{if } i = n-(j-1) \text{ or } j = l \\ 2q+1 & \text{if } r < l-j \\ 2q+2 & \text{if } r \ge l-j \end{cases}.$$

Now we can prove the main theorem for this chapter.

Theorem 5.15. Let $A = KQ_n/I_l$, for some n and l such that $2 \le l < n$. Further let d be the global dimension of A. Then A has a d-cluster tilting module if and only if $l = 2$ or $l \ge 3$ and $n \equiv 1 \mod l$.

Proof. Since we know from Proposition 5.11 that there always exists a d-cluster tilting module if $l = 2$, we may assume $l \ge 3$.

(if) Suppose first that $n \equiv 1 \mod l$. First we want to prove that all A-modules which are injective but not projective have the same projective dimension. By Lemma 5.12, these modules are exactly those of the form $(n-(j-1), j)$ for $1 \le j \le l-1$. Since $n \equiv 1 \mod l$, there exists q such that

$$n = ql + 1 \text{ and thus } q = \frac{n-1}{l}$$

and therefore

$$n-(j-1)-2 = ql+1-(j-1)-2 = ql-j = (q-1)l+(l-j)$$

where $0 \leq l-j \leq l-1$ since

$$1 \leq j \leq l-1 \Rightarrow 1-l \leq -j \leq -1 \Rightarrow 1 \leq l-j \leq l-1.$$

Therefore, by applying Lemma 5.13 we get $\text{pd}(n-(j-1), j) = 2(q-1)+2 = d$ which is independent of j. So all injective but not projective modules have the same projective dimension.

Next we need to prove that $\text{Ext}^i(\mathcal{I}, \mathcal{P}) = 0$ for $1 \leq i \leq d-1$. Since $\text{Ext}^i(\mathcal{P}, M) = \text{Ext}^i(M, \mathcal{I}) = 0$ for any $M \in \Gamma(\text{mod}A)_0$, it is enough to prove $\text{Ext}^i(\mathcal{I} \setminus \mathcal{P}, \mathcal{P} \setminus \mathcal{I}) = 0$.

By Lemma 5.12, the indecomposable injective non-projective modules are of the form $(n-j+1, j)$ for $1 \leq j \leq l-1$. Then, by Proposition 3.10, there exists a k such that $\text{pd}(\Omega^k M_{n-(j-1),j}) = 1$. Suppose $\Omega^k M_{n-(j-1),j} = M_{x,y}$. Then a projective resolution of $\Omega^k M_{n-(j-1),j}$ is

$$0 \to P_{1,y_{x+y}-y} \to P_{1,y_{x+y}} \to M_{x,y}$$

because $\text{pd}M_{x,y} = 1$ means that $M_{x,y}$ is not projective. Therefore, we get a projective resolution of $(n-(j-1), j)$:

$$0 \to P_{1,y_{x+y}-y} \to P_{1,y_{x+y}} \to P_{s_1,s_2} \to \cdots \to P_{s_{t_1},s_{t_2}} \to M_{n-(j-1),j}.$$

Now, again by Lemma 5.12, the indecomposable projective non-injective modules in $\Gamma(\text{mod}A)_0$ are of the form $P_{1,\beta}$ for $1 \leq \beta \leq l-1$. Therefore, we need to calculate

$$\text{Ext}^i(M_{n-(j-1),j}, P_{1,\beta}).$$

By applying $\text{Hom}(\cdot, P_{1,\beta})$ to the above projective resolution of $M_{n-(j-1),j}$ we get

$$\text{Hom}(P_{s_{t_1},s_{t_2}}, P_{1,\beta}) \to \cdots \to \text{Hom}(P_{s_1,s_2}, P_{1,\beta}) \to \text{Hom}(P_{1,y_{x+y}}, P_{1,\beta})$$

$$\to \text{Hom}(P_{1,y_{x+y}-y}, P_{1,\beta}) \to 0$$

and since $\text{Hom}(P_{a,b}, P_{1,c}) = 0$ for $a > 1$, the above complex is

$$0 \to \cdots \to 0 \to \text{Hom}(P_{1,y_{x+y}}, P_{1,\beta}) \to \text{Hom}(P_{1,y_{x+y}-y}, P_{1,\beta}) \to 0.$$

If $\beta < y_{x+y} - y$, the above becomes

$$0 \to \cdots \to 0 \to 0 \to 0 \to 0$$

so $\text{Ext}^i(M_{n-(j-1),j}, P_{1,\beta}) = 0$, $1 \leq i \leq d$. If $y_{x+y} - y \leq \beta < y_{x+y}$, the above becomes

$$0 \to \cdots \to 0 \to 0 \to K \to 0$$

so $\text{Ext}^i(M_{n-(j-1),j}, P_{1,\beta}) = 0$, $1 \leq i \leq d-1$, $\text{Ext}^d(M_{n-(j-1),j}, P_{1,\beta}) \neq 0$. Finally, if $y_{x+y} \leq \beta$ the above becomes

$$0 \to \cdots \to 0 \to K \xrightarrow{\sim} K \to 0$$

so again $\mathrm{Ext}^i(M_{n-(j-1),j}, P_{1,\beta}) = 0$, $1 \leq i \leq d$. In any case, we have proved that $\mathrm{Ext}^i(M_{n-(j-1),j}, P_{1,\beta}) = 0$ for $1 \leq i \leq d-1$ and therefore, a candidate for a d-cluster tilting module is the module C, where C is the direct sum of all indecomposable projective and injective modules in $\Gamma(\mathrm{mod}A)_0$. We claim that C is a d-cluster tilting module.

To prove this, we prove that for every other indecomposable module M there exist i, j such that $1 \leq i, j \leq d-1$ and $\mathrm{Ext}^i(\mathcal{I}, M) \neq 0$, $\mathrm{Ext}^j(M, \mathcal{P}) \neq 0$. For this to hold, by Lemma 3.14, it is enough to prove that M has projective and injective dimension strictly less than d. We prove this for the projective dimensions then, by symmetry, it will be true for injective dimensions too.

Let $M = M_{a,b}$ and assume that M is not projective or injective. Then, write $a - 2 = q'l + r$ with $0 \leq r \leq l-1$. If $r < l - b$ then $\mathrm{pd}(M_{a,b}) = 2q' + 1$. Then $2q' + 1 = d = 2q$ yields a contradiction, since the left hand side is odd and the right hand side is even. Since $\mathrm{pd}(M_{a,b}) \leq d$ this implies that $\mathrm{pd}(M_{a,b}) < d$.

Assume now that $r \geq l - b$. Then $\mathrm{pd}(M_{a,b}) = 2q' + 2$ and $2q' + 2 = d$ gives:

$$2q'+2 = d \Rightarrow 2q'+2 = 2(q-1)+2 \Rightarrow q' = \frac{n-1}{l} - 1 \Rightarrow q'l + r + 2 = n + 1 + r - l$$

$$\Rightarrow a = n + 1 + r - l \geq n + 1 + (l - b) - l = n + 1 - b$$

so we get $a + b = n + 1$, a contradiction since in this case $M_{a,b}$ is injective. Therefore, $2q' + 2 < d$ or $\mathrm{pd}(M_{a,b}) < d$.

So in any case, for any non-injective $M \in \Gamma(\mathrm{mod}A)_0$, we have $\mathrm{pd}(M) < d$ as claimed. This completes the proof that C is a d-cluster tilting module.

(only if) Suppose now that a $n \not\equiv 1 \mod l$ and we will show that a d-cluster tilting module does not exist. We consider the cases $n \equiv 0 \mod l$ and $n \equiv r \mod l$ with $2 \leq r \leq l-1$ separately.
If $n \equiv r \mod l$ with $2 \leq r \leq l-1$, then consider the injective modules $(n - (r-1), r)$ and $(n - (r-2), r-1)$. We have

$$n \equiv r \mod l \Rightarrow n = ql + r \Rightarrow n - (r-1) - 2 = ql + r - (r-1) = ql - 1 = (q-1)l + (l-1)$$

$$n \equiv r \mod l \Rightarrow n = ql + r \Rightarrow n - (r-2) = ql + r - (r-2) - 2 = ql + 0$$

and by Lemma 5.13 we have $\mathrm{pd}(n - (r-1), r) = 2(q-1) + 2 = 2q$ while $\mathrm{pd}(n - (r-2), r-1) = 2q + 1$. Since we have two injective modules which are not projective with different projective dimensions, by Lemma 3.35 there can be no d-cluster tilting module.

Finally, suppose $n \equiv 0 \mod l$. Then for all injective modules, that is $(n - (j-1), j)$ for $1 \leq j \leq l-1$, we have

$$n \equiv 0 \mod l \Rightarrow n = ql \Rightarrow n - (j-1) - 2 = ql - (j-1) - 2 = (q-1)l + (l - (j+1))$$

so $n - (j-1) - 2 = q'l + r'$ with $q' = q - 1$ and $r' = l - (j+1)$. We can see that $0 \leq l - (j+1) \leq l - 1$ since

$$1 \leq j \leq l-1 \Rightarrow 2 \leq j+1 \leq l \Rightarrow -l \leq -(j+1) \leq -2 \Rightarrow 0 \leq l-(j+1) \leq l-2 \leq l-1$$

and moreover $l-j-1<l-j$, so by Lemma 5.13 we have

$$\mathrm{pd}(n-(j-1),j) = 2q'+1 = 2(q-1)+2 = 2q-1$$

which means that all injective modules have the same projective dimension. So if M is the sum of all indecomposable projective and injective modules on $\Gamma(\mathrm{mod}A)_0$ and $\mathcal{C} = \mathrm{add}M$, M is a candidate for a d-cluster tilting module with $d = 2q-1$. Consider now the module $(n-1,1)$ and note that

$$n-1-2 = ql-3 = (q-1)l + (l-3) = q'l + r'$$

with $q' = q-1$ and $r' = l-3$. Since $l-3 < l-1$, we have by Lemma 5.13 $\mathrm{pd}(n-1,1) = 2q'+1 = 2(q-1)+1 = d$. Moreover, the same argument that we gave in the "(if)" part of the proof, where we proved that $\mathrm{Ext}^i(\mathcal{I},\mathcal{P}) = 0$ for $1 \leq i \leq d-1$ works for $(n-1,1)$, so we have that $\mathrm{Ext}^i(M_{n-1,1},\mathcal{P}) = 0$ for $1 \leq i \leq d-1$. Therefore, we have $(n-1,1) \in {}^{\perp_d}\mathcal{C}$. On the other hand, the injective dimension of $(n-1,1)$ can be calculated by Lemma 5.14. Since $n-(n-1+1) = 0 = 0\cdot l + 0$, we have that $\mathrm{id}(n-1,1) = 2\cdot 0 + 1 = 1$. By Lemma 3.14, this means that $\mathrm{Ext}^1(I, M_{n-1,1}) \neq 0$ for some $I \in \mathcal{I}$ and therefore $(n-1,1) \notin \mathcal{C}^{\perp_d}$. Since $\mathcal{C}^{\perp_d} \neq {}^{\perp_d}\mathcal{C}$, $M = \mathrm{add}\mathcal{C}$ is not a d-cluster tilting module.

Therefore, if there is to be a d-cluster tilting subcategory $\mathcal{C}$, it must have more elements than just the indecomposable projective and injective modules. Note that if an indecomposable non-projective and non-injective module N has projective or injective dimensions lower than the global dimension, it can not be an element of $\mathcal{C}$, by Lemma 3.14. So it must satisfy $\mathrm{pd}N = \mathrm{id}N = d$. But a similar argument as in the previous case, proves that $\mathrm{pd}N = d \Rightarrow \mathrm{id}N = 1$. Hence there exists no d-cluster tilting module in this case, which completes the proof. □

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