Abstract

We introduce an algorithm using interval arithmetic with directed rounding to rigorously compute interval enclosures of rotation numbers of diffeomorphisms on the circle. The algorithm is implemented as a C++ program using the CAPD library to handle the interval arithmetic. It is parallelized via OpenMP for increased efficiency. By utilizing multiple precision to reduce round-off errors, the enclosures can be made arbitrarily small.

Contents

1 Introduction

The theory of dynamical systems has been of interest to scientists and mathematicians for many centuries. It is a field that is extensively used to model physical, biological and economical systems. With the advent of computers and the explosion of calculation capacity in the last thirty years, dynamical systems have become a very active subject of research. It is an enchanting field where many simple-looking problems turn out to have immensely deep structure.

Another subject, known as validated or rigorous numerics, has also seen an enormous growth with the advent of computers. Its aim is to provide a way of doing computations in a rigorous way so that the result of a computation is mathematically proven to be correct. In this thesis, interval arithmetic is used to compute various quantities. It takes into account the limitations of the floating point-precision arithmetic used in modern computers and allows one to do computation in a rigorous way. At the heart of interval arithmetic lies the idea of *enclosure*, that is, instead of looking for a precise solution to some problem (which will never be obtainable using finite precision arithmetic), we look for some set which we can be certain contains our actual solution. It is then the goal to make this set as small as possible.

This thesis concerns specifically one-dimensional maps from the circle to itself. These maps are interesting for a variety of reasons. One of them is that they can be used to model several different physical phenomena [1]. There are examples of maps being used in the study of so called mode-locked circuits in electronics and the kicked rotor Hamiltonian in quantum mechanics. They are also used in biology to construct models of cardiac arrhythmias [2]. Additionally, the theory of one-dimensional maps has deep connections to other areas of mathematics. Finally, higher-dimensional systems can sometimes be studied as one-dimensional systems to gain insight into their dynamics. This approach was used by Guckenheimer to understand the first return map of the Lorenz attractor.

To a large extent, the thesis is an application of rigorous numerics to a problem in dynamical systems concerning one-dimensional maps of the circle to itself. The dynamics of such maps are essentially determined by a real number in the interval $[0,1]$, called the rotation number, which Poincaré introduced in 1885. Thus, given a circle map f, it is of interest to be able to calculate its rotation number. The goal of this thesis is to develop an algorithm for computing a tight enclosure of the rotation number given circle a specific kind of map. The algorithm is to use the developments in the field of validated numerics, rigorously taking calculation errors into account, ensuring with certainty that the rotation number of the map will always lie in the interval returned. Thus, while it might not be the most efficient of its kind, its strength will lie in that the result of a computation is always correct with absolute certainty.

2 Problem setting and formulation

In the following section, we will introduce our setting, formulate and motivate the problem which is the focus on the thesis. We study orientation preserving diffeomorphisms from the circle $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ to itself. That is, maps $f: \mathbb{S}^1 \to \mathbb{S}^1$, which are continuous, bijective and have continuous inverses. These maps preserve the cyclic order of $\mathbb{S}^1$, so for any $x, y, z \in \mathbb{S}^1$, $x < y < z \iff f(x) < f(y) < f(z)$.

We look at iterates f^k of f, inductively defined as $f^0(x) = x, f^k(x) = f\left(f^{k-1}(x)\right)$. We are interested in describing how the orbit $O(x) \coloneqq \{x, f(x), f^2(x), \ldots\}$ for some arbitrary x is distributed along $\mathbb{S}^1$. This task turns out to be a lot more complicated than one may expect. For example, take the most basic circle map, the rigid rotation $R_\alpha: x \longmapsto x + \alpha$ for some $\alpha \in \mathbb{R}$. This a common example of a circle map which looks very plain but shares some interesting features, of which the following proposition deals with the most famous one.

Proposition 1. *If $\alpha \in \mathbb{Q}$, $|O(x)|$ is finite for all x, i.e. R_α is periodic. If $\alpha \notin \mathbb{Q}$, $|O(x)|$ is infinite for all x and $O(x)$ is dense in $\mathbb{S}^1$.*

Proof. We begin by proving the first part of the statement. Let $\alpha = \frac{p}{q}$ with $p \in \mathbb{Z}, q \in \mathbb{Z}_+ \gcd(p, q) = 1$. Then

$$R_\alpha^q(x) = x + \frac{p}{q}q = x + p = x \bmod 1.$$

So we have $|O(x)| = q$ for every $x \in \mathbb{S}^1$. For the second part, assume $\alpha \notin \mathbb{Q}$. To see that there are no periodic orbits, notice that $R_\alpha^n(x) = x \iff x + n\alpha = x \iff n\alpha = 0 \bmod 1$. This is clearly impossible for α irrational.

That $O(x)$ is dense follows by contradiction, assume there is some open interval $I \subset \mathbb{S}^1$ of length ε such that $I \cap O(x) = \emptyset$. This requires that no points of $O(x)$ are closer than ε, since if there were two points in $O(x)$ closer than ε, say $n\alpha, m\alpha$ for $m \neq n$, we would be able to rotate x by $(m-n)\alpha$ until we landed in I (since I is of length ε).

However, this implies that $|O(x)| \leq 1/\varepsilon$ which is clearly a contradiction, thus $O(x)$ intersects every interval on the circle with nonzero length.

□

From this, it is clear that the dynamics of even very simple circle maps can have very rich structure indeed. To help us describe the distribution of orbits of more complicated maps, we define the *rotation number* of a given map f through the following process.

For a given map f, pick some point $x \in \mathbb{S}^1$ and let I_0 denote the half-open arc $[x, f(x)$ and I_1 its complement $\mathbb{S}^1 \backslash I_0 = [f(x), x)$. For any other point $y \in \mathbb{S}^1$, we define $\Gamma(y, N)$ as

$$\Gamma(y, N) = |\{k: 0 \leq k < N, f^k(y) \in I_0\}|.$$

Finally, the rotation number of f can now be defined as the limit

$$\rho_{x,y}(f) = \lim_{N \to \infty} \frac{\Gamma(y, N)}{N}.$$

Poincaré showed in 1885 that this limit always exists, and in fact is independent of the choice of y. We give a short proof similar to that in [3].

Proposition 2. If $f: \mathbb{S}^1 \to \mathbb{S}^1$ is an order-preserving diffeomorphism, $\rho(f)$ exists and is independent of y.

Proof. First, notice that the definition of Γ immediately yields $\Gamma(y, k + l) = \Gamma(y, k) + \Gamma(f^k(y), l)$.

Secondly, by looking at the discontinuities of $\Gamma(y, k)$ for some $k \in \mathbb{N}$, we see that for every pair $y, z \in \mathbb{S}^1$, $|\Gamma(y, z) - \Gamma(z, k)| \leq 1$ must hold. The reason for this is that the discontinuities must lie at one of the points of the type $f^{-i}(x)$ for $-1 \leq i < k$. This is because $N(y, z)$ changes value exactly when y crosses between I_0, I_1, who both have the boundary $\{x, f(x)\}$. Assume that y, z lies on opposite sides of one of these points, so $y < f^{-i}(x) < z$ and due to f being order-preserving, $f^i(y) < x < f^i(z)$. Then, we must also have $f^{i+1}(y) < f(x) < f^{i+1}(z)$, so the only possible discontinuities are at the smallest and largest possibilities, $f^{-k+1}(x)$ and $f(x)$. The discontinuities can only exist assuming that $f^{-k+1}(x) \neq f(x)$, in which case they divide the circle into two regions whose value differ by one.

From these two observations, we have that for all N, k, l that

$$|\Gamma(y, Nk + l) - \left(n\Gamma(y, Nk) + \ \Gamma(y, l)\right)| \leq N$$

Dividing by n and letting $N \to \infty$, we obtain $\left|\rho(f) - \frac{\Gamma(y,k)}{k}\right| \leq \frac{1}{k}$. Thus the limit always converges, and since $\Gamma(y, N)$ differs by either zero or one, it is independent of y, and we will henceforth denote it simply by $\rho(f)$.

$\square$

The rotation number intuitively measures the proportion of points of the orbit $O(y) :=$ $\{y, f(y), f^2(y), \ldots\}$ that are contained in I_0, and is always number between 0 and 1. It can also be shown that it is independent of the choice of x (assuming $\rho_x \neq 0$), and thus, it is purely a property of the map f and will henceforth be denoted simply by $\rho(f)$. It thereby gives us an asymptotic measure of how much the map f rotates points on $\mathbb{S}^1$ on average.

It is also sometimes useful to extend the map from $\mathbb{S}^1$ to $\mathbb{R}$. We do this by defining a *lift* F such that if π denotes the canonical transformation $\pi: \mathbb{R} \to \mathbb{R}\frac{}{\mathbb{Z}} = \mathbb{S}^1,\ \pi(x) = x \bmod 1$, then

$\pi \circ F = f \circ \pi$. These lifts are unique up to an integer shift. We restrict ourselves to orientation preserving maps, which are exactly maps for which $F(x+1) = F(x) + 1$. Using the lift F of f, we have an alternative but equivalent definition of the rotation number $\rho = \lim_{N\to\infty} \frac{F^N(x)-x}{N}$. This definition is often introduced as the standard definition of the rotation number (as an example, see [4]).

To stress the importance of the rotation number in understanding the dynamics of circle maps, we will give an exposition of some of its properties and how it determines the characteristics of a given map. We begin by proving an intuitive and reassuring proposition.

Proposition 3. The rigid rotation $R_\alpha(x) = x + \alpha$ has rotation number α.

Proof. Using the lift definition of ρ, we have $\rho = \lim_{N\to\infty} \frac{F^N(x)-x}{N} = \lim_{N\to\infty} \frac{x+N\alpha-x}{N} = \lim_{N\to\infty} \alpha = \alpha$.

□

The following proposition relates the rotation number to periodic points of the map in a straightforward way.

Proposition 4. An order-preserving diffeomorphism $f: \mathbb{S}^1 \to \mathbb{S}^1$ has a periodic point if and only if $\rho_f \in \mathbb{Q}$.

Proof. We give a proof similar to that in [4]. For the right implication, assume f has a periodic point. That is, there exists $q \in \mathbb{N}$ and $x_0 \in \mathbb{S}^1$ such that $f^q(x_0) = x_0$. This implies that the lift F satisfies $F^q(x_0) = x_0 + p$ for some $p \in \mathbb{Z}$. Thus, using the lifting definition and factoring $N = nq$ for $q \in \mathbb{Z}_+$, letting $n \to \infty$, we have

$$\rho(f) = \lim_{n\to\infty} \frac{F^{nq}(x) - x}{nq} = \lim_{n\to\infty} \frac{(x+np) - x}{nq} = \lim_{n\to\infty} \frac{np}{nq} = \frac{p}{q} \in \mathbb{Q}$$

For the left implication, assume that $\rho(f) = \frac{p}{q} \in \mathbb{Q}$. We first note that $\rho(f^q) = \lim_{N\to\infty} \frac{F^{qN}(x)-x}{N} =$ $q\rho(f) \bmod 1$. Thus $\rho(f) = p/q$ is equivalent to $\rho(f^q) = 0$. We now just need to show that if

$\rho(g) = 0$ for some map, it implies that g has a fixed point. We do this by contraposition, assuming g has no fixed points and showing that we then must have $\rho(g) \neq 0$ and $\rho(g) \neq 1$. Take a lift G and normalize it so $G(0) \in [0,1)$. Since g has no fixed point, there can be no points x such that $G(x) - x \in \mathbb{Z}$ since that would mean $\pi(x)$ was a fixed point. Thus, since $G(x) - x$ is a continuous function, we have, using the Intermediate Value Theorem,

$$0 < G(x) - x < 1.$$

Since [0,1] is compact and the function continuous, it must achieve its extreme values inside the set. Therefore, we can bound the function from above and below using some $\delta > 0$ such that

$$0 < \delta < G(x) - x < 1 - \delta < 1$$

This inequality holds for all $x \in \mathbb{R}$ (since $G(x) - x$ is periodic), so we can set $x = G^k(0)$ for some k. Summing the inequality from $k = 1$ to n then yields

$$\sum_{i=1}^{n} \delta < \sum_{i=1}^{n} G\left(G^k(0)\right) - G^k(0) < \sum_{i=1}^{n} 1 - \delta$$

$$n\delta < G^n(0) < n(1 - \delta)$$

$$\delta < \frac{G^n(0)}{n} < 1 - \delta$$

By letting $n \to \infty$, we see that the rotation number is strictly contained in $[\delta, 1 - \delta]$, and thus it cannot be 0 or 1.

□

Theorem 1. (Denjoy 1932). If an order-preserving diffeomorphism $f: \mathbb{S}^1 \to \mathbb{S}^1$ has irrational rotation number α, and $\log f'$ has bounded variation over $\mathbb{S}^1$,then f is topologically conjugate to the rotation R_α.

For proof see [3]. The theorem explicitly states that there exists a continuous surjection h such that $h \circ f = R_\alpha \circ h$. This means that the iterated dynamics of f are essentially the same as for R_α, since

$$f^k = (h^{-1} \circ R_\alpha \circ h)^k = h^{-1} \circ R_\alpha^k \circ h.$$

By now, we hope it is clear that the rotation number plays a central role in understanding the dynamics of circle maps. This motivates main problem of focus for this thesis which is the following:

Given any order-preserving diffeomorphism $f: \mathbb{S}^1 \to \mathbb{S}^1$, find an enclosure of its rotation number $\rho(f)$.

By enclosure, we explicitly mean a closed interval $I \subset [0,1]$ such that $\rho(f) \in I$. Perhaps the question should be further refined as asking for an enclosure *as small as possible*, or it becomes somewhat trivial ($I = [0,1]$ is surely an enclosure of $\rho(f)$, but not a particularly tight or useful one). The main focus of the algorithm we develop will be that it will give validated results. In other words, we wish to control any errors made in the process of computation is a rigorous way so that the end result is proven to contain $\rho(f)$. To accomplish this, we will use methods from validated numerics.

As a last remark, we will introduce one particularly important family of maps. The *standard circle map* the is mapping $f(x) = x + \alpha + \varepsilon \sin(2\pi x) \mod 1$. This map is a diffeomorphism exactly when $f'(x) > 0$ on $\mathbb{S}^1$, that is, $1 + 2\pi\varepsilon \sin(2\pi x) > 0 \Leftrightarrow \varepsilon < \frac{1}{2\pi}$. It is widely studied not only for various models in applied mathematics, but also for the remarkable structure one obtains when viewing ρ as a function of (α, ε). In figure 1, the value of the standard circle map with parameters $\alpha = 0.3, \varepsilon = 0.1$ is shown for $0 \leq x \leq 1$.

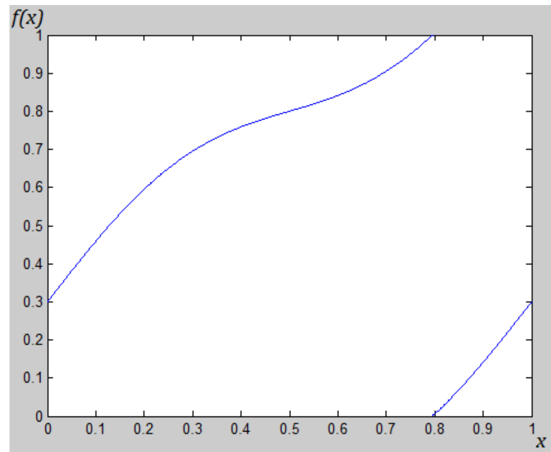

Figure 1. *The standard circle map for* $\alpha = 0.3, \varepsilon = 0.1$.

3 Interval arithmetic

As mentioned previously, one of the main goals of the algorithm is to generate an enclosure in a rigorous way so that we are certain the result is always correct. In this section, we will elaborate on exactly how this is done. Essentially, the idea is to introduce a way of keeping track of the possible rounding errors at each computation. This is done by replacing floating point computations with floating point interval computations. The result of any operation is an interval which is calculated in such a way as to ensure that the actual result is guaranteed to be contained in the interval. This technique of rigorous computations is known as interval arithmetic with directed rounding.

We begin by giving an elementary introduction to the interval arithmetic that will be necessary for our algorithm (for a thorough development, see [5]). Let $\mathbb{IR}$ denote all closed intervals on the real line, that is $\mathbb{R} = \{[a,b]: a \leq b, a, b \in \mathbb{R}\}$. We will use bracket notation $[a] = [\overline{a}, \underline{a}]$ for intervals, with $\overline{a}, \underline{a}$ denoting the upper and lower endpoints of the interval, respectively.

Let $\star$ be one of the standard arithmetic operations, $\star \in \{+, -, \cdot, /\}$. We define these operations for $[a], [b] \in \mathbb{IR}$ as $[a] \star [b] := \{a \star b: a \in [a], b \in [b]\}$, where we require that $0 \notin [b]$ for division.

An immediate and useful consequence of this definition is what is known as *inclusion isotonicity* of intervals, that is

$$[a] \subseteq [a'] \text{ and } [b] \subseteq [b'] \Longrightarrow [a] \star [b] \subseteq [a'] \star [b']$$

The result of an arithmetic operation is always a new interval, whose endpoints can be characterized by the following relations:

$$[a] + [b] = [\underline{a} + \underline{b}, \overline{a} + \overline{b}]$$

$$[a] - [b] = [\underline{a} - \overline{b}, \overline{a} - \underline{b}]$$

$$[a] \cdot [b] = [\min(\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b}), \max(\underline{a}\underline{b}, \underline{a}\overline{b}, \overline{a}\underline{b}, \overline{a}\overline{b})]$$

$$[a] \div [b] = [a] \cdot \left[1/\overline{b}, 1/\underline{b}\right], 0 \notin b$$

We can extend this to characterize the results of evaluations of standard functions, i.e. functions in the class $\mathfrak{G} = \{a^x, \log_a x, x^{\frac{p}{q}}, |x|, \sin x, \cos x, \tan x, \arcsin x, \ldots\}$. We can then extend it further, to combinations of standard functions using constants, arithmetic operations, and compositions. The class of such functions will be denoted $\mathfrak{E}$. Extending our interval arithmetic to this class, we can evaluate expressions like $[x]^2 + [x]$, $e^{\sin[x]}$, etc. Next, we introduce a definition and a theorem which will be essential for extending our circle maps to interval maps.

Let $f \in \mathfrak{E}$. We will denote the range of f over $[a]$ as $R(f, [a]) = \{x \in \mathbb{R}: \exists a \in [a]: f(a) = x\}$. We denote by $f([x])$ the *interval extension* of f, which is obtained by replacing every occurrence of the variable x with the interval $[x]$ and evaluating using the rules of interval arithmetic.

Theorem 2 (Range Enclosure). *For any function $f(x) \in \mathfrak{E}$ composed of standard functions, we have*

$$R(f, [x]) \subseteq f([x]).$$

For proof see [5]. This essentially says that an evaluation of an interval extension of a function will always yield an interval that contains the actual range of the corresponding function over that interval. This is the central property that allows us to use interval arithmetic to enclose the result of a function evaluation.

4 Directed rounding

When doing actual computations, we will no longer be working with the field $\mathbb{R}$, but with the floating point numbers $\mathbb{F}$. Since $\mathbb{F}$ is a finite set that is not closed under the arithmetic operations, we must implement careful rounding to ensure that the result of any operation encloses the true result. Thus, we need a specific arithmetic for floating point intervals using rounding in such a way that the interval always encloses the result had the operation been computed in $\mathbb{IR}$.

Our goal is to restrict the arithmetic on $\mathbb{IR}$ to the set $\mathbb{IF} = \{[a, b]: a \leq b, a, b \in \mathbb{F}\}$ in a way so that any operation on $\mathbb{IF}$-intervals always contains the result of the corresponding operation on the same intervals in $\mathbb{IR}$. To provide a practical way of handling numbers that are too large for our number system to handle, we will extend $\mathbb{R}$ to $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$ and $\mathbb{F}$ to $\mathbb{F}^* = \mathbb{F} \cup \{-\infty, +\infty\}$. The interval sets are extended analogously. We will use two modes of rounding, *toward* ∞, $\Delta(x)$ and *toward* $-\infty$, $\nabla(x)$. These are maps from $\mathbb{R}^*$ to $\mathbb{F}^*$ defined as $\Delta(x) = \max\{y \in \mathbb{F}^* : y \leq x\}, \nabla(x) = \min\{y \in \mathbb{F}^* : y \geq x\}$. We can now define our closed arithmetic operators on $\mathbb{IF}^*$

$$[a] + [b] = [\nabla(\underline{a} + \underline{b}), \Delta(\overline{a} + \overline{b})]$$

$$[a] - [b] = [\nabla(\underline{a} - \overline{b}), \Delta(\overline{a} - \underline{b})]$$

$$[a] \cdot [b] = [\min\left(\nabla(\underline{ab}), \nabla(\underline{a}\overline{b}), \nabla(\overline{a}\underline{b}), \nabla(\overline{a}\overline{b})\right), \max\left(\Delta(\underline{ab}), \Delta(\underline{a}\overline{b}), \Delta(\overline{a}\underline{b}), \Delta\left(\overline{a}\overline{b}\right)\right)]$$

$$[a] \div [b] = [a] \cdot [\nabla(1/\overline{b}), \Delta(1/\underline{b})]$$

As before, this can be extended to evaluation of other functions, and there are many software libraries that allow one to easily evaluate elementary functions using interval arithmetic with directed rounding.

We see that the result of an operation on $\mathbb{IF}^*$ always encloses the result of the corresponding operation on $\mathbb{IR}^*$. This is known as interval arithmetic with directed rounding, and it provides us with a rigorous way of doing computations that we can be certain will enclose the actual answer. This remains true for function evaluations, and thus, we can safely use the arithmetic to generate intervals which are certain to contain the actual results.

5 Construction of the algorithm

We are now ready to formulate our algorithm to calculate an enclosure of $\rho(f)$. Let f and F denote our given diffeomorphism and its lifting, respectively. The algorithm uses the initial definition of the rotation number, namely$\rho(f) = \lim_{N\to\infty} \frac{\Gamma(y,N)}{N}$, where $\Gamma(y, N)$ is the cardinality of the set $\{k: 0 \le k < N, f^k(y) \in I\}$. It assumes we are given two points $x, y \in \mathbb{S}^1$ and generates the half-open arc $I = \left[x, f(x)\right)$. The idea of the algorithm is to iterate f on y and count the number of iterates that are contained in I. We know that after k iterations, $\left|\rho(f) - \frac{\Gamma(y,k)}{k}\right| \le \frac{1}{k}$, equivalent to $\rho(f) \in [\frac{\Gamma-1}{k}, \frac{\Gamma+1}{k}]$. Thus, the algorithm depends on rigorous calculation of Γ. We introduce both a rigorous and a nonrigorous algorithm for comparison.

Nonrigorous algorithm (given two points x, y and a maximum number of iterations N):

Initialize $k \leftarrow 0, \Gamma \leftarrow 0, I \leftarrow \left[x, f(x)\right)$

While $k < N$

If $y \in I, \Gamma \leftarrow \Gamma + 1$

$y \leftarrow f(y), k \leftarrow k + 1$

Return $[\frac{\Gamma-1}{k}, \frac{\Gamma+1}{k}]$

Rigorous algorithm (given two points x, y and a maximum number of iterations N):

Initialize $k \leftarrow 0, \Gamma \leftarrow 0, I \leftarrow \left[x, \overline{f([x])}\right], [y] \leftarrow [y, y]$

While $k < N$

If $y \cap I \neq \emptyset$ and $y \cap (\mathbb{S}^1 \backslash I) \neq \emptyset$, Break loop

If $y \subset I, \Gamma \leftarrow \Gamma + 1$

$y \leftarrow f([y]), k \leftarrow k + 1$

Return $[\frac{\Gamma-1}{k}, \frac{\Gamma+1}{k}]$

Note that in the rigorous algorithm, the supremum of I, is computed as the supremum of $f([x, x])$.) Additionally, in the rare case that the algorithm terminates with $k = 0$, the whole circle (the interval [0,1]) is returned instead.

The main difference between the two algorithms, beyond the second one using analogous steps with intervals, is that the rigorous one contains a check at each iteration to see if $y \cap I \neq \emptyset$ and $y \cap (\mathbb{S}^1 \setminus I) \neq \emptyset$. If this kind of intersection with both I and $\mathbb{S}^1$ happens, we need to stop our algorithm and return whatever result we have. This is because the only certain information we have is that the actual $f^k(y)$ is to be contained in $[y]$. However, if $[y]$ spans over the boundary of I, there is no way to determine whether we should increase Γ or not. This is what makes our algorithm rigorous, by making sure we are never doing iterations in an area where computation errors could have messed up the result. Due to practical difficulties of dealing with intervals containing 0 when working modulo 1, we also chose to terminate the algorithm and return if $0 \in [y]$.

However, this check also introduces a clear limitation for how many iterations we will be able to perform. This is because the width $w([y]) = \overline{y} - \underline{y}$ grows very rapidly. As an example, take the standard circle map $f(x) = x + \alpha + \varepsilon \sin(2\pi x)$ with some $[y_0]$ on the circle and $[y_n] = f([y_{n-1}])$. Since the width is additive, we have that

$$w([y_n]) = w([y_{n-1}]) + w([\alpha, \alpha]) + w(\varepsilon \sin 2\pi [y_{n-1}])$$

$$= w([y_{n-1}]) + 0 + \beta_n w([y_{n-1}]) = (1 + \beta_n) w([y_{n-1}]).$$

For some small number β_n (dependent on where on the circle $[y_{n-1}]$ is situated. However, $\beta_n > 0$ always, so we have essentially exponential growth of the interval width. Thus, depending on the precision of the computation, after about 10^3 iterations the interval will usually be large enough so that a “bad” intersection is inevitable and the algorithm must halt. This puts an effective upper bound of how tight enclosure we are able to achieve.

One way of decreasing the width of the enclosure without increasing the precision of the system is to repeat the algorithm for a sequence of different pairs $\{(x_n, y_n\}_{n=1}^{M}$ and we will obtain a sequence of intervals $\{I_n\}_{n=1}^{M}$ in which each interval contains ρ. Thus, we can obtain a (hopefully) smaller interval by intersection, and our final result will be $I = \bigcap_{n=1}^{M} I_n$. The idea at work here is that different pairs of initial points result in different intervals, and the hope is that by iterating on many different points, we have a high probability of finding a good choice that will result in a small interval. The sequence $\{(x_n, y_n\}_{n=1}^{M}$ is chosen randomly from the circle. Due to the mentioned difficulties with intervals containing 0, it is often desirable to choose the x_n:s from some smaller subset, like [0,0.5] or [0,0.1] to avoid the case where $0 \in I$.

6 Implementation, results and conclusions

The algorithm was implemented in C++ code, using the CAPD library [6] for interval arithmetic. Since the same algorithm was to be run multiple times with different inputs, OpenMP was used to parallelize the process and increase performance. To investigate the effect of changing the number of initial points, the code was run for the standard circle map $f(x) = x + \alpha + \varepsilon \sin(2\pi x)$ with $\alpha = 0.25, \varepsilon = 0.125$. The effect of increasing the number of initial points, while the precision is held constant at 64bits, is displayed in Figure 1. In Figure 2, we can see the effect of increasing the precision of computation, while the number of initial points is held constant at 100.

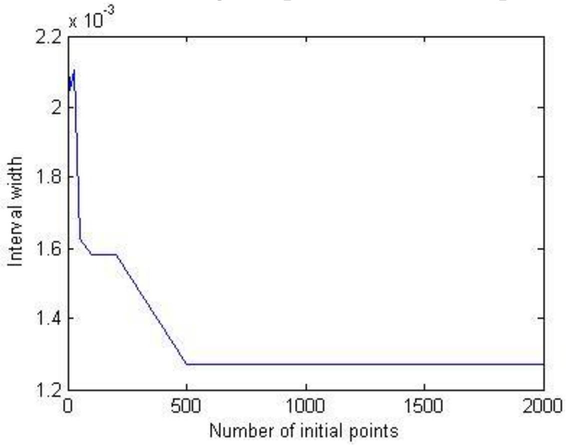

Figure 1. *Interval width as number of initial points increases (constant 64-bit precision)*

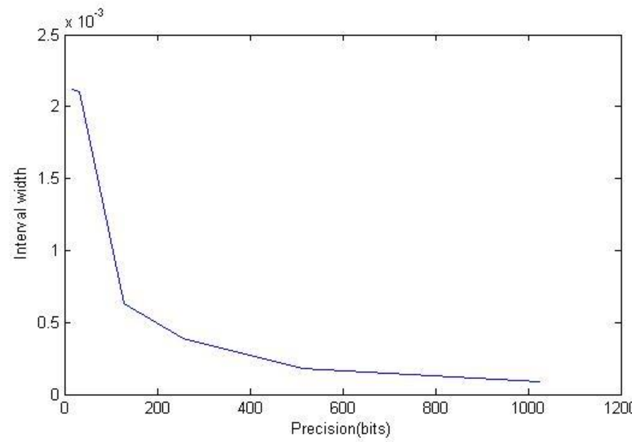

Figure 2. *Interval width as bit precision increases (constant number of 100 initial points).*

We can observe that as we increase the number of initial points the interval width decreases. However, at around 500 points, the width does not decrease any further. This is because the precision sets a limit for how narrow the final interval can get. Therefore, by increasing the precision, we can narrow the interval size down to any requested width.

In summary the algorithm works as intended, returning intervals which are certain to contain the rotation number of the map. The intervals are usually of width between 10^{-3} and 10^{-5}. By increasing the number of initial points we can decrease the final interval width, but only up to a certain point. Increasing the precision however, will always decrease the final interval width, at the cost of computation time. The main goal of the algorithm was however achieved, allowing one to generate validated enclosures with certain containment of the rotation number.

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