Different Replicator Equations in Symmetric and Asymmetric Games

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DIFFERENT REPLICATOR EQUATIONS IN
SYMMETRIC AND ASYMMETRIC GAMES

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1 Introduction

The dynamics of populations are very interesting and is a useful thing to study. It covers many areas; how the numbers of predators and prey relate, how a symbiosis or an arms race between two parts can evolve. It is a field both studied by biologists, economists, medical- and political scientists in order to better understand how animals, humans and corporations interact and affect each other.

In biology it is used to model the evolution of species and ecosystems [Smith, J.M. 1976]. How will different species influence each other in terms of health, food and reproduction? Which mutations will be successful, and how will different mutations impact the interaction between species?

In economics you can model how corporations and individuals behave [Mailath, G.J. 1998]. We can model monopolies and other market failures, or see when people act rationally or not.

In medical science it is used to model the spreading of viruses [Turner, P.E. and Chao, L. 1999]. This to get ideas about how fast a virus might spread, if there are any thresholds for the virus to spread efficiently, and how long it will take until an epidemic will halt given certain countermeasures.

All these problems can be analyzed from a game theory perspective, and give solutions or clues thereof. Our aim is to look at the classical replicator function and its dynamic properties, and then expand and change the function to model other dynamics. We will start with a population made up of two types of agents that interact with one another and analyze how the numbers of the different agents depend on how we set up the payoffs of the interactions. This can be used to model interactions between herbivores and carnivores, different players in a casino et cetera.

We will expand this to involve two different populations that each consists of two types of agents, where agents only interact with agents of the other population, and thus creates a dynamic between the ratios of agents within each population. This can be likened to a market consisting of different sellers and buyers (the populations), with different types of sellers and buyers (types of agents), where sellers only interacts with buyers, and the ratios of the different types will influence one another through this interaction. These will be our main cases, and we will study both using the replicator equation and our own equation.

1.1 Basic game theory

To model evolutionary behavior, game theory is a useful tool to start with. Game theory was introduced by von Neuman and Morgenstern [Nowak 2006, p. 45] as a tool to analyze strategic and economic decisions. It has many branches, but we will focus on the so called normal form game. In a normal form game you have two or more agents that are equipped with two or more strategies. Depending on what strategies the players choose they will get a different payoff, which is usually represented by a matrix where the element $a_{i,j}$ is
the payoff that an agent using strategy $i$ gets when the opponent uses strategy $j$. In the
game called “prisoners dilemma” we imagine two burglars being individually interrogated
by the police. Their payoff matrix becomes

<table>
<thead>
<tr>
<th></th>
<th>Sell out</th>
<th>Cooperate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sell out</td>
<td>-3</td>
<td>-1</td>
</tr>
<tr>
<td>Cooperates</td>
<td>-4</td>
<td>-2</td>
</tr>
</tbody>
</table>

Table 1: Prisoners dilemma

The burglars can either cooperate with each other and stay silent or sell out and tell the
police. If both decide to cooperate with the other and stay silent during the interrogation
they will both serve two years each due to lack of evidence. If one decided to sell out the
other while one decides to cooperate the police will reward the informer with a reduced
sentence of one year while punishing the silent cooperator with four years of prison. If
both decide to sell out they will both get three years of prison since the evidence against
them is stronger. The idea of the game is to show that the maximization of the individual
payoff results in a worse payoff for the group, since the best response for the burglars
as a unit is to cooperate, but the individual best choice is to sell out. A normal form
game can have different properties depending on their payoff matrices. In the prisoners
dilemma above, the so called dominating strategy is to sell out, since no matter what your
opponent’s strategy is, you always maximize your individual payoff by choosing the defect
strategy. We find a completely different dynamic if we look at the so called “Hawk-Dove”
game. Here we imagine two animals that are competing over a meal. Each animal can
choose to either be aggressive (i.e. “Hawk”) or friendly (i.e. “Dove”). An aggressor will
either fight another aggressor and gain nothing, or chase a friendly animal away and gain
the whole meal. A friendly stance will either flee from an aggressor and have to find
slimmer pickings, or share the meal with another friendly animal.

<table>
<thead>
<tr>
<th></th>
<th>Hawk</th>
<th>Dove</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hawk</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>Dove</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

Table 2: Hawk and dove

Here no dominant strategy exists and it is instead wisest to try and choose the opposite
of what your opponent chooses, which creates a whole other dynamic than the “prisoners
dilemma” game.

1.2 The replicator equation

In evolutionary game theory we switch the idea of people choosing strategies to the idea
of a large group of people where individuals have only one strategy. If we take the “Hawk-
Dove” game as an example then instead of two animals meeting and choosing a strategy,
we will have a big population of animals made up of the two types “Hawks” and “Doves”.
The individual animals will now randomly meet one another and play the game and get
their respective payoff. The outcome of all these interactions will then influence the rate
of reproduction for each type, which is modeled by a replicator equation (4).
In our population we denote the percentage of “Hawks” $x$, and the percentage of “Doves”
Table 3: Payoff matrix

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>1-x</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>1-x</td>
<td>c</td>
<td>d</td>
</tr>
</tbody>
</table>

We will also introduce a more general payoff matrix (Table 3). This means that for a hawk, the average payoff will depend on the ratio of hawks to doves in the population, and can be written

\[ f_x = xa + (1-x)b \]  

(1)

Analogous for a dove we get a payoff of

\[ f_{1-x} = xc + (1-x)d \]  

(2)

Now we can also express the average payoff of the whole population

\[ \phi(x) = xf_x + (1-x)f_{1-x} \]  

(3)

Given our example the “Hawk-Dove” game, we see that similarly to the normal form game, this evolutionary game does not have a clear winner. If we have a big population made of “Hawks” and introduce a few “Doves”, which can be done by setting \( x = 1 - \epsilon \), \( \epsilon > 0 \) for some small \( \epsilon \), the “Doves” will have a higher average payoff than the “Hawks” (and vice versa for a few “Hawks” in a big population of “Doves”). Their respective payoffs would be close to 0 for the hawks, and close to 1 for the doves, since our \( x \) would be close to 1. Since the “Doves” are doing better than the “Hawks” in this setting, it would be reasonable to assume that the population of doves grew as time passed. One way to model this is with a so called replicator equation

\[ \dot{x} = x(f_x - \phi(x)) = x(1-x)(f_x - f_{1-x}) \]

= \[ x(1-x)(x(a-c) - (1-x)(d-b)) \]  

(4)

Here \( \dot{x} \) represents the the change in \( x \) over time, i.e. \( \frac{dx}{dt} \). The change at any given time is equal to the current amount \( x \) times the difference between \( x \)’s payoff and the average population payoff.

The replicator equation was first introduced in [Taylor and Jonker 1978] and described in detail in [Hofbauer and Sigmund 1998, p. 85].

We note that the last step makes the replicator equation for our given payoff matrix (Table 3) equal to the replicator equation for the payoff matrix

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>1-X</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>a-c</td>
<td>0</td>
</tr>
<tr>
<td>1-X</td>
<td>0</td>
<td>d-b</td>
</tr>
</tbody>
</table>

Table 4: Transitional payoff matrix

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>1-X</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>a'</td>
<td>0</td>
</tr>
<tr>
<td>1-X</td>
<td>0</td>
<td>b'</td>
</tr>
</tbody>
</table>

Table 5: Simplified payoff matrix

This means we can simplify our replicator equation by introducing a new payoff matrix
(Table 5) and a companion replicator equation (6) that is equal to our original equation (4) by defining two new payoffs (5)

\[ a' = a - c, \quad b' = d - b \]  

\[ \dot{x}' = x(1 - x)(f_x - f_{1-x}) = x(1 - x)(xa' - (1 - x)b') = x(1 - x)(x(a - c) - (1 - x)(d - b)) \]

We can see how the ratio of the two types change with our simplified replicator equation. We want to know when the change in \( x \) is positive, negative or zero. If \( x \in \{0, 1\} \) then \( \dot{x} = 0 \), so the population will not change if it is made up of only one type of agent. What happens when \( x \in (1, 0) \)? Can we have some stable ratio of \( x \) to \( 1 - x \), a point for \( x \in (0, 1) \) where \( x \) rests? To find out we set \( \dot{x}' = 0 \) and, taking away the factors for roots \( x_1 = 0, x_2 = 1 \), we solve for \( x \)

\[ 0 = x(1 - x)(xa' - (1 - x)b') = x - \frac{b'}{a' + b'} \]  

This gives us the root \( x_3 = b'/(a' + b') \), so we conclude that we have three roots: \( x_1 = 0, x_2 = 1, x_3 = b'/(a' + b') \). The third root however is only defined if it lies in \((0, 1)\), which it does during certain conditions

\[
0 < \frac{b'}{a' + b'} \Rightarrow \begin{cases} 
0 < b', 0 < a + b \\
0 > b', 0 > a' + b'
\end{cases} \\
1 > \frac{b'}{a' + b'} \Rightarrow \begin{cases} 
0 < a' + b', 0 < a' \\
0 > a' + b', 0 > a'
\end{cases} \\
\text{thus } 0 < \frac{b'}{a' + b'} < 1 \Rightarrow \begin{cases} 
0 < a', b' \\
0 > a', b'
\end{cases}
\]

We can translate (8) to our payoff matrix (Table 3)

\[
\begin{cases} 
0 < a', b' \Rightarrow 0 < (a - c), (d - b) \Rightarrow c < a, b < d \\
0 > a', b' \Rightarrow 0 > (a - c), (d - b) \Rightarrow c > a, b > d
\end{cases}
\]

We will denote such a rest point

\[ x = \frac{b'}{a' + b'} = x^*. \]

There are now two possible rest points, stable and unstable.

If it is stable, any population that is sufficiently close to this point will move towards it. For us, sufficiently close means in the interval \((0,1)\). That is, if \( x < x^* \) then \( \dot{x} > 0 \), so that \( x \) will increase until it reaches \( x^* \) where it will rest since \( \dot{x}(x^*) = 0 \). Analogously if
Figure 1: Stable rest point in $(0, 1)$ for the replicator equation

$x > x^*$ then $\dot{x} < 0$ and $x$ will decrease until $x = x^*$. This is illustrated in (Figure 1).

However, if it is unstable then the reverse is true, and any $x \neq x^*$ will move away from $x^*$, since

$$x < x^* \Rightarrow \dot{x} < 0 \quad \land \quad x > x^* \Rightarrow \dot{x} > 0$$  \hspace{1cm} (11)

To determine whether we have a system with a stable or unstable rest point we take the derivate of our replicator equation and set $x = x^*$ and see whether it is positive or negative. A positive derivative at $x^*$ means a positive slope, thus our $\dot{x}$ will have the properties of an unstable rest point, and a negative slope indicates a stable rest point. Starting with the derivative of $\dot{x}$ with respect to $x$

$$\frac{d}{dx} \dot{x} = 2x(a' + b') - b' - 3x^2(a' + b') + 2xb'$$  \hspace{1cm} (12)

which for $x = x^* = \frac{b'}{a'+b'}$ becomes

$$\frac{d}{dx} \dot{x}(x^*) = b'(1 - \frac{b'}{a'+b'})$$  \hspace{1cm} (13)

Since

$$\frac{b'}{a'+b'} = x^* \in (0, 1) \quad \Rightarrow \quad (1 - \frac{b'}{a'+b'}) > 0 \quad \Leftrightarrow \quad \text{sgn}(\frac{d}{dx} \dot{x}(x^*)) = \text{sgn}(b')$$  \hspace{1cm} (14)

(14) tells us that the rest point will be stable for $d - b = b' < 0$ and unstable for $d - b = b' > 0$. Since we can only have one rest point in $(0, 1)$, its type will determine whether $x_1 = 0, x_2 = 1$ are stable or unstable rest points themselves. A stable rest point in $(0, 1)$ means that they are both unstable, and vice versa. The unknown case is when we do not
have a stable rest point between them, so we check what sign the derivative will have at $x_1 = 0$, $x_2 = 1$ respectively

$$\frac{d}{dx} \dot{x}(0) = -b' = b - d \quad (15)$$

$$\frac{d}{dx} \dot{x}(1) = -a' = c - a \quad (16)$$

So $x_1 = 0$ is unstable when $b > d$ and stable when $b < d$, and $x_2 = 1$ is stable when $c < a$ and unstable when $c > a$. When one endpoint is stable and the other one is not, then we will not have a rest point in between. When both are of the same kind there will exist one. If either of $a', b'$ is zero then we will not have a rest point between the endpoints and the stability of the endpoints will be determined by the sign of the non-zero payoff value. If both are zero we have a straight line between the endpoints, i.e. infinitely many rest points between the end points. This sections results are in accordance with [Nowak 2006] and [Hofbauer and Sigmund 1998].

Our results are summed up by (Figure 2), showing the general behavior of the replicator equation for different payoff values.

1.3 Two populations

In our next example we imagine a market with two different populations, buyers and sellers. These two populations both consists of two types of individuals, we have the honest and dishonest sellers, and the naive and critical buyers. In a market dominated by naive buyers dishonest sellers will do better than honest ones, but increase the dishonest seller and the buyers will catch up and become more critical, and so on. The catch is that the types within a population are directly independent of each other, but indirectly influencing each other by affecting the other population. We can model such a scenario with the help of the replicator equation, as was first shown by [Schuster 1981] cited by
Let us denote percentage of each type within their respective population by $x$ for the honest sellers, $(1-x)$ for the dishonest sellers, $y$ for the naive buyers and $(1-y)$ for the critical buyers. Their payoff matrixes are then given by

\[
\begin{array}{c|cc}
\text{x: Honest} & y: \text{Naive} & 1-y: \text{Critical} \\
\hline
1-x: \text{Dishonest} & a & b \\
\end{array}
\]

Table 6: Sellers payoff matrix

\[
\begin{array}{c|cc}
\text{x: Honest} & y: \text{Naive} & 1-x: \text{Dishonest} \\
\hline
1-y: \text{Critical} & e & f \\
\end{array}
\]

Table 7: Buyers payoff matrix

and their replicator functions are

\[
\dot{x} = x(1-x)(f_x - f_{1-x}) = x(1-x)(ya + (1-y)b - yc - (1-y)d) \\
= x(1-x)(y(a-c) - (1-y)(d-b)) \\
\dot{y} = y(1-y)(f_y - f_{1-y}) = y(1-y)(xe + (1-x)f - xg - (1-x)h) \\
= y(1-y)(x(e - g) - (1-x)(h - f))
\]

Similarly to our single population case, we can simplify our equations by setting

\[
a' = a - c, \quad b' = d - b, \quad c' = e - g, \quad d' = h - f
\]

getting replicator equations

\[
\dot{x} = x(1-x)(y(a') - (1-y)(b')) \\
\dot{y} = y(1-y)(x(c') - (1-x)(d'))
\]

Again we look for solutions to $\dot{x} = 0$, $\dot{y} = 0$ to find rest points, and then finding out when they are stable or unstable. Since we are now dealing with a

By our earlier investigations we know that we have three roots for each replicator equation, the two trivial roots being $x, y \in \{0, 1\}$. The third solution depends on the other population, and are

\[
\dot{x}(x, \frac{b'}{a' + b'}) = 0, \quad \dot{y}\left(\frac{d'}{c' + d'}, y\right) = 0
\]

but since both replicator equations depend on each other, both must be zero in order for us to have a rest point. Thus we have a total of five possible rest points that are

\[
(x, y) \in \left\{(0, 0), (0, 1), (1, 0), (1, 1), \left\{\frac{d'}{c' + d'}, \frac{b'}{a' + b'}\right\}\right\}
\]

where the points $x = \frac{d'}{c' + d'}, y = \frac{b'}{a' + b'}$ will only be rest points for the system if they are in $(0, 1)$ which, by (8), will happen when

\[
\text{sgn}(a') = \text{sgn}(b') \quad \text{and} \quad \text{sgn}(c') = \text{sgn}(d')
\]
If so, we will denote

\[ (x^*, y^*) = \left( \frac{d'}{c' + d'}, \frac{b'}{a' + b'} \right) \]  

(23)

Evaluating when the respective replicator equations will be negative or positive is now done similarly to the one population case

\[
\begin{align*}
  y \in (0, 1) & \implies y(1-y) > 0 \implies \\
  \Rightarrow & \begin{cases} 
    \dot{y} = y(1-y)(xc' - (1-x)d') > 0 & \Rightarrow x > \frac{d'}{c' + d'} \\
    \dot{y} = y(1-y)(xc' - (1-x)d') < 0 & \Rightarrow x < \frac{d'}{c' + d'}
  \end{cases}
\]
\]

(24)

The case for \( \dot{x} \) follows analogously.

A rest point for our single population case could take two forms, unstable or stable, but since we have two populations, a rest point can one of three types: stable-, unstable- and saddle point. A stable rest point, also called a sink, is a rest point which both populations, if close to, will move towards by time. An unstable rest point, also called source, is a rest point that both population, if close to, will draw away from by time. A saddle point will have both populations, if sufficiently close, move both towards it and away from it. A useful method to find out what type a rest point is to evaluate the eigenvalues of the system using the Jacobian determinant [Hilborn 2000, chapter 2, section 3.14]

\[
\det(J) = \begin{vmatrix}
  \frac{dx}{dx} & \frac{dx}{dy} \\
  \frac{dy}{dx} & \frac{dy}{dy}
\end{vmatrix}
\]

by solving \( \frac{d\dot{x}}{dx} - \lambda_1 \left( \frac{d\dot{y}}{dy} - \lambda_2 \right) - \frac{d\dot{y}}{dx} \frac{dx}{dy} = 0 \)  

(25)

Inserting the critical points from (23) into (25) we get the following eigenvalues:

\begin{align*}
  J_{0,0} & \Rightarrow (\lambda_1, \lambda_2) = (-b', -d') \\
  J_{0,1} & \Rightarrow (\lambda_1, \lambda_2) = (a', d') \\
  J_{1,0} & \Rightarrow (\lambda_1, \lambda_2) = (b', c') \\
  J_{1,1} & \Rightarrow (\lambda_1, \lambda_2) = (-a', -c')
\end{align*}

(26)

The eigenvalues of a rest point fall into five cases with a corresponding type of rest point

\begin{align*}
  \lambda_1, \lambda_2 > 0 & \Rightarrow \text{unstable rest point, also called source} \\
  \lambda_1, \lambda_2 < 0 & \Rightarrow \text{stable rest point, also called sink} \\
  \lambda_1 > 0 \text{ or } \lambda_1 < 0 < \lambda_2 & \Rightarrow \text{unstable rest point, called saddle} \\
  \lambda_1, \lambda_2 \text{ complex conjugates, no real part} & \Rightarrow \text{orbits}
\end{align*}

(27)

When dealing with two separate populations it is helpful to visualize the dynamics between them with a vector field

\[ V(x, y) = \dot{x}(x, y)i + \dot{y}(x, y)j \]  

(28)
where a point \((x, y)\) has \(\dot{x}\) as its horizontal component and \(\dot{y}\) as its vertical component, forming a vector whose direction is towards the new population composition that \((x, y)\) will move towards as time flows. Selecting a few points and drawing their vectors will give us an idea of how the dynamics of our populations will look given specific payoff matrices [Simmons, G. F. and Krantz, S. G. 2011, pp. 446].

The general behavior of our system can, with a schematic of its vector field, be well described in (Figure 3) where you can easily determine whether your payoff matrices will result in a central saddle or orbit, and which endpoints (corners in the figure) will be stable or unstable. We can also easily produce payoff matrices given a wanted behavior of our vector field. For example, lets say we want to model our earlier market of sellers and buyers, and we want to know when the system will create a loop like this

\[
\ldots \text{Naive} \uparrow \rightarrow \text{Dishonest} \uparrow \rightarrow \text{Critical} \uparrow \rightarrow \text{Honest} \uparrow \rightarrow \text{Naive} \uparrow \ldots
\]

(29)

By looking at (Figure 3) we realize that we need to fulfill \(d', c' > 0, a', b' < 0\), so we set up our payoff matrices

\begin{table}
<table>
<thead>
<tr>
<th>(x: \text{Honest})</th>
<th>(y: \text{Naive})</th>
<th>(1-y: \text{Critical})</th>
<th>(x: \text{Honest})</th>
<th>(1-x: \text{Dishonest})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1-x: \text{Dishonest})</td>
<td>3</td>
<td>2</td>
<td>(y: \text{Naive})</td>
<td>4</td>
</tr>
<tr>
<td>6</td>
<td>-1</td>
<td>(1-x: \text{Dishonest})</td>
<td>3</td>
<td>2</td>
</tr>
</tbody>
</table>
\end{table}

Table 8: Sellers payoff matrix

Table 9: Buyers payoff matrix
Figure 4: Vector field for two population replicator equation using the payoff matrices in Table 8 and Table 9

Giving us the parameters

\[ a' = 3 - 6 = -3, \quad b' = -1 - 2 = -3, \quad c' = 4 - 3 = 1, \quad d' = 2 - (-1) = 3 \]

\[ x^* = \frac{d'}{c' + d'} = \frac{3}{4} = 0.75, \quad y^* = \frac{b'}{a' + b'} = \frac{-3}{-6} = 0.5 \]  

This results in the vector field seen in (Figure 4) which, as our analysis told us, is consisting of orbits around a rest point \((x^* = 0.75, y^* = 0.5)\), going counter clockwise as shown in (Figure 3).
2 Changing the replicator equation

2.1 Concept and first analysis

Now that we know how the classical replicator equation works for our examples, we might think of things that are left out by the model, but that we could imagine having an impact on the changes in the population. The main idea is to introduce imitation or peer pressure into the model. The idea is that a person with a certain strategy, but with the choice of changing to another strategy, will be influenced by both the strategy of her peers and the payoff of the available strategies. This means that a strategy that has a high payoff might not be successful, immediately or ever, because it is not used by the majority. One popular example of this behavior was made by Solomon Asch, showing that individuals go against their own beliefs when under peer pressure [Rowe, M. 2013].

The changed replicator equation that we are introducing is of the following form

\[
\dot{x} = (1 - x) \frac{xf_x}{xf_x + (1 - x)f_{1-x}} - x \frac{(1-x)f_{1-x}}{xf_x + (1 - x)f_{1-x}} + \gamma(1 - x) - \gamma x \quad (31)
\]

The leftmost term is the rate at which the type \((1 - x)\) agents switch to become \(x\) agents. Since the important factors are the payoff \(f_x\) and the relative size of the \(x\) population, we set the rate of change to be a weighted product of these two factors, times the amount of \((1 - x)\) agents. The next term will then be the rate of change of \(x\) agents to \((1 - x)\), determined by the same weighted factors. The two rightmost terms represent changes by chance or whim. The \(\gamma\) parameter is the chance that an agent of either sort will change type due to some other factor that is random. The two rightmost terms represent the the addition of agents from \((1 - x)\) and the loss of \(x\) agents with a rate of \(\gamma\), thus \(\gamma \in (0, 1)\) is a reasonable assumption.

This model does not allow for much simplification of the payoff matrix compared to the replicator equation. We can divide both the weighted payoffs by one of the entries in the payoff matrix

\[
\frac{1/a}{1/a xf_x + (1 - x)f_{1-x}} = \frac{x(x + (1 - x)\frac{b}{a})}{x(x + (1 - x)\frac{b}{a}) + (1 - x)(x\frac{c}{a} + (1 - x)\frac{d}{a})} \quad \text{for } a \neq 0 \quad (32)
\]

and remove one variable, leaving four. This makes our analysis easier, but still quite difficult, why we will use a simplified payoff matrix (Table 10) based on the “Hawk-Dove” game. (Table 10) is a “Hawk-Dove” payoff matrix if one assumes \(T > 1\), \(1 > R > 0\), but we will also look at how our equation behaves outside these assumptions.

<table>
<thead>
<tr>
<th></th>
<th>x</th>
<th>1-x</th>
</tr>
</thead>
<tbody>
<tr>
<td>x</td>
<td>0</td>
<td>T</td>
</tr>
<tr>
<td>1-x</td>
<td>R</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 10: General “Hawk-Dove” matrix

Rewriting the equation we get

\[
\dot{x} = \frac{xf_x}{xf_x + (1 - x)f_{1-x}} - x + \gamma(1 - 2x) \quad (33)
\]
Figure 5: Areas when $\lim_{x \to 1} \dot{x} < 0$

which, given payoff matrix (Table 10), becomes

$$\dot{x} = \frac{x(1-x)T}{x(1-x)T + (1-x)(xR + (1-x))} - x + \gamma(1-2x)$$  \hspace{1cm} (34)

Looking at the endpoints

$$x = 1 \Rightarrow \dot{x} = \text{undefined} \hspace{1cm} (35)$$
$$x = 0 \Rightarrow \dot{x} = 0 - 0 + \gamma > 0 \hspace{1cm} (36)$$

we see that our $\gamma$ parameter contributes to make one endpoint unstable, since $\gamma > 0$. We can interpret this property as the introduction of a mutation or a newly invented strategy in an otherwise homogenous population. We do encounter a problem with the endpoint $x = 1$ since we will divide by 0. This point will be undefined for us given our payoff matrix, but does not necessarily stay undefined for all payoff matrixes. We can however evaluate the limit

$$\lim_{x \to 1} \frac{x(1-x)T}{x(1-x)T + (1-x)(xR + (1-x))} - x + \gamma(1-2x) = \frac{T}{T + R} - 1 - \gamma$$ \hspace{1cm} (37)

thus $\lim_{x \to 1} \dot{x} < 0$ $\iff$ $\frac{T}{T + R} < 1 + \gamma$ $\iff$ $\frac{T}{T + R} < 1$ $\iff$ $\begin{cases} T > -R, R > 0 \\ T < -R, R < 0 \end{cases}$ \hspace{1cm} (38)

This means that we will have unstable endpoints not only for $R, T > 0$ but also whenever (38) is fulfilled, which is illustrated in (Figure 5). This is not only useful as a way of modeling the introduction of mutations, but it also tells us that, for $\dot{x}$ continuous on $(0, 1)$, we will have a stable rest point in $(0, 1)$.

To find said stable rest point we need to solve for $x$ when $\dot{x} = 0$. The solutions are not easily analyzed, and finding out when they are defined (i.e. when a solution is $\in (0, 1)$) proves difficult. Instead we look at how the model behaves, and see if we can find out whether our equation has any more rest points. Firstly we observe that our equation is
not defined when
\[ x(R + T - 1) + 1 = 0 \quad \Rightarrow \quad x = \frac{-1}{R + T - 1} \quad \Rightarrow \]
\[ 0 < \frac{-1}{R + T - 1} < 1 \quad \Rightarrow \quad R + T - 1 < -1 \quad \Rightarrow \quad T < -R \] (39)

We note that the inequality (39), similarly to (38), never occurs in a "hawk-dove" game, but it is useful to know that the equation will be well defined as long as (39) does not hold. Assuming that (39) does not hold, we can rewrite \( \dot{x} \) as a polynomial of third degree. Thus we have two critical points which implies three possible rest points, as pictured in (Figure 6).

In order for both roots to be defined as in (Figure 6) we need both the local minimum and maximum to be within \((0, 1)\). Thus we can derivate (34) and set the derivate to zero to find our minimum and maximum, and find out if and when they are both in \((0, 1)\).

Taking the derivate with respect to \( x \)
\[ \frac{d}{dx} \dot{x} = \frac{T}{(x(T + R - 1) + 1)^2} - (1 + 2\gamma) \] (40)

and setting \( \frac{d}{dx} \dot{x} = 0 \) to find critical points
\[ 0 = \frac{xT(T + R - 1) + T - xT(T + R - 1)}{(x(T + R - 1) + 1)^2} - (1 + 2\gamma) \] (41)
\[ = x^2(-T - R + 1)^2 + x2(-T - R + 1) + \left(\frac{T}{1 + 2\gamma} - 1\right) \] (42)

The roots for the quadratic equation are given by
\[ x_{1,2} = \frac{-1 \pm \sqrt{T}}{(T + R - 1)} \] (43)
We will not have both our minimum and maximum within \((0,1)\) since for this to occur we need, due to symmetry, both

\[
0 < \frac{-1}{T + R - 1} < 1 \quad \text{and} \quad 0 < \frac{\sqrt{T}}{1 + 2\gamma} < 0.5 \quad \Rightarrow
\]

\[
(T + R - 1) < -1 \quad \text{and} \quad -0.5 < \frac{T}{1 + 2\gamma} < 0 \quad \Rightarrow \quad \frac{x}{\gamma}
\]

therefor we will only have one defined rest point with our new replicator equation. Additionally we see that we again have undefined roots both when \(T < 0\), since this leads to complex roots (since \(\gamma > 0\) we will not have to worry about it contributing to any complex roots), and when \((T + R - 1) = 0\), since this leads to division by 0. This will, similar to (38) and (39), not occur for a “Hawk-Dove” game, for which \(T > 0\). Plotting our replicator equation in (Figure 7) we see how our equation behaves under a “Hawk-Dove” game, and also its behavior with negative payoff values shown in (Figure 8). Our analysis indicates that the equation does not behave substantially different from the original replicator equation under the “Hawk-Dove” game, but it can differ when using negative values which produces the vertical asymptotes that we see in (Figure 8) where \(T = 0.5\) and we let \(R\) take on values from 0 to -3. These asymptotes are not necessarily bad, and are not occurring for all negative values either. An all negative values payoff matrix will not cause these asymptotes as they are due to the denominator \((xf_x + (1 - x)f_{1-x})\) being zero, which can and will occur when

\[
x = \frac{f_{1-x}}{f_{1-x} - f_x} \quad \Rightarrow \quad 0 < \frac{f_{1-x}}{f_{1-x} - f_x} < 1 \quad \Leftrightarrow \quad \begin{cases} f_{1-x} > 0 > f_x & \text{where } x \in (0,1) \\ f_{1-x} < 0 < f_x & \end{cases}
\]

Since the payoff functions \(f_x, f_{1-x}\) are determined by \(x\), which is positive, and of the payoff values, the sign of the payoff function is determined by the sign of the payoff values. Thus using payoff values with mixed signs could result in a non-continous equation.

Looking at the gamma function we realize that not only does it make an endpoint unstable, it also shifts the rest point towards \(x = 0.5\). This is of course expected since the larger the \(\gamma\) is, the more the equation is gonna be determined by chance. We see this clearly when setting \(\dot{x} = 0\) and taking the limit as \(\gamma\) goes to infinity, as the rest point becomes \(x = 0.5\)

\[
\lim_{\gamma \to \infty} \dot{x} = 0
\]

\[
\Leftrightarrow \lim_{\gamma \to \infty} \left( \frac{xf_x}{xf_x + (1 - x)f_{1-x}} - x + \gamma(1 - 2x) \right) = 0
\]

\[
\Leftrightarrow \lim_{\gamma \to \infty} \left( \frac{1}{\gamma} \left( \frac{xf_x}{xf_x + (1 - x)f_{1-x}} - x \right) + (1 - 2x) \right) = 0 \quad \Rightarrow \quad x = 0.5
\]

We will see another example of how our chance parameter effects the dynamics of the system later on.
Figure 7: $\dot{x}$ for a “Hawk-Dove” game

Figure 8: Vertical asymptotes of $\dot{x}$ as the payoff R gets negative
2.2 Tweaking imitation

With our new replicator equation we have not yet seen any clear imitation dynamic occur. What we would want from our model is to see rest points similar to those in (Figure 6) where we, despite a clearly dominating strategy, have a tipping point in the population where the imitation trump the larger payoff. Trying to emphasize the limitation we will square the population weight in our equation, which will have the form

$$\dot{x} = (1 - x) \frac{x^2f_x}{x^2f_x + (1 - x)^2f_{1-x}} - x \frac{(1 - x)^2f_{1-x}}{x^2f_x + (1 - x)^2f_{1-x}} + \gamma(1 - x) - \gamma x$$ \hfill (47)

The squared function will weigh down the payoff of the types that are in small numbers even more, and weigh down types in larger number much less than before. The difference in percentage of the older weight and the new one is

$$\frac{x - x^2}{x} = 1 - x$$ \hfill (48)

which is linear and increases as $x$ goes to zero, i.e. increasingly weighing down the payoff of a population as it grows smaller, which seems to attain the effect we are pursuing.

This however complicates our model further, and now both equations

$$\dot{x} = 0, \quad \frac{d}{dx} \dot{x} = 0$$ \hfill (49)

have solutions that are difficult to analyze. Since we can express the equation as a polynomial of fourth degree one might be able to find bounds for its roots. If such a bound could be more readily analyzed, it would clearly help us to understand the dynamic of this model. Some of our earlier conclusions still apply, like the fact that it will become undefined if (45) holds, since

$$x = \sqrt{\frac{f_{1-x}}{f_{1-x} - f_x}} \quad \Rightarrow \quad 0 < \sqrt{\frac{f_{1-x}}{f_{1-x} - f_x}} < 1 \quad \Leftrightarrow \quad 0 < \frac{f_{1-x}}{f_{1-x} - f_x} < 1 \quad \text{where} \quad x \in (0, 1)$$

From here on we will focus on the numerical analysis of this system, comparing it to both the replicator equation and our original improved equation. Since we are now dealing with three different equations, we will name and refer to them as follows

$$\dot{x} = x(f_x - \phi(x)) \quad \text{the replicator equation, or Rep}$$ \hfill (50)

$$\dot{x} = \frac{x f_x}{x f_x + (1 - x) f_{1-x}} - x + \gamma(1 - 2x) \quad \text{the new replicator equation, or New}$$ \hfill (51)

$$\dot{x} = \frac{x^2 f_x}{x^2 f_x + (1 - x)^2 f_{1-x}} - x + \gamma(1 - 2x) \quad \text{the tweaked replicator equation, or Tweak}$$ \hfill (52)

Since our replicator equation had a rest point in $(0, 1)$ whenever we have

$$\begin{cases} 0 < a', b' \Rightarrow & 0 < (a - c), (d - b) \Rightarrow \quad c < a, b < d \\ 0 > a', b' \Rightarrow & 0 > (a - c), (d - b) \Rightarrow \quad c > a, b > d \end{cases}$$ \hfill (9)
it is a good place to see if New and Tweak behave similarly when using the same payoff matrix. Note that we are now using New without restricting it to the previous “Hawk-Dove” payoff matrix (Table 10), and thus it can have more than one rest point. We will still avoid vertical asymptotes through the use of our earlier results. We will in the following graphs set $\gamma = 0.1$.

Table 11:
"Hawk-Dove"
Figure 9

Table 12:
"Coordination"
Figure

Table 13:
"Deviation"
Figure

Table 14:
"Prisoners dilemma"
Figure

Figure 9: Rep, New and Tweak for a “Hawk-Dove” game, payoff matrix (Table 11)

Figure 10: Rep, New and Tweak for a “Coordination” game, payoff matrix (Table 12)
Starting at (Figure 9) we see that Tweak has three stable points for the “Hawk-Dove” game, where New, as by our earlier results, does not. Thus we see that the imitation in Tweak takes over at around $x = 0.4$ and makes all populations with a lower $x$ go to the leftmost stable point. We still see that the stronger strategy is “Hawk”, which is clear in Rep and New, and this is still reflected in Tweak since the unstable point, a sort of imitation barrier, is left oriented.

Continuing with the “Coordination game” in (Figure 10) where either type $x, (1-x)$ benefits from a larger population of its own type, we see that all three equations are similar. In fact, adding the chance parameter to Rep would make all three equations have a very similar behavior, since Rep only deviates by not having unstable endpoints. We should note that the minimum and maximum of Rep and Tweak are further apart than those for New, making the change away from the unstable rest point towards the endpoints, or the stable rest points respectively, much more rapid.

In (Figure 11) we have the opposite situation of the “Cooperation” game, the “Deviation” game rewards interaction between different types of individuals, and is therefore similar to the “Hawk-Dove” game, although here with a symmetric payoff matrix (Table 13). Both Rep and New display this well by having a central stable point with a 50-50
mix of $x$ and $1-x$, maximizing the average payoff. Tweak, however, is again demonstrating the imitation taking overhand, making it so that even close to the 50-50 mark we see that the imitation weighs down the higher payoff of the minority type, making the payoff of the majority enough to push it towards its stable rest point. Note that the rate of movement away from the unstable rest point, the imitation barrier, is dampened by the nature of the game. Compared to the strong movement towards the stable rest point in Rep and New, the movement away from the unstable point in Tweak is quite weak.

Finally we have the “Prisoners dilemma” in (Figure 12), where we have a clear dominating strategy in $1-x$, which is the “defect and tell the police” strategy. This is well portrayed in Rep and New where $1-x$ is the winning type, with the stable rest point being very close to $1-x$ for New, only kept from the pure strategy rest point by the chance parameter. For tweak however, this is not the case and we yet again see an imitation barrier around $x = 0.7$ which means that whenever we have a higher amount of $x$ the imitation weight will downplay the large benefit of the $1-x$ payoff, and we can actually have a stable, high $x$ ratio rest point.

Since we saw that Tweak can have an unstable rest point for the “Hawk-Dove” game where New could not, we wonder if Tweak will always have an unstable rest point for this game. Would it be possible to raise the payoff of “Hawks” to lift the imitation barrier? In (Figure 13) we see that this is very much possible. It turns out it is also possible to change the values in “Deviation” (Table 13) and “Prisoners dilemma” (Table 14) so that the underlying idea of the game does not change, but the imitation barrier of Tweak is no longer present. Note that the nature of the “Coordination” game makes any such attempt futile, since the basis of the game revolves around a single strategy winning, thus creating the unstable rest point for all our equations.

Since we did not analyze New with more than two variables in the payoff matrix, we want to know if we, in a similar way, can introduce an imitation barrier into New for the “Deviation” game. It seems this is not the case as seen in (Figure 14) where we used the payoff matrix (Table 15). Our earlier analysis saying that New will only have one rest point for a “Hawk-Dove” game seems to be valid for more games, as the only game among our examples to have more rest points for New is “Coordination”, which, as we pointed out, could be produced with Rep plus the chance parameter.
2.3 The effects of chance

At the end of section 2.1 we briefly discussed the chance parameter $\gamma$ and how it would affect the endpoints by making them unstable, and the rest points by pushing them towards $x = 0.5$. We can now look at how the chance parameter affects New and Tweak when they have their imitation barrier. Since our result (46) can be applied to both New and Tweak (and Rep for that matter), we realize that our three rest points must either merge or disperse to become this single rest point $x = 0.5$. Plotting the rest points for “Deviation” (Figure 15) and “Hawk-Dove” (Figure 17) while changing $\gamma$ shows us that Tweak will morph towards a dynamic similar to Rep. In “Deviation” (Figure 15) we see the two stable rest points, as seen in (Figure 14), move towards the unstable point and merge with it, becoming a single stable rest point (since our endpoints have different signs). (Figure 16) shows us this process for three different levels of chance.

The intuition is that in a “Deviation” game you want to meet someone of the opposite type, so the population will gravitate towards a 50-50 ratio (if the game is symmetric, which it is here). Tweak will introduces an imitation barrier, which will be in the middle because of symmetry, and will slowly become weaker as switching from one type to another will be more and more influenced by chance rather than imitation, until a stable chance based rest point is established.

A similar process is shown for the “Hawk-Dove” game (Table 11) in (Figure 17 and 18). Note that the rest point that remains at higher chance levels never merges with the two other rest points. Interpreting this, we see that the main differences between the “Deviation” game and the “Hawk-Dove” game are that one is symmetric and the other one is not, and that one has in this case a stronger strategy. Looking at “Hawk-Dove” for Rep in (Figure 9) we see that $x$ is the stronger strategy having a somewhat larger
part of the population at the rest point. Introducing imitation translates into moving the existing rest point further towards $x = 1$ and introducing an imitation barrier and a rest point for $(1 - x)$ on the other side. By increasing the chance parameter we effectively weaken this imitation in the same manner as with “Deviation”, but here the stable rest point closer to $x = 0$ is the weaker one, and is thus lifted along with the imitation barrier.
2.4 Two populations

Now that we have a better understanding of New and Tweak we can formulate some thoughts on how they might behave when adding a population with similar or different payoff matrices. Note that we now again have a second population made up $y$ and $1-y$ types, where the growth of $y$ is determined by the New or Tweak function

New for $y$: \[
\dot{y} = (1-y) \frac{y f_y}{y f_y + (1-y) f_{1-y}} - y \frac{(1-y) f_{1-y}}{y f_y + (1-y) f_{1-y}} + \gamma (1-y) - \gamma y
\]

Tweak for $y$: \[
\dot{y} = (1-y) \frac{y^2 f_y}{y^2 f_y + (1-y)^2 f_{1-y}} - y \frac{(1-y)^2 f_{1-y}}{y^2 f_y + (1-y)^2 f_{1-y}} + \gamma (1-y) - \gamma y (53)
\]

where the payoff functions for each population is now solely dependent on the other population (and their payoff matrix)

\[f_x(y), f_{1-x}(y), f_y(x), f_{1-y}(x) \]  

(54)

In the following graphs we have introduced nullclines for each populations equation in order to make the graphs easier for interpretation. A nullcline of a differential equation is made up of all points for which that equation is zero, which for our equations translates in zero growth for any type in that population along its nullcline, which is made up of all points where $\dot{x} = 0$. Since a rest point is a point for which both equations are zero, they will occur whenever the nullclines of both equations intercept.

Since New behaved similarly to Rep, a reasonable assumption would be that it will behave similarly to New with two populations, the exception being the unstable endpoints. The main difference that we could expect is the orbits to behave differently, perhaps turning into stable or unstable spirals. Looking at (Figure 19) we see that New for the game (Table 16) has a stable spiral towards the rest point. This could very be due to the chance parameter, pushing the system towards the rest point just enough to turn a orbit into a stable spiral. Setting $\gamma = 0$ and running the same game again, the vector field for New looks identical to that of Rep in (Figure 19) and the stable spiral is replaced by what looks like orbits.
Looking at Tweak for the same game (Figure 20) with different values of $\gamma$ we see a very different dynamic. The imitation barrier is very present in the middle as an unstable spiral rest point. This imitation follows along the nullclines splitting the vector field into four areas, each with its own stable rest point, separated by saddle points. With a nonzero chance parameter we see that the stable rest points get pushed towards the middle in a way similar to the one population case.

In the same way that different payoff matrices lead to different dynamics when changing
Figure 21: Vector fields of Tweak for the game in (Table 17) with different chance parameters

Table 17: Example game for Tweak in (Figure 21)

<table>
<thead>
<tr>
<th>x</th>
<th>y</th>
<th>1-y</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>4</td>
</tr>
<tr>
<td>1-y</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

the chance parameter, as we saw in section 2.3, we can give the two populations different
payoff matrices that react differently to the chance parameter, and get very asymmetric
dynamics. In (Figure 21) we see that the different payoff matrixes (Table 17) lead to a
very asymmetric dynamic for $\gamma = 0.3$ where the $y$ population has basically a single stable
point whereas the $x$ population has two and an imitation barrier in between. These inter-
esting dynamics can occur when the two equations go towards more chance dependence.
interpreting the two graphs in (Figure 21), combining the weakening imitation and the
great payoff of $x$ interacting with $1 - y$, the effect is great enough to eliminate the stable
rest point close to $(0,0)$, and the payoff of $y$ interacting with $x$ is high enough to also
eliminate the rest point close to $(1,0]$, resulting in the dynamic of the right graph. With-
out the increased chance the imitation takes hold and creates all four stable rest points
seen in the left graph.

Comparing the three equations like this we clearly see the appeal of Tweak as it is
able to model very complex interactions between groups that clearly involves both the
imitation we were seeking and the element of chance that can dampen the pure drive
towards efficiency in a model and make it less rational. Comparing Rep and New we
can however not see any clear motivation of using New as Rep plus an eventual chance
parameter would produce very similar dynamics to those of New, although we have not
fully analyzed New and can not say anything definite.
3 Conclusions and reflexions

With the help of game theory we formulated the replicator equation, a powerful tool to model changes in populations depending on the interactions between the its different agents. Depending on the interactions, defined by a payoff matrix, the replicator equation behaves differently, and by summing up all its behaviors we can easily analyze either a given payoff matrix, or come up with a payoff matrix to fit a behavior sought. This means that we can either tell how a population with given interactions will behave according to the replicator equation, or quickly try to pinpoint what form unknown interactions might have by looking at the dynamics of a population.

Wanting to include the concept of chance and imitation into our model, we formulated a new equation, the idea of which is to model the behavior of peer pressure. How to model growth of Individuals who take into account not just larger payoffs, but also the composition of the population it is in, and who wants to be part of the majority. We also want our model to reflect irrational or random behavior of individuals. Our new model appears to not reflect a strong imitation behavior, closely resembling the original replicator equation. By changing the equation slightly to emphasize peer pressure further we got a much different result.

With the change we manage to produce the sought imitation behavior for several games. For our improved equation we no longer always see the rational, payoff maximizing being a stable equilibrium. Instead we have stable rest points near homogeneous compositions of the population. The equation could therefor model populations in which the usage of the majority’s strategy could grow compared to the minority’s strategy, despite the minority’s strategy having a higher payoff. For example in a group where conformity is the norm, and trying out new methods is not worthwhile. This could not occur with the replicator equation, which will always favor the stronger strategy.

It is important that “dominant enough” strategies will not result in this behavior, and that our equation will be similar to that of the replicator equation given high enough differences in payoff. Otherwise the dynamic would be completely determined by the composition of the population, and only weakly affected by the payoffs, which is not realistic for most social beings.

The equations also takes randomness or irrational decisions into account, which results in equilibriums that are less influenced by both peer pressure and payoff maximization. As individuals seldom has all information, and can make choices based on false logic or things not accounted for in our model, this is a reasonable addition to our model.

Although our model is difficult to analyze, it displays useful modeling qualities that the replicator equation is lacking, and is worth investigating further.
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