Construction of Two-Dimensional Topological Field Theories

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Abstract

In this thesis I thoroughly review the construction of topological quantum field theories (TQFT), with particular emphasis on the two dimensional examples. In the first part of the thesis, we consider the mathematical foundations of TQFT’s by introducing notions such as categories, cobordisms and Frobenius algebras. In this setting, an \( n \)-dimensional topological quantum field theory is a functor between the categories of cobordisms and vector spaces. In the two dimensional case, we show that to give a TQFT is equivalent to giving a Frobenius algebra. In the last part of the thesis, we investigate the Dijkgraaf-Witten model, which is a TQFT over principal \( G \)-bundles with \( G \) being a finite group.

Sammanfattning

Jag undersöker i detta arbete noggrant konstruktionen hos topologiska kvantfältteorier (TQFT), med särskild betoning på det tvådimensionella fallet. I första delen behandlar vi den matematiska grunden hos TQFT och introducerar begrepp som kategorier, kobordismer och Frobeniusalgebror. Genom att använda det kategoriteoretiska språket, definieras en \( n \)-dimensionell topologisk kvantfältteori som en functor mellan kategorierna av kobordismer och vektorrum. I det tvådimensionella fallet, visar vi att att ge en tvådimensionell topologisk kvantfältteori är ekvivalent med att ge en Frobeniusalgebra. I den avslutande delen undersöker vi Dijkgraaf-Wittenmodellen, som är en TQFT över principalknippen med diskret strukturgrupp.
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1 Introduction

The world of quantum physics is a vast subject. *Quantum field theory* (abbreviated QFT), is one of the main pillars of modern physics providing the fundamental framework for high energy physics. QFT is a very successful theory; it has lead to several exciting discoveries such as the standard model of particle physics. Despite all these considerable achievements, we are still far from a complete mathematical understanding of QFT. In fact, one of the central notions, the *functional integral*, is equipped with an ill-defined measure over the infinite dimensional space of fields. This problem has been a curse for physicist and mathematicians alike who has attempted to obtain a rigorous mathematical description of quantum field theory. Only in some very special cases has the struggle proved to be fruitful (typically in spacetime dimension 2). A possible way out from this conundrum is obtained by "loosening" the structures of a quantum field theory in order to generate a topological theory. *Topological (quantum) field theories* (typically abbreviated TQFT or TFT) in this thesis are independent of the metric. Furthermore, the observables of these theories are topological invariants. The TQFT's not only provide an interesting toolbox for calculating topological invariants but they also give a way to test different properties of QFT's in a mathematically "controlled" surrounding. In this setting the problems with the functional integration are absent. We believe that the study of these mathematically beautiful theories will yield deeper insight about realistic examples.

TQFT started with the pioneering papers of E. Witten 1989 [18], and many works since has been based on this subject. The first attempt to give an axiomatization of TQFT was made by Atiyah [3], inspired by the earlier works of Witten. Other developments were obtained by G.B. Segal [16] which treated the mathematical definition of conformal field theory and by Dijkgraaf, with his Ph.D. thesis [8] containing very closely related ideas.

In this thesis I will review thoroughly the construction of a two dimensional TQFT and in the final part work through some calculations of the two dimensional Dijkgraaf-Witten model [9].

This thesis is mainly based on the works by Aly [1], Bartlett [5] and Kock [13]. We begin, in section 2, by considering the mathematical foundations governing the TQFT's. In 2.1, we give some basic definitions found in category theory, amongst them the notion of *monoidal categories* and *monoidal functors* (the precise definition of the latter one however not given) being the ones which perhaps is the most used throughout the thesis. We proceed in 2.2 by discussing the notion of *cobordisms*, which can be thought of as some kind of manifold with an in-boundary propagating to an out-boundary. Afterwards, we put them in the context of categories. Since the thesis main focus is two dimensional theories, we also take a closer look at the geometry of 2-manifolds, in other words surfaces. The section 2.3 is devoted to the description of so called *Frobenius algebras*, which has an intimate relation with TQFT's in two dimensions. In this section, various definitions of a Frobenius algebra will be given, which then will be considered in categorical terms. Furthermore, a graphical representation of these mathematical structures will be developed. When this section ends, we have developed the mathematical language needed to describe a TQFT. This is done in section 3. In 3.1, we define a TQFT twice, first in terms of the mathematical theory described in section 2 and secondly the definition given by Atiyah. We look at some properties of the two definitions in order to compare them and later also look at the physical interpretation of TQFT's. When this is done, we move on to the two dimensional case in section 3.2 where we explore the link between cobordisms and Frobenius algebras. As previously, we look at some properties specific to the two dimensional case. In the last part of the thesis, in section 3.3, we investigate a particular TQFT known as the *Dijkgraaf-Witten model*, first the general definition and then move to the two dimensional case.

As a final remark I would like to stress the fact that these constructions have many interesting applications. For instance, TQFT can be applied to string theory and also very recently, TQFT has proven useful for the classification of topological phases of matter [12].

The reader is presumed to have some basic orientation in differential geometry and topology (mostly the notion of smooth manifolds, which otherwise can be thought of as some kind of generalised curve or surface which locally behaves like \( \mathbb{R}^n \)), linear algebra (e.g. basic definitions as vector spaces over a field (examples of a fields are \( \mathbb{R} \) and \( \mathbb{C} \)) and linear maps) and abstract algebra (with the notion of homomorphisms, i.e. structure preserving maps, being the most important). In order to understand all the physical interpretations of
this otherwise mathematically formulated thesis, some knowledge of quantum mechanics is required.

A remark on the terminology used, when we write "linear map" between vector spaces over a field $\mathbb{K}$ it should be understood as "$\mathbb{K}$-linear map". Also, throughout the thesis, all manifolds will be smooth.

The thesis was typeset in \LaTeX\ using Texmaker 4.4.1. The commutative diagrams in the thesis was made with the package "diagrams" (version 4) written by Paul Taylor. The cobordisms were drawn using the "tqft" package (version 2) provided by Andrew Stacey.

2 Mathematical Foundations

In this section we define the mathematical notions needed to describe topological quantum field theories. We begin by discussing some general notions from category theory, in which language the definition of a TQFT takes a short and elegant form. We then proceed by considering cobordisms and Frobenius algebras which will be used when we define a TQFT, in particular the two dimensional case.

2.1 Categories

Category theory is a very general field in mathematics with vast applications, not least in physics. Here, we only go through some basic definitions and build up the theory we need to make sense of our approach to TQFT. In the following sections, we mostly follow Awodey [4] and Mac Lane [14].

2.1.1 Preliminary Notions

A category is a collection of objects and arrows between objects such that the arrows compose in a nice way. More precisely, a category $\mathbf{C}$ consists of the following data:

(i) Objects $\text{Ob}(\mathbf{C})$ ($A, B, C, \ldots$)

(ii) Arrows (or morphisms) $\text{Ar}(\mathbf{C})$ ($f, g, h, \ldots$)

(iii) For each arrow $f$, we have assigned an object $\text{dom}(f)$ called the domain of $f$ and an object $\text{cod}(f)$ called the codomain of $f$. This is written $f : A \to B$, where $A = \text{dom}(f)$ and $B = \text{cod}(f)$.

(iv) For each arrow $f : A \to B$ and $g : B \to C$ (i.e. $\text{cod}(f) = \text{dom}(g)$), there exists an arrow $g \circ f : A \to C$ known as the composite of $f$ and $g$.

(v) For every object $A$, there exists an arrow $\text{id}_A$ called the identity arrow. Given two objects $A$ and $B$ with identity arrows $\text{id}_A$ and $\text{id}_B$ and an arrow $f : A \to B$, we have $f \circ \text{id}_A = f = \text{id}_B \circ f$.

(vi) We require the composition to be associative: $h \circ (g \circ f) = (h \circ g) \circ f$ for all $f : A \to B$, $g : B \to C$ and $h : C \to D$.

We begin by looking at some examples of simple finite categories in order to warm up and then proceed to larger categories (in fact, some categories are so "large" their collection of objects or arrows cannot be regarded as sets) and other different kinds of structures. Later in this text, we shall encounter other examples of categories, but with additional structure.

Example 2.1. The category $\mathbf{1}$ consists of one object and its identity arrow (which will be omitted in the diagrams from here on):

\begin{center}
\begin{tikzpicture}
  \node (a) at (0,0) {$\bullet$};
\end{tikzpicture}
\end{center}

(2.1)

The category $\mathbf{2}$ consists of two objects and their respective two identity arrows, together with an arrow between the objects:

\begin{center}
\begin{tikzpicture}
  \node (a) at (0,0) {$\bullet$};
  \node (b) at (1,0) {$\ast$};
  \draw (a) -- (b);
\end{tikzpicture}
\end{center}

(2.2)
The category $3$:

\[ \begin{array}{c}
\bullet \\
\downarrow \\
\ast \\
\end{array} \quad \begin{array}{c}
\downarrow \\
\ast \\
\end{array} \]

(2.3)

Note that the diagonal arrow is the composite of the vertical and horizontal ones. The category $0$ has no objects or arrows, so the picture is trivial.

**Example 2.2.** The category $\textbf{Set}$ of sets and mappings between sets, the category $\textbf{Top}$ of topological spaces and continuous maps, the category $\textbf{Grp}$ of groups and group homomorphisms, $\textbf{Vect}_K$, the category of vector spaces over a field $K$ and linear transformations are examples of large categories. Objects in each respective category are sets, topological spaces, groups and vector spaces and arrows are set maps, continuous maps, group homomorphisms and linear transformations respectively. The composition will be the usual composition we are accustomed to and the identity arrows are the respective identity mappings.

We may also consider mappings between categories which preserve the categorical structure. Such entities are known as **functors**. Given two categories $\mathcal{C}$ and $\mathcal{D}$, a functor $F : \mathcal{C} \to \mathcal{D}$ between them maps objects to objects and arrows to arrows such that

(i) $F(f : A \to B) = F(f) : F(A) \to F(B)$

(ii) $F(g \circ_C f) = F(g) \circ_D F(f)$

(iii) $F(\text{id}_A) = \text{id}_{F(A)}$

where $\circ_C$ and $\circ_D$ are the compositions in respective category.

There are various constructions on categories, one of them being the **product** of two categories; given two categories $\mathcal{C}$ and $\mathcal{D}$, the product between them $\mathcal{C} \times \mathcal{D}$ consists of objects $(C, D)$ for $C$ in $\mathcal{C}$ and $D$ in $\mathcal{D}$, with arrows $(f, g) : (C, D) \to (C', D')$. The composition and unit arrows are defined pointwise. From this we also have the **projection functors** $\pi_1 : \mathcal{C} \times \mathcal{D} \to \mathcal{C}$ and $\pi_2 : \mathcal{C} \times \mathcal{D} \to \mathcal{D}$ such that $\pi_1((C, D)) = C$ and $\pi_1((f, g)) = f$ and analogously with $\pi_2$ for the second entry.

### 2.1.2 Monoidal Categories

We need the concept of monoidal categories in order to give the nice definition of a TQFT as promised in the beginning of the chapter. We start by describing monoids.

A **monoid** $X$ is a set equipped with a binary operation $\ast : X \times X \to X$ such that the operation is associative and there is an identity element in $X$. In terms of commutative diagrams, following Kock [13] and Bartlett [5], a monoid $X$ is a set together with maps $\mu : X \times X \to X$ and $\eta : 1 \to X$ (here, $1$ is the set with only one element) such that the following diagrams commute:

\[ \begin{array}{c}
X \times X \times X \\
\downarrow \mu \times \text{id}_X \quad \downarrow \text{id}_X \times \mu \\
X \times X \\
\downarrow \mu \\
X \\
\end{array} \quad \begin{array}{c}
X \times X \\
\downarrow \mu \\
X \\
\end{array} \]

(2.4)
Here $\text{id}_X$ is the identity function on $X$ and the unlabeled arrows are canonical identifications. A monoid itself can be thought of as a category with one object, the object being the monoid itself and the arrows being the elements in the monoid. Composition will be the binary operation and the identity arrow will be the identity element. In the approach of commutative diagrams, we are very close to the definition of a (strict) monoidal category.

**Definition 2.1.** A strict monoidal category is a category $\mathcal{C}$ together with functors $\mu : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and $\eta : 1 \to \mathcal{C}$ (1 as in 2.1) such that the following diagrams commute:

$$
\begin{array}{ccc}
\mathcal{C} \times \mathcal{C} \times \mathcal{C} & \xrightarrow{\mu \times \text{id}_\mathcal{C}} & \mathcal{C} \\
\text{id}_\mathcal{C} \times \mu & & \mu \\
\mathcal{C} \times \mathcal{C} & \xrightarrow{\mu} & \mathcal{C} \\
\end{array}
$$

(2.6)

$$
\begin{array}{ccc}
1 \times \mathcal{C} & \xrightarrow{\eta \times \text{id}_\mathcal{C}} & \mathcal{C} \times \mathcal{C} \\
\mu & & \mu \\
\mathcal{C} & \xrightarrow{\mu} & \mathcal{C} \\
\end{array}
$$

(2.7)

where $\text{id}_\mathcal{C}$ is the identity functor on $\mathcal{C}$ and the unlabeled arrows are projections.

We will write $\mu((C, C'))$ and $\mu((f, f'))$ as $C \otimes C'$ and $f \otimes f'$, respectively. We will also adapt the notation $\eta(1) = 1$ for the neutral element (meaning we denote the object in 1 and its image $\eta(1)$ with the same symbol). The axioms could also be stated without the help of diagrams. The first diagram merely expresses the associativity condition,

$$(C \otimes C') \otimes C'' = \mu((\mu(C, C'), C'')) = \mu((C, \mu(C', C''))) = C \otimes (C' \otimes C'')$$

(2.8)

(the same with arrows $f$, $f'$ and $f''$). The two other diagrams expresses the unit condition

$$1 \otimes C = \mu(\eta(1), \text{id}_\mathcal{C}(C)) = C = \mu(\text{id}_\mathcal{C}(C), \eta(1)) = C \otimes 1$$

(2.9)

The fact that $\mu$ is a functor is a compatibility condition on composition and on identities,

$$(f \circ f') \otimes (g \circ g') = \mu((f \circ f', g \circ g')) = \mu((f, g)) \circ \mu((f', g')) = (f \otimes g) \circ (f' \otimes g')$$

(2.10)

and

$$\text{id}_C \otimes \text{id}_{C'} = \mu((\text{id}_C, \text{id}_{C'})) = \mu_{(C, C')} = \text{id}_{C \otimes C'}$$

(2.11)

In the future, we will stick to the notation with $\otimes$ for the sake of simplicity. As this is the definition of a strict monoidal category, we could also consider (nonstrict) monoidal categories (usually simply known as...
monoidal categories, but we reserve that word here for the strict case). In those we only require, roughly speaking, associativity up to isomorphism and unit up to isomorphism. The nonstrict version is actually more correct to use. However Mac Lane [14] has proved that every (nonstrict) monoidal category is equivalent (the definition of two categories being equivalent we do not show here) to a strict one. For us, practically, this means we could without the thesis crumbling apart work with only strict monoidal categories.

We can right away give examples of monoidal categories. The category $\text{Set}$ together with the cartesian product $\times$ as $\otimes$ and with the singleton set $1$ as the neutral element, and the category $\text{Vect}_K$ together with the usual tensor product (which is denoted by $\otimes$) and the field $K$ as neutral element are monoidal categories. The monoidal categories will be written as $(\text{Set}, \times, 1)$ and $(\text{Vect}_K, \otimes, K)$, respectively. Other examples are $(\text{Top}, \sqcup, \emptyset)$ and $(\text{Set}, \sqcup, \emptyset)$, where $\sqcup$ denotes disjoint union.

Since there are monoidal categories, there are of course functors between them that preserves the monoidal structures. Such functors are known as monoidal functors. There are different types of such functors depending on the requirements. In our case, we use the notion of strict monoidal functors, which respect all structure.

We will now introduce symmetry in our monoidal categories. We follow [13].

**Definition 2.2.** A monoidal category $(C, \otimes, 1)$ is symmetric if for each pair of objects, there is a twist map

$$\tau_{X,Y} : X \otimes Y \to Y \otimes X$$

(2.12)

such that the following three conditions are satisfied:

(i) For each pair arrows $f : X \to X'$ and $g : Y \to Y'$, the following diagram commutes:

$$\begin{array}{cccc}
X \otimes Y & \xrightarrow{\tau_{X,Y}} & Y \otimes X \\
\downarrow & & \downarrow \\
X' \otimes Y' & \xrightarrow{\tau_{X',Y'}} & Y' \otimes X'
\end{array}$$

(2.13)

We call this the naturality condition.

(ii) For every triple of objects $X, Y$ and $Z$, the following diagrams commute:

$$\begin{array}{ccc}
X \otimes Y \otimes Z & \xrightarrow{\tau_{X,Y} \otimes Z} & Y \otimes Z \otimes X \\
\tau_{X,Y} \otimes \text{id}_Z & & \text{id}_Y \otimes \tau_{X,Z} \\
Y \otimes X \otimes Z & \xrightarrow{\tau_{X,Z} \otimes \text{id}_Y} & X \otimes Z \otimes Y
\end{array}$$

(2.14)

(iii) The following holds for every pair of objects $X, Y$:

$$\tau_{X,Y} \tau_{Y,X} = \text{id}_{X \otimes Y}$$

(2.15)

Axiom (i) tells us that if we have two arrows $f$ and $g$, it makes no difference if we twist first and then apply the arrows or if apply the arrows first and then use the twist. The first diagram in (ii) states that if we twist $X$ with $Y$ and $Z$ together, it should be the same as twisting $X$ and $Y$ first, keeping $Z$ and then twisting $X$ and $Z$, keeping $Y$. The second diagram is analogous. Axiom (iii) just gives the condition that twisting twice is the same as doing nothing. These axioms can be illustrated very nicely graphically (which unfortunately is not done here, it is however done in [5, 13]. As an example of a symmetric monoidal category, one can
consider \((\mathbf{Vect}_{\mathbb{K}}, \otimes, \mathbb{K})\) together with the canonical symmetry \(\sigma : V \otimes W \to W \otimes V\) defined by \(v \otimes w \mapsto w \otimes v\). Then \((\mathbf{Vect}_{\mathbb{K}}, \otimes, \mathbb{K}, \sigma)\) becomes a symmetric monoidal category.

Since we have symmetric monoidal categories, we might also consider functors preserving this symmetry. Such a functor is known as a symmetric monoidal functor. Given two symmetric monoidal categories \((C, \otimes, I, \tau)\) and \((C', \otimes', I', \tau')\), a symmetric monoidal functor \(F : (C, \otimes, I, \tau) \to (C', \otimes', I', \tau')\) should satisfy

\[
F(\tau_X, Y) = \tau'_F(X), F(Y)
\]

for each pair of objects \(X\) and \(Y\). These kind of functors turn out to be the ones we are looking for, which will be clear as we move on in the thesis.

### 2.2 Cobordism Theory

The notion of cobordisms play a central role in this text. We use them in the formulation of TQFT and it will be later evident that in two dimensions, cobordisms and Frobenius algebras are closely related.

#### 2.2.1 Preliminary Notions

We begin by describing the notion of a manifold with boundary together with the concepts of in-boundary and out-boundary in order to be able to give a definition of a cobordism. We also take the empty set to be an \(n\)-dimensional manifold, which we simply denote by \(\emptyset_n\) - the \(n\)-dimensional empty manifold. Since \(\emptyset_n\) is a topological space and every point of \(\emptyset_n\) has an open neighbourhood homeomorphic to \(\mathbb{R}^n\) (in fact \(\emptyset_n\) has no points so this statement is always true for any \(n\)), we are on safe grounds.

An \(n\)-manifold with boundary \(M\) is a Hausdorff, second countable, topological space equipped with an open covering such that each open set is homeomorphic to \(H^n := \{(x^1, \ldots, x^n) \in \mathbb{R}^n | x^n \geq 0\}\). The boundary of \(M\), denoted by \(\partial M\), consists of all the points in \(M\) mapped to points in \(H^n\) through the coordinate functions. The set of points mapped to points in \(H^n\) with \(x^n > 0\) is called the interior of \(M\), denoted by \(\text{Int}(M)\). A closed manifold is a compact manifold with empty boundary.

As an example of a manifold with boundary, consider a cylinder. A product between the unit circle, denoted \(S^1\), and the compact unit interval \(I = [0, 1]\) is a cylinder \(C\) in two dimensions, \(C = S^1 \times I\). The boundary \(\partial C\) consists of two copies of the unit circle, \(\partial C_0 = S^1 \times \{0\}\) and \(\partial C_1 = S^1 \times \{1\}\). On the other hand, \(S^3\) is an example of a closed manifold.

![Figure 2.1: The cylinder is an example of a manifold with boundary.](image)

Since we want to be able to speak about what is inwards and outwards of the boundary of a manifold, we need to make sense some idea of orientation of a manifold. We follow the ideas given in [13]. Given a vector space \(V\), the orientation of \(V\) is determined by assigning a sign to each ordered basis such that the determinant of the linear transformation between the bases have positive determinant. Thus there are two possible orientations, which will be determined as soon as we choose the sign of one ordered basis of \(V\). We say that a basis is positive if the determinant of the linear transformation between the bases have positive Jacobian determinant. This means that the orientation of the tangent spaces of the manifold should be preserved when moving between patches. For example, the cylinder is an orientable manifold, whereas the Möbius strip is not. We denote the reverse orientation of \(M\) by \(\overline{M}\).

We now fix the idea of in- and out-boundaries. Let \(M\) be an \(n\)-dimensional orientable manifold with boundary and \(\Sigma\) a closed orientable \((n-1)\)-dimensional submanifold of \(M\). Consider the tangent space \(T_p\Sigma\) at \(p \in \Sigma\) and choose a positive basis \((v_1, \ldots, v_{n-1})\). At the same point \(p \in M\) consider a basis \((v_1, \ldots, v_{n-1}, v_n)\)
of $T_p M$ such that this basis also is positive. If now $\Sigma$ is a connected component of $\partial M$, we consider the fact that $v_n$ points either inwards or outwards with respect to $M$ (locally, we consider a vector of $\mathbb{R}^n$ which either points inwards or outwards of $H^n$). If $v_n$ points inwards, we say that $\Sigma$ is an in-boundary and if it points outwards, $\Sigma$ is an out-boundary. The boundary $\partial M$ of $M$ will consist of the union of different in-boundaries and out-boundaries. We write $\partial M_{in}$ and $\partial M_{out}$ for the collection (as disjoint union) of in-boundaries and out-boundaries respectively. From here on, we will draw in-boundaries to the left and out-boundaries to the right. For example, a cylinder with one in-boundary and one out-boundary, we will draw as (a) in Figure 2.2, while in the case when both boundary components are in-boundaries the cylinder will take the form (b), and analogously, for the case with two out-boundaries we draw as in (c).

![Figure 2.2: The three cases of cylinders.](image)

### 2.2.2 Cobordisms

We are now ready to give a satisfactory definition of a cobordism. Only the oriented case is relevant to us, hence the definition given will be of such nature. We will simply refer to them as cobordisms, when we actually mean oriented cobordisms.

**Definition 2.3.** Let $\Sigma_1$ and $\Sigma_2$ be two closed and oriented $(n-1)$-manifolds. An oriented cobordism from $\Sigma_1$ to $\Sigma_2$ is a compact oriented $n$-manifold together with orientation preserving smooth maps $f_1 : \Sigma_1 \to M$ and $f_2 : \Sigma_2 \to M$

\[ \Sigma_1 \overset{f_1}{\to} M \overset{f_2}{\leftarrow} \Sigma_2 \]

such that $f_1$ maps $\Sigma_1$ diffeomorphically onto the in-boundary of $M$ and $f_2$ maps $\Sigma_2$ diffeomorphically onto the out-boundary of $M$.

From this definition, we can think of the cobordism as some kind of propagation from $\Sigma_1$ to $\Sigma_2$. We will hence write $\Sigma_1 \overset{M}{\to} \Sigma_2$ for a cobordism $M$ from $\Sigma_1$ to $\Sigma_2$ (cobordisms are absolutely not functions, despite the slightly confusing notation). In order to illustrate the definition, let us consider some examples. The cylinder (with both an in- and out-boundary)

![cylinder](image)

is a cobordism from one circle to another circle. If we instead consider the case when we have two in-boundaries

![two in-boundaries](image)

we see that it becomes a cobordism from two circles to the empty $(n-1)$-manifold $\emptyset_{n-1}$, and analogously for the case with a cylinder with two out-boundaries (when the cobordism propagates from $\emptyset_{n-1}$ to two circles).

Let us investigate some other important examples. A disc $D^2$ is a two dimensional manifold with boundary. As a cobordism, the disc $D^2$ gives us two cases: either the boundary component $S^1$ is an in-boundary or it is an out-boundary. In the case when $S^1$ is an out-boundary, the disc becomes a cobordism from the empty $(n-1)$-manifold $\emptyset_{n-1}$ to $S^1$.

$\emptyset_{n-1} \overset{D^2}{\to} S^1$
In this case we call it the *left-cap* (see Figure 2.3 (a)). In the other case when we have
\[ S^1 \xrightarrow{\partial_2} \emptyset_{n-1} \]
we call it the *right-cap* (see Figure 2.3 (b)).

![Figure 2.3: The left-cap (a) and right-cap (b).](image)

We may also consider a cobordism \( M \) between two circles and one circle. When two circles are mapped to the in-boundary and one circle to the out-boundary of the cobordism,
\[ S^1 \sqcup S^1 \xrightarrow{M} S^1, \]
we call it the *left-pair of pants* (see Figure 2.4 (a)). In the other situation, we reverse the orientation
\[ S^1 \xrightarrow{M} S^1 \sqcup S^1, \]
and receive the *right-pair of pants* (see Figure 2.3 (b)).

![Figure 2.4: The left-pair of pants (a) and right-pair of pants (b).](image)

There are several ways to construct cobordism and we would like to know when two cobordisms are equivalent. Precisely, two cobordisms \( M \) and \( M' \) from \( \Sigma_1 \) to \( \Sigma_2 \) are *equivalent* if there exists an orientation preserving diffeomorphism \( \psi : M \xrightarrow{\sim} M' \) such that the following diagram commutes [13]:

![Diagram](image)

Moreover, this defines an equivalence relation meaning we can gather equivalent cobordisms in equivalence classes, the so called *cobordism classes*, and quotient out unneeded data.

### 2.2.3 The Category nCob

It turns out that one very important feature of cobordisms is that they can be glued together to form new cobordisms. By *gluing* (or *sewing*), we mean that given two cobordisms \( \Sigma_1 \xrightarrow{f_1} M \xleftarrow{f_2} \Sigma_2 \) and \( \Sigma_2 \xrightarrow{f_3} M' \xleftarrow{f_4} \Sigma_3 \), i.e.,
\[ \Sigma_1 \xrightarrow{f_1} M \xleftarrow{f_2} \Sigma_2 \]
and
\[ \Sigma_2 \xrightarrow{f_3} M' \xleftarrow{f_4} \Sigma_3 \]


where \( \Sigma_2 \) is non empty, we form the composite cobordism

\[
\Sigma_1 \xrightarrow{f_2} M' \circ M \xrightarrow{f_2} \Sigma_3
\]  
(2.20)

by identifying their common boundary components using \( f_2' \circ f_2^{-1} : \partial M_{\text{out}} \to \partial M'_{\text{in}} \). By common, we of course mean that \( \partial M_{\text{out}} \) and \( \partial M'_{\text{in}} \) are not necessarily equal, only that they are diffeomorphic to \( \Sigma_2 \).

From here on, by cobordism, we actually mean cobordism class. We will now start the construction of the category \( \text{nCob} \), following [13]. The objects in this category are closed oriented \((n-1)\)-dimensional manifolds and the arrows are cobordism classes between them. The composition in this category consists of gluing, and the identity arrows are cylinders (given a closed oriented \((n-1)\)-dimensional manifold \( \Sigma \), a cylinder is constructed as the product \( \Sigma \times [0,1] \), which is of course just one representative of the cobordism class in question). The composition is associative; given three cobordisms \( \Sigma_1 \xrightarrow{M} \Sigma_2, \Sigma_2 \xrightarrow{M'} \Sigma_3 \) and \( \Sigma_3 \xrightarrow{M''} \Sigma_4 \), it does not matter whether we glue \( M \) to \( M' \) first and then glue \( M'' \) to \( M' \circ M \) or if we make the gluing \( M'' \circ M' \) and then glue it with \( M \):

\[
M'' \circ (M' \circ M) = (M'' \circ M') \circ M.
\]  
(2.21)

The cylinder is indeed the identity, take a cobordism \( \Sigma_1 \xrightarrow{M} \Sigma_2 \) and the cylinder \( C \) over \( \Sigma_1 \). Note that we can decompose \( M \) into two parts \( M = M_{[\epsilon,1]} \circ M_{[0,\epsilon]} \), where \( M_{[\epsilon,1]} \) is diffeomorphic to a cylinder over \( \Sigma_1 \). We have (up to diffeomorphism)

\[
M \circ C = (M_{[\epsilon,1]} \circ M_{[0,\epsilon]}) \circ C = M_{[\epsilon,1]} \circ (M_{[0,\epsilon]} \circ C) = M_{[\epsilon,1]} \circ M_{[0,\epsilon]} = M.
\]  
(2.22)

In the second equality, we used associativity of gluing and in the third, we use the fact that if we glue two cylinders over \( \Sigma_1 \), we get another cylinder over \( \Sigma_1 \).

The category \( \text{nCob} \) can be endowed with a monoidal structure. Given two cobordisms \( \Sigma_1 \xrightarrow{M} \Sigma_2 \) and \( \Sigma'_1 \xrightarrow{M'} \Sigma'_2 \), their disjoint union \( M \sqcup M' \) is a cobordism from \( \Sigma_1 \sqcup \Sigma_2 \) to \( \Sigma'_1 \sqcup \Sigma'_2 \) (drawn as two cobordism parallel to each other). The empty cobordism \( \emptyset \xrightarrow{T} \emptyset \) will work as unit. Together, the disjoint union \( \sqcup \) and the empty cobordism will with \( \text{nCob} \) satisfy the axioms (2.8), (2.9), (2.10) and (2.11). \( \text{nCob} \) will hence take the form of a monoidal category \( \langle \text{nCob} , \sqcup , \emptyset \rangle \). By also introducing the twist cobordism \( \Sigma \sqcup \Sigma' \xrightarrow{T_{\Sigma,\Sigma'}} \Sigma' \sqcup \Sigma \), \( \langle \text{nCob} , \sqcup , \emptyset , T \rangle \) becomes a symmetric monoidal category. The twist satisfies \( T_{\Sigma',\Sigma} \circ T_{\Sigma,\Sigma'} = (\Sigma \sqcup \Sigma') \times [0,1] \), i.e. gluing with the twist amounts to gluing with a cylinder over \( \Sigma \sqcup \Sigma' \). By composing with a twist, the resulting cobordism should, as a manifold, be of the same type as before (since the twist cobordism should act as a twist map in the symmetric monoidal sense, take \( f \) and \( g \) as identities in axiom (i) (2.13)). Hence the twist is built up of two (connected) cylinders "crossing each other". Gluing with the twist cobordism can be seen as permuting two in-boundary components our out-boundary components. For example, given a cobordism with out-boundary \( \Sigma \) and \( \Sigma' \), by gluing the twist to the out-boundary, the new out-boundary of the cobordism becomes \( \Sigma' \sqcup \Sigma \) (instead of \( \Sigma \sqcup \Sigma' \) as in the case of the identity). This works analogously for in-boundary-components. We will draw the twist as

\[
\begin{array}{c}
\includegraphics[height=1cm]{twist.png}
\end{array}
\]  
(2.23)

where the intersection in this picture is not an actual intersection. It is important to stress that the twist cobordism is not the disjoint union of two cylinders in the context of cobordisms. \( \langle \text{nCob} , \sqcup , \emptyset , T \rangle \) satisfies all conditions to be a symmetric monoidal category ((2.13), (2.14), and (2.15)).

### 2.2.4 The Geometry of Surfaces

We will in this section restrict ourselves to two dimensions, i.e. we only work with \( 2\text{Cob} \), which is the main focus of this text. The surfaces considered mainly are compact and oriented. The objects in \( 2\text{Cob} \) are closed
oriented 1-manifolds. There are however not many choices in the sense that every closed oriented 1-manifold is diffeomorphic to a finite disjoint union of circles. For the cobordisms (arrows) in question, we need to develop some more machinery before we can describe them with the precision we want. We will study some aspects of surfaces (i.e. 2-manifolds) and use that to deepen our understanding of the category 2Cob. In this section, we will mostly follow [13].

The genus of a closed connected orientable surface is basically the number of holes in that surface. For instance, the torus has genus 1, whereas the sphere has genus 0. If the 2-manifold has boundary, the genus is defined to be the genus of the closed surface obtained by gluing discs at each boundary component. As an example the disc has genus 0 since if we sew another disc along its boundary, we obtain a sphere.

There is a classification theorem which states that every two connected closed, oriented surface are diffeomorphic if and only if they have the same genus. In our case however, the surfaces will have oriented boundaries which are diffeomorphic to a finite disjoint union of circles. Moreover, we need to distinguish between in-boundaries and out-boundaries. It turns out that two such surfaces (i.e. connected, compact, oriented surfaces with oriented boundary) are diffeomorphic if and only if the genus, the number of in-boundaries and the number of out-boundaries coincide for the two surfaces.

Using the above result we can describe every connected, compact, oriented surfaces with genus $g$, $m$ in-boundaries and $n$ out-boundaries in terms of a normal form of the surface. The only quantities needed to be specified are the genus and the number of in-boundaries an out-boundaries, meaning that the order of which the holes appear and the boundaries appear is not important. The normal form consists mainly of three parts. The first part (the in-part) consists of a cobordism from $m$ circles to one circle in the following manner: for $m > 0$, take $m - 1$ left-pair of pants and the required number of cylinders and glue them so that each out boundary of the pair of pants is glued to the lower in-boundary of the next pair of pants. The cylinders are glued above the pair of pants. In the case $m = 0$, the in-part only consists of a left-cap.

![Figure 2.5: The in-part for the case $m = 4$.](image)

The third part (the out-part) of the normal form will enjoy an analogous description but with right-pair of pants and right-caps. The second part (the topological part) of the normal form consists of all the holes; the topological part can be built up from $g$ left-pair of pants and $g$ right pair of pants so that the in-boundaries of the left-pair of pants connects to the out-boundaries of the right pair of pants (creating a hole). Then the out-boundaries of the left-pair of pants gets connected to the in-boundaries of the right-pair of pants (forming a chain of holes). At last, we connect the three pieces and obtain the normal form of the surface.

![Figure 2.6: The topological part for the case $g = 3$.](image)

**Example 2.3.** We give the example of the case when we have a connected, compact, oriented surface with genus $g = 3$, $m = 4$ in-boundaries and $n = 3$ out-boundaries:
We have now a complete description of the connected case; every connected 2-cobordism can be constructed from gluing and the disjoint union of the left- and right-pair of pants, left-and right-caps and cylinders. For the disconnected case though, we have to be more cautious since the twist cobordism might be involved. Fortunately, it turns out that we can use the twist twice to factor our cobordism in three parts \[ [5]. \] Suppose we have a cobordism \( \Sigma \xrightarrow{M} \Sigma' \) such that \( \Sigma = \Sigma_1 \sqcup \Sigma_2 \cdots \sqcup \Sigma_m \) and \( \Sigma' = \Sigma_1' \sqcup \Sigma_2' \cdots \sqcup \Sigma_n' \). Assume moreover that \( M \) has two connected components \( M_1 \) and \( M_2 \) (we can use this to describe the case with \( r \) components inductively). Then let \( \sigma_1 \subseteq \Sigma \) be the collection of in-boundaries for \( M_1 \) and \( \sigma_2 \subseteq \Sigma \) be the complement of \( \sigma_1 \), the collection of in-boundaries of \( M_2 \). Likewise, let the collection of out-boundaries for the respective components be denoted by \( \sigma_1' \subseteq \Sigma' \) and \( \sigma_2' \subseteq \Sigma' \). Note that there is no guarantee that the circles in e.g. \( \sigma_1 \) all come before the ones in \( \sigma_2 \), they could be mixed up altogether. However, this can be fixed by applying the twist twice (and hence doing nothing) on the appropriate places to permute the circles until we can factor the cobordisms into \( M = T \circ M' \circ S \), where \( M' \) consists of the disjoint union of two connected components \( M_1' \) and \( M_2' \) such that the in-boundary of \( M' \) is the disjoint union of the in-boundary of \( M_1' \) and \( M_2' \) and likewise for the out-boundary. \( S \) and \( T \) (which we call permutation cobordisms) are the gluing and disjoint union of twist cobordisms and cylinders (due to properties of permutations) glued to the in-boundary and out-boundary of \( M' \), respectively. Hence, we have arrived to the fact that every 2-cobordism can be built up by a permutation cobordism glued to a disjoint union of connected cobordisms, which is glued to another permutation cobordism. We illustrate this rather long description with a concrete example:

**Example 2.4.** Consider the case pictured below with four in-boundaries and two out-boundaries. The cobordism is disconnected, although it is not the disjoint union of the two connected components.
However, by doing the following operations, the cobordism will be factored in three parts as described before

\begin{equation}
\begin{aligned}
\text{\includegraphics[width=0.4\textwidth]{cobordism1.png}} &= \text{\includegraphics[width=0.4\textwidth]{cobordism2.png}} \\
\text{\includegraphics[width=0.4\textwidth]{cobordism3.png}} &= \text{\includegraphics[width=0.4\textwidth]{cobordism4.png}}
\end{aligned}
\end{equation}

(2.25) (2.26)

where the first part corresponds to $S$, the middle part to $M'$ and the end part to $T$. Of course, they are not really separated.

We may conclude that the symmetric monoidal category \((2\text{Cob}, \sqcup, \emptyset, T)\) is generated, under gluing (composition) and disjoint union, by the cobordisms

\begin{equation}
\text{\includegraphics[width=0.2\textwidth]{cobordism5.png}} \quad \text{\includegraphics[width=0.2\textwidth]{cobordism6.png}} \quad \text{\includegraphics[width=0.2\textwidth]{cobordism7.png}} \quad \text{\includegraphics[width=0.2\textwidth]{cobordism8.png}} \quad \text{\includegraphics[width=0.2\textwidth]{cobordism9.png}} \quad \text{\includegraphics[width=0.2\textwidth]{cobordism10.png}}
\end{equation}

(2.27)

We have found the cobordisms which builds up all arrows. What is left to find are the relations for the structure to really become a symmetric monoidal category, i.e. relations such that all equalities in \((2\text{Cob}, \sqcup, \emptyset, T)\) can be obtained from the relations. One possible set of relations are

\begin{equation}
\begin{aligned}
\text{\includegraphics[width=0.2\textwidth]{relation1.png}} &= \text{\includegraphics[width=0.2\textwidth]{relation2.png}} = \text{\includegraphics[width=0.2\textwidth]{relation3.png}} \\
\text{\includegraphics[width=0.2\textwidth]{relation4.png}} &= \text{\includegraphics[width=0.2\textwidth]{relation5.png}} = \text{\includegraphics[width=0.2\textwidth]{relation6.png}} \\
\text{\includegraphics[width=0.2\textwidth]{relation7.png}} &= \text{\includegraphics[width=0.2\textwidth]{relation8.png}} = \text{\includegraphics[width=0.2\textwidth]{relation9.png}} \\
\text{\includegraphics[width=0.2\textwidth]{relation10.png}} &= \text{\includegraphics[width=0.2\textwidth]{relation11.png}} = \text{\includegraphics[width=0.2\textwidth]{relation12.png}} \\
\text{\includegraphics[width=0.2\textwidth]{relation13.png}} &= \text{\includegraphics[width=0.2\textwidth]{relation14.png}} \quad \left(\text{or} \quad \text{\includegraphics[width=0.2\textwidth]{relation15.png}} = \text{\includegraphics[width=0.2\textwidth]{relation16.png}}\right)
\end{aligned}
\end{equation}

(2.28) (2.29) (2.30) (2.31)

together with the twist relations

\begin{equation}
\begin{aligned}
\text{\includegraphics[width=0.2\textwidth]{twist1.png}} &= \text{\includegraphics[width=0.2\textwidth]{twist2.png}} \\
\text{\includegraphics[width=0.2\textwidth]{twist3.png}} &= \text{\includegraphics[width=0.2\textwidth]{twist4.png}} \\
\text{\includegraphics[width=0.2\textwidth]{twist5.png}} &= \text{\includegraphics[width=0.2\textwidth]{twist6.png}}
\end{aligned}
\end{equation}

(2.32) (2.33)
Relations (2.28), (2.29), (2.30) and (2.31) are true since for each relation, the 2-manifolds on each side of the equalities are connected, compact and oriented and have the same number of genera (plural for genus) and the same number of in-boundaries and the same number of out-boundaries. They are hence by the classification theorem diffeomorphic. The twist relations are due to the fact that the twist cobordism should be the twist map making our monoidal category symmetric. For instance, (2.32) corresponds to (2.15) (axiom (iii)), (2.36) is an expression of (2.13) (axiom (i)) and (2.37) can be obtained from (2.14) (axiom (ii)).

We need to show sufficiency of the relations. Given a cobordism $M$, we can apply a Morse function $f : M \to I$ to it, where $I$ is an interval. A Morse function is a smooth map whose critical points are non-degenerate (the Hessian matrix $\frac{\partial^2 f}{\partial x^i \partial x^j}$ is non-singular in some coordinate system) with the requirement that $f^{-1}(\partial I) = \partial M$. In our case, the neighbourhood of a critical point will take one of four forms:

- $\bigcirc$ (local minimum)
- $\bigtriangledown$ (local maximum)
- (saddle point I)
- (saddle point II)

We arrange the critical points such that each critical point has different images. The Morse function can detect which type of critical point we are dealing with and divides the cobordism into intervals, each interval containing only one critical point (we can think of the Morse function as a timeline slicing our cobordism into pieces representing a time interval. Each such interval can then be represented as a combination of the generators (2.27).
We obtain a word (as an analogy, the generators can be seen as letters in an alphabet) $w(M)$, which can then be reduced to a normal form with the help of the relations. It can in this way be shown that the relations we have are enough to convert every word into normal form, thus proving the sufficiency of the relations.

Note that (2.28), (2.29) and (2.30) together imply

\begin{equation}
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{diagram1.png}} \\
= \text{\includegraphics[width=0.3\textwidth]{diagram2.png}}
\end{array}
\end{equation}

and

\begin{equation}
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{diagram3.png}} \\
= \text{\includegraphics[width=0.3\textwidth]{diagram4.png}}
\end{array}
\end{equation}

Moreover, the relation

\begin{equation}
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{diagram5.png}} \\
= \text{\includegraphics[width=0.3\textwidth]{diagram6.png}}
\end{array}
\end{equation}

known as the *snake relation*, holds since the surfaces are, again, of the same topological type by the classification theorem. This will be useful to consider later in the text.

### 2.3 Frobenius Algebras

Frobenius algebras are intimately related to TQFT in two dimensions. The relation between cobordisms and Frobenius algebras will be evident as we move on in the text. Here the main references are [1], [5] and especially [13].

#### 2.3.1 Preliminary notions

An *associative algebra* $\mathcal{A}$ over a field $\mathbb{K}$ is a vector space $\mathcal{A}$ with a bilinear multiplication (written as juxtaposition) with identity, defined between the vectors such that it is compatible with scalars; for all $a, b, c \in \mathcal{A}$ and $k \in \mathbb{K}$,

\begin{align}
    a(bc) &= (ab)c & \text{(associativity)} \\
    \exists 1 \in \mathcal{A} : 1x &= x = x1 & \text{(identity)} \\
    a(b + c) &= ab + ac & \text{(left distributivity)} \\
    (a + b)c &= ac + ab & \text{(right distributivity)} \\
    (ka)b &= k(ab) &= a(kb) & \text{(compatibility of scalars)}
\end{align}

We could have equally stated the definition in terms of commutative diagrams (this will help us later). In this case an algebra $\mathcal{A}$ is a vector space together with linear maps $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ (called multiplication)
and $\eta: K \to A$ (called unit) such that the following diagrams commute:

\begin{equation}
\begin{array}{c}
A \otimes A \otimes A \\
\mu \otimes id_A & id_A \otimes \mu \\
A \otimes A & A \otimes A \\
\mu & \mu
\end{array}
\end{equation}

(2.46)

\begin{equation}
\begin{array}{c}
K \otimes A \\
\eta \otimes id_A \\
A \otimes A \\
\mu
\end{array} \quad \begin{array}{c}
A \otimes A \\
id_A \otimes \eta \\
A \otimes K \\
\mu
\end{array}
\end{equation}

(2.47)

As usual, $id_A$ is the identity map and the unlabelled arrows are scalar multiplication, which are canonical isomorphisms [13]. The linearity of $\mu$ encodes (2.43), (2.44) and (2.45). The commutative diagram (2.46) expresses associativity and lastly, (2.47) gives us (2.42).

For one of the definitions of a Frobenius algebra to make sense, some concepts of modules over algebras needs to be clarified. A left $A$-module $M$ is a vector space $M$ over $K$ with a multiplication $:\cdot: A \times M \to M$ such that for all $a, a^{'}, m, m^{'} \in M$ and all $k \in K$, we have

\begin{equation}
(\psi.)(m + m^{'}) = \psi.(m) + \psi.(m^{'})
\end{equation}

(2.48)

\begin{equation}
(a + a^{'}) \cdot m = a \cdot m + a^{'} \cdot m
\end{equation}

(2.49)

\begin{equation}
(a^{'}) \cdot (m + m^{'}) = a^{'} \cdot (m) + a^{'} \cdot (m^{'})
\end{equation}

(2.50)

\begin{equation}
1. m = m
\end{equation}

(2.51)

\begin{equation}
(k a) \cdot m = k(a \cdot m) = a(k m)
\end{equation}

(2.52)

where 1 is the identity element in $A$. A right $A$-module is defined analogously. Note that $A$ is itself a left module with the canonical multiplication (called the left regular $A$-module), denoted by $A A$. In the same manner, $A$ is a right module (the right regular $A$-module), denoted by $A A$. As this type of objects have structure, it is of great interest to consider maps that preserve the structure. For left $A$-modules $M$ and $N$, such a map $\psi: M \to N$ is known as a left $A$-homomorphism and satisfies, for $m, m^{'} \in M$ and $a, a^{'} \in A$, $\psi(a \cdot m + a^{'} \cdot m^{'}) = \psi(a \cdot m) + \psi(a^{'} \cdot m^{'})$. If it in addition is bijective, it is a left $A$-isomorphism. This principle is the same for the right case, note however that the order is important. Here we actually have another example of a category - the category with left (right) $A$ - modules as objects and left (right) $A$-homomorphisms as arrows, $\text{IMod}_A$ ($\text{rMod}_A$) since the composition of two left (right) $A$-homomorphisms is again a left (right) $A$-homomorphism and the identity map is of course a left (right) $A$-homomorphism.

Since a left $A$-module is a vector space, we could consider its dual vector space $M^* = \text{Hom}(M, K)$ ($\psi \in \text{Hom}(M, K)$ is a left $A$-homomorphisms from $M$ to $K$, i.e. $\text{Hom}(M, K)$ is the set of arrows from $M$ to $K$ in $\text{IMod}_A$). This space has a canonical right $A$-module structure; for $\psi \in M^*, m \in M$ and $a \in A$

\begin{equation}
(\psi \cdot a)(m) = \psi(a \cdot m)
\end{equation}

(2.53)

This works similarly for right $A$-modules (where we get the relation $(a \cdot \psi)(m) = \psi(m \cdot a)$ instead).

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2.3.2 Frobenius Algebras

We are now in position to give three equivalent definitions of a Frobenius algebra.

**Definition 2.4.** A Frobenius algebra is a finite dimensional algebra \(A\) over \(K\) satisfying one (and hence all) of the following equivalent conditions:

(i) There exists a left \(A\)-isomorphism \(\lambda : A \rightarrow (A_A)^*\).

(ii) There exists a non-degenerate associative bilinear pairing, a so called Frobenius pairing, \(\beta : A \otimes A \rightarrow K\), with associative meaning \(\beta(a \otimes bc) = \beta(ab \otimes c)\) for \(a, b, c \in A\).

(iii) There exists a linear functional, a so called Frobenius form, \(\epsilon : A \rightarrow K\) whose kernel contains no nontrivial left or right ideals.

Note that condition (i) could be equivalently expressed with the right case. Also, we will denote a Frobenius algebra with Frobenius form \(\epsilon\) by \((A, \epsilon)\). Furthermore, a Frobenius algebra is symmetric if the given Frobenius form \(\epsilon : A \rightarrow K\) satisfies the trace condition: \(\epsilon(ab) = \epsilon(ba)\).

Given a non-degenerate associative bilinear pairing \(\beta\) on \(A\), we can construct a linear form \(\epsilon\) by

\[
\epsilon(a) := \beta(1 \otimes a) = \beta(1 \otimes a1) = \beta(1a \otimes 1) = \beta(a \otimes 1) \quad (2.54)
\]

where we use associativity of \(\beta\) to establish the equality. \(\text{Ker}(\epsilon)\) has no nontrivial left ideals since if \(0 = \epsilon(ab) = \beta(1 \otimes ab) = \beta(a \otimes b)\) for all \(a \in A\), then \(b = 0\) since \(\beta\) is non-degenerate. In particular, this means that the only left ideal in the kernel is trivial. This works similarly for right ideals, \(\epsilon(ab) = 0\) for all \(b \in A\), then \(a = 0\). The converse also works. Suppose \(\epsilon\) is a linear functional with no nontrivial ideals in its kernel. Then define a bilinear pairing by

\[
\beta(a \otimes b) := \epsilon(ab) \quad (2.55)
\]

This pairing is associative; the associativity of \(\beta\) is inherited from \(A\): \(\beta(a \otimes bc) = \epsilon(abc) = \epsilon((ab)c) = \beta(ab \otimes c)\). Moreover, \(\epsilon\) having no left ideals means that if \(0 = \beta(a \otimes b) = \epsilon(ab)\) for all \(a \in A\), then \(b = 0\). But this is exactly the definition for a bilinear pairing to be non-degenerate (at least for the finite dimensional case, in which we work in). As for the right case, we proceed analogously.

In order to more firmly grasp the definition, we give some examples.

**Example 2.5.** The simplest example of a Frobenius algebra is \((K, \text{id}_K)\) since \(K\) is itself a finite dimensional algebra over \(K\) and the kernel of \(\text{id}_K\) contain no nontrivial left or right ideals.

**Example 2.6.** The complex numbers \(\mathbb{C}\) together with a linear functional defined by \(\epsilon(a + bi) := a\) is a Frobenius algebra over \(\mathbb{R}\). Since \(\mathbb{C}\) is a commutative algebra over \(\mathbb{R}\), the discussed Frobenius algebra is symmetric. We could however choose another linear functional which satisfies the conditions to be a Frobenius form, for example the one defined by \(\epsilon(2 + 3i) = 7\) and \(\epsilon(1 - i) = 4\). By linearity of the Frobenius form, we obtain that \(1 + 19i\) gets mapped to 0 and hence \(\text{Ker}(\epsilon) = \{z \in \mathbb{C}\mid z = r(1 + 19i), r \in \mathbb{R}\}\) (the only ideal in the kernel is \(\{0\}\), i.e. it contains no nontrivial ideals).

**Example 2.7.** Matrix algebras \(\text{Mat}(n, K)\) of \(n \times n\)-matrices together with the trace \(\text{tr}\) as Frobenius form is a Frobenius algebra. It has no nontrivial left (or right) ideals since if \(\text{tr}(AB) = 0\) for all \(B \in \text{Mat}(n, K)\), then \(A = 0\) meaning that the only ideal in the kernel is trivial. It is furthermore symmetric since \(\text{tr}(AB) = \text{tr}(BA)\) for \(A, B \in \text{Mat}(n, K)\). \(\text{Mat}(n, K)\) is, however, not commutative.

**Example 2.8.** A perhaps more involved example, for a closed and oriented \(n\)-manifold \(M\), the de Rham cohomology ring \(H^*(M) = \bigoplus_{i=0}^n H^i(M)\) is an algebra under the wedge product and together with integration over \(M\) (with a chosen volume form) as the Frobenius form it becomes a Frobenius algebra. The corresponding pairing \(\beta : H^*(M) \otimes H^*(M) \rightarrow \mathbb{R}\) is non-degenerate due to the Poincaré duality, which states that the inner product which defines the duality of \(H^*(M)\) and \(H^{n-\tau}(M)\), \(\beta : H^*(M) \times H^{n-\tau}(M) \rightarrow \mathbb{R}\) defined by \(\beta(\omega, \eta) := \int_M \omega \wedge \eta\), is non-degenerate.
The following two examples will be used later in the thesis.

**Example 2.9.** Let $G$ be a finite group. The group algebra $\mathbb{C}[G]$ is the set of formal linear combinations

$$x = \sum_g x_g g$$

(2.56)

with $x_g \in \mathbb{C}$ and $g \in G$. The algebra multiplication (see Grillet [11]) is given by

$$\left( \sum_g x_g g \right) \left( \sum_h y_h h \right) = \sum_k z_k k,$$

(2.57)

where $z_k = \sum_{g,h:gh=k} x_g y_h$ (we can think of it as a multiplication of polynomials, but where the indeterminates do not necessarily commute). Define

$$\epsilon \left( \sum_g x_g g \right) := \sum_g x_g g^0_e,$$

(2.58)

where $e \in G$ is the identity element. Now, consider when $\epsilon(xy) = 0$ for all $x \in \mathbb{C}[G]$. Suppose $y \neq 0$, i.e. $y = \sum_g y_g g$ such that $y_p \neq 0$ for some $p \in G$. Then there exists an $x$ containing the corresponding inverses such that

$$xy = \sum_{g^{-1},g} x_{g^{-1}} y_g g^{-1} + R = \left( \sum_{g^{-1},g} x_g y_g^{-1} \right) e + R,$$

(2.59)

where $\sum_{g^{-1},g} x_{g^{-1}} y_g \neq 0$ and $R = \sum_h R_h h$ such that $h \neq e$. But then

$$\epsilon(xy) = \epsilon \left( \left( \sum_{g^{-1},g} x_g y_g^{-1} \right) e + R \right) = \left( \sum_{g^{-1},g} x_g y_g^{-1} \right) \epsilon(e) + \sum_h R_h \epsilon(h) \neq 0$$

(2.60)

which means we have a contradiction. Hence $y = 0$ for all $x$ and thus the kernel contains no nontrivial left ideals (the right case works with analogous arguments). Therefore, $\epsilon$ is a Frobenius form and $\mathbb{C}[G]$ is together with $\epsilon$ a Frobenius algebra. It is furthermore symmetric since $gg^{-1} = g^{-1}g$ for all $g \in G$. One may also define $\epsilon$ as

$$\epsilon \left( \sum_g x_g g \right) := \frac{1}{|G|} \sum_g x_g g^0_e$$

(2.61)

**Example 2.10.** The centre of the group algebra $\mathbb{C}[G]$, denoted by $\hat{Z}(\mathbb{C}[G])$, is the set of all elements in $\mathbb{C}[G]$ which commutes with all elements in $\mathbb{C}[G]$, i.e. $Z(\mathbb{C}[G]) = \{ x \in \mathbb{C}[G] | xy = yx, \forall y \in \mathbb{C}[G] \}$. Since $p \in Z(\mathbb{C}[G])$, in particular, it commutes with $h \in G$, so

$$hp = ph \iff p = hph^{-1}.$$  

(2.62)

Let now $p = \sum_{i=0}^n p_i g_i$ (note the switch of notation), where $g_i \neq g_j$ if $i \neq j$ and $n = |G|$. Then this means that

$$p = hph^{-1} = h \left( \sum_{i=0}^n p_i g_i \right) h^{-1}$$

(2.63)

$$= hp_1 g_1 h^{-1} + hp_2 g_2 h^{-1} + \cdots + hp_n g_n h^{-1}$$

(2.64)

$$= p_1 h g_1 h^{-1} + p_2 h g_2 h^{-1} + \cdots + p_n h g_n h^{-1}$$

(2.65)
which is equivalent to the statement
\[ g_i = hg_jh^{-1} \implies p_i = p_j \] (2.66)
meaning that all elements in the same conjugacy class have the same coefficients (since \( p \) commutes with all \( h \in G \)). We can hence rewrite \( p \) as
\[ p = \sum_\alpha p_\alpha e^\alpha \] (2.67)
where we sum over the conjugacy classes \( \alpha \) and \( e^\alpha = \sum_{g \in \alpha} g \). Since \( \alpha \) are equivalence classes, \( \{e^\alpha\} \) is a basis for \( \tilde{Z}(\mathbb{C}[G]) \).

Consider now the space of class functions on \( G \) (i.e. functions constant over the conjugacy classes)
\[ C_{\text{class}}(G) := \{ f : G \to \mathbb{C} | f(hgh^{-1}) = f(g) \} \] (2.68)
which is the same as \( Z(\mathbb{C}[G]) \), the center of the algebra of functions on \( G \). Moreover \( C_{\text{class}}(G) \cong \tilde{Z}(\mathbb{C}[G]) \); the mapping that maps \( e^\alpha \mapsto f_\alpha \), where \( f_\alpha \) is 1 on \( \alpha \) and 0 everywhere else, is an isomorphism since \( \{f_\alpha\} \) is a basis for \( C_{\text{class}}(G) \).

The algebra multiplication is defined by \( e^\alpha e^{\alpha'} = \sum_\gamma N^{\alpha \alpha'}_\gamma e^\gamma \), where
\[ N^{\alpha \alpha'}_\gamma = |\{ h \in \alpha' | gh^{-1} \in \alpha \text{ for } g \in \gamma \}|. \] (2.69)
The Frobenius form we define by
\[ \epsilon(f) = \frac{1}{|G|} f(e). \] (2.70)

### 2.3.3 Coalgebras
A coalgebra is a vector space \( A \) over \( K \) together with linear maps \( \Delta : A \to A \otimes A \) (called comultiplication) and \( \epsilon : A \to K \) (called counit) such that the following diagrams commute:

\[
\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{\Delta \otimes \text{id}_A} & A \otimes A \\
\downarrow \text{id}_A \otimes \Delta & & \downarrow \Delta \\
A \otimes A & \xrightarrow{\Delta} & A \otimes A \\
\end{array}
\] (2.71)
\[
\begin{array}{ccc}
K \otimes A & \xrightarrow{\epsilon \otimes \text{id}_A} & A \otimes A \\
\downarrow \Delta & & \downarrow \Delta \\
A & \xrightarrow{\Delta} & A \otimes K \\
\end{array}
\] (2.72)

Axiom (2.71) is known as coassociativity and (2.72) is the counit condition. We have arrived at another definition of a Frobenius algebra:
Definition 2.5. A *Frobenius algebra* \( \mathcal{A} \) is a finite dimensional vector space together with linear maps 
\( \mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \), \( \Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \), \( \eta : K \to \mathcal{A} \) and \( \epsilon : \mathcal{A} \to K \) subject to 

\[ \mu(\eta \otimes \text{id}_\mathcal{A}) = \text{id}_\mathcal{A} = \mu(\text{id}_\mathcal{A} \otimes \eta) \quad \text{(Unit)} \]  
\[ (\epsilon \otimes \text{id}_\mathcal{A})(\Delta) = \text{id}_\mathcal{A} = (\text{id}_\mathcal{A} \otimes \epsilon)(\Delta) \quad \text{(Counit)} \]  
\[ (\text{id}_\mathcal{A} \otimes \mu)(\delta \otimes \text{id}_\mathcal{A}) = \Delta \mu = (\mu \otimes \text{id}_\mathcal{A})(\text{id}_\mathcal{A} \otimes \Delta) \quad \text{(Frobenius condition)} \]

We will return to the equivalence of the definitions later, with the graphical tools developed later. This definition together with the previous ones, allows us to define the graphical calculus in such a way that the graphical relations will be analogous to the relations in \((2\text{Cob}, \sqcup, \emptyset, T)\).

2.3.4 Categorical Aspects

We keep using [13] as reference. An *algebra homomorphism* \( \varphi : \mathcal{A} \to \mathcal{B} \) between two algebras \( \mathcal{A} \) and \( \mathcal{B} \) over a field \( K \) is a linear map satisfying \( \varphi(ab) = \varphi(a)\varphi(b) \) and \( \varphi(1_\mathcal{A}) = 1_\mathcal{B} \). Let \((\mathcal{A}, \epsilon)\) and \((\mathcal{A}', \epsilon')\) be Frobenius algebras. The maps preserving the Frobenius algebra structure are called *Frobenius algebra homomorphism* and are algebra homomorphisms which are simultaneously coalgebra homomorphisms (maps preserving the coalgebra structure). Also, it can be shown that the comultiplication admits \( \epsilon \) as counit, which in particular means that the Frobenius forms will be preserved; given a Frobenius algebra homomorphism \( \varphi : (\mathcal{A}, \epsilon) \to (\mathcal{A}', \epsilon') \), we have \( \epsilon = \epsilon' \circ \varphi \). If we take the Frobenius algebra homomorphisms as arrows and Frobenius algebras as objects, we obtain a category which we denote by \( \mathbf{FA}_K \). If we instead have commutative Frobenius algebras as objects, we obtain the full subcategory (full meaning the set of arrows in a subcategory \( C' \) of \( C \) is the same as the set of arrows in the category \( C \)) of commutative Frobenius algebras and Frobenius algebra homomorphisms which we denote by \( \mathbf{cFA}_K \). Frobenius algebras are closed under taking tensor products, and we have the fact that \((K, \text{id}_K)\) is a Frobenius algebra. Thus \((\mathbf{FA}_K, \otimes, K)\) is in fact a monoidal category.

2.3.5 Graphical Approach

The advantage of having expressed the axioms in terms of commutative diagrams is that we can in a natural way convert them into a graphical formulation. We first begin to define the symbols we are going to use. The symbols should be seen as formal mathematical symbols. The multiplication map \( \mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \) we define as 

\[ \text{(2.76)} \]

This symbol is justified by the fact that multiplication can be seen as merging two items into one; here the two circles to the left goes to the one circle to the right. Similarly, we define the unit map \( \eta : K \to \mathcal{A} \) as 

\[ \text{(2.77)} \]

and the identity map \( \text{id}_\mathcal{A} : \mathcal{A} \to \mathcal{A} \)

\[ \text{(2.78)} \]

The composition of two maps will simply be joining symbols at the appropriate border and the tensor product will be represented by putting two symbols in parallel. As an example, \( \mu \circ (\text{id}_\mathcal{A} \otimes \eta) \) will be drawn as 

\[ \text{(2.79)} \]
We will sometimes omit drawing the identity (unless it is really necessary or illustrative). We are now able to state the axioms of an algebra over $\mathbb{K}$. The commutative diagrams (2.46) and (2.47) give us the graphical representation of associativity and the unit condition:

\[(2.80)\]

and

\[(2.81)\]

In order for our algebra to become a Frobenius algebra, we need to equip it with a Frobenius form $\epsilon : A \to \mathbb{K}$ or a Frobenius pairing $\beta : A \otimes A \to \mathbb{K}$. Symbolically, $\epsilon$ is defined as

\[(2.82)\]

and $\beta$ as

\[(2.83)\]

Since we should be able to construct a Frobenius form from a Frobenius pairing (and vice versa, see (2.54) and (2.55)), the equalities $\beta(1 \otimes a) = \epsilon(a) = \beta(a \otimes 1)$ and $\beta(a \otimes b) = \epsilon(ab)$ should hold for the pictures as well. The respective equalities are given by

\[(2.84)\]

and

\[(2.85)\]

In the case of the pairing we may express the associativity condition as

\[(2.86)\]

For the case of non-degeneracy, we use another definition of non-degeneracy (which is more general than the previous used in 2.55). It states, in our case, that a pairing $\beta : A \otimes A \to \mathbb{K}$ is non-degenerate if there exists a copairing $\gamma : \mathbb{K} \to A \otimes A$ such that the following diagrams commute:

\[(2.87)\]
We draw the copairing $\gamma$ as

![Diagram of copairing $\gamma$]

The non-degeneracy condition will then become

\begin{align}
\quad = \quad =
\end{align}

We have now introduced enough components to describe a Frobenius algebra. However, we are not content here. We want to obtain the coalgebra structure as well, in doing so, we have to define some more quantities. Define the three-point function $\phi : \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \rightarrow \mathbb{K}$

![Diagram of three-point function $\phi$]

and define the comultiplication as

\begin{align}
\quad = \quad = \quad =
\end{align}

which is well-defined due to properties of the three-point function and the non-degeneracy condition. Furthermore, it is unique and has $\epsilon$ as counit

\begin{align}
\quad = \quad = \quad =
\end{align}

The comultiplication also satisfies the Frobenius condition

\begin{align}
\quad = \quad = \quad =
\end{align}
and is coassociative

Recall that we had an alternative definition of Frobenius algebras in terms of coalgebras. We collect our newly developed graphical tools in a theorem. The proof can be found in [5, 13].

**Theorem 2.1.** The following are equivalent:

(i) A finite dimensional algebra \( A \) equipped with an associative, non-degenerate bilinear pairing \( \beta : A \otimes A \to K \) (equivalently, equipped with a linear form \( \epsilon : A \to K \) whose kernel contains no left or right ideals).

Graphically: maps

\[ \begin{array}{c}
\text{subject to associativity (2.80), the unit condition (2.81) and the non-degeneracy condition (2.89).}
\end{array} \]

(ii) A finite dimensional vector space \( A \) equipped with linear maps \( \mu : A \otimes A \to A \), \( \Delta : A \to A \otimes A \), \( \eta : K \to A \) and \( \epsilon : A \to K \) subject to

\[ \begin{align*}
\mu(\eta \otimes \text{id}_A) &= \text{id}_A = \mu(\text{id}_A \otimes \eta) \quad &\text{(Unit) (2.96)} \\
(\epsilon \otimes \text{id}_A)(\Delta) &= \text{id}_A = (\text{id}_A \otimes \epsilon)(\Delta) \quad &\text{(Counit) (2.97)} \\
(\text{id}_A \otimes \mu)(\delta \otimes \text{id}_A) &= \Delta \mu = (\mu \otimes \text{id}_A)(\text{id}_A \otimes \Delta) \quad &\text{(Frobenius condition) (2.98)}
\end{align*} \]

Graphically: maps

\[ \begin{array}{c}
\text{satisfying the unit condition (2.81), the counit condition (2.92) and the Frobenius condition (2.93).}
\end{array} \]

**Proof.** A quick remark on (i): strictly speaking, we only need to have one of either the pairing together with relation (2.84) or the right-cap with relation (2.85), due to the nature of \( \beta \) and \( \epsilon \) discussed above. We use the latter one.

The proof utilizes the graphical toolbox to prove that both statements have the same behaviour and hence give rise to the same structure on \( A \). Assume that (i) holds. Firstly, algebras are by definition vector spaces, so that part is taken care of. Observe also that we have already, due to prior discussions, established the comultiplication map. Furthermore, the unit condition is already satisfied. Using the non-degeneracy condition, we prove the counit condition:

\[ \begin{array}{c}
\text{The left hand side of (2.92) is hence established. The right hand side is analogous, using the left hand side of the non-degeneracy condition (2.89). We will henceforth omit drawing the identity maps unless it is really}
\end{array} \]

24
needed. To prove the Frobenius condition is satisfied, use (2.91) and (2.80):

\[ (2.91) = (2.80) = (2.91) = (2.101) \]

The left hand side of the Frobenius relation is thus proven. For the right hand side, the situation is similar, but with the other equality in (2.91). The proof of this part is now complete.

Assume now that \((ii)\) holds. To show associativity of the multiplication, we use the Frobenius condition together with caps:

\[ (2.102) \]

Similarly, we obtain

\[ (2.103) \]

We now have

\[ (2.104) \]

The associativity means we have an algebra. Now, define the pairing as

\[ (2.105) \]

To show non-degeneracy, we have to show there exists a copairing such that (2.89) is satisfied. It turns out that defining the copairing as

\[ (2.106) \]

25
and using the Frobenius condition will solve the problem:

\[
\begin{array}{ccc}
\text{\includegraphics[width=0.3\textwidth]{frobenius_condition}} & (2.93) & \text{\includegraphics[width=0.3\textwidth]{frobenius_condition}} \\
\end{array}
\]

The right hand side of the non-degeneracy condition is obtained similarly. The proof is now complete. \(\square\)

Recall that we earlier mentioned that the monoidal category of vector spaces carries a canonical symmetry \(\sigma_{V,W} : V \otimes W \to W \otimes V\), which permutes the factors, i.e. \(\sigma(v \otimes w) = w \otimes v\). Pictorially, we draw \(\sigma\) as

\[
\text{\includegraphics[width=0.1\textwidth]{symmetry_diagram}}
\]

The twist map satisfies \(\sigma_{W,V} \circ \sigma_{V,W} = \text{id}_{V \otimes W}\), which is pictured as

\[
\text{\includegraphics[width=0.2\textwidth]{twist_map_diagram}}
\]

As we are mostly interested in the case \(V = W = \mathcal{A}\) when we have multiplication, the naturality condition can be drawn as

\[
\text{\includegraphics[width=0.4\textwidth]{naturality_diagram}}
\]

We are interested in the commutative Frobenius algebras. The commutativity condition can be stated as

\[
\text{\includegraphics[width=0.2\textwidth]{commutativity_diagram}}
\]

In fact it can be shown that a Frobenius algebra is commutative if and only if it is cocommutative. The cocommutativity condition is pictured analogously. As one might suspect, a symmetric Frobenius algebra should satisfy

\[
\text{\includegraphics[width=0.2\textwidth]{cocommutativity_diagram}}
\]

3 Two Dimensional Topological Quantum Field Theory

In this section we will discuss topological quantum field theories; we begin by giving the general definition and some properties before focusing on the case of two dimensions, the main goal of the thesis. Afterwards we will look at a specific TQFT known as the Dijkgraaf-Witten model [9].
3.1 Topological Quantum Field Theories

A cobordism \( \Sigma_1 \xrightarrow{M} \Sigma_2 \) can be seen as spacetime with boundary components \( \Sigma_1 \) and \( \Sigma_2 \) as space. \( M \) is then an evolution of space from its initial state \( \Sigma_1 \) to its final state \( \Sigma_2 \). A quantum field theory gives us the recipe on how to construct Hilbert spaces \( \mathbb{A}(\Sigma) \) from space-like manifolds \( \Sigma \) and rules on how to calculate the time evolution operator \( U(M) : \Sigma_1 \rightarrow \Sigma_2 \) to every cobordism \( \Sigma_1 \xrightarrow{M} \Sigma_2 \). The time evolution operator can be constructed via

\[
\langle \hat{A}_2 | U | \hat{A}_1 \rangle = \int_{\hat{A}|\Sigma_1 = \hat{A}_1}^{\hat{A}|\Sigma_2 = \hat{A}_2} \mathcal{D}A \exp(iS[A])
\]  

which calculates the amplitude for a system in state \( \hat{A}_1 \) on \( \Sigma_1 \) to be in the state \( \hat{A}_2 \) on \( \Sigma_2 \). A theory is topological if it only depends on the topology. This is the focus of the thesis, thus we define a topological quantum field theory.

3.1.1 Definitions

A linear representation of a symmetric monoidal category \( (\mathcal{S}, \circ, I, \tau) \) is a symmetric monoidal functor \( (\mathcal{S}, \circ, I, \tau) \rightarrow (\mathbf{Vect}_k, \otimes, k, \sigma) \), where \( \sigma \) is the canonical symmetry discussed previously. This gives us precisely one of our main definitions in the thesis:

**Definition 3.1.** An \( n \)-dimensional topological quantum field theory is a linear representation of \( (\mathbf{nCob}, \sqcup, \emptyset, T) \).

In other words, an \( n \)-dimensional topological quantum field theory is a symmetric monoidal functor

\[
Z : (\mathbf{nCob}, \sqcup, \emptyset, T) \rightarrow (\mathbf{Vect}_k, \otimes, k, \sigma).
\]  

This is one definition of a TQFT. It is worth mentioning that there are several different definitions with different properties. Hereafter, we will let our ground field be \( k = \mathbb{C} \). We could especially consider closed oriented \( n \)-manifolds \( M \). These would then in \( 2\mathbf{Cob} \) be regarded as cobordisms \( \emptyset_{n-1} \xrightarrow{M} \emptyset_{n-1} \) and through a TQFT give rise to a map \( Z(M) : \mathbb{C} \rightarrow \mathbb{C} \); since \( Z \) is monoidal functor, it will necessarily send \( \emptyset \) to \( \mathbb{C} \) and also preserve the structure of the categories (recall the definition of a functor, axiom (i): \( \mathcal{F}(f : A \rightarrow B) = \mathcal{F}(f) : \mathcal{F}(A) \rightarrow \mathcal{F}(B) \)). In particular, \( Z(M) \) will be a complex number (the map will be given by multiplication by a fixed complex number; if \( f : \mathbb{C} \rightarrow \mathbb{C} \) is linear, then \( f(z_1 + z_2) = f(1)(z_1 + z_2) \) since \( \mathbb{C} \) is a vector space over \( \mathbb{C} \) in \( \mathbf{Vect}_\mathbb{C} \)). Moreover, since we are working with cobordism classes, this number is a topological invariant (meaning it does not change under homeomorphisms). To summarize: an \( n \)-dimensional TQFT calculates topological invariant complex numbers for \( n \)-dimensional closed oriented manifolds. The invariants are the partition functions of the quantum field theory and are of great interest.

The definition of a TQFT given by Atiyah in [2, 3] has a slight different viewpoint which we should investigate closer.

**Definition 3.2** (Atiyah). An \( n \)-dimensional topological quantum field theory is a functor \( Z \) which

(a) to each compact oriented \( (n-1) \)-manifold \( \Sigma \), assigns a finite dimensional complex vector space \( Z(\Sigma) \).

(b) to each compact oriented \( n \)-manifold \( M \) with boundary \( \Sigma \), assigns a vector \( Z(M) \in Z(\Sigma) \).

The functor is subject to the following axioms:

(i) \( Z \) is involutory, i.e. \( Z(\overline{\Sigma}) = Z(\Sigma)^* \) (where \( \overline{\Sigma} \) denotes orientation reversion and \( Z(\Sigma)^* \) denotes the dual space of \( Z(\Sigma) \), as usual).

(ii) \( Z \) is multiplicative, i.e. \( Z(\Sigma_1 \sqcup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2) \).

(iii) \( Z \) is associative, i.e. given a composite cobordism \( M = M'' \circ M' \) with \( \Sigma_1 \xrightarrow{M'} \Sigma_2 \) and \( \Sigma_2 \xrightarrow{M''} \Sigma_3 \),

\[
Z(M) = Z(M'')Z(M') \in \text{Hom}(Z(\Sigma_1), Z(\Sigma_3))
\]
We also require non-triviality by the following axioms:

(iv) $Z(\emptyset_{n-1}) = \mathbb{C}$.

(v) $Z(\Sigma \times I) = \text{id}_{Z(\Sigma)}$, where $I$ is an interval.

Note that axioms (ii) and (iv) actually states that $Z$ is a monoidal functor. Moreover, the theory is topological since, together with axiom (v), if we consider cobordism classes, two cobordisms in the same class must have the same image since $Z$ is a functor. As we look closer to the properties of this definition, it will become clearer how the two definitions are related.

### 3.1.2 Properties

We state some results which can be found in [1]. Firstly, note that given two finite dimensional vector spaces $V$ and $W$, the following holds:

$$V^* \otimes W \cong \Hom_{\mathbb{C}}(V,W)$$

(3.3)

since

$$\dim(V^* \otimes W) = \dim(V^*)\dim(W) = \dim(V)\dim(W) = \dim(\Mat_{\mathbb{C}}(\dim(V), \dim(W))) = \dim(\Hom_{\mathbb{C}}(V,W))$$

and finite dimensional vector spaces of the same dimension are isomorphic. We then use (3.3) to show that for a cobordism $\Sigma \xrightarrow{M} \Sigma_2$, the image of $M$ is a linear map, i.e. $Z(M) \in \Hom(Z(\Sigma_1), Z(\Sigma_2))$:

$$Z(\partial M) = Z(\Sigma_1 \sqcup \Sigma_2) \overset{(i),(ii)}{=} Z(\Sigma_1)^* \otimes Z(\Sigma_2) \cong \Hom_{\mathbb{C}}(Z(\Sigma_1), Z(\Sigma_2))$$

meaning we can regard

$$Z(M) \in \Hom_{\mathbb{C}}(Z(\Sigma_1), Z(\Sigma_2))$$

(3.4)

Note that this works well with axiom (iii). From this point of view, if we work with $\mathbf{nCob}$, we see that $Z$ is actually a monoidal functor from $(\mathbf{nCob}, \sqcup, \emptyset)$ to $(\mathbf{Vect}_\mathbb{C}, \otimes, \mathbb{C})$. However, there is no requirement that the symmetry should be preserved, so we actually need to postulate it. Now, axiom (iv) and (v) together imply that if the empty set $\emptyset_n$ is regarded as a closed $n$-manifold written as $\emptyset_{n-1} \times I$, where $I$ is an interval, then

$$Z(\emptyset_n) = 1$$

(3.5)

In fact, if we consider any closed oriented $n$-manifold $M$, the image will be a linear map $f : \mathbb{C} \to \mathbb{C}$, which by previous arguments is basically a fixed complex number $Z(M) \in \mathbb{C}$ which is a topological invariant. Also if we cut $M$ along a $(n-1)$-manifold $\Sigma$, using (iii) we may calculate the number via

$$Z(M) = \langle Z(M_1), Z(M_2) \rangle = Z(M_2)(Z(M_1))$$

(3.6)

where the bracket denote the natural pairing $\langle ., . \rangle : Z(\Sigma) \otimes Z(\Sigma)^* \to \mathbb{C}$. This means the number can be calculated for any decomposition $M = M_2 \sqcup M_1$.

With all this said, we see that the axioms with some additional details corresponds to the first given definition. We now check the converse. Given a symmetric monoidal functor $Z : (\mathbf{nCob}, \sqcup, \emptyset, T) \to (\mathbf{Vect}_K, \otimes, K, \sigma)$, we see that (a), (ii), (iii), (iv) and (v) are automatically satisfied. To verify (i), consider a cylinder and reverse one of its boundaries such that we have two in-boundaries, meaning we get a cobordism $\Sigma \otimes \Sigma \xrightarrow{M} \emptyset_{n-1}$. The image under this cobordism is a pairing $Z(M) : Z(\Sigma) \otimes Z(\Sigma) \to \mathbb{C}$. Likewise, by reversing the in-boundary so we get two out-boundaries, we receive a copairing $Z(N) : \mathbb{C} \to Z(\Sigma) \otimes Z(\Sigma)$. Now, by considering the image of the left hand side of the snake relation (2.40)

$$Z(\Sigma) \xrightarrow{\text{id}_Z(\Sigma) \otimes Z(N)} Z(\Sigma) \otimes Z(\Sigma) \otimes Z(\Sigma) \xrightarrow{Z(M) \otimes \text{id}_Z(\Sigma)} Z(\Sigma)$$

(3.7)
which by the relation should be equal to the cylinder, whose image is the identity map, (3.7) must also be the identity. This, together with the relation obtained with the right hand side of the snake relation, amounts to saying that the pairing \( Z(M) \) is non-degenerate (see (2.87)). This means precisely that there is a canonical isomorphism between \( Z(\Sigma) \) and \( Z(\Sigma) \) given by the map \( Z(M)_{\text{left}} : Z(N) \rightarrow Z(M)^* \) defined by \( w \mapsto Z(M)(\cdot \otimes w) \). Thus, we can identify \( Z(\Sigma) \) with \( Z(\Sigma)^* \). To justify (b), view all cobordisms \( \Sigma_1 \overset{M}{\rightarrow} \Sigma_2 \) as cobordisms \( \emptyset_{n-1} \overset{M}{\rightarrow} \Sigma_1 \sqcup \Sigma_2 \). The image of \( M \) then becomes \( Z(M) : \mathbb{C} \rightarrow Z(\Sigma_1 \sqcup \Sigma_2) \). With the same reasoning as before, the map \( Z(M) \) is completely specified by its value on \( 1 \in \mathbb{C} \), hence we can regard it as giving a vector \( Z(M)(1) \in Z(\Sigma_1 \sqcup \Sigma_2) \). Thus, this type of TQFT gives rise to a TQFT defined by Atiyah.

We now would like to calculate the invariant produced by a the cobordism \( \Sigma \times S^1 \), where \( \Sigma \) is a closed \((n - 1)\)-manifold. This manifold can be seen as joining the two ends of a cylinder \( \Sigma \times I \). We do it by decomposing it into two parts, i.e. cut it along \( \Sigma \sqcup \Sigma \),

\[
\text{(3.8)}
\]

We can then view this as a composite cobordism \( M_2 \circ M_1 \), with \( \emptyset_{n-1} \overset{M_1}{\rightarrow} \Sigma \sqcup \Sigma \) and \( \Sigma \sqcup \Sigma \overset{M_2}{\rightarrow} \emptyset_{n-1} \) which will have the image

\[
Z(M_2 \circ M_1) = Z(M_2)Z(M_1)
\]

(3.9)

with

\[
Z(M_1) : \mathbb{C} \rightarrow Z(\Sigma) \otimes Z(\Sigma)
\]

(3.10)

and

\[
Z(M_2) : Z(\Sigma) \otimes Z(\Sigma) \rightarrow \mathbb{C}.
\]

(3.11)

We use the notation \( Z(M_1) := \gamma \) and \( Z(M_2) := \beta \). Now we use the fact that a linear map is determined by what the basis of the domain gets mapped to. Let the basis for \( Z(\Sigma) \otimes Z(\Sigma) \) be denoted by \( e_i \otimes e_j \) with \( i, j = 1, \ldots, m \), \( m \) being the dimension of \( Z(\Sigma) \). A basis for \( \mathbb{C} \) is \( 1 \). For the calculations to be smoother, we use the Einstein summation convention. Let now \( \gamma(1) = \gamma^{ij} e_i \otimes e_j \) and \( \beta(e_i \otimes e_j) := \beta_{ij} \). This gives us

\[
Z(M_2 \circ M_1)(1) = Z(M_2)Z(M_1)(1) = \beta \circ \gamma(1) = \beta(\gamma^{ij} e_i \otimes e_j) = \gamma^{ij} \beta(e_i \otimes e_j) = \gamma^{ij} \beta_{ij}.
\]

(3.12)

Now, we have the fact that the snake relation (2.40) holds not only for two, but for arbitrary dimensions. In particular, this means that the relation has to be respected by \( Z \) which gives us

\[
\gamma^{ij} \beta_{jk} = \delta^k_i
\]

(3.13)

\[
\beta^{ij} \gamma^{jk} = \delta^i_j
\]

(3.14)

which can be shown by considering the image of the relation: use \((\beta \otimes \text{id}_A) \circ (\text{id}_A \otimes \gamma) = \text{id}_A = (\text{id}_A \otimes \beta) \circ (\gamma \otimes \text{id}_A)\) and evaluate at \((a^i e_i \otimes 1)\) and \((1 \otimes a^i e_i)\) on respective sides. Moreover, \( Z \) is a symmetric monoidal functor, meaning the symmetry has to be preserved, in particular \( \beta_{ij} = \beta_{ji} \) and \( \gamma^{ij} = \gamma^{ji} \). Altogether, it is just the statement that \( (\beta_{ij}) \) is an invertible symmetric matrix with inverse \( (\gamma^{ij}) \). Thus

\[
\beta^{ij} \gamma^{ij} = \delta^j_i
\]

(3.15)

We have obtained the following result:

\[
Z(\Sigma \times S^1) = \dim(Z(\Sigma)).
\]

(3.16)

It is quite remarkable that the invariant produced by closing the ends of a cylinder is actually the dimension of the space \( Z(\Sigma) \).

29
3.1.3 Physical Interpretations

A TQFT is *Hermitian* if for every cobordism \( \Sigma_1 \xrightarrow{M} \Sigma_2 \), \( Z(M) = Z(M)^\dagger \) is satisfied, where \( \dagger \) is the Hermitian conjugate. In \( \text{nCob} \), the Hermitian conjugate precisely the orientation reversal, meaning that the Hermitian conjugate of \( \Sigma_1 \xrightarrow{M} \Sigma_2 \) is \( \Sigma_2 \xrightarrow{M^\dagger} \Sigma_1 \), which is the time reversal operation in \( \text{nCob} \). In particular, if not all values are real, the theory can detect the orientation reversal. A theory which has that detection property is called *unitary*. These are the TQFT’s of greatest interest in physics.

We describe now the physical picture of a unitary TQFT, i.e. a unitary symmetric monoidal functor \( Z \). Being a functor means that given a composite cobordism \( M = M_2 \circ M_1 \), \( Z(M_2 \circ M_1) = Z(M_2)Z(M_1) \) is satisfied. This corresponds to the fact that time elapsing through \( M_1 \) first and then through \( M_2 \), should be the same as time elapsing through \( M \). Moreover, functors maps identities to identities, i.e. \( Z(\Sigma \times I) = \text{id}_{Z(\Sigma)} \) meaning that if the time elapses without a change in the topology, the state of the universe will remain unchanged.

\( Z \) being a monoidal functor is related to non-interacting systems in quantum mechanics. If we have a space of states of non-interacting systems, it can be written with tensor products between the space of states of individual systems.

The functor \( Z \) being symmetric is related to the statistics of of the particles in question. In our case, we send the symmetry in \( \text{nCob} \) to the symmetry in \( (\text{Vect}_\mathbb{C}, \otimes, \mathbb{C}, \sigma) \), meaning the interchange of two states \( \phi \) and \( \psi \) corresponds to the interchange \( \phi \otimes \psi \mapsto \psi \otimes \phi \). The particles we consider are hence bosons.

For a fermionic TQFT, one considers \( \text{grVect}_\mathbb{C} \) instead of \( \text{Vect}_\mathbb{C} \).

Lastly, being unitary means that, given a time evolution operator \( U \) for a process, the time evolution operator corresponding to the reversed process is \( U^\dagger \) (it does not mean, however, that \( U \) is unitary). In \( \text{nCob} \), we can define a cobordism \( \Sigma_1 \xrightarrow{M} \Sigma_2 \) to be unitary if \( M \circ M^\dagger \) is the identity over \( \Sigma_2 \) and \( M^\dagger \circ M \) is the identity over \( \Sigma_1 \). In the case \( n \leq 3 \), it turns out that only cylinders satisfies this condition. This means that if there is no change in the topology, the time evolution is unitary.

3.2 Two-Dimensional Topological Quantum Field Theories

3.2.1 The Link between Two-Dimensional Topological Field Theories and Frobenius Algebras

Recall that a *two-dimensional topological quantum field theory* (2dTQFT) is a symmetric monoidal functor

\[
Z : (\text{2Cob}, \sqcup, \emptyset, T) \rightarrow (\text{Vect}_\mathbb{C}, \otimes, \mathbb{C}, \sigma).
\] (3.17)

As we might suspect from earlier discussions, 2dTQFT’s seem to be closely related to Frobenius algebras. In fact, we state it in a theorem:

**Theorem 3.1.** To give a two-dimensional topological quantum field theory \( Z : (\text{2Cob}, \sqcup, \emptyset, T) \rightarrow (\text{Vect}_\mathbb{C}, \otimes, \mathbb{C}, \sigma) \) is equivalent to giving a commutative Frobenius algebra \( \mathcal{A} \) in \( (\text{Vect}_\mathbb{C}, \otimes, \mathbb{C}, \sigma) \). Also, there is a one-to-one correspondence between two-dimensional topological quantum field theories and commutative Frobenius algebras.

We are going to follow the proof given in [13]. For an alternative proof, consult [5].

**Proof.** The meaning of the word "equivalent" will become evident in due proof. First, let there be given a 2dTQFT \( Z \). Recall that the only closed oriented 1-manifolds, i.e. the objects in \( \text{2Cob} \), are circles \( S^1 \) and the disjoint union of \( n \) copies of \( S^1 \). Let the image vector space of \( S^1 \) be denoted by \( \mathcal{A} \); \( Z(S^1) = \mathcal{A} \). Let \( \mathcal{A}^n := \mathcal{A} \otimes \cdots \otimes \mathcal{A} \) for \( n \) factors of \( \mathcal{A} \) and \( n \) denote the disjoint union of \( n \) copies of circles. Then, since \( Z \) is a symmetric monoidal functor, we will automatically have

\[
Z(\mathcal{A}) = \mathcal{A}^n \quad \text{(3.18)}
\]

\[
Z(\emptyset) = \text{id}_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A} \quad \text{(3.19)}
\]

\[
Z(\mathcal{A} \otimes \cdots \otimes \mathcal{A}) = \sigma : \mathcal{A}^2 \rightarrow \mathcal{A}^2 \quad \text{(3.20)}
\]
The generators will have the following images with the following notation:

\[ Z(\bigodot) = \eta : \mathbb{C} \to \mathcal{A} \]  \hspace{1cm} (3.21)

\[ Z(\bigcirc) = \epsilon : \mathcal{A} \to \mathbb{C} \] \hspace{1cm} (3.22)

\[ Z\left(\begin{array}{c}
\text{\scalebox{0.8}{\text{\textcircled{\scriptsize 0}}}}
\end{array}\right) = \mu : \mathcal{A}^2 \to \mathcal{A} \] \hspace{1cm} (3.23)

\[ Z\left(\begin{array}{c}
\text{\scalebox{0.8}{\text{\textcircled{\scriptsize 0}}}}
\end{array}\right) = \Delta : \mathcal{A} \to \mathcal{A}^2 \] \hspace{1cm} (3.24)

Now, using again the fact that \( Z \) is a symmetric monoidal functor, we can see that the relations in \textbf{2Cob} described in section 2.2.4 will translate to the axioms for commutative Frobenius algebras. This can be seen especially by comparing the relations described graphically in section 2.3.5. Hence, given a 2dTQFT \( Z \), the image of \( S^1 \), \( Z(S^1) = \mathcal{A} \), is a commutative Frobenius algebra.

Conversely, suppose we have given a Frobenius algebra \((\mathcal{A}, \epsilon)\) with multiplication \( \mu \), comultiplication \( \Delta \), unit \( \eta \) and counit \( \epsilon \). We use the recipe given above and define the images as given in the equation above to design the functor. However, in this case, we must check if the relations are intact. In fact they are since, again, the relations have an exact correspondence to the axioms of a commutative Frobenius algebra. Thus the functor defined is well-defined.

Also, these two construction are inverse to each other; by first having a 2dTQFT \( Z \) given, \( Z(S^1) = \mathcal{A} \) is a Frobenius algebra. But from \( \mathcal{A} \), we can recover the same 2dTQFT by the same construction given above. Thus, there is a one-to-one correspondence between 2dTQFT’s and commutative Frobenius Algebras.

\[ 3.2.2 \text{ Properties of Two Dimensional Topological Quantum Field Theories} \]

From the classification theorem for closed oriented surfaces, we know that the genus of a surface will completely classify it, e.g. if the genus is 0 it will be a sphere, if it is 1 it will be a torus, and so on. As we said earlier, we are interested in the numerical invariants, the partition functions, produced by a \( n \)-dimensional TQFT on closed \( n \)-manifolds. We work in the functorial picture. Let \( T \) be the manifold defined by cutting out a disc of a torus and making the boundary formed to an out-boundary (we could have equally made it an in-boundary). Let also \( Z \) be a 2dTQFT such that \( Z(S^1) = \mathcal{A} \) and \( Z(T)(1) = \alpha \in \mathcal{A} \) \( (Z(T) : \mathbb{C} \to \mathcal{A} \) is completely specified from its image on \( 1 \in \mathbb{C} \)). Let \( K_g \) be surface of genus \( g \). Given these data, we want to determine \( Z(K_g) \) for each \( g = 1, \ldots, n \). This can be done by induction. Let \( Z(\bigodot) = \epsilon : \mathcal{A} \to \mathbb{C} \) be the Frobenius form. Also, let the multiplication be the image of the right-pair of pants \( \begin{array}{c}
\text{\scalebox{0.8}{\text{\textcircled{\scriptsize 0}}}}
\end{array} \) with a right-cap \( \begin{array}{c}
\text{\scalebox{0.8}{\text{\textcircled{\scriptsize 0}}}}
\end{array} \) glued on its boundary. Since \( Z \) is a functor, we have

\[ Z(K_1)(1) = (Z(\bigodot) \circ Z(T))(1) = \epsilon(\alpha). \] \hspace{1cm} (3.25)

Suppose now \( Z(K_m) = \epsilon(\alpha^m) \), i.e. if \( Z(K'_m) = \alpha^m \) if we cut away a disc from \( K_m \) and make the cut into an out-boundary. Consider now for the case \( g = m + 1 \). Then the new surface will be \( (K'_m \sqcup T) \) glued to the in-boundary of the right-pair of pants \( \begin{array}{c}
\text{\scalebox{0.8}{\text{\textcircled{\scriptsize 0}}}}
\end{array} \) with a right-cap \( \begin{array}{c}
\text{\scalebox{0.8}{\text{\textcircled{\scriptsize 0}}}}
\end{array} \) glued to the out-boundary. Then

\[ Z(K_{m+1})(1) = (Z(\bigodot) \circ Z\left(\begin{array}{c}
\text{\scalebox{0.8}{\text{\textcircled{\scriptsize 0}}}}
\end{array}\right) \circ Z((K'_m \sqcup T)))(1) \]
\[ = (Z(\bigodot) \circ Z\left(\begin{array}{c}
\text{\scalebox{0.8}{\text{\textcircled{\scriptsize 0}}}}
\end{array}\right) \circ (Z(K_m) \otimes Z(T)))(1) \]
\[ = \epsilon((Z(K_m)(1)) \otimes Z(T))(1)) \]
\[ = \epsilon(\alpha^m \alpha) = \epsilon(\alpha^{m+1}) \] \hspace{1cm} (3.26)

since the multiplication is the image of the right pair of pants. We can hence conclude that

\[ Z(K_g)(1) = \epsilon(\alpha^g), \ g \in \mathbb{Z}^+. \] \hspace{1cm} (3.27)
Note that we could have done it without evaluating at 1, then we would have pointwise multiplication of linear maps instead (which are basically just numbers either way). Note that we have the following due to (3.16):

$$Z(K_1)(1) = \epsilon(\alpha) = \dim(Z(S^1)).$$  
(3.28)

In the case $g = 0$, $K_0$ is a sphere, which can be decomposed into a left cap and a right cap, meaning

$$Z(K_0)(1) = (Z(\Box) \circ Z(\Box))(1) = (\epsilon \circ \eta)(1).$$  
(3.29)

We have that $\eta(1) = 1_A$, where $1_A$ is the identity in $A$, in order for the diagrams in (2.47) to commute. Hence

$$Z(K_0)(1) = \epsilon(1_A).$$  
(3.30)

### 3.3 The Dijkgraaf-Witten Model

The Dijkgraaf-Witten model [9] (DW-model) is a TQFT over principal $G$-bundles, where $G$ is a finite group. Quite remarkably, this model turns out to be a discrete model for several important theories. For instance, in two dimensions, it is a model for BF-theory. In three dimensions, it is a model for Chern-Simons theory.

#### 3.3.1 Principal Bundles

In order to describe the DW-model, we need to introduce principal $G$-bundles and concepts related to it. Firstly, a (smooth) fibre bundle $(E, \pi, M, F, G)$ consists of the following data:

1. A smooth manifold $E$ called the total space.
2. A smooth manifold $M$ called the base space.
3. A smooth manifold $F$ called the fibre (or the typical fibre).
4. A surjection $\pi : E \to M$ (the projection). If $p \in M$, then $\pi^{-1}(p) = F_p \cong F$, where $F_p$ is called the fibre at $p$.
5. A Lie Group $G$ acting on $F$ from the left called the structure group.
6. An open covering $\{U_i\}$ of $M$ together with diffeomorphisms $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ called the local trivializations such that the following digram commutes:

$$\begin{array}{ccc}
\pi^{-1}(U_i) & \xleftarrow{\phi_i} & U_i \times F \\
\pi \downarrow & & \downarrow \\
U_i & & \\
\end{array}$$

(3.31)

where the unlabelled arrow is the canonical projection.

7. Let $\phi_i(p, f) := \phi_{i,p}(f)$ which is a diffeomorphism $\phi_{i,p}(f) : F \to F_p$. On every non-empty intersection of open neighbourhoods $U_i \cup U_j \neq \emptyset$, we demand that $t_{ij}(p) := \phi_{i,p}^{-1} \circ \phi_{j,p} : F \to F$, called a transition function, is an element of $G$. Then for $t_{ij} : U_i \cup U_j \to G$, we have

$$\phi_j(p, f) = \phi_i(p, t_{ij}(p)f).$$  
(3.32)
A fibre bundle can be seen as a geometrical object which locally looks like a product manifold \( U_i \times F \), where every fibre \( F \) is modelled after a typical fibre \( F \). This is actually the definition for a coordinate bundle, which is just a representative in the equivalence class which is the actual fibre bundle. This distinction will however not be done here. We denote a fibre bundle by \( E \) and if we know the base space \( M \), we say that \( E \) is a fibre bundle over \( M \). A principal \( G \)-bundle (or simply \( G \)-bundle) \( P \) is a fibre bundle whose fibre \( F \) is identical to the structure group \( G \). We consider a free right action \( R_g : P \times G \to P \) of \( G \) on \( P \), with \((p, g) \to pg\) and free meaning if \( pg_1 = pg_2 \) for some \( p \in P \), then \( g_1 = g_2 \). We also want to have some notion of equivalence between the principal bundles; two \( G \)-bundles \( P \) and \( P' \) are equivalent if there is a homeomorphism \( h : P \to P' \) such that \( h(p \cdot g) = h(p) \cdot g \) and such that the diagram

\[
\begin{array}{ccc}
P & \xrightarrow{h} & P' \\
\pi & & \pi' \\
\downarrow & & \downarrow \\
M & & M
\end{array}
\]

commutes, where \( \pi \) is the projection of \( P \) and \( \pi' \) is the projection of \( P' \). We call the map \( h \) a bundle isomorphism. From this, we may construct equivalence classes, isomorphism classes, \([P]\) of \( G \)-bundles. If \( P \) is a \( G \)-bundle over a closed \((n-1)\)-manifold \( \Sigma \) we denote the set of isomorphism classes by \( \mathbb{C} \Sigma \).

In order to have some notion of transport of data on a principal bundle, we define a connection on a principal bundle. Let \( P \) be a principal \( G \)-bundle. A connection on \( P \) is a decomposition of the tangent space \( T_pP \) at every point \( p \in P \) in a vertical part \( V_pP = \ker(\pi_\ast p) \), where \( \pi_\ast : T_pP \to \pi^{-1}(p)M \) is the differential map of the projection \( \pi \) at \( p \), and horizontal part \( H_pP \) such that

(i) \( T_pP = V_pP \oplus H_pP \)

(ii) Every smooth vector field \( X \) on \( P \) decomposes into smooth vector fields \( X^V \in V_pP \) and \( X^H \in H_pP \)

\[X = X^V + X^H\]

(iii) \( R_{pg}H_pP = H_{pg}P \) for every \( p \in P \) and \( g \in G \), where \( R_{g\ast} : T_pP \to T_{pg}G \) is the differential map of the right action \( R_g \).

Through the connection, we may define the notion of a horizontal lift of a curve. Let \( \gamma : [0,1] \to M \) be a curve. A curve \( \tilde{\gamma} : [0,1] \to P \) is a horizontal lift of \( \gamma \) if \( \pi \circ \tilde{\gamma} = \gamma \) and the tangent vector to \( \tilde{\gamma}(t) \) is in \( H_{\tilde{\gamma}(t)}P \) for every \( t \in [0,1] \). It turns out that the horizontal lift is unique, which has great consequences. There is hence given a point \( p_0 \in P \) and a curve \( \gamma \) on \( M \), a unique point \( p_1 = \tilde{\gamma}(1) \) since the horizontal lift \( \tilde{\gamma} \) is unique. The point \( p_1 \) is the parallel transport of \( p_0 \) along \( \gamma \).

Let now \( \gamma \) be a loop based at \( \gamma(0) = \gamma(1) = x \). The parallel transport of \( p_0 = \tilde{\gamma}(0) \in \pi^{-1}(x) \) will in general not be \( p_0 \), i.e. \( \tilde{\gamma}(0) \neq \tilde{\gamma}(1) \) in general. This means a loop \( \gamma \) based at \( x \) will define a transformation \( \tau_\gamma : \pi^{-1}(x) \to \pi^{-1}(x) \), such that \( \tau_\gamma(p_0) = p_0g \) for some \( g \in G \). Furthermore, \( \tau_\gamma \) is compatible with the right action of \( G \),

\[
\tau(ph) = \tau(p)h \tag{3.34}
\]

for \( p \in P \) and \( h \in G \).

Consider now a point \( p \in P \) and the set of loops based at \( x \in M \) such that \( \pi(p) = x \), denoted by \( C_x(M) = \{ \gamma : [0,1] \to M | \gamma(0) = \gamma(1) = x \} \). The set

\[
\Phi_p = \{ g \in G | \tau_\gamma(p) = pg, \ \gamma \in C_x(M) \} \tag{3.35}
\]

forms a group known as the holonomy group at \( p \).

We will also make use of the notion of the fundamental group. Let \( I = [0,1] \), \( X \) a topological space (in our case manifolds) and \( \gamma, \gamma' : I \to X \) be two loops at \( x_0 \). Then \( \gamma \) and \( \gamma' \) are homotopic if there exists a continuous map \( F : I \times I \to X \), called a homotopy, such that
(i) \( F(s, 0) = \gamma(s) \forall s \in I \).

(ii) \( F(s, 1) = \gamma'(s) \forall s \in I \).

(iii) \( F(0, t) = F(1, t) = x_0 \forall t \in I \).

The relation "\( \gamma \) is homotopic to \( \gamma' \)" is furthermore an equivalence relation; a homotopy class is the equivalence class of all loops at \( x_0 \) homotopic to each other. The set of all homotopy classes of loops based at \( x_0 \) forms a group known as the fundamental group (or the first homotopy group) and is denoted by \( \pi_1(X, x_0) \).

If, in addition, \( X \) is (arcwise) connected, the fundamental group is independent of the base point, i.e. \( \pi_1(X, x_0) \cong \pi_1(X, x_1) \).

Moreover, it can be shown that the fundamental group is in fact a topological invariant.

3.3.2 The Dijkgraaf-Witten Model

As stated earlier, the DW-model is built around \( G \)-bundles with the structure group \( G \) being a finite group. As a consequence of the structure group being finite, there will only be one flat connection (i.e. the curvature of the connection is vanishing, the precise meaning of curvature will not be discussed in this thesis). Let \( \Sigma \) be a closed oriented \((n-1)\)-manifold. We denote the space of \( G \)-bundles over \( \Sigma \) by \( C_\Sigma \). This is the space of fields in the DW-model. Recall that to give a TQFT (within the framework of the definition by Atiyah), we need to specify a vector field \( Z(\Sigma) \) and a vector \( Z(M) \in Z(\Sigma) \), \( M \) being a compact oriented \( n \)-manifold with boundary \( \partial M = \Sigma \). In the DW-model, we define the space of wave functions

\[
Z(\Sigma) := \{ \text{functions } f : C_\Sigma \to \mathbb{C} \}. 
\]

(3.36)

This space has a basis \( \{ \hat{f} \} \) of functions which takes the value 1 on \( [P] \) and 0 otherwise. For \( Z(M) \), we define it as

\[
Z(M)(Q) := \sum_{[P] \in C_M : P|_{\partial M} = Q} \frac{1}{|\text{Aut}(P)|} 
\]

(3.37)

where \( C_M \) is the (finite) set of all isomorphism classes of \( G \)-bundles over \( M \). This rather obscure summation should be interpreted as follows: sum over all equivalence classes \( [P] \) in \( C_M \) such that \( [P] \) has a representative \( P \) whose restriction to the boundary of \( M \) is equal to \( Q \). Each term is weighted by the reciprocal size of the automorphism group of \( P \). Later on, this choice of weight will prove to be useful.

In the case when \( \Sigma = \Sigma_1 \sqcup \cdots \sqcup \Sigma_k \), \( Q \) will be determined by \( Q_1, \ldots, Q_k \), each \( Q_i \) restricted to the corresponding boundary component \( \Sigma_i \). This allows us to view \( Z(M) : Z(\Sigma_1) \otimes \cdots \otimes Z(\Sigma_k) \to \mathbb{C} \otimes \cdots \otimes \mathbb{C} \cong \mathbb{C} \) as a \( k \)-point function \( (Q_1 \cdots Q_k) \) or a \((k, 0)\)-tensor \( Z(M)Q_1 \cdots Q_k \). An important case is the cylinder \( C = \Sigma \times I \) with boundary \( \Sigma \sqcup \Sigma \). This gives us \( Z(C) : Z(\Sigma) \otimes Z(\Sigma) \to \mathbb{C} \) which defines a metric \( g^{QR} \) on \( Z(\Sigma) \)

\[
(3.38)
\]

The inverse metric,

\[
(3.39)
\]

we define as \( g_{QR} := (g^{QR})^{-1} \) (compare with the discussion which yielded (3.13) and (3.14)).

With the help of the metrics defined above, we can view \( M \) as a cobordism \( \Sigma_{\text{in}} \rightarrow \Sigma_{\text{out}} \) by choosing

\[
\Sigma_{\text{out}} = \Sigma_{j_1} \sqcup \cdots \sqcup \Sigma_{j_t} 
\]

(4.0)

and

\[
\Sigma_{\text{in}} = \Sigma_{j_{t+1}} \sqcup \cdots \sqcup \Sigma_{j_k}. 
\]

(3.41)
We can then use a sequence of metrics to raise the indices of $Z_{Q_1...Q_k}$:

$$Z_{Q_1...Q_k}^{Q_{j_1+1}...Q_{j_k}} = g^{Q_{j_1+1} R_{j_1+1} \ldots g^{Q_{j_k} R_{j_k}} Z_{Q_1...Q_k}}$$  (3.42)

This can be seen as turning the out-boundaries into in-boundaries by applying the u-tube (3.38). Using this, we define $Z(M) : Z(\Sigma_{j_1}) \otimes \cdots \otimes Z(\Sigma_{j_i}) \rightarrow Z(\Sigma_{j_{i+1}}) \otimes \cdots \otimes Z(\Sigma_{j_k})$, i.e. we define $Z(M) : Z(\Sigma_{\text{in}}) \rightarrow Z(\Sigma_{\text{out}})$ by

$$\langle [P_2] | Z(M) | [P_1] \rangle = \sum_{P|\Sigma_{\text{in}}=P_2} g(\partial M) [\text{Aut}(P)]^{-1}$$  (3.43)

where $g(\partial M)$ is a sequence of factors corresponding to the metric raising operations for each component in $\Sigma_{\text{out}}$. The summation is over equivalence classes $[P]$ in $C_M$ such that $[P]$ has a representative $P$ which restricted to the in-boundary of $M$ is $P_1$ and restricted to the out-boundary of $M$ is $P_2$. It is interpreted as the probability for a $G$-bundle over $\Sigma_{\text{in}}$ to evolve to a $G$-bundle over $\Sigma_{\text{out}}$. Note that this formula has the same form as (3.1).

As before, we are in particular interested in calculating the partition functions produced from closed $n$-manifolds. In the case $M$ is a closed and oriented $n$-manifold, (3.37) takes the form

$$Z(M) = \sum_{[P]} \frac{1}{|\text{Aut}(P)|}$$  (3.44)

where we simply sum over all isomorphism classes of $G$-bundles over $M$ (since in 3.37, here $Q$ will be empty and any $P$ restricted to $\partial M$ will also be empty). We would like to investigate the space of fields and the holonomy group will effectively be

$$\Phi_{p_0} = \{ g \in G | [\tau_1] (p_0) = p_0 g, \ [\gamma] \in \pi_1(M,x) \}.$$  (3.45)

This in particular means every $[\gamma] \in \pi_1(M,x)$ will be associated to a $g \in G$. Furthermore, if $M$ is arcwise connected, the fundamental group will not depend on the base point meaning each principal bundle determines a homomorphism $\phi \in \text{Hom}(\pi_1(M),G)$. Conversely each homomorphism $\phi \in \text{Hom}(\pi_1(M),G)$ will determine $P$. We may hence adopt the notation $P_\phi$ for the principal bundle determined by $\phi$. If we had chosen another reference point $p'_0 = p_0 h$, the holonomy would have been conjugated:

$$\tau(p'_0) = \tau(p_0 h) = \tau(p_0) h = p_0 g h = p'_0 h^{-1} g h.$$  (3.46)

Hence there is a right action on $\text{Hom}(\pi_1(M),G)$ given by

$$(\phi.g)(\sigma) = g^{-1} \phi(\sigma).g.$$  (3.47)

It turns out that $P_{\phi_1}$ and $P_{\phi_2}$ are isomorphic if and only if $\phi_2 = \phi_1.g$ for some $g \in G$. Altogether, we may associate each isomorphism class to an orbit in $\text{Hom}(\pi_1(M),G)$, i.e.

$$C_M \cong \text{Hom}(\pi_1(M),G)/G.$$  (3.48)

Taken a point $p_0 \in P$, an automorphism $f : P \rightarrow P$ should satisfy

$$f(p_0) = p_0 g$$  (3.49)

for some $g \in G$ and for every $h \in G$

$$f(p_0 h) = f(p_0) h.$$  (3.50)
The automorphism will be uniquely determined by what it maps \( p_0 \) to, i.e. it will be determined by \( g \) as in (3.49). Given any other point on the fibre \( p'_0 = p_0h \), then \( f(p'_0) = f(p_0h) = p_0gh \), meaning the image of \( p'_0 \) is decided when the image of \( p_0 \) is specified. To extend \( f \) to whole of \( P \) with continuity, this in particular means we must have consistency if we parallel transport around a loop \( \gamma \), i.e.

\[
f(p_0\phi(\gamma)) = f(p_0)g^{-1}\phi(\gamma)g
\]

(3.51)
since changing the reference point will conjugate the holonomy. In order for this map to work with (3.50), we require that \( g \) must commute with the image of \( \phi \) since this this equality should hold

\[
f(p_0)\phi(\gamma) = f(p_0\phi(\gamma)) = f(p_0)g^{-1}\phi(\gamma)g.
\]

(3.52)
Hence \( \text{Aut}(P_\phi) \) is the subgroup of elements commuting with the image of \( \phi \),

\[
\text{Aut}(P_\phi) = \{ g. \in G | g\phi(\cdot) = \phi(\cdot)g \}.
\]

(3.53)
The commutativity condition can be rewritten as \( \phi(\cdot) = g^{-1}\phi(\cdot)g = (\phi.g)(\cdot) \), which means \( \text{Aut}(P_\phi) \) is the stabiliser subgroup of \( \phi \),

\[
\text{Aut}(P_\phi) = \text{Stab}(\phi).
\]

(3.54)
By using the orbit-stabiliser theorem

\[
|G| = |\text{Orbit}(\phi)||\text{Stab}(\phi)|,
\]

(3.55)
we may rewrite (3.44) as follows:

\[
Z(M) = \sum_{[P]} \frac{1}{|\text{Aut}(P)|}
\]

\[
= \sum_{\text{orbits}} \frac{1}{|\text{Stab}(\phi)|}
\]

\[
= \sum_{\text{orbits}} \frac{|\text{Orbit}(\phi)|}{|G|}
\]

\[
= \frac{1}{|G|} \sum_{\text{orbits}} |\text{Orbit}(\phi)|
\]

\[
= \frac{|\text{Hom}(\pi_1(M), G)|}{|G|}
\]

(3.56)
Here we see the advantage of having weighted with the automorphism group of \( P \); had we not, (3.56) would have resulted in \( |\text{Hom}(\pi_1(M), G)/G| \) instead, which is more difficult to calculate.

### 3.3.3 The Dijkgraaf-Witten Model in Two Dimensions

We will now briefly investigate the DW-model in two dimensions. In this section, we make use of examples 2.9 and 2.10 as promised.

In two dimensions, the only closed oriented manifold is the circle \( S^1 \). Since \( S^1 \) is arcwise connected, it is independent of basepoint. The fundamental group of \( S^1 \) is \( \pi_1(S^1) = \mathbb{Z} \). Hence, to give a homomorphism \( \phi \in \text{Hom}(\pi_1(S^1), G) = \text{Hom}(\mathbb{Z}, G) \) is the same as specifying an element \( g \in G \) since \( \phi \) is determined by the image of 1 (due to the fact that 1 solely generates \( \mathbb{Z} \)). This in particular means that \( G \cong \text{Hom}(\pi_1(S^1), G) \). Hence, the conjugation of \( \phi \) is the same as conjugating the corresponding group element \( g \in G \). By (3.48),

\[
\overline{C_{S^1}} \cong \text{Cl}(G)
\]

(3.57)
where $\text{Cl}(G)$ is the set of conjugacy classes of $G$. From this, we conclude that

$$Z(S^1) = C_{\text{class}}(G) = \tilde{Z}(\mathbb{C}[G]), \quad (3.58)$$

where $\tilde{Z}(\mathbb{C}[G])$ is the centre of $\mathbb{C}[G]$ as before.

From here however, we are going to use a different approach. We will mostly follow seminar notes from lectures given by Baez [6, 7] (the original ideas can be found in the paper [10] by Fukuma, Hosono and Kawai). It can be shown that a semisimple algebra $A$ (e.g. $\mathbb{C}[G]$) gives rise to a 2dTQFT $Z$, which in turn will take the unit circle to the centre of $A$, i.e. $Z(S^1) = \tilde{Z}(A)$. Hence, since the DW-model computes for $S^1$ (3.58), we investigate the (semisimple) group algebra $\mathbb{C}[G]$ further. We have that the bilinear pairing

$$g(a, b) = \text{tr}(L_a L_b) \quad (3.59)$$

is non-degenerate and associative (meaning it is a Frobenius pairing), where $L_x$ is the left multiplication by $x \in \mathbb{C}[G]$. We now choose a basis $\{e^g\}$ of $\mathbb{C}[G]$ such that $e^g \in \mathbb{C}[G]$ corresponds to $g \in G$. Then the multiplication is given by

$$e^i e^j = m_{ij}^k e^k \quad (3.60)$$

where

$$m_{ij}^k = \delta_{ij}^k = \begin{cases} 1, & \text{if } ij = k \\ 0, & \text{if } ij \neq k. \end{cases} \quad (3.62)$$

In order to be able to raise and lower indices, we need a metric. We define the metric by

$$g_{ij} = |G| \delta_{ij} = |G| \delta_{i-j} \quad (3.64)$$

meaning the metric will be given by

$$g^{ij} = \frac{1}{|G|} \delta_{ij} = \frac{1}{|G|} \delta_{i-j}^{-1} \quad (3.65)$$

Using the inverse metric, we may calculate

$$m_{jk}^i m_{ij}^l g_{lk} = \delta_{ij}^l \frac{1}{|G|} \delta_{j-k}^{-1} = \frac{1}{|G|} \delta_{i-k}^{-1} = \frac{1}{|G|} \delta_{i-j} \quad (3.66)$$

Lastly, we would like to calculate the unit and the Frobenius form. Recall that we may construct a Frobenius form via (2.54), i.e.

$$\epsilon^i := \epsilon(e^i) := g(e^i, 1) = \text{tr}(L_i) = |G| \delta^i_1 \quad (3.67)$$

For the unit, we use the fact that the unit condition must be satisfied, i.e. if $\eta_i$ is the unit, then

$$\eta_i m_{ij}^i = \delta_{i-k} \quad (3.68)$$

meaning

$$\eta_i = \delta_{i-k} \quad (3.69)$$
4 Concluding Remarks

We conclude the thesis by giving a quick summary of the main results. In our setting, a topological quantum field theory is a symmetric monoidal functor $Z : (n\text{Cob}, \sqcup, \emptyset, T) \to (\text{Vect}_\mathbb{C}, \otimes, \sigma)$. We have shown that in two dimensions, to give a TQFT $Z : (2\text{Cob}, \sqcup, \emptyset, T) \to (\text{Vect}_\mathbb{C}, \otimes, \sigma)$ is equivalent to giving a commutative Frobenius algebra $\mathcal{A}$ in $(\text{Vect}_\mathbb{C}, \otimes, \sigma)$. Furthermore, an important feature of TQFT’s is the fact that they compute topological invariant complex numbers from closed oriented $n$-manifolds. In particular, if $\Sigma$ is a closed oriented $(n - 1)$-manifold, the invariant produced from the cylinder $\Sigma \times S^1$ is the dimension of the vector space $Z(\Sigma)$ for a given TQFT $Z$.

The following are a few suggestions of interesting topics which are not covered in this thesis. This work is focused on cobordisms over closed manifolds; the case when also cobordisms over non-closed manifolds are allowed is not taken into account here, meaning we have another source category than $n\text{Cob}$. Another modification would be to study TQFT’s with other target categories, for instance the category of graded vector spaces $\text{grVect}_\mathbb{K}$. Further extensions would be to consider TQFT’s over dimensions other than two. At last, one could of course continue exploring the two dimensional case or investigating the general properties deeper.
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References


