UPPSALA
UNIVERSITET

U.U.D.M. Project Report 2015:25

The subgraph containment problem in random graphs

Niklas Fastlund

Examensarbete i matematik, 15 hp
Handledare och examinator: Vera Koponen
Augusti 2015

Department of Mathematics
Uppsala University

The subgraph containment problem in random graphs

Author:
Niklas Fastlund
nfastlund@gmail.com

Department of Mathematics
Uppsala university

Supervisor: Vera Koponen

August 17, 2015

Typeset in LaTeX

Abstract

In this thesis, the necessary introductions of the binomial and uniform random graph is given. The concept of asymptotically almost surely is explained and some asymptotic notation is presented. With this the thesis proceeds with obtaining the threshold-theorem for the binomial model, which states when the random graph asymptotically almost surely contains a given subgraph with at least one edge. Afterwards by asymptotic equivalence, an analogue threshold-theorem for the uniform model is obtained.

Contents

Chapter 1

Introduction

The subject of random graphs is an area of mathematics which this paper focuses on. Random graphs is considered by many to originate from a series of papers published in the period 1959-1968 by two mathematicians, Paul Erdös and Alfred Rényi. One example is a paper published in 1959 by Erdös and Rényi [1] which begins by introducing the uniform random graph. The uniform random graph is one of two different models that this thesis will focus on, the other one being the binomial random graph.

Random graphs are often used to model real-world-networks of different types such as social networks, collaboration graphs and power grids. Also in the field of epidemiology, the spread of a disease throughout a community can be modeled by letting the individuals be represented by vertices and the possibility of transmitting the disease between two individuals by edges. Even though the classical random graphs (uniform and binomial) may be lacking to incorporate certain behaviour of more modern network problems, one may generalize the mathematics of random graphs to solve this. [2]

Let n be a positive integer and let p be a real number satisfying the inequalities $0 \leq p \leq 1$. The binomial random graph $\mathbb{G}(n,p)$ is defined by taking Ω as the set of all possible graphs on the vertex set $[n] = \{1, 2, ..., n\}$ and letting

$$\mathbb{P}(G) = p^{e_G}(1-p)^{\binom{n}{2}-e_G}, \qquad G \in \Omega \tag{1.1}$$

where $e_G = |E(G)|$ is the number of edges of G. Intuitively one may view it as the result of $\binom{n}{2}$ independent coin flippings, one for each possible pair of vertices, with probability p of successfully drawing an edge between them. A nice property of the binomial model is the independency of each edge. However the number of edges are not fixed. If one conditions on the number of edges (i.e the event $|E(\mathbb{G}(n,p))| = M$) the uniform space emerges. Let M be an integer satisfying $0 \leq M \leq \binom{n}{2}$. Define the uniform random graph, denoted $\mathbb{G}(n,M)$ by taking Ω as the family of all graphs on the vertex set $[n]$ and $\mathbb{P}$ as the uniform probability on Ω,

$$\mathbb{P}(G) = \binom{\binom{n}{2}}{M}^{-1}, \qquad G \in \Omega \tag{1.2}$$

Note that $\binom{\binom{n}{2}}{M}$ is the number of ways of choosing M unordered pairs, i.e edges, from the set $[n]$. The parameter p and M can be fixed. However in this thesis the interest lies in when p and M depends on the number of vertices n. In other words we view p and M as functions of n. The models of random graphs which have been introduced are actually probability spaces with respective measure $\mathbb{P}$ and sample space Ω.

A second example of a paper published in 1960 by Erdös and Rényi [3] brings up the following problem: given a graph G does it exist at least one copy of G in the random graph $\mathbb{G}(n, M)$? Erdös and Rényi [3] found a threshold for certain special cases of G. Later Bollobás 1981 solved it in full generality. Even later, a simpler proof was given by Rucinski and Vince 1985. This is the proof we present in this thesis for the binomial model, and all the necessary work that surrounds it. Later, by asymptotic equivalence, we obtain an analogue theorem for the uniform model. The proof of the theorem and the surrounding work that is necessary follows the book Random graphs [4]. The proof for the analogue theorem follows the book as well. However everything is done in more explanatory steps and exercises that have been left to the reader in the book are presented in this thesis.

Chapter 2

Preliminaries

In this chapter we will go through the notion of asymptotically almost surely, asymptotic notation and some results in the area of probability theory.

2.1 Notation and the notion 'asymptotically almost surely'

We begin with some notations that will be used in this thesis.

- $a_n = O(b_n)$ as $\lim n \to \infty$ if there exist constants $C \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that $|a_n| \leq Cb_n$ for all $n \geq n_0$
- $a_n = \Theta(b_n)$ as $\lim n \to \infty$ if there exist constants $C, c \in \mathbb{R}$, $C, c > 0$ and n_0 such that $cb_n \leq a_n \leq Cb_n$ for all $n \geq n_0$. This can be thought as a_n and b_n having the same order of magnitude.
- $a_n \asymp b_n$ if $a_n = \Theta(b_n)$
- $a_n = o(b_n)$ if for every $\epsilon > 0$ there exists $N(\epsilon)$ such that $|a_n| < \epsilon b_n$ for all $n \geq N(\epsilon)$. (i.e., $\lim a_n/b_n \to 0$)
- $a_n \ll b_n$ or $b_n \gg a_n$ if $a_n \geq 0$ and $a_n = o(b_n)$

Now we define *asymptotically almost surely* (abbreviated a.a.s).

Let A_n be the event describing a property of a random structure depending on n in the sequence of probability spaces $(P_n)_{n\in\mathbb{N}}$. We say that A_n holds asymptotically almost surely if $\lim \mathbb{P}(A_n) \to 1$ as $\lim n \to \infty$. Note that this is not the same as almost surely (abbreviated a.s.) in probablity theory.

2.2 Moment methods

Theorem 2.2.1 Let X be a random variable and $h : \mathbb{R} \to [0, \infty)$ be a non-negative function. Then

$$\mathbb{P}(h(X) \geq a) \leq \frac{\mathbb{E}(h(X)))}{a} \; for \; all \; a > 0 \tag{2.1}$$

Proof. Denote by A the event $\{h(X) \geq a\}$, so that $h(X) \geq aI_A$. Where I_A is the indicator function. Recall that a random variable I_A is called a indicator function if it is 1 when the event A occurs with say, probability p and 0 otherwise with probability $(1-p)$. Now taking the expectation on both sides give us

$$\mathbb{E}(h(X)) \geq \mathbb{E}(aI_A) = a\mathbb{E}(I_A) = a\mathbb{P}(h(X) \geq a).$$

Dividing both sides with a gives us the theorem.

□

We are interested in two particular cases of $h(X)$ of this theorem. First one is when $h(X) = |X|$. Then we get

$$\mathbb{P}(|X| \geq a) \leq \frac{\mathbb{E}(|X|)}{a} \; for\ all\ a > 0$$

The above equation is called *Markov's inequality*. If X is a discrete non-negative random variable then we get that

$$\mathbb{P}(X > 0) \leq \mathbb{E}(X) \tag{2.2}$$

If one let the random variable X_n be the number of copies of G in $\mathbb{G}(n,p)$ or in $\mathbb{G}(n,M)$ and can show that $\mathbb{E}(X_n) = o(1)$. Then one can use the above equation to conclude that $X_n = 0$ a.a.s. This method is what we will refer to as the *first moment method*. The second case we obtain by letting $h(X) = X^2$ we then get

$$\mathbb{P}(X^2 \geq a^2) \leq \frac{\mathbb{E}(X^2)}{a^2} \Leftrightarrow \mathbb{P}(|X| \geq a) \leq \frac{\mathbb{E}(X^2)}{a^2} \; if\ a > 0.$$

This inequality is called *Chebyshev's inequality*. We are interested again with a special case of this. Let X be a random variable where $Var(X)$ exists and $\mathbb{E}(X) > 0$. Let $X' = X - \mathbb{E}(X)$. Now put X' into above inequality with $a = \mathbb{E}(X)$

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq \mathbb{E}(X)) \leq \frac{\mathbb{E}((X - \mathbb{E}(X))^2)}{(\mathbb{E}(X))^2}.$$

Now $\mathbb{E}((X - \mathbb{E}(X))^2) = Var(X)$ and $\mathbb{P}(|X - \mathbb{E}(X)| \geq \mathbb{E}(X))$ is true whenever $X \leq 0$ or $X \geq 2\mathbb{E}(X)$ so it is definitely larger or equal to $\mathbb{P}(X = 0)$. Hence we get

$$\mathbb{P}(X = 0) \leq \mathbb{P}(|X - \mathbb{E}(X)| \geq \mathbb{E}(X)) \leq \frac{Var(X)}{(\mathbb{E}(X))^2} \tag{2.3}$$

By showing that the right hand side of inequality (2.3) where X is replaced with X_n tends to 0 when $\lim n \to \infty$ one asserts that $X_n > 0$ a.a.s. This is what we will refer to as the *second moment method*.

Let us recall that the covariance of two random variables X, Y is defined as $Cov(X,Y) := \mathbb{E}[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))] = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$. With this definition we can present the following lemma which is used in the proof of Theorem 3.2.1

Lemma 2.2.2 Let $\sum_i X_i$ be a finite sum of random variables. Then the following holds

$$Var\left(\sum_i X_i\right) = \sum_i \sum_j Cov(X_i, X_j).$$

We will not prove this lemma in this thesis. However we will prove it for the case $i = 2$ and leave the necessary induction work if one wishes to the reader. A more formalized setting of this result can be found in the book Stokastik [5].

Proof for $i = 2$. First we need to recall that if X, Y, Z are random variables then $Cov(X + Y, Z) = Cov(X, Z) + C(Y, Z)$. This can be shown directly by using the definition of covariance and the linearity of the expectation.

$$\begin{aligned}
Cov(X+Y,Z) &= \mathbb{E}\left[(X+Y-(\mathbb{E}(X+Y)))\cdot(Z-\mathbb{E}(Z))\right] \\
&= \mathbb{E}\left[((X-\mathbb{E}(\mathbb{X}))+(Y-\mathbb{E}(Y)))\cdot(Z-\mathbb{E}(Z))\right] \\
&= \mathbb{E}\left[(X-\mathbb{E}(\mathbb{X}))\cdot(Z-\mathbb{E}(Z))+(Y-\mathbb{E}(Y))\cdot(Z-\mathbb{E}(Z))\right] \\
&= \mathbb{E}\left[(X-\mathbb{E}(\mathbb{X}))\cdot(Z-\mathbb{E}(Z))\right]+\mathbb{E}\left[(Y-\mathbb{E}(\mathbb{Y}))\cdot(Z-\mathbb{E}(Z))\right] \\
&= Cov(X,Z)+Cov(Y,Z).
\end{aligned}$$

This can be shown for sums of more than two random variables by using induction. However by just using the above and that $Cov(X, Y) = Cov(Y, X)$ (this follows from the definition of covariance) one can see that if X, Y, V, Z are random variables then

$$Cov(X+Y, V+Z) = Cov(X,V) + Cov(X,Z) + Cov(Y,V) + Cov(Y,Z).$$

Now if X and Y are two random variables then the following holds

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y).$$

This can be shown by using the above fact and that $Cov(X, X) = Var(X)$ (this follows from the definition of covariance)

$$\begin{aligned}
V(X+Y) = Cov(X+Y,X+Y) &= Cov(X,X)+Cov(X,Y)+Cov(Y,X)+Cov(Y,Y) \\
&= Var(X)+Var(Y)+2Cov(X,Y).
\end{aligned}$$

Now using these two properties and letting $i = 1, 2$ we get

$$\begin{aligned}
Var\left(\sum_{i=1}^{2} X_i\right) &= Var(X_1+X_2) = Var(X_1)+Var(X_2)+2Cov(X,Y) \\
&= Cov(X_1,X_1)+Cov(X_2,X_2)+Cov(X_1,X_2)+Cov(X_2,X_1) \\
&= \sum_{i,j} Cov(X_i,X_j) \quad i,j = 1,2.
\end{aligned}$$

By using induction to show the property for $Cov(X+Y, V+Z)$ for larger sums and then using induction on i one completes the proof of Lemma 2.2.2. This will however be left to the reader.

□

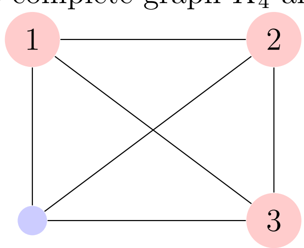
Chapter 3

Containment of small subgraphs

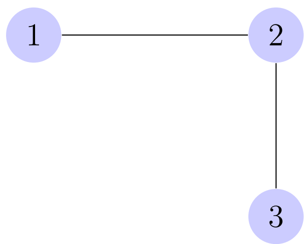
In this chapter the main theorem of the thesis will be presented. Given a graph G with at least one edge the theorem states a threshold where if $p(n)$ is above this threshold the probability $\mathbb{P}(\mathbb{G}(n, p) \supset G)$ will converge to 1 when $n \to \infty$. Then by definition it holds that asymptotically almost surely there is a copy of G in $\mathbb{G}(n, p(n))$. Also all necessary work to prove this theorem will enter in this chapter. We will work solely with binomial model in this chapter but in chapter 4 we will show by asymptotic equivalence that this theorem has an analogue version for the uniform model. Finally since G is a fixed (but arbitrary) random graph (with at least one edge) and the number of vertices for the random graph $\mathbb{G}(n, p(n))$ grows as $n \to \infty$, we call G a small subgraph.

3.1 First threshold

Two graphs G_1 and G_2 are isomorphic if there exists an bijection f between both sets of vertices $V(G_1)$,$V(G_2)$ such that $\{x, y\}$ is an edge of G_1 iff $\{f(x), f(y)\}$ is an edge of G_2. A mapping $\sigma : V(G) \longrightarrow V(G)$ that satisfies above is called an automorphism. Consider the complete graph K_4 and the subgraph G

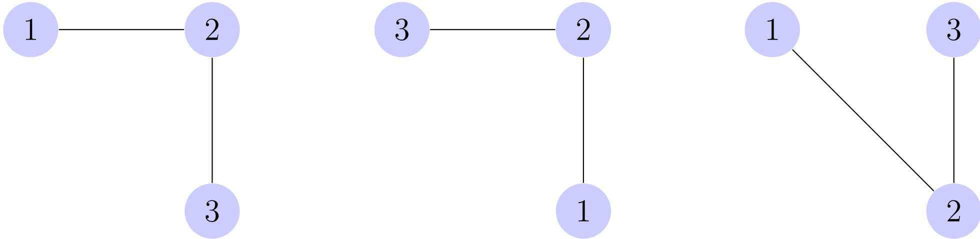


K_4



G

We are interested in counting the number of copies of G in K_n. We can choose 3 vertices $4 \cdot 3 \cdot 2$ number of ways, but we need to divide with the number of permutations on the selected vertex set which we do not wish to count. Imagine for simplicity that these vertices were choosen in this particular order 1,2,3. We could permute the vertices in the order 3,2,1, but that uses the same edges from K_3 (see figure below) as before and the induced bijective mapping satisfies the isomorphic requirement. However if we permute them in the order 1,3,2 we use different edges from K_3 and the induced bijective mapping does not satisfy the isomorphic requirement. The number of possible ways to permute this vertice set so it uses the same edges from K_3 is the same number of bijections $\sigma : V(G) \longrightarrow V(G)$ which satisfies the above definition of isomorphic graphs. The



number is denoted $|Aut(G)|$ which is the size of the automorphism group of G (i.e the number of isomorphisms from G to G). In the above example $|Aut(G)| = 2$, hence we get that the number of copies of G in K_4 is $\frac{4!}{2} = 12$.
In general, if v_G denotes the number of vertices in G we have that the number of copies of G in K_n is

$$\frac{n!}{(n-v_G)!|Aut(G)|} = \frac{\binom{n}{v_G} v_G!}{|Aut(G)|} := f(n, G).$$

Now after having obtained the number $f(n, G)$ we proceed to our first useful threshold.

Let the random variable X_G be the number of copies of G in the binomial random graph $\mathbb{G}(n, p)$. For each copy G' of G in K_n we define the indicator random variable $I_{G'} = \mathbf{1}[\mathbb{G}(n, p) \supseteq G']$. This random variable is 1 with probability $\mathbb{P}(\mathbb{G}(n, p) \supseteq G')$ and 0 otherwise. We have that for $f(n, G)$

$$f(n, G) = \frac{n!}{(n-v_G)!|Aut(G)|} = \frac{n(n-1)...(n-v_G+1)}{|Aut(G)|} \asymp n^{v_G}.$$

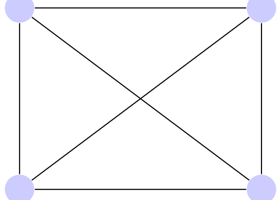
With this and by the linearity of the expectation we get

$$\mathbb{E}(X_G) = \sum_{G'} \mathbb{E}(\mathbf{1}[\mathbb{G}(n, p) \supseteq G']) = f(n, G)p^{e_G} = \Theta(n^{v_G} p^{e_G}) \to \begin{cases} 0 & \text{if } p \ll n^{-v_G/e_G} \\ \infty & \text{if } p \gg n^{-v_G/e_G} \end{cases} \tag{3.1}$$

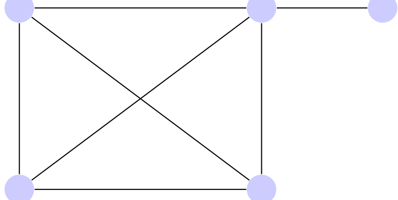
and by the first moment method (i.e. using (2.2))

$$\mathbb{P}(X_G > 0) \leq \mathbb{E}(X_G) = o(1) \quad \text{if } p \ll n^{-v_G/e_G}. \tag{3.2}$$

The result is that if $p \ll n^{-v_G/e_G}$ then the event $\{\mathbb{G}(n, p) \supseteq G\}$ which describes the property of $\mathbb{G}(n, p)$ having G as a subgraph equals 0 a.s.s. Does this imply that $\mathbb{P}(X_G > 0) = 1 - o(1)$ if $p \gg n^{-v_G/e_G}$? We will by example now show that this is not true. Let H be the complete graph $H = K_4$ and let G be the graph by adding one vertex and connecting this vertex with one vertex from H. (See figure below).



H



G

Choose p such that p satisfies $n^{-5/7} \ll p \ll n^{-4/6}$, for example $p = n^{-29/41}$. Now note that $5/7$ is the ratio of the graph G's number of vertices and edges and $4/6$ is the same for H. By our previous result, $\mathbb{E}(X_G > 0) = \Theta(n^5 p^7) \to \infty$ but at the same time we have that, $\mathbb{E}(X_H > 0) = \Theta(n^4p^6) \to 0$ and it follows that a.a.s no copy of H exists in $\mathbb{G}(n,p)$. Therefore no copy of G exists either a.a.s since H is a subgraph of G. The reason is that G contains a subgraph which is more dense than G which makes the expectation a bit tricky regarding G. By more dense we mean that there exists a subgraph for which the ratio e_H/v_H is larger than e_G/v_G. This problem motivates us to consider the densest subgraph in G and using that ratio to find a better threshold.

3.2 Main result

Bollobás solved this threshold problem in full generality with the following theorem in which this thesis revolves around. We first define the number $m(G)$ which were hinted at before

$$m(G) := max\left\{\frac{e_H}{v_H} : H \subseteq G, v_H > 0\right\}. \tag{3.3}$$

Theorem 3.2.1 For an arbitrary graph G with at least one edge,

$$\lim_{n\to\infty} \mathbb{P}(\mathbb{G}(n,p) \supset G) = \begin{cases} 0 & \text{if } p \ll n^{-1/m(G)} \\ 1 & \text{if } p \gg n^{-1/m(G)} \end{cases}$$

Proof. The proof consists of two parts, proving the 0-statement and the 1-statement. The first one uses the first threshold developed in the section 3.1 by means of the first moment method.
Proof of $0-statement$. Assume that $p \ll n^{-1/m(G)}$ and let Q be the densest subgraph of G i.e. $\frac{E(Q)}{V(Q)} = m(G)$. Then by (3.2), we have that there is no copy a.a.s. of Q in $\mathbb{G}(n,p)$ and therefore no copy of G.

□

For the proof of the $1-statement$ we want to use the second moment method (i.e 2.3) and need to find a bound from above for $Var(X_G)$. For that we are going to need a new quantity and two lemmas. Begin by defining the following quantity:

$$\Phi_G = \Phi_G(n,p) = min\left\{\mathbb{E}(X_H) : H \subseteq G, e_H > 0\right\}. \tag{3.4}$$

In (3.1) we had that for sufficiently large n that

$$\Phi_G \asymp \min_{H\subseteq G, e_H>0} n^{v_H}p^{e_H}. \tag{3.5}$$

Lemma 3.2.2 Let G be a graph with at least one edge. Then

$$\begin{aligned} Var(X_G) \asymp (1-p)\sum_{H\subseteq G, e_H>0} n^{2v_G-v_H}p^{2e_G-e_H} &\asymp (1-p)\max_{H\subseteq G, e_H>0}\frac{(\mathbb{E}(X_G))^2}{\mathbb{E}(X_H)} \\ &= (1-p)\frac{(\mathbb{E}(X_G))^2}{\Phi_G} \end{aligned} \tag{3.6}$$

where the constants used in the relation $\asymp$ depends on G but not on p or n.

Proof. Define as before $I_{G'}, I_{G''}$ to be two indicator random variables. If G' and G'' do not share any edges, i.e. $E(G') \cap E(G'') = \emptyset$, they are independent. Let v_G and e_G

denote the number of vertices and edges of the graph G and of course $v_{G'} = v_{G''} = v_G$ and $e_{G'} = e_{G''} = e_G$ since they are copies of G. For each subgraph $H \subseteq G$ we wish to count the number of pairs (G', G'') of copies of G in the complete graph K_n with the property $G' \cap G''$ being isomorphic to H. First we choose H, then G' and then G''. For sufficiently large n we have that

$$\binom{n}{v_H} = \frac{n!}{v_H!(n-v_H)!} = \frac{n(n-1)...(n-v_H+1)}{v_H!} \asymp nn...n = n^{v_H}.$$

Then we choose the rest of G' from the remaining $n - v_H$ vertices. For sufficiently large n we get

$$\binom{n-v_H}{v_{G'}-v_H} = \frac{(n-v_H)!}{(v_G-v_H)!(n-v_G)!} = \frac{(n-v_H)(n-v_H-1)...(n-v_H-(v_G-v_H-1))}{(v_G-v_H)!} \asymp n^{v_G-v_H}.$$

Now choose G'' from the remaining $n-(v_{G'}-v_H)-v_H = n-v_G$ vertices. For sufficiently large n we have that

$$\begin{aligned}\binom{n-v_G}{v_{G''}-v_H} = \binom{n-v_G}{v_G-v_H} &= \frac{(n-v_G)!}{(n-2v_G+v_H)!(v_G-v_H)!} \\ &= \frac{(n-v_G)(n-v_G-1)...(n-v_G-(v_G-v_H-1))}{(v_G-v_H)!} \\ &\asymp n^{v_G-v_H}.\end{aligned}$$

Hence the number of pairs of copies (G', G'') with this property is $\Theta(n^{v_H}n^{2(v_G-v_H)}) = \Theta(n^{2v_G-v_H})$. Now using Lemma 2.2.2 which says that the variance of a sum of random variables can be written using Cov, we then get

$$Var(X_G) = \sum_{G',G''} Cov(I_{G'}, I_{G''}) = \sum_{E(G')\cap E(G'')\neq\emptyset} [\mathbb{E}(I_{G'}I_{G''}) - \mathbb{E}(I_{G'})\mathbb{E}(I_{G''})].$$

The double sum turns into one sum over the pairs (G', G'') which satisfies that the intersection of the two sets of edges $E(G'), E(G'')$ is nonempty. Because if the intersection is empty the covariance is zero. Now, $\mathbb{E}(I_{G'}) = \mathbb{E}(I_{G''}) = p^{e_G}$ and since $I_{G'}$ and $I_{G''}$ are two indicator random variables, only when both are 1 the corresponding term contributes to the expectation of the product. Hence $\mathbb{E}(I_{G'}I_{G''}) = p^{e_H}p^{e_G-e_H}p^{e_G-e_H} = p^{2e_G-e_H}$. Using $\Theta(n^{2v_G-v_H})$, the above and iterating over all possible H´s,

$$\begin{aligned}\sum_{E(G')\cap E(G'')\neq\emptyset} [\mathbb{E}(I_{G'}I_{G''}) - \mathbb{E}(I_{G'})\mathbb{E}(I_{G''})] &\asymp \sum_{H\subseteq G, e_H>0} n^{2v_G-v_H}(p^{2e_G-e_H} - p^{2e_G}) \\ = \sum_{H\subseteq G, e_H>0} n^{2v_G-v_H}p^{2e_G-e_H}(1-p^{e_H}) &\asymp \sum_{H\subseteq G, e_H>0} n^{2v_G-v_H}p^{2e_G-e_H}(1-p)\end{aligned}$$

We have from (3.1) that $\mathbb{E}(X_G) \asymp n^{v_G}p^{e_G}$ and $\mathbb{E}(X_H) \asymp n^{v_H}p^{e_H}$, so

$$\begin{aligned}\sum_{H\subseteq G, e_H>0} n^{2v_G-v_H}p^{2e_G-e_H}(1-p) &= (1-p)\sum_{H\subseteq G, e_H>0} n^{2v_G}p^{2e_G}n^{-v_H}p^{-e_H} \\ &\asymp (1-p)\sum_{H\subseteq G, e_H>0} \frac{\mathbb{E}((X_G))^2}{\mathbb{E}(X_H)}.\end{aligned}$$

Last we need to motivate the final step in the proof regarding the implicit constants in

$$(1-p)\sum_{H\subseteq G, e_H>0}\frac{\mathbb{E}((X_G))^2}{\mathbb{E}(X_H)} \asymp (1-p)\max_{H\subseteq G, e_H>0}\frac{\mathbb{E}((X_G))^2}{\mathbb{E}(X_H)} = (1-p)\frac{\mathbb{E}((X_G))^2}{\Phi_G}$$

At least $Var(X_G) = O(\frac{(\mathbb{E}(X_G))^2}{\Phi_G})$ since $|1-p| \leq 1$ and for example take the constant $C(G) = 2^{e_G - 1}$. C bounds the number of terms of the sum and we multiply the largest number in the sum with this constant. Thus

$$|(1-p)\sum_{H\subseteq G, e_H>0}\frac{(\mathbb{E}(X_G))^2}{\mathbb{E}(X_H)}| \leq C\frac{(\mathbb{E}(X_G))^2}{\Phi_G}$$

This is true for $n \geq n_0 = 2$. Now if $p = p(n)$ is bounded away from 1. Depending on the graph G, one may choose a constant $0 \leq p < d < 1$ such that d is sufficiently close to 1 to make the following hold

$$(1-d)\cdot C\frac{(\mathbb{E}(X_G))^2}{\Phi_G} \leq |(1-p)\sum_{H\subseteq G, e_H>0}\frac{(\mathbb{E}(X_G))^2}{\mathbb{E}(X_H)}|$$

Take c to be $c = (1-d)C$ and we have the relation $\asymp$ and the proof is complete. □

Lemma 3.2.3 The following statements are equivalent, for any graph G with $e_G > 0$.

(i) $np^{m(G)} \to \infty$.

(ii) $n^{v_H}p^{e_H} \to \infty$ for every $H \subseteq G$ with $v_H > 0$.

(iii) $\mathbb{E}(X_H) \to \infty$ for every $H \subseteq G$ with $v_H > 0$.

(iv) $\Phi_G \to \infty$.

Proof. $(i) \Leftarrow (ii)$. If $n^{v_H}p^{e_H} \to \infty$ for every $H \subseteq G$ then this is especially true for the densest H in G. $(i) \Rightarrow (ii)$. Assume $np^{m(G)} \to \infty$. For $0 \leq p < 1$ it follows that $n^{v_H}p^{e_H} = (np^{e_H/v_H})^{v_H} \geq (np^{m(G)})^{v_H} \to \infty$. For $p = 1$ it is trivial.
$(ii) \Leftrightarrow (iii)$. Since $\mathbb{E}(X_H) \asymp n^{v_H}p^{e_H}$ it is clear.
$(iv) \Rightarrow (iii)$. Assume $\Phi_G \to \infty$. By definition of $\Phi_G = min\{\mathbb{E}(X_H) : H \subseteq G, e_H > 0\}$ it is clear that any larger expectation of must as well $\to \infty$ or it will become smaller than Φ_G, contradicting the assumption. The case when $v_H > 0$ and $e_H = 0$ is trivial.
$(iv) \Leftarrow (iii)$ follows immediatly since Φ_G is a special case. This completes the proof of the lemma. □

To tie everything together and complete the proof of the Theorem 3.2.1 we observe that if $p \gg n^{-1/m(G)}$ then by definition we have that for all $\epsilon > 0$ there is $N(\epsilon)$ such that $|\frac{1}{n^{1/m(G)}}| \leq p\epsilon$ for all $n \geq N$. Thus $|\frac{1}{n^{1/m(G)}}| \leq p\epsilon \Leftrightarrow np^{m(G)} \geq \frac{1}{\epsilon^{m(G)}} \to \infty$ when $\epsilon \to 0$ $\Rightarrow np^{m(G)} \to \infty$. Then by Lemma 3.2.3 this is equivalent to $\Phi_G \to \infty$. Using this and Lemma 3.2.2 the second moment method yields

$$\mathbb{P}(\mathbb{G}(n,p) \not\supset G) = \mathbb{P}(X_G = 0) \leq \frac{Var(X_G}{(\mathbb{E}(X_G))^2} \asymp (1-p)\frac{\mathbb{E}((X_G))^2}{\Phi_G}\frac{1}{(\mathbb{E}(X_G))^2} = O(1/\Phi_G) = o(1).$$

This finishes the proof of the $1 - statement$ and completes the proof of Theorem 3.2.1

□

As it stands the theorem is only for the binomial model, however the asymptotic equivalence between the two models will give us a analogue theorem with different thresholds for the uniform model in the next chapter. Before we will state two corollaries that follow directly from Theorem 3.2.1.

First note that the threshold for the random graph $\mathbb{G}(n,p)$ to a.a.s. contain a triangle is $1/n$.

Corollary 3.2.4 Let $k \geq 3$. The threshold for $\mathbb{G}(n,p)$ to asymptotically almost surely contain a k-cycle is $1/n$.

Proof. $m(G)$ for a $k - cycle$ is always 1, so the corollary follows then from Theorem 3.2.1.

□

The interesting part of this corollary is that regardless of the cycle length, all cycles of fixed length appear somewhat simultaneously in the evolution of the random graph $\mathbb{G}(n,p)$. That is to say when p hits above this threshold a 'typical' random graph from Ω have this property.

Corollary 3.2.5 Let $k \geq 2$. The threshold for $\mathbb{G}(n,p)$ to asymptotically almost surely contain the complete graph K_k is $n^{-2/(k-1)}$.

Proof. We can not make the quotient e_H/v_H any larger than when $H = G$. Hence $m(G) = \frac{\binom{k}{2}}{k} = \frac{k(k-1)/2}{k} = (k-1)/2$.

□

Chapter 4

Asymptotic equivalence

Before we establish when the convergence of $\mathbb{P}(\mathbb{G}(n,p) \supset G)$ implies the convergence of $\mathbb{P}(\mathbb{G}(n,M) \supset G)$. We introduce the notion of random subsets which inludes the random graphs as a special case. Then we introduce general definitions regarding sets and work under this more general framework to show certain results which as well hold for our random graphs.

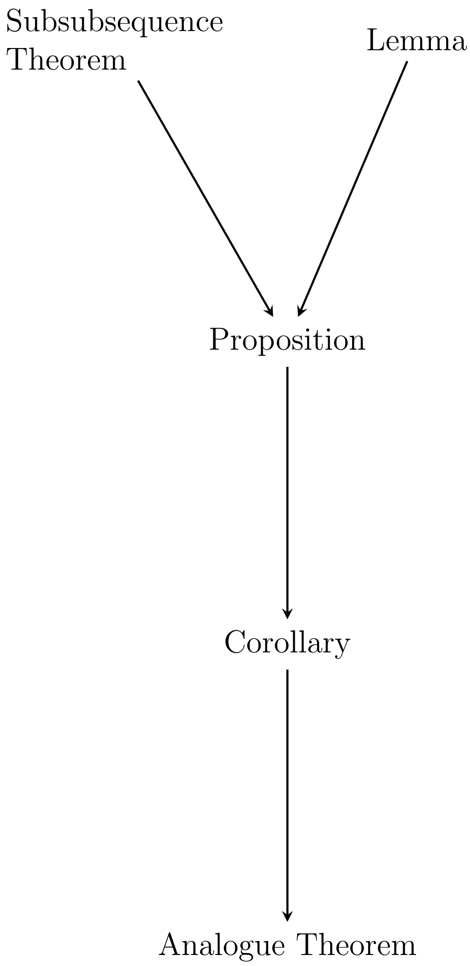
4.1 Random subsets

Let X be an arbitrarly set and k a integer. We let $[X]^k$ to be the family of all possible k-element subsets of X. In particular, $[n]^k$ denotes the set of alla k-element subsets of $[n] = \{1, ..., n\}$. Let Γ be a finite set, $|\Gamma| = N$, let $0 \leq p \leq 1$ and $0 \leq M \leq N$. We define the random subset Γ_p of Γ to be the result of N coin flips, one for each element in Γ with probability p to include it and $1-p$ not to include it. The distribution of Γ_p is given by the probability distribution on 2^Ω where $\mathbb{P}(F) = p^{|F|}(1-p)^{|\Gamma|-|F|}$ for $F \subseteq \Gamma$. Likewise let Γ_M be a randomly chosen element from $[\Gamma]^M$ with probability $1/\binom{N}{M}$. That is Γ_M has the uniform distribution with $\mathbb{P}(F) = 1/\binom{N}{M}$ for every $F \in [\Gamma]^M$. If one chooses $\Gamma = [n]^2$ then the random subset Γ_p contains elements which represents edges in the binomial random graph $\mathbb{G}(n,p)$. Likewise for the uniform model since Γ_M can be viewed as a graph with exactly M edges.

To connect random subsets to graph properties we begin with the powerset 2^Γ. This set contains all possible subsets of Γ and in the case where $\Gamma = [n]^2$ it is the set of all possible graphs on the vertex set $[n]$. Then any family of subsets $Q \subseteq 2^{[n]^2}$ will be a family of graphs. If this family is closed under isomorphism we can identify it as a graph property. That $Q \subseteq 2^{[n]^2}$ is closed under isomorphism means that if $G \in Q$, $H \in 2^{[n]^2}$ and G and H are isomorphic, then $H \in Q$. We will work under the more general framework letting Γ be any finite set to show results regarding our random graphs. A family of subsets $Q \subseteq 2^\Gamma$ is called

- Increasing if $A \subseteq B$, $B \in 2^\Gamma$ and $A \in Q \Rightarrow B \in Q$
- Decreasing if $A \supseteq B$, $B \in 2^\Gamma$ and $A \in Q \Rightarrow B \in Q$
- Monotone if it is either increasing or decreasing
- Convex if $A \subseteq B \subseteq C$ and $A, C \in Q \Rightarrow B \in Q$

One example of an increasing graph property Q is if Q contains all graphs $G \in 2^{[n]^2}$ which contain a triangle. Since if a graph H is a subgraph of G and H contains a triangle, any larger graph G would as well contain a triangle. We will now present a lemma and the subsubsequence principle for sequences of real numbers. These two will be proven and used for the proof of a proposition presented after them. This proposition gives us a corollary which immediatly gives us an analogue theorem for the uniform model $\mathbb{G}(n, M)$ of Theorem 3.2.1. This flowchart will help the readers to orient themselves.



Lemma 4.1.1 Let Q be a convex property of subsets of Γ, and let M_1, M, M_2 be three integer functions of N satisfying $0 \leq M_1 \leq M \leq M_2 \leq N$. Then

$$\mathbb{P}(\Gamma_M \in Q) \geq \mathbb{P}(\Gamma_{M_1} \in Q) + \mathbb{P}(\Gamma_{M_2} \in Q) - 1.$$

Worth mentioning is that if $\mathbb{P}(\Gamma_{M_i} \in Q) \to 1$ as $N \to \infty$, for $i = 1, 2$, then $\mathbb{P}(\Gamma_M \in Q) \to 1$. This will be used later.

Proof. First we will show that Q is convex if and only if Q is the intersection of an increasing property Q_1 and a decreasing property Q_2. First the implication $\Leftarrow$ can be seen easily. Let $Q_1 \cap Q_2$ be as above and take $B \in 2^\Gamma$ such that $A \subseteq B \subseteq C$ where $A, C \in Q_1 \cap Q_2$. This implies that $B \in Q_1$ by increasing property of Q_1 and $B \in Q_2$ by decreasing property of Q_2. Thus $B \in Q_1 \cap Q_2$ and hence $Q_1 \cap Q_2$ is convex. The implication $\Rightarrow$ requires a little more effort. Choose Q_1 to consist of all $A \in 2^\Gamma$ such that

A includes some $B \in Q$. Let Q_2 be the set of all $A \in 2^\Gamma$ such that A is included in some $B \in Q$. We need to show that Q_1 is increasing and Q_2 is decreasing and that Q is the intersection of Q_1 and Q_2. Let $B' \in 2^\Gamma$ be such that $A' \subseteq B'$ for some $A' \in Q_1$. We want to show that B' includes some $B \in Q$ implying that $B' \in Q_1$ and thus concluding that Q_1 is an increasing property. This is easy since by definition of A', there is some B'' such that $B'' \subseteq A'$ and $B'' \in Q$. At the same time we have $B'' \subseteq A' \subseteq B'$ implying that $B' \in Q_1$.
To show that Q_2 is a decreasing property take $B' \in 2^\Gamma$ such that $B' \subseteq A'$ where $A' \in Q_2$. We want to show that B' is included in some $B'' \in Q$ and by definition having that $B' \in Q_2$. We have by definition of Q_2 that, there is some B'' such that $A' \subseteq B''$ and $B'' \in Q$. Thus we have that $B' \subseteq A' \subseteq B''$ which implies that $B' \in Q_2$. Now it remains to show that Q really is the intersection of Q_1 and Q_2. To see this take an element $B \in Q_1 \cap Q_2$. Now since B is in both Q_1 and Q_2 we have that $A \subseteq B \subseteq C$ for some $A, C \in Q$. By the convexity of Q this implies that $B \in Q$. Finally the reverse inclusion is trivially true by definitions of Q_1 and Q_2 thus completing the proof of the statement.

Now let $A = \{\Gamma_M \in Q_1\}$ and $B = \{\Gamma_M \in Q_2\}$ and $Q = Q_1 \cap Q_2$. We then have

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1.$$

Writing it out explicitly

$$\mathbb{P}(\Gamma_M \in Q) = \mathbb{P}(\Gamma_M \in Q_1 \cap Q_2) \geq \mathbb{P}(\Gamma_M \in Q_1) + \mathbb{P}(\Gamma_M \in Q_2) - 1 \tag{4.1}$$

Now consider a random subset process $\{\Gamma_M\}_M$ which starts with no elements and adds new elements, one by one; each new element is picked at random, uniformly among all elements not yet chosen. The time (M) goes through the discrete set $\{0, 1, ..., N\}$. Take for example the subset during $M = 3$, what is the probability for a certain subset of size three to be choosen? Well, we do not care in what order the individual elements were picked, since for example $\{4, 2, 9\} = \{2, 4, 9\}$. The probability becomes

$$\frac{1}{N}\frac{1}{N-1}\frac{1}{N-2}3! = \frac{1}{\binom{N}{3}}$$

and we clearly see that the random subset Γ_M can be identified with the random process during time M. Now consider the random process at a particular time M_1 (i.e when $|\Gamma_{M_1}| = M_1$), and view Γ_M as the set to which $M - M_1$ elements are added as previously described to the set Γ_{M_1}. Then the following inclusion is motivated: $\Gamma_{M_1} \subseteq \Gamma_M$. Now assume that $\Gamma_{M_1} \in Q$, since Q_1 is increasing, we have that adding any new elements does not change the fact that it will belong to Q_1. Therefore the probability $\mathbb{P}(\Gamma_{M_1} \in Q)$, can only increase or stay the same. Hence we have the following inequality: $\mathbb{P}(\Gamma_{M_1} \in Q_1) \leq \mathbb{P}(\Gamma_M \in Q_1)$. Likewise, consider a similar random process where we start with all elements possible and remove elements, one by one; each element to be removed is picked at random, uniformly among all elements not yet removed. One can see again that Γ_M can be viewed as the random process during the time M. Now if we consider this random process at the time M_2, (i.e when $|\Gamma_{M_2}| = M_2$), we can then view Γ_M as the set where we remove $M_2 - M$ elements from Γ_{M_2}. The following inclusion is then motivated: $\Gamma_M \subseteq \Gamma_{M_2}$. Now assume that $\Gamma_{M_2} \in Q_2$, since Q_2 is decreasing we have that removing any edges not yet removed, does not change the fact that the set will belong to Q_2.

Therefore we have the following inequality of probabilities: $\mathbb{P}(\Gamma_M \in Q_2) \geq \mathbb{P}(\Gamma_{M_2} \in Q_2)$. Continuing on equation (4.1) with our two new inequalities we then get

$$\mathbb{P}(\Gamma_M \in Q_1) + \mathbb{P}(\Gamma_M \in Q_2) - 1 \geq \mathbb{P}(\Gamma_{M_1} \in Q_1) + \mathbb{P}(\Gamma_{M_2} \in Q_2) - 1.$$

Now since Q is the intersection of Q_1 and Q_2 we get

$$\mathbb{P}(\Gamma_{M_1} \in Q_1) + \mathbb{P}(\Gamma_{M_2} \in Q_2) - 1 \geq \mathbb{P}(\Gamma_{M_1} \in Q) + \mathbb{P}(\Gamma_{M_2} \in Q) - 1.$$

Thus giving us

$$\mathbb{P}(\Gamma_M \in Q) \geq \mathbb{P}(\Gamma_{M_1} \in Q) + \mathbb{P}(\Gamma_{M_2} \in Q) - 1$$

which completes the proof of Lemma 4.1.1. □

4.2 Subsubsequence principle

The subsubsequence theorem is valid in different settings than just the reals. However the setting of sequences of real numbers is sufficient for us. The sequences in question are of the form $\mathbb{P}(\mathbb{G}(n,p) \supset G)$ or more generally $\mathbb{P}(\Gamma_p \in Q)$, where Q is the family of all graphs containing G. Note that $Q = Q(n)$ since the family of graphs which contains G changes with n. This will explained more in detail after the theorem.

Theorem 4.2.1 Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and let $x \in \mathbb{R}$ be a fixed point. If for every subsequence of $(x_n)_{n\in\mathbb{N}}$ there exists a subsubsequence that converges to x, then the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x.

Proof. Let $p = \limsup x_n$. Then $p \in E$ where E is the set of all numbers $r \in \mathbb{R} \cup \{-\infty, +\infty\}$ such that $(x_{n_k})_{n_k\in\mathbb{N}} \to r$ for some subsequence $(x_{n_k})_{n_k}$ and p is then $p = supE$. This implies that there exists a subsequence call it (y_{n_i}) s.t. $(y_{n_i}) \to p$. Now by hypothesis there exists a subsequence of (y_{n_i}), call it $(y_{n_{i_k}})$, which converges to x. Now a sequence converges to p if and only if every subsequence of it converges to p. This implies that $\limsup x_n = p = x$. Now we only need to show that $\liminf x_n = q = x$ and the proof is complete. That part is completely analogous to the first one. Now we have that $\liminf x_n = q = x = p = \limsup x_n$ which implies that $(x_n)_{n\in\mathbb{N}} \to x$. □

Before the actual work towards an analogue theorem we remind ourselves about the Central limit theorem which will be needed later on.

Central limit theorem Let $X_1, X_2, ...$ be independent and similarly distributed random variables with $\mathbb{E}(X_i) = \mu$ and standard deviation $D(X_i) = \sigma$, where $0 < \sigma < \infty$, and let $\bar{X}_n := \sum_{i=1}^{n} X_i/n$. For arbitrary $a < b$ it then holds that

$$\mathbb{P}(a < \frac{\sqrt{n}}{\sigma}(\bar{X}_n - \mu) < b) \to \Phi(b) - \Phi(a), \quad when\ n \to \infty,$$

where Φ is the distribution function of the normal distribution $N(0,1)$.

4.3 Analogue theorem of Theorem 3.2.1

To present and prove the following proposition we need to be more specific setting with things up. Let $\Gamma(n)$ be a sequence of sets of size $N(n) = |\Gamma(n)| \to \infty$. (Our interest is when $\Gamma = [n]^2$ and the size then becomes $|\Gamma| = \binom{n}{2}$.) Let $Q(n) \subseteq 2^{\Gamma(n)}$ be a sequence of families of subsets of $\Gamma(n)$, $n = 1, 2, ...$ Let $p(n)$, $M(n)$ be two given sequences, one consisting of real numbers with $0 \leq p(n) \leq 1$. The other being a sequence of integers $M(n)$ with $0 \leq M(n) \leq N(n)$. To make things easier to read we omit the argument n and write Γ, N, Q, p and M. Finally, $q = 1 - p$.

Proposition 4.3.1 Let $Q = Q(n)$ be a sequence of families of subsets of $\Gamma = \Gamma(n)$ which are convex and let $0 \leq M \leq N$. If $\mathbb{P}(\Gamma_{M/N} \in Q) \to 1$ as $n \to \infty$ then $\mathbb{P}(\Gamma_M \in Q) \to 1$.

Proof. It suffices to consider the following cases of the expression $\frac{M(N-M)}{N}$: $\frac{M(N-M)}{N} \to \infty$ as $n \to \infty$, $M = O(1)$ as $n \to \infty$ and $N - M = O(1)$ as $n \to \infty$. At the end of the proof we explain why this is so, and for this we use Theorem 4.2.1.

Case 1 : $M(N - M)/N \to \infty$
Let M_1 and M_2 maximize $\mathbb{P}(\Gamma_{M'} \in Q)$ for $M' \leq M$ and $M' \geq M$. By the law of total probability,

$$\mathbb{P}(\Gamma_{M/N} \in Q) = \sum_{k=0}^{N} \mathbb{P}(\Gamma_{M/N} \in Q \mid |\Gamma_{M/N}| = k)\mathbb{P}(|\Gamma_{M/N}| = k).$$

Now if one conditions on the number of elements in $\Gamma_{M/N}$ the binomial probability measure becomes the uniformal probability measure. That is $\mathbb{P}(\Gamma_{M/N} \in Q \mid |\Gamma_{M/N}| = M) = \mathbb{P}(\Gamma_M \in Q)$ and we get

$$\sum_{k=0}^{N} \mathbb{P}(\Gamma_{M/N} \in Q \mid |\Gamma_{M/N}| = k)\mathbb{P}(|\Gamma_{M/N}| = k)$$
$$= \sum_{k=0}^{N} \mathbb{P}(\Gamma_k \in Q)\mathbb{P}(|\Gamma_{M/N}| = k) \leq \mathbb{P}(\Gamma_{M_1} \in Q)\mathbb{P}(|\Gamma_{M/N}| \leq M) + \mathbb{P}(|\Gamma_{M/N}| > M).$$

Now one can view the subset $\Gamma_{M/N}$ as the sum $X_1 + ... + X_N$ of independent, similar distributed random variables with $X_i \sim Ber(M/N)$ which have finite expectation and variance. Since the expected value $\mathbb{E}(\Gamma_{M/N}) = N\frac{M}{N} = M$, we get by the central limit theorem that $\mathbb{P}(|\Gamma_{M/N}| \leq M) \to 1/2$. Hence as well $\mathbb{P}(|\Gamma_{M/N}| > M) \to 1/2$. It then follows that

$$1 = \lim_{n\to\infty} \mathbb{P}(\Gamma_{M/N} \in Q) \leq \frac{1}{2} \liminf_{n\to\infty} \mathbb{P}(\Gamma_{M_1} \in Q) + \frac{1}{2} \Rightarrow \lim_{n\to\infty} \mathbb{P}(\Gamma_{M_1} \in Q) = 1.$$

Similarly

$$\mathbb{P}(\Gamma_{M/N} \in Q) \leq \mathbb{P}(|\Gamma_{M/N}| \leq M) + \mathbb{P}(\Gamma_{M_2} \in Q)\mathbb{P}(|\Gamma_{M/N}| > M).$$

By the same argument we get

$$1 = \lim_{n\to\infty} \mathbb{P}(\Gamma_{M/N} \in Q) \leq \frac{1}{2} + \liminf_{n\to\infty} \mathbb{P}(\Gamma_{M_2} \in Q)\frac{1}{2} \Rightarrow \lim_{n\to\infty} \mathbb{P}(\Gamma_{M_2} \in Q) = 1.$$

We now have by Lemma 4.1.1

$$\mathbb{P}(\Gamma_M \in Q) \geq \mathbb{P}(\Gamma_{M_1} \in Q) + \mathbb{P}(\Gamma_{M_2} \in Q) - 1.$$

Since $\lim_{n\to\infty} \mathbb{P}(\Gamma_{M_1} \in Q) = 1$ and $\lim_{n\to\infty} \mathbb{P}(\Gamma_{M_2} \in Q) = 1$ we get $\lim_{n\to\infty} \mathbb{P}(\Gamma_M \in Q) = 1$.

□

Case 2 : $M = O(1)$
We then have, for some constant C,

$$\lim_{n\to\infty} (1 - \frac{M}{N})^{N-M} \geq \lim_{n\to\infty} (1 - \frac{C}{N})^{N-M} = \lim_{n\to\infty} \left((1 - \frac{C}{N})^N (1 - \frac{C}{N})^{-M} \right).$$

The first limit is a known one, that is e^{-C}, and the second one converges to 1 as $n \to \infty$. By the product rule for limits we then get that

$$\lim_{n\to\infty} (1 - \frac{C}{N})^N \lim_{n\to\infty} (1 - \frac{C}{N})^{-M} \to e^{-C}.$$

Again by the law of total probability and $\binom{n}{k} \geq (\frac{n}{k})^k$ we get

$$\mathbb{P}(\Gamma_{M/N} \notin Q) = \sum_{k=0}^{N} \mathbb{P}(\Gamma_{M/N} \notin Q \mid |\Gamma_{M/N}| = k)\mathbb{P}(|\Gamma_{M/N}| = k)$$

$$= \sum_{k=0}^{N} \mathbb{P}(\Gamma_k \notin Q)\mathbb{P}(|\Gamma_{M/N}| = k) \geq \mathbb{P}(\Gamma_M \notin Q)\binom{N}{M} \left(\frac{M}{N}\right)^M (1 - \frac{M}{N})^{N-M}$$

$$\geq \mathbb{P}(\Gamma_M \notin Q)(1 - \frac{C}{N})^{N-M}.$$

Hence

$$0 = \lim_{n\to\infty} \mathbb{P}(\Gamma_{M/N} \notin Q) \geq \lim_{n\to\infty} \left((\mathbb{P}(\Gamma_M \notin Q)(1 - \frac{C}{N})^{N-M} \right)$$

$$= \lim_{n\to\infty} \mathbb{P}(\Gamma_M \notin Q) \cdot \lim_{n\to\infty} (1 - \frac{C}{N})^{N-M} \geq 0$$

Now we have by the Squeeze theorem from analysis that

$$\lim_{n\to\infty} \mathbb{P}(\Gamma_M \notin Q) \cdot \lim_{n\to\infty} (1 - \frac{C}{N})^{N-M} \to 0.$$

Furthermore since $\lim_{n\to\infty}(1 - \frac{C}{N})^{N-M} \to e^{-C}$, we must have that $\lim_{n\to\infty} \mathbb{P}(\Gamma_M \notin Q) \to 0$. Thus concluding that

$$\lim_{n\to\infty} \mathbb{P}(\Gamma_M \notin Q) \to 0 \Rightarrow \lim_{n\to\infty} \mathbb{P}(\Gamma_M \in Q) \to 1$$

which ends the proof of the second case.

□

Case 3 : $N - M = O(1)$
By using the law of total probability and the following facts $\binom{N}{M} = \binom{N}{N-M}$, $\binom{N}{M} \geq (\frac{N}{M})^M$, $M \geq N - C$ for some constant $C \in \mathbb{R}$ we get

$$\begin{aligned}
\mathbb{P}(\Gamma_{M/N} \notin Q) &= \sum_{k=0}^{N} \mathbb{P}(\Gamma_{M/N} \notin Q \mid |\Gamma_{M/N}| = k)\mathbb{P}(|\Gamma_{M/N}| = k) \\
&= \sum_{k=0}^{N} \mathbb{P}(\Gamma_k \notin Q)\mathbb{P}(|\Gamma_{M/N}| = k) \\
&\geq \mathbb{P}(\Gamma_M \notin Q)\binom{N}{M}\left(\frac{M}{N}\right)^M\left(1 - \frac{M}{N}\right)^{N-M} \\
&\geq \mathbb{P}(\Gamma_M \notin Q)\left(\frac{N}{N-M}\right)^{N-M}\left(\frac{M}{N}\right)^M\left(\frac{N-M}{N}\right)^{N-M} \\
&\geq \mathbb{P}(\Gamma_M \notin Q)\left(\frac{N-C}{N}\right)^M = \mathbb{P}(\Gamma_{K/N} \notin Q)\left(1 - \frac{C}{N}\right)^M \geq 0.
\end{aligned}$$

Since $\left(1 - \frac{C}{N}\right)^M \to 1$ as $n \to \infty$, we get by the same reasoning as in the second case

$$\begin{aligned}
0 = \lim_{n\to\infty} \mathbb{P}(\Gamma_{M/N} \notin Q) &\geq \lim_{n\to\infty}\left(\mathbb{P}(\Gamma_M \notin Q)(1 - \frac{C}{N})^M\right) \geq 0 \\
&\Rightarrow \lim_{n\to\infty} \mathbb{P}(\Gamma_M \notin Q) = 0 \\
&\Rightarrow \lim_{n\to\infty} \mathbb{P}(\Gamma_M \in Q) = 1.
\end{aligned}$$

Thus ending the proof for this case.

□

Now the proof Proposition 4.3.1 is complete if it suffices to consider these three cases. Theorem 4.2.1 (i.e. a special case of the subsubsequence principle) implies this. To show this let $\mathbb{P}(\varepsilon_n) = \mathbb{P}(\Gamma_M \in Q(n)) = x_n$. We divide all subsequences $(x_{n_k})_{n_k \in \mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ into two groups. Either we have

$$\limsup_{k\to\infty} \frac{M(n_k)(N(n_k) - M(n_k))}{N(n_k)} = \infty$$

or

$$\limsup_{k\to\infty} \frac{M(n_k)(N(n_k) - M(n_k))}{N(n_k)} \leq \alpha$$

In the first group there exists a subseqence of $(x_{n_k})_{n_k\in\mathbb{N}}$ such that $\frac{M(n_{k_q})(N(n_{k_q})-M(n_{k_q}))}{N(n_{k_q})} \to \infty$ as $q \to \infty$ and $\mathbb{P}(\varepsilon_{n_{k_q}}) \to 1$ for this sequence by proof of the first case. The second group occurs if either $M(n) = O(1)$ or $N(n) - M(n) = O(1)$. We have that for sufficiently large $n > C$, $\frac{M(n_k)(N(n_k)-M(n_k))}{N(n_k)} \leq \alpha$, since n_k are picked after this point $n > C$. So there exists a subsequence of $(x_{n_k})_{n_k\in\mathbb{N}}$ such that $\frac{M(n_{k_q})(N(n_{k_q})-M(n_{k_q}))}{N(n_{k_q})} \leq \alpha$ and $\mathbb{P}(\varepsilon_{n_{k_q}}) \to 1$ in this sequence by proof of case 2 and 3 above. Now since all subsequences of $(x_n)_{n\in\mathbb{N}}$ have a subsequence $(x_{n_{k_q}})_{n_{k_q}\in\mathbb{N}} = \mathbb{P}(\varepsilon_{n_{k_q}})$ such that $\mathbb{P}(\varepsilon_{n_{k_q}}) \to 1$ this implies by Theorem 4.2.1 that $(x_n)_{n\in\mathbb{N}} \to 1$. This completes the proof of Proposition 4.3.1.

□

Finally we arrive at the corollary that follows from Proposition 4.3.1

Corollary 4.3.2 Let $Q = Q(n)$ be a sequence of increasing properties of subsets Γ, and let $M = M(n) \to \infty$.

(i) If $\mathbb{P}(\Gamma_{M/N} \in Q) \to 1$, then $\mathbb{P}(\Gamma_M \in Q) \to 1$.

(ii) If $\mathbb{P}(\Gamma_{M/N} \in Q) \to 0$, then $\mathbb{P}(\Gamma_M \in Q) \to 0$.

Proof. It follows from proposition 4.3.1. To see this consider an increasing property Q. Assume $A \in Q$ and take $B, C \in 2^\Gamma$ such that $A \subseteq B \subseteq C$. $A \in Q$ implies that $B \in Q$ and this implies that $C \in Q$. Hence Q is as well convex, proving (i). If Q is increasing, then the family of complements in 2^Γ is a decreasing. A decreasing property is as well a convex property. To see this reverse the implications in the increasing case. The assumption in (ii) gives us

$$\mathbb{P}(\Gamma_{M/N} \in Q) \to 0 \Rightarrow \mathbb{P}(\Gamma_{M/N} \in \overline{Q}) \to 1.$$

Now since $\overline{Q}$ is a decreasing and convex property we can use Proposition 4.3.1 to show the following

$$\begin{aligned}\mathbb{P}(\Gamma_{M/N} \in \overline{Q}) \to 1 &\Rightarrow \mathbb{P}(\Gamma_M \in \overline{Q}) \to 1\\ &\Rightarrow \mathbb{P}(\Gamma_M \in Q) \to 0.\end{aligned}$$

Proving (ii) and completing the proof of this corollary.

□

Finally we can deduce the analogue theorem of Theorem 3.2.1. Since if a graph R contains a graph H then any larger graph that contains R will as well contain H. Hence the property of containing a subgraph is a increasing one. Let $\Gamma(n) = \binom{n}{2}$. By taking p to be $p = M/N = M/\binom{n}{2}$ in Theorem 3.2.1 we get

$$\mathbb{P}(G(n, \frac{M}{N}) \supset G) \to 0 \quad \text{if} \quad \frac{M}{\binom{n}{2}} \ll n^{-1/m(G)}$$

and Corollary 4.3.2 gives us

$$\mathbb{P}(G(n, M) \supset G) \to 0 \quad \text{if} \quad \frac{M}{\binom{n}{2}} \ll n^{-1/m(G)}$$

For sufficiently large n becomes

$$\mathbb{P}(G(n, M) \supset G) \to 0 \quad \text{if} \quad M \ll n^{2-1/m(G)}$$

Likewise for the upper result

$$\mathbb{P}(G(n, M) \supset G) \to 1 \quad \text{if} \quad M \gg n^{2-1/m(G)}$$

The result above is now formalized and stated as a theorem.

Theorem 4.3.3 For an arbitrary graph G with at least one edge,

$$\lim_{n\to\infty} \mathbb{P}(\mathbb{G}(n, M) \supset G) = \begin{cases} 0 & \text{if } M \ll n^{2-1/m(G)} \\ 1 & \text{if } M \gg n^{2-1/m(G)} \end{cases}$$

We have now presented and proven in full the two main theorems of this thesis.

Bibliography

[1] P.Erdös and A.Rényi, "On random graphs I", Publ. Math. Debrecen (1959), 290-297

[2] M.E.J. Newman et al, "Random graphs with arbitrary distributions and their applications", Physical Review E, Volume 64

[3] P.Erdös and A.Rényi, "On the evolution of random graphs", Publ. Math. Inst. Hung. Acad. Sci. (1960), Ser. A 5, 17-61

[4] S.Jansson et al, "Random graphs", John Wiley and Sons, Inc, ISBN 0-471-17541-2

[5] S.E Alm and T.Britton, "Stokastik" (2008), Liber AB, ISBN 978-91-47-05351-3