Quasi-linear PDEs and low-dimensional sets

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Abstract

In this paper we establish new results concerning boundary Harnack inequalities and the Martin boundary problem, for non-negative solutions to equations of p-Laplace type with variable coefficients. The key novelty is that we consider solutions which vanish only on a low-dimensional set Σ in $\mathbb{R}^n$ and this is different compared to the more traditional setting of boundary value problems set in the geometrical situation of a bounded domain in $\mathbb{R}^n$ having a boundary with (Hausdorff) dimension in the range $[n-1, n)$. We establish our quantitative and scale-invariant estimates in the context of low-dimensional Reifenberg flat sets.

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1 Introduction

Let $D \subset \mathbb{R}^n$, $n \geq 2$, be a bounded domain, i.e., a bounded, open and connected set, and let K be a compact subset of D. Let $\Omega := D \setminus K$, and let p, $1 < p < \infty$, be fixed. Given D and K the p-capacity of K relative to D, $\mathrm{Cap}_p(K, D)$ for short, is defined as

$$\mathrm{Cap}_p(K, D) = \inf\{ \int_D |\nabla \phi|^p dy : \ \phi \in C_0^\infty(D), \ \phi \geq 1 \text{ in } K\}. \tag{1.1}$$

If $\mathrm{Cap}_p(K, D) > 0$, then the set K is not removable for the p-Laplace equation and given $f \in W^{1,p}(\mathbb{R}^n) \cap C(\bar{\Omega})$ there exists a unique p-harmonic function u in Ω satisfying $u = f$ on $\partial\Omega$ in the weak sense. Furthermore, if all points on $\partial\Omega$ are regular in the Dirichlet problem for the p-Laplace operator, then $u \in C(\bar{\Omega})$ and hence $u = f$ continuously on $\partial\Omega$. In particular, assuming that $\mathrm{Cap}_p(K, D) > 0$, and that all points on $\partial\Omega$ are regular, one can conclude that there exists, given a non-negative function $f \in C(\partial D)$ which is not identically zero, a unique positive p-harmonic function u in Ω such that $u = f$ on ∂D and $u = 0$ on ∂K. A sufficient condition for $w \in \partial\Omega$ being regular in this Dirichlet problem is that $\mathbb{R}^n \setminus \Omega$ is p-thick at w in the sense that

$$\int_0^1 \left[\frac{\mathrm{Cap}_p((\mathbb{R}^n \setminus \Omega) \cap B(w, t), B(w, 2t))}{\mathrm{Cap}_p(B(w, t), B(w, 2t))} \right]^{1/(p-1)} \frac{dt}{t} = \infty. \tag{1.2}$$

It is well known that if $p > n$ then the p-capacity of a point is positive and for $1 < p \leq n$ conditions on the set K which imply $\mathrm{Cap}_p(K, D) = 0$, can be formulated using Hausdorff measure and Hausdorff dimension. In particular, if $p = 2$, $n \geq 3$, and if the Hausdorff dimension of K is m, then the only cases which are non-trivial occur when $m \in (n-2, n]$. Hence, focusing on sets with integer dimension, the only non-trivial low-dimensional case is $m = n - 1$. For more general p we see, assuming that the Hausdorff dimension of K is m, that given K the set-up is interesting whenever $p > n-m$. In particular, all low-dimensional cases are interesting as long as we consider p large enough. Phrased in another way, while the Laplace operator can not be used as a vehicle for the extension of a function from a set of dimension $n - 2$ or lower, to neighbourhoods of the set, the p-Laplace operator, for p sufficiently large, can always achieve such an extension. The conclusion is that the p-Laplacian, and p-harmonic functions, can be studied in many interesting geometrical situations beyond the traditional set up of a bounded domain in $\mathbb{R}^n$, having a $(n - 1)$-dimensional boundary.

The purpose of this paper is to pursue the lines of thoughts outlined above in one direction by establishing certain refined boundary Harnack estimates for non-negative solutions to operators of p-Laplace type, assuming that the set K is well approximated by m-dimensional hyperplanes in the Hausdorff sense. To further put our work into perspective we recall that in [LN], [LN1], [LN2], see also [LN3], a number of results concerning the boundary behavior of positive p-harmonic functions, $1 < p < \infty$, in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$ were proved. In particular, the boundary Harnack inequality and Hölder continuity for ratios of positive p-harmonic functions, $1 < p < \infty$, vanishing on a portion of $\partial\Omega$ were established. Furthermore, the p-Martin boundary problem at $w \in \partial\Omega$ was resolved under the assumption that Ω is either convex, C^1-regular or a Lipschitz domain with small constant. Also, in [LN4] these questions were resolved for p-harmonic functions vanishing on a portion of certain Reifenberg flat and Ahlfors regular NTA-domains. The results and techniques developed in [LN], [LN1], [LN2] and [LN4] concerning p-harmonic functions have also been used and further developed in [LN5], [LN6], in the context of free boundary regularity in general two-phase free boundary problems for the p-Laplace operator, and in [LN7] in the context of regularity and free boundary regularity, below the continuous threshold, for the p-Laplace equation in Reifenberg flat and Ahlfors regular NTA-domains. In addition, in [LLuN] boundary Harnack inequalities and the Martin boundary problem was studied for more general operators of p-Laplace type with variable coefficients in Reifenberg flat domains. Further generalizations and applications can also be found in [ALuN], [ALuN1], [AN].

All papers mentioned above are set in the traditional geometrical situation of a bounded domain in $\mathbb{R}^n$ having a boundary with dimension in the range $[n-1, n)$. In this paper we begin the development of the corresponding results in the rich low-dimensional geometrical setting outlined above. This paper can be seen as a novel generalization of [LLuN] to the setting of non-negative solutions, to equations of p-Laplace type, vanishing on low-dimensional Reifenberg flat sets in $\mathbb{R}^n$. To our knowledge this paper is the first serious attack on problems of this type.

1.1 A-harmonic functions

Points in Euclidean n-space $\mathbb{R}^n$ will be denoted by $y = (y_1, \ldots, y_n)$ or (y', y_n) where $y' = (y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n-1}$. $\mathbb{S}^k$ will denote the unit sphere in $\mathbb{R}^k$. We let $\bar{E}, \partial E$, diam E, be the closure, boundary, diameter, of the set $E \subset \mathbb{R}^n$ and we define $d(y, E)$ to equal the distance from $y \in \mathbb{R}^n$ to E. $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$ and we let $|y| = \langle y, y \rangle^{1/2}$ be the Euclidean norm of y. $B(y, r) = \{z \in \mathbb{R}^n : |z - y| < r\}$ is defined whenever $y \in \mathbb{R}^n$, $r > 0$, and dy denotes Lebesgue n-measure on $\mathbb{R}^n$. Let

$$h(E, F) = \max(\sup\{d(y, E) : y \in F\}, \sup\{d(y, F) : y \in E\})$$

be the Hausdorff distance between the sets $E, F \subset \mathbb{R}^n$. If $O \subset \mathbb{R}^n$ is open and $1 \leq q \leq \infty$, then by $W^{1,q}(O)$ we denote the space of equivalence classes of functions f with distributional gradient $\nabla f = (f_{y_1}, \ldots, f_{y_n})$, both of which are q th power integrable on O. Let $\|f\|_{1,q} = \|f\|_q + \| \, |\nabla f| \, \|_q$ be the norm in $W^{1,q}(O)$ where $\| \cdot \|_q$ denotes the usual Lebesgue q norm in O. Next let $C_0^\infty(O)$ be the set of infinitely differentiable functions with compact support in O and let $W_0^{1,q}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,q}(O)$. By $\nabla \cdot$ we denote the divergence operator.

Definition 1.1 *Let $p, \beta, \alpha \in (1, \infty)$ and $\gamma \in (0, 1)$. Let $A = (A_1, ..., A_n) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, assume that $A = A(y, \eta)$ is continuous in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ and that $A(y, \eta)$, for fixed $y \in \mathbb{R}^n$, is continuously differentiable in η_k, for every $k \in \{1, ..., n\}$, whenever $\eta \in \mathbb{R}^n \setminus \{0\}$. We say that the function A belongs to the class $M_p(\alpha, \beta, \gamma)$ if the following conditions are satisfied whenever y, x, $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^n \setminus \{0\}$:*

$$\begin{aligned}
&(i) \qquad \alpha^{-1}|\eta|^{p-2}|\xi|^2 \leq \sum_{i,j=1}^n \frac{\partial A_i}{\partial \eta_j}(y, \eta)\xi_i\xi_j \leq \alpha|\eta|^{p-2}|\xi|^2, 1 \leq i, j \leq n,\\
&(ii) \qquad |A(x, \eta) - A(y, \eta)| \leq \beta|x - y|^\gamma|\eta|^{p-1},\\
&(iii) \qquad A(y, \eta) = |\eta|^{p-1}A(y, \eta/|\eta|).
\end{aligned}$$

For short, we write $M_p(\alpha)$ for the class $M_p(\alpha, 0, \gamma)$.

Definition 1.2 *Let $p \in (1,\infty)$ and let $A \in M_p(\alpha,\beta,\gamma)$ for some (α,β,γ). Given a bounded domain G we say that u is A-harmonic in G provided $u \in W^{1,p}(G)$ and*

$$\int \langle A(y,\nabla u(y)), \nabla\theta(y)\rangle\, dy = 0, \tag{1.3}$$

whenever $\theta \in W_0^{1,p}(G)$. We say that $u \in W^{1,p}(G)$ is an A-subsolution (A-supersolution) in G if (1.3) holds with $=$ replaced by $\leq$ ($\geq$) whenever $\theta \in W_0^{1,p}(G)$, $\theta \geq 0$. If $A(y,\eta) = |\eta|^{p-2}(\eta_1,\dots,\eta_n)$, and u is a function satisfying (1.3), then u is said to be p-harmonic in G. As a short notation for (1.3) we write $\nabla \cdot A(y,\nabla u) = 0$ in G. Finally, A-subharmonic function (A-superharmonic function) is a function which is upper (lower) semi-continuous and which satisfies the standard comparison principle with respect to A-harmonic functions.

Remark 1.3 *Let $G \subset \mathbb{R}^n$ be an open set, suppose that p, $1 < p < \infty$, is given and let $A \in M_p(\alpha,\beta,\gamma)$ for some (α,β,γ). Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be the composition of a translation, a rotation and a dilation $z \to rz$, $r \in (0,1]$. Suppose that u is A-harmonic in G and define $\hat{u}(z) = u(F(z))$ whenever $F(z) \in G$. Then $\hat{u}$ is $\hat{A}$-harmonic in $F^{-1}(G)$ and $\hat{A} \in M_p(\alpha,\beta,\gamma)$. For a proof of this, see Lemma 2.15 in [LLuN].*

1.2 Geometry: low-dimensional Reifenberg flat sets

Definition 1.4 *Let n, m, be integers such that $1 \leq m \leq n-1$. Given $w \in \mathbb{R}^n$ we let $\Lambda_m(w)$ denote the set of all m-dimensional hyperplanes which pass through w.*

Definition 1.5 *Let n, m, be integers such that $1 \leq m \leq n-1$. Let $\Sigma \subset \mathbb{R}^n$ be a closed set and let $r_0, \delta > 0$ be given. We say that Σ is (m, r_0, δ)-Reifenberg flat (in $\mathbb{R}^n$) if there exists, whenever $w \in \Sigma$ and $0 < r < r_0$, a hyperplane $\Lambda = \Lambda_m(w,r) \in \Lambda_m(w)$, such that*

$$h(\Sigma \cap B(w,r), \Lambda \cap B(w,r)) \leq \delta r.$$

Definition 1.6 *Let Σ be (m, r_0, δ)-Reifenberg flat (in $\mathbb{R}^n$) for some $r_0, \delta > 0$ and suppose $w \in \Sigma$, $0 < r < r_0$. We say that $\Sigma \cap B(w,r)$ is m-Reifenberg flat with vanishing constant if, for each $\epsilon > 0$, there exists $\tilde{r} = \tilde{r}(\epsilon) > 0$ with the following property. If $x \in \Sigma \cap B(w,r)$ and $0 < \rho < \tilde{r}$, then there exists a hyperplane $\Lambda' = \Lambda'_m(x,\rho) \in \Lambda_m(x)$ such that*

$$h(\Sigma \cap B(x,\rho), \Lambda' \cap B(x,\rho)) \leq \epsilon\rho.$$

Remark 1.7 *For our purposes the class of (m, r_0, δ)-Reifenberg flat sets supply a rich class of sets for our analysis. However, the literature devoted to this type of sets seems very limited. We are only aware of one paper, see [PTT], where analytic question are considered in the same framework as ours. In particular, in [PTT] the authors are concerned with the quantity*

$$R_t(w,r) = \frac{\mu(B(w,tr))}{\mu(B(w,r))} - t^m, \tag{1.4}$$

where $w \in \Sigma$, $r > 0$, $t \in (0,1]$, and μ is a measure supported on Σ. The authors prove results concerning the relation between the regularity and flatness of Σ and the asymptotic behavior of $R_t(w,r)$ as $r \to 0$.

Remark 1.8 *In [LLuN] all theorems were established for A-harmonic functions and in the context of $(n-1, r_0, \delta)$-Reifenberg flat domains in $\mathbb{R}^n$ Consequently, in this paper we will only consider the case when Σ is (m, r_0, δ)-Reifenberg flat (in $\mathbb{R}^n$) for some m, $1 \leq m \leq n-2$.*

1.3 Main results

We here state the main results established in the paper and in light of Remark 1.8 we consider A-harmonic functions, $A \in M_p(\alpha, \beta, \gamma)$, and we assume that Σ is (m, r_0, δ) -Reifenberg flat for some m, $1 \leq m \leq n-2$. As it turns out, for $m = 1$ we are able to establish a complete analog of the results in [LLuN] while for $2 \leq m \leq n-2$ we have to impose additional assumptions on A. We first prove the following two theorems.

Theorem 1.9 *Let $m = 1, n \geq 3$, and let $n - 1 < p < \infty$, be given. Let $\Sigma \subset \mathbb{R}^n$ be a closed set and assume that Σ is $(1, r_0, \delta)$-Reifenberg flat (in $\mathbb{R}^n$) for some $r_0, \delta > 0$. Let $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ). Let $w \in \Sigma, 0 < r < r_0$. Assume that u, v are positive A-harmonic functions in $B(w, 4r) \setminus \Sigma$, continuous on $B(w, 4r)$ with $u = 0 = v$ on $\Sigma \cap B(w, 4r)$. Then there exist $\tilde{\delta} = \tilde{\delta}(p, n, m, \alpha, \beta, \gamma) > 0$, $c = c(p, n, m, \alpha, \beta, \gamma) \geq 1$ and $\sigma = \sigma(p, n, m, \alpha, \beta, \gamma) > 0$, such that if $0 < \delta < \tilde{\delta}$, then*

$$\left| \log \frac{u(y_1)}{v(y_1)} - \log \frac{u(y_2)}{v(y_2)} \right| \leq c \left(\frac{|y_1 - y_2|}{r} \right)^{\sigma},$$

whenever $y_1, y_2 \in B(w, r/c) \setminus \Sigma$.

Theorem 1.10 *Let n, m, be integers such that $2 \leq m \leq n-2$ and let p, $n - m < p < \infty$, be given. Let $\Sigma \subset \mathbb{R}^n$ be a closed set and assume that Σ is (m, r_0, δ)-Reifenberg flat (in $\mathbb{R}^n$) for some $r_0, \delta > 0$. Let $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ) and assume, in addition, that A satisfies one of the following conditions.*

(*a*) *There exists $0 < \lambda < \infty$ such that $|\frac{\partial A_i}{\partial \eta_j}(y, \eta) - \frac{\partial A_i}{\partial \eta_j}(y, \eta')| \leq \lambda\, |\eta - \eta'| |\eta|^{p-3}$ whenever $y \in \mathbb{R}^n, 1 \leq i, j \leq n$ and $\eta, \eta' \in \mathbb{R}^n \setminus \{0\}$ with $\frac{1}{2}|\eta| \leq |\eta'| \leq 2|\eta|$.*

(*b*) *$A(y, \eta) = \kappa(y, \eta)\, |\langle C(y)\eta, \eta\rangle|^{p/2-1}\, C(y)\eta,\ y \in \mathbb{R}^n, \eta \in \mathbb{R}^n \setminus \{0\}$, where $C(y)$ is a linear transformation of $\mathbb{R}^n$ and $\kappa(y, \cdot)$, is homogeneous of degree 0 in η, whenever $y \in \mathbb{R}^n$.*

Let $w \in \Sigma, 0 < r < r_0$ and let u, v be as in Theorem 1.9 (relative to Σ). Then the conclusion of Theorem 1.9 holds with the only difference that in case of (a), the constants may also depend on λ.

Let n, m, be integers such that $1 \leq m \leq n - 2$ and let p, $n - m < p < \infty$, be given. Let $\Sigma \subset \mathbb{R}^n$ be a closed set and assume that Σ is (m, r_0, δ)-Reifenberg flat (in $\mathbb{R}^n$) for some $r_0, \delta > 0$. Let $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ). Let $w \in \Sigma, r$ and u, v be as in Theorem 1.9 (relative to Σ). Then there exist, see Lemma 3.7 stated below, positive Borel measures μ and ν on $\mathbb{R}^n$, with support contained in $\Sigma \cap B(w, 4r)$, such that

$$\int \langle A(y, \nabla u), \nabla \phi\rangle\, dx = -\int \phi d\mu, \quad \int \langle A(y, \nabla v), \nabla \phi\rangle\, dx = -\int \phi d\nu, \tag{1.5}$$

when $\phi \in C_0^\infty(B(w, 4r))$. We deduce the following corollaries to Theorem 1.9 and Theorem 1.10.

Corollary 1.11 *Let n, m, p, Σ, r_0, A, be as above. Let $w \in \Sigma, r$ and u, v be as in Theorem 1.9 or 1.10 (relative to Σ). Let μ and ν be measures associated to u, v, in the sense of* (1.5).

Let σ be as in the conclusion of Theorem 1.9, Theorem 1.10. Then $d\mu = k\,d\nu$, for some $k \in L^1(\Sigma \cap B(w,2r), d\nu)$, and there exists $c \geq 1$, depending at most on $p, n, m, \alpha, \beta, \gamma, \lambda$, such that

$$|\log k(y_1) - \log k(y_2)| \leq c \left(\frac{|y_1 - y_2|}{r}\right)^{\sigma}, \tag{1.6}$$

whenever $y_1, y_2 \in \Sigma \cap B(w, r/c)$.

Corollary 1.12 *Let n, m, p, Σ, r_0, A, w, u, μ, be as in Corollary 1.11 and suppose, in addition, that $\Sigma \cap B(w,4r)$ is m-Reifenberg flat with vanishing constant. Then*

$$\lim_{r \to 0} \frac{\mu(B(x,tr))}{\mu(B(x,r))} = t^m \text{ uniformly for } x \in \Sigma \cap \overline{B(w,r)} \text{ and } t \in [1/2, 1].$$

We note that in the language of [PTT], a measure μ is said to be asymptotically optimally doubling on $\Sigma \cap \overline{B(w,r)}$ if the conclusion of Corollary 1.12 holds.

Finally we prove a theorem which implies that the Martin boundary of $B(w,4r) \setminus \Sigma$ agrees with the topological boundary of this set when $\Sigma \cap B(w,4r)$ is (m, r_0, δ)-Reifenberg flat.

Theorem 1.13 *Let n, m, be integers such that $1 \leq m \leq n-2$ and let p, $n - m < p < \infty$, be given. Let $\Sigma \subset \mathbb{R}^n$ be a closed set and assume that $\Sigma \cap B(w,4r)$ is (m, r_0, δ)-Reifenberg flat. Let $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ) and assume, in addition, that either (a) or (b) of Theorem 1.10 hold in the case $2 \leq m \leq n-2$. Then there exists $\delta^* = \delta^*(p, m, n, \alpha, \beta, \gamma)$, or $\delta^* = \delta^*(p, m, n, \alpha, \beta, \gamma, \lambda)$, such that the following is true, whenever $0 < \delta < \delta^*$, $w \in \Sigma, 0 < r < r_0$. Suppose that $\hat u, \hat v$ are positive A-harmonic functions in $B(w,4r) \setminus \Sigma$, continuous on $\overline{B(w,4r)} \setminus \{w\}$ and $\hat u = 0 = \hat v$ on $\partial(B(w,4r) \setminus \Sigma) \setminus \{w\}$. If $0 < \delta < \delta^*$, then $\hat u(y) = \tau \hat v(y)$ for all $y \in B(w,4r) \setminus \Sigma$ and for some constant τ.*

Remark 1.14 *We emphasize that Theorem 1.9-Theorem 1.13, are completely new and that there is currently essentially no competing literature. Theorem 1.9, Theorem 1.10, Theorem 1.13, are proved in [LLuN] in the setting of $(n-1, r_0, \delta)$-Reifenberg flat domains in $\mathbb{R}^n$, assuming only that $A \in M_p(\alpha, \beta, \gamma)$ for some (α, β, γ).*

Remark 1.15 *Theorem 1.10 applies in the case $A(y, \eta) = |\eta|^{p-2}(\eta_1, \ldots, \eta_n)$, i.e., in the case of the p-Laplace operator. In particular, using [LN4], or [LLuN], and Theorem 1.9-Theorem 1.13, we can conclude that the conclusions of Theorem 1.9-Theorem 1.13 hold in the context of p-harmonic functions whenever $1 \leq m \leq n-1$ and p, $n - m < p < \infty$.*

Remark 1.16 *The condition in Theorem 1.10 (a) is an additional regularity condition on $A = A(y, \eta)$ in the η-variables. The condition in Theorem 1.10 (b) is a structural restriction on A. In particular, if*

$$\nabla \cdot A(y, \nabla u) = \nabla \cdot ((A(y)\nabla u \cdot \nabla u)^{p/2-1} A(y) \nabla u),$$

and $A \in M_p(\alpha, \beta, \gamma)$, then Theorem 1.10 (b) holds. The class $M_p(\alpha, \beta, \gamma)$ is invariant with respect to translations, rotations and dilations $z \to rz$, $r \in (0,1]$, as discussed in Remark 1.3. The same applies to the classes which include the conditions in Theorem 1.10 (a) and (b).

Remark 1.17 *As discussed below, in the case $2 \leq m \leq n-2$, $n-m < p < \infty$, and in the proof of Theorem 1.10, the additional assumption on A, beyond $A \in M_p(\alpha, \beta, \gamma)$, see Theorem 1.10* (*a*) *and* (*b*), *is only used in one crucial estimate. Indeed, consider the geometrical baseline configuration for our results*

$$\Sigma = \{y = (y', y'') : y' = (y_1, \dots, y_m), y'' = (y_{m+1}, \dots, y_n) = 0\}, \tag{1.7}$$

and let $C_r(0) = \{y = (y', y'') : \ |y'| < r, \ |y''| < r\}$ whenever $r > 0$. Let $A \in M_p(\alpha)$, i.e., A has constant coefficients, and assume that u is a positive A-harmonic function in $C_4(0) \setminus \Sigma$, continuous on $C_4(0)$ with $u = 0$ on $\Sigma \cap C_4(0)$. Assume that $u(0, y'') = 1$ for some $|y''| = 1$. We then need to prove that there exists $c \geq 1$, depending only on the data, such that

$$c^{-1}|y''|^{\xi} \leq u(y', y'') \ \text{ whenever } y \in C_1(0) \setminus \Sigma, \tag{1.8}$$

and where $\xi = (p-n+m)/(p-1)$. In particular, the function $|y''|^{\xi}$ gives a lower bound of the growth away from the low-dimensional set Σ in analogy with the linear growth established in the case $m = n-1$ in the corresponding baseline configuration, see Lemma 2.8 in [LLuN]. The estimate in (1.8) *is the only place where we have been unable to push our arguments through in the same generality as in [LLuN] and it is in the proof of* (1.8) *that Theorem 1.10* (*a*) *and* (*b*) *are used.*

1.4 Outline of proofs and organization of the paper

As mentioned in Remark 1.14, Theorem 1.9, Theorem 1.10, Theorem 1.13 are proved in [LLuN] in the more traditional setting of $(n-1, r_0, \delta)$-Reifenberg flat domains Ω in $\mathbb{R}^n$. In the introduction in [LLuN] some effort is given to explain and expose the key steps in the proof, stated as Step A-Step D in [LLuN]. The proof of our main results, in particular Theorem 1.9 and Theorem 1.10, proceed, structurally, also along the lines of these steps but details are considerably more involved and often require some ingenuity.

Section 2 and section 3 are motivated by the fact that many of the basic estimates used in [LLuN] have to be derived in the low-dimensional case. For example, if δ is small enough, then a δ-Reifenberg flat domain Ω in $\mathbb{R}^n$ is an NTA-domain in the sense of [JK]. In particular, from the outer corkscrew condition it then immediately follows that $\mathbb{R}^n \setminus \overline{\Omega}$ satisfies a uniform capacity density condition at every point $w \in \partial\Omega$ based on which one can conclude that the continuous Dirchlet problem for A-harmonic functions is uniquely solvable and that weak solutions with continuous boundary data are Hölder continuous up to the boundary. In our case, we first have to find a substitute for this argument, due to the lack of complement, and in Lemma 2.9 we prove, for n, m, p, Σ as in Theorem 1.9 or Theorem 1.10, that there exists $\hat{\delta} = \hat{\delta}(p, n, m)$ such that if $0 < \delta < \hat{\delta}$, then $\Sigma \cap B(w, 4r)$ is uniformly p-thick with constant $\eta = \eta(p, n, m) > 0$ (see Definition 2.8) whenever $w \in \Sigma$. Using this result we can then establish, see Lemma 3.2 and Lemma 3.3, Hölder continuity for A-harmonic functions up to Σ.

In section 4 we consider solutions to elliptic PDEs whose degeneracy is given in terms of an A_2-weight λ (see (4.1)). In case $\lambda = (|\nabla u| + |\nabla v|)^{p-2}$, where u, v are A-harmonic and $A \in M_p(\alpha, \beta, \gamma)$, we in Lemma 4.7 prove the existence of $\bar{\delta} = \bar{\delta}(p, n, m, \alpha, \beta, \gamma) > 0$ and $c = c(n, m) \geq 1$, such that if $0 < \delta < \bar{\delta}$ and $\tilde{r} = r/c$, then $\Sigma \cap B(w, 4\tilde{r})$ is uniformly $(2, \lambda)$-thick (see Definition 4.3) for some constant $\eta = \eta(p, n, m, \alpha, \beta, \gamma) > 0$. Using results in [FJK] we can then guarantee Hölder continuity of solutions to these degenerate elliptic PDEs up to Σ. We also prove, see Lemma 4.10, that if n, m, p, u, v, Σ, are as in Theorem 1.9 or Theorem 1.10, and $(a|\nabla u|+b|\nabla v|)^{p-2}$ is an A_2-weight with A_2-constant independent of $a, b \in [0,\infty)$, then Theorem 1.9 or Theorem 1.10 is valid. In subsection 4.2 we also list several other assumptions and prove that these assumptions imply Theorem 1.9 and Theorem 1.10 when Σ is a m-dimensional hyperplane.

In section 5 we prove, for $A \in M_p(\alpha)$ and $\tilde{A}$ with $\tilde{A}_j = A_{m+j}, 1 \leq j \leq n-m$, $p > n-m$, the existence and uniqueness of a 'fundamental solution', say $\tilde{u}$, to $\nabla \cdot \tilde{A}(\nabla \tilde{u}) = 0$ with pole at 0 in $\mathbb{R}^{n-m}$. It turns out that

$$\tilde{u}(z) = |z|^{\xi}\, \tilde{u}(z/|z|), z \in \mathbb{R}^{n-m} \setminus \{0\}, \text{ and } |\nabla \tilde{u}|(z) \approx \tilde{u}(z)/|z| \approx |z|^{\xi-1}, \tag{1.9}$$

where $\xi = (p-n+m)/(p-1)$ and $\approx$ means the ratio of the two quantities is bounded above and below by constants depending only on the data, i.e., the structure constants in Definition 1.1 and n, m, p. Let $\bar{u}(y) = \tilde{u}(\pi(y))$, when $y \in \mathbb{R}^n$ and where $\pi(y)$ denotes the projection of y onto $z \in \mathbb{R}^{n-m}$. Then $\bar{u}$ is an A-harmonic function on $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$, vanishing on $\mathbb{R}^m \times \{0\} \in \mathbb{R}^m \times \mathbb{R}^{n-m}$. In our arguments, $\bar{u}$ plays the same role as the function y_n does in [LLuN].

In section 6 we prove Theorem 1.9 and Theorem 1.10 in the special case when $A \in M_p(\alpha)$ and Σ is as stated in (1.7) in Remark 1.17. Indeed, let u, v be positive A-harmonic functions in $B(0,4) \setminus \Sigma$, continuous on $B(0,4)$ and $u = 0 = v$ on $\Sigma \cap B(0,4)$. Assume that $u(0,y'') \approx v(0,y'') \approx 1$ for some $|y''| = 1$. The crucial estimate is to prove there exists $c \geq 1$ (depending only on the data) such that

$$c^{-1} \; \leq u(y)/v(y) \; \leq \; c \; \text{ whenever } y \in C_1(0) \setminus \Sigma, \tag{1.10}$$

where the sets $C_\cdot(0)$ were introduced in Remark 1.17. To prove (1.10) in the case $m = 1$ we use an argument from [BL]. In fact, see Remark 6.3 below, this argument is also applicable in the case of the p-Laplace operator in the full range $1 \leq m \leq n-2$, but the proof in this case relies heavily on the p-Laplacian being invariant under rotations. For general $A \in M_p(\alpha)$, in the case $2 \leq m \leq n-2$, we first note, in view of (1.9), that to prove (1.10) it suffices to establish (1.10) with $v = \bar{u}$, and in particular to establish the existence of $c \geq 1$, depending only on the data, such that

$$c^{-1}|y''|^{\xi} \; \leq u(y', y'') \; \leq \; c|y''|^{\xi} \; \text{ whenever } (y', y'') \in C_1(0) \setminus \Sigma. \tag{1.11}$$

To get the upper estimate in (1.11) we consider the function u' which is defined to be A-harmonic in $B(0,8) \setminus (\Sigma \cap \overline{B(0,4)})$ with continuous boundary values $u' \equiv 1$ on $\partial B(0,8)$ and $u' \equiv 0$ on $\Sigma \cap \overline{B(0,4)}$. Then, using Harnack's inequality we have $u \leq cu'$, and we prove, see (6.9), that u' satisfies the fundamental inequality

$$c^{-1}\frac{u'(y)}{d(y,\Sigma)} \leq |\nabla u'(y)| \leq c\frac{u'(y)}{d(y,\Sigma)}, \tag{1.12}$$

whenever $y \in C_1(0) \setminus \Sigma$ and where $c \geq 1$ depends only on the data. Using (1.12) and (1.9) we then conclude from our work in section 4 that (1.10) holds with $u = u'$ and $v = \bar{u}$, implying the upper bound in (1.11).

To get the lower bound in estimate (1.11), for a general A as in Definition 1.1, turns out to be a more difficult problem and, as discussed in Remark 1.17, for $2 \leq m \leq n-2$ this is the only place in the proof of Theorem 1.10 where we require Theorem 1.10 (a) and (b). Our proof of the lower estimate in (1.11) is based on the construction of appropriate A-subsolutions (barriers). The constructions are rather subtle and make essential use of (1.9) and the $\bar u$ introduced above. In particular, in the case of Theorem 1.10 (a) and (b), both the constructions rely on the function

$$f(y) = f(y', y'') = (1 - |y'|^2)\,(e^{\bar u(y)} - 1) = (1 - |y'|^2)\,(e^{\bar u(0,y'')} - 1). \tag{1.13}$$

Note that f has a product structure which facilitates computations.

In section 7 we prove Theorem 1.9 and Theorem 1.10 in general, as well as Corollaries 1.11, 1.12. Theorem 1.9 and Theorem 1.10, for $A \in M_p(\alpha, \beta, \gamma)$ and Σ as in (1.7), follow from the corresponding results established in section 6 in the baseline configuration and by a technique which can loosely be described as 'freezing the coefficients'. Indeed, given our results from section 6, as well as our preliminary work in sections 2-5, we at this stage can (with modifications) invoke the A-harmonic machine developed in [LLuN]. In particular, based on the validity of Theorem 1.9 and Theorem 1.10, in the case when Σ is as in (1.7), we can prove, for $u, v, \Sigma, m, n, p, \delta, \tilde\delta$ as in Theorems 1.9 and 1.10, that (1.12) holds with u' replaced by u, v in $B(w, r/c) \setminus \Sigma$, with $c \geq 1$ depending only on the data, provided $\tilde\delta > 0$ is small enough. We then use this result to prove that $(|\nabla u| + |\nabla v|)^{p-2}$ is an A_2-weight with A_2-constant bounded independently of u, v. In view of this fact we can once again invoke boundary Harnack and Hölder continuity results from [FJK1] to conclude Theorems 1.9 and 1.10 based on our work in section 4. Finally, in section 7 we easily obtain Corollaries 1.11, 1.12, as a consequence of Theorem 1.9 and Theorem 1.10. In the proof of Corollary 1.12 we also use a compactness and blow up type argument for A-harmonic functions.

In section 8 we prove Theorem 1.13. To do this we first prove Theorem 1.13 in the baseline case when $\Sigma = \mathbb{R}^m \cup \{0\}$. Once this is done we can use Theorems 1.9, 1.10, and Theorem 1.13 in the baseline case, to argue as earlier in order to eventually obtain Theorem 1.13.

Acknowledgement 1.18 *The first author would like to thank Benny Avelin for some stimulating conversations regarding the construction of the barrier in Theorem 1.10.*

2 Geometry of (m, r_0, δ)-Reifenberg flat sets in $\mathbb{R}^n$

In this section we develop a number of results concerning the geometry of (m, r_0, δ)-Reifenberg flat sets in $\mathbb{R}^n$. In particular, we assume $1 \leq m \leq n-2$ and we let $\Sigma \subset \mathbb{R}^n$ be a closed set which is (m, r_0, δ)-Reifenberg flat for some $r_0, \delta > 0$. Given $w \in \mathbb{R}^n$ and $\Lambda_m(w)$ we can always introduce coordinates $y = (y', y'')$, $y' \in \mathbb{R}^m$, $y'' \in \mathbb{R}^{n-m}$, such that

$$\Lambda_m(w) = \{y = (y' + w', y'' + w'') \in \mathbb{R}^m \times \mathbb{R}^{n-m} : \ y'' = 0\},$$

where $w = (w', w'')$. Using this coordinate system and r, $0 < r$, we let

$$a_\Lambda(w, r) = (a'_\Lambda(w, r), a''_\Lambda(w, r))$$

be any point satisfying $a'_\Lambda(w, r) = w'$, $|a''_\Lambda(w, r) - w''| = r$.

Lemma 2.1 *Let $1 \leq m \leq n-2$ and suppose $\Sigma \subset \mathbb{R}^n$ is a closed set which is (m, r_0, δ)-Reifenberg flat for some $r_0, \delta > 0$. Then there exists $\delta_0 = \delta_0(n, m) > 0$, and a constant $M =$*

$M(n,m) \geq 2$, *such that the following is true whenever* $0 < \delta < \delta_0$. *Given* $w \in \Sigma, 0 < r < r_0$, *there is a point* $a_r(w) \in \mathbb{R}^n \setminus \Sigma$ *such that*

$$d(a_r(w), \Sigma) > M^{-1}r,\ M^{-1}r < |a_r(w) - w| \leq r.$$

Proof. Consider $w \in \Sigma, 0 < r < r_0$. Then using Definition 1.5 we see that there exists $\Lambda = \Lambda_m(w,r) \in \Lambda_m(w)$ such that

$$h(\Sigma \cap B(w,r), \Lambda \cap B(w,r)) \leq \delta r.$$

For $\eta \in (1/4, 1)$ fixed we now let, using coordinates with respect to $\Lambda = \Lambda_m(w,r)$ as introduced above,

$$a_r(w) := a_\Lambda(w, \eta r). \tag{2.1}$$

It then immediately follows that there exist $\delta_0 = \delta_0(n,m) > 0$, and a constant $M = M(n,m)$, such that the conclusion of the lemma holds whenever $0 < \delta < \delta_0$. ■

Lemma 2.2 *Assume* $1 \leq m \leq n-2$, *let* $\Sigma \subset \mathbb{R}^n$ *be a closed set and suppose that* Σ *is* (m, r_0, δ)*-Reifenberg flat (in* $\mathbb{R}^n$*) for some* $r_0, \delta > 0$. *Then there exists* $\delta_0 = \delta_0(n,m) > 0$ *such that the following is true whenever* $0 < \delta < \delta_0$, $0 < r < r_0/2$. *There exists* $c = c(n,m)$, $1 \leq c < \infty$, *such that*

$$\begin{aligned} &(i) \quad h(\Lambda_m(w,r) \cap B(w,1), \Lambda_m(w,r/2) \cap B(w,1)) \leq c\delta, \\ &(ii) \quad h(\Lambda_m(\tilde{w},r) \cap B(\tilde{w},1), \Lambda_m(\tilde{w},r) \cap B(\tilde{w},1)) \leq c\delta, \end{aligned} \tag{2.2}$$

whenever $w, \hat{w}, \tilde{w} \in \Sigma$ *and* $r/2 \leq |\hat{w} - \tilde{w}| \leq 2r$.

Proof. Let $w \in \Sigma$. Then, using Definition 1.5 we see that

$$\begin{aligned} &(i') \quad h(\Sigma \cap B(w,r), \Lambda_m(w,r) \cap B(w,r)) \leq \delta r, \\ &(ii') \quad h(\Sigma \cap B(w,r/2), \Lambda_m(w,r/2) \cap B(w,r/2)) \leq \delta r/2. \end{aligned} \tag{2.3}$$

Hence, using (2.3) (i') and (ii') we find that

$$h(\Lambda_m(w,r) \cap B(w,r/2), \Lambda_m(w,r/2) \cap B(w,r/2)) \leq 2\delta r. \tag{2.4}$$

(2.2) (i) now follows from (2.4) by scaling and elementary geometry. To prove (2.2) (ii) we first note, using the definitions and the assumption $r/2 \leq |\hat{w} - \tilde{w}| \leq 2r$, $\hat{w}, \tilde{w} \in \Sigma$, that

$$\begin{aligned} &(i'') \quad h(\Sigma \cap B(\hat{w},4r), \Lambda_m(\hat{w},4r) \cap B(\hat{w},4r)) \leq 4\delta r, \\ &(ii'') \quad h(\Sigma \cap B(\tilde{w},r), \Lambda_m(\tilde{w},r) \cap B(\tilde{w},r)) \leq \delta r. \end{aligned} \tag{2.5}$$

Since $B(\tilde{w},r) \subset B(\hat{w},4r)$ we conclude from (2.5) that

$$h(\Lambda_m(\hat{w},4r) \cap B(\tilde{w},r), \Lambda_m(\tilde{w},r) \cap B(\tilde{w},r)) \leq 5\delta r. \tag{2.6}$$

(2.2) (ii) follows from this observation, (2.2) (i), and scaling. ■

Definition 2.3 *Let $\Sigma \subset \mathbb{R}^n$ be a closed set. Given $M \geq 2$, we say that a ball $B(y,r)$, $y \in \mathbb{R}^n$, $0 < r < \infty$, is a M-non-tangential ball (in $\mathbb{R}^n$ and with respect to Σ) if*

$$M^{-1}r < d(B(y,r),\Sigma) < Mr.$$

Furthermore, given $y, y' \in \mathbb{R}^n \setminus \Sigma$ we say that a sequence of M-non-tangential balls (in $\mathbb{R}^n$ and with respect to Σ), $B(y_1,r_1)$,..., $B(y_p,r_p)$, is a M-Harnack chain of length p (in $\mathbb{R}^n$ and with respect to Σ), joining y to y', if $y \in B(y_1,r_1)$, $y' \in B(y_p,r_p)$, and $B(y_i,r_i) \cap B(y_{i+1},r_{i+1}) \neq \varnothing$ for $i \in \{1,...,p-1\}$.

Lemma 2.4 *Assume $1 \leq m \leq n-2$, let $\Sigma \subset \mathbb{R}^n$ be a closed set and suppose that Σ is (m,r_0,δ)-Reifenberg flat (in $\mathbb{R}^n$) for some $r_0, \delta > 0$. Then there exists $\delta_0 = \delta_0(n,m) > 0$, and a constant $M = M(n,m) \geq 2$, such that the following is true. Assume $0 < \delta < \delta_0$, $w \in \Sigma$, $0 < r < \tilde{r}_0$, $\tilde{r}_0 = r_0/M$. Consider $y \in B(w,r) \setminus \Sigma$, let $\epsilon = d(y,\Sigma)$, and let $\hat{y} \in \Sigma$ be such that $\epsilon = d(y,\hat{y})$. Then y, $a_\epsilon(\hat{y})$, and $a_{2\epsilon}(\hat{y})$, can all be joined by M-Harnack chains (in $\mathbb{R}^n$ and with respect to Σ), which are contained in $B(\hat{y},M\epsilon) \setminus \Sigma$ and which have a length depending only on n, m.*

Proof. Lemma 2.4 can be proved using Lemma 2.2 and elementary observations. ■

Lemma 2.5 *Assume $1 \leq m \leq n-2$, let $\Sigma \subset \mathbb{R}^n$ be a closed set and suppose that Σ is (m,r_0,δ)-Reifenberg flat (in $\mathbb{R}^n$) for some $r_0, \delta > 0$. Then there exists $\delta_0 = \delta_0(n,m) > 0$, and a constant $M = M(n,m) \geq 2$, such that the following is true. Assume $0 < \delta < \delta_0$, $w \in \Sigma$, $0 < r < \tilde{r}_0$ and $\tilde{r}_0 = r_0/M$. Consider $y, y' \in B(w,r) \setminus \Sigma$, such that $d(y,\Sigma) \geq \epsilon$, $d(y',\Sigma) \geq \epsilon$, and $d(y,y') \leq C\epsilon$, for some $\epsilon > 0$, $C \geq 1$. Then there exists a M-Harnack chain (in $\mathbb{R}^n$ and with respect to Σ), joining y and y', which is contained in $B(w,Mr) \setminus \Sigma$ and which has a length depending only on C, M, i.e., a length depending only on C, n, m.*

Proof. Lemma 2.5 can be proved by proceeding along the lines of the proof in [KT] of the corresponding statements in the more traditional setting of Reifenberg flat domains in $\mathbb{R}^n$. ■

Remark 2.6 *Let $1 \leq m \leq n-2$ be given. Throughout the paper we will always assume, given a (m,r_0,δ)-Reifenberg flat set $\Sigma \subset \mathbb{R}^n$, that $0 < \delta < \delta_0$ with $\delta_0 = \delta_0(n,m) > 0$, so that Lemma 2.1, Lemma 2.4, and Lemma 2.5 are all valid. We will sometimes refer to M, r_0, as parameters defining (i) non-tangential approach regions to Σ as well as (ii) the connectivity of $\mathbb{R}^n \setminus \Sigma$.*

2.1 An estimate of p-capacity

Definition 2.7 *Let $O \subset \mathbb{R}^n$ be open and let K be a compact subset of O. Given p, $1 < p < \infty$, we let*

$$Cap_p(K,O) = \inf\{\int_O |\nabla\phi|^p \, dy \ \phi \in C_0^\infty(O),\ \phi \geq 1 \text{ in } K\}.$$

$Cap_p(K,O)$ is referred to as the p-capacity of K relative to O. The p-capacity of an arbitrary set $E \subset O$ is defined by

$$Cap_p(E,O) = \inf_{E\subset G\subset O,\ G \text{ open}} \ \sup_{K\subset G,\ K \text{ compact}} Cap_p(K,O). \tag{2.7}$$

Definition 2.8 *Let $\Sigma \subset \mathbb{R}^n$ be a closed set, let $w \in \Sigma$, $0 < r < \infty$. Let p, $1 < p < \infty$, be given and assume that there exists a constant $\eta > 0$ such that*

$$\frac{Cap_p(\Sigma \cap B(\hat{w}, \hat{r}), B(\hat{w}, 2\hat{r}))}{Cap_p(B(\hat{w}, \hat{r}), B(\hat{w}, 2\hat{r}))} \geq \eta,$$

whenever $\hat{w} \in \Sigma \cap B(w, 4r)$, $0 < \hat{r} < r$. We then say that $\Sigma \cap B(w, 4r)$ is uniformly p-thick with constant η.

Lemma 2.9 *Assume $1 \leq m \leq n-2$, let $\Sigma \subset \mathbb{R}^n$ be a closed set and assume that Σ is (m, r_0, δ)-Reifenberg flat (in $\mathbb{R}^n$) for some $r_0, \delta > 0$. Let p, $n - m < p < \infty$, be given. Then there exists $\hat{\delta} = \hat{\delta}(p, n, m)$ such that if $0 < \delta < \hat{\delta}$, then $\Sigma \cap B(w, 4r)$ is uniformly p-thick for some constant $\eta = \eta(p, n, m)$ whenever $w \in \Sigma, 0 < r < r_0/4$.*

Proof. Let $\hat{w} \in \Sigma \cap B(w, 4r)$, $0 < \hat{r} < r$, let $\hat{\delta} = \hat{\delta}(p, n, m)$ be a degree of freedom to be chosen and consider $0 < \delta < \hat{\delta}$. As uniform p-thickness is invariant under translation and dilation, we may in the following proof assume, without loss of generality, that $\hat{w} = 0$ and $\hat{r} = 1$. We may also assume that p is fixed and that $n - m < p < n - m/2$, as Lemma 2.9 for other values of p follows from this case and inclusion relations for Riesz capacities (see [AH] , Theorem 5.51). To start the argument we note that there exists a hyperplane $\Lambda = \Lambda_m(0, 1)$ such that

$$h(\Sigma \cap B(0, 1), \Lambda \cap B(0, 1)) \leq \delta \leq \hat{\delta}. \tag{2.8}$$

In the following argument we let $N := \hat{\delta}^{-m}/(10^{10}A)$, where $A \geq 1$ is a large but fixed degree of freedom, depending on m, and to be chosen. Using (2.8) we find, for A large enough, that $B(0, 1/8)$ contains at least $\tilde{N} \geq N$ disjoint balls of radius $\hat{\delta}$, $\{B(y_i, \hat{\delta})\}_{i=1}^{\tilde{N}}$, with $y_i \in \Sigma \cap B(0, 1)$. Let Γ_1 denote a sub collection of these balls consisting of exactly N balls. In particular, $\Gamma_1 = \{B(z_i, \hat{\delta})\}_{i=1}^{N}$ for some $\{z_1, ..., z_N\} \subset \{y_1, ..., y_{\tilde{N}}\}$. Given a ball $B(z_i, \hat{\delta})$ in Γ_1 we can now repeat this construction with $B(0, 1)$, $B(0, 1/8)$, replaced by $B(z_i, \hat{\delta})$, $B(z_i, \hat{\delta}/8)$. Doing this for every ball in Γ_1 the result is a new collection, denoted Γ_2, of N^2 balls of radius $\hat{\delta}^2$. Inductively we can in this way construct $\{\Gamma_l\}_{l=1}^{\infty}$ where Γ_l is a collection of N^l disjoint balls of radius $\hat{\delta}^l$ and such that each ball in Γ_{l+1} is contained in a ball in Γ_l. Furthermore, it follows, for $\hat{\delta}$ small enough, that the closure of any ball in Γ_l is contained in $B(0, 1/4)$.

Next let

$$E_l := \{y \in \mathbb{R}^n : \ d(y, \Sigma) \leq \hat{\delta}^l\} \cap B(0, 1),$$

let l_0 be a large but fixed integer, and let ν_{l_0} denote the n-dimensional Lebesgue measure restricted to the balls in Γ_{l_0}. Then

$$\nu_{l_0}(E_{l_0} \cap B(0, 1)) = N^{l_0} \hat{\delta}^{n l_0} \gamma(n), \tag{2.9}$$

where $\gamma(n)$ is the volume of the unit ball in $\mathbb{R}^n$. Let $\tilde{\nu}_{l_0} = \nu_{l_0}/\nu_{l_0}(E_{l_0} \cap B(0, 1))$ and let

$$W_{1,p}^{\tilde{\nu}_{l_0}}(y) = \int_0^\infty \left(\frac{\tilde{\nu}_{l_0}(B(y, t))}{t^{n-p}} \right)^{1/(p-1)} \frac{dt}{t}, \ y \in \mathbb{R}^n, \tag{2.10}$$

denote the Wolff potential associated to $\tilde{\nu}_{l_0}$. We intend to prove for some small fixed $\hat{\delta} = \hat{\delta}(p, n, m) > 0$ that

$$W_{1,p}^{\tilde{\nu}_{l_0}}(y) \leq c \text{ whenever } y \in \mathbb{R}^n, \tag{2.11}$$

where $c = c(p,n,m)$, $1 \le c < \infty$. Using (2.11), the dual formulation of capacity proved in Theorem 2.2.7 in [AH], as well as Theorem 4.5.4 in [AH], we can then conclude that

$$\mathrm{Cap}_p(E_{l_0} \cap B(0,1), B(0,2)) \ge \hat{c}^{-1},$$

for yet another $\hat{c} = \hat{c}(p,n,m)$, $1 \le \hat{c} < \infty$. In particular, letting $l_0 \to \infty$ we then deduce that

$$\mathrm{Cap}_p(\Sigma \cap B(0,1), B(0,2)) \ge \hat{c}^{-1}/2.$$

Furthermore, since $\mathrm{Cap}_p(B(0,1), B(0,2)) \approx 1$ we see that Lemma 2.9, for $n-m < p < n-m/2$, follows immediately once (2.11) is proved.

To start the proof of (2.11) we first note that

$$\begin{aligned} W_{1,p}^{\tilde{\nu}_{l_0}}(y) &\le \int_1^\infty \left(\frac{\tilde{\nu}_{l_0}(B(y,t))}{t^{n-p}}\right)^{1/(p-1)} \frac{dt}{t} + \int_{\hat{\delta}^{l_0}}^1 \left(\frac{\tilde{\nu}_{l_0}(B(y,t))}{t^{n-p}}\right)^{1/(p-1)} \frac{dt}{t} \\ &\quad + \int_0^{\hat{\delta}^{l_0}} \left(\frac{\tilde{\nu}_{l_0}(B(y,t))}{t^{n-p}}\right)^{1/(p-1)} \frac{dt}{t} \\ &:= I_1(y) + I_2(y) + I_3(y). \end{aligned} \tag{2.12}$$

Using $\tilde{\nu}_{l_0}(\mathbb{R}^n) = 1$ and integrating in $I_1(y)$ we obtain $I_1(y) \le c$, since $n-m < p < n-m/2$. Next, consider $l \le l_0$, $\hat{\delta}^l \le t < \hat{\delta}^{l-1}$, and note, for $y \in \mathbb{R}^n$, that if $\tilde{\nu}_{l_0}(B(y,t)) \ne 0$, then $B(y,t)$ intersects at most $c(n)$ balls in Γ_{l-1}. Moreover, each of these balls has ν_{l_0} measure at most $N^{l_0-l+2}\hat{\delta}^{nl_0}\gamma(n)$. Hence, using this and (2.9), we see that

$$\tilde{\nu}_{l_0}(B(y,t)) \le c(n)N^{-l+2} = c(n)\hat{\delta}^{m(l-2)}(10^{10}A)^{l-2} \le \frac{(10^{10}A)^{l-2}}{\hat{\delta}^{3m}} t^m, \tag{2.13}$$

whenever $\hat{\delta}^l \le t < \hat{\delta}^{l-1}$, provided $\hat{\delta}$ is small enough. Furthermore, given $\varepsilon \in (0,1)$ it follows from (2.13) that these exist $\hat{\delta} = \hat{\delta}(n,m,\varepsilon)$ and $c = c(n,m,\varepsilon) \ge 1$, such that

$$\tilde{\nu}_{l_0}(B(\hat{y},t)) \le ct^{m\varepsilon} \text{ whenever } \hat{\delta}^{l_0} \le t \le 1. \tag{2.14}$$

Let $\varepsilon = (1 + (n-p)/m)/2 \in (0,1)$ and fix $\hat{\delta} = \hat{\delta}(p,n,m) > 0$ to be the largest number so that the above inequalities hold. Then, using (2.14) we see that

$$I_2(y) \le \int_{\hat{\delta}^{l_0}}^1 t^{(m\varepsilon+p-n)/(p-1)} \frac{dt}{t} \le c(p,n,m).$$

Finally, using the trivial estimate $\nu_{l_0}(B(y,t)) \le \gamma(n)t^n$, whenever $0 < t < \hat{\delta}^{l_0}$, we get

$$I_3(y) \le c(N^{l_0}\hat{\delta}^{nl_0}\gamma(n))^{1/(1-p)} \int_0^{\hat{\delta}^{l_0}} t^{p/(p-1)} \frac{dt}{t} \le c(p,n,m),$$

whenever $n-m < p < n-m/2$. Putting together the estimates for $I_1(y), I_2(y), I_3(y)$ we obtain (2.11) in the case $n-m < p < n-m/2$. From our earlier remarks we now conclude Lemma 2.9. ■

3 A-harmonic functions

In this section we first state and prove some fundamental estimates for non-negative A-harmonic functions. Throughout the section we assume, unless otherwise stated, that

$$
\begin{aligned}
&(i) && p,\ n-m<p<\infty,\ 1\leq m\leq n-2,\\
&(ii) && \Sigma\subset\mathbb{R}^n \text{ is a closed set and } \Sigma \text{ is } (m,r_0,\delta)\text{-Reifenberg flat},\\
&(iii) && A\in M_p(\alpha,\beta,\gamma) \text{ or } A\in M_p(\alpha) \text{ for some } (\alpha,\beta,\gamma).
\end{aligned}
\tag{3.1}
$$

Furthermore, assuming (3.1) we let $\bar\delta=\min\{\delta_0,\hat\delta\}$ where δ_0 is as stated in Lemma 2.1, Lemma 2.4, and Lemma 2.5, and where $\hat\delta$ is as stated in Lemma 2.9. Then $\bar\delta=\bar\delta(p,n,m)$. In particular, when we in the following assume (3.1), and state that $0<\delta<\bar\delta$, then we ensure that

$$
\begin{aligned}
&(i) && \text{Lemma 2.1, Lemma 2.4, and Lemma 2.5 are valid for some } M=M(n,m)\geq 2, \text{ and}\\
&(ii) && \text{there exists } \eta=\eta(p,n,m)>0 \text{ such that } \Sigma\cap B(w,4r) \text{ is uniformly } p\text{-thick}\\
& && \text{with constant } \eta \text{ whenever } w\in\Sigma,\ 0<r<r_0/4.
\end{aligned}
\tag{3.2}
$$

Concerning constants, unless otherwise stated, in this section, and throughout the paper, c will denote a positive constant ≥ 1, not necessarily the same at each occurrence, depending at most on $p,n,m,\alpha,\beta,\gamma,\lambda$, which sometimes we refer to as depending on the data. In general, $c(a_1,\dots,a_m)$ denotes a positive constant ≥ 1, which may depend at most on the data and $a_1,\dots,a_m$, not necessarily the same at each occurrence. If $A\approx B$ then A/B is bounded from above and below by constants which, unless otherwise stated, depends at most on the data. Moreover, we let $\max\limits_{B(z,s)} u$, $\min\limits_{B(z,s)} u$ be the essential supremum and infimum of u on $B(z,s)$ whenever $B(z,s)\subset\mathbb{R}^n$ and whenever u is defined on $B(z,s)$.

3.1 Basic estimates

Lemma 3.1 *Given $p, 1<p<\infty$, assume that $A\in M_p(\alpha,\beta,\gamma)$ for some (α,β,γ). Let u be a positive A-harmonic function in $B(w,2r)$. Then*

$$
\begin{aligned}
&(i) && r^{p-n}\int\limits_{B(w,r/2)}|\nabla u|^p\,dy\ \leq\ c\,(\max_{B(w,r)}u)^p,\\
&(ii) && \max_{B(w,r)}u\leq c\min_{B(w,r)}u.
\end{aligned}
$$

Furthermore, there exists $\sigma=\sigma(p,n,\alpha,\beta,\gamma)\in(0,1)$ such that if $x,y\in B(w,r)$, then

$$
(iii)\qquad |u(x)-u(y)|\leq c\left(\tfrac{|x-y|}{r}\right)^{\sigma}\max_{B(w,2r)}u.
$$

Lemma 3.2 *Assume* (3.1) *and that $0<\delta<\bar\delta$. Let $w\in\Sigma$ and consider $0<r<r_0$. Then, given $f\in W^{1,p}(B(w,4r))$ there exists a unique A-harmonic function $u\in W^{1,p}(B(w,4r)\setminus\Sigma)$ such that $u-f\in W^{1,p}_0(B(w,4r)\setminus\Sigma)$. Furthermore, let $u,v\in W^{1,p}_{loc}(B(w,4r)\setminus\Sigma)$ be an A-superharmonic function and an A-subharmonic function in Ω, respectively. If $\inf\{u-v,0\}\in W^{1,p}_0(B(w,4r)\setminus\Sigma)$, then $u\geq v$ a.e in $B(w,4r)\setminus\Sigma$. Finally, every point $\hat w\in\Sigma\cap B(w,4r)$ is regular for the continuous Dirichlet problem for $\nabla\cdot A(x,\nabla u)=0$.*

Proof. The first part of the lemma is a standard maximum principle so we only prove the statement that every point $\hat{w} \in \Sigma \cap B(w, 4r)$ is regular in the continuous Dirichlet problem for $\nabla \cdot A(x, \nabla u) = 0$ and to prove this we use results established in section 6 of [HKM]. Indeed, given $\hat{w} \in \Sigma \cap B(w, 4r)$, from (3.1) and the assumption that $0 < \delta < \bar{\delta}$ we have, see (3.2), that there exist $r_{\hat{w}} > 0$ and $\eta = \eta(p, n, m) > 0$ such that

$$\frac{\mathrm{Cap}_p(\Sigma \cap B(\hat{w}, \rho), B(\hat{w}, 2\rho))}{\mathrm{Cap}_p(B(\hat{w}, \rho), B(\hat{w}, 2\rho))} \geq \eta,$$

whenever $0 < \rho < r_{\hat{w}}/2$. In particular,

$$\int_0^{r_{\hat{w}}/2} \left[\frac{\mathrm{Cap}_p(\Sigma \cap B(\hat{w}, \rho), B(\hat{w}, 2\rho))}{\mathrm{Cap}_p(B(\hat{w}, \rho), B(\hat{w}, 2\rho))}\right]^{1/(p-1)} \frac{d\rho}{\rho} = \infty,$$

and hence $\hat{w}$ is regular in the Dirichlet problem for $\nabla \cdot A(x, \nabla u) = 0$. ■

Lemma 3.3 *Assume* (3.1), $0 < \delta < \bar{\delta}$, *and that* $w \in \Sigma$. *Assume also that* u *is a positive* A*-harmonic function in* $B(w, 4r) \setminus \Sigma$, *continuous on* $B(w, 4r)$ *and* $u = 0$ *on* $\Sigma \cap B(w, 4r)$. *Then*

$$(i) \qquad r^{p-n} \int_{B(w, r/2)} |\nabla u|^p \, dy \;\leq\; c\,(\max_{B(w,r)} u)^p.$$

Furthermore, there exists $\sigma = \sigma(p, n, m, \alpha, \beta, \gamma) \in (0, 1)$ *such that if* $x, y \in B(w, r)$, *then*

$$(ii) \qquad |u(x) - u(y)| \leq c \left(\tfrac{|x-y|}{r}\right)^{\sigma} \max_{B(w,2r)} u.$$

Proof. (i) is a standard Caccioppoli inequality so we only prove (ii). We note, using Lemma 3.1, the triangle inequality and elementary arguments, that it suffices to prove there exist $1 \leq c < \infty$ and $\sigma \in (0, 1)$, depending only on the data, such that

$$\max_{B(w,\rho)} u \leq c \left(\frac{\rho}{r}\right)^{\sigma} \max_{B(w,r)} u, \qquad \text{whenever } 0 < \rho \leq r. \tag{3.3}$$

To prove (3.3) we again use results established in section 6 in [HKM]. Indeed, using Theorem 6.18 in [HKM] we immediately see that there exists a constant $c > 0$, depending only on the data such that

$$\max_{B(w,\rho)} u \leq \exp\left(-c \int_\rho^r \left[\frac{\mathrm{Cap}_p(\Sigma \cap B(w, t), B(w, 2t))}{\mathrm{Cap}_p(B(w, t), B(w, 2t))}\right]^{1/(p-1)} \frac{dt}{t}\right) \max_{B(w,r)} u.$$

Furthermore, using (3.1) and the assumption that $0 < \delta < \bar{\delta}$, we have

$$\exp\left(-c \int_\rho^r \left[\frac{\mathrm{Cap}_p(\Sigma \cap B(w, t), B(w, 2t))}{\mathrm{Cap}_p(B(w, t), B(w, 2t))}\right]^{1/(p-1)} \frac{dt}{t}\right) \leq \exp(-\hat{c} \ln(r/\rho)).$$

Putting these inequalities together we obtain (3.3). ■

Lemma 3.4 *Assume* (3.1) *and that* $0 < \delta < \bar{\delta}$. *Assume also that* u *is a positive* A*-harmonic function in* $B(w, 4r) \setminus \Sigma$. *There exists* $c = c(p, n, m)$, $1 \leq c < \infty$, *such that if* $\tilde{r} = r/c$, $w_1, w_2 \in B(w, \tilde{r}) \setminus \Sigma$, $\min\{d(w_1, \Sigma), d(w_2, \Sigma)\} > \epsilon$ *and* $|w_1 - w_2| \leq C\epsilon$, *for some* $\epsilon > 0$, *then*

$$u(w_1) \leq \hat{c} u(w_2) \text{ for some } \hat{c} \geq 1 \text{ depending only on the data and } C.$$

Proof. The lemma is elementary and follows from Lemma 2.5 and Lemma 3.1. ■

Lemma 3.5 *Assume* (3.1) *and that* $0 < \delta < \bar{\delta}$. *Assume also that* u *is a positive* A*-harmonic function in* $B(w, 4r) \setminus \Sigma$, *continuous on* $B(w, 4r)$ *and* $u = 0$ *on* $\Sigma \cap B(w, 4r)$. *There exists* $c \geq 1$, *depending only on the data, such that if* $\tilde{r} = r/c$, *then*

$$\max_{B(w,\tilde{r})} u \leq c\, u(a_{\tilde{r}}(w)).$$

Proof. A proof of Lemma 3.5 for linear elliptic PDE can be found in [CFMS]. The proof uses only analogues of Lemma 3.1, Lemma 3.3 and Lemma 3.4 for linear PDE. In particular, the proof also applies in our situation. ■

Lemma 3.6 *Assume* (3.1) *and that* $0 < \delta < \bar{\delta}$. *Let* $w \in \Sigma$, $0 < r < r_0$, *and suppose that* u *is a non-negative* A*-harmonic function in* $B(w, 4r) \setminus \Sigma$, *continuous on* $B(w, 4r)$ *and* $u = 0$ *on* $\Sigma \cap B(w, 4r)$. *Then* u *has a representative in* $W^{1,p}(B(w, 4r))$ *with Hölder continuous partial derivatives in* $B(w, 4r) \setminus \Sigma$. *Furthermore, there exists* $\hat{\sigma} \in (0, 1]$, *depending only on* p, n, m, α, β, γ, *such that if* $x, y \in B(\hat{w}, \hat{r}/2)$, $B(\hat{w}, 4\hat{r}) \subset B(w, 4r) \setminus \Sigma$, *then*

$$(i)\ c^{-1} |\nabla u(x) - \nabla u(y)| \leq (|x-y|/\hat{r})^{\hat{\sigma}} \max_{B(\hat{w},\hat{r})} |\nabla u| \leq c\, \hat{r}^{-1} (|x-y|/\hat{r})^{\hat{\sigma}} \max_{B(\hat{w},2\hat{r})} u.$$

Furthermore, if $A \in M_p(\alpha)$,

$$\frac{u(y)}{d(y, \Sigma)} \approx |\nabla u|(y), y \in B(\hat{w}, 3\hat{r}),$$

and if A *also satisfies Theorem 1.10 (a), then* u *has continuous second derivatives in* $B(\hat{w}, 3\hat{r})$, *and there exists* $\bar{c} \geq 1$, *depending only on the data such that*

$$(ii)\ \max_{B(\hat{w},\frac{\hat{r}}{2})} \sum_{i,j=1}^{n} |\hat{u}_{y_i y_j}| \leq \bar{c} \left(\hat{r}^{-n} \int_{B(\hat{w},\hat{r})} \sum_{i,j=1}^{n} |\hat{u}_{y_i y_j}|^2 \, dy \right)^{1/2} \leq \bar{c}^2\, \hat{u}(\bar{w})/d(\bar{w}, \Sigma)^2.$$

Proof. A proof of (i) can be found in [T]. (ii) follows from the first display, the added assumptions, and Schauder type estimates (see [GT]). ■

Lemma 3.7 *Assume* (3.1) *and that* $0 < \delta < \bar{\delta}$. *Let* $w \in \Sigma$, $0 < r < r_0$, *and suppose that* u *is a non-negative* A*-harmonic function in* $B(w, 4r) \setminus \Sigma$, *continuous on* $B(w, 4r)$ *and* $u = 0$ *on* $\Sigma \cap B(w, 4r)$. *There exists a unique finite positive Borel measure* μ *on* $\mathbb{R}^n$, *with support in* $\Sigma \cap B(w, 4r)$, *such that whenever* $\theta \in C_0^\infty(B(w, 4r))$, *then*

$$(i) \qquad \int \langle A(y, \nabla u(y)), \nabla \phi(y) \rangle \, dy = -\int \phi \, d\mu.$$

Moreover, there exists $c = c(p, n, m, \alpha, \beta, \gamma)$, $1 \leq c < \infty$, *such that if* $\tilde{r} = r/c$, *then*

$$(ii) \qquad c^{-1} r^{p-n} \mu(\Sigma \cap B(w, \tilde{r})) \leq (u(a_{\tilde{r}}(w)))^{p-1} \leq c\, r^{p-n}\, \mu(\Sigma \cap B(w, \tilde{r}/2)).$$

Proof. See [KZ]. ■

3.2 Technical lemmas

To start this section, assume that $1 \leq m \leq n-2$. Given $0 \in \mathbb{R}^m \times \mathbb{R}^{n-m}$, r_1, r_2, $0 < r_1, r_2 < \infty$, we let

$$C_{r_1,r_2}(0) = \{y = (y', y'') : \ |y'| < r_1, \ |y''| < r_2\}.$$

If $r_1 = r_2 = r$ we simply write $C_r(0)$. Given $w \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ we assume that $\Sigma = \Lambda_m(w)$. Let T be the composition of a translation and a rotation, which maps $0 \in \mathbb{R}^n$ to w and $\{(y', y'') \in \mathbb{R}^m \times \mathbb{R}^{n-m}, \ y'' = 0\}$ to Σ. Based on T we let

$$C_{r_1,r_2}(w) = T(C_{r_1,r_2}(0)), \, C_r(w) = T(C_r(0)). \tag{3.4}$$

Furthermore, we let, whenever $0 < r_1 < \infty$,

$$\Sigma_{r_1}(w) = T(\{y = (y', y'') : \ |y'| < r_1, \ y'' = 0\}). \tag{3.5}$$

Lemma 3.8 *Let p, $n - m < p < \infty$, $1 \leq m \leq n-2$, and assume that $A_1, A_2 \in M_p(\alpha, \beta, \gamma)$ with*

$$|A_1(y, \eta) - A_2(y, \eta)| \leq \epsilon |\eta|^{p-1} \text{ whenever } y \in C_1(0),$$

for some $0 < \epsilon < 1/2$. Let u_2 be a non-negative A_2-harmonic function in $C_1(0) \setminus \Sigma_1(0)$, continuous on the closure $C_1(0) \setminus \Sigma_1(0)$, and with $u_2 = 0$ on $\Sigma_1(0)$. Furthermore, let u_1 be the A_1-harmonic function in $C_{1/2}(0) \setminus \Sigma_{1/2}(0)$ which is continuous on the closure of $C_{1/2}(0) \setminus \Sigma_{1/2}(0)$ and which coincides with u_2 on $\partial(C_{1/2}(0) \setminus \Sigma_{1/2}(0))$. Then there exist, given $\rho \in (0, 1/16)$, c, $\tilde{c}$, θ, and τ, all depending only on $p, n, \alpha, \beta, \gamma$, such that

$$|u_2(y) - u_1(y)| \leq c\epsilon^\theta u_2(a_{1/2}(w)) \leq \tilde{c}\epsilon^\theta \rho^{-\tau} u_2(y) \text{ whenever } y \in C_{1/4}(0) \setminus C_{1/4,\rho}(0).$$

Proof. The statement of the lemma and its proof is similar to Lemma 3.1 in [LLuN] but we here include a proof for completion. To start with we observe that the existence and uniqueness of u_1, as stated in the lemma and given u_2, follows from Lemma 3.2. Next we note that if $y \in \mathbb{R}^n, \lambda \in \mathbb{R}^n, \xi \in \mathbb{R}^n \setminus \{0\}$, and $A \in M_p(\alpha, \beta, \gamma)$, then

$$A_i(y, \lambda) - A_i(y, \xi) = \sum_{j=1}^{n} (\lambda_j - \xi_j) \int_0^1 \frac{\partial A_i}{\partial \eta_j}(y, t\lambda + (1-t)\xi) dt \tag{3.6}$$

for $i \in \{1, .., n\}$. Using (3.6) and Definition 1.1 we see that

$$c^{-1} (|\lambda| + |\xi|)^{p-2} |\lambda - \xi|^2 \ \leq \ \langle A(y, \lambda) - A(y, \xi), \lambda - \xi \rangle \ \leq c\, (|\lambda| + |\xi|)^{p-2} |\lambda - \xi|^2. \tag{3.7}$$

In particular, using (3.7) we deduce that if

$$I = \int_{C_{1/2}(0) \setminus \Sigma_{1/2}(0)} |\nabla u_2 - \nabla u_1|^p dy,$$

then,

$$I \le cJ,\ J := \int\limits_{C_{1/2}(0)\setminus\Sigma_{1/2}(0)} \langle A_1(y,\nabla u_1(y)) - A_1(y,\nabla u_2(y)), \nabla u_2(y) - \nabla u_1(y)\rangle dy, \tag{3.8}$$

since $p \ge 2$. As $\nabla\cdot(A_1(y,\nabla u_1(y))) = 0 = \nabla\cdot(A_2(y,\nabla u_2(y)))$ whenever $y \in C_{1/2}(0)\setminus\Sigma_{1/2}(0)$, and as $\theta = u_2 - u_1 \in W_0^{1,p}(C_{1/2}(0)\setminus\Sigma_{1/2}(0))$, we see from the definition of J in (3.8) that

$$J = \int\limits_{C_{1/2}(0)\setminus\Sigma_{1/2}(0)} \langle A_2(y,\nabla u_2(y)) - A_1(y,\nabla u_2(y)), \nabla u_2(y) - \nabla u_1(y)\rangle dy. \tag{3.9}$$

Hence, using (3.8), (3.9), the assumption on the difference $|A_1(y,\eta) - A_2(y,\eta)|$ stated in the lemma and Hölder's inequality, we can conclude that

$$I \le c\epsilon \int\limits_{C_{1/2}(0)\setminus\Sigma_{1/2}(0)} (|\nabla u_1|^p + |\nabla u_2|^p)dx. \tag{3.10}$$

Now from the observation above (3.9), (3.7) with $\xi = 0$, and Hölder's inequality we see that

$$\begin{aligned}\int\limits_{C_{1/2}(0)\setminus\Sigma_{1/2}(0)} |\nabla u_1|^p dy &\le c \int\limits_{C_{1/2}(0)\setminus\Sigma_{1/2}(0)} \langle A_1(y,\nabla u_1(y)),\ \nabla u_2(y)\rangle dy \\ &\le (1/2) \int\limits_{C_{1/2}(0)\setminus\Sigma_{1/2}(0)} |\nabla u_1|^p dy + c \int\limits_{C_{1/2}(0)\setminus\Sigma_{1/2}(0)} |\nabla u_2|^p dy.\end{aligned}$$

Thus,

$$\int\limits_{C_{1/2}(0)\setminus\Sigma_{1/2}(0)} |\nabla u_1|^p dy \le c \int\limits_{C_{1/2}(0)\setminus\Sigma_{1/2}(0)} |\nabla u_2|^p dy. \tag{3.11}$$

In particular, using (3.11) in (3.10), and Lemma 3.1, Lemma 3.3, Lemma 3.5, for u_2, we obtain

$$I \le c\epsilon(u_2(e_n/2))^p. \tag{3.12}$$

Next using the Poincáre inequality for functions in $C_{1/2}(0)\setminus\Sigma_{1/2}(0))$ we deduce from (3.12) that

$$\int\limits_{C_{1/2}(0)\setminus\Sigma_{1/2}(0)} |u_2 - u_1|^p\, dy \le c \int\limits_{C_{1/2}(0)\setminus\Sigma_{1/2}(0)} |\nabla u_2 - \nabla u_1|^p\, dy \le c\epsilon(u_2(e_n/2))^p. \tag{3.13}$$

In the following we let $\eta = 1/(p+2)$ and we introduce the sets

$$E = \{y \in C_{1/2}(0) :\ |u_2(y) - u_1(y)| \le \epsilon^\eta u_2(e_n/2)\},\ F = C_{1/2}(0)\setminus E. \tag{3.14}$$

Moreover, for a measurable function f defined on $C_{1/2}(0)$ we introduce, whenever $y \in C_{1/2}(0)$, the Hardy-Littlewood maximal function

$$M(f)(y) := \sup_{\{r>0,\ C_r(y)\subset C_{1/2}(0)\setminus\Sigma_{1/2}(0)\}} \frac{1}{|C_r(y)|} \int\limits_{C_r(y)} |f(z)|dz. \tag{3.15}$$

Let

$$G = \{y \in C_{1/2}(0) :\ M(\chi_F)(y) \leq \epsilon^\eta\}, \tag{3.16}$$

where χ_F is the indicator function for the set F. Then using weak (1,1)-estimates for the Hardy-Littlewood maximal function, (3.13) and (3.14) we see that

$$|C_{1/2}(0) \setminus G| \leq c\epsilon^{-\eta}|F| \leq c\epsilon^{-\eta}\epsilon^{-p\eta}\epsilon = c\epsilon^\eta, \tag{3.17}$$

by our choice for η. Also, using continuity of $u_2(y) - u_1(y)$ we find for $y \in G$ that

$$|u_2(y) - u_1(y)| = \lim_{r\to 0} \frac{1}{|B(y,r)|} \int\limits_{B(y,r)} |u_2(z) - u_1(z)|dz \leq c\epsilon^\eta u_2(e_n/2). \tag{3.18}$$

If $y \in C_{1/4}(0) \setminus G$, then from (3.17) we see there exists $\hat{y} \in G$ such that $|y - \hat{y}| \leq c(n)\epsilon^{\eta/n}$. Using Lemma 3.1 and Lemma 3.3 we hence get that

$$\begin{aligned} |u_2(y) - u_1(y)| &\leq |u_2(\hat{y}) - u_1(\hat{y})| + |u_2(y) - u_2(\hat{y})| + |u_1(y) - u_1(\hat{y})| \\ &\leq c(\epsilon^\eta + \epsilon^{\sigma\eta/n})u_2(e_n/2). \end{aligned} \tag{3.19}$$

This completes the proof of the first inequality stated in Lemma 3.8. Finally, using the Harnack inequality we see that there exists $\tau \geq 1$, depending only on the data such that $u_2(e_n/2) \leq c\rho^{-\tau}u_2(y)$ whenever $y \in C_{1/4}(0) \setminus C_{1/4,\rho}(0)$. ■

Lemma 3.9 *Let $O \subset \mathbb{R}^n$ be an open set, suppose $1 < p < \infty$, and that $A_1, A_2 \in M_p(\alpha, \beta, \gamma)$. Also, suppose that $\hat{u}_1, \hat{u}_2$ are non-negative functions in O, that $\hat{u}_1$ is A_1-harmonic in O and that $\hat{u}_2$ is A_2-harmonic in O. Let $\tilde{a} \geq 1, y \in O$ and assume that*

$$\frac{1}{\tilde{a}}\frac{\hat{u}_1(y)}{d(y,\partial O)} \leq |\nabla \hat{u}_1(y)| \leq \tilde{a}\frac{\hat{u}_1(y)}{d(y,\partial O)}.$$

Let $\tilde{\epsilon}^{-1} = (c\tilde{a})^{(1+\hat{\sigma})/\hat{\sigma}}$, where $\hat{\sigma}$ is as in Lemma 3.6. If

$$(1-\tilde{\epsilon})\hat{L} \leq \frac{\hat{u}_2}{\hat{u}_1} \leq (1+\tilde{\epsilon})\hat{L} \text{ in } B(y, \tfrac{1}{100}d(y,\partial O))$$

for some $\hat{L}, 0 < \hat{L} < \infty$, then for $c = c(p, n, \alpha, \beta, \gamma)$ suitably large,

$$\frac{1}{c\tilde{a}}\frac{\hat{u}_2(y)}{d(y,\partial O)} \leq |\nabla \hat{u}_2(y)| \leq c\tilde{a}\frac{\hat{u}_2(y)}{d(y,\partial O)}.$$

Proof. This is Lemma 3.18 in [LLuN] and we refer the reader to [LLuN] for the proof. ■

4 Linear degenerate elliptic equations

Let $w \in \mathbb{R}^n, r > 0$, and let $\lambda(y)$ be a real valued, non-negative, Lebesgue measurable function defined almost everywhere on $B(w, 2r)$. $\lambda(y)$ is said to belong to the class $A_2(B(w,r))$ if there exists a constant Γ such that

$$\tilde{r}^{-2n} \int\limits_{B(\tilde{w},\tilde{r})} \lambda \, dy \cdot \int\limits_{B(\tilde{w},\tilde{r})} \lambda^{-1} dy \leq \Gamma, \tag{4.1}$$

whenever $\tilde{w} \in B(w,r)$ and $0 < \tilde{r} \leq r$. If $\lambda(y)$ belongs to the class $A_2(B(w,r))$ then λ is referred to as an $A_2(B(w,r))$-weight. The smallest Γ such that (4.1) holds is referred to as the constant of the weight. Throughout the section we assume that

$$\begin{aligned} &(i) && 1 \leq m \leq n-2, \\ &(ii) && \Sigma \subset \mathbb{R}^n \text{ is a closed set and } \Sigma \text{ is } (m, r_0, \delta)\text{-Reifenberg flat (in } \mathbb{R}^n) \text{ for some } r_0, \delta > 0, \\ &(iii) && 0 < \delta < \delta_0 \text{ where } \delta_0 \text{ is as stated in Lemma 2.1, Lemma 2.4, and Lemma 2.5.} \end{aligned} \tag{4.2}$$

We let $w \in \Sigma$, $0 < r < r_0$, and we consider the operator

$$\hat{L} = \sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}\left(\hat{a}_{ij}(y)\frac{\partial}{\partial y_j}\right), \tag{4.3}$$

in $B(w, 16r) \setminus \Sigma$. We assume that the coefficients $\{\hat{a}_{ij}(y)\}$ are bounded, Lebesgue measurable functions defined almost everywhere in $B(w, 16r)$ and that

$$c^{-1}\lambda(y)|\xi|^2 \leq \sum_{i,j=1}^{n} \hat{a}_{ij}(y)\xi_i\xi_j \leq c|\xi|^2\lambda(y), \tag{4.4}$$

for almost every $y \in B(w, 16r)$, where $\lambda \in A_2(B(w, 8r))$. By definition $\hat{L}$ is a degenerate elliptic operator (in divergence form) in $B(w, 8r)$ with ellipticity measured by the function λ and c. If $O \subset B(w, 8r) \setminus \Sigma$ is open, then we let $\tilde{W}^{1,2}(O)$ be the weighted Sobolev space of equivalence classes of functions v with distributional gradient ∇v and norm

$$\|v\tilde{\|}_{1,2}^2 = \int\limits_O v^2 \lambda dy + \int\limits_O |\nabla v|^2 \lambda dy < \infty. \tag{4.5}$$

Let $\tilde{W}_0^{1,2}(O)$ be the closure of $C_0^\infty(O)$ in the norm $\tilde{W}^{1,2}(O)$. We say that v is a weak solution to $\hat{L}v = 0$ in O provided $v \in \tilde{W}^{1,2}(O)$ and

$$\int\limits_O \sum_{i,j} \hat{a}_{ij} v_{y_i} \phi_{y_j} dy = 0, \tag{4.6}$$

whenever $\phi \in C_0^\infty(O)$. $u \in \tilde{W}^{1,2}(O)$ is called a subsolution of $\hat{L}$ if (4.6) holds with $=$ replaced by $\leq$ for all $\phi \in \tilde{W}^{1,2}(O)$ such that $\phi \geq 0$. u is called a supersolution if $-u$ is a subsolution.

For the proof of the following lemma we refer to [FKS].

Lemma 4.1 *Let $w \in \Sigma$, $0 < r < r_0$ and let λ be an $A_2(B(w, 8r))$-weight with constant Γ. Suppose that v is a positive weak solution to $Lv = 0$ in $B(w, 4r) \setminus \Sigma$. Then there exists a constant $c = c(n, \Gamma)$, $1 \leq c < \infty$ such that if $\hat{w} \in \mathbb{R}^n$, $0 < \hat{r}$, $B(\hat{w}, 2\hat{r}) \subset B(w, 4r) \setminus \Sigma$, then*

$$(i) \qquad \hat{r}^2 \int_{B(\hat{w},\hat{r}/2)} |\nabla v|^2 \lambda dy \leq c \int_{B(\hat{w},\hat{r})} |v|^2 \lambda dy,$$

$$(ii) \qquad \max_{B(\hat{w},\hat{r})} v \leq c \min_{B(\hat{w},\hat{r})} v.$$

Furthermore, there exists $\alpha = \alpha(n, \Gamma) \in (0, 1)$ such that if $x, y \in B(\hat{w}, \hat{r})$, then

$$(iii) \qquad |v(x) - v(y)| \leq c \left(\frac{|x-y|}{\hat{r}} \right)^{\alpha} \max_{B(\hat{w},2\hat{r})} v.$$

Definition 4.2 *Let $w \in \mathbb{R}^n$, $0 < r < r_0$, let $O \subset B(w, 8r)$ be open, let K be a compact subset of O and assume that λ is a real valued, non-negative, Lebesgue measurable function defined almost everywhere on $B(w, 8r)$. We define,*

$$Cap_{2,\lambda}(K, O) = \inf\{ \int_O |\nabla \phi|^2 \lambda \, dy \;\; \phi \in C_0^\infty(O), \;\; \phi \geq 1 \;\, in \; K \}.$$

Then $Cap_{2,\lambda}(K, O)$ is referred to as the $(2, \lambda)$-capacity K relative O. The $(2, \lambda)$-capacity of an arbitrary set $E \subseteq O$ is defined by

$$Cap_{2,\lambda}(E, O) = \inf_{E \subset G \subset O, \; G \; open} \;\; \sup_{K \subset G, \; K \; compact} Cap_{2,\lambda}(K, O). \tag{4.7}$$

Definition 4.3 *Let $\Sigma \subset \mathbb{R}^n$ be a closed set, let $w \in \Sigma$, $0 < r < \infty$, assume that λ is a real valued, non-negative, Lebesgue measurable function defined almost everywhere on $B(w, 8r)$. Also assume there exists a constant $\eta > 0$ such that*

$$\frac{Cap_{2,\lambda}(\Sigma \cap B(\hat{w}, \hat{r}), B(\hat{w}, 2\hat{r}))}{Cap_{2,\lambda}(B(\hat{w}, \hat{r}), B(\hat{w}, 2\hat{r}))} \geq \eta$$

whenever $\hat{w} \in \Sigma \cap B(w, 4r)$, $0 < \hat{r} < r$. We then say that $\Sigma \cap B(w, 4r)$ is uniformly $(2, \lambda)$-thick with constant η.

Lemma 4.4 *Let $w \in \Sigma$, $0 < r < r_0$, and suppose that λ is an $A_2(B(w, 8r))$-weight. Furthermore, assume (4.2) and that $\Sigma \cap B(w, 4r)$ is uniformly $(2, \lambda)$-thick for some constant $\eta > 0$. Then, given $f \in \tilde{W}^{1,2}(B(w, 4r))$ there exists a unique weak solution $u \in \tilde{W}^{1,2}(B(w, 4r) \setminus \Sigma)$ to $\hat{L}u = 0$ in $B(w, 4r) \setminus \Sigma$ such that $u - f \in \tilde{W}_0^{1,2}(B(w, 4r) \setminus \Sigma)$. Furthermore, let $u, v \in \tilde{W}_{loc}^{1,2}(B(w, 4r) \setminus \Sigma)$ be a $\hat{L}$-supersolution and a $\hat{L}$-subsolution in $B(w, 4r) \setminus \Sigma$, respectively. If $\inf\{u - v, 0\} \in \tilde{W}_0^{1,2}(B(w, 4r) \setminus \Sigma)$, then $u \geq v$ a.e in $B(w, 4r) \setminus \Sigma$. Finally, every point $\hat{w} \in \Sigma \cap B(w, 4r)$ is regular for the continuous Dirichlet problem for $\hat{L}u = 0$.*

Proof. The proof is essentially identical to the proof of Lemma 3.2, see also [FJK]. ■

The following lemmas, Lemma 4.5 and Lemma 4.6, are tailored to our situation and based on results in [FKS], [FJK] and [FJK1]. We note that these authors assumed $\hat{L}$ to be symmetric, i.e., $\hat{a}_{ij} = \hat{a}_{ji}, 1 \leq i, j \leq n$, but, as pointed out in [LLuN], this assumption was not needed in the proof of these lemmas.

Lemma 4.5 *Let $w \in \Sigma$, $0 < r < r_0$, and suppose that λ is an $A_2(B(w, 8r))$-weight. Let v be a positive solution to $\hat{L}v = 0$ in $B(w, 2r) \setminus \Sigma$, continuous on $B(w, 2r)$ and $v = 0$ on $\Sigma \cap B(w, 2r)$. Furthermore, assume (4.2) and that $\Sigma \cap B(w, 4r)$ is uniformly $(2, \lambda)$-thick for some constant $\eta > 0$. Then there exists $c = c(n, \Gamma, \eta)$, $1 \leq c < \infty$, such that the following holds with $\tilde{r} = r/c$.*

$$(i) \qquad r^2 \int\limits_{B(w,r/2)} |\nabla v|^2 \lambda dy \leq c \int\limits_{B(w,r)} |v|^2 \lambda dy,$$

$$(ii) \qquad \max_{B(w,\tilde{r})} v \leq c v(a_{\tilde{r}}(w)).$$

Moreover, there exists $\alpha = \alpha(n, \Gamma, \eta) \in (0, 1)$ such that if $x, y \in B(w, \tilde{r})$, then

$$(iii) \qquad |v(x) - v(y)| \leq c \left(\tfrac{|x-y|}{r} \right)^{\alpha} \max_{B(w,2\tilde{r})} v.$$

Lemma 4.6 *Let $w \in \Sigma$, $0 < r < r_0$, and suppose that λ is an $A_2(B(w, 8r))$-weight. Also let v_1, v_2, be two positive solution to $\hat{L}v = 0$ in $B(w, 2r) \setminus \Sigma$, continuous on $B(w, 2r)$ and $v_1 = 0 = v_2$ on $\Sigma \cap B(w, 2r)$. Furthermore, assume (4.2) and that $\Sigma \cap B(w, 4r)$ is uniformly $(2, \lambda)$-thick for some constant $\eta > 0$. Then there exist $c = c(n, \Gamma, \eta)$, $1 \leq c < \infty$, and $\alpha = \alpha(n, \Gamma, \eta) \in (0, 1)$, such that*

$$\left| \log \frac{v_1(y_1)}{v_2(y_1)} - \log \frac{v_1(y_2)}{v_2(y_2)} \right| \leq c \left(\frac{|y_1 - y_2|}{r} \right)^{\alpha},$$

whenever $y_1, y_2 \in B(w, r/c) \setminus \Sigma$.

4.1 A-harmonic functions: linearization and weighted capacity

Recall that we are assuming (3.1) and $0 < \delta < \bar{\delta}$ so that also (3.2) holds, see (4.2). Assume that $\hat{u}, \hat{v}$ are two positive A-harmonic functions in $B(w, 4r) \setminus \Sigma$, continuous on $B(w, 4r)$ and satisfying $\hat{u} = 0 = \hat{v}$ on $\Sigma \cap B(w, 4r)$. We define

$$e(y) = \hat{u}(y) - \hat{v}(y) \text{ whenever } y \in B(w, 2r), \tag{4.8}$$

and put

$$u(y, \tau) = \tau \hat{u}(y) + (1 - \tau)\hat{v}(y) \text{ whenever } y \in B(w, 2r) \text{ and } \tau \in [0, 1]. \tag{4.9}$$

Clearly, $e(y) = u(y, 1) - u(y, 0)$ and it follows from (3.6) that e is a weak solution to

$$\hat{L}e := \sum_{i,j=1}^{n} \frac{\partial}{\partial y_i} \left(\hat{a}_{ij}(y) \frac{\partial}{\partial y_j} \right) = 0 \text{ in } B(w, 2r) \setminus \Sigma, \tag{4.10}$$

where, whenever $y \in B(w, 2r) \setminus \Sigma$ and $1 \leq i, j \leq n$,

$$\begin{aligned} \hat{a}_{ij}(y) &= \int_0^1 a_{ij}(y, \tau) d\tau, \\ a_{ij}(y, \tau) &= \frac{\partial A_i}{\partial \eta_j}(\nabla u(y, \tau)). \end{aligned} \tag{4.11}$$

In particular, using the structure assumptions in Definition 1.1, we observe from (4.10), (4.11), that $e = \hat{u} - \hat{v}$ is the solution to a divergence form PDE with ellipticity constant, at $y \in B(w, 2r) \setminus \Sigma$, estimated by

$$\min\{p-1, 1\}|\xi|^2 \lambda(y) \leq \sum_{i,j=1}^{n} \hat{a}_{ij}(y)\xi_i\xi_j \leq \max\{p-1, 1\}|\xi|^2\lambda(y), \tag{4.12}$$

whenever $\xi \in \mathbb{R}^n$. Here,

$$\lambda(y) = \int_0^1 |\nabla u(y, \tau)|^{p-2} d\tau \;\; \approx \;\; \Big(|\nabla \hat{u}(y)| + |\nabla \hat{v}(y)|\Big)^{p-2}, \tag{4.13}$$

whenever $y \in B(w, 2r) \setminus \Sigma$. In (4.13) $\approx$ means that the constants of proportionality only depend on p, n, α. We prove the following lemma.

Lemma 4.7 *Assume* (3.1) *and that* $0 < \delta < \bar{\delta}$. *Also suppose that* $\hat{u}, \hat{v}$ *are two positive* A-*harmonic functions in* $B(w, 4r) \setminus \Sigma$, *continuous on* $B(w, 4r)$ *and satisfying* $\hat{u} = 0 = \hat{v}$ *on* $\Sigma \cap B(w, 4r)$. *Let* $\hat{\lambda} = \hat{\lambda}(y) = (|\nabla \hat{u}(y)| + |\nabla \hat{v}(y)|)^{p-2}$ *and suppose that* $\hat{\lambda} \neq 0$, *almost everywhere in* $B(w, 4r)$. *There exists* $c = c(n, m) \geq 1$, *such that if* $\tilde{r} = r/c$ *then* $\Sigma \cap B(w, 4\tilde{r})$ *is uniformly* $(2, \hat{\lambda})$-*thick for some constant* $\eta = \eta(p, n, m, \alpha, \beta, \gamma) > 0$.

Proof. In the following we simply choose $c = c(n, m) \geq 1$, $\tilde{r} = r/c$, such that if $\hat{w} \in \Sigma \cap B(w, 4\tilde{r})$, $0 < \hat{r} < \tilde{r}$, then $a_{\hat{r}}(\hat{w})$ and the point realizing $\sup_{B(\hat{w}, 4\hat{r})} \hat{u}$ can be connected with a Harnack chain contained in $B(w, r)$ and of length independent of $\hat{w}$, $\hat{r}$. Using this choice for $\tilde{r}$ we want to prove, for $\tilde{r}$ and η as stated, that

$$\frac{\mathrm{Cap}_{2,\hat{\lambda}}(\Sigma \cap B(\hat{w}, \hat{r}), B(\hat{w}, 2\hat{r}))}{\mathrm{Cap}_{2,\hat{\lambda}}(B(\hat{w}, \hat{r}), B(\hat{w}, 2\hat{r}))} \geq \eta,$$

whenever $\hat{w} \in \Sigma \cap B(w, 4\tilde{r})$, $0 < \hat{r} < \tilde{r}$. By scaling we can assume that $\hat{w} = 0$, $\hat{r} = 1$ and hence we want to bound the quotient

$$\frac{\mathrm{Cap}_{2,\hat{\lambda}}(\Sigma \cap B(0, 1), B(0, 2))}{\mathrm{Cap}_{2,\lambda}(B(0, 1), B(0, 2))}, \tag{4.14}$$

from below, with a positive constant depending at most on only on $p, n, m, \alpha, \beta, \gamma$. Furthermore, in the following we can, without loss of generality, assume that

$$\max\{\hat{u}(a_1(0)), \hat{v}(a_1(0))\} = \hat{u}(a_1(0)).$$

Let now $\phi \in C_0^\infty(B(0, 2))$, $\phi \geq 1$ on $\Sigma \cap B(0, 1)$, be an admissible test function used in the definition of $\mathrm{Cap}_{2,\hat{\lambda}}(\Sigma \cap B(0, 1), B(0, 2))$. Let $\hat{\mu}$ be the measure corresponding to $\hat{u}$ as in Lemma 3.7. Then

$$\int \langle A(y, \nabla \hat{u}(y)), \nabla \phi(y) \rangle \, dy \; = -\int \phi \, d\hat{\mu}. \tag{4.15}$$

In particular,

$$\hat{\mu}(B(0, 1)) \leq \int |\langle A(y, \nabla \hat{u}(y)), \nabla \phi(y) \rangle| \, dy \leq c \int |\nabla \hat{u}|^{p-1} |\nabla \phi| \, dy, \tag{4.16}$$

and hence, simply using the Hölder inequality, we find that

$$\hat{\mu}(B(0,1)) \leq c\Big(\int |\nabla\phi|^2\hat{\lambda}(y)\,dy\Big)^{1/2}\Big(\int_{B(0,2)} |\nabla\hat{u}|^p\,dy\Big)^{1/2}.$$

Next, applying Lemma 3.1, the Harnack inequality and Lemma 3.5, we have

$$\Big(\int_{B(0,2)} |\nabla\hat{u}|^p\,dy\Big)^{1/2} \leq \hat{u}(a_1(0))^{p/2}.$$

Furthermore, using Lemma 3.7 (*ii*) and arguing as above we see that $\hat{\mu}(B(0,1)) \approx \hat{u}(a_1(0))^{p-1}$. In particular, using this fact and the above displays we deduce that

$$\hat{u}(a_1(0))^{p-2} \leq c\Big(\int |\nabla\phi|^2\hat{\lambda}(y)\,dy\Big). \tag{4.17}$$

As ϕ is an arbitrary admissible test function used in the definition of $\mathrm{Cap}_{2,\hat{\lambda}}(\Sigma\cap B(0,1), B(0,2))$, we conclude that

$$\hat{u}(a_1(0))^{p-2} \leq c\mathrm{Cap}_{2,\hat{\lambda}}(\Sigma\cap B(0,1), B(0,2)), \tag{4.18}$$

and this is a lower bound for $\mathrm{Cap}_{2,\hat{\lambda}}(B(0,1), B(0,2))$.

Next, to establish an upper bound for $\mathrm{Cap}_{2,\hat{\lambda}}(B(0,1), B(0,2))$ we simply note that

$$\begin{aligned}\int_{B(0,2)} |\nabla\phi|^2\hat{\lambda}(y)dy &= \int_{B(0,2)} |\nabla\phi|^2(|\nabla\hat{u}|+|\nabla\hat{v}|)^{p-2}dy\\ &\leq c\Big(\int_{B(0,2)} (|\nabla\hat{u}|+|\nabla\hat{v}|)^p\,dy\Big)^{1-2/p}\Big(\int_{B(0,2)} |\nabla\phi|^p\,dy\Big)^{2/p}. \end{aligned}\tag{4.19}$$

Choosing ϕ as the p-capacitary function for $B(0,2)\setminus B(0,1)$ we can therefore conclude that

$$\begin{aligned}\mathrm{Cap}_{2,\hat{\lambda}}(B(0,1), B(0,2)) &\leq c\Big(\int_{B(0,2)} (|\nabla\hat{u}|+|\nabla\hat{v}|)^p\,dy\Big)^{1-2/p}\\ &\leq c\Big(\max\{\hat{u}(a_1(0)),\hat{v}(a_1(0))\}\Big)^{p-2} = c\hat{u}(a_1(0))^{p-2}. \end{aligned}\tag{4.20}$$

(4.18) and (4.20) now give the bound from below for the quotient in (4.14) and hence the proof of Lemma 4.7 is complete. ■

4.2 A-harmonic functions: estimates based on linearization

In the following we again assume (3.1) and $0<\delta<\bar{\delta}$ so that also (3.2) holds. We also put

$$\tilde{\theta} = 1 \text{ when } m=1 \text{ and } \tilde{\theta}=\lambda, \text{ as in Theorem 1.10, when } 2\leq m\leq n-2. \tag{4.21}$$

Let $\hat{u},\hat{v}$, $\hat{\lambda}=\hat{\lambda}_{\hat{u},\hat{v}}$, be as in the statement of Lemma 4.7. Then, by Lemma 4.7 we see that there exists $c=c(n,m)\geq 1$, such that if $\varrho_0=r/c$, then $\Sigma\cap B(w,4\varrho_0)$ is uniformly $(2,\hat{\lambda})$-thick for some constant $\eta = \eta(p, n, m, \alpha, \beta, \gamma) > 0$. The analysis in this subsection is based on the following assumption.

Assumption 1. There exists $c_1 = c_1(p, n, m, \alpha, \beta, \gamma, \tilde{\theta}) \geq 1$ such that if $\varrho_1 = \varrho_0/c_1$, $a, b \in [0, \infty)$, and $\hat{u}, \hat{v}$ are as above, then $\hat{\lambda}(y) := \hat{\lambda}(y, a, b, \hat{u}, \hat{v}) = (a|\nabla \hat{u}(y)| + b|\nabla \hat{v}(y)|)^{p-2}$ is an $A_2(B(w, 4\varrho_1))$-weight with constant $\Gamma = \Gamma(p, n, m, \alpha, \beta, \gamma, \tilde{\theta})$.

Lemma 4.8 *Assume* (3.1), $0 < \delta < \bar{\delta}$, *and Assumption 1. Let* $\hat{u}, \hat{v}$ *and* ϱ_1 *be as in Lemma 4.7 with* $\hat{v} \leq \hat{u}$. *There exists* $c \geq 1$, $c = c(p, n, m, \alpha, \beta, \gamma, \Gamma)$ *such that if* $\varrho_2 = \varrho_1/c$, *then*

$$c^{-1} \frac{\hat{u}(a_{\varrho_2}(w)) - \hat{v}(a_{\varrho_2}(w))}{\hat{v}(a_{\varrho_2}(w))} \leq \frac{\hat{u}(y) - \hat{v}(y)}{\hat{v}(y)} \leq c \frac{\hat{u}(a_{\varrho_2}(w)) - \hat{v}(a_{\varrho_2}(w))}{\hat{v}(a_{\varrho_2}(w))},$$

whenever $y \in B(w, \varrho_2) \setminus \Sigma$.

Proof. We first prove the left hand inequality in Lemma 4.8. To do so we show the existence of T, $1 \leq T < \infty$, and $\hat{c} \geq 1$, such that if $\varrho_2 = \varrho_1/\hat{c}$, and if

$$e(y) = T\left(\frac{\hat{u}(y) - \hat{v}(y)}{\hat{u}(a_{\varrho_1}(w)) - \hat{v}(a_{\varrho_1}(w))}\right) - \frac{\hat{v}(y)}{\hat{v}(a_{\varrho_1}(w))}, \tag{4.22}$$

for $y \in B(w, \varrho_1) \setminus \Sigma$, then

$$e(y) \geq 0 \text{ whenever } y \in B(w, 2\varrho_2) \setminus \Sigma. \tag{4.23}$$

To do this, we initially allow $T, \hat{c} \geq 1$ to vary in (4.22). $T, \hat{c}$, are then fixed near the end of the argument. Put

$$u'(y) = \frac{T\, \hat{u}(y)}{\hat{u}(a_{\varrho_1}(w)) - \hat{v}(a_{\varrho_1}(w))},$$

$$v'(y) = \frac{T\, \hat{v}(y)}{\hat{u}(a_{\varrho_1}(w)) - \hat{v}(a_{\varrho_1}(w))} + \frac{\hat{v}(y)}{\hat{v}(a_{\varrho_1}(w))}.$$

Observe from (4.22) that $e = u' - v'$. Let L be defined as in (4.10) using u', v', instead of $\hat{u}, \hat{v}$, and let e_1, e_2 be the solutions to $Le_i = 0, i = 1, 2$, in $B(w, \varrho_1) \setminus \Sigma$, with continuous boundary values

$$e_1(y) = \frac{\hat{u}(y) - \hat{v}(y)}{\hat{u}(a_{\varrho_1}(w)) - \hat{v}(a_{\varrho_1}(w))},\ e_2(y) = \frac{\hat{v}(y)}{\hat{v}(a_{\varrho_1}(w))}, \tag{4.24}$$

whenever $y \in \partial(B(w, \varrho_1) \setminus \Sigma)$. Note that by construction, and by Lemma 4.7 and Lemma 4.4, that e_1, e_2 are well defined. Furthermore, now using Assumption 1 we see that Lemma 4.6 can be applied and we get, for some $c_+ \geq 1$ and $r_+ = \varrho_1/c_+$, that

$$c_+^{-1} \frac{e_1(a_{r_+}(w))}{e_2(a_{r_+}(w))} \leq \frac{e_1(y)}{e_2(y)} \leq c_+ \frac{e_1(a_{r_+}(w))}{e_2(a_{r_+}(w))} \tag{4.25}$$

whenever $y \in B(w, 2r_+) \setminus \Sigma$. We now put

$$\hat{c} = c_+,\ \varrho_2 = r_+,\ \text{and } T = \hat{c}\, \frac{e_2(a_{\varrho_2}(w))}{e_1(a_{\varrho_2}(w))},$$

and we observe from (4.25) that

$$Te_1(y) - e_2(y) \geq 0 \text{ whenever } y \in B(w, 2\varrho_2) \setminus \Sigma. \tag{4.26}$$

Let $\hat{e} = T\,e_1 - e_2$ and note from linearity of L that $\hat{e}, e$, both satisfy the same linear locally uniformly elliptic sub-elliptic PDE in $B(w, \varrho_1) \setminus \Sigma$ and also that these functions have the same continuous boundary values on $\partial(B(w, \varrho_1) \setminus \Sigma)$. Hence, using the maximum principle for the operator L, it follows that $e = \hat{e}$ and then, by (4.26), that $e(y) \geq 0$ in $B(w, 2\varrho_2) \setminus \Sigma$. To complete the proof of the left-hand inequality in Lemma 4.8 we prove that

$$T \leq c(p, n, m, \alpha, \beta, \gamma, \Gamma) = c(p, n, m, \alpha, \beta, \gamma, \tilde{\theta}). \tag{4.27}$$

To do this, let $\hat{L}$ denote the operator corresponding to $\hat{u} - \hat{v}$ and defined as in (4.10). Then from the Harnack inequality in Lemma 4.1 (ii) for $\hat{L}$, applied to $\hat{u} - \hat{v}$, and the definition of ϱ_2, we deduce the existence of $\zeta \in \partial B(w, \varrho_1) \setminus \Sigma$ with $d(\zeta, \Sigma) \geq r/c$ and such that $e_1 \geq c^{-1}$ on $\partial B(w, \varrho_1) \cap B(\zeta, d(\zeta, \Sigma)/4)$. Using this we get, essentially just using Lemma 4.5 (iii) and the Harnack inequality in Lemma 4.1 applied to the function e_1, that $e_1(a_{\varrho_2}(w)) \geq \bar{c}^{-1}$. Also from Lemma 3.5 and the Harnack inequality applied to $\hat{v}$ we get $e_2(a_{\varrho_2}(w)) \leq \bar{c}$ for some $\bar{c} = \bar{c}(p, n, m, \alpha, \beta, \gamma, \Gamma)$. Thus (4.27) is true and the proof of the left hand inequality in Lemma 4.8 is complete. To prove the right hand inequality in Lemma 4.8, one can proceed similarly and in this case one needs to prove, for e_1, e_2 as above, that $e_1(a_{\varrho_2}(w)) \leq \bar{c}$ and $e_2(a_{\varrho_2}(w)) \geq \bar{c}$. The proof of the second inequality follows, as above, essentially from Lemma 4.5 (iii) and the Harnack inequality in Lemma 4.1 applied to the function e_2. The first inequality follows from Lemma 4.5 (iii), (ii) for $\hat{L}$, applied to $\hat{u} - \hat{v}$, and the Harnack inequality. This completes the proof of Lemma 4.8. ■

Lemma 4.9 *Assume* (3.1), $0 < \delta < \bar{\delta}$, *and Assumption 1. Let* $\hat{u}, \hat{v}$ *and* ϱ_1 *be as in Lemma 4.7. There exists* $c \geq 1$, $c = c(p, n, m, \alpha, \beta, \gamma, \tilde{\theta}, \Gamma)$ *such that if* $\varrho_2 = \varrho_1/c$, *then*

$$c^{-1}\frac{\hat{u}(a_{\varrho_2}(w))}{\hat{v}(a_{\varrho_2}(w))} \leq \frac{\hat{u}(y)}{\hat{v}(y)} \leq c\frac{\hat{u}(a_{\varrho_2}(w))}{\hat{v}(a_{\varrho_2}(w))},$$

whenever $y \in B(w, \varrho_2) \setminus \Sigma$.

Proof. Note that we do not assume $\hat{v} \leq \hat{u}$ in Lemma 4.9. Our proof is similar to the proof of Lemma 4.8. To prove the left hand inequality in Lemma 4.9 we set

$$e(y) = \frac{T\hat{u}(y)}{\hat{u}(a_{\varrho_1}(w))} - \frac{\hat{v}(y)}{\hat{v}(a_{\varrho_1}(w))} \text{ for } y \in B(w, \varrho_1) \setminus \Sigma, \tag{4.28}$$

and show that

$$e(y) \geq 0 \text{ whenever } y \in B(w, 2\varrho_2) \setminus \Sigma \tag{4.29}$$

where $T, \hat{c}, \varrho_2$ are as in Lemma 4.9. In this case we let

$$u'(y) = \frac{T\,\hat{u}(y)}{\hat{u}(a_{\varrho_1}(w))} \text{ and } v'(y) = \frac{\hat{v}(y)}{\hat{v}(a_{\varrho_1}(w))}.$$

Put $e = u' - v'$ and let L be defined as in (4.10) relative to u', v'. Repeating the argument in Lemma 4.8 from above (4.24), through the discussion below (4.27), we get the lefthand inequality in Lemma 4.9. To prove the righthand inequality in Lemma 4.9 we argue as above with $\hat{u}, \hat{v}$ interchanged. ■.

Lemma 4.10 *Assume* (3.1), $0 < \delta < \bar{\delta}$, *and Assumption 1. Let* $\hat{u}$, $\hat{v}$ *be as in Lemma 4.7 and let* ϱ_2 *be as in Lemma 4.8. Then there exist* $c \geq 1$, $c = c(p, n, m, \alpha, \beta, \gamma, \tilde{\theta}, \Gamma)$, *and* $\sigma = \sigma(p, n, m, \alpha, \beta, \tilde{\theta}, \gamma, \Gamma)$, $\sigma \in (0, 1)$ *such that if* $\varrho_3 = \varrho_2/c$, *then*

$$\left|\log \frac{\hat{u}(y_1)}{\hat{v}(y_1)} - \log \frac{\hat{u}(y_2)}{\hat{v}(y_2)}\right| \leq c \left(\frac{d(y_1, y_2)}{r}\right)^{\sigma},$$

whenever $y_1, y_2 \in B(w, \varrho_3) \setminus \Sigma$.

Proof. From Lemma 4.9, we have

$$c^{-1} \frac{\hat{u}(a_{\varrho_2}(w))}{\hat{v}(a_{\varrho_2}(w))} \leq \frac{\hat{u}(y)}{\hat{v}(y)} \leq c \frac{\hat{u}(a_{\varrho_2}(w))}{\hat{v}(a_{\varrho_2}(w))},$$

whenever $y \in B(w, \varrho_2) \setminus \Sigma$. Using this inequality we see that

$$\frac{\hat{u}(y_1)}{\hat{v}(y_1)} \leq c \frac{\hat{u}(y_2)}{\hat{v}(y_2)} \text{ whenever } y_1, y_2 \in B(w, \varrho_2) \setminus \Sigma. \tag{4.30}$$

Next if $\hat{w} \in B(w, \varrho_2/8) \cap \Sigma$, then we let

$$M(\rho) = \sup_{B(\hat{w},\rho)} \frac{\hat{u}}{\hat{v}} \text{ and } m(\rho) = \inf_{B(\hat{w},\rho)} \frac{\hat{u}}{\hat{v}},$$

whenever $0 < \rho < \varrho_2/2$. We also let $\operatorname{osc}(\rho) := M(\rho) - m(\rho)$ for $0 < \rho < \varrho_2/2$. Then, if ρ is fixed we can apply Lemma 4.8, with $m(\rho)\hat{v}$ replacing $\hat{v}$ in $B(w, \rho) \setminus \Sigma$ to find that if $c_* \geq 1$ is large enough and $\tilde{\rho} = \rho/c_*$, then

$$M(\tilde{\rho}) - m(\rho) \leq c_*(m(\tilde{\rho}) - m(\rho)).$$

Likewise applying Lemma 4.8 with $M(\rho)\hat{v}, \hat{u}$, playing the roles of $\hat{u}, \hat{v}$, respectively we obtain after multiplication by $\hat{u}/\hat{v}$ in view of (4.30) that

$$M(\rho) - m(\tilde{\rho}) \leq c_*(M(\rho) - M(\tilde{\rho})).$$

Adding these inequalities we obtain after some arithmetic that

$$\operatorname{osc}(\tilde{\rho}) \leq \frac{c_* - 1}{c_* + 1} \operatorname{osc}(\rho) \tag{4.31}$$

where c_* has the same dependence as c in Lemma 4.10. Iterating (4.31) we conclude that

$$\operatorname{osc}(s) \leq c(s/t)^{\phi} \operatorname{osc}(t) \text{ whenever } 0 < s < t \leq \varrho_2/2, \tag{4.32}$$

for some $\phi > 0, c \geq 1$. For slightly more details in the proof of (4.32), see (6.16)-(6.20) in [LLuN]. (4.32), (4.30), the arbitrariness of $\hat{w} \in B(w, \varrho_2/8) \cap \Sigma$ and the interior Hölder continuity-Harnack inequalities in Lemma 3.1 applied to $\hat{u}, \hat{v}$, are now easily seen to imply Lemma 4.10. ■

Next we consider the following alternative assumptions to Assumption 1 .

Assumption $1'$**.** Let $\hat u$, $\hat v$ be as in Lemma 4.7. Assume that there exists $\hat c_1 = \hat c_1(p,n,m,\alpha,\beta,\gamma,\tilde\theta) \geq 1$ such that if $\hat\varrho_1 = \varrho_0/\hat c_1$, then for $y \in B(w, 4\hat\varrho_1)\setminus\Sigma$:

$$\hat c_1^{-1}\frac{\tilde u(y)}{d(y,\Sigma)} \leq |\nabla \tilde u(y)| \leq \hat c_1 \frac{\tilde u(y)}{d(y,\Sigma)} \quad \text{for } \tilde u \in \{\hat u, \hat v\}\,.$$

Assumption $1''$**.** Let $\hat u$, $\hat v$ be as in Lemma 4.7. Assume that there exists $\breve c_1 = \breve c_1(p,n,m,\alpha,\beta,\gamma,\tilde\theta) \geq 1$ such that if $\tilde\varrho_1 = \varrho_0/\breve c_1$, then for $y \in B(w, 4\tilde\varrho_1)\setminus\Sigma$:

$$(i) \quad \breve c_1^{-1}\frac{\hat u(a_{\varrho_1}(w))}{\hat v(a_{\varrho_1}(w))} \leq \frac{\hat u(y)}{\hat v(y)} \leq \breve c_1 \frac{\hat u(a_{\varrho_1}(w))}{\hat v(a_{\varrho_1}(w))},$$

$$(ii) \quad \breve c_1^{-1}\frac{\hat u(y)}{d(y,\Sigma)} \leq |\nabla \hat u(y)| \leq \breve c_1 \frac{\hat u(y)}{d(y,\Sigma)}\,.$$

We end the section by proving that Assumption $1'$ as well as Assumption $1''$ imply Assumption 1 when Σ is a m-dimensional hyperplane and $A \in M_p(\alpha)$. Thus, in this particular case Lemma 4.10 is valid under either assumption.

Lemma 4.11 *Assume* (3.1)*,* $A \in M_p(\alpha)$*, and that* Σ *is a* m*-dimensional hyperplane. Assume either Assumption* $1'$ *or Assumption* $1''$*. Then Assumption 1 holds, for some* c_1, Γ*, depending only on the data and either* $\hat c_1$ *or* $\breve c_1$*.*

Proof. We first prove that Assumption $1'$ implies Assumption 1. To do so, let $x \in B(w, \hat\varrho_1)\setminus\Sigma$ and consider $0 < \rho \leq c_*^{-1}\hat\varrho_1$ where $c_* \geq 100$, will eventually be chosen to depend only on the data. If $\rho \leq 3d(x,\Sigma)/4$, then from Assumption $1'$, Lemma 2.4, and Harnack's inequality in Lemma 3.1 applied to $\hat u, \hat v$, we see that $\lambda = (a|\nabla \hat u| + b|\nabla \hat v|)^{p-2}$ satisfies

$$\int_{B(x,\rho)} \lambda^t\, dx \approx \left(\frac{a\hat u(x) + b\hat v(x)}{d(x,\Sigma)}\right)^{t(p-2)} \rho^n, \text{ whenever } a, b \in [0,\infty) \text{ and } t = \pm 1. \tag{4.33}$$

If $\rho \geq 3d(x,\Sigma)/4$ let $z \in \Sigma$ with $|x-z| = d(x,\Sigma)$ and put $\bar\rho = c_*\rho$. Let P be a $(n-1)$-dimensional hyperplane with $z \in P$ and $\Sigma \subset P$. Let Ω be the component of $B(z,\bar\rho)\setminus P$ containing x and let $\Omega' = B(z,\bar\rho)\setminus\bar\Omega$ be the other component. Choose $y \in \Omega\cap\partial B(z,\bar\rho), y' \in \Omega'\cap\partial B(z,\bar\rho)$ with $\tilde u(y') \approx \tilde u(a_{\bar\rho}(z)) \approx \tilde u(y)$ whenever $\tilde u \in \{\hat u, \hat v\}$. Also choose $\hat\rho \approx \rho$ with $B(y, 2\hat\rho) \subset \Omega$ and $B(y', 2\hat\rho) \subset \Omega'$. Existence of $y, y', \hat\rho$ follows from elementary geometry and Harnack's inequality in Lemma 3.1 applied to $\hat u, \hat v$. Let u', v' be the A-harmonic functions in $B(z,\bar\rho)\setminus[P\cup\overline{B(y,\hat\rho)}\cup\overline{B(y',\hat\rho)}]$ with continuous boundary values $u' = v' = 0$ on $P\cup\partial B(z,\bar\rho)$ while $u' = \hat u(a_{\bar\rho}(z))$ and $v' = \hat v(a_{\bar\rho}(z))$ on $\partial B(y,\hat\rho)\cup\partial B(y',\hat\rho)$.

We remark that linear functions are A-harmonic when $A \in M_p(\alpha)$. Using this remark, and either Lemma 2.8 in [LLuN] or just the barrier argument in this lemma, we deduce, for c_* large enough, that

$$u'(\hat y)/d(\hat y, P) \geq c^{-1}\hat u(a_{\bar\rho}(z))/\bar\rho \text{ whenever } \hat y \in B(z, 4\rho)\setminus P, \tag{4.34}$$

where c depends only on p, n, α. With c_* now fixed we use (4.34) and the maximum principle for A-harmonic functions to find that

$$\hat{u}(\hat{y}) \geq u'(\hat{y}) \geq c^{-1} d(y,P)\, \hat{u}(a_{\bar{\rho}}(z))/\bar{\rho} \geq c^{-2}\, d(y,P)\, \tilde{u}(a_\rho(z))/\rho,\ \hat{y} \in B(z,4\rho) \setminus P. \tag{4.35}$$

(4.34), (4.35) are also valid with $\hat{u}, u'$ replaced by $\hat{v}, v'$. Let

$$E = E(P) = \{\hat{y} \in B(z,4\rho) \setminus P : d(\hat{y},P) \geq \tfrac{1}{4} d(\hat{y},\Sigma)\}.$$

Using (4.35), Assumption $1'$ and the fact that $p > 2$, we see that

$$\int_E \lambda^{-1} dx \,\leq\, c \left(\frac{a\hat{u}(a_\rho(z)) + b\hat{v}(a_\rho(z))}{\rho} \right)^{2-p} \rho^n\,. \tag{4.36}$$

From basic geometry we can choose $(n-1)$-dimensional hyperplanes $P_1, \dots, P_N$, where $N = N(n)$, so that $B(z,4\rho) \setminus \Sigma \subset \cup_{i=1}^N E(P_i)$. Using this fact, and that $B(x,\rho) \subset B(z,4\rho)$, we conclude from (4.36) that

$$\int_{B(x,\rho)} \lambda^{-1} dx \leq c' \left(\frac{a\hat{u}(a_\rho(z)) + b\hat{v}(a_\rho(z))}{\rho} \right)^{2-p} \rho^n, \tag{4.37}$$

where c' depends only on $\hat{c}_1$ and the data. Finally observe from Lemmas 3.3, 3.5 for $\hat{u}, \hat{v}$, and Hölder's inequality that

$$\int_{B(x,\rho)} \lambda\, dx \leq \int_{B(z,4\rho)} \lambda\, dx \leq c'' \left(\frac{a\hat{u}(a_\rho(z)) + b\hat{v}(a_\rho(z))}{\rho} \right)^{p-2} \rho^n, \tag{4.38}$$

where c'' has the same dependence as c' . Combining (4.37) (4.38) we find, in view of (4.33) and the arbitrariness of x that Lemma 4.11 is true when Assumption $1'$ holds.

To prove Lemma 4.11 under Assumption $1''$ we assume, as we may, that

$$\hat{u}(a_{\tilde{\varrho}_1}(w)) = \hat{v}(a_{\tilde{\varrho}_1}(w)) = 1, \tag{4.39}$$

since otherwise we can multiply $\hat{u}, \hat{v}$ by appropriate constants to get (4.39) and then observe that the resulting functions satisfy the same PDE as $\hat{u}, \hat{v}$. From (4.39) and Assumption $1''$ we see that

$$c_+^{-1} \leq \frac{\hat{u}(y)}{\hat{v}(y)} \leq c_+ \text{ in } B(w,\tilde{\rho}_1) \setminus \Sigma, \tag{4.40}$$

where $c_+ \geq 1$ depends only on $\breve{c}_1$ in Assumption $1''$. Hence, if $2c_+\bar{u} = \hat{u}$, then

$$\bar{u} \leq \hat{v}/2 \leq c_+^2 \bar{u} \tag{4.41}$$

Let now $\{u(\cdot,\tau)\}, 0 \leq \tau \leq 1$, be the sequence of A-harmonic functions in $B(w,\tilde{\varrho}_1) \setminus \Sigma$ with continuous boundary values,

$$u(y,\tau) = \tau\hat{v}(y) + (1-\tau)\bar{u}(y), \text{ for } y \in \partial(B(w,\tilde{\varrho}_1) \setminus \Sigma), 0 \leq \tau \leq 1. \tag{4.42}$$

Existence of $u(\cdot,\tau), \tau \in (0,1)$, is a consequence of Lemma 3.2. Also using the maximum principle for A-harmonic functions in Lemma 3.2, Assumption $1''$, (4.41), and (4.42), we find for some $\tilde{c}$, depending on $\breve{c}_1$ and the data, that

$$\tilde{c}^{-1}\, u(\cdot,\tau_1) \,\leq\, \frac{u(\cdot,\tau_2) - u(\cdot,\tau_1)}{\tau_2 - \tau_1} \leq \tilde{c} u(\cdot,\tau_1) \tag{4.43}$$

on $B(w,\tilde\rho_1)\setminus\Sigma$, whenever $0\le\tau_1<\tau_2\le 1$. Let $\epsilon_0=\tilde\epsilon$ where $\tilde\epsilon$ is as in Lemma 3.9 with $\tilde a$ replaced by $\breve c_1$. From (4.43) we find the existence of $\epsilon_0', 0<\epsilon_0'\le\epsilon_0$, with the same dependence as ϵ_0, such that if $|\tau_2-\tau_1|\le\epsilon_0'$, then

$$1-\epsilon_0/2\le\frac{u(\cdot,\tau_2)}{u(\cdot,\tau_1)}\le 1+\epsilon_0/2 \text{ in } B(w,\rho_1)\setminus\Sigma. \tag{4.44}$$

Let $\xi_1=0<\xi_2<...<\xi_l=1$ and consider [0,1] as divided into $\{[\xi_k,\xi_{k+1}]\}, 1\le k\le l-1$. We assume that all of these intervals have a length of $\epsilon_0'/2$ with the possible exception of the interval containing $\xi_l=1$ which is of length $\le\epsilon_0'/2$.

Using Assumption $1''$, $u(\cdot,\xi_1)=\bar u=(2c_+)^{-1}\hat u$, and (4.44) we see that Lemma 3.9 can be applied with $\hat u_1=u(\cdot,\xi_1)$ and $\hat u_2=u(\cdot,\xi_2)$. Doing this we first find, for some $c_-\ge 1$ depending only on $\breve c_1$ and the data, that

$$c_-^{-1}\frac{u(y,\xi_2)}{d(y,\Sigma)}\le|\nabla u(y,\xi_2)|\le c_-\frac{u(y,\xi_2)}{d(y,\Sigma)}, \tag{4.45}$$

whenever $y\in B(w,\tilde\rho_1/200)\setminus\Sigma$. Hence Assumption $1'$ applies to $u(\cdot,\xi_1),u(\cdot,\xi_2)$ with $\hat\varrho_1$ replaced by $\tilde\rho_1/200$. Second from the first part of our proof it follows that Assumption 1 is satisfied for these functions, so we can use Lemma 4.10 to conclude that

$$\left|\log\left(\frac{u(y_1,\xi_2)}{u(y_1,\xi_1)}\right)-\log\left(\frac{u(y_2,\xi_2)}{u(y_2,\xi_1)}\right)\right|\le c\left(\frac{|y_1-y_2|}{\tilde\rho}\right)^\sigma \text{ whenever } y_1,y_2\in B(w,\tilde\rho/c), \tag{4.46}$$

where c depends on $p,n,m,\alpha,\tilde\theta,\breve c_1$. We can now continue by induction, as in the proof of (4.24) - (4.28) in Theorem 2 of [LN1] to eventually obtain (see [LN1] Lemma 4.28) that (4.45) holds wirh $u(\cdot,\xi_2)$ replaced by $u(\cdot,\xi_l)=\hat v$ whenever $y\in B(w,\tilde\rho/\bar c)$. Here $\bar c$ depends only on $\breve c_1$ and the data. Thus $\hat u,\hat v$ satisfy the hypotheses of Assumption $1'$ and so Assumption 1 is also valid. The proof of Lemma 4.11 is now complete. ■

5 Existence and uniqueness of fundamental solutions

Let n, m, be integers such that $1\le m\le n-2$ and let p, $n-m<p<\infty$, be given. In this section we assume that $A\in M_p(\alpha)$ for some $\alpha\in[1,\infty)$, i.e., we consider operators with constant coefficients. Furthermore, we consider coordinates $y=(y',y'')\in\mathbb R^m\times\mathbb R^{n-m}$, and let $\Sigma=\{y=(y',y'')\in\mathbb R^m\times\mathbb R^{n-m}:\ y''=0\}$. We are here interested in constructing $u=u_{n-m}$ defined on $\mathbb R^n$ such that $u\in W^{1,p}_{\rm loc}(\mathbb R^n\setminus\Sigma)$, u is continuous on $\mathbb R^n$, $u=0$ on Σ, $u>0$ on $\mathbb R^n\setminus\Sigma$, and such that u is a weak solution to $\nabla\cdot A(\nabla u)=0$ in $\mathbb R^n\setminus\Sigma$. To start the construction we in the following let $k=n-m$ and we define $\tilde A=(\tilde A_1,...,\tilde A_k):\mathbb R^k\to\mathbb R^k$, by setting $\tilde A_j(\eta)=A_{m+j}(0,\eta)$ whenever $\eta\in\mathbb R^k$ and for $j\in\{1,...,k\}$. Then also $\tilde A\in M_p(\alpha)$ in the sense of Definition 1.1 but with $\mathbb R^n$ replaced by $\mathbb R^k$. In the following points in $\mathbb R^k$ will be denoted by $z=(z_1,...,z_k)$. We now say that $\tilde u$ is a fundamental solution to the equation $\nabla\cdot\tilde A(\nabla\tilde u)=0$ in $\mathbb R^k$, with pole at $0\in\mathbb R^k$, if

(i) $\quad\tilde u\in W^{1,p}_{\rm loc}(\mathbb R^k)$, $\tilde u$ is continuous in $\mathbb R^k$, $\tilde u(0)=0$, $\tilde u>0$ in $\mathbb R^k\setminus\{0\}$,

(ii) $\quad\theta\in C_0^\infty(\mathbb R^k)$, then $\displaystyle\int\langle\tilde A(\nabla\tilde u(z)),\nabla\theta(z)\rangle\,dz=-\theta(0).\qquad (5.1)$

Note that (5.1) (ii) implies that $\tilde{u}$ is a weak solution to $\nabla \cdot \tilde{A}(\nabla \tilde{u}) = 0$ in $\mathbb{R}^k \setminus \{0\}$. We first prove the following lemma.

Lemma 5.1 *Let k be an integer, $2 \leq k < \infty$, and let p, $k < p < \infty$, be given. Let $\xi = (p-k)/(p-1)$. Assume that $\tilde{A} \in M_p(\alpha)$ for some $\alpha \in [1, \infty)$ with $\mathbb{R}^k$ as the underlying space. Then there exists a fundamental solution $\tilde{u}$ to the equation $\nabla \cdot \tilde{A}(\nabla \tilde{u}) = 0$ in $\mathbb{R}^k$, with pole at $0 \in \mathbb{R}^k$, in the sense of (5.1), and a constant $c = c(p, k, \alpha)$, $1 \leq c < \infty$, such that*

$$\begin{aligned} (i') \qquad & c^{-1}|z|^{\xi} \leq \tilde{u}(z) \leq c|z|^{\xi}, \\ (ii') \qquad & c^{-1}|z|^{\xi-1} \leq |\nabla \tilde{u}(z)| \leq c|z|^{\xi-1}, \end{aligned} \tag{5.2}$$

whenever $z \in \mathbb{R}^k \setminus \{0\}$.

Proof. Assume that $\tilde{u}$ is a fundamental solution to the equation $\nabla \cdot \tilde{A}(\nabla \tilde{u}) = 0$ in $\mathbb{R}^k$, with pole at 0, i.e., $\tilde{u}$ is an $\tilde{A}$-harmonic function in $\mathbb{R}^k \setminus \{0\}$ satisfying (5.1) (i), (ii). Using $p > k$, $\tilde{u}(0) = 0$, we find as in Lemma 3.7 with Σ replaced by $\{0\}$ that there exists a unique finite positive Borel measure $\tilde{\mu}$ on $\mathbb{R}^k$, with support at $\{0\}$, such that

$$\int \langle \tilde{A}(\nabla \tilde{u}(z)), \nabla \theta(z) \rangle \, dz \; = \; -\int \theta \, d\tilde{\mu}, \tag{5.3}$$

whenever $\theta \in C_0^\infty(\mathbb{R}^k)$. In particular, from uniqueness and (5.1) (ii) we find that $\tilde{\mu}(\mathbb{R}^k) = 1$. Also using Lemma 3.7 we immediately deduce that $\tilde{u}$ satisfies (5.2) (i'). In particular, any fundamental solution to the equation $\nabla \cdot \tilde{A}(\nabla \tilde{u}) = 0$ in $\mathbb{R}^k$, with pole at 0, satisfies, by construction, (5.2) (i'). Hence, in the following it suffices to prove the existence of a $\tilde{u}$ satisfying (5.1) (i), (ii), and (5.2) (ii'). Note that in the following all balls $B(0, \varrho)$ are standard Euclidean k-dimensional balls. To start the proof of the existence of $\tilde{u}$ we in the following let, for $\epsilon > 0$ given and small,

$$\tilde{A}(\eta, \epsilon) = \int_{\mathbb{R}^k} \tilde{A}(\eta - \zeta) \theta_\epsilon(\zeta) d\zeta \text{ whenever } \eta \in \mathbb{R}^k, \tag{5.4}$$

where $\theta \in C_0^\infty(B(0,1))$ with $\int_{\mathbb{R}^k} \theta d\zeta = 1$ and $\theta_\epsilon(\zeta) = \epsilon^{-k}\theta(\zeta/\epsilon)$ whenever $\zeta \in \mathbb{R}^k$. Using the definition of the class $M_p(\alpha)$ and standard properties of approximations to the identity, we deduce for some $c = c(p, k) \geq 1$, that

$$\begin{aligned} (i) \qquad & (c\alpha)^{-1}(\epsilon + |\eta|)^{p-2}|\xi|^2 \leq \sum_{i,j=1}^{k} \frac{\partial \tilde{A}_i}{\partial \eta_j}(\eta, \epsilon)\xi_i\xi_j, \\ (ii) \qquad & \left| \frac{\partial \tilde{A}_i}{\partial \eta_j}(\eta, \epsilon) \right| \leq c\alpha(\epsilon + |\eta|)^{p-2}, 1 \leq i, j \leq k, \end{aligned} \tag{5.5}$$

whenever $\eta \in \mathbb{R}^k$, $\xi \in \mathbb{R}^k$. Moreover, $\tilde{A}(\cdot, \epsilon)$ is, for fixed ϵ, infinitely differentiable. We now let $w(\cdot, \epsilon)$ be the unique solution to $\nabla \cdot (\tilde{A}(\nabla w(z, \epsilon), \epsilon)) = 0$ in $B(0,1) \setminus \{0\}$ which is continuous on the closure of $B(0,1)$, and satisfies $w(\cdot, \epsilon) = 1$ on $\partial B(0,1)$, $w(0, \epsilon) = 0$. Note that, using [T], [T1], [Li], it follows that $w(\cdot, \epsilon)$ is in $C^{1,\hat{\sigma}}(B(0,1) \setminus \{0\})$ for some $\hat{\sigma} > 0$ with constants independent of ϵ. Letting $\epsilon \to 0$, using the definition of the class $M_p(\alpha)$, one can prove that subsequences of $\{w(\cdot,\epsilon)\}$, $\{\nabla w(\cdot,\epsilon)\}$, converge pointwise to w, ∇w on $\overline{B(0,1)}$ and $B(0,1)\setminus\{0\}$, respectively, where w is the unique solution to $\nabla\cdot(\tilde{A}(\nabla w))=0$ in $B(0,1)\setminus\{0\}$ which is continuous on the closure of $B(0,1)$, and satisfies $w=1$ on $\partial B(0,1)$, $w(0)=0$. To proceed we let

$$\tilde{A}^*_{ij}(z,\epsilon)=\tfrac{1}{2}(\epsilon+|\nabla w(z,\epsilon)|)^{2-p}\left[\frac{\partial \tilde{A}_i}{\partial \eta_j}(\nabla w(z,\epsilon),\epsilon)+\frac{\partial \tilde{A}_j}{\partial \eta_i}(\nabla w(z,\epsilon),\epsilon)\right],$$

whenever $z\in B(0,1)\setminus\{0\}$ and $1\leq i,j\leq k$. From (5.5) (ii) and Schauder type estimates we see that $w(\cdot,\epsilon)$ is a classical solution to the non-divergence form uniformly elliptic equation,

$$L^*\zeta=\sum_{i,j=1}^{n}\tilde{A}^*_{ij}(z,\epsilon)\zeta_{z_iz_j}=0, \tag{5.6}$$

for $z\in B(0,1)\setminus\{0\}$. Note also from (5.5) that the ellipticity constant for $(\tilde{A}^*_{ij}(z,\epsilon))$ and the L^∞-norm for $\tilde{A}^*_{ij}(z,\epsilon), 1\leq i,j\leq k$, in $B(0,1)\setminus\{0\}$, depend only on p,k,α. To continue we again note that it follows from the assumption $p>k$ that points are uniformly p-thick. In particular, using this, the Harnack inequality, Lemma 3.3, and Lemma 3.5, we immediately see that

$$c(1-w(z,\epsilon))\geq 1 \text{ whenever } z\in B(0,1/2), \tag{5.7}$$

for some $c=c(p,k,\alpha)$, $1\leq c<\infty$. We now let

$$\psi(z)=\frac{e^{-N|z|^2}-e^{-N}}{e^{-N/4}-e^{-N}}, \tag{5.8}$$

whenever $z\in B(0,1)\setminus\overline{B(0,1/2)}$ and where N is a non-negative integer. Then ψ is a subsolution to L^* in $B(0,1)\setminus\overline{B(0,1/2)}$ if $N=N(p,n,\alpha)$ is sufficiently large, and $\psi\equiv 1$ on $\partial B(0,1/2)$ while $\psi\equiv 0$ on $\partial B(0,1)$. Hence, using the comparison principle we see that

$$c(1-w(z,\epsilon))\geq \psi(z) \text{ on } B(0,1)\setminus\overline{B(0,1/2)}, \tag{5.9}$$

where c is independent of ϵ. Furthermore, it is easily seen that

$$c\psi(z)\geq 1-|z| \text{ on } B(0,1)\setminus\overline{B(0,3/4)}, \tag{5.10}$$

for some $c=c(p,k,\alpha)$. We can therefore conclude that

$$\hat{c}(1-w(z,\epsilon))\geq (1-|z|) \text{ on } B(0,1)\setminus\overline{B(0,3/4)}, \tag{5.11}$$

for some $\hat{c}=\hat{c}(p,k,\alpha)$. Furthermore, letting $\epsilon\to 0$ we also have, by the above argument, that

$$\hat{c}(1-w(z))\geq (1-|z|) \text{ on } B(0,1)\setminus\overline{B(0,3/4)}. \tag{5.12}$$

Next, given $R\geq 1$ let $\tilde{w}_R$ be the unique solution to $\nabla\cdot(\tilde{A}(\nabla\tilde{w}_R))=0$ in $B(0,R)\setminus\{0\}$ which is continuous on the closure of $B(0,R)$, and satisfies $\tilde{w}_R=1$ on $\partial B(0,R)$, $\tilde{w}_R(0)=0$.

We observe, using Definition 1.1 (iii) for $\tilde{A}$, and the maximum principle in Lemma 3.2, that $w(z/R) = \tilde{w}_R(z), z \in B(0,R)$. Thus we can apply (5.12) to conclude that

$$\hat{c}(1 - \tilde{w}_R(z)) \geq \frac{(R - |z|)}{R} \text{ on } B(0,R) \setminus \overline{B(0, 3R/4)}. \tag{5.13}$$

Using (5.13) and the comparison principle it follows, for $\lambda > 1$ given, that

$$\frac{\tilde{w}_R(\lambda z) - \tilde{w}_R(z)}{\lambda - 1} \geq c^{-1}\tilde{w}_R(z), \tag{5.14}$$

in $B(0, R/\lambda) \setminus \{0\}$ and for some constant c which can be chosen independent of λ whenever $1 < \lambda < 9/8$. Next, letting $\lambda \to 1$ in (5.14) we obtain that

$$|z|\langle \nabla \tilde{w}_R(z), z/|z|\rangle \geq c^{-1}\tilde{w}_R(z) \text{ whenever } z \in B(0,R) \setminus \{0\}. \tag{5.15}$$

Let $\hat{w}_R = \tilde{w}_R/\tilde{w}_R(1, 0, \ldots, 0)$. From Harnack's inequality and Hölder $1 - k/p$ continuity of Sobolev functions in $W^{1,p}$ when $p > k$, as well as the basic estimates in section 3, we see that a certain subsequence of $(\hat{w}_R)$ converges uniformly on compact subsets of $\mathbb{R}^k$ to u' satisfying (5.1) (i) and $\nabla \cdot \tilde{A}(\nabla u') = 0$ in $\mathbb{R}^k \setminus \{0\}$, weakly. Arguing as in (5.3) it now follows that $\tilde{u} = cu'$ satisfies (5.1) $(i), (ii)$ for some $c = c(p, k, \alpha)$. Also the lower bound in (5.2) (ii') is a consequence of (5.15). The upper bound follows immediately from (5.2) (i') and interior regularity, see Lemma 3.6. This completes the proof of Lemma 5.1. ■

Lemma 5.2 *Let k be an integer, $2 \leq k < \infty$, and let p, $k < p < \infty$, be given. Let $\xi = (p-k)/(p-1)$. Assume that $\tilde{A} \in M_p(\alpha)$ for some $\alpha \in [1, \infty)$ with $\mathbb{R}^k$ as the underlying space. Then there exists a unique fundamental solution $\tilde{u}$ to the equation $\nabla \cdot \tilde{A}(\nabla \tilde{u}) = 0$ in $\mathbb{R}^k$, with pole at $0 \in \mathbb{R}^k$, in the sense of (5.1). Furthermore, there exist $\sigma = \sigma(p, k, \alpha)$, $\sigma \in (0,1)$, and $\psi \in C^{1,\sigma}(\mathbb{S}^k)$ such that $u(z) = |z|^\xi \psi(z/|z|)$ whenever $z \in \mathbb{R}^k \setminus \{0\}$.*

Proof. By Lemma 5.1 we have the existence of a fundamental solution $\tilde{u}$ to the equation $\nabla \cdot \tilde{A}(\nabla \tilde{u}) = 0$ in $\mathbb{R}^k$, with pole at $0 \in \mathbb{R}^k$, in the sense of (5.1), satisfying also (5.2). We want to prove that $\tilde{u}$ is the unique fundamental solution in the sense of (5.1). To do this let $\tilde{v}$ be another fundamental solution to $\nabla \cdot \tilde{A}(\nabla \tilde{v}) = 0$ in $\mathbb{R}^k$, with pole at $0 \in \mathbb{R}^k$, in the sense of (5.1). Then, as in the proof of Lemma 5.1 we see that $\tilde{v}$ also satisfies (5.2) (i'). In particular, $\tilde{u} \approx \tilde{v}$ in $\mathbb{R}^k$. From this fact and (5.2) (ii') for $\tilde{u}$ we observe that $\tilde{u}, \tilde{v}$ satisfy the hypotheses of Assumption $1''$ in $\mathbb{R}^k \setminus \{0\}$. Using this observation and arguing as in the proof of Lemma 4.11 it follows first that $\tilde{v}$ also satisfies (5.2) (ii'), with constants depending only on the data, and thereupon that $\lambda(\cdot, a, b, u, v) = (a|\nabla \tilde{u}| + b|\nabla \tilde{v}|)^{p-2}$ is an A_2-weight on $\mathbb{R}^k$ with constants independent of $a, b \in [0, \infty)$. Now arguing as earlier, we get that Lemma 4.10 holds on $\mathbb{R}^k \setminus \{0\}$ with $\hat{u}, \hat{v}$ replaced by $\tilde{u}, \tilde{v}$. Exponentiating both sides of the inequality in this lemma we conclude the existence of $c \geq 1$, $c = c(p, k, \alpha)$, and $\sigma = \sigma(p, k, \alpha)$, $\sigma \in (0, 1)$, such that

$$\left| \frac{\tilde{u}(z'')}{\tilde{v}(z'')} - \frac{\tilde{u}(z')}{\tilde{v}(z')} \right| \leq c(|z'' - z'|/R)^\sigma \max_{\partial B(0,R)} \frac{\tilde{u}}{\tilde{v}} \leq c^2(|z'' - z'|/R)^\sigma, \tag{5.16}$$

whenever $z', z'' \in B(0, R/4) \setminus \{0\}$. In particular, letting $R \to \infty$ we see that $\tilde{u} \equiv \tilde{v}$ on $\mathbb{R}^k$ and this completes the proof of uniqueness in Lemma 5.2. To prove the structural statement in this lemma, let $\tilde{u}$ be as in the statement of the lemma, and let $\tilde{v}(z) = \tilde{u}(tz)$ for some $t > 0$. Then, again using homogeneity in Definition 1.1 (iii) we see that $\nabla \cdot \tilde{A}(\nabla \tilde{v}) = 0$ weakly in $\mathbb{R}^k \setminus \{0\}$ and also we easily deduce for fixed $t \in (0, \infty)$, that $t^{-\xi}\tilde{u}(tz)$ satisfies both conditions in (5.1). Hence, by uniqueness we have $\tilde{u}(tz) = t^{\xi}\tilde{u}(z)$ whenever $z \in \mathbb{R}^k \setminus \{0\}$ or equivalently

$$\tilde{u}(z) = |z|^{\xi}\tilde{u}(z/|z|) \text{ whenever } z \in \mathbb{R}^k \setminus \{0\}. \tag{5.17}$$

The proof of Lemma 5.2 is now complete. ■

Lemma 5.3 *Let n, m, be integers such that $1 \leq m \leq n-2$ and let p, $n-m < p < \infty$, be given. Let $\xi = (p-n+m)/(p-1)$. Assume that $A \in M_p(\alpha)$ for some $\alpha \in [1, \infty)$, consider coordinates $y = (y', y'') \in \mathbb{R}^m \times \mathbb{R}^{n-m}$ and let $\Sigma = \{y = (y', y'') \in \mathbb{R}^m \times \mathbb{R}^{n-m} : \; y'' = 0\}$. Then there exists a function $\bar{u} = u_{n-m}$, defined on $\mathbb{R}^n$, which satisfies*

$$\begin{aligned} &(i) \quad && \bar{u} \in W^{1,p}_{loc}(\mathbb{R}^n \setminus \Sigma),\ \bar{u} \text{ is continuous on } \mathbb{R}^n, \\ &(ii) \quad && \bar{u} = 0 \text{ on } \Sigma,\ \bar{u} > 0 \text{ on } \mathbb{R}^n \setminus \Sigma, \\ &(iii) \quad && \bar{u} \text{ is a weak solution to } \nabla \cdot A(\nabla \bar{u}) = 0 \text{ in } \mathbb{R}^n \setminus \Sigma, \end{aligned} \tag{5.18}$$

and the quantitative estimates

$$\begin{aligned} &(i') \quad && c^{-1}|y''|^{\xi} \leq \bar{u}(y) \leq c|y''|^{\xi}, \\ &(ii') \quad && c^{-1}|y''|^{\xi-1} \leq |\nabla \bar{u}(y)| \leq c|y''|^{\xi-1}, \end{aligned} \tag{5.19}$$

for some constant $c = c(p, n, m, \alpha)$, $1 \leq c < \infty$, whenever $y \in \mathbb{R}^n \setminus \Sigma$. Moreover, $\bar{u}(y) = |y''|^{\xi}\psi(y''/|y''|)$ for all $y \in \mathbb{R}^n \setminus \Sigma$ where $\sigma = \sigma(p, n, m, \alpha)$, $\sigma \in (0, 1)$, and $\psi \in C^{1,\sigma}(\mathbb{S}^{n-m})$.

Proof. To construct $\bar{u} = u_{n-m}$ we simply let

$$\bar{u}(y) = \bar{u}(y', y'') := \tilde{u}(y'') \text{ whenever } y \in \mathbb{R}^n \setminus \Sigma,$$

where $\tilde{u}$ is as in Lemma 5.2. Then obviously $\bar{u}$ satisfies (5.18) and (5.19). Also the last statement of the lemma follows from Lemma 5.2. ■

6 Proof of Theorems 1.9 and 1.10 in the baseline case

In this section we prove Theorem 1.9 and Theorem 1.10 in the special case when Σ is an m-dimensional hyperplane, passing through 0, and $A \in M_p(\alpha)$, i.e., we consider only operators with constant coefficients. We note that if h is a weak solution to $\nabla \cdot A(\nabla h) = 0$ in $\mathbb{R}^n \setminus \Sigma$, and T is a rotation of $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$ which maps $\mathbb{R}^m \times \{0\}$ onto Σ, then, as follows by straightforward calculation, $\tilde{h}(x) = h(Tx)$ is a weak solution to a PDE, $\nabla \cdot \tilde{A}(\nabla \tilde{h}) = 0$, in $\mathbb{R}^n \setminus (\mathbb{R}^m \times \{0\})$, with $\tilde{A} \in M_p(\alpha)$. Thus, in the following we can assume that $\Sigma = \mathbb{R}^m \times \{0\}$ since otherwise we can change coordinate systems. As usual we write $y = (y', y'')$ when $y \in \mathbb{R}^n$ where $y' \in \mathbb{R}^m$ and $y'' \in \mathbb{R}^{n-m}$. Furthermore, given $w = (w', w'') \in \mathbb{R}^n$, r_1, r_2, $0 < r_1, r_2 < \infty$, we let $C_{r_1,r_2}(w)$ be as defined as in (3.4) and if $r_1 = r_2 = r$, then we write $C_r(w)$.

Lemma 6.1 *Let n, m, be integers such that $1 \leq m \leq n-2$ and let p, $n-m < p < \infty$, be given. Let $\Sigma = \mathbb{R}^m \times \{0\}, 0 < r < \infty$, and assume that $A \in M_p(\alpha)$. Let u, v be positive A-harmonic functions in $C_{4r}(0) \setminus \Sigma$, continuous on $C_{4r}(0)$, with $u = 0 = v$ on $\Sigma \cap C_{4r}(0)$. If $m = 1$, then there exists $c = c(p, n, m, \alpha)$, $1 \leq c < \infty$, such that*

$$c^{-1}\frac{u(a_r(0))}{v(a_r(0))} \leq \frac{u(y)}{v(y)} \leq c\frac{u(a_r(0))}{v(a_r(0))} \text{ whenever } y \in C_r(0) \setminus \Sigma. \tag{6.1}$$

If $2 \leq m \leq n-2$, and if Theorem 1.10 (a) or (b) hold, then there exists c, $1 \leq c < \infty$, depending at most on p, n, m, α, λ, such that (6.1) holds.

Lemma 6.2 *Theorems 1.9 and 1.10 are valid for p, n, m, A, u, v as in Lemma 6.1.*

Proof. Theorem 1.9 and Theorem 1.10 in this baseline case follow immediately from Lemma 6.1, Lemma 5.3, Lemma 4.11, and Lemma 4.10. ■

Below we give the proof of Lemma 6.1 divided into cases. As the PDE's satisfied by u, v are invariant under dilation and scaling, we may, without loss of generality and to this end, assume that

$$r = 1,\ u(a_1(0)) = 1 = v(a_1(0)). \tag{6.2}$$

Hence, we want to prove that there exists $c \geq 1$, depending only on the data, such that

$$c^{-1} \leq u(y)/v(y) \leq c \text{ whenever } y \in C_1(0) \setminus \Sigma. \tag{6.3}$$

In light of Lemma 5.3 it is sufficient to prove (6.3) with v replaced by $\bar{u} = u_{n-m}$. Equivalently, it suffices to establish the existence of $c \geq 1$, depending only on the data, such that

$$c^{-1}|y''|^{\xi} \leq u(y) \leq c|y''|^{\xi} \text{ whenever } y \in C_1(0) \setminus \Sigma. \tag{6.4}$$

6.1 The case $m = 1$

In this case we can complete the proof without making use of the explicit structure of $v = \bar{u}$. Indeed, to estimate u/v, suppose $u/v \geq \zeta$ on $\partial C_1(0)$ for some large $\zeta > 0$. Let $s \in (1, 3)$ and observe from Harnack's inequality for A-harmonic functions that for ζ large enough, we have $u/v > \zeta$ at some point in $\partial C_s(0)$ with $y' = \pm s$. This observation implies there exists a closed interval $I \subset [1, 3] \cup [-3, -1]$ of length 1 such that for all $t \in I$ there exists $y'' = y''(t)$ with $|y''| \leq 1$ and $(u/v)(t, y'') > \zeta$. Indeed, if we for some $z' \in (1, 2) \cup (-2, -1)$ have $(u/v)(z', z'') \leq \zeta$, whenever $|z''| \leq 2$, then we can apply the above analysis to cylinders of radius 2 whose boundary contains $\{(z', z'') : |z''| \leq 2\}$ in order to conclude the existence of $I \subset [2, 3] \cup [-3, -2]$. Otherwise we choose $I = [1, 2]$.

Let μ, ν be the measures corresponding to u, v as in Lemma 3.7. Note from (ii) of Lemma 3.7, and Harnack's inequality for u, v that μ, ν are doubling measures in the sense that

$$\theta(B(y, 2s)) \leq c\,\theta(B(y, s)) \text{ whenever } y = (y', 0) \text{ with } |y'| + 4s < 4 \text{ and } \theta \in \{\mu, \nu\}. \tag{6.5}$$

Given $t \in I$, choose $y''(t)$ as above and put $\rho(t) = |y''(t)|, \tau = (t, 0)$. Using Lemma 3.7 (ii) we deduce, for some $c \geq 1$ depending only on p, n, α, that

$$\zeta^{p-1} \leq \left(\frac{u(t, y''(t))}{v(t, y''(t))} \right)^{p-1} \leq c \, \frac{\mu(B(\tau, \rho(t))}{\nu(B(\tau, \rho(t)))} \,. \tag{6.6}$$

Using a standard covering lemma we see there exists $\{t_j\}, 0 < t_j < 1/2$, for which (6.6) holds with $t, y''(t), \rho(t), \tau$ replaced by $t_j, y''(t_j), \rho(t_j), \tau_j$. Also

$$I \subset \bigcup_j B(\tau_j, \rho_j) \text{ and } B(\tau_k, \rho_k/5) \cap B(\tau_l, \rho_l/5) = \emptyset \text{ when } l \neq k. \tag{6.7}$$

From (6.5), (6.6), (6.7) and Lemma 3.7 it follows, for some $\tilde{c} \geq 1$ depending only on the data, that

$$\begin{aligned} 1 \approx \nu(B(0, 7/2)) \quad &\leq \quad \tilde{c}\nu(\bigcup_j B(\tau_j, \rho_j)) \\ &\leq \quad \tilde{c}^2 \zeta^{1-p} \mu(\bigcup_j B(\tau_j, \rho_j/5)) \leq \tilde{c}^2 \zeta^{1-p} \mu(B(0, 7/2)) \approx \zeta^{1-p} \,. \end{aligned} \tag{6.8}$$

Thus ζ cannot be too big (depending on the data). This completes the proof of Lemma 6.1 when $m = 1$.

Remark 6.3 *We remark that Lemma 6.1 can be proved, using the above argument, also when u, v are solutions to the p-Laplace equation and $1 \leq m \leq n - 2$. Indeed in this case one can construct p-harmonic $\tilde{v}, \tilde{u}$ that are rotationally symmetric in y', y'' and satisfy $u \leq c\tilde{u}, \tilde{v} \leq cv$. Then, using the two dimensional character of $\tilde{u}, \tilde{v}$, one can essentially repeat the above argument to get Lemma 6.1 for $\tilde{u}, \tilde{v}$ and so also for u, v. We emphasize that this argument uses heavily that the p-Laplacian is invariant under rotations. For another proof of Lemma 6.1 when u, v are p-harmonic, see [Lu].*

6.2 The upper bound in (6.4) for $1 \leq m \leq n - 2$

For $1 \leq m \leq n-2$, and $A \in M_p(\alpha)$, let u' be the A-harmonic function in $B(0, 8) \setminus (\Sigma \cap \overline{B(0, 4)})$ with continuous boundary values $u' \equiv 1$ on $\partial B(0, 8)$ and $u' \equiv 0$ on $\Sigma \cap \overline{B(0, 4)}$. We will first prove, for some $\breve{c} = \breve{c}(p, n, m, \alpha)$, that

$$\breve{c}^{-1} \frac{u'(y)}{|y''|} \leq |\nabla u'|(y) \leq \breve{c} \, \frac{u'(y)}{|y''|} \text{ when } y \in C_4(0) \setminus \Sigma. \tag{6.9}$$

In order to prove (6.9) we observe from Lemma 3.3, and Harnack's inequality applied to $1 - u'$, that $1 - u' \geq c^{-1}$ in $B(0, 6)$. Using this fact, and a barrier argument as in (5.7)-(5.12), we obtain that

$$1 - u'(y) \geq \bar{c}^{-1} d(y, \partial B(0, 8)) \text{ when } y \in B(0, 8) \setminus B(0, 6). \tag{6.10}$$

Given $\hat{x} \in \Sigma \cap \overline{B(0, 4)}$, put $u_+(y) = u'(\hat{x} + y)$ when $y \in \Omega := \{z : z + \hat{x} \in B(0, 8)\}$. Let $\Sigma' = \{z : z + \hat{x} \in \Sigma \cap \overline{B(0, 4)}\}$. Since A-harmonic functions, $A \in M_p(\alpha)$, are invariant under translation and dilation it follows first that u_+ is A-harmonic in $\Omega \setminus \Sigma'$ and second that if $s > 1$, then the function $y \to u_+(sy)$ is A-harmonic in $\Omega(s)$ where $\Omega(s) = \{y \in \Omega : sy \in \Omega \setminus \Sigma'\}$. Using (6.10) and comparing boundary values we deduce, for $1 < s < 1.01$, that

$$\frac{u_+(sy) - u_+(y)}{s-1} \geq c^{-1} u_+(y) \text{ when } y \in \partial\Omega(s), \tag{6.11}$$

where c depends on p, n, m, α. From the maximum principle for A-harmonic functions we see that (6.11) holds in $\Omega(s)$. Letting $s \to 1$ and using Lemma 3.6 we find that

$$\langle \nabla u'(y), y - \hat{x} \rangle \geq c^{-1} u'(y) \text{ for } y \in B(0,8) \setminus \Sigma \,. \tag{6.12}$$

From arbitrariness of $\hat{x} \in \Sigma \cap \overline{B(0,4)}$, (6.12), and the fact that $|y''| = d(y, \Sigma)$, we deduce that the left hand inequality in (6.9) is valid. The right hand inequality in (6.9) follows from Lemma 3.6. Thus (6.9) is valid.

Next, let ξ be as in Lemma 5.3 and let $\bar{u} = u_{n-m}$ denote the A-harmonic function in this lemma. Then $u', \bar{u}$ satisfy the hypotheses of Assumption $1'$ in section 4. Hence, using Lemma 4.10 and Lemma 4.11, we have

$$c_*^{-1} \leq \frac{u'(y)}{|y''|^\xi} \leq c_*, \tag{6.13}$$

whenever $y \in C_{1/\hat{c}}(0)$ for some c_* depending only on p, n, m, α. Repeating this argument with $C_1(0)$ replaced by $C_1(w)$, whenever $w \in \Sigma \cap \overline{B(0,1)}$, and using Harnack's inequality again it follows that (6.13) holds for $y \in C_1(0)$, with c_* replaced by a larger constant also depending only on the data. Moreover, if u is as in (6.3), then $u \leq cu'$ in $C_4(0)$ so the right hand inequality in (6.13) holds with u' replaced by u. In particular, we can conclude the validity of the upper bound in (6.4) for $1 \leq m \leq n-2$.

6.3 The lower bound in (6.4): A as in Theorem 1.10 (a)

Let $2 \leq m \leq n-2, A \in M_p(\alpha)$, and assume that $A \in M_p(\alpha)$ satisfies Theorem 1.10 (a). We here prove the lower bound in (6.4), i.e., assuming (6.2) we prove that

$$|y''|^\xi \leq c\, u \text{ on } C_1(0) \setminus \Sigma. \tag{6.14}$$

This then completes the proof of Lemma 6.1 in the case considered. To prove (6.14) we first observe, by the same argument as in (4.34), (4.35), that

$$d(y, \Sigma) = |y''| \leq \tilde{c}_1 u(y) \text{ when } y \in C_1(0) \setminus \Sigma, \tag{6.15}$$

for some $\tilde{c}_1 = \tilde{c}_1(p, n, m, \alpha) \geq 1$. Let $\bar{u} = u_{n-m}$ be as in Lemma 5.3, and put

$$f(y) = (1 - |y'|^2)\,(e^{\bar{u}(y)} - 1), \tag{6.16}$$

whenever $y \in C_1(0)$. We claim that

$$f \leq \tilde{c}_2 u \text{ on } C_1(0) \setminus \Sigma, \tag{6.17}$$

and for some $\tilde{c}_2 = \tilde{c}_2(p, n, m, \lambda) \geq 1$. To prove this claim we first observe that $f \leq cu$ on $\partial(C_1(0) \setminus \Sigma)$, as follows from the facts that $u(a_1(0)) = 1$ and that $f(y) \equiv 0$ when $|y'| = 1$ or

$y \in \Sigma \cap \overline{B(0,1)}$. Hence, using this, (6.15), the maximum principle and Lemma 3.6, we see that in order to prove (6.17) it suffices to show, for some $\tilde{c}_3 = \tilde{c}_3(p,n,m,\alpha,\lambda)$, that if

$$y \in C_1(0) \setminus \Sigma \text{ and } f(y) \geq \tilde{c}_3 |y''|, \tag{6.18}$$

then

$$\nabla \cdot A(\nabla f)(y) \geq 0, \tag{6.19}$$

where the latter inequality is taken in the strong or classical sense. In order to prove that (6.18) implies (6.19) we let $\tilde{c}_3$ be a degree of freedom to be fixed and depending only on p,n,m,α,λ.

Let

$$\nabla' f(y) = \left(\frac{\partial f}{\partial y_1}, \ldots, \frac{\partial f}{\partial y_m} \right)(y) \text{ and } \nabla'' f(y) = \left(\frac{\partial f}{\partial y_{m+1}}, \ldots, \frac{\partial f}{\partial y_n} \right)(y), \tag{6.20}$$

when $y \in C_1(0) \setminus \Sigma$. We write $\nabla f(y) = (\nabla' f(y), \nabla'' f(y))$. Note that

$$\nabla \cdot A(\nabla f)(y) = \sum_{i,j=1}^{n} \frac{\partial A_i}{\partial \eta_j}(\nabla f(y)) f_{y_i y_j} = T_1 + T_2 + T_3, \tag{6.21}$$

where

$$\begin{aligned} T_1 &:= \hat{\sum}_{i,j} \frac{\partial A_i}{\partial \eta_j}(\nabla f(y)) f_{y_i y_j}, \\ T_2 &:= \sum_{m+1 \leq i,j \leq n} \left(\frac{\partial A_i}{\partial \eta_j}(\nabla f(y)) - \frac{\partial A_i}{\partial \eta_j}(0, \nabla'' f(y)) \right) f_{y_i y_j}, \\ T_3 &:= \sum_{m+1 \leq i,j \leq n} \frac{\partial A_i}{\partial \eta_j}(0, \nabla'' f(y))\, f_{y_i y_j}, \end{aligned} \tag{6.22}$$

where $\hat{\sum}_{i,j}$ means that the sum is taken over all i,j for which at least one of $i,j \leq m$. To estimate T_1 we note that if either i and/or $j \leq m$, then we obtain from Lemma 5.3 that

$$|f_{y_i y_j}| \leq c|y''|^{\xi - 1} \text{ when } y \in C_1(0) \setminus \Sigma. \tag{6.23}$$

Hence, from (6.23) and Definition 1.1 (i) it follows that

$$|T_1| \leq c|y''|^{\xi-1} |\nabla f(y)|^{p-2}. \tag{6.24}$$

We next estimate T_2 and T_3. From the definition of f, Lemma 5.3, and (6.18) we see that

$$1 - |y'|^2 \geq c^{-1}\, \tilde{c}_3\, |y''|^{1-\xi}, \tag{6.25}$$

where $c \geq 1$ depends only on p,n,m,α. From (6.25) and Lemma 5.3 we observe that

$$|\nabla' f(y)| \leq c'\, |y''|^{\xi} \text{ and } |\nabla'' f(y)| \geq \tilde{c}_3 / c', \tag{6.26}$$

where c' has the same dependence as c in (6.25). From (6.26) and condition (a) in Theorem 1.10, with $\eta = (\nabla' f(y), \nabla'' f(y))$, and $\eta' = (0, \nabla'' f(y))$, we see that if $\tilde{c}_3$ is large enough, then

$$\begin{aligned} \left| \frac{\partial A_i}{\partial \eta_j}(\nabla f(y)) - \frac{\partial A_i}{\partial \eta_j}(0, \nabla'' f(y)) \right| &\leq \lambda\, |\nabla' f(y)| |\nabla f(y)|^{p-3} \\ &\leq \lambda\, (c^* / \tilde{c}_3)\, |y''|^{\xi}\, |\nabla f(y)|^{p-2}, \end{aligned} \tag{6.27}$$

where again c^* depends only on p, n, m, α. Note that

$$f_{y_i y_j} = e^{\bar{u}}\,(1 - |y'|^2)\,(\,\bar{u}_{y_i}\bar{u}_{y_j} + \bar{u}_{y_i y_j}\,), \tag{6.28}$$

whenever $y \in C_1(0) \setminus \Sigma$ and $m+1 \leq i, j \leq n$. Using Lemma 3.6, (6.28), and Lemma 5.3 we find that

$$|f_{y_i y_j}| \leq c|y''|^{\xi-2}(1 - |y'|^2) \text{ when } y \in C_1(0) \setminus \Sigma \text{ and } m+1 \leq i, j \leq n. \tag{6.29}$$

Hence, using (6.27) and (6.29) we see that

$$|T_2| \leq (c/\tilde{c}_3)(1 - |y'|^2)|y''|^{2\xi-2}|\nabla f(y)|^{p-2}. \tag{6.30}$$

To estimate T_3 we first deduce, using (6.28), Lemma 5.2 and Lemma 5.3, as well as $(p-2)$ homogeneity of derivatives of A, that

$$T_3 = (1 - |y'|^2)\,e^{\bar{u}} \sum_{m+1 \leq i,j \leq n} \frac{\partial A_i}{\partial \eta_j}(0, \nabla'' f)\,\bar{u}_{y_i}\bar{u}_{y_j}. \tag{6.31}$$

Now, using Definition 1.1 (i), Lemma 5.3, (6.26), and the above equality it follows, for some $c = c(p, n, m, \alpha, \lambda) \geq 1$, that

$$T_3 \geq c^{-1}(1 - |y'|^2)|\nabla \bar{u}(y)|^2|\nabla'' f(y)|^{p-2} \geq c^{-2}\,(1 - |y'|^2)\,|y''|^{2\xi-2}|\nabla f(y)|^{p-2}. \tag{6.32}$$

In view of (6.30), (6.32), and (6.25) we see for $\tilde{c}_3$ large enough, depending on p, n, m, α, λ, that

$$\begin{aligned} \sum_{m+1 \leq i,j \leq n} \frac{\partial A_i}{\partial \eta_j}(\nabla f(y)) f_{y_i y_j} = T_2 + T_3 \;&\geq\; c^{-1}(1 - |y'|^2)|y''|^{2\xi-2}|\nabla f(y)|^{p-2} \\ &\geq\; \tilde{c}_3 c^{-2}|y''|^{\xi-1}|\nabla f(y)|^{p-2}. \end{aligned} \tag{6.33}$$

Combining (6.24) and (6.33) we conclude that if $\tilde{c}_3$ is sufficiently large, depending only on p, n, m, α, λ, then (6.19) holds. As a consequence, (6.17) is valid. (6.17) implies (6.14).

6.4 The lower bound in (6.4): A as in Theorem 1.10 (b)

Let $2 \leq m \leq n-2, A \in M_p(\alpha)$, and assume that $A \in M_p(\alpha)$ satisfies Theorem 1.10 (b). To complete the proof of Lemma 6.1 in the case we again have to prove (6.14). Since A now has constant coefficients in the y variable we write C for $C(y)$ and κ for $\kappa(y, \cdot)$. In the proof we assume, as we may, that C is a symmetric linear transformation since otherwise we can replace C by $(C + C^t)/2$, where C^t denotes the transpose of C, and note that the weak formulation of solutions is unchanged. Also since rotations preserve $M_p(\alpha)$, and functions homogeneous of degree 0, we may assume that C has a representation in the standard basis as a diagonal matrix. Finally, observe that dilations in the coordinate directions change $M_p(\alpha)$ into $M_p(\tilde{\alpha})$ with $\tilde{\alpha} \approx \alpha$ while κ remains homogeneous of degree 0. Thus we assume, as we may, that C is the identity transformation so that

$$A(\eta) = \kappa(\eta)|\eta|^{p-2}\eta \text{ and } \nabla \cdot (\kappa(\nabla v)|\nabla v|^{p-2}\nabla v) = 0 \text{ in } C_4(0) \setminus \Sigma, \text{ weakly}. \tag{6.34}$$

Let $\tilde{u}(y'') = |y''|^\xi$. Since this function is also a solution to the p-Laplace equation in $\mathbb{R}^{n-m} \setminus \{0\}$ we see from (6.34) that

$$\nabla \cdot (\kappa(\nabla \tilde{u})|\nabla \tilde{u}|^{p-2}\nabla \tilde{u}) = \langle \nabla\kappa(\nabla\tilde{u}), \nabla\tilde{u}\rangle |\nabla\tilde{u}|^{p-2} \text{ at } y'' \in \mathbb{R}^{n-m} \setminus \{0\}. \tag{6.35}$$

Moreover, using the degree zero homogeneity of κ, and Euler's equation, it follows that

$$\langle \nabla\kappa(\nabla\tilde{u}), \nabla\tilde{u}\rangle |\nabla\tilde{u}|^{p-2} = \xi |y''|^{(\xi-2)} \langle \nabla\kappa(y''), y''\rangle |\nabla\tilde{u}|^{p-2} = 0 \text{ at } y'' \in \mathbb{R}^{n-m} \setminus \{0\}. \tag{6.36}$$

In particular, $\tilde{u}$ is A-harmonic in $\mathbb{R}^{n-m} \setminus \{0\}$ and we can conclude, by uniqueness in Lemma 5.2, that if u_{n-m} is the fundamental solution on $\mathbb{R}^{n-m}$ in Lemma 5.3, relative to the A in (6.34), then

$$u_{n-m}(y'') = c|y''|^\xi, \;\; y'' \in \mathbb{R}^{n-m}, \tag{6.37}$$

for some $c = c(n, m, p)$.

We now proceed as in the proof lower bound in (6.4) in the case of Theorem 1.10 (a). Indeed, in this case we let, based on (6.37), $\bar{u}(y) = |y''|^\xi$ and we define f as in (6.16) using this $\bar{u}$. Again we prove (6.17), for sufficiently large $c_3 = c_3(p, n, m, \alpha)$, by proving that (6.19) is valid for A as in (6.34). In this case we let, using (6.28)

$$\nabla \cdot A(\nabla f)(y) = \sum_{i,j=1}^{n} \frac{\partial A_i}{\partial \eta_j}(\nabla f(y)) f_{y_i y_j} = S_1 + S_2 + S_3, \tag{6.38}$$

where now

$$\begin{aligned} S_1 &:= \hat{\sum}_{i,j} \frac{\partial A_i}{\partial \eta_j}(\nabla f(y)) f_{y_i y_j}, \\ S_2 &:= (1 - |y'|^2)\, e^{\bar{u}} \sum_{m+1 \le i,j \le n} \frac{\partial A_i}{\partial \eta_j}(\nabla f(y))\, \bar{u}_{y_i y_j}, \\ S_3 &:= (1 - |y'|^2)\, e^{\bar{u}} \sum_{m+1 \le i,j \le n} \frac{\partial A_i}{\partial \eta_j}(\nabla f(y))\, \bar{u}_{y_i} \bar{u}_{y_j}, \end{aligned} \tag{6.39}$$

where again $\hat{\sum}_{i,j}$ means the sum is taken over all i, j for which at least one of $i, j \le m$. Arguing as in the proofs of (6.24), (6.32), we see that

$$|S_1| \le c|y''|^{\xi-1} |\nabla f(y)|^{p-2}, \tag{6.40}$$

and

$$S_3 \ge c^{-1} (1 - |y'|^2)\, |y''|^{2\xi-2} |\nabla f(y)|^{p-2}, \tag{6.41}$$

at $y \in C_1(0) \setminus \Sigma$, respectively.

To estimate S_2 we note, for $1 \le i, j \le n$, that $\frac{\partial A_i}{\partial \eta_j}(\nabla f(y)) = b_{ij}(y) + c_{ij}(y)$, where at y,

$$\begin{aligned} b_{ij} &= \kappa(\nabla f)|\nabla f|^{p-4}[(p-2) f_{y_i} f_{y_j} + \delta_{ij}|\nabla f|^2], \\ c_{ij} &= |\nabla f|^{p-2} \kappa_{\eta_j}(\nabla f) f_{y_i}. \end{aligned} \tag{6.42}$$

In (6.42), δ_{ij} denotes the Kronecker delta. We write at $y \in C_1(0) \setminus \Sigma$,

$$S_2 = (1-|y'|^2)e^{\bar u} \sum_{m+1 \le i,j \le n} b_{ij} \bar u_{y_i y_j} + (1-|y'|^2)e^{\bar u} \sum_{m+1 \le i,j \le n} c_{ij} \bar u_{y_i y_j} = S_{21} + S_{22}. \tag{6.43}$$

Since $\bar u$ is also a solution to the p-Laplace equation it follows that, at $y \in C_1(0) \setminus \Sigma$,

$$S_{21} = (1-|y'|^2)e^{\bar u} |\nabla' f|^2 |\nabla f|^{p-4} \sum_{m+1 \le i,j \le n} \bar u_{y_i y_i}, \tag{6.44}$$

where $\nabla' f$ was defined in (6.20). Using (6.26) and (6.29) in (6.44) we obtain, for $y \in C_1(0) \setminus \Sigma$, that

$$|S_{21}(y)| \le (c/c_3)^2\,(1-|y'|^2)|\nabla f|^{p-2}|y''|^{3\xi-2}. \tag{6.45}$$

To estimate S_{22} we first observe, for $m+1 \le i,j \le n$, that

$$\bar u_{y_i} = \xi\, y_i |y''|^{\xi-2} \text{ and } \bar u_{y_i y_j} = \xi(\xi-2)\, y_i\, y_j |y''|^{\xi-4} + \xi \delta_{ij} |y''|^{\xi-2}. \tag{6.46}$$

We rewrite (6.46) as

$$\bar u_{y_i y_j} = e^{-2\bar u(y)}(1-2/\xi)\,(1-|y'|^2)^{-2}|y''|^{-\xi}\, f_{y_i}\, f_{y_j} + \xi\, \delta_{ij} |y''|^{\xi-2}. \tag{6.47}$$

Putting this expression for $\bar u_{y_i y_j}$ into S_{22}, and using the definition of c_{ij}, we have

$$\begin{aligned} S_{22} \;=\; & |\nabla f|^{p-2}\, e^{-\bar u(y)}(1-2/\xi)\,(1-|y'|^2)^{-1}|y''|^{-\xi} \sum_{m+1 \le i,j \le n} \kappa_{\eta_j}(\nabla f)\, f_{y_i}^2\, f_{y_j} \\ & + |\nabla f|^{p-2}\, e^{\bar u(y)}\, \xi\, (1-|y'|^2)|y''|^{\xi-2} \sum_{i=m+1}^{n} \kappa_{\eta_i}(\nabla f) f_{y_i}, \end{aligned} \tag{6.48}$$

whenever $y \in C_1(0) \setminus \Sigma$. Now using Definition 1.1 (i) it is not difficult to show that

$$|k(\eta)| + |\eta| \sum_{i=1}^{n} |\kappa_{\eta_i}| \;\le\; c$$

where c depends only on p, n, m, α. From this fact, 0 homogeneity of κ, and (6.26) we see that

$$\left| \sum_{i=m+1}^{n} \kappa_{\eta_i}(\nabla f) f_{y_i} \right| = \left| \sum_{i=1}^{m} \kappa_{\eta_i}(\nabla f) f_{y_i} \right| \le (c/c_3)|y''|^{\xi}. \tag{6.49}$$

Using (6.49) in (6.48) we arrive at

$$|S_{22}| \le (c/c_3)|(1-|y'|^2)|y''|^{2\xi-2}|\nabla f|^{p-2} \text{ for } y \in C_1(0) \setminus \Sigma. \tag{6.50}$$

Putting (6.50) and (6.45) into (6.43) we find (6.50) holds with S_{22} replaced by S_2. We can now complete the proof as in the proof of the lower bound in (6.4) in the case Theorem 1.10 (a). We omit further details.

Remark 6.4 *Note that if A, for fixed y, satisfies Theorem 1.10 (b), then A does not in general give rise to a rotationally symmetric solution in y', y'' even when $C(y) = I$ = the identity transformation. However, as explored in the proof, the fundamental solution in Lemma 5.2 for $C(y) = I$ is a radial solution having an extension to $\mathbb{R}^n$ that is symmetric in y', y''.*

7 Proof of Theorems 1.9, 1.10 and Corollaries 1.11, 1.12

In this section we prove Theorem 1.9, Theorem 1.10 and Corollary 1.11. As in section 4, we will use the convention that

$$\tilde{\theta} = 1 \text{ when } m = 1 \text{ and } \tilde{\theta} = \lambda \text{ when } 2 \leq m \leq n-2,$$

where λ is the constant appearing in Theorem 1.10 (a). The proofs of Theorem 1.9, Theorem 1.10 are based on the following two lemmas: Lemma 7.1 and Lemma 7.2.

Lemma 7.1 *Assume* (3.1) *and* $0 < \delta < \bar{\delta}$ *so that also* (3.2) *holds. If* $2 \leq m \leq n-2$, *assume in addition that either Theorem 1.10* (*a*) *or* (*b*) *hold. Let* $w \in \Sigma, 0 < r < r_0$. *Assume that* u *is a positive* A*-harmonic functions in* $B(w, 4r) \setminus \Sigma$, *continuous on* $B(w, 4r)$ *and* $u = 0$ *on* $\Sigma \cap B(w, 4r)$. *Then there exist* $\hat{\delta} = \hat{\delta}(p, n, m, \alpha, \beta, \gamma, \tilde{\theta})$, $\hat{c} = \hat{c}(p, n, m, \alpha, \beta, \gamma, \tilde{\theta})$ *and* $\bar{\lambda} = \bar{\lambda}(p, n, m, \alpha, \beta, \gamma, \tilde{\theta})$, *such that if* $0 < \delta \leq \hat{\delta}$, *then*

$$\bar{\lambda}^{-1} \frac{u(y)}{d(y, \Sigma)} \leq |\nabla u(y)| \leq \bar{\lambda} \frac{u(y)}{d(y, \Sigma)},$$

whenever $y \in B(w, r/\hat{c}) \setminus \Sigma$.

Lemma 7.2 *Assume* (3.1) *and* $0 < \delta < \bar{\delta}$ *so that also* (3.2) *holds. If* $2 \leq m \leq n-2$, *assume in addition that either Theorem 1.10* (*a*) *or* (*b*) *hold. Let* $w \in \Sigma$, $0 < r < \min\{r_0, 1\}$. *Assume that* u, v, *are positive* A*-harmonic functions in* $B(w, 4r) \setminus \Sigma$, *continuous on* $B(w, 4r)$ *and* $u = 0 = v$ *on* $\Sigma \cap B(w, 4r)$. *Then there exist* $\delta' = \delta'(p, n, m, \alpha, \beta, \gamma, \tilde{\theta})$, *and* $c = c(p, n, m, \alpha, \beta, \gamma, \tilde{\theta}) \geq 1$ *such that if* $0 < \delta < \delta'$, *and* $\hat{r} = r/c$, *then* $\hat{\lambda}_{u,v} := (|\nabla u| + |\nabla v|)^{p-2}$ *is an* $A_2(B(w, \hat{r}))$*-weight with constant depending only on* $p, n, m, \alpha, \beta, \gamma, \tilde{\theta}$.

7.1 Non-degeneracy of $|\nabla u|$: proof of Lemma 7.1

Given $w = (w', w'') \in \mathbb{R}^n$, r_1, r_2, $0 < r_1, r_2 < \infty$, recall the notation introduced in (3.4), (3.5). Using Lemma 3.8 and Lemma 3.9 we first prove the following lemma in the baseline case.

Lemma 7.3 *Assume* p, $n - m < p < \infty$, $1 \leq m \leq n-2$. *Assume that* $A \in M_p(\alpha, \beta, \gamma)$ *for some* (α, β, γ). *If* $2 \leq m \leq n-2$, *assume in addition that either Theorem 1.10* (*a*) *or* (*b*) *hold. Let* $\Sigma = \mathbb{R}^m \times \{0\}$ *and suppose that* u *is a positive* A*-harmonic function in* $C_1(0) \setminus \Sigma$, *continuous on the closure of* $C_1(0) \setminus \Sigma$, *and that* $u = 0$ *on* Σ. *Then there exist* $\hat{c} = \hat{c}(p, n, m, \alpha, \beta, \gamma, \tilde{\theta})$ *and* $\bar{\lambda} = \bar{\lambda}(p, n, m, \alpha, \beta, \gamma, \tilde{\theta})$, *such that*

$$\bar{\lambda}^{-1} \frac{u(y)}{d(y, \Sigma)} \leq |\nabla u(y)| \leq \bar{\lambda} \frac{u(y)}{d(y, \Sigma)} \text{ whenever } y \in C_{1/\hat{c}}(0) \setminus \Sigma.$$

Proof. Let $A \in M_p(\alpha, \beta, \gamma)$, $A = A(y, \eta)$, be given as in the statement of the lemma. Put $A_2(y, \eta) = A(y, \eta)$, $A_1(\eta) = A(0, \eta)$. Clearly, $A_1, A_2 \in M_p(\alpha, \beta, \gamma)$. We first note that Lemma 7.3 holds for the operator A_1. Indeed, assume that u is a positive A_1-harmonic function in $C_1(0) \setminus \Sigma$, continuous on the closure of $C_1(0) \setminus \Sigma$, and that $u = 0$ on Σ. Let $\hat{u}_1(y', y'') =$

$\bar{u}(y', y'') = u_{n-m}(y', y'')$ be as in Lemma 5.3. Then $\hat{u}_1$ is A_1-harmonic in $C_1(0) \setminus \Sigma$ and $\hat{u}_1 = 0$ on Σ. Let $\hat{u}_2 = u$. Then, as a consequence of Lemma 6.2 applied to the pair $\hat{u}_1, \hat{u}_2$, we see that

$$\left| \log\left(\frac{\hat{u}_1(y_1)}{\hat{u}_2(y_1)}\right) - \log\left(\frac{\hat{u}_1(y_2)}{\hat{u}_2(y_2)}\right) \right| \leq c|y_1 - y_2|^\sigma, \tag{7.1}$$

whenever $y_1, y_2 \in C_{1/16}(0) \setminus \Sigma$. Exponentiation of this inequality yields the equivalent inequality

$$\left| \frac{\hat{u}_1(y_1)}{\hat{u}_2(y_1)} - \frac{\hat{u}_1(y_2)}{\hat{u}_2(y_2)} \right| \leq c' \frac{\hat{u}_1(y_2)}{\hat{u}_2(y_2)} |y_1 - y_2|^\sigma, \tag{7.2}$$

whenever $y_1,\ y_2 \in C_{1/16}(0) \setminus \Sigma$ and for some c' depending at most on $p, n, m, \alpha, \tilde{\theta}$. Let $O = C_{1/16}(0) \setminus \Sigma$ and note that if $y_2 \in C_{1/32}(0) \setminus \Sigma$ then, see Lemma 5.3,

$$\frac{1}{\tilde{a}} \frac{\hat{u}_1(y_2)}{d(y_2, \partial O)} \leq |\nabla \hat{u}_1(y_2)| \leq \tilde{a} \frac{\hat{u}_1(y_2)}{d(y_2, \partial O)}, \tag{7.3}$$

for some $\tilde{a} = \tilde{a}(p, n, m, \alpha)$. Let r be defined through the relation $c' r^\sigma = \frac{1}{2}\tilde{\epsilon}$ where $\tilde{\epsilon}$ is as in Lemma 3.9. Using (7.2) we then see that

$$(1 - \tilde{\epsilon}/2) \frac{\hat{u}_1(y_2)}{\hat{u}_2(y_2)} \leq \frac{\hat{u}_1(y_1)}{\hat{u}_2(y_1)} \leq (1 + \tilde{\epsilon}/2) \frac{\hat{u}_1(y_2)}{\hat{u}_2(y_2)}, \tag{7.4}$$

whenever $y_1 \in B(y_2, r)$. From (7.3), (7.4), and Lemma 3.9 we conclude that Lemma 7.3 holds for the operator A_1.

Using the established conclusion for A_1 we now establish Lemma 7.3 for the operator A_2 using comparison principles. We let $\varrho \in (0, 1/16)$ and $\bar{\varrho} \in (0, 1/8)$ be degrees of freedom to be chosen below. Let $\hat{u}_1$ be the A_1-harmonic function in $C_{\bar{\varrho}/2}(0) \setminus \Sigma$ which is continuous on the closure of $C_{\bar{\varrho}/2}(0) \setminus \Sigma$ and which satisfies $\hat{u}_1 = u$ on $\partial(C_{\bar{\varrho}/2}(0) \setminus \Sigma)$. Then, using Lemma 7.3 for the operator A_1, we see that there exist $\lambda_1 = \lambda_1(p, n, m, \alpha, \tilde{\theta})$, $\hat{c}_1 = \hat{c}_1(p, n, m, \alpha, \tilde{\theta}) \geq 1$, such that

$$\lambda_1^{-1} \frac{\hat{u}_1(y)}{d(y, \Sigma)} \leq |\nabla \hat{u}_1(y)| \leq \lambda_1 \frac{\hat{u}_1(y)}{d(y, \Sigma)} \text{ whenever } y \in C_{\bar{\varrho}/\hat{c}_1}(0) \setminus \Sigma. \tag{7.5}$$

Moreover, using Definition 1.1 (*iii*) we have

$$|A_2(y, \eta) - A_1(y, \eta)| \leq \epsilon |\eta|^{p-2} \text{ with } \epsilon = 2\beta \bar{\varrho}^\gamma \text{ whenever } y \in C_{\bar{\varrho}}(0). \tag{7.6}$$

Let $\hat{u}_2 = u$. Using Lemma 3.8 we see there exist c', θ, τ, each depending only on $p, n, m, \alpha, \beta, \tilde{\theta}$, such that

$$|\hat{u}_2(y) - \hat{u}_1(y)| \leq c' \epsilon^\theta \varrho^{-\tau}\, \hat{u}_2(y) \text{ whenever } y \in C_{\bar{\varrho}/4}(0) \setminus C_{\bar{\varrho}/4, \varrho\bar{\varrho}}(0). \tag{7.7}$$

Let $\tilde{\epsilon}$ be as in the statement of Lemma 3.9 relative to λ_1 and put $\varrho = 1/(32\hat{c}_1)$. Fix $\bar{\varrho}$ subject to $c' \epsilon^\theta \varrho^{-\tau} = c' \, (2\beta \bar{\varrho}^\gamma)^\theta \varrho^{-\tau} = \min\{\tilde{\epsilon}/2, 10^{-8}\}$. In particular, we note that $\bar{\varrho} = \bar{\varrho}(p, n, m, \alpha, \beta, \tilde{\theta})$. Then from (7.7) we see that

$$1 - \tilde{\epsilon} \leq \frac{\hat{u}_2(y)}{\hat{u}_1(y)} \leq 1 + \tilde{\epsilon} \text{ whenever } y \in C_{\bar{\varrho}/4}(0) \setminus C_{\bar{\varrho}/4, \varrho\bar{\varrho}}(0). \tag{7.8}$$

Using (7.5), (7.8), and Lemma 3.9 we therefore conclude that

$$\lambda_2^{-1}\frac{\hat{u}_2(y)}{d(y,\Sigma_1(0))}\leq|\nabla\hat{u}_2(y)|\leq\lambda_2\frac{\hat{u}_2(y)}{d(y,\Sigma_1(0))}\text{ whenever } y\in C_{\bar{\varrho}/\hat{c}_1}(0)\setminus C_{\bar{\rho}/\hat{c}_1,2\varrho\bar{\varrho}}(0),\tag{7.9}$$

for some $\lambda_2=\lambda_2(p,n,m,\alpha,\beta,\gamma,\tilde{\theta})$. Moreover, if $y\in C_{\bar{\varrho}/\hat{c}_1,2\varrho\bar{\varrho}}(0)$, then we can also prove that (7.9) is valid at y by essentially repeating the previous argument and by making use of the invariance of the class $M_p(\alpha,\beta,\tilde{\theta})$, as well as of the conditions in Theorem 1.10 (a) and (b), with respect to translations and dilations. This completes the proof of Lemma 7.3. ■

Proof of Lemma 7.1. Let $A\in M_p(\alpha,\beta,\gamma)$, $A=A(y,\eta)$, be given as in the statement of the lemma. Let $w\in\Sigma$, $0<r<r_0$, suppose that u is a positive A-harmonic function in $B(w,4r)\setminus\Sigma$, continuous on $B(w,4r)$ and $u=0$ on $\Sigma\cap B(w,4r)$. We use Lemma 7.3 and Lemma 3.8 to prove Lemma 7.1. Let $c_1=\hat{c}$ be as in Lemma 7.3 and choose $c'\geq 100c_1$ so that if $\hat{y}\in B(w,r/c')\setminus\Sigma$, $s=4c_1 d(\hat{y},\Sigma)$, and $z\in\Sigma$ with $|\hat{y}-z|=d(\hat{y},\Sigma)$, then

$$\max_{B(z,4s)} u\ \leq\ cu(\hat{y}),\tag{7.10}$$

for some $c=c(p,n,m,\alpha,\beta,\gamma)$. Using Definition 1.5 with w,r replaced by $z,4s$, we see that there exists a m-dimensional hyperplane $\Lambda=\Lambda_m(z,4s)$, $z\in\Lambda$, such that

$$h(\Sigma\cap B(z,4s),\Lambda\cap B(z,4s))\leq 4\delta s.\tag{7.11}$$

For the moment we allow $\hat{\delta}$ in the statement of the lemma to vary but shall later fix it as a number satisfying several conditions. First, using that the class $M_p(\alpha,\beta,\gamma)$, as well as the conditions in Theorem 1.10 (a) and (b), are invariant under rotations, we again see that we may without loss of generality assume that $z=0$, $\Lambda=\{(y',y'')\in\mathbb{R}^m\times\mathbb{R}^{n-m},\ y''=0\}$. Thus if s' is the largest s' such that $C_{4s'}(0)\subset\overline{B(0,4s)}$. then, we have

$$h(\Sigma\cap C_{4s'}(0),\Lambda\cap C_{4s'}(0))\leq 4c'\delta s,\tag{7.12}$$

for some harmless constant c'. Next, we let v be a non-negative A-harmonic function in $C_{4s'}(0)$ with continuous boundary values on $\partial(C_{4s'}(0)\setminus\Lambda)$ defined as follows. We construct v such that $v=0$ on $C_{4s'}(0)\cap\Lambda$,

$$\begin{aligned} v(y) &= u(y) \text{ whenever } y\in\partial C_{4s'}(0)\setminus\partial C_{4s',30c'\delta s}(0),\\ v(y) &= 0 \quad\text{ whenever } y\in\partial C_{4s'}(0)\cap\partial C_{4s',20c'\delta s}(0),\end{aligned}$$

and

$$v\leq u \text{ on } \partial C_{4s'}(0)\cap(\partial C_{4s',30c'\delta s}(0)\setminus\partial C_{4s',20c'\delta s}(0)).$$

Then, by construction, using Lemma 3.3, we see that

$$u\leq v+c\delta^\sigma u(\hat{y}) \text{ on } \partial(C_{4s'}(0)\setminus C_{4s',20c'\delta s}(0)),\tag{7.13}$$

and hence the same holds, again by the maximum principle for A-harmonic functions, in $C_{4s'}(0)\setminus C_{4s',20c'\delta s}(0)$. Similarly,

$$v\leq u+c\delta^\sigma u(\hat{y}) \text{ on } C_{4s'}(0)\setminus C_{4s',20c'\delta s}(0).\tag{7.14}$$

In particular, using the Harnack inequality we can conclude that

$$(1+c\delta^{\sigma})^{-1} \leq \frac{u(y)}{v(y)} \leq (1-c\delta^{\sigma})^{-1} \text{ whenever } y \in B(\hat{y}, d(\hat{y},\Sigma)/4). \tag{7.15}$$

Furthermore, using Lemma 7.3, and the construction, we have

$$\hat{\lambda}^{-1}\frac{v(\hat{y})}{d(\hat{y},\Sigma)} \leq |\nabla v(\hat{y})| \leq \hat{\lambda}\frac{v(\hat{y})}{d(\hat{y},\Sigma)}, \tag{7.16}$$

for some $\hat{\lambda} = \hat{\lambda}(p,n,m,\alpha,\beta,\gamma,\tilde{\theta})$. In particular, from (7.15), (7.16), we see, if $0 < \delta < \hat{\delta}$ and if we fix $\hat{\delta} = \hat{\delta}(p,n,m,\alpha,\beta,\gamma)$ to be small enough, that the hypotheses of Lemma 3.9 are satisfied with $O = B(\hat{y}, d(\hat{y},\Sigma)/4)$ and $\tilde{a} = \hat{\lambda}$. Now, using Lemma 3.9 we can conclude that

$$\bar{\lambda}_1^{-1}\frac{u(\hat{y})}{d(\hat{y},\Sigma)} \leq |\nabla u(\hat{y})| \leq \bar{\lambda}_1\frac{u(\hat{y})}{d(\hat{y},\Sigma)},$$

for some $\bar{\lambda}_1 = \bar{\lambda}_1(p,n,m,\alpha,\beta,\gamma,\tilde{\theta})$. As $\hat{y} \in B(w, r/c') \setminus \Sigma$ is arbitrary, the proof of Lemma 7.1 is complete. ■

7.2 $(|\nabla u| + |\nabla v|)^{p-2}$ is an A_2-weight: proof of Lemma 7.2

Our proof of Lemma 7.2 is based on the following lemma.

Lemma 7.4 *Assume* (3.1) *and* $0 < \delta < \bar{\delta}$ *so that also* (3.2) *holds. If* $2 \leq m \leq n-2$, *assume in addition that either Theorem 1.10* (a) *or* (b) *hold. Let* $w \in \Sigma, 0 < r < r_0$. *Assume that* u *is a positive* A*-harmonic functions in* $B(w,4r) \setminus \Sigma$, *continuous on* $B(w,4r)$ *and* $u = 0$ *on* $\Sigma \cap B(w,4r)$. *Then there exist, for* $\epsilon^* > 0$ *given,* $\hat{\delta} = \hat{\delta}(p,n,m,\alpha,\beta,\gamma,\tilde{\theta},\epsilon^*) > 0$, $\hat{\delta} \leq \bar{\delta}$, *and* $c = c(p,n,m,\alpha,\beta,\gamma,\tilde{\theta},\epsilon^*)$, $1 \leq c < \infty$, *such that*

$$c^{-1}\left(\frac{\hat{r}}{r}\right)^{\xi(1+\epsilon^*)} \leq \frac{u(a_{\hat{r}}(w))}{u(a_r(w))} \leq c\left(\frac{\hat{r}}{r}\right)^{\xi(1-\epsilon^*)}$$

whenever $0 < \delta \leq \hat{\delta}$, $0 < \hat{r} < r/4$, *and where* $\xi = (p-n+m)/(p-1)$.

Proof. In the more traditional setting of Reifenberg flat domains in $\mathbb{R}^n$ a version of Lemma 7.4 is proved in Lemma 4.8 in [LLuN]. The proof is based on some rather straightforward, but still delicate, comparisons of non-negative solutions. Let $A \in M_p(\alpha,\beta,\gamma)$, $A = A(y,\eta)$, be given as in the statement of the lemma. Set $A_2(y,\eta) = A(y,\eta)$, $A_1(\eta) = A(w,\eta)$. Then $A_1, A_2 \in M_p(\alpha,\beta,\gamma)$. Let u be a A_2-harmonic function as in the statement of the lemma. Observe, using Definition 1.5, that it suffices to prove Lemma 7.4 for $\delta = \hat{\delta}$. Moreover, we can without loss of generality assume that $r = 4$, $w = 0$ and $u(a_1(0)) = 1$. In the following we let $\check{\delta}$, $\check{\delta} \leq \hat{\delta}$, and ϱ be small constants to be chosen below. In particular, $\check{\delta}, \varrho$ will be fixed to depend only on $p,n,m,\alpha,\beta,\gamma,\tilde{\theta}$. For ϱ fixed we can again, without loss of generality, also assume that

$$h(\Sigma \cap B(0,4\varrho), \Lambda \cap B(0,4\varrho)) \leq 4\check{\delta}\varrho,$$

and where $\Lambda = \{(y', y'') \in \mathbb{R}^m \times \mathbb{R}^{n-m},\ y'' = 0\}$. In particular, we see that to prove Lemma 7.4 it suffices to prove that

$$c^{-1}\hat{r}^{\xi(1+\epsilon^*)} \leq u(a_{\hat{r}}(0)) \leq c\hat{r}^{\xi(1-\epsilon^*)} \text{ whenever } 0 < \hat{r} < \varrho. \tag{7.17}$$

To begin the proof of (7.17) we introduce an auxiliary function u^+. In particular, we define u^+ to be A_2-harmonic in $C_\varrho(0) \setminus \Lambda$ with continuous boundary values on $\partial(C_\varrho(0) \setminus \Lambda)$ defined as follows. We let $u^+ = 0$ on $C_\varrho(0) \cap \Lambda$,

$$\begin{aligned} u^+(y) &= u(y) \ \text{ if } y \in \partial(C_\varrho(0)) \setminus \partial(C_{\varrho,16\check{\delta}\varrho}(0)), \\ u^+(y) &= 0 \quad\ \ \text{ if } y \in \partial(C_\varrho(0)) \cap \partial(C_{\varrho,8\check{\delta}\varrho}(0)). \end{aligned}$$

Furthermore, on $\partial(C_\varrho(0)) \cap (\partial(C_{\varrho,16\check{\delta}\varrho}(0)) \setminus \partial(C_{\varrho,8\check{\delta}\varrho}(0)))$ we define u^+ so that $u^+ \leq u$. Now, arguing as in the proof of (7.13), (7.14), we see that

$$u \leq u^+ + c\check{\delta}^\sigma u(a_{\varrho/4}(0)),\ u^+ \leq u + c\check{\delta}^\sigma u(a_{\varrho/4}(0)) \text{ on } C_\varrho(0) \setminus C_{\varrho,8\check{\delta}\varrho}(0), \tag{7.18}$$

for some $\sigma = \sigma(p, n, m, \alpha, \beta, \gamma)$, $\sigma \in (0, 1)$. Using Definition 1.5 (iii) we next note that

$$|A_2(y, \eta) - A_1(y, \eta)| \leq \epsilon|\eta|^{p-1} \text{ whenever } y \in C_\varrho(0),\ \epsilon = 2\beta\varrho^\gamma. \tag{7.19}$$

To proceed we let $\bar{u}^+$ be the A_1-harmonic function in $C_{\varrho/2}(0) \setminus \Lambda$ which is continuous on the closure of $C_{\varrho/2}(0) \setminus \Lambda$ and which coincides with u^+ on $\partial(C_{\varrho/2}(0) \setminus \Lambda)$. Finally, we define $v^+(y) := |y''|^\xi$ whenever $y \in \mathbb{R}^n$. To prove the right hand inequality in (7.17), we first see, using (7.19) and Lemma 3.8, that

$$u^+(y) \leq (1 - \tilde{c}\epsilon^\theta\check{\delta}^{-\tau})^{-1}\bar{u}^+(y) \text{ for } y \in C_{\varrho/4}(0) \setminus C_{\varrho/4,4\check{\delta}\varrho}(0), \tag{7.20}$$

for constants $\tilde{c}$, θ, τ, depending only on $p, n, m, \alpha, \beta, \gamma$. Then, as a consequence of Lemma 6.2 and Lemma 5.3, see (6.4), we can conclude that there exists a constant $\bar{c} = \bar{c}(p, n, m, \alpha, \tilde{\theta})$, $1 \leq \bar{c} < \infty$, such that

$$u^+(y) \leq (1 - \tilde{c}\epsilon^\theta\check{\delta}^{-\tau})^{-1}\bar{u}^+(y) \leq c(1 - \tilde{c}\epsilon^\theta\check{\delta}^{-\tau})^{-1}\,\bar{u}^+(a_{\varrho/4}(0))\frac{v^+(y)}{\varrho^\xi}, \tag{7.21}$$

whenever $y \in C_{\varrho/\bar{c}}(0) \setminus C_{\varrho/\bar{c},4\check{\delta}\varrho}(0)$. In particular, using (7.18), (7.21) and the Harnack inequality we see that

$$u(y) \leq c(1 - \tilde{c}\epsilon^\theta\check{\delta}^{-\tau})^{-1}\,\bar{u}^+(a_{\varrho/8}(0))\frac{v^+(y)}{\varrho^\xi} + c\check{\delta}^\sigma u(a_{\varrho/8}(0)), \tag{7.22}$$

whenever $y \in C_{\varrho/\bar{c}}(0) \setminus C_{\varrho/\bar{c},4\check{\delta}\varrho}(0)$. We now let $\tilde{\delta}$ be defined through the relation

$$\tilde{\delta}^\xi = \max\{\check{\delta}^\xi, \check{\delta}^\sigma\} \tag{7.23}$$

Note that $\tilde{\delta} \geq \check{\delta}$ and applying (7.22) for $y = a_{8\tilde{\delta}\varrho}(0)$ we see, as long as

$$a_{8\tilde{\delta}\varrho}(0) \in C_{\varrho/\bar{c}}(0) \setminus C_{\varrho/\bar{c},4\check{\delta}\varrho}(0), \tag{7.24}$$

that

$$\begin{aligned} u(a_{8\tilde\delta\varrho}(0)) &\leq c(1-\tilde c\epsilon^\theta\check\delta^{-\tau})^{-1}\,\bar u^+(a_{\varrho/8}(0))(8\tilde\delta)^\xi + c\check\delta^\sigma u(a_{\varrho/8}(0))\\ &\leq \biggl(c(1-\tilde c\epsilon^\theta\check\delta^{-\tau})^{-1}(8\tilde\delta)^\xi + c\tilde\delta^\xi\biggr)u(a_{\varrho/8}(0)), \end{aligned} \tag{7.25}$$

where we have also used that $\bar u^+(a_{\varrho/8}(0)) \approx u(a_{\varrho/8}(0))$. In particular, simply using the Harnack inequality once more, and the normalization $u(a_1(0)) = 1$, we see that

$$u(a_{\tilde\delta\varrho}(0)) \leq \biggl(c(1-\tilde c\epsilon^\theta\check\delta^{-\tau})^{-1}(8\tilde\delta)^\xi + c\tilde\delta^\xi\biggr). \tag{7.26}$$

Next, let $\check\delta < 1/(16\bar c)$ and let ϱ be defined though the relation

$$1/2 = \tilde c\epsilon^\theta\check\delta^{-\tau} = \tilde c(2\beta\varrho^\gamma)^\theta\ \check\delta^{-\tau}. \tag{7.27}$$

Then $\varrho = \varrho(p,n,m,\alpha,\beta,\gamma,\tilde\theta,\check\delta) = \varrho(p,n,m,\alpha,\beta,\gamma,\tilde\theta,\tilde\delta)$ and

$$u(a_{\tilde\delta\varrho}(0)) \leq \hat c\tilde\delta^\xi. \tag{7.28}$$

We now proceed by induction and we suppose that we have shown, for some $k \in \{1,2,...\}$, that

$$u(a_{\tilde\delta^k\varrho}(0))\ \leq (\hat c\tilde\delta^\xi)^k, \tag{7.29}$$

for some $\hat c$ depending at most on $p,n,m,\alpha,\beta,\gamma,\tilde\theta$. Then, again using Definition 1.5 we see there exists $\Lambda' \in \Lambda_m(0)$ such that

$$h(\Sigma\cap B(0,4\tilde\delta^k\varrho), \Lambda'\cap B(0,4\tilde\delta^k\varrho)) \leq 4\check\delta\tilde\delta^k\varrho.$$

We can now repeat the above argument with Λ replaced by Λ' and 4 replaced by $4\tilde\delta^k$ and with cylinders of size defined by $\tilde\delta^k\varrho$ instead of ϱ. As a result we see that

$$u(a_{\tilde\delta^{k+1}\varrho}(0)) \leq \hat c\tilde\delta^\xi u(a_{\tilde\delta^k\varrho}(0)) \leq (\hat c\tilde\delta^\xi)^{k+1}, \tag{7.30}$$

by the induction hypothesis. In particular, by induction we see that the inequality in (7.29) is true for all positive integers k. Next we fix $\tilde\delta$ through the relation

$$\tilde\delta^{-\xi\epsilon^*} = \hat c, \tag{7.31}$$

where $\hat c$ is the constant in (7.30). Then $\tilde\delta$, as well as ρ, depend only on $p,n,m,\alpha,\beta,\gamma,\tilde\theta$ and ϵ^*. Moreover, given $0 < \hat r < \varrho$, let k be the smallest integer such that $\tilde\delta^k\varrho \leq \hat r$. Then, simply using the Harnack inequality, (7.29), and our choice of $\tilde\delta$ in (7.31) we see that $u(a_{\hat r}(0)) \leq c\hat r^{\xi(1-\epsilon^*)}$, for some $c = c(p,n,m,\alpha,\beta,\gamma,\tilde\theta,\epsilon^*)$ and hence the proof of the right-hand side inequality in (7.17) is complete.

To prove the left-hand side inequality in (7.17) we argue in a similar manner. Indeed, in this case we first see that

$$u(y) \geq u^+(y) - c\check\delta^\sigma u(a_{\varrho/4}(0)) \geq (1+\tilde c\epsilon^\theta\check\delta^{-\tau})^{-1}\bar u^+(y) - c\check\delta^\sigma u(a_{\varrho/4}(0)), \tag{7.32}$$

for $y \in C_\varrho(0) \setminus C_{\varrho,8\check{\delta}_\varrho}(0)$ and then, again as a consequence Lemma 6.2 and Lemma 5.3, see (6.4), and familiar arguments, we deduce that

$$u(a_{32\tilde{\delta}_\varrho}(0)) \geq \hat{c}^{-1}\tilde{\delta}^\xi, \tag{7.33}$$

for some $\hat{c}$ depending only on $p, n, m, \alpha, \beta, \gamma, \tilde{\theta}$. The left-hand side inequality in (7.17) then follows as above by induction. We omit further details. ■

Proof of Lemma 7.2. Assume (3.1) and $0 < \delta < \bar{\delta}$ so that also (3.2) holds. Let $w \in \Sigma$, $0 < r < \min\{r_0, 1\}$. Assume that u, v, are positive A-harmonic functions in $B(w, 4r) \setminus \Sigma$, continuous on $B(w, 4r)$ and $u = 0 = v$ on $\Sigma \cap B(w, 4r)$. We want to prove that there exist $\delta' = \delta'$ and $c \geq 1$ depending only on the data (i.e, $p, n, m, \alpha, \beta, \gamma, \tilde{\theta}$) such that if $0 < \delta < \delta'$, and $\hat{r} = r/c$, then $\hat{\lambda}_{u,v}(y) := (|\nabla u(y)| + |\nabla v(y)|)^{p-2}$ is an $A_2(B(w, \hat{r}))$-weight with constant depending only on the data. To start the proof we first see, using Lemma 7.1 that there exist $\hat{\delta}$, $\hat{c}$ and $\bar{\lambda}$, depending on the data such that if $0 < \delta \leq \hat{\delta}$, then

$$\bar{\lambda}^{-1}\tilde{\lambda}_{u,v} \leq \hat{\lambda}_{u,v} \leq \bar{\lambda}\tilde{\lambda}_{u,v}, \text{ whenever } y \in B(w, r/\hat{c}) \setminus \Sigma, \tag{7.34}$$

and where

$$\tilde{\lambda}_{u,v}(y) := \left(\frac{u(y)}{d(y, \Sigma)} + \frac{v(y)}{d(y, \Sigma)}\right)^{p-2}. \tag{7.35}$$

We now simply let $\hat{r} = r/(100\hat{c}^2)$ and we consider $\tilde{w} \in B(w, \hat{r})$ and $\tilde{r} \leq \hat{r}$. We want to prove

$$\Gamma(\tilde{w}, \tilde{r}) := \tilde{r}^{-2n} \int_{B(\tilde{w},\tilde{r})} \hat{\lambda}_{u,v}\, dy \cdot \int_{B(\tilde{w},\tilde{r})} \hat{\lambda}_{u,v}^{-1}\, dy \leq c^*, \tag{7.36}$$

where c^* depends only on the data. To do this we first note from Harnack's inequality that if $d(\tilde{w}, \Sigma) \geq 2\tilde{r}$, then $\Gamma(\tilde{w}, \tilde{r}) \leq c$, and hence we can assume that $d(\tilde{w}, \Sigma) \leq 2\tilde{r}$. In the latter case we let $\hat{w} \in \Sigma$ be such that $|\tilde{w} - \hat{w}| = d(\tilde{w}, \Sigma)$. Now, from the definition of $\hat{\lambda}_{u,v}$ Lemma 3.1-Lemma 3.5, and Hölder's inequality it follows that

$$\int_{B(\tilde{w},\tilde{r})} \hat{\lambda}_{u,v}\, dy \leq c\tilde{A}\,\tilde{r}^{n+2-p}, \tag{7.37}$$

where $\tilde{A} := (u(a_{\tilde{r}}(\hat{w}))^{p-2} + v(a_{\tilde{r}}(\hat{w}))^{p-2})$. Next, we let

$$\eta = \min\{1, (n - m + (1 - \xi)(p - 2))/(\xi(p - 2))\}/20.$$

Then, using Lemma 7.4 we see, for $\hat{\delta}$, small enough, that

$$c\tilde{u}(y) \geq \tilde{u}(a_{\tilde{r}}(\hat{w}))\left(\frac{d(y, \Sigma)}{\tilde{r}}\right)^{\xi(1+\eta)}, \ \tilde{u} \in \{u, v\}, \tag{7.38}$$

whenever $y \in B(\hat{w}, 50\tilde{r}) \setminus \Sigma$. Using (7.38) and (7.34), we deduce that

$$\int_{B(\tilde{w},\tilde{r})} \hat{\lambda}_{u,v}^{-1}\, dy \ \leq \ c\tilde{r}^{\xi(1+\eta)(p-2)}\tilde{A}^{-1} \int_{B(\hat{w},50\tilde{r})} d(y, \Sigma)^{(1-\xi(1+\eta))(p-2)}\, dy. \tag{7.39}$$

In particular from (7.36), (7.37), we see that

$$\Gamma(\tilde{w},\tilde{r}) \leq c\tilde{r}^{-2n}\tilde{r}^{n+2-p}\tilde{r}^{\xi(1+\eta)(p-2)} \int\limits_{B(\hat{w},50\tilde{r})} d(y,\Sigma)^{(1-\xi(1+\eta))(p-2)}\, dy. \tag{7.40}$$

To complete the estimate in (7.40) we define

$$I(z,s) = \int\limits_{B(z,s)} d(y,\Sigma)^{(1-\xi(1+\eta))(p-2)}\, dy,$$

whenever $z \in \Sigma \cap B(w, r/100)$, $0 < s < r/100$. Let

$$E_k = B(z,s) \cap \{y : d(y,\partial\Omega) \leq \delta^k s\} \text{ for } k = 1,2,\ldots$$

and recall that $1 \leq m \leq n-2$, $\Sigma \subset \mathbb{R}^n$ is a closed set, Σ is (m, r_0, δ)-Reifenberg flat (in $\mathbb{R}^n$) for some $r_0, \delta > 0$. We prove that

$$\int_{E_k} dy \leq c_+^k \delta^{(n-m)k} s^n \text{ for } k = 1,2,\ldots. \tag{7.41}$$

Indeed, using that Σ is (m, r_0, δ)-Reifenberg flat we see that E_1 can be covered by at most c/δ^m balls of radius $100\delta s$, and with centers in $\Sigma \cap B(z,s)$, and hence (7.41) follows readily for $k = 1$. One can then repeat this argument in each of the balls to get that (7.41) holds for E_2. Continuing in this way, arguing by induction, we get (7.41) for all positive integers k. Using (7.41) and writing $I(z,s)$ as a sum over $E_k \setminus E_{k+1}$, $k = 1,2,\ldots$ we get

$$\begin{aligned} I(z,s) &\leq cs^{n+(1-\xi(1+\eta))(p-2)} + \sum_{k=1}^{\infty} (c_+^k \delta^{(n-m)k} s^n)(\delta^k s)^{(1-\xi(1+\eta))(p-2)} \\ &\leq \tilde{c} s^{n+(1-\xi(1+\eta))(p-2)}, \end{aligned} \tag{7.42}$$

where $\tilde{c} = \tilde{c}(p,n,m)$, provided δ' is small enough by the choice of η. Using this estimate with $s = \tilde{r}$, we can continue our calculation in (7.40) and conclude that

$$\Gamma(\tilde{w},\tilde{r}) \leq c\tilde{r}^{-2n}\tilde{r}^{n+2-p}\tilde{r}^{\xi(1+\eta)(p-2)}\tilde{r}^{n+(1-\xi(1+\eta))(p-2)} \leq c. \tag{7.43}$$

The proof of Lemma 7.2 is now complete. ■

7.3 The final proof of Theorem 1.9 and Theorem 1.10

Assuming (3.1) and $0 < \delta < \bar{\delta}$, and using Lemma 7.2, we see that Theorem 1.9 and Theorem 1.10 follow immediately from Lemma 4.10.

7.4 Proof of Corollary 1.11

Let $u, v, n, m, p, \Sigma, w, r_0, A, \sigma$ be as in Theorem 1.9 or Theorem 1.10 and let μ, ν, be the corresponding measures as in (1.5). If $z \in B(w, 2r) \setminus \Sigma$, then from these theorems, with w replaced by z, we see that

$$\left| \frac{u(x)}{v(x)} - \frac{u(y)}{v(y)} \right| \leq c \frac{u(x)}{v(x)} \left(\frac{|x-y|}{r} \right)^{\sigma}, \tag{7.44}$$

whenever $x, y \in B(z, r/c) \setminus \Sigma$. From (7.44) we deduce that

$$0 < f(z) = \lim_{y \to z} \frac{u(y)}{v(y)} \text{ exists,}$$

and that (7.44) holds with $u(y)/v(y)$ replaced by $f(z)$. Hence there exists c', depending only on the data, such that if $0 < s < r/c$ and $x \in B(z, s) \setminus \Sigma$, then

$$u(x)(1 - c'(s/r)^{\sigma}) < f(z)\, v(x) < u(x)(1 + c'(s/r)^{\sigma}). \tag{7.45}$$

Set

$$\tau_1 = \frac{f(z)}{(1 + c'(s/r)^{\sigma})}, \quad \tilde{v}(x) = \tau_1 v, \text{ and } h = u - \tilde{v} > 0 \text{ in } B(z, s) \setminus \Sigma.$$

Given $\psi \in C_0^{\infty}(B(z, s))$ and small positive numbers θ_1, θ_2, we put $\phi = \max\{h - \theta_1, 0\}^{\theta_2} \psi$. Arguing as in (3.8) we see that

$$0 \leq \int \langle A(x, \nabla u) - A(x, \nabla \tilde{v}), \nabla(\max\{h - \theta_1, 0\}^{\theta_2}) \rangle \psi \, dx. \tag{7.46}$$

Also from the usual limiting argument we find that ϕ can be used as a test function in the weak formulation of A-harmonicity for both $u, \tilde{v}$. Doing this, using (7.46), and letting first $\theta_1 \to 0$, and then $\theta_2 \to 0$, we conclude from (7.46) and (1.5) that

$$\int \psi \, (\tau_1^{p-1} d\nu - d\mu) \leq \int_{B(z,s)} \langle A(x, \nabla u) - A(x, \nabla \tilde{v}), \nabla \psi \rangle \, dx \, \leq 0, \tag{7.47}$$

where we have also used $(p-1)$-homogeneity of A in Definition 1.1 (iii) to deduce the measure corresponding to $\tilde{v}$. From arbitrariness of ψ it follows that $\tau_1^{p-1} \nu \leq \mu$ on $B(z, s) \cap \Sigma$. Similarly if $\tau_2 = \frac{f(z)}{(1 - c'(s/r)^{\sigma})}$ then $\mu \leq \tau_2^{p-1} \nu$ on $B(z, s) \cap \Sigma$. From this discussion we see that μ, ν are mutually absolutely continuous on $B(w, 4r_0)$ and if $d\mu = k \, d\nu$, then

$$\tau_1^{p-1} \leq k(\hat{z}) \leq \tau_2^{p-1} \text{ when } \hat{z} \in B(z, s) \cap \Sigma \text{ and } k(z) = f(z)^{p-1}. \tag{7.48}$$

Taking logarithms it follows that

$$c^{-1}(s/r)^{\sigma} \; \leq \; |\log(k(\hat{z})/k(z))| \; \leq c(s/r)^{\sigma}, \tag{7.49}$$

for some $c \geq 1$ depending only on the data. From (7.49) and arbitrariness of s, z we conclude that Corollary 1.11 is valid.

7.5 Proof of Corollary 1.12

The proof of Corollary 1.12 is by contradiction. If Corollary 1.12 is false there exist $\epsilon > 0$ and $t_j \in [1/2, 1], x_j \in \Sigma \cap \overline{B(w,r)}, 0 < r_j \leq 10^{-j} r$, for $j = 1, \dots,$ such that

$$\epsilon \leq \left| \frac{\mu(B(x_j, t_j r_j))}{\mu(B(x_j, r_j))} - t_j^m \right|. \tag{7.50}$$

We assume, as we may, that $t_j \to t \in [1/2, 1]$ and $x_j \to \hat{x} \in \Sigma \cap \overline{B(w,r)}$ as $j \to \infty$. Let

$$u_j(x) = \frac{u(x_j + r_j x)}{u(a_{r_j}(x_j))} \text{ whenever } x \in \Omega_j = \{x : x_j + r_j x \in B(w, 2r) \setminus \Sigma\}.$$

Let $A_j(x, \eta) = A(x_j + r_j x, \eta)$ when $x, \eta \in \mathbb{R}^n$. Using the $(p-1)$-homogeneity of A, see Definition 1.1, we see that u_j is a weak solution to $\nabla \cdot A_j(x, \nabla u_j) = 0$ in Ω_j. Note that A_j has the same structure constants as do A in $(i), (iii)$, of Definition 1.1, while β in (ii) is replaced by βr_j^γ. From the vanishing Reifenberg flat assumption in Corollary 1.12 we see, for a subsequence of (Ω_j) (also denoted (Ω_j)), that $\partial\Omega_j \to \Lambda$, where Λ is a m-dimensional hyperplane through 0, as $j \to \infty$, uniformly in the Hausdorff distance sense on compact subsets of $\mathbb{R}^n$. From Lemmas 3.1, 3.3, and 3.4, as well as Harnack's inequality, and the NTA property of Ω_j we see, given $R > 0$, that there exists j_0 such that whenever $j \geq j_0$, then u_j is Hölder continuous with exponent σ and the Hölder norm of u_j in $B(0, R)$ is uniformly bounded. Also given K, a compact subset of $\mathbb{R}^n \setminus \Lambda$, we find from Lemma 3.6 that ∇u_j is $\hat{\sigma}$ Hölder continuous on K with a uniformly bounded Hölder norm for j large enough. Moreover, from these Lemmas, we conclude that (u_j) is bounded in the norm of $W^{1,p}(B(0, R))$.

Using these facts we obtain from Ascoli's theorem that subsequences of $(u_j), (\nabla u_j)$ (also denoted $(u_j), (\nabla u_j)$), converge uniformly on compact subsets of $\mathbb{R}^n, \mathbb{R}^n \setminus \Lambda$, to $\hat{u}, \nabla \hat{u}$. From weak compactness of $W^{1,p}$ we may also assume that $u_j \to \hat{u}$ weakly in $W^{1,p}(B(0, R))$ for each $R > 0$. By construction, $\hat{u}$ is σ Hölder continuous in $\mathbb{R}^n$ and $\hat{u} \equiv 0$ on Λ. It is also easily seen that $\hat{u}$ is $\hat{A}$-harmonic in $\mathbb{R}^n \setminus \Lambda$ with $\hat{A}(\eta) = A(\hat{x}, \eta), \eta \in \mathbb{R}^n \setminus \{0\}$. To reach a contradiction we assume, as we may, that $\Lambda = \mathbb{R}^m \times \{0\}$. Indeed, otherwise we first rotate the coordinate system so that Λ becomes $\mathbb{R}^m \times \{0\}$ and so that $\hat{u}$ becomes u', u' being a weak solution to $\nabla \cdot A'(\nabla u') = 0$. We then apply the following argument to u'.

Applying Theorem 1.9 or 1.10 with u, v replaced by $\hat{u}, u_{n-m}$, u_{n-m} as in Lemma 5.3, and then letting $r \to \infty$, we see that $\hat{u}$ is a constant multiple of u_{n-m}. Using this and Lemma 5.3 we deduce that the measure, say $\hat{\mu}$, corresponding to $\hat{u}$, is a constant multiple of Lebesgue measure on $\mathbb{R}^m \times \{0\}$. Let μ_j be the measure corresponding to u_j, for $j = 1, 2, \dots$ Using the above convergence results, we easily deduce that $\mu_j \to \hat{\mu}$ weakly as measures. From weak convergence and the fact that $\hat{\mu}(B(0, s))$ is a constant multiple of s^m when $s \in (0, 1]$, we conclude

$$\lim_{j \to \infty} \frac{\mu_j(B(0, t_j))}{\mu_j(B(0, 1))} = \frac{\hat{\mu}(B(0, t))}{\hat{\mu}(B(0, 1))} = t^m. \tag{7.51}$$

Finally we note from $(p-1)$-homogeneity of A that

$$\frac{\mu_j(B(0, t_j))}{\mu_j(B(0, 1))} = \frac{\mu(B(x_j, t_j r_j))}{\mu(B(x_j, r_j))} \text{ for } j = 1, 2, \dots \tag{7.52}$$

Using (7.50)-(7.52) we deduce that

$$\epsilon \leq \lim_{j\to\infty} \left| \frac{\mu(B(x_j, t_j r_j))}{\mu(B(x_j, r_j))} - t_j^m \right| = \lim_{j\to\infty} \left| \frac{\mu_j(B(0, t_j))}{\mu_j(B(0,1))} - t_j^m \right| = 0. \tag{7.53}$$

We have reached a contradiction. Hence Corollary 1.12 is valid.

8 Proof of Theorem 1.13

To begin the proof of Theorem 1.13 we assume that Σ is (m, r_0, δ)-Reifenberg flat with $0 < \delta < \tilde{\delta}$, where $\tilde{\delta}$ is the constant appearing in Theorem 1.9 or 1.10. We start by making several key observations. First, if $\hat{u}, \hat{v}, w, r, \Sigma$ are as in the statement of Theorem 1.13, then by Theorem 1.9 or 1.10, and Harnack's inequality, it follows that

$$\sup_{\partial B(w,s)\setminus\Sigma} \frac{\hat{u}}{\hat{v}} \leq c \inf_{\partial B(w,s)\setminus\Sigma} \frac{\hat{u}}{\hat{v}} \text{ as } s\to 0. \tag{8.1}$$

In particular, there exists $K > 0$ such that

$$K \leq \hat{u}/\hat{v} \leq cK \text{ in } B(w,r) \setminus \Sigma, \tag{8.2}$$

where c depends only on the data. Indeed, suppose $\hat{u}/\hat{v}$ is unbounded in $B(w,r) \setminus \Sigma$. Then from the maximum principle for A-harmonic functions we see that

$$\sup_{\partial B(w,s)\setminus\Sigma} \frac{\hat{u}}{\hat{v}} \to \infty \text{ as } s\to 0.$$

Using (8.1) it follows that

$$\sup_{\partial B(w,s)\setminus\Sigma} \frac{\hat{u}}{\hat{v}} \leq c \inf_{\partial B(w,s)\setminus\Sigma} \frac{\hat{u}}{\hat{v}} \to \infty \text{ as } s\to 0. \tag{8.3}$$

The maximum principle for A-harmonic functions then implies that $\hat{v} \equiv 0$ in $B(w,r)$. From this contradiction and the same argument as in (8.3) we conclude the validity of (8.2). Second, suppose $0 < s << 4r < \tilde{r}_0$, where $\tilde{r}_0 = \min\{r_0, 1\}$, and suppose that $\bar{u}$ is an A-harmonic function in $B(w,4r) \setminus (\Sigma \cup \overline{B(w,s)})$ with $\bar{u} = 0$ continuously on $\Sigma \setminus \overline{B(w,s)}$. We can apply Lemma 7.1 to conclude that there exist $\delta^*, 0 < \delta^* < 1, \bar{c}, \bar{\lambda} \geq 1$, depending only on the data such that if $0 < \delta \leq \delta^*$, and $\hat{y} \in (\Sigma \cap B(w,2r)) \setminus B(w,2s)$, then the 'fundamental inequality',

$$\bar{\lambda}^{-1} \frac{\bar{u}(y)}{d(y,\Sigma)} \leq |\nabla \bar{u}(y)| \leq \bar{\lambda} \frac{\bar{u}(y)}{d(y,\Sigma)}, \tag{8.4}$$

holds whenever $y \in B(\hat{y}, |\hat{y} - w|/\bar{c}) \setminus \Sigma$. Using this fact we see that if $0 < \delta \leq \delta^*$, then there exists $\tilde{\eta}$, depending only on the data such that if we define a non-tangential approach region at w, denoted $\tilde{\Omega}(w, \tilde{\eta})$, by $\tilde{\Omega}(w, \tilde{\eta}) = \{y \in B(w, \tilde{r}_0) : d(y, \Sigma) \geq \tilde{\eta}|y - w|\}$, then

$$\bar{u} \text{ satisfies (8.4) } \quad \text{for } y \in B(w, 2r) \setminus [\tilde{\Omega}(w, \tilde{\eta}) \cup B(w, 2s)]. \tag{8.5}$$

To prove Theorem 1.13 we now use (8.2)-(8.5), and proceed essentially along the same proof scheme as used in the proofs of Theorems 1.9, 1.10. In particular, we first prove that the quantitative estimates underlying the conclusion in Theorem 1.13 are true in the baseline case when $\Sigma = \mathbb{R}^m \times \{0\}$. We then use this to complete the proof in the general case. We will use the following lemma.

Lemma 8.1 *Under the structure assumptions of either Theorem 1.9 or Theorem 1.10 suppose* $0 < s < r/100, w = 0$, $\Sigma = \mathbb{R}^m \times \{0\}$, $A \in M_p(\alpha)$. *Let* $\Sigma_1 = \Sigma \setminus B(0, 2s)$ *and let* u *be* A*-harmonic in* $\Omega = B(0, 4r) \setminus [\Sigma_1 \cup \overline{B(0, s)}]$ *with continuous boundary values* $u = 0$ *on* $\partial\Omega \setminus \overline{B(0, s)}$ *and* $u = 1$ *on* $\partial B(0, s)$. *Then for some* $\bar{\lambda}$, *depending only on the data,* (8.4) *is valid with* $\bar{u}$ *replaced by* u *in* $\Omega \cap [B(0, 2r) \setminus B(0, 2s)]$.

Proof. We first argue as in the proof of (5.13). Consider $\lambda > 1$ given with $\lambda - 1$ small. We assert that

$$\frac{u(x) - u(\lambda x)}{\lambda - 1} \geq c^{-1} u(x) \text{ whenever } x \in \Omega(\lambda) = \{x \in \Omega : \lambda x \in \Omega\}. \tag{8.6}$$

Indeed, from basic geometry it follows that this inequality holds trivially on $\partial\Omega(\lambda) \setminus \overline{B(0, s)}$ as $u \equiv 0$ on $\partial\Omega \setminus \overline{B(0, s)}$. In the case $x \in \partial\Omega(\lambda) \cap \partial B(0, s)$, we use Lemma 3.3 applied to u, and Harnack's inequality applied to $1 - u$, to conclude that

$$1 - u \geq c^{-1} \text{ on } \partial B(0, 7s/4). \tag{8.7}$$

As in (5.8) we set

$$\hat{\psi}(z) = \frac{e^{N|z|^2} - e^N}{e^{49N/16} - e^N}, \tag{8.8}$$

whenever $z \in B(0, 7/4) \setminus \overline{B(0, 1)}$ and where N is a non-negative integer. Put $\psi(x) = \hat{\psi}(x/s)$ whenever $x \in B(0, 7s/4) \setminus \overline{B(0, s)}$. Using (8.7), (8.8), and repeating the argument leading up to (5.13), we see that there exists $c_+ \geq 1$, depending only on the data, such that

$$c_+(1 - u(z)) \geq \psi(z) \geq c_+^{-1} \frac{d(z, \partial B(0, s))}{s} \text{ for } z \in B(0, 7s/4) \setminus \overline{B(0, s)}. \tag{8.9}$$

If $x \in \partial B(0, s)$ we can use (8.9), with z replaced by λx, to obtain that (8.6) is also valid on $\partial B(0, s)$. From the maximum principle for A-harmonic functions, we conclude that (8.6) holds in $\Omega(\lambda)$. Letting $\lambda \to 1$ in (8.6) we have

$$-\langle x, \nabla u(x) \rangle \geq c^{-1} u(x) \text{ whenever } x \in B(0, 2r) \setminus [B(0, 2s) \cup \Sigma]\,. \tag{8.10}$$

In view of (8.10), (8.5) and basic geometry we can conclude the validity of Lemma 8.1. ■

8.1 Proof of Theorem 1.13 in the baseline case when $A \in M_p(\alpha)$.

We here prove the following lemma.

Lemma 8.2 *Theorem 1.13 is valid when* $w = 0, \Sigma = \mathbb{R}^m \times \{0\}$, *and* $A \in M_p(\alpha)$.

Proof. Let u_k, for $k = 1, 2, \dots$, be the A-harmonic function u defined in Lemma 8.1 with $w = 0, A \in M_p(\alpha)$, but with s replaced by $s_k = 10^{-4k} r$. Set $\tilde{u}_k = u_k / u_k(a_r(0))$ for $k = 1, 2, \dots$ From Lemma 8.1, and our work in sections 3 and 4, we deduce, as in the proof of Corollary 1.12, that subsequences of $(\tilde{u}_k), (\nabla \tilde{u}_k)$, converge uniformly on compact subsets of $B(0, 4r) \setminus \{0\}, B(0, 4r) \setminus \Sigma$, respectively, to $\tilde{u}$, $\nabla \tilde{u}$, where $\tilde{u}$ is an A-harmonic function in $B(0, 4r) \setminus \Sigma$ with $\tilde{u}(a_r(0)) = 1$. Furthermore, the fundamental inequality (8.4) holds for $\tilde{u}$ in $B(0, 2r) \setminus \Sigma$, and

$\tilde{u} \equiv 0$ continuously on the boundary of $B(0,4r) \setminus \Sigma$ except at $\{0\}$. Fix $s, 0 < s < 10^{-4}r, A \in M_p(\alpha)$, recall that $\Sigma = \mathbb{R}^m \times \{0\}$, and let $\bar{v} \not\equiv 0$ be a A-harmonic in $D = B(0,4r) \setminus [\Sigma \cup \overline{B(0,s)}]$. Assume also that $\bar{v}$ has continuous boundary values with $\bar{v} \equiv 0$ on $\partial D \setminus \partial B(0,s)$. We will use the fundamental inequality for $\tilde{u}$, and the same argument as in the proof of Lemma 4.11, under Assumption 1$''$, to first prove that if $0 < c's \leq r/100$, $D_1 = B(0,4r) \setminus [\Sigma \cup \overline{B(0,c's)}]$, and c' is large enough, then

$$\begin{aligned}&(8.4) \text{ is valid with } \bar{u} \text{ replaced by } \bar{v}, \text{ in } D_1 \cap B(0,r), \\ &\text{and with constants depending only on the data.}\end{aligned} \tag{8.11}$$

Using this we will prove that if $t \in (0,r)$,

$$m(t) = \inf_{\partial B(0,t)} \frac{\tilde{u}}{\bar{v}},\ M(t) = \sup_{\partial B(0,t)} \frac{\tilde{u}}{\bar{v}}, \text{ and } \operatorname{osc}(t) = M(t) - m(t),$$

and if s is as above, then for some $\breve{c} \geq 1, a \in (0,1)$, depending only on the data,

$$\operatorname{osc}(t) \leq \breve{c} \left(\frac{s}{t}\right)^a \operatorname{osc}(s) \text{ whenever } s \leq t \leq r. \tag{8.12}$$

Armed with (8.12) the proof of Lemma 8.2 can be completed. Indeed, suppose that $\tilde{v}$ is a positive A-harmonic functions in $B(0,4r) \setminus \Sigma$, continuous on $\overline{B(0,4r)} \setminus \{0\}$ and that $\tilde{v} = 0$ on $\partial(B(0,4r) \setminus \Sigma) \setminus \{0\}$. For $s > 0$ fixed as above, let $\bar{v}$ denote the restriction of $\tilde{v}$ to D. Applying (8.12) and letting $s \to 0$ in this inequality we obtain that $\tilde{v}$ is a constant multiple of $\tilde{u}$.

To prove (8.11) and (8.12) we assume, as we may by the same argument as in (8.2), that

$$2 \leq \bar{v}/\tilde{u} \leq c_+ \text{ in } D \setminus B(0,2s) \text{ where } c_+ \text{ depends only on the data.} \tag{8.13}$$

Also let $u(\cdot,\tau), \tau \in [0,1]$, be A-harmonic functions in $D_2 = B(0,4r) \setminus [\Sigma \cup \overline{B(0,2s)}]$ with continuous boundary values,

$$u(y,\tau) = \tau\bar{v}(y) + (1-\tau)\tilde{u}(y), \text{ for } y \in \partial D_2, 0 \leq \tau \leq 1. \tag{8.14}$$

Existence of $u(\cdot,\tau), \tau \in (0,1)$, is a consequence of Lemma 3.2. Using the maximum principle for A-harmonic functions and (8.13) we find, for some $\tilde{c} \geq 1$ depending only on the data, that

$$\tilde{c}^{-1} u(\cdot,\tau_1) \leq \frac{u(\cdot,\tau_2) - u(\cdot,\tau_1)}{\tau_2 - \tau_1} \leq \tilde{c}u(\cdot,\tau_1) \tag{8.15}$$

in D_2 whenever $0 \leq \tau_1 < \tau_2 \leq 1$. Copying the argument after (4.43) we deduce, since $\tilde{u}$ satisfies the fundamental inequality in D_2, that there exists ϵ_0', depending only on the data, such that if $\xi_2 = \epsilon_0'$, then

$$c_-^{-1} \frac{u(y,\xi_2)}{d(y,\Sigma)} \leq |\nabla u(y,\xi_2)| \leq c_- \frac{u(y,\xi_2)}{d(y,\Sigma)}, \tag{8.16}$$

whenever $y \in D_2$. Using (8.16), as well as the fundamental inequality for $\tilde{u}$, and arguing as in the proof of Lemma 4.11 under Assumption 1$'$ we find that Assumption 1 in section 4 holds with $\hat{u}, \hat{v}$, replaced by $\tilde{u}, u(\cdot,\xi_2)$ in $D_2 \setminus B(0,2s)$, and with constants depending only on the data. Next we use this fact and argue as in (4.31), (4.32), to obtain (8.12) with $\bar{v}$ replaced by $u(\cdot,\xi_2)$ and s by $2s$. Continuing this argument by induction, as in the proof of (4.46), we eventually get (8.11) in $D \cap B(0,r) \setminus B(0,c's)$ and then (8.12) with s replaced by $2c's$, where c' depends only on the data. Since osc $(\cdot)$ is decreasing on $(0,r)$ we also have (8.12). ∎

8.2 Final Proof of Theorem 1.13

To prove Theorem 1.13 in the general case, assuming that $A \in M_p(\alpha, \beta, \gamma)$ and that $\hat{u}, \hat{v}$ are functions as in the statement of Theorem 1.13, we note, for some $b \in (0,1), c \geq 1$, depending only on the data, that

$$u^*(a_{t_2}(w)) \leq c(t_1/t_2)^b\, u^*(a_{t_1}(w)) \text{ whenever } 0 < t_1 < t_2 < 4r, \tag{8.17}$$

and $u^* \in \{\hat{u}, \hat{v}\}$. Also from the Harnack inequality we have, for some $\hat{b} \geq 2$, depending only on the data, that

$$u^*(a_{t_2}(w)) \geq (t_1/t_2)^{\hat{b}}\, u^*(a_{t_1}(w)) \text{ whenever } 0 < t_1 < t_2 < 4r, \tag{8.18}$$

and $u^* \in \{\hat{u}, \hat{v}\}$. Let $s_1 \leq r$ and let $\bar{c}$ be a large positive constant such that $0 < \bar{c}s \leq s_1 \leq r$. Let $A_1(\eta) = A(w, \eta)$, $\eta \in \mathbb{R}^n \setminus \{0\}$, and let u_1, v_1 be A_1-harmonic functions in $D_3 = B(w, \bar{c}s) \setminus (\Sigma \cup \overline{B(w,s)})$ having continuous boundary values, $u_1 = \hat{u}, v_1 = \hat{v}$ on ∂D_3. We first show that if $\bar{c}$ is large enough, then there exist $c_1, c_2 \geq 1$, such that

$$c_1^{-1}\frac{u^*(y)}{d(y,\Sigma)} \leq |\nabla u^*(y)| \leq c_1\frac{u^*(y)}{d(y,\Sigma)} \text{ whenever } y \in B(w, 6c_2 s) \setminus [\Sigma \cup B(w, 2c_2 s)], \tag{8.19}$$

and $u^* \in \{u_1, v_1\}$. To outline the argument we can without loss of generality assume that $w = 0$ and that

$$h[B(0,\bar{c}s) \cap \Sigma, B(0,\bar{c}s) \cap (\mathbb{R}^m \times \{0\})] \leq 2\bar{c}\delta s.$$

For u^* as above, let $v^* \geq 0$ be the A_1-harmonic function in $D_3 = B(0,\bar{c}s) \setminus [(\mathbb{R}^m \times \{0\}) \cup \overline{B(0,s)}]$ with continuous boundary values, $v^* \equiv 0$ on $\partial D_3 \setminus \partial B(0,s)$ while $v^* \leq u^*$ on $\partial B(0,s)$, and $v^* \equiv u^*$ at points z in this set with $d(z, \mathbb{R}^m \times \{0\}) \geq 20\bar{c}\delta s$. Using (8.17) and Lemma 3.3, we deduce for c large enough, depending only on the data, that

$$u^* \leq c[(\bar{c})^{-b} + (\bar{c}\delta)^\sigma]u^*(a_s(0)) + v^* \text{ and } v^* \leq c(\bar{c}\delta)^\sigma u^*(a_s(0)) + u^* \text{ on } D_3. \tag{8.20}$$

Also using (8.18), we see that if $c_2 << \bar{c}$ is large enough, depending only on the data, then

$$\min\{v^*(x) : x \in \tilde{\Omega}(w, \tilde{\eta}/2) \cap B(0, 8c_2 s) \setminus B(0, c_2 s)\} \geq (c_2)^{-2\hat{b}}u^*(a_s(0)). \tag{8.21}$$

We can, without loss of generality, also assume that $c_2 > 2c'$, where c' is the constant in (8.11). Using this assumption we see that the fundamental inequality in (8.11) holds for v^* in $D_3 \cap [B(0,\bar{c}s/4) \setminus B(0,c_2 s)]$, with $\bar{c}s$ playing the role of $4r$. With c_2 now fixed, we observe from (8.20), (8.21), that the ratio of u^*/v^* in $\tilde{\Omega}(w,\tilde{\eta}/2) \cap [B(0,8c_2 s) \setminus B(0,c_2 s)]$ can be made arbitrarily close to 1 by first choosing $\bar{c}$ large, and then choosing $\delta < \delta^*$ small enough depending on $\bar{c}$. In view of (8.11) for v^*, we see that these constants can in fact be chosen to depend only on the data and in such a way that Lemma 3.9 can be applied to u^*, v^*. Hence, applying Lemma 3.9 we can conclude (8.19) for u_1, v_1 in $\tilde{\Omega}(w,\tilde{\eta}) \cap B(0, 6c_2 s) \setminus B(0, 2c_2 s)$. From this conclusion and (8.5) we obtain (8.19).

Armed with (8.19) we can now repeat the argument in Lemma 3.8 with A_1, A_2 replaced by A, A_1, and with cylinders replaced by balls, in order to conclude that

$$|u_1(x) - \hat{u}(x)| \leq cs_1^\theta\, u_1(x), x \in \tilde{\Omega}(w,\tilde{\eta}/2) \cap \bar{B}(w, 6c_2 s) \setminus [\Sigma \cup B(w, 2c_2 s)], \tag{8.22}$$

for some c, θ, depending only on the data. (8.22) also holds for $v_1, \hat{v}$. From (8.22), and Lemma 3.9 we obtain, for s_1 small enough, that (8.19) is valid for $\hat{u}, \hat{v}$ on $\tilde{\Omega}(w, \tilde{\eta}) \cap [B(w, 5c_2) \setminus B(w, 3c_2)]$ with c_1 replaced by $c_4 \geq c_1$, depending only on the data. Using this fact and once more (8.5) we get the fundamental inequality for $\hat{u}, \hat{v}$ on $B(w, 5c_2 s) \setminus [\Sigma \cup B(w, 3c_2 s)]$ provided $s_1 \leq r/c^*$ and c^* is large enough.

From arbitrariness of s we deduce that the fundamental inequality holds for $\hat{u}, \hat{v}$ in $B(0, r/c) \setminus \Sigma$ with constants depending only on the data. This deduction and Theorems 1.9, 1.10, easily imply that if $a, b \in (0, \infty)$, then $(a|\nabla \hat{u}| + b|\nabla \hat{v}|)^{p-2}$ is an A_2-weight on cubes $\subset B(0, r/c) \setminus B(0, s), 0 < s \leq r/c$, with constants that can be chosen independent of a, b. Using this fact, and the same argument as in the proof of (8.12), we see that if

$$m(t, w) = \inf_{\partial B(w,t)} \frac{\hat{u}}{\hat{v}},\ M(t, w) = \sup_{\partial B(w,t)} \frac{\hat{u}}{\hat{v}},\ \text{and}\ \ \text{osc}\ (t, w) = M(t, w) - m(t, w),$$

then for some $\hat{c} \geq 1$, and $\hat{a} \in (0, 1)$, depending only on the data, we have

$$\text{osc}\ (t, w)\ \leq \hat{c} \left(\frac{s}{t}\right)^{\hat{a}}\ \text{osc}\ (s, w),\ s \leq t \leq r. \tag{8.23}$$

Theorem 1.13 now follows from (8.23) if we let $s \to 0$.

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