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The distribution of free path lengths in a one-dimensional quasicrystal

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Abstract

The Lorentz gas describes the dynamics of a point particle in an array of scatterers, and has been studied for different distributions of scatterers. One numerical study by Wennberg compared the distributions of free path lengths for small scatterer size, in the case of a random, periodic or quasicrystalline scatterer distribution. Due to the practical restrictions of numerical simulations, one-dimensional models were used. This paper focuses on the one-dimensional quasicrystal model in the limit of small scatterer size and provides both proofs of existence of free path lengths as well as derivations of explicit formulas for the distribution for small free path lengths. The methods used are based on previous work on Lorentz gas in several dimensions by Marklof and Strömbergsson; of special importance are equidistribution theorems on homogenous spaces that follow from Ratner's measure classification.

Contents

1 Introduction

The Lorentz gas is a dynamical system describing the motion of non-interacting point particles in a d-dimensional space, $\mathbb{R}^d$, with a fixed array of spherical scatterers placed at the elements of a given point set $\mathcal{P} \subset \mathbb{R}^d$. The particles travel in straight lines with constant velocity until they collide elastically with a scatterer. Since the particles are non-interacting the model can be reduced to studying one-particle systems. Let $\mathcal{K}_\rho \subset \mathbb{R}^d$ be the "billiard domain" defined by $\mathcal{K}_\rho = \mathbb{R}^d \setminus (\mathcal{B}_\rho^d + \mathcal{P})$, where $\mathcal{B}_\rho^d$ is the d-dimensional open ball of radius ρ, centered at the origin. Let $\boldsymbol{q}(t)$ and $\boldsymbol{v}(t)$ denote the position and velocity of the particle at time t. The collision is assumed to be elastic, so the speed of the particles remain constant, and for simplicity we can assume $\|\boldsymbol{v}(t)\| = 1$ for all t. The phase space for the Lorentz gas is then given by the unit tangent bundle $\mathrm{T}^1(\mathcal{K}_\rho)$, which we can parametrize by $(\boldsymbol{q}, \boldsymbol{v}) \in \mathcal{K}_\rho \times S_1^{d-1}$, and we use the convention that for $\boldsymbol{q} \in \partial\mathcal{K}_\rho$ the velocity $\boldsymbol{v}$ points away from the scatterer.
Given an initial position and velocity $(\boldsymbol{q}, \boldsymbol{v}) \in \mathrm{T}^1(\mathcal{K}_\rho)$, we see that the first collision of a particle with this initial condition is

$$\tau(\boldsymbol{q}, \boldsymbol{v}; \rho) = \inf\{t > 0 : \boldsymbol{q} + t\boldsymbol{v} \notin \mathcal{K}_\rho\},$$

which we will call the first collision time. Note that since the particles are moving at unit speed, $\tau(\boldsymbol{q}, \boldsymbol{v}; \rho)$ also describes the first free path length for a particle starting at $(\boldsymbol{q}, \boldsymbol{v})$. We will be mostly concerned with the distribution of $\tau(\boldsymbol{q}, \boldsymbol{v}; \rho)$ in the so called Boltzmann-Grad limit. This refers to a scaling of time and length units so that the mean collision time and mean free path length remains constant as ρ tends to 0. This can be achieved by introducing the *macroscopic* coordinates

$$(\boldsymbol{Q}(t), \boldsymbol{V}(t)) = (\rho^{d-1}\boldsymbol{q}(\rho^{-(d-1)}t), \boldsymbol{v}(\rho^{-(d-1)}t)),$$

cf. [2, Section 4]. Note that $\boldsymbol{Q}'(t) = \boldsymbol{V}(t)$ and $\|\boldsymbol{V}(t)\| = 1$ for all t. This means that for a Borel probability measure Λ on $\mathrm{T}^1(\mathcal{K}_\rho)$ we are interested in $\Lambda(\{(\boldsymbol{q}, \boldsymbol{v}) \in \mathrm{T}^1(\mathcal{K}_\rho) : \rho^{d-1}\tau(\boldsymbol{q}, \boldsymbol{v}; \rho) \geq \xi\})$ as $\rho \to 0$. For a visual comparison of regular *microscopic* coordinates with the macroscopic coordinates see Figure 1.
The behavior of the Lorentz gas in the Boltzmann-Grad limit will differ for different scatterer distributions $\mathcal{P}$. The most well studied examples are when $\mathcal{P}$ is taken as a fixed realization of a random point process, and when $\mathcal{P}$ is a Euclidian lattice. Some of the analysis for lattices has been extended to a class of point sets called cut-and-project sets, see [5]. Of particular interest are cut-and-project sets exhibiting forbidden symmetries, called quasicrystals. One way to compare the distributions of free path length for different choices of $\mathcal{P}$ is to preform a numerical investigation. This is done in [8], but due to the large computational demands for simulating long point particle trajectories in quasicrystal Lorentz gas, the above model has to be modified to a one-dimensional setting. This will demand a new definition of free path length, which is closely related to the above definition. Our goal in this report is to derive an explicit formula for the limiting distribution of the free path lengths in the one-dimensional quasicrystal considered in [8], when the free path lengths are small.
In Section 2 we present a general definition of a cut-and-project set, and in Section 3 construct the quasicrystal $\mathcal{F}$ presented in [8, Section 3]. In Section 4 we introduce the notion of free path length for one-dimensional scatterers and state the existence of limiting distributions for the free path lengths. In Section 5 we present a few different spaces of

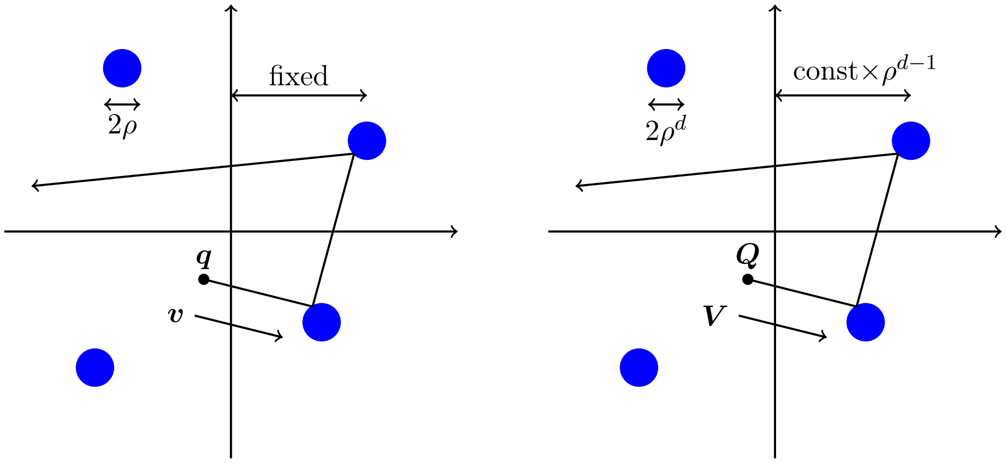

Figure 1: Left: The Lorentz gas in *microscopic* coordinates where each scatterer has a fixed position and radius ρ. Right: The Lorentz gas in *macroscopic* coordinates where the radius and coordinates tend to zero in such a way that the mean free path length remains finite.

lattices that are then used in Section 6 where we prove the Theorems in Section 4. In Section 7 we calculate our explicit formula. Section 8 examines the limiting distributions numerically and compare the numerical results to the exact formula.

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2 Cut-and-project sets

We will define a cut-and-project set in a d-dimensional space $\mathbb{R}^d$ as in [5]. Call $\mathbb{R}^d$ the physical space and let another space $\mathbb{R}^m$ be called the internal space, where m can be any positive integer. Take the cartesian product of these spaces $\mathbb{R}^n := \mathbb{R}^d \times \mathbb{R}^m$ and let $\pi : \mathbb{R}^n \to \mathbb{R}^d$ and $\pi_{int} : \mathbb{R}^n \to \mathbb{R}^m$ be the projections onto the physical space and internal space respectively. Now we let $\mathcal{L}$ be a lattice of full rank in $\mathbb{R}^n$. We see that the closure of the projection of the lattice on the internal space $\mathcal{A} := \overline{\pi_{int}(\mathcal{L})}$ is an abelian subgroup of $\mathbb{R}^m$ under addition. Let $\mathcal{A}^\circ$ be the connected subgroup of $\mathcal{A}$ containing the origin. $\mathcal{A}^\circ$ will then be a linear subspace of $\mathbb{R}^m$. If m_1 is the dimension of $\mathcal{A}^\circ$ then since $\mathcal{L}$ is of full rank, there are $m_2 = m - m_1$ vectors $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{m_2} \in \mathcal{L}$ such that $\pi_{int}(\boldsymbol{b}_1), \ldots, \pi_{int}(\boldsymbol{b}_{m_2})$ are linearly independent in $\mathbb{R}^m \setminus \mathcal{A}^\circ$ and

$$\mathcal{A} = \mathcal{A}^\circ + \mathbb{Z}\pi_{int}(\boldsymbol{b}_1) + \ldots + \mathbb{Z}\pi_{int}(\boldsymbol{b}_{m_2}).$$

We can now define a cut-and-project set

$$\mathcal{P}(\mathcal{W}, \mathcal{L}) = \{\pi(\boldsymbol{y}) : \boldsymbol{y} \in \mathcal{L}, \pi_{int}(\boldsymbol{y}) \in \mathcal{W}\} \subset \mathbb{R}^d$$

for $\mathcal{W} \subset \mathcal{A}$, a bounded subset with non-empty interior called the window.

We let $\mu_{\mathcal{A}}$ denote the Haar measure of $\mathcal{A}$, normalized so that its restriction to $\mathcal{A}^\circ$ is the standard m_1-dimensional Lebesgue measure. $\mathcal{P}(\mathcal{W}, \mathcal{L})$ is said to be regular if the boundary of $\mathcal{W}$ has measure zero with respect to $\mu_{\mathcal{A}}$. If we set $\mathcal{V} = \mathbb{R}^d \times \mathcal{A}^\circ$ then $\mathcal{L}_{\mathcal{V}} = \mathcal{L} \cap \mathcal{V}$ is a lattice of full rank in $\mathcal{V}$ and we let $\mu_{\mathcal{V}} = \mathrm{vol} \times \mu_{\mathcal{A}}$ be the natural volume measure on $\mathbb{R}^d \times \mathcal{A}$. In particular we have that for any regular cut-and-project set $\mathcal{P}$ and any bounded $\mathcal{D} \subset \mathbb{R}^d$ with boundary of Lebesgue measure zero,

$$\lim_{T\to\infty} \frac{\#\{\boldsymbol{b} \in \mathcal{L} : \pi(\boldsymbol{b}) \in \mathcal{P} \cap T\mathcal{D}\}}{T^d} = \delta_{d,m}(\mathcal{L})\mathrm{vol}(\mathcal{D})\mu_{\mathcal{A}}(\mathcal{W})$$

where

$$\delta_{d,m}(\mathcal{L}) = \frac{1}{\mu_{\mathcal{V}}(\mathcal{V}/\mathcal{L}_{\mathcal{V}})}. \tag{2.1}$$

This is a consequence of Weyl equidistribution, see [1] or [5, Prop. 3.2]. Making the further assumption that the map $\pi\big|_{\mathcal{L}}$ is injective we get the relation

$$\lim_{T\to\infty} \frac{\#(\mathcal{P} \cap T\mathcal{D})}{T^d} = \delta_{d,m}(\mathcal{L})\mathrm{vol}(\mathcal{D})\mu_{\mathcal{A}}(\mathcal{W}).$$

3 Construction of the quasicrystal $\mathcal{F}$

We now proceed with the construction the one-dimensional quasicrystal studied in [8, Section 3]. Let

$$\tau := \frac{\sqrt{5}+1}{2} \qquad \varphi := \arctan(\tau^{-1})$$

and take

$$K := \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} \tau/\nu & -1/\nu \\ 1/\nu & \tau/\nu \end{pmatrix}$$

where $\nu := \sqrt{1+\tau^2}$. If we let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection onto the first coordinate and $\pi_{int} : \mathbb{R}^2 \to \mathbb{R}$ be the projection onto the last coordinate, then they can be written as

$$\pi((y_1, y_2)K) = y_1\frac{\tau}{\nu} + y_2\frac{1}{\nu} \quad \text{and} \quad \pi_{int}((y_1, y_2)K) = y_2\frac{\tau}{\nu} - y_1\frac{1}{\nu}.$$

Let both the physical space and internal space be one-dimensional and take $\mathcal{L} = \mathbb{Z}^2 K$. If we choose the window as

$$\mathcal{W} = \pi_{int}([-1/2, 1/2]^2 K) = \left[-\frac{1+\tau}{2\nu}, \frac{1+\tau}{2\nu}\right]$$

we get the following cut-and-project set

$$\begin{aligned} \mathcal{F} &:= \mathcal{P}(\mathcal{W}, \mathbb{Z}^2 K) \\ &= \{\pi(\boldsymbol{y}) : \boldsymbol{y} \in \mathbb{Z}^2 K, \pi_{int}(\boldsymbol{y}) \in \mathcal{W}\} \\ &= \left\{ n_1\frac{\tau}{\nu} + n_2\frac{1}{\nu} : n_1, n_2 \in \mathbb{Z}, n_2\frac{\tau}{\nu} - n_1\frac{1}{\nu} \in \left[-\frac{1+\tau}{2\nu}, \frac{1+\tau}{2\nu}\right] \right\}, \end{aligned}$$

corresponding to the quasicrystal in [8]. First we can note that if

$$n_2\frac{\tau}{\nu} - n_1\frac{1}{\nu} = \frac{1+\tau}{2\nu}$$

then

$$\tau = \frac{2n_1+1}{2n_2-1}$$

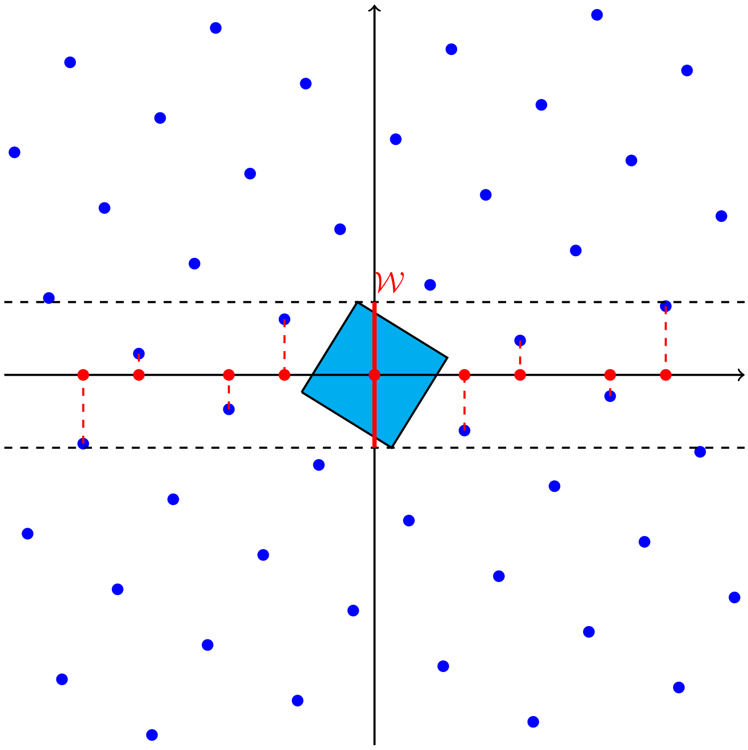

Figure 2: Construction of the quasicrsytal using the cut-and-project method.

so τ would be rational. Since τ is not rational we have

$$n_2\frac{\tau}{\nu} - n_1\frac{1}{\nu} \neq \frac{1+\tau}{2\nu}$$

for all $n_1, n_2 \in \mathbb{Z}$. We can prove the same for $-(1+\tau)/(2\nu)$, so that

$$\mathcal{F} \quad = \quad \left\{ n_1\frac{\tau}{\nu} + n_2\frac{1}{\nu} : n_1, n_2 \in \mathbb{Z}, n_2\frac{\tau}{\nu} - n_1\frac{1}{\nu} \in \left(-\frac{1+\tau}{2\nu}, \frac{1+\tau}{2\nu} \right) \right\}.$$

This allows us to redefine $\mathcal{W}$ as the open set $(-(1+\tau)/(2\nu), (1+\tau)/(2\nu))$, which will simplify notation in Section 7. For a visual representation of the construction of the quasicrystal see Figure 2.

Lemma 3.1. *We can write $\mathcal{F}$ as*

$$\mathcal{F} = \{x_m\}_{m\in\mathbb{Z}} \quad \textit{with} \quad x_m = \frac{m}{\nu} + \frac{1}{\tau\nu}\left\|\frac{m}{\tau}\right\|,$$

where $\|x\|$ denotes the nearest integer to $x \in \mathbb{R} \setminus (\mathbb{Z} + 1/2)$.

Here we see that the distance between two consecutive points take one of only two values:

$$x_m - x_{m-1} = \frac{1}{\nu} \quad \text{or} \quad \frac{1}{\nu} + \frac{1}{\tau\nu}.$$

Proof. First we note that $\pi((n_1, n_2)K) \in \mathcal{F}$ if and only if

$$-\frac{1+\tau}{2\nu} < n_2\frac{\tau}{\nu} - n_1\frac{1}{\nu} < \frac{1+\tau}{2\nu},$$

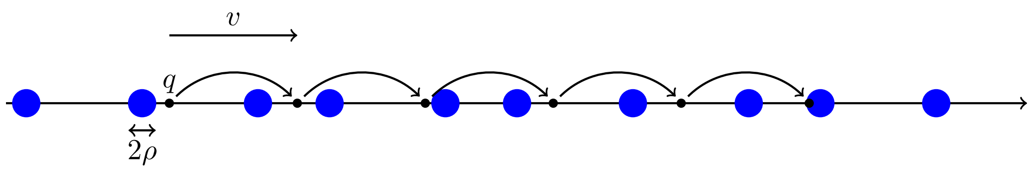

Figure 3: In the one-dimensional Lorentz gas we look at discrete "jump" paths instead of continuous paths.

which we can rewrite as

$$n_1 - \frac{1}{2} < \frac{n_1+n_2}{\tau} < n_1 + \frac{1}{2}.$$

Then we get that

$$n_1 = \left\| \frac{n_1+n_2}{\tau} \right\|$$

and we can write

$$\mathcal{F} \;=\; \left\{ n_1\frac{\tau}{\nu} + n_2\frac{1}{\nu} : n_1, n_2 \in \mathbb{Z}, n_1 = \left\| \frac{n_1+n_2}{\tau} \right\| \right\}.$$

Finally we see that

$$n_1\frac{\tau}{\nu} + n_2\frac{1}{\nu} = \frac{n_1+n_2}{\nu} + \frac{1}{\tau\nu}\left\| \frac{n_1+n_2}{\tau} \right\|$$

so taking $n_1 + n_2$ as m we get the desired formula. □

4 Free path lengths in one dimension

Since $\mathcal{F}$ is a one-dimensional quasicrystal we cannot define the free path length the usual way. Instead of looking at continuous paths parametrized by $\mathbb{R}^+$ we will look at “jump” paths parametrized by $\mathbb{Z}^+$, as in Figure 3. We will see that this is similar to regular free paths in 2 dimensions with scatterers at $\mathbb{Z} \times \mathcal{F}$. Given initial position and velocity $(q, v) \in \mathbb{R} \times \mathbb{R}$ we let

$$k_0(q, v; \rho) = \min\{j \in \mathbb{Z}^+ : q + jv \in \mathcal{F} + (-\rho, \rho)\}.$$

Theorem 4.1. *There is a non-increasing continuous function $F_{\mathcal{F}} : [0, \infty] \to [0, 1]$ with $F_{\mathcal{F}}(0) = 1$, $F_{\mathcal{F}}(\infty) = 0$, such that for any Borel probability measure Λ on $\mathbb{R} \times \mathbb{R}$ which is absolutely continuous with respect to Lebesgue measure and any $\xi > 0$, we have*

$$\Lambda(\{(q, v) \in \mathbb{R} \times \mathbb{R} : k_0(q, v, \rho) \geq \xi/\rho\}) \to F_{\mathcal{F}}(\xi) \quad \text{as } \rho \to 0.$$

We can also consider the problem where the particle has initial data on, or near, the boundary of a scatterer. To do this we would first specify the location $q \in \mathcal{F}$ of a scatterer and take $(q + \rho\beta(v), v)$ as initial data, where $\beta : \mathbb{R} \to \mathbb{R}$ is a continuous function. It is worth noting that in the usual Lorentz gas model with continuous trajectories, one must take care to avoid pathologies where the particle might hit the scatterer it started at, but in this model where we have discrete trajectories the probability of that happening will tend to zero as $\rho \to 0$.

Theorem 4.2. *Given any $q \in \mathcal{F}$, there is a continuous function $F_{\mathcal{F},q} : [0,\infty] \times \mathbb{R}_{\geq 0} \to [0,1]$ with $F_{\mathcal{F},q}$ non-increasing, $F_{\mathcal{F},q}(0,r) = 1$, $F_{\mathcal{F},q}(\infty, r) = 0$ for all $r \in \mathbb{R}_{\geq 0}$, such that for any Borel probability measure λ on $\mathbb{R}$ which is absolutely continuous with respect to Lebesgue measure, any continuous function $\beta : \mathbb{R} \to \mathbb{R}$, and any $\xi > 0$, we have*

$$\lambda(\{v \in \mathbb{R} : k_0(q + \rho\beta(v), v, \rho) \geq \xi/\rho\}) \to \int_{\mathbb{R}} F_{\mathcal{F},q}(\xi, |\beta(v)|) d\lambda(v) \quad \text{as } \rho \to 0. \tag{4.1}$$

The convergence in (4.1) *is uniform over all $q \in \mathcal{F}$.*

Before we can prove Theorems 4.1 and 4.2 we will need a way to represent lattices.

5 Spaces of lattices and quasicrystals

Our goal is to find formulas for the limit distribution in Theorem 4.1. We will do this in terms of certain homogenous spaces of the form

$$\Gamma \backslash G = \{\Gamma g : g \in G\}$$

where G is a connected Lie group and Γ is a lattice in G. In this context lattice means that Γ is a discrete subgroup of G such that there exists a fundamental domain of Γ in G with finite left Haar measure. In our case we will take G to be the group of all invertible affine operators over $\mathbb{R}^n$, that is $G = \mathrm{ASL}(n, \mathbb{R}) = \mathrm{SL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$ and $\Gamma = \mathrm{ASL}(n, \mathbb{Z})$, where multiplication is defined by

$$(M, \boldsymbol{x})(M', \boldsymbol{x}') = (MM', \boldsymbol{x}M' + \boldsymbol{x}').$$

We will also make use of $G^1 = \mathrm{SL}(n, \mathbb{R})$, and $\Gamma^1 = \mathrm{SL}(n, \mathbb{Z})$. Note that $\Gamma \backslash G$ is a manifold of dimension $d^2 + d - 1$ and $\Gamma^1 \backslash G^1$ a manifold of dimension $d^2 - 1$. We know that there exist (left and right) Haar measures on G and G^1. Let μ and μ_1 denote push-forwards of the Haar measures to $\Gamma \backslash G$ and $\Gamma^1 \backslash G^1$ respectively, normalized so that $\mu(\Gamma \backslash G) = 1$ and $\mu_1(\Gamma^1 \backslash G^1) = 1$. These will be the unique G- and G^1-invariant probability measures on $\Gamma \backslash G$ and $\Gamma^1 \backslash G^1$.

5.1 Representing lattices

We see that every Euclidian lattice $\mathcal{L} \subset \mathbb{R}^n$ with covolume 1 (hence of full rank) can be written as

$$\mathcal{L} = \mathbb{Z}^n M$$

for some $M \in G^1$. One can then show that there is a bijection

$$\begin{aligned} \Gamma^1 \backslash G^1 &\xrightarrow{\sim} \{\text{Euclidian lattices with covolume } 1\} \\ \Gamma^1 M &\mapsto \mathbb{Z}^n M. \end{aligned}$$

This is due to Γ^1 being the stabilizer of $\mathbb{Z}^n$ under right multiplication by G^1. The map is well defined since $\Gamma^1 M = \Gamma^1 M'$ if and only if for all $\gamma \in \Gamma^1$ there is $\gamma' \in \Gamma^1$ such that $\gamma M = \gamma' M'$. Therefore $\mathbb{Z}^n M = \mathbb{Z}^n \gamma^{-1} \gamma' M' = \mathbb{Z}^n M'$ since $\gamma^{-1}\gamma' \in \Gamma^1$. We saw from before that the map was surjective and it is injective since if $\mathbb{Z}^n M = \mathbb{Z}^n M'$ then $\mathbb{Z}^n = \mathbb{Z}^n M' M^{-1}$, so $M' M^{-1} \in \Gamma^1$. Hence for all $\gamma \in \Gamma^1$ there is $\gamma' := \gamma M (M')^{-1} \in \Gamma^1$ such that $\gamma M = \gamma M (M')^{-1} M' = \gamma' M'$. Similarly there is a bijection between $\Gamma \backslash G$ and the space of affine lattices.

5.2 Submanifolds of $\Gamma\backslash G$ and $\Gamma^1\backslash G^1$

In Section 6 we will see that $F_{\mathcal{F}}$ and $F_{\mathcal{F},q}$ can be expressed in terms of the measures μ and μ_1 and in Section 7 that they can also be expressed as integrals over measures of certain submanifolds of $\Gamma\backslash G$ and $\Gamma^1\backslash G^1$ which we will briefly describe here.
First we will consider the family of submanifolds $X(\boldsymbol{y})$ of $\Gamma\backslash G$ defined, for each $\boldsymbol{y} \in \mathbb{R}^n$, as

$$X(\boldsymbol{y}) = \{\Gamma(M, \boldsymbol{x}) \in \Gamma\backslash G : \boldsymbol{y} \in \mathbb{Z}^n M + \boldsymbol{x}\},$$

that is to say $X(\boldsymbol{y})$ is the space of affine lattices with a point at $\boldsymbol{y}$. It is simple to check that $X(\boldsymbol{y})$ can be identified with $\Gamma^1\backslash G^1$ through

$$\begin{array}{ccc} \Gamma^1\backslash G^1 & \xrightarrow{\sim} & X(\boldsymbol{y}) \\ \Gamma^1 M & \mapsto & \Gamma(M, \boldsymbol{y}). \end{array}$$

This gives $X(\boldsymbol{y})$ the structure of an (n^2-1)-dimensional submanifold of $\Gamma\backslash G$. Under this identification μ_1 induces a Borel probability measure on $X(\boldsymbol{y})$ which we will denote $\nu_{\boldsymbol{y}}$. Two properties of $\nu_{\boldsymbol{y}}$ will be particularly important in Section 7 and are described in the following propositions.

Proposition 5.1. *Let $\mathscr{E} \subset \Gamma\backslash G$ be any Borel set; then $\boldsymbol{y} \mapsto \nu_{\boldsymbol{y}}(\mathscr{E} \cap X(\boldsymbol{y}))$ is a measurable function from $\mathbb{R}^n$ to $\mathbb{R}$. If $U \subset \mathbb{R}^n$ is any Borel set such that $\mathscr{E} \in \bigcup_{\boldsymbol{y}\in U} X(\boldsymbol{y})$, then*

$$\mu(\mathscr{E}) \leq \int_U \nu_{\boldsymbol{y}}(\mathscr{E} \cap X(\boldsymbol{y}))d\boldsymbol{y}. \tag{5.1}$$

Furthermore, if for all $\boldsymbol{y}_1 \neq \boldsymbol{y}_2 \in U$: $X(\boldsymbol{y}_1) \cap X(\boldsymbol{y}_2) \cap \mathscr{E} = \emptyset$, then equality holds in (5.1).

Proposition 5.2. *Let $n \geq 2$ and $C > 1$ and write $U = \boldsymbol{z} + \mathfrak{Z}_{cyl}(c_1, c_2, C)$, where $\mathfrak{Z}_{cyl}(c_1, c_2, C) := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : c_1 < x_1 < c_2, ||(x_2, \ldots, x_n)|| < C\}$. Then*

$$\int_U \nu_{\boldsymbol{y}}\left(\{\Gamma(M, \boldsymbol{x}) \in X(\boldsymbol{y}) : \#(U \cap \mathbb{Z}^n M + \boldsymbol{x}) \geq 2\}\right) d\boldsymbol{y} \ll (c_2 - c_1)^2,$$

uniformly over all $\boldsymbol{z} \in \{0\} \times \mathbb{R}^{n-1}$ and $c_1 < c_2$. (The implied constant depends only on C, d.)

Proofs can be found in [3, Proposition 7.10] and [3, Lemma 7.13] respectively.
Similarly to how we defined $X(\boldsymbol{y})$ as a subset of $\Gamma\backslash G$ we can define $X_1(\boldsymbol{y})$ as a subset of $\Gamma^1\backslash G^1$ for any given $\boldsymbol{y} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$ through

$$X_1(\boldsymbol{y}) = \{\Gamma^1 M \in \Gamma^1\backslash G^1 : \boldsymbol{y} \in \mathbb{Z}^n M\}.$$

It is possible to split $X_1(\boldsymbol{y})$ in terms of $X_1(\boldsymbol{k}, \boldsymbol{y})$ as

$$X_1(\boldsymbol{y}) = \sqcup_{\boldsymbol{k}\in S} X_1(\boldsymbol{k}, \boldsymbol{y}) \tag{5.2}$$

where we can take $S = \{k\boldsymbol{e}_1 : k \in \mathbb{Z}^+\}$. It is now possible to give $X_1(\boldsymbol{y})$ the structure of a submanifold of $\Gamma^1\backslash G^1$, where $X_1(\boldsymbol{k}, \boldsymbol{y})$ in (5.2) are the countably many connected components of $X_1(\boldsymbol{y})$. As with $X(\boldsymbol{y})$ we have that $X_1(\boldsymbol{y})$ can be endowed with a Borel measure $\nu_{\boldsymbol{y}}$. It will always be clear from the context wether $\nu_{\boldsymbol{y}}$ refers to the measure on $X(\boldsymbol{y})$ or $X_1(\boldsymbol{y})$. Similarly to before we have the following properties of $\nu_{\boldsymbol{y}}$ on $X_1(\boldsymbol{y})$.

Proposition 5.3. *Let $\mathscr{E} \subset X_1$ be any Borel set; then $\boldsymbol{y} \mapsto \nu_{\boldsymbol{y}}(\mathscr{E} \cap X_1(\boldsymbol{y}))$ is a measurable function of $\boldsymbol{y} \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$. If $U \subset \mathbb{R}^n \setminus \{\boldsymbol{0}\}$ is any Borel set such that $\mathscr{E} \subset \bigcup_{\boldsymbol{y}\in U} X_1(\boldsymbol{y})$, then*

$$\mu_1(\mathscr{E}) \leq \int_U \nu_{\boldsymbol{y}}(\mathscr{E} \cap X_1(\boldsymbol{y}))d\boldsymbol{y}. \tag{5.3}$$

Furthermore, if for all $\boldsymbol{y}_1 \neq \boldsymbol{y}_2 \in U$: $X_1(\boldsymbol{y}_1) \cap X_1(\boldsymbol{y}_2) \cap \mathscr{E} = \emptyset$, then the equality holds in (5.3).

Proposition 5.4. *Let $n \geq 3$ and $C > 1$ and write $U = \boldsymbol{z} + \mathfrak{Z}_{cyl}(c_1, c_2, C)$, where $\mathfrak{Z}_{cyl}(c_1, c_2, C)$ is defined as before. Then*

$$\int_U \nu_{\boldsymbol{y}} \left(\{\Gamma M \in X_1(\boldsymbol{y}) : \#(U \cap \mathbb{Z}^n M) \geq 2\}\right) d\boldsymbol{y} \ll (c_2 - c_1)^2,$$

uniformly over all $\boldsymbol{z} \in \{0\} \times \mathbb{R}^{n-1}$ and $C^{-1} \leq c_1 < c_2$. (The implied constant depends only on C, d.)

For details regarding the manifold structure of $X_1(\boldsymbol{y})$ and measure $\nu_{\boldsymbol{y}}$ we refer to [3, Section 7.1], and for proofs of Proposition 5.3 and 5.4 see [3, Proposition 7.3] and [3, Lemma 7.12] respectively.
The space $X_1(\boldsymbol{e}_1, \boldsymbol{y})$ will be of particular interest later so we will write it in another, more useful form. First we choose a matrix $M_{\boldsymbol{y}} \in G^1$ such that $\boldsymbol{e}_1 M_{\boldsymbol{y}} = \boldsymbol{y}$. We introduce the new space

$$H = \{M \in G^1 : \boldsymbol{e}_1 M = \boldsymbol{e}_1\}.$$

Now we see that for $\Gamma^1 M \in X_1(\boldsymbol{e}_1, \boldsymbol{y})$

$$\boldsymbol{e}_1 M = \boldsymbol{y} = \boldsymbol{e}_1 M_{\boldsymbol{y}}$$

and this is if and only if $MM_{\boldsymbol{y}}^{-1} \in H$, i.e. $M \in HM_{\boldsymbol{y}}$. Therefore we can write

$$X_1(\boldsymbol{e}_1, \boldsymbol{y}) = \{\Gamma^1 M \in \Gamma^1 \backslash G^1 : M \in HM_{\boldsymbol{y}}\}.$$

We want to reduce the redundancy in $X_1(\boldsymbol{e}_1, \boldsymbol{y})$, so assume that $M, M' \in HM_{\boldsymbol{y}}$ and $\Gamma^1 M = \Gamma^1 M'$. This implies that $M'M^{-1} \in \Gamma^1$, and since $M'M^{-1} \in H$ we get that $M'M^{-1} \in \Gamma^1 \cap H$, i.e. $(\Gamma^1 \cap H)M = (\Gamma^1 \cap H)M'$. This in turn means that we can identify $X_1(\boldsymbol{e}_1, \boldsymbol{y})$ with a "smaller" space and we write $X_1(\boldsymbol{e}_1, \boldsymbol{y}) = (\Gamma^1 \cap H)\backslash HM_{\boldsymbol{y}}$.

5.3 Quasicrystals

We can also describe quasicrystals in terms of homogenous spaces in a similar way to lattices. Let $g \in G$ and define an embedding of $\mathrm{ASL}(d, \mathbb{R})$ in G by

$$\varphi_g : \mathrm{ASL}(d, \mathbb{R}) \to G, \quad (A, \boldsymbol{x}) \mapsto g\left(\begin{pmatrix} A & 0 \\ 0 & 1_m \end{pmatrix}, (\boldsymbol{x}, \boldsymbol{0})\right) g^{-1}.$$

The following statements follow from Ratner's work [6], [7], though we will state them in the form found in [5, Section 1.4]. There exists a unique closed connected subgroup H_g of G such that $\Gamma \cap H_g$ is a lattice in H_g, $\varphi_g(\mathrm{SL}(d, \mathbb{R})) \subset H_g$, and the closure of $\Gamma\backslash\Gamma\varphi_g(\mathrm{SL}(d, \mathbb{R}))$ in $\Gamma\backslash G$ is given by $\Gamma\backslash\Gamma H_g$. Furthermore we have that $\Gamma\backslash\Gamma H_g$ can be naturally identified with the homogenous space $(\Gamma \cap H_g)\backslash H_g$, and we denote the unique right-H_g invariant probability measure on either of these spaces by μ_g.

We also have that there exists a unique closed connected subgroup $\tilde{H}_g$ of G such that $\Gamma \cap \tilde{H}_g$ is a lattice in $\tilde{H}_g$, $\varphi_g(\mathrm{ASL}(d,\mathbb{R})) \subset \tilde{H}_g$, and the closure of $\Gamma\backslash\Gamma\varphi_g(\mathrm{ASL}(d,\mathbb{R}))$ in $\Gamma\backslash G$ is given by $\Gamma\backslash\Gamma\tilde{H}_g$. Similarly to before we have that $\Gamma\backslash\Gamma\tilde{H}_g$ can be identified with the homogenous space $(\Gamma \cap \tilde{H}_g)\backslash\tilde{H}_g$, and we denote the unique right-$\tilde{H}_g$ invariant probability measure on either of these spaces by $\mu_{\tilde{H}_g}$.

6 Existence of limit distributions

In Section 5 we looked at spaces of lattices of dimension n. For the rest of this paper we will fix $n = 3$, $d = 2$ and $m = 1$.
For $v \in \mathbb{R}$ and $t > 0$ we will use the notation $n(v)$ and Φ^t for the following elements in $\mathrm{SL}(2,\mathbb{R})$:

$$n(v) = \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix}, \quad \Phi^t = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^t \end{pmatrix}.$$

6.1 Preparation

The following proposition is a special case of Theorem 4.2 in [5].

Proposition 6.1. *Fix $g \in G$ and let λ be a Borel probability measure on $\mathbb{R}$ which is absolutely continuous with respect to Lebesgue measure. Then for any $f \in C_b(\mathbb{R} \times \Gamma\backslash\Gamma H_g)$ we have*

$$\lim_{t\to\infty} \int_{\mathbb{R}} f\left(v, \Gamma\varphi_g\left(n(v)\Phi^t\right)\right) d\lambda(v) = \int_{\mathbb{R}\times\Gamma\backslash\Gamma H_g} f(v,p) d\lambda(v) d\mu_g(p).$$

We will also need the following restatement of Proposition 4.5 in [5] for $d = 2$, $n = 3$.

Proposition 6.2. *Let $g \in G$ be fixed. Then for (Lebesgue)-almost all $\boldsymbol{q} \in \mathbb{R}^2 \times \{\mathbf{0}\}$ we have $H_{g(1_3,\boldsymbol{q})} = \tilde{H}_g$.*

In the proof of existence of limit distributions we will utilize the periodic extension of $\mathcal{F}$ from $\mathbb{R}$ to $\mathbb{R}^2$ defined by

$$\mathcal{F}' = \mathbb{Z} \times \mathcal{F} = \{(x,y) \in \mathbb{R}^2 : x \in \mathbb{Z}, y \in \mathcal{F}\},$$

and we note that $\mathcal{F}'$ can be written as a cut-and-project set with $d = 2$ and $m = 1$ by taking

$$\mathcal{F}' = \mathcal{P}(\mathcal{W}, \mathbb{Z}^3 K'),$$

where

$$K' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} \in \mathrm{SO}(3,\mathbb{R}).$$

Lemma 6.3. *Let $g = K' \in \mathrm{SO}(3,\mathbb{R})$. Then $H_g = G^1$ and $\tilde{H}_g = G$.*

Proof. Let $\mathcal{L} = \mathbb{Z}^3 K'$ and see that we get the desired result from Proposition 2.1 and Lemma 2.2 in [5], if we show that the map $\pi_{|\mathcal{L}}$ is injective and $\mathcal{A} = \overline{\pi_{\mathrm{int}}(\mathcal{L})} = \mathbb{R}$.
To prove injectivity we assume there are integers $m_1, m_2, m_3, m_1', m_2', m_3'$ such that

$$\pi((m_1,m_2,m_3)K') = \pi((m_1',m_2',m_3')K').$$

We can write this as

$$m_1(1,0) + m_2(0,\cos\varphi) + m_3(0,\cos\varphi) = m_1'(1,0) + m_2'(0,\cos\varphi) + m_3'(0,\cos\varphi)$$

so

$$m_1 - m_1' = 0 \quad \text{and} \quad m_2 - m_2' + (m_3 - m_3')\tan\varphi = 0.$$

Hence $(m_1, m_2, m_3) = (m_1', m_2', m_3')$ since $\tan\varphi = \tau^{-1}$ is irrational.
To see that $\overline{\pi_{\text{int}}(\mathcal{L})} = \mathbb{R}$ we note

$$\overline{\pi_{\text{int}}(\mathcal{L})} = \mathbb{Z}(-\sin\varphi) + \mathbb{Z}(\cos\varphi) = \cos\varphi\,(\mathbb{Z}\tan\varphi + \mathbb{Z})\,.$$

One way to see that $\mathbb{Z}\tan\varphi + \mathbb{Z}$ is dense in $\mathbb{R}$ is to realize that this is equivalent to saying that $\mathbb{Z}\tan\varphi \mod 1$ is dense in $[0,1)$. This of course follows directly from the fact that the orbit of an irrational circle rotation is dense on the circle. □

Note that in the proof of Lemma 6.3 we get that $\mathcal{A} = \mathbb{R}$, so that in this case we get $\mathcal{V} = \mathbb{R}^3$, $\mathcal{L}_{\mathcal{V}} = \mathcal{L} = \mathbb{Z}^3 K'$ and $\mu_{\mathcal{V}} = \text{vol}$.
Now we can prove the following equidistribution result which will be a key ingredient in the proof of Theorem 4.1.

Proposition 6.4. *Take $K' \in \mathrm{SO}(3,\mathbb{R})$ as before and let Λ be any Borel probability measure on $\mathbb{R}\times\mathbb{R}$ which is absolutely continuous with respect to Lebesgue measure. For any $f \in C_b(\Gamma\backslash G)$ we have that*

$$\lim_{\rho\to 0}\int_{\mathbb{R}} f\left(\Gamma\left(K', -(0,q,0)\right)\begin{pmatrix} 1 & -v & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix}\rho & & \\ & 1/\rho & \\ & & 1\end{pmatrix} K'^{-1}\right) d\Lambda(q,v) = \int_{\Gamma\backslash G} f\,d\mu.$$

Proof. If we let $g = K'(1_3, \boldsymbol{q})$, $\boldsymbol{q} = (q_1, q_2, 0) \in \mathbb{R}^2 \times \{\mathbf{0}\}$, then Proposition 6.2 together with Lemma 6.3 gives us that $H_g = \tilde{H}_{K'} = G$ for almost all $\boldsymbol{q} \in \mathbb{R}^2 \times \{\mathbf{0}\}$. This gives us that $\Gamma\backslash\Gamma H_g = \Gamma\backslash G$.
Let $\tilde{\Lambda}$ be a Borel probability measure on $\mathbb{R}^3$ which is absolutely continuous with respect to three dimensional Lebesgue measure and let $\tilde{\lambda}$ be a Borel probability measure on $\mathbb{R}$ that is equivalent to one dimensional Lebesgue measure. Then by the Radon-Nikodym Theorem there exists a function $L \in L^1(\mathbb{R}^3)$ such that $d\tilde{\Lambda}(q_1, q_2, v) = L(q_1, q_2, v)d\text{vol}(q_1, q_2)d\tilde{\lambda}(v)$. For f and g given as above we fix $q_1, q_2 \in \mathbb{R}$ and define $\tilde{f} \in C_b(\mathbb{R}\times\Gamma\backslash G)$ through

$$\tilde{f}(v,p) := f(p(K', (q_1, q_2, 0))K'^{-1})L(q_1, q_2, v)$$

With $g = K'(1_3, (q_1, q_2, 0))$ Proposition 6.1 gives

$$\lim_{t\to\infty}\int_{\mathbb{R}} \tilde{f}\left(v, \Gamma\varphi_g\left(n(v)\Phi^t\right)\right) d\tilde{\lambda}(v) = \int_{\mathbb{R}\times\Gamma\backslash G} \tilde{f}(v,p)d\tilde{\lambda}(v)d\mu(p),$$

for almost all $q_1, q_2 \in \mathbb{R}$. Using the definitions of $\tilde{f}$ and g we can write this as

$$\lim_{t\to\infty}\int_{\mathbb{R}} f\left(\Gamma(K', (q_1, q_2, 0))\varphi_1\left(n(v)\Phi^t\right)K'^{-1}\right) L(q_1, q_2, v)d\tilde{\lambda}(v)$$
$$= \int_{\mathbb{R}\times\Gamma\backslash G} f(p(K', (q_1, q_2, 0))K'^{-1})L(q_1, q_2, v)d\tilde{\lambda}(v)d\mu(p). \tag{6.1}$$

Since μ is right G-invariant we can simplify the right hand side of equation (6.1) to

$$\int_{\mathbb{R}\times\Gamma\backslash G} f(p)L(q_1,q_2,v)d\tilde{\lambda}(v)d\mu(p).$$

We note that equation (6.1) is valid for Lebesgue-almost all $q_1,q_2 \in \mathbb{R}$ and the right hand side is a Lebesgue-integrable function in (q_1,q_2). Integrating both sides of equation (6.1) with respect to $\mathrm{vol}(q_1,q_2)$ over $\mathbb{R}^2$ we see that since $f \in C_b(\Gamma\backslash G)$ and $L \in L^1(\mathbb{R}^3)$, Lebesgue's Bounded Convergence Theorem gives us

$$\lim_{t\to\infty}\int_{\mathbb{R}^3} f\left(\Gamma(K',(q_1,q_2,0))\varphi_1\left(n(v)\Phi^t\right)K'^{-1}\right)d\tilde{\Lambda}(q_1,q_2,v) = \int_{\Gamma\backslash G} f(p)d\mu(p). \tag{6.2}$$

Now we will choose $\tilde{\Lambda}$ to be a have a certain form. Let ν be a Borel probability measure on $\mathbb{R}$ which is absolutely continuous with respect to the Lebesgue measure. Then let $\tilde{\Lambda}$ be the measure satisfying the equation

$$\int_{\mathbb{R}^3} h(q_1,q_2,v)d\tilde{\Lambda}(q_1,q_2,v) = \int_{\mathbb{R}^3} h(y,yv-q,v)d\Lambda(q,v)d\nu(y)$$

for each $h \in C_c(\mathbb{R}^3)$. Then we can write equation (6.2) as

$$\lim_{t\to\infty}\int_{\mathbb{R}^3} f\left(\Gamma(K',(y,yv-q,0))\varphi_1\left(n(v)\Phi^t\right)K'^{-1}\right)d\Lambda(q,v)d\nu(y) = \int_{\Gamma\backslash G} f d\mu.$$

Now we make the additional assumption that $f \in C_c(\Gamma\backslash G)$, so in particular we know that f is uniformly continuous. Note that

$$(K',(y,yv-q,0))\varphi_1(n(v)\Phi^t) = (K',-(0,q,0))\varphi_1(n(v)\Phi^t)(1_3,(ye^{-t},0,0))$$

and $(1_3,(ye^{-t},0,0))$ approaches 1_3 as $t\to\infty$. Since f is uniformly continuous it follows that

$$\begin{aligned}&\lim_{t\to\infty}\int_{\mathbb{R}^3} f\left(\Gamma(K',(y,yv-q,0))\varphi_1\left(n(v)\Phi^t\right)K'^{-1}\right)d\Lambda(q,v)d\nu(y)\\ &\quad= \lim_{t\to\infty}\int_{\mathbb{R}^2} f\left(\Gamma(K',-(0,q,0))\varphi_1\left(n(v)\Phi^t\right)K'^{-1}\right)d\Lambda(q,v).\end{aligned}$$

To extend this to $f \in C_b(\Gamma\backslash G)$ we take a sequence of functions $\{f_n\} \subset C_c(\Gamma\backslash G)$ such that $|f_n(p)| \le |f(p)|$ for all $p \in \Gamma\backslash G$ and $f_n(p) \to f(p)$ as $n\to\infty$ for all $p\in\Gamma\backslash G$. By Lebesgue's Dominated Convergence Theorem we get that

$$\lim_{t\to\infty}\int_{\mathbb{R}^2} f\left(\Gamma(K',-(0,q,0))\varphi_1\left(n(v)\Phi^t\right)K'^{-1}\right)d\Lambda(q,v) = \int_{\Gamma\backslash G} f d\mu$$

and our desired result follows by changing the variable t to ρ by taking $\rho = e^{-t}$. □

Before we go on to prove Theorem 4.1 we will need the following two versions of the Siegel integral formula, which are special cases of Theorem 5.1 (corrected) and Corollary 5.2 in [5] respectively. Recall from (2.1) that $\delta_{d,m}(\mathcal{L}) = 1/\mu_{\mathcal{V}}(\mathcal{V}/\mathcal{L}_{\mathcal{V}})$, so for $\mathcal{L} = \mathbb{Z}^3K'$ we have that $\delta_{2,1}(\mathbb{Z}^3K') = 1/\mathrm{vol}(\mathbb{R}^3/\mathbb{Z}^3K') = 1/\det(K') = 1$, so we don't need the factor $\delta_{2,1}(\mathbb{Z}^3K')$ in the formulas below.

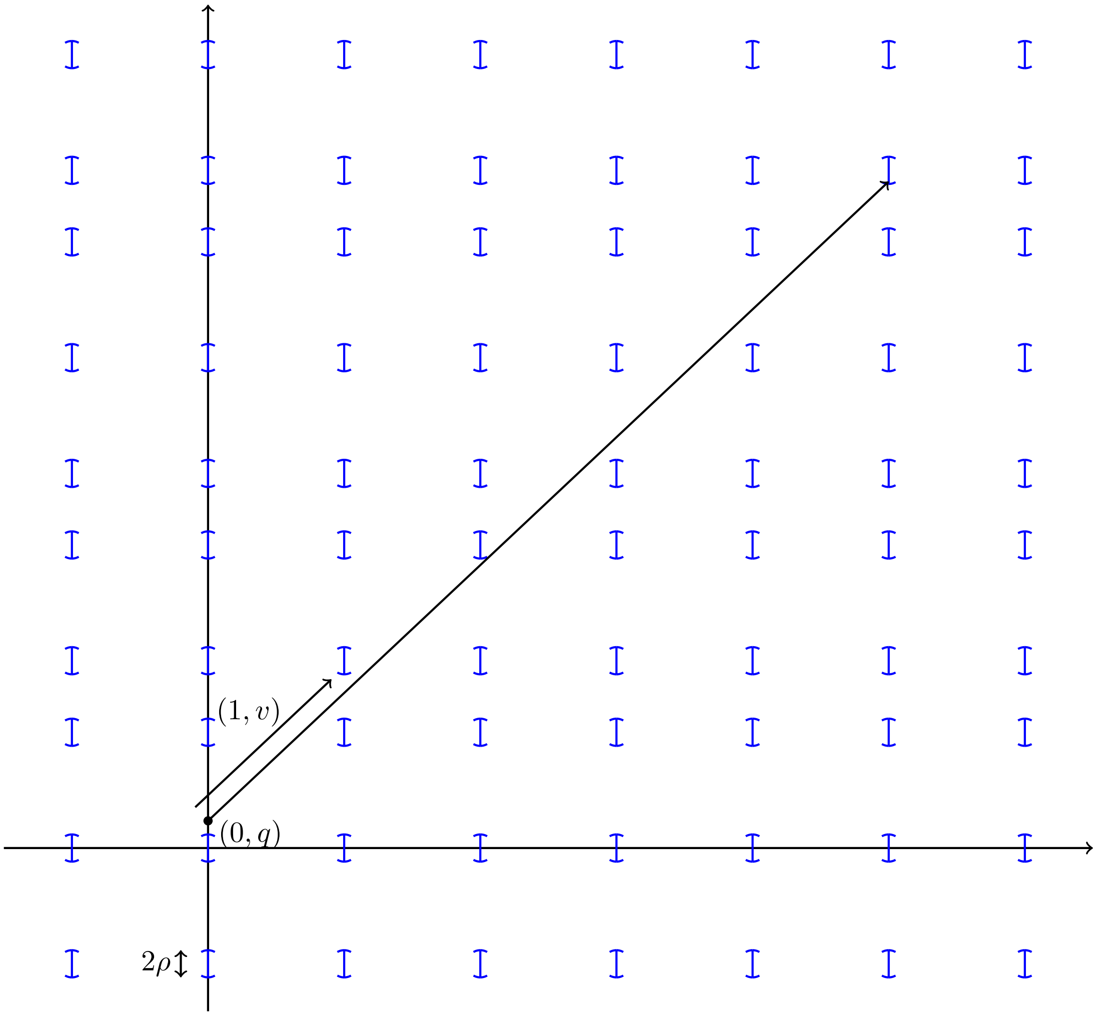

Figure 4: We extend the one-dimension Lorentz gas to a two-dimensional model in order to apply the methods used for multi-dimension Lorentz gas.

Proposition 6.5. *For any* $f \in L^1(\mathbb{R}^3)$,

$$\int_{\Gamma^1\backslash G^1} \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}^3 hK' \\ \pi(\boldsymbol{m})\neq \boldsymbol{0}}} f(\boldsymbol{m}) d\mu_1(h) = \int_{\mathbb{R}^3} f d\mathrm{vol}.$$

Proposition 6.6. *For any* $f \in L^1(\mathbb{R}^3)$,

$$\int_{\Gamma\backslash G} \sum_{\boldsymbol{m}\in\mathbb{Z}^3 hK'} f(\boldsymbol{m}) d\mu(h) = \int_{\mathbb{R}^3} f d\mathrm{vol}.$$

6.2 Proof of Theorem 4.1

Proof. Recall our extension of $\mathcal{F}$ given by

$$\mathcal{F}' = \mathbb{Z} \times \mathcal{F} = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Z}, y \in \mathcal{F}\}.$$

This extension allows us to write the free path length as

$$k_0(q,v,\rho) = \inf\{t \in \mathbb{R}_{>0} : (0,q) + t(1,v) \in \mathcal{F}' + (\{0\} \times (-\rho,\rho))\},$$

see Figure 4 for a visual explanation. Multiplying both sides of the equation by ρ and changing the coordinates in $\mathbb{R}^2$ by the linear map $\begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix}$ we get that

$$\rho k_0(q,v,\rho) = \inf\left\{t \in \mathbb{R}_{>0} : (0,q) + t/\rho(1,0) \in \mathcal{F}' \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} + (\{0\} \times (-\rho,\rho))\right\}.$$

We see that $\rho k_0(q,v,\rho)$ is the smallest positive number t such that the rectangle $(0,t/\rho) \times (-\rho,\rho)$ contains some point in the point set $(\mathcal{F}' - (0,q))\begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix}$, i.e.

$$\rho k_0(q,v,\rho) = \inf\left\{t \in \mathbb{R}_{>0} : ((0,t/\rho) \times (-\rho,\rho)) \cap (\mathcal{F}' - (0,q)) \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} \neq \emptyset\right\}.$$

Rescaling everything by the linear map $\begin{pmatrix} \rho & 0 \\ 0 & 1/\rho \end{pmatrix}$ makes the rectangle depend only on t,

$$\rho k_0(q,v,\rho) = \inf\left\{t \in \mathbb{R}_{>0} : ((0,t) \times (-1,1)) \cap (\mathcal{F}' - (0,q)) \begin{pmatrix} 1 & -v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \rho & 0 \\ 0 & 1/\rho \end{pmatrix} \neq \emptyset\right\}.$$

Recall that $\mathcal{F}'$ can be written as a cut-and-project set with $d = 2$ and $m = 1$ by writing

$$\mathcal{F}' = \mathcal{P}(\mathcal{W}, \mathbb{Z}^3 K').$$

Using this formula we write $\rho k_0(q,v,\rho)$ as

$$\rho k_0(q,v,\rho) = \inf\{t \in \mathbb{R}_{>0} : \mathfrak{Z}_t \cap \mathcal{P} \neq \emptyset\}$$

where

$$\mathfrak{Z}_t = (0,t) \times (-1,1) \times \mathcal{W} = \{(x_1,x_2,x_3) \in \mathbb{R}^3 : 0 < x_1 < t, |x_2| < 1, x_3 \in \mathcal{W}\}$$

and

$$\mathcal{P} = (\mathbb{Z}^3 K' - (0,q,0)) \begin{pmatrix} 1 & -v & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} \rho & & \\ & 1/\rho & \\ & & 1 \end{pmatrix}.$$

With this formula for $k_0(q,v,\rho)$ we get that

$$\Lambda\left(\{(q,v) \in \mathbb{R} \times \mathbb{R} : k_0(q,v,\rho) \geq \xi/\rho\}\right) = \Lambda\left(\{(q,v) \in \mathbb{R} \times \mathbb{R} : \mathfrak{Z}_\xi \cap \mathcal{P} = \emptyset\}\right).$$

We see that $\mathfrak{Z}_\xi$ is Jordan measurable, so for any given $\varepsilon > 0$ there exists nonnegative continuous functions a^- and a^+ on $\mathbb{R}^3$ satisfying $a^- \leq \chi_{\mathfrak{Z}_\xi} \leq a^+ \leq 1$ such that

$$\operatorname{vol}\left(\operatorname{supp}(a^+ - a^-)\right) < \varepsilon. \tag{6.3}$$

Let f^+ and f^- in $C_b(\Gamma\backslash G)$ be defined by

$$f^{\pm}(\Gamma h) = \max\left(0, 1 - \sum_{\boldsymbol{m} \in \mathbb{Z}^3 h K'} a^{\pm}(\boldsymbol{m})\right).$$

Then we see that for all $\Gamma h \in \Gamma\backslash G$

$$f^+(\Gamma h) \le \max\left(0, 1 - \sum_{\boldsymbol{m}\in\mathbb{Z}^3 hK'} \chi_{\mathfrak{Z}_\xi}(\boldsymbol{m})\right)$$

$$= \max\left(0, 1 - \sum_{\boldsymbol{m}\in\mathbb{Z}^3 hK'} I(\boldsymbol{m}\in\mathfrak{Z}_\xi)\right) = I(\mathfrak{Z}_\xi \cap \mathbb{Z}^3 hK' = \emptyset) \tag{6.4}$$

and similarly

$$f^-(\Gamma h) \ge I(\mathfrak{Z}_\xi \cap \mathbb{Z}^3 hK' = \emptyset). \tag{6.5}$$

From equations (6.4) and (6.5) and the fact that μ is G-invariant we get that

$$\int_{\Gamma\backslash G} f^+(\Gamma h)d\mu(h) \le \mu(\{\Gamma h \in \Gamma\backslash G : \mathbb{Z}^3 h \cap \mathfrak{Z}_\xi = \emptyset\}) \le \int_{\Gamma\backslash G} f^-(\Gamma h)d\mu(h). \tag{6.6}$$

Now from Proposition 6.4 we get

$$\liminf_{\rho\to 0} \Lambda\left(\{(q,v)\in\mathbb{R}\times\mathbb{R} : \mathfrak{Z}_\xi \cap \mathcal{P} = \emptyset\}\right)$$

$$\ge \lim_{\rho\to 0}\int_{\mathbb{R}\times\mathbb{R}} f^+\left(\Gamma(K' - (0,q,0))\begin{pmatrix} 1 & -v & \\ & 1 & \\ & & 1\end{pmatrix}\begin{pmatrix} \rho & & \\ & 1/\rho & \\ & & 1\end{pmatrix} K'^{-1}\right) d\Lambda(q,v)$$

$$= \int_{\Gamma\backslash G} f^+(\Gamma h)d\mu(h) \tag{6.7}$$

and

$$\limsup_{\rho\to 0} \Lambda\left(\{(q,v)\in\mathbb{R}\times\mathbb{R} : \mathfrak{Z}_\xi \cap \mathcal{P} = \emptyset\}\right) \le \int_{\Gamma\backslash G} f^-(\Gamma h)d\mu(h). \tag{6.8}$$

We see that

$$0 \le f^-(\Gamma h) - f^+(\gamma h) \le \sum_{\boldsymbol{m}\in\mathbb{Z}^3 hK'} \big(a^+(\boldsymbol{m}) - a^-(\boldsymbol{m})\big)$$

so by Proposition 6.6 and the condition in (6.3) we get

$$0 \le \int_{\Gamma\backslash G} \big(f^-(\Gamma h) - f^+(\gamma h)\big)\, d\mu(h) \le \int_{\Gamma\backslash G} \sum_{\boldsymbol{m}\in\mathbb{Z}^3 hK'} \big(a^+(\boldsymbol{m}) - a^-(\boldsymbol{m})\big)\, d\mu(h)$$

$$= \int_{\mathbb{R}^3} (a^+ - a^-) d\mathrm{vol} \le \mathrm{vol}\big(\mathrm{supp}(a^+ - a^-)\big) < \varepsilon. \tag{6.9}$$

Now (6.6) and (6.9) gives us that the left hand sides of (6.7) and (6.8) are within $\varepsilon > 0$ of $\mu(\{\Gamma h \in \Gamma\backslash G : \mathbb{Z}^3 h \cap \mathfrak{Z}_\xi = \emptyset\})$ and since ε was arbitrary we get that

$$\lim_{\rho\to 0} \Lambda\left(\{(q,v)\in\mathbb{R}\times\mathbb{R} : \mathfrak{Z}_\xi \cap \mathcal{P} = \emptyset\}\right) = \mu(\{\Gamma h \in \Gamma\backslash G : \mathbb{Z}^3 h \cap \mathfrak{Z}_\xi = \emptyset\}),$$

which completes our proof. □

6.3 Proof of Theorem 4.2

Proof. This proof is very similar to the proof of Theorem 4.1 so we will not detail all the steps in this proof. We begin by noticing that since $\pi\big|_{\mathbb{Z}^2K}$ is injective there is a function $w : \mathcal{F} \to \mathbb{R}$ such that $(q, w(q)) \in \mathbb{Z}^2K$ for all $q \in \mathcal{F}$. This also implies that $(0, q, w(q)) \in \mathbb{Z}^3K'$ so

$$\mathbb{Z}^3K' - (0, q+\rho\beta(v), 0) = \mathbb{Z}^3K' - (0, q, w(q)) + (0, -\rho\beta(v), w(q)) = \mathbb{Z}^3K' + (0, -\rho\beta(v), w(q))$$

and

$$\begin{aligned}(\mathbb{Z}^3K' - (0, q+\rho\beta(v), 0)) &\begin{pmatrix} 1 & -v & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} \rho & & \\ & 1/\rho & \\ & & 1 \end{pmatrix} \\ = (0, -\beta(v), w(q)) + \mathbb{Z}^3K' &\begin{pmatrix} 1 & -v & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} \rho & & \\ & 1/\rho & \\ & & 1 \end{pmatrix}.\end{aligned}$$

Then we can write the free path length as

$$\rho k_0(q+\rho\beta(v), v, \rho) = \inf\{t \in \mathbb{R}_{>0} : (\mathfrak{Z}_t + (0, \beta(v), -w(q))) \cap \mathcal{P}\}$$

where

$$\mathcal{P} = \mathbb{Z}^3K' \begin{pmatrix} 1 & -v & \\ & 1 & \\ & & 1 \end{pmatrix}\begin{pmatrix} \rho & & \\ & 1/\rho & \\ & & 1 \end{pmatrix}.$$

So we get

$$\lambda\left(\{v \in \mathbb{R} : k_0(q+\rho\beta(v), v, \rho) \geq \xi/\rho\}\right) = \lambda\left(\{v \in \mathbb{R} : (\mathfrak{Z}_\xi + (0, \beta(v), -w(q))) \cap \mathcal{P} = \emptyset\}\right).$$

Now given $\varepsilon > 0$ we can take a^- and a^+ as in the proof of Theorem 2. Then for $v \in \mathbb{R}$ and $z \in \mathbb{R}$ we let $a^-_{v,z}$ and $a^+_{v,z}$ be defied by

$$a^\pm_{v,z}(\boldsymbol{x}) = a^\pm\left(\boldsymbol{x} + (0, -\beta(v), z)\right), \quad \boldsymbol{x} \in \mathbb{R}^3.$$

We see that $a^-_{v,z}$ and $a^+_{v,z}$ are jointly continuous in $v, z, \boldsymbol{x}$, and $a^-_{v,w(q)}(\boldsymbol{x}) \leq \chi_{\mathfrak{Z}_\xi + (0,\beta(v),-w(q))}(\boldsymbol{x}) \leq a^+_{v,w(q)}(\boldsymbol{x}) \leq 1$ for any $v \in \mathbb{R}$. Similar to before we define f^+ and f^- in $C_b(\mathbb{R}, \Gamma^1\backslash G^1)$ by

$$f^\pm(v, \Gamma^1 h) = \max\left(0, 1 - \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}^3hK' \\ \pi(\boldsymbol{m})\neq\boldsymbol{0}}} a^\pm_{v,w(q)}(\boldsymbol{m})\right).$$

If $\pi(\boldsymbol{m}) = \boldsymbol{0}$ then $\boldsymbol{m} = (0, 0, m_3) \notin \mathfrak{Z}_\xi + (0, \beta(v), -w(q))$ for all $v \in \mathbb{R}$ so we get that

$$f^+(v, \Gamma^1 h) \leq I\left((\mathfrak{Z}_\xi + (0, \beta(v), -w(q))) \cap \mathbb{Z}^3hK'\right) \leq f^-(v, \Gamma^1 h), \quad \forall \Gamma^1 h \in \Gamma^1\backslash G^1.$$

Now Lemma 6.3 with $g = K'$ gives us that $H_g = G^1$ so $\Gamma\backslash\Gamma H_g = \Gamma\backslash\Gamma G^1$ and it is known that $\Gamma\backslash\Gamma G^1$ can identified with $\Gamma^1\backslash G^1$. Hence by Proposition 6.1 we get that

$$\limsup_{\rho\to 0} \lambda\left(\{v \in \mathbb{R} : (\mathfrak{Z}_\xi + (0, \beta(v), -w(q))) \cap \mathcal{P} = \emptyset\}\right) \leq \int_{\mathbb{R}\times\Gamma^1\backslash G^1} f^-(v, \Gamma^1 h)\, d\lambda(v)\, d\mu_1(h)$$

and

$$\liminf_{\rho\to 0} \lambda\left(\{v \in \mathbb{R} : (\mathfrak{Z}_\xi + (0,\beta(v),-w(q))) \cap \mathcal{P} = \emptyset\}\right) \geq \int_{\mathbb{R}\times\Gamma^1\backslash G^1} f^+(v,\Gamma^1 h)d\lambda(v)d\mu_1(h).$$

Proposition 6.5 then yields

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}\times\Gamma^1\backslash G^1} \left(f^-(v,\Gamma^1 h) - f^+(v,\Gamma^1 h)\right) d\lambda(v)d\mu_1(h) \\ &\leq \int_{\mathbb{R}} \operatorname{vol}\left(\operatorname{supp}(a^+_{v,w(q)} - a^-_{v,w(q)})\right) d\lambda(v) < \varepsilon, \end{aligned}$$

and since $\varepsilon > 0$ was arbitrary we conclude that

$$\begin{aligned} &\lim_{\rho\to 0} \lambda\left(\{v \in \mathbb{R} : (\mathfrak{Z}_\xi + (0,\beta(v),-w(q))) \cap \mathcal{P} = \emptyset\}\right) \\ &= \int_{\mathbb{R}} \mu_1\left(\{\Gamma^1 h \in \Gamma^1\backslash G^1 : \mathbb{Z}^3 h \cap (\mathfrak{Z}_\xi + (0,\beta(v),-w(q))) = \emptyset\}\right) d\lambda(v). \qquad (6.10) \end{aligned}$$

We see that by applying $\begin{pmatrix} 1 & & \\ & \operatorname{sgn}(\beta(v)) & \\ & & 1 \end{pmatrix} \in G^1$ to $\mathfrak{Z}_\xi + (0,\beta(v),-w(q))$ from the right and using the G^1-invariance of μ^1 we get that the right hand side of equation (6.10) equals

$$\int_{\mathbb{R}} \mu_1\left(\{\Gamma^1 h \in \Gamma^1\backslash G^1 : \mathbb{Z}^3 h \cap (\mathfrak{Z}_\xi + (0,|\beta(v)|,-w(q))) = \emptyset\}\right) d\lambda(v) = \int_{\mathbb{R}} F_{\mathcal{F},q}(\xi,|\beta(v)|)d\lambda(v).$$

It remains to prove that the convergence in equation (4.1) is uniform over all $q \in \mathcal{F}$. To do this we will impose stronger conditions on a^- and a^+. Let $\varepsilon > 0$ be given and take

$$\mathfrak{Z}^-_{\xi,\eta} = (0,\xi) \times (-1,1) \times \left(-\frac{1+\tau}{2\nu} + \eta, \frac{1+\tau}{2\nu} - \eta\right)$$

and

$$\mathfrak{Z}^+_{\xi,\eta} = (0,\xi) \times (-1,1) \times \left(-\frac{1+\tau}{2\nu} - \eta, \frac{1+\tau}{2\nu} + \eta\right).$$

Now we can choose non-negative continuous functions a^- and a^+ such that for some $\eta = \eta(\varepsilon) > 0$, we have that $a^- \leq \chi_{\mathfrak{Z}^-_{\xi,\eta}+(0,\beta(v),-w(q))}$ and $\chi_{\mathfrak{Z}^+_{\xi,\eta}+(0,\beta(v),-w(q))} \leq a^+ \leq 1$ holds in addition to the condition in (6.3). We proceed by choosing points $z_1,\ldots,z_s \in \mathbb{R}$ such that each $z \in \mathcal{W}$ lies in the η-nieghborhood of some z_j, and for each $j = 1,\ldots,s$ we define $f^\pm_j \in C_b(\mathbb{R}\times\Gamma^1\backslash G^1)$ by

$$f^\pm_j(v,\Gamma^1 h) = \max\left(0, 1 - \sum_{\substack{\boldsymbol{m}\in\mathbb{Z}^3 hK' \\ \pi(\boldsymbol{m})\neq\boldsymbol{0}}} a^\pm_{v,z_j}(\boldsymbol{m})\right).$$

Since s is finite we can use Proposition 6.1 to get that there is some $\rho_0 > 0$ such that for every $\rho \in (0,\rho_0]$ and every $j = 1,\ldots,s$,

$$\left|\int_{\mathbb{R}} f^\pm_j\left(v, \Gamma\varphi_{K'}(n(v)\Phi^{-\log\rho})\right) d\lambda(v) - \int_{\mathbb{R}\times\Gamma^1\backslash G^1} f(v,p)d\lambda(v)d\mu_1(p)\right| < \varepsilon. \qquad (6.11)$$

Now let $q \in \mathcal{F}$ be given and take j such that $|w(q) - z_j| < \eta$. Then we get that

$$f_j^+(v, \Gamma^1 h) \leq I\left((\mathfrak{Z}_\xi + (0, \beta, -w(q))) \cap \mathbb{Z}^3 hK' = \emptyset\right) \leq f_j^-(v, \Gamma^1 h) \quad \forall (v, h) \in \mathbb{R} \times G^1.$$

This together with equation (6.11) gives us that for all $\rho \in (0, \rho_0]$

$$\lambda\left(\{v \in \mathbb{R} : (\mathfrak{Z}_\xi + (0, \beta(v), -w(q))) \cap \mathcal{P} = \emptyset\}\right) \leq \int_{\mathbb{R} \times \Gamma^1 \backslash G^1} f_j^-(v, \Gamma^1 h) d\lambda(v) d\mu_1(h) + \varepsilon$$

and

$$\lambda\left(\{v \in \mathbb{R} : (\mathfrak{Z}_\xi + (0, \beta(v), -w(q))) \cap \mathcal{P} = \emptyset\}\right) \geq \int_{\mathbb{R} \times \Gamma^1 \backslash G^1} f_j^+(v, \Gamma^1 h) d\lambda(v) d\mu_1(h) - \varepsilon.$$

Using the same argument as before we see that the integrals on the right hand sides differ by at most ε, hence we get that for all $\rho \in (0, \rho_0]$

$$\Bigg| \lambda\left(\{v \in \mathbb{R} : (\mathfrak{Z}_\xi + (0, \beta(v), -w(q))) \cap \mathcal{P} = \emptyset\}\right)$$
$$- \int_{\mathbb{R}} \mu_1\left(\{\Gamma^1 h \in \Gamma^1 \backslash G^1 : \mathbb{Z}^3 h \cap (\mathfrak{Z}_\xi + (0, |\beta(v)|, -w(q))) = \emptyset\}\right) d\lambda(v) \Bigg| < 2\varepsilon.$$

Since $q \in \mathcal{F}$ and $\varepsilon > 0$ were arbitrary we get that the convergence is uniform in the desired manner. □

7 Formula for $F_{\mathcal{F}}(\xi)$ for small ξ

In Section 6 we saw that the distribution function $F_{\mathcal{F}}$ is given by

$$F_{\mathcal{F}}(\xi) = \mu(\{\Gamma M \in \Gamma \backslash G : \mathbb{Z}^3 M \cap \mathfrak{Z}_\xi = \emptyset\}), \tag{7.1}$$

where $\mathfrak{Z}_\xi$ is the following box

$$\mathfrak{Z}_\xi = (0, \xi) \times (-1, 1) \times \mathcal{W} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_1 < \xi, |x_2| < 1, x_3 \in \mathcal{W}\}.$$

Similarly, $F_{\mathcal{F},q}(\xi, r)$ is given by

$$F_{\mathcal{F},q}(\xi, r) = \mu_1(\{\Gamma^1 M \in \Gamma^1 \backslash G^1 : \mathbb{Z}^3 M \cap (\mathfrak{Z}_\xi + (0, r, -w(q))) = \emptyset\}), \tag{7.2}$$

where $w(q)$ is the unique point in $\mathcal{W}$ such that $(q, w) \in \mathbb{Z}^2 K$.
For $r_1, r_2 > 0$ we will denote by $\mathcal{R}(r_1, r_2)$ the rectangle $(-r_1, r_1) \times (-r_2, r_2)$.
Let $\mathbf{\Phi} : [0, \infty] \times \mathbb{R}^2 \to [0, 1]$ be defined as

$$\mathbf{\Phi}(\xi, \boldsymbol{w}) = \mu_1(\{\Gamma^1 M \in \Gamma^1 \backslash G^1 : \mathbb{Z}^3 M \cap (\mathfrak{Z}_\xi + (0, \boldsymbol{w})) = \emptyset\}), \tag{7.3}$$

and let $\mathcal{R} = \mathcal{R}(1, (\tau + 1)/(2\nu)) = (-1, 1) \times \mathcal{W}$.
In particular we see that taking equation (7.2) together with (7.3) we get that

$$F_{\mathcal{F},q}(\xi, r) = \mathbf{\Phi}(\eta, (r, -w(q))). \tag{7.4}$$

7.1 Integral formulas

Proposition 7.1. *The function* $F_{\mathcal{F}}$ *defined in* (7.1) *is* C^1 *and can be written as*

$$F_{\mathcal{F}}(\xi) = \int_{\xi}^{\infty} \int_{\mathcal{R}} \boldsymbol{\Phi}(\eta, \boldsymbol{w}) d\boldsymbol{w} d\eta. \tag{7.5}$$

Proof. We will use the notation

$$\mathfrak{Z}(c_1, c_2) = (c_1, c_2) \times \mathcal{R}.$$

We see that the difference quotient for $F_{\mathcal{F}}(\xi)$ with step size $h > 0$ can be written as

$$h^{-1}(F_{\mathcal{F}}(\xi + h) - F_{\mathcal{F}}(\xi))$$

$$= -h^{-1}\mu\left(\left\{\Gamma(M, \boldsymbol{x}) \in \Gamma\backslash G : \#((\mathbb{Z}^3 M + \boldsymbol{x}) \cap \mathfrak{Z}(0, \xi)) = 0, \#((\mathbb{Z}^3 M + \boldsymbol{x}) \cap \mathfrak{Z}(\xi, \xi + h)) \geq 1\right\}\right).$$

Now let $U = \mathfrak{Z}(\xi, \xi + h)$ and let

$$\mathscr{E} = \left\{\Gamma(M, \boldsymbol{x}) \in \Gamma\backslash G : \#((\mathbb{Z}^3 M + \boldsymbol{x}) \cap \mathfrak{Z}(0, \xi)) = 0, \#((\mathbb{Z}^3 M + \boldsymbol{x}) \cap \mathfrak{Z}(\xi, \xi + h)) \geq 1\right\}$$

and

$$\mathscr{E}' = \left\{\Gamma(M, \boldsymbol{x}) \in \Gamma\backslash G : \#((\mathbb{Z}^3 M + \boldsymbol{x}) \cap \mathfrak{Z}(0, \xi)) = 0, \#((\mathbb{Z}^3 M + \boldsymbol{x}) \cap \mathfrak{Z}(\xi, \xi + h)) = 1\right\}.$$

Now we see that for all $\boldsymbol{y}_1 \neq \boldsymbol{y}_2 \in U$: $X(\boldsymbol{y}_1) \cap X(\boldsymbol{y}_2) \cap \mathscr{E}' = \emptyset$, so by Proposition 5.1 we have that

$$\limsup_{h\to 0} h^{-1}(F_{\mathcal{F}}(\xi + h) - F_{\mathcal{F}}(\xi)) = \limsup_{h\to 0} -h^{-1}\mu(\mathscr{E})$$

$$\leq \limsup_{h\to 0} -h^{-1}\mu(\mathscr{E}') = \limsup_{h\to 0} -h^{-1} \int_U \nu_{\boldsymbol{y}}(\mathscr{E}' \cap X(\boldsymbol{y})) d\boldsymbol{y}. \tag{7.6}$$

We see that we can write the integral in the right hand side of (7.6) as

$$\int_U \nu_{\boldsymbol{y}}\big(\{\Gamma(M, \boldsymbol{x}) \in X(\boldsymbol{y}) : (\mathbb{Z}^n M + \boldsymbol{x}) \cap \mathfrak{Z}(0, \xi) = \emptyset,$$

$$(\mathbb{Z}^n M + \boldsymbol{x}) \cap \mathfrak{Z}(\xi, \xi + h) = \{\boldsymbol{y}\}\}\big) d\boldsymbol{y};$$

making a linear coordinate transformation reflecting everything in the point $\boldsymbol{y}/2$ and using the identification of $X(\boldsymbol{y})$ with $\Gamma^1\backslash G^1$ we can write the integral as

$$\int_U \mu_1\big(\{\Gamma^1 M \in \Gamma^1\backslash G^1 : \mathbb{Z}^3 M \cap (\mathfrak{Z}(y_1 - \xi, y_1) - (0, y_2, y_3)) = \emptyset,$$

$$\mathbb{Z}^3 M \cap (\mathfrak{Z}(y_1 - \xi - h, y_1 - \xi) - (0, y_2, y_3)) = \{\boldsymbol{0}\}\}\big) d\boldsymbol{y}.$$

Hence we get that the right hand side of (7.6) equals

$$\lim_{h\to 0} -h^{-1} \int_{\xi}^{\xi+h} \int_{\mathcal{R}} \mu_1\big(\{\Gamma^1 M \in \Gamma^1\backslash G^1 : \mathbb{Z}^3 M \cap (\mathfrak{Z}(y_1 - \xi, y_1) - (0, y_2, y_3)) = \emptyset,$$

$$\mathbb{Z}^3 M \cap (\mathfrak{Z}(y_1 - \xi - h, y_1 - \xi) - (0, y_2, y_3)) = \{\boldsymbol{0}\}\}\big) d(y_2, y_3) dy_1.$$

$$= -\int_{\mathcal{R}} \mu_1\big(\{\Gamma^1 M \in \Gamma^1\backslash G^1 : \mathbb{Z}^3 M \cap (\mathfrak{Z}(0, \xi) + (0, \boldsymbol{w})) = \emptyset\}\big) d\boldsymbol{w},$$

where we renamed (y_2, y_3) to $\boldsymbol{w}$.
Proposition 5.1 also gives us that

$$\liminf_{h\to 0} h^{-1}(F_{\mathcal{F}}(\xi+h) - F_{\mathcal{F}}(\xi)) \geq \liminf_{h\to 0} -h^{-1}\int_U \nu_{\boldsymbol{y}}(\mathscr{E}\cap X(\boldsymbol{y}))d\boldsymbol{y}$$

$$\geq \liminf_{h\to 0} -h^{-1}\bigg(\int_U \nu_{\boldsymbol{y}}(\mathscr{E}'\cap X(\boldsymbol{y}))d\boldsymbol{y}$$

$$-\int_U \nu_{\boldsymbol{y}}(\{\Gamma(M,\boldsymbol{x})\in X(\boldsymbol{y}) : \#((\mathbb{Z}^3M+\boldsymbol{x})\cap \mathfrak{Z}(\xi,\xi+h))\geq 2\})d\boldsymbol{y}\bigg). \tag{7.7}$$

Now from Proposition 5.2 we see that the second integral in the right hand side of (7.7) can be bounded by C_1h^2 since $\mathfrak{Z}(\xi,\xi+h)$ can be contained in $\mathfrak{Z}_{cyl}(\xi,\xi+h,C)$ for some fixed $C>0$. Hence we get that the right hand side of (7.7) equals

$$\lim_{h\to 0} -h^{-1}\int_U \nu_{\boldsymbol{y}}(\mathscr{E}'\cap X(\boldsymbol{y}))d\boldsymbol{y}$$

$$= -\int_{\mathcal{R}} \mu_1\big(\{\Gamma^1M\in\Gamma^1\backslash G^1 : \mathbb{Z}^3M\cap(\mathfrak{Z}(0,\xi)+(0,\boldsymbol{w}))=\emptyset\}\big)d\boldsymbol{w}.$$

Therefore we have found that

$$\frac{d}{d\xi}F_{\mathcal{F}}(\xi) = -\int_{\mathcal{R}} \boldsymbol{\Phi}(\xi,\boldsymbol{w})d\boldsymbol{w},$$

and using the fact that $F_{\mathcal{F}}(\infty)=0$ we get the desired equation (7.5). □

Furthermore let $\boldsymbol{\Phi_0} : [0,\infty]\times\mathbb{R}^2\times\mathbb{R}^2\to[0,1]$ be defined as

$$\boldsymbol{\Phi_0}(\xi,\boldsymbol{w},\boldsymbol{z}) = \nu_{\boldsymbol{y}}(\{\Gamma^1M\in X_1(\boldsymbol{y}) : \mathbb{Z}^3M\cap(\mathfrak{Z}_\xi+(0,\boldsymbol{w}))=\emptyset\}),$$

where $\boldsymbol{y}=(\xi,\boldsymbol{w}+\boldsymbol{z})$.

Proposition 7.2. *The function* $\boldsymbol{\Phi}$ *defined in* (7.3) *is* C^1 *and can be written as*

$$\boldsymbol{\Phi}(\xi,\boldsymbol{w}) = \int_\xi^\infty\int_{\mathcal{R}} \boldsymbol{\Phi_0}(\eta,\boldsymbol{w},\boldsymbol{z})d\boldsymbol{z}d\eta. \tag{7.8}$$

Proof. This proof is very similar to the proof of Proposition 7.1. In this case the difference quotient with $h>0$ becomes

$$h^{-1}(\boldsymbol{\Phi}(\xi+h,\boldsymbol{w})-\boldsymbol{\Phi}(\xi,\boldsymbol{w})) = -h^{-1}\mu_1(\mathscr{E}), \tag{7.9}$$

where

$$\mathscr{E} = \{\Gamma^1M\in\Gamma^1\backslash G^1 : \#(\mathbb{Z}^3M\cap(\mathfrak{Z}(0,\xi)+(0,\boldsymbol{w})))=0, \#(\mathbb{Z}^3M\cap(\mathfrak{Z}(\xi+h,\xi)+(0,\boldsymbol{w})))\geq 1\}.$$

If we let $U=\mathfrak{Z}(\xi,\xi+h)+(0,\boldsymbol{w})$ and

$$\mathscr{E}' = \{\Gamma^1M\in\Gamma^1\backslash G^1 : \#(\mathbb{Z}^3M\cap(\mathfrak{Z}(0,\xi)+(0,\boldsymbol{w})))=0, \#(\mathbb{Z}^3M\cap(\mathfrak{Z}(\xi+h,\xi)+(0,\boldsymbol{w})))=1\},$$

then $X_1(\boldsymbol{y}_1)\cap X_1(\boldsymbol{y}_2)\cap\mathscr{E}'=\emptyset$ for all $\boldsymbol{y}_1\neq\boldsymbol{y}_2\in U$. So Proposition 5.3 gives

$$\limsup_{h\to 0} h^{-1}(\boldsymbol{\Phi}(\xi+h,\boldsymbol{w})-\boldsymbol{\Phi}(\xi,\boldsymbol{w})) \leq \limsup_{h\to 0} -h^{-1}\mu_1(\mathscr{E}')$$

$$= \limsup_{h\to 0} -h^{-1}\int_{\xi}^{\xi+h}\int_{\mathcal{R}} \boldsymbol{\Phi_0}(\eta,\boldsymbol{w},\boldsymbol{z})d\boldsymbol{z}d\eta = -\int_{\mathcal{R}} \boldsymbol{\Phi_0}(\xi,\boldsymbol{w},\boldsymbol{z})d\boldsymbol{z},$$

and also

$$\liminf_{h\to 0} h^{-1}(\boldsymbol{\Phi}(\xi+h,\boldsymbol{w}) - \boldsymbol{\Phi}(\xi,\boldsymbol{w})) \geq \liminf_{h\to 0} -h^{-1}\int_U \nu_{\boldsymbol{y}}(\mathscr{E}\cap X_1(\boldsymbol{y}))d\boldsymbol{y}$$

$$\geq \liminf_{h\to 0}\Bigg(\int_U \nu_{\boldsymbol{y}}(\mathscr{E}'\cap X_1(\boldsymbol{y}))d\boldsymbol{y}$$

$$-\int_U \nu_{\boldsymbol{y}}(\{\Gamma^1 M\in X_1(\boldsymbol{y}) : \#(\mathbb{Z}^3 M\cap(\mathfrak{Z}(\xi+h,\xi)+(0,\boldsymbol{w})))\geq 2\})d\boldsymbol{y}\Bigg). \tag{7.10}$$

Now we can apply Proposition 5.4 since for large enough C we have that $\mathfrak{Z}(\xi,\xi+h)\subset\mathfrak{Z}_{cyl}(\xi,\xi+h,C)$ and $C^{-1}<\xi<\xi+h$, so the right hand side of (7.10) equals

$$-\int_{\mathcal{R}} \boldsymbol{\Phi_0}(\xi,\boldsymbol{w},\boldsymbol{z})d\boldsymbol{z},$$

and we get that

$$\frac{d}{d\xi}\boldsymbol{\Phi}(\xi,\boldsymbol{w}) = -\int_{\mathcal{R}} \boldsymbol{\Phi_0}(\xi,\boldsymbol{w},\boldsymbol{z})d\boldsymbol{z}.$$

Since $\lim_{\xi\to\infty}\boldsymbol{\Phi_0}(\xi,\boldsymbol{w},\boldsymbol{z}) = 0$ we get the desired result. □

7.2 Preparation

In Section 5 we introduced the splitting $X_1(\boldsymbol{y}) = \sqcup_{k\in\mathbb{Z}^+} X_1(k\boldsymbol{e}_1,\boldsymbol{y})$, and we see that if $M\in G^1$ is such that $k\boldsymbol{e}_1 M = \boldsymbol{y}$ for $k\geq 2$, then $\boldsymbol{e}_1 M = k^{-1}\boldsymbol{y}\in\mathfrak{Z}_\xi + (0,\boldsymbol{w})$. Therefore $\boldsymbol{\Phi_0}$ can be expressed as

$$\boldsymbol{\Phi_0}(\xi,\boldsymbol{w},\boldsymbol{z}) = \nu_{\boldsymbol{y}}(\{\Gamma^1 M\in X_1(\boldsymbol{e}_1,\boldsymbol{y}) : \mathbb{Z}^3 M\cap(\mathfrak{Z}_\xi+(0,\boldsymbol{w})) = \emptyset\})$$

It is shown in [4] that there is a convenient parametrization of $X_1(\boldsymbol{e}_1,\boldsymbol{y})$, and $\nu_{\boldsymbol{y}}$ can be written in terms of this parametrization. We state the parametrization here without proof. As before we have that

$$X_1(\boldsymbol{e}_1,\boldsymbol{y}) = (\Gamma^1\cap H)\backslash HM_{\boldsymbol{y}},$$

where $\boldsymbol{y}\in\mathbb{R}^3\setminus\{\boldsymbol{0}\}$ and $M_{\boldsymbol{y}}\in G = G^{(3)}$ such that $\boldsymbol{e}_1 M_{\boldsymbol{y}} = \boldsymbol{y}$. Taking an arbitrary matrix M in $HM_{\boldsymbol{y}}$ we can write it as

$$M = \begin{pmatrix}\boldsymbol{y}\\ \boldsymbol{q}\\ \boldsymbol{p}\end{pmatrix} = \begin{pmatrix} y_1 & y_2 & y_3\\ q_1 & q_2 & q_3\\ p_1 & p_2 & p_3\end{pmatrix}$$

where $\boldsymbol{q} = (q_1,q_2,q_3)$ and $\boldsymbol{p} = (p_1,p_2,p_3)$ are two vectors in $\mathbb{R}^3$ satisfying $\boldsymbol{q}\cdot(\boldsymbol{p}\times\boldsymbol{y}) = 1$. Here "×" denotes the vector product on $\mathbb{R}^3$. We see that if $\boldsymbol{p}\in\mathbb{R}\boldsymbol{y}$ then $\boldsymbol{q}\cdot(\boldsymbol{p}\times\boldsymbol{y}) = 0$ so we have to choose $\boldsymbol{p}\in\mathbb{R}^3\setminus\mathbb{R}\boldsymbol{y}$. Now we see that $\{\boldsymbol{y},\boldsymbol{p},\boldsymbol{p}\times\boldsymbol{p}\}$ constitute a basis in $\mathbb{R}^3$, hence there are $x_1,x_2,x_3\in\mathbb{R}^3$ such that $\boldsymbol{q} = x_1\boldsymbol{y}+x_2\boldsymbol{p}+x_3\boldsymbol{p}\times\boldsymbol{y}$. We see that $\boldsymbol{q}\cdot(\boldsymbol{p}\times\boldsymbol{y}) = x_3\|\boldsymbol{p}\times\boldsymbol{y}\|^2$, so we have to have that $x_3 = \|\boldsymbol{p}\times\boldsymbol{y}\|^{-2}$. Therefore we can write $\boldsymbol{q}$ as a function of $\boldsymbol{y}$, $\boldsymbol{p}$ and $\boldsymbol{x} = (x_1,x_2)\in\mathbb{R}^2$ as follows

$$\boldsymbol{q} = \boldsymbol{q}_{\boldsymbol{y},\boldsymbol{p}}(\boldsymbol{x}) = x_1\boldsymbol{y}+x_2\boldsymbol{p}+\|\boldsymbol{p}\times\boldsymbol{y}\|^{-2}\boldsymbol{p}\times\boldsymbol{y}.$$

If we write $[\boldsymbol{p},\boldsymbol{x}]_{\boldsymbol{y}}$ for the matrix M obtained in this way, then we have a (surjective) diffeomorphism

$$\begin{array}{rcl}(\mathbb{R}^3\setminus\mathbb{R}\boldsymbol{y})\times\mathbb{R}^2 & \to & HM_{\boldsymbol{y}}\\ \langle\boldsymbol{p},\boldsymbol{x}\rangle & \mapsto & [\boldsymbol{p},\boldsymbol{x}]_{\boldsymbol{y}}.\end{array}$$

We see that the lattice corresponding to $[\boldsymbol{p},\boldsymbol{x}]_{\boldsymbol{y}}$ can be written as

$$\mathbb{Z}^3[\boldsymbol{p},\boldsymbol{x}]_{\boldsymbol{y}}=\mathbb{Z}\boldsymbol{y}+\mathbb{Z}\boldsymbol{p}+\mathbb{Z}\boldsymbol{q}_{\boldsymbol{y},\boldsymbol{p}}(\boldsymbol{x}).$$

Since $\nu_{\boldsymbol{y}}$ is a measure on $X_1(\boldsymbol{y})$ the restriction of $\nu_{\boldsymbol{y}}$ to $X_1(\boldsymbol{e}_1,\boldsymbol{y})$ is also a measure. Let $\nu_{\boldsymbol{y}}$ also denote the lift of this measure to $HM_{\boldsymbol{y}}$. We can then express the measure in terms of the $[\boldsymbol{p},\boldsymbol{x}]_{\boldsymbol{y}}$-parametrization as

$$d\nu_{\boldsymbol{y}}=\frac{6}{\pi^2\zeta(3)}d\boldsymbol{p}d\boldsymbol{x}.$$

To prove our main result we will need the following lemma.

Lemma 7.3. *Let ℓ be any line in $\mathbb{R}^2$. If $\boldsymbol{0}\notin\ell$ let $H^+(\ell)\subset\mathbb{R}^2$ be the open half space which has boundary ℓ and contains $\boldsymbol{0}$, otherwise let $H^+(\ell)$ be any of the two open half spaces with boundary ℓ. Let $H^-(\ell)$ be the other open half space with boundary ℓ. Furthermore let $L_{\boldsymbol{v}}:\mathbb{R}^2\to\mathbb{R}^2$ be the reflection in the point $\boldsymbol{v}$, i.e.*

$$L_{\boldsymbol{v}}(\boldsymbol{x})=\boldsymbol{x}+2(\boldsymbol{v}-\boldsymbol{x})=2\boldsymbol{v}-\boldsymbol{x}.$$

Then for any $r_1,r_2>0$ and any point $\boldsymbol{v}\in\mathcal{R}(r_1,r_2)\subset\mathbb{R}^2$ there exists a line ℓ going through $\boldsymbol{v}$ such that

$$L_{\boldsymbol{v}}(\mathcal{R}(r_1,r_2)\cap H^-(\ell))\subset\mathcal{R}(r_1,r_2)\cap H^+(\ell) \tag{7.11}$$

and

$$\mathrm{Area}(\mathcal{R}(r_1,r_2)\cap H^+(\ell))=4r_1r_2-2(r_1-|v_1|)(r_2-|v_2|). \tag{7.12}$$

Proof. Recall that $\mathcal{R}(r_1,r_2)=(-r_1,r_1)\times(-r_2,r_2)$, and let $\boldsymbol{v}=(v_1,v_2)$ be any point in $\mathcal{R}(r_1,r_2)$. W.l.o.g. we can assume $v_1,v_2\geq 0$, and we let $\boldsymbol{a}=(r_1-2(r_1-v_1),r_2)$ and $\boldsymbol{b}=(r_1,r_2-2(r_2-v_2))$. We choose $\ell=\{\boldsymbol{a}+t(\boldsymbol{b}-\boldsymbol{a}):t\in\mathbb{R}\}=\{(x_1,x_2)\in\mathbb{R}^2:(x_1-v_1)(v_2-r_2)=(x_2-v_2)(r_1-v_1)\}$. If $\boldsymbol{0}\in\ell$ then we get that $v_1=-(r_1-v_1)/(r_2-v_2)v_2$, and since we assumed $v_1,v_2\geq 0$ we get that $\boldsymbol{v}=\boldsymbol{0}$. Let $H^+(\ell)=\{(x_1,x_2)\in\mathbb{R}^2:(x_1-v_1)(v_2-r_2)>(x_2-v_2)(r_1-v_1)\}$. Obviously ℓ is the boundary of $H^+(\ell)$, and we see that if $\boldsymbol{v}\neq\boldsymbol{0}$ then $\boldsymbol{0}\in H^+(\ell)$ since $v_1(r_2-v_2)>0>-v_2(r_1-v_1)$. For the same reason we let $H^-(\ell)=\{(x_1,x_2)\in\mathbb{R}^2:(x_1-v_1)(v_2-r_2)<(x_2-v_2)(r_1-v_1)\}$.
Now assume that $\boldsymbol{x}\in\mathcal{R}(r_1,r_2)\cap H^-(\ell)$, and let $\boldsymbol{y}=L_{\boldsymbol{v}}(\boldsymbol{x})=2\boldsymbol{v}-\boldsymbol{x}$. This immediately implies that $\boldsymbol{y}\in H^+(\ell)$. We begin by looking at the first coordinate and see that since $v_1\geq 0$ and $x_1<r_1$ we have that $y_1=2v_1-x_1>2v_1-r_1\geq -r_1$. Furthermore since $\mathcal{R}(r_1,r_2)\cap H^-(\ell)$ is the interior of the convex hull of the points $\boldsymbol{a},\boldsymbol{b},(r_1,r_2)$, we see that $r_1-2(r_1-v_1)<x_1$, implying $y_1<r_1$. In the same way we see that $-r_2<y_2<r_2$. This gives that $\boldsymbol{y}\in\mathcal{R}(r_1,r_2)\cap H^+(\ell)$, and we conclude that (7.11) holds.
To see that (7.12) holds we note that

$$\mathrm{Area}(\mathcal{R}(r_1,r_2)\cap H^+(\ell))=\mathrm{Area}(\mathcal{R}(r_1,r_2))-\mathrm{Area}(\mathcal{R}(r_1,r_2)\cap H^-(\ell))$$

where $\mathcal{R}(r_1,r_2)$ has area $4r_1r_2$, and $\mathcal{R}(r_1,r_2)\cap H^-(\ell)$ is a triangle with area $2(r_1-v_1)\times 2(r_2-v_2)/2$. This was for the assumption that $v_1,v_2\geq 0$, and for the general case we have to exchange v_1,v_2 with $|v_1|,|v_2|$. □

For the case $\mathcal{R} = \mathcal{R}(1, (\tau+1)/(2\nu))$ we let $F : \mathcal{R} \to \mathbb{R}$ be the function defined by

$$F(\boldsymbol{v}) = \text{Area}(\mathcal{R} \cap H^+(\ell)) = \frac{2(\tau+1)}{\nu} - 2(1 - |v_1|)\left(\frac{\tau+1}{2\nu} - |v_2|\right)$$

for $\boldsymbol{v} = (v_1, v_2) \in \mathcal{R}$ and ℓ is taken as in the Lemma 7.3.
Before we can compute $\boldsymbol{\Phi_0}$ we define $\xi_1 : \mathcal{R} \times \mathcal{R} \to \mathbb{R}$ as the continuous function given by $\xi_1(\boldsymbol{w}, \boldsymbol{z}) = (6V)^{-1}$, where V is the largest possible volume of a tetrahedron which is contained in box $[0,1] \times [-1,1] \times [-(\tau+1)/(2\nu), (\tau+1)/(2\nu)]$ and which has one vertex at $(0, -\boldsymbol{w})$ and one at $(1,\boldsymbol{z})$.

Lemma 7.4. *For $\xi_1 : \mathcal{R} \times \mathcal{R} \to \mathbb{R}$ defined as above we have that*

$$\bar{\xi} := \inf_{\boldsymbol{w},\boldsymbol{z} \in \mathcal{R}} \xi_1(\boldsymbol{w}, \boldsymbol{z}) = \frac{\nu}{4(\tau+1)}.$$

Proof. We immediately see that $\bar{\xi} = (6\bar{V})^{-1}$ where $\bar{V}$ is the largest possible volume of a tetrahedron which is contained in the box $\mathcal{B} := [0,1] \times [-1,1] \times [-(\tau+1)/(2\nu), (\tau+1)/(2\nu)]$ and has on vertex in $\{0\} \times [-1,1] \times [-(\tau+1)/(2\nu), (\tau+1)/(2\nu)]$ and one in $\{1\} \times [-1,1] \times [-(\tau+1)/(2\nu), (\tau+1)/(2\nu)]$. Now we see that the volume is the absolute value of a linear function of the vertices, so the maximal volume will be either a maximum or a minimum of this function. Furthermore we see that the vertices must lie in $\mathcal{B}$ which is bounded and can be given by a finite number of linear constraints. It is a fundamental result of linear optimization that the maximum and minimum will be achieved with the vertices placed in the corners of $\mathcal{B}$. So to find the maximum volume we need only consider a finite number of cases. Before we do that we see that if $\hat{V}$ is the maximal volume of a tetrahedron which is contained in the box $\hat{\mathcal{B}} = [0,1]^3$, then $\bar{V} = 2\frac{\tau+1}{\nu}\hat{V}$ by a linear transformation and the vertices of the tetrahedron will be located at the corners of $\hat{\mathcal{B}}$. So we calculate $\hat{V}$ first. It is immediately clear that not all vertices can be located at one side of the box $\hat{\mathcal{B}}$ since then the volume will be zero. We then have two cases, either one side has three vertices and the opposite side has one, or one side has two and the opposite side has two. In the first case we get a tetrahedron with a hight of 1 and a base area that is the maximal area of a triangle in $[0,1]^2$, which is 1/2. So we get a volume of $1/3 \times 1 \times 1/2 = 1/6$. The second case is a bit more involved and we begin by letting $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4$ denote the vertices. W.l.o.g. we can assume that $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \{0\} \times [0,1]^2$ and $\boldsymbol{v}_3, \boldsymbol{v}_4 \in \{1\} \times [0,1]^2$. If $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are connected by one of the edges of the square $\{0\} \times [0,1]^2$ we see that either $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are sharing a side of $\hat{\mathcal{B}}$ with one of the vertices $\boldsymbol{v}_3, \boldsymbol{v}_4$ or the volume becomes zero. However if three of the vertices share one side then we revert back to the previous case and get the maximum volume of 1/6. The same goes for $\boldsymbol{v}_3$ and $\boldsymbol{v}_4$ so the only possibility left is that none of the vertices are connected by edges. The simplest way to calculate this volume is to take the total volume of the cube and subtract the volume of the space not contained in the tetrahedron, which happens to be four other tetrahedrons each with the volume 1/6. So the volume becomes $1 - 4 \times 1/6 = 1/3$ which is larger than 1/6 and hence our maximal volume. Using this we get that $\bar{\xi} = \frac{\nu}{4(\tau+1)}$. □

7.3 Explicit formulas

Theorem 7.5. *For all $\boldsymbol{w}, \boldsymbol{z} \in \mathcal{R}$ and all $0 < \xi < \xi_1(\boldsymbol{w}, \boldsymbol{z})$ we have that*

$$\boldsymbol{\Phi_0}(\xi, \boldsymbol{w}, \boldsymbol{z}) = \frac{1}{\zeta(3)}\left(1 - \frac{6}{\pi^2} F\left(\frac{\boldsymbol{w} - \boldsymbol{z}}{2}\right)\xi\right).$$

Proof. Let $\boldsymbol{w} = (w_1, w_2)$ and $\boldsymbol{z} = (z_1, z_2)$. Using $\boldsymbol{v} = \frac{1}{2}(\boldsymbol{w} - \boldsymbol{z})$ in Lemma 7.3 we get a line ℓ in the plane $\mathbb{R}^2$. Let V be the affine plane

$$V = \{(x_1, x_2, x_3) : x_1 \in \mathbb{R}, (x_2, x_3) \in \boldsymbol{w} + \ell\}.$$

If $\boldsymbol{w} \neq \boldsymbol{z}$ let $V^+ \subset \mathbb{R}^3$ be that open half space which has boundary V and which contains $(0, \boldsymbol{w})$. If $\boldsymbol{w} = \boldsymbol{z}$ then let V^+ be any of the two open half spaces determined by V. This is a similar construction to the one in Lemma 7.3, but this is in three dimensions as opposed to two.

Now we look at the map J defined by

$$\begin{aligned} (V^+ \cap (\mathfrak{Z}_\xi + (0, \boldsymbol{w})) \setminus \mathbb{R}\boldsymbol{y}) \times [0,1)^2 &\to X_1(\boldsymbol{e}_1, \boldsymbol{y}) \\ \langle \boldsymbol{p}, \boldsymbol{x} \rangle &\mapsto (\Gamma^1 \cap H)[\boldsymbol{p}, \boldsymbol{x}]_{\boldsymbol{y}} \end{aligned}$$

and prove that the image of J equals

$$\{M \in X_1(\boldsymbol{e}_1, \boldsymbol{y}) : \mathbb{Z}^3 M \cap (\mathfrak{Z}_\xi + (0, \boldsymbol{w})) \neq \emptyset\}, \tag{7.13}$$

up to a set of $\nu_{\boldsymbol{y}}$-measure zero.

For any M in the image of J we have that $M = [\boldsymbol{p}, \boldsymbol{x}]_{\boldsymbol{y}}$. We see that $\boldsymbol{p} = (0,0,1)[\boldsymbol{p}, \boldsymbol{x}]_{\boldsymbol{y}} \in \mathbb{Z}^3[\boldsymbol{p}, \boldsymbol{x}]_{\boldsymbol{y}}$, and since $\boldsymbol{p} \in V^+ \cap (\mathfrak{Z}_\xi + (0, \boldsymbol{w})) \setminus \mathbb{R}\boldsymbol{y}$ we get that $\mathbb{Z}^3[\boldsymbol{p}, \boldsymbol{x}]_{\boldsymbol{y}} \cap (\mathfrak{Z}_\xi + (0, \boldsymbol{w})) \neq \emptyset$, for any choice of $\boldsymbol{p}$ and $\boldsymbol{x}$. Therefore the image of J is a subset of (7.13).

Furthermore, let $M \in X_1(\boldsymbol{e}_1, \boldsymbol{y})$ be any element satisfying $\mathbb{Z}^3 M \cap (\mathfrak{Z}_\xi + (0, \boldsymbol{w})) \neq \emptyset$. We have that $\mathbb{Z}^3 M \cap (\mathfrak{Z}_\xi + (0, \boldsymbol{w})) \cap V = \emptyset$ for $\nu_{\boldsymbol{y}}$-almost all $M \in X_1(\boldsymbol{e}_1, \boldsymbol{y})$. Since $\mathbb{Z}^3 M$ is discrete and $\mathfrak{Z}_\xi + (0, \boldsymbol{w})$ is bounded, $\mathbb{Z}^3 M \cap (\mathfrak{Z}_\xi \cap (0, \boldsymbol{w}))$ is finite. Choose $\boldsymbol{p}'$ such that it is the point in $\mathbb{Z}^3 M \cap (\mathfrak{Z}_\xi \cap (0, \boldsymbol{w}))$ with minimal distance to $\mathbb{R}\boldsymbol{y}$. Note that $\boldsymbol{p}'$ cannot be on V since $\mathbb{Z}^3 M \cap (\mathfrak{Z}_\xi + (0, \boldsymbol{w})) \cap V$ is assumed to be empty. If $\boldsymbol{p}' \in V^+$ set $\boldsymbol{p} := \boldsymbol{p}'$, otherwise set $\boldsymbol{p}$ to be the reflection of $\boldsymbol{p}'$ in the point $\frac{1}{2}\boldsymbol{y}$, i.e. $\boldsymbol{p} := \boldsymbol{p}' - 2(\boldsymbol{p}' - \frac{1}{2}\boldsymbol{y}) = \boldsymbol{y} - \boldsymbol{p}'$. Note that $\boldsymbol{p} \in \mathbb{Z}^3 M$ and $\boldsymbol{p}$ has the same distance to the line $\mathbb{R}\boldsymbol{y}$ as the point $\boldsymbol{p}'$. In both cases we see that we must have $\boldsymbol{p} \in V^+ \cap (\mathfrak{Z}_\xi + (0, \boldsymbol{w}))$. To see this, we note that the reflection in the first coordinate is simple, i.e. $p_1' \in (0, \xi)$ implies $p_1 = y_1 - p_1' = \xi - p_1' \in (0, \xi)$. For the two other coordinates we see that lemma 7.3 is directly applicable from our choice of reflection point. Furthermore we see that $\boldsymbol{p} \notin \mathbb{R}\boldsymbol{y}$, since the points in $\mathbb{R}\boldsymbol{y} \cap \mathbb{Z}^3 M = \mathbb{Z}\boldsymbol{y}$ all have $\boldsymbol{e}_1$-coordinates in $\mathbb{Z}\xi$, while $\boldsymbol{p} \in \mathfrak{Z}_\xi + (0, \boldsymbol{w})$ has $\boldsymbol{e}_1$-coordinate in $(0, \xi)$.

It has to be the case that $\mathbb{Z}^3 M \cap (\mathbb{R}\boldsymbol{y} + \mathbb{R}\boldsymbol{p}) = \mathbb{Z}\boldsymbol{y} + \mathbb{Z}\boldsymbol{p}$, because if there is some $\boldsymbol{r} \in \mathbb{Z}^3 M \cap (\mathbb{R}\boldsymbol{y} + \mathbb{R}\boldsymbol{p})$ so that $\boldsymbol{r} \notin \mathbb{Z}\boldsymbol{y} + \mathbb{Z}\boldsymbol{p}$ then we can choose $n_1, n_2 \in \mathbb{Z}$ such that $\boldsymbol{r}' = \boldsymbol{r} + n_1\boldsymbol{y} + n_2\boldsymbol{p}$ is in the parallelogram spanned by $\boldsymbol{y}$ and $\boldsymbol{p}$, and $\boldsymbol{r}'$ will have the same properties as $\boldsymbol{r}$. We can assume that $\boldsymbol{r}'$ is in the closed triangle $\Delta(\boldsymbol{0}, \boldsymbol{y}, \boldsymbol{p})$ with vertices $\boldsymbol{0}, \boldsymbol{y}, \boldsymbol{p}$, since otherwise we can take $\boldsymbol{y} + \boldsymbol{p} - \boldsymbol{r}'$ which will be in the triangle if $\boldsymbol{r}'$ is not. Either way we have found a point $\boldsymbol{r}' \in \Delta(\boldsymbol{0}, \boldsymbol{y}, \boldsymbol{p}) \setminus \{\boldsymbol{0}, \boldsymbol{y}, \boldsymbol{p}\}$, however this point will be in $\mathbb{Z}^3 M \cap (\mathfrak{Z}_\xi + (0, \boldsymbol{w}))$ and be closer to $\mathbb{R}\boldsymbol{y}$ than $\boldsymbol{p}$, which is a contradiction.

Since $\mathbb{Z}^3 M \cap (\mathbb{R}\boldsymbol{y} + \mathbb{R}\boldsymbol{p}) = \mathbb{Z}\boldsymbol{y} + \mathbb{Z}\boldsymbol{p}$ we see there must exist a point $\boldsymbol{q} \in \mathbb{Z}^3 M$ such that $\mathbb{Z}^3 M = \mathbb{Z}\boldsymbol{y} + \mathbb{Z}\boldsymbol{p} + \mathbb{Z}\boldsymbol{q}$ and $\boldsymbol{q} \cdot (\boldsymbol{p} \times \boldsymbol{y}) = 1$. So for some $\boldsymbol{x} \in \mathbb{R}^2$ we have that $\boldsymbol{q} = \boldsymbol{q}_{\boldsymbol{y},\boldsymbol{p}}(\boldsymbol{x})$, and we may assume $\boldsymbol{x} \in [0,1)^2$ since if not we can replace $\boldsymbol{q}$ by $\boldsymbol{q} + n_1\boldsymbol{y} + n_2\boldsymbol{p}$, for some appropriate choice of $n_1, n_2 \in \mathbb{Z}$. This means we have that $\mathbb{Z}^3 M = \mathbb{Z}\boldsymbol{y} + \mathbb{Z}\boldsymbol{p} + \mathbb{Z}\boldsymbol{q}_{\boldsymbol{y},\boldsymbol{p}}(\boldsymbol{x}) = \mathbb{Z}^3[\boldsymbol{p}, \boldsymbol{x}]_{\boldsymbol{y}}$, and therefore $(\Gamma \cap H)M = J(\langle \boldsymbol{p}, \boldsymbol{x} \rangle)$, which completes the proof that the image of J equals (7.13).

Now we need to prove that J is injective, so assume that $J(\langle \boldsymbol{p}, \boldsymbol{x} \rangle) = J(\langle \boldsymbol{p}', \boldsymbol{x}' \rangle)$ for some $\boldsymbol{p}, \boldsymbol{p}' \in V^+ \cap (\mathfrak{Z}_\xi + (0, \boldsymbol{w})) \setminus \mathbb{R}\boldsymbol{y}$ and $\boldsymbol{x}, \boldsymbol{x}' \in [0,1)^2$. This means that

$$\mathbb{Z}\boldsymbol{y} + \mathbb{Z}\boldsymbol{p} + \mathbb{Z}\boldsymbol{q}_{\boldsymbol{y},\boldsymbol{p}}(\boldsymbol{x}) = \mathbb{Z}\boldsymbol{y} + \mathbb{Z}\boldsymbol{p}' + \mathbb{Z}\boldsymbol{q}_{\boldsymbol{y},\boldsymbol{p}'}(\boldsymbol{x}').$$

First we claim that

$$(\mathfrak{Z}_\xi + (0,\boldsymbol{w})) \cap (\mathbb{R}\boldsymbol{y} + \mathbb{R}\boldsymbol{p} + n\boldsymbol{q}_{\boldsymbol{y},\boldsymbol{p}}(\boldsymbol{x})) = \emptyset \quad \text{for all } n \in \mathbb{Z} \setminus \{0\}. \tag{7.14}$$

To show this we assume that there is a point $\boldsymbol{r} \in (\mathfrak{Z} + (0,\boldsymbol{w})) \cap (\mathbb{R}\boldsymbol{y} + \mathbb{R}\boldsymbol{p} + n\boldsymbol{q}_{\boldsymbol{y},\boldsymbol{p}}(\boldsymbol{x}))$ for some integer $n \in \mathbb{Z} \setminus \{0\}$. Let $\mathcal{T} = \mathcal{T}(\boldsymbol{0},\boldsymbol{y},\boldsymbol{p},\boldsymbol{r})$ be the tetrahedron with vertices $\boldsymbol{0},\boldsymbol{y},\boldsymbol{p},\boldsymbol{r}$. Then the volume of $\mathcal{T}$ becomes $\mathrm{Vol}\mathcal{T}(\boldsymbol{0},\boldsymbol{y},\boldsymbol{p},\boldsymbol{r}) = \frac{1}{3}\mathrm{Area}(\Delta(\boldsymbol{0},\boldsymbol{y},\boldsymbol{p}))\frac{|n|}{\|\boldsymbol{p}\times\boldsymbol{y}\|} = \frac{|n|}{6} \geq \frac{1}{6}$. After translation and scaling we get a tetrahedron that is contained in the box $[0,1]\times[-1,1]\times[-(\tau+1)/(2\nu),(\tau+1)/(2\nu)]$ with one vertex at $(0,-\boldsymbol{w})$, another vertex at $(1,\boldsymbol{z})$, and has volume $\geq (6\xi)^{-1}$. But if V is the largest possible volume of a tetrahedron in $[0,1]\times[-1,1]\times[-(\tau+1)/(2\nu),(\tau+1)/(2\nu)]$ with vertices at $(0,-\boldsymbol{w})$ and $(1,\boldsymbol{z})$, then $V \geq (6\xi)^{-1}$ meaning $\xi \geq (6V)^{-1} = \xi_1(\boldsymbol{w},\boldsymbol{z})$, which contradicts our assumption on ξ. Hence we have proved (7.14).

Next we see that $\boldsymbol{p}' \in \mathbb{Z}\boldsymbol{y} + \mathbb{Z}\boldsymbol{p} + \mathbb{Z}\boldsymbol{q}_{\boldsymbol{y},\boldsymbol{p}}(\boldsymbol{x})$ and $\boldsymbol{p}' \in \mathfrak{Z}_\xi + (0,\boldsymbol{w})$ so by (7.14) we get that $\boldsymbol{p}' \in \mathbb{Z}\boldsymbol{y} + \mathbb{Z}\boldsymbol{p}$. In the same way we have that $\boldsymbol{p} \in \mathbb{Z}\boldsymbol{y}' + \mathbb{Z}\boldsymbol{p}'$. From this we get that $\boldsymbol{p}' = \varepsilon\boldsymbol{p} + m\boldsymbol{y}$ for some $\varepsilon = \pm 1$ and $m \in \mathbb{Z}$. Since $\boldsymbol{p},\boldsymbol{p}' \in \mathfrak{Z}_\xi + (0,\boldsymbol{w})$ we get that $\boldsymbol{e}_1\cdot\boldsymbol{p}, \boldsymbol{e}_1\cdot\boldsymbol{p}' \in (0,\xi)$ and $\boldsymbol{e}_1\cdot\boldsymbol{p}' = \varepsilon\boldsymbol{e}_1\cdot\boldsymbol{p} + m\xi$. If $\varepsilon = -1$ then $m\xi - \boldsymbol{e}_1\cdot\boldsymbol{p} \in (0,\xi)$, i.e. $\boldsymbol{e}_1\cdot\boldsymbol{p} \in (\xi(m-1),\xi m)$, so we have to have $m = 1$. This give us that $\boldsymbol{p}' = \boldsymbol{y} - \boldsymbol{p}$. As before we see that $\boldsymbol{p}$ is the point we get if we reflect $\boldsymbol{p}'$ in the point $\frac{1}{2}\boldsymbol{y}$, so we cannot have that both $\boldsymbol{p}$ and $\boldsymbol{p}'$ are in V^+. Therefore we must have $\varepsilon = 1$. This gives us that $m\xi + \boldsymbol{e}_1\cdot\boldsymbol{p} \in (0,\xi)$, i.e. $\boldsymbol{e}_1\cdot\boldsymbol{p} \in (-\xi m, -\xi(m-1))$, so we have that $m = 0$. This proves that $\boldsymbol{p} = \boldsymbol{p}'$.

We now have that

$$\mathbb{Z}\boldsymbol{y} + \mathbb{Z}\boldsymbol{p} + \mathbb{Z}\boldsymbol{q}_{\boldsymbol{y},\boldsymbol{p}}(\boldsymbol{x}) = \mathbb{Z}\boldsymbol{y} + \mathbb{Z}\boldsymbol{p} + \mathbb{Z}\boldsymbol{q}_{\boldsymbol{y},\boldsymbol{p}}(\boldsymbol{x}')$$

implying $\boldsymbol{q}_{\boldsymbol{y},\boldsymbol{p}}(\boldsymbol{x}) = n_1\boldsymbol{y} + n_2\boldsymbol{p} + \boldsymbol{q}_{\boldsymbol{y},\boldsymbol{p}}(\boldsymbol{x}')$ for some $n_1, n_2 \in \mathbb{Z}$. But then we have $x_1\boldsymbol{y} + x_2\boldsymbol{p} = (n_1 + x_1')\boldsymbol{y} + (n_2 + x_2')\boldsymbol{p}$, implying $x_1 = n_1 + x_1'$ and $x_2 = n_2 + x_2'$. Since $x_1, x_2, x_1', x_2' \in [0,1)$we must have that $n_1 = n_2 = 0$, and so $\boldsymbol{x} = \boldsymbol{x}'$. This concludes the proof that J is injective.

From this it follows that J is a diffeomorphism from $(V^+ \cap (\mathfrak{Z}_\xi + (0,\boldsymbol{w})) \setminus \mathbb{R}\boldsymbol{y}) \times (0,1)^2$ onto an open subset of full $\nu_{\boldsymbol{y}}$-measure in $\{\Gamma^1 M \in X_1(\boldsymbol{e}_1,\boldsymbol{y}) : \mathbb{Z}^3 M \cap (\mathfrak{Z}_\xi + (0,\boldsymbol{w})) \neq \emptyset\}$. This gives that

$$\begin{aligned}
\boldsymbol{\Phi}_{\boldsymbol{0}}(\xi,\boldsymbol{w},\boldsymbol{z}) &= \nu_{\boldsymbol{y}}(\{\Gamma^1 M \in X_1(\boldsymbol{e}_1,\boldsymbol{y}) : \mathbb{Z}^3 M \cap (\mathfrak{Z}_\xi + (0,\boldsymbol{w})) = \emptyset\}) \\
&= \nu_{\boldsymbol{y}}(X_1(\boldsymbol{e}_1,\boldsymbol{y})) - \frac{6}{\pi^2\zeta(3)}\int_{V^+\cap(\mathfrak{Z}_\xi+(0,\boldsymbol{w}))}\int_{(0,1)^2} d\boldsymbol{p}d\boldsymbol{x} \\
&= \frac{1}{\zeta(3)} - \frac{6}{\pi^2\zeta(3)}F\left(\frac{\boldsymbol{w}-\boldsymbol{z}}{2}\right)\xi.
\end{aligned}$$

□

In order to get formulas for $\boldsymbol{\Phi}(\xi,\boldsymbol{w})$ and $F_{\mathcal{F}}(\xi)$ we see that we can not use equation (7.8) as it is, since it depends on $\boldsymbol{\Phi}_{\boldsymbol{0}}(\eta,\boldsymbol{w},\boldsymbol{z})$ for $\xi < \eta < \infty$. To amend this we have to calculate $\boldsymbol{\Phi}(\xi,\boldsymbol{w})$ as $\xi \to 0$. For this we see that

$$\begin{aligned}
1 - \boldsymbol{\Phi}(\xi,\boldsymbol{w}) &= 1 - \mu_1(\{\Gamma^1 M \in \Gamma^1\backslash G^1 : \mathbb{Z}^3 M \cap (\mathfrak{Z}_\xi + (0,\boldsymbol{w})) = \emptyset\}) \\
&= \mu_1(\{\Gamma^1 M \in \Gamma^1\backslash G^1 : \mathbb{Z}^3 M \cap (\mathfrak{Z}_\xi + (0,\boldsymbol{w})) \neq \emptyset\}) \\
&\leq \int_{\Gamma^1\backslash G^1} \#(\mathbb{Z}^3 M \cap (\mathfrak{Z}_\xi + (0,\boldsymbol{w})))d\mu^1(\Gamma^1 M) \\
&= \mathrm{Vol}(\mathfrak{Z}_\xi + (0,\boldsymbol{w}))
\end{aligned}$$

where the last result is a formula due to C. L. Siegel. Since $\mathrm{Vol}(\mathfrak{Z}_\xi + (0, \boldsymbol{w})) = 2\xi(\tau+1)/\nu$ we get that $\boldsymbol{\Phi}(\xi, \boldsymbol{w}) \to 1$ as $\xi \to 0$. This allows us to write

$$\boldsymbol{\Phi}(\xi, \boldsymbol{w}) = 1 - \int_0^\xi \int_{\mathcal{R}} \boldsymbol{\Phi}_{\mathbf{0}}(\eta, \boldsymbol{w}, \boldsymbol{z}) d\boldsymbol{z} d\eta.$$

This expression can be evaluated exactly for $0 < \xi < \inf_{\boldsymbol{z} \in \mathcal{R}} \xi_1(\boldsymbol{w}, \boldsymbol{z})$ as

$$\boldsymbol{\Phi}(\xi, \boldsymbol{w}) = 1 - \frac{2(\tau+1)}{\zeta(3)\nu}\xi + \frac{3}{\pi^2\zeta(3)}\left(\frac{\tau+1}{2\nu}\right)^2 \left(16 - \frac{1}{2}(3 - w_1^2)\left(3 - \left(\frac{2\nu w_2}{\tau+1}\right)^2\right)\right)\xi^2.$$

From Theorem 4.1 we have that $F_{\mathcal{F}}(0) = 1$ and equation (7.5) gives us

$$\begin{aligned} F_{\mathcal{F}}(\xi) &= 1 - \int_0^\xi \int_{\mathcal{R}} \boldsymbol{\Phi}(\eta, \boldsymbol{w}) d\boldsymbol{w} d\eta \qquad (7.15) \\ &= 1 - \frac{2(\tau+1)}{\nu}\xi + \frac{8}{\zeta(3)}\left(\frac{\tau+1}{2\nu}\right)^2 \xi^2 - \frac{448}{9\pi^2\zeta(3)}\left(\frac{\tau+1}{2\nu}\right)^3 \xi^3 \end{aligned}$$

for $0 \le \xi \le \bar{\xi} = \inf_{\boldsymbol{w}, \boldsymbol{z} \in \mathcal{R}} \xi_1(\boldsymbol{w}, \boldsymbol{z}) = \frac{\nu}{4(\tau+1)} \approx 0.1816\ldots$, c.f. Lemma 7.4. Furthermore from equation (7.4) we get that

$$F_{\mathcal{F},q}(\xi, r) = 1 - \frac{2(\tau+1)}{\zeta(3)\nu}\xi + \frac{3}{\pi^2\zeta(3)}\left(\frac{\tau+1}{2\nu}\right)^2 \left(16 - \frac{1}{2}(3 - r^2)\left(3 - \left(\frac{2\nu w(q)}{\tau+1}\right)^2\right)\right)\xi^2$$

for $0 \le \xi \le \inf_{\boldsymbol{z} \in \mathcal{R}} \xi_1((r, -w(q)), \boldsymbol{z})$.

8 Numerical investigations

Equation (7.15) gives us the values for $F_{\mathcal{F}}$ in the interval $[0, 0.1816\ldots]$, but if we wish to investigate the values of $F_{\mathcal{F}}$ for larger values of ξ we have to resort to numerical approximations. We will now do so following the method outlined in [4], Section 2.6. As such we will not look at the distribution function $F_{\mathcal{F}}$ but rather the probability density function $f_{\mathcal{F}}$ defined as

$$F_{\mathcal{F}}(\xi) = \int_\xi^\infty f_{\mathcal{F}}(\eta) d\eta,$$

where $f_{\mathcal{F}}$ can be taken as continuous. In fact we get that

$$f_{\mathcal{F}}(\xi) = \int_{\mathcal{R}} \Phi(\xi, \boldsymbol{w}) d\boldsymbol{w}.$$

In particular we see that for $\xi \in [0, \nu/(4(\tau+1))]$ we have

$$f_{\mathcal{F}}(\xi) = \frac{2(\tau+1)}{\nu} - \frac{16}{\zeta(3)}\left(\frac{\tau+1}{2\nu}\right)^2 \xi + \frac{448}{3\pi^2\zeta(3)}\left(\frac{\tau+1}{2\nu}\right)^3 \xi^2.$$

To numerically calculate $f_{\mathcal{F}}$ we will use the fact the for any $0 \le \alpha < \beta$

$$\int_\alpha^\beta f_{\mathcal{F}}(\xi) d\xi = \int_{\Gamma^1 \backslash G^1} \int_{\mathbb{R}^3 \backslash \mathbb{Z}^3} I(\{(\mathbb{Z}^3 + \boldsymbol{x})M \cap \mathfrak{Z}_\alpha = \emptyset,$$

$$(\mathbb{Z}^3 + \boldsymbol{x})M \cap \mathfrak{Z}_\beta \neq \emptyset\})d\boldsymbol{x}d\mu(\Gamma^1 M) \tag{8.1}$$

from [4]. This formula can be used to evaluate $\delta^{-1}\int_{n\delta}^{(n+1)\delta} f_{\mathcal{F}}(\xi)d\xi$, which can be taken as approximation of $f_{\mathcal{F}}((n+1/2)\delta)$ when δ is small and $n = 0, 1, 2, \ldots, N$, for some $N > 0$. To numerically evaluate (8.1) we must discretize the integrals. Discretization of the integral over $\Gamma^1\backslash G^1$ is done by taking M as the Hecke points corresponding to a large prime p, shifted by a fixed rotation. This can be written as $M = p^{-1/3}Tk$ where $k \in SO(3)$ is fixed and T runs through the set

$$S(p) = \left\{ \begin{pmatrix} p & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & a & \\ & p & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & a \\ & 1 & b \\ & & p \end{pmatrix} : a, b \in \{0, 1, 2, \ldots, p-1\} \right\}.$$

To discretize the integral over $\mathbb{R}^3\backslash\mathbb{Z}^3$ we simply take $\boldsymbol{x}$ as the points $\boldsymbol{x}_0 + (m^{-1}\mathbb{Z}^3)\backslash\mathbb{Z}^3$ for some fixed $\boldsymbol{x}_0 \in \mathbb{R}^3\backslash\mathbb{Z}^3$ and integer $m \geq 0$.

In Figure 5 we see the resulting curve when using $\delta = 0.01$, $N = 100$, $p = 311$, $m = 19$,

$$k = \begin{pmatrix} \cos(1/2) & \sin(1/2) & 0 \\ -\sin(1/2) & \cos(1/2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(1) & \sin(1) \\ 0 & -\sin(1) & \cos(1) \end{pmatrix} \begin{pmatrix} \cos(3/2) & \sin(3/2) & 0 \\ -\sin(3/2) & \cos(3/2) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\boldsymbol{x}_0 = (\sqrt{2}, \sqrt{3}, \sqrt{5})$. Comparing the numerical results to the exact formula for $\xi \leq \nu/(4(\tau+1))$ shows that the approximation deviates at most 0.0333 from the exact values. This gives an indication of the accuracy of the numerical method.

Furthermore we may be interested in comparing our integral approximation (8.1) with numerical simulation of the distribution of free path lengths in a quasicrystal for a small choice of ρ. Similar to the simulations done in [8] we choose a starting point q_0 from a uniformly random distribution over the interval $[0, 10000\nu/\tau^2]$ and jump length v chosen from a uniformly random distribution over $[0, \nu/\tau^2]$. Note that these initial distributions should not matter for small enough ρ as long as they are Borel probability distributions which are absolutely continuous with respect to the Lebesgue measure. In Figure 6 we see the graph for our numerical approximation of $f_{\mathcal{F}}(\xi)$, $0 < \xi \leq 5$, and the distribution of 1000000 randomly generated free path lengths for $\rho = 0.0001$.

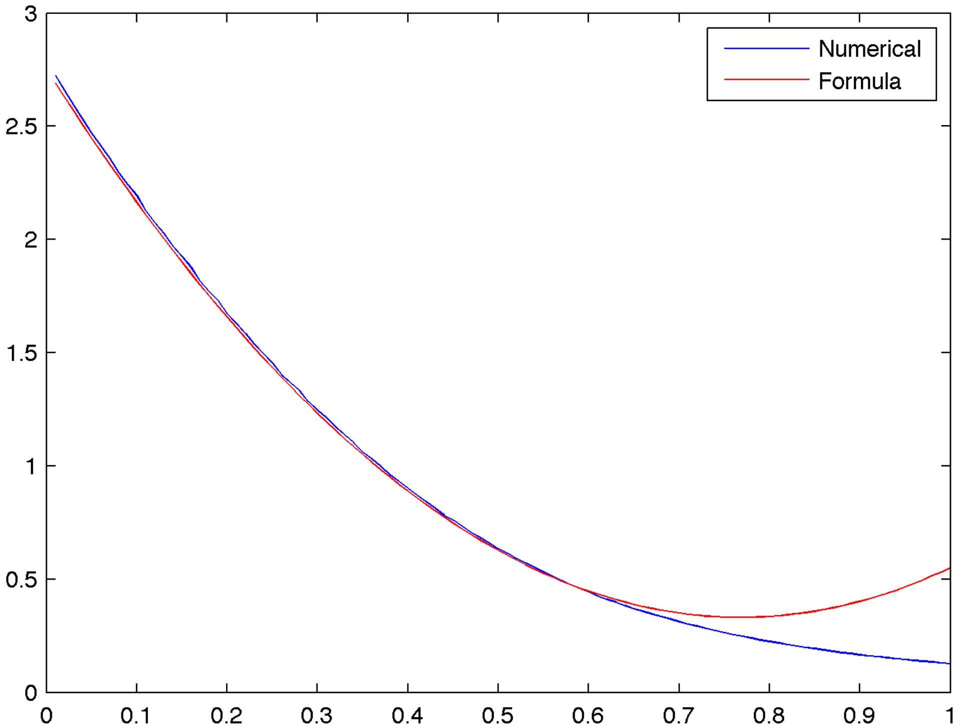

Figure 5: The numerical approximation of $f_{\mathcal{F}}(\xi)$ plotted for $0 < \xi \leq 1$, along with the exact formula for $f_{\mathcal{F}}(\xi)$. Here the polynomial in the formula is plotted for all $0 < \xi \leq 1$, but is still only valid for $0 < \xi \leq \nu/(4(\tau+1)) \approx 0.1816\ldots$.

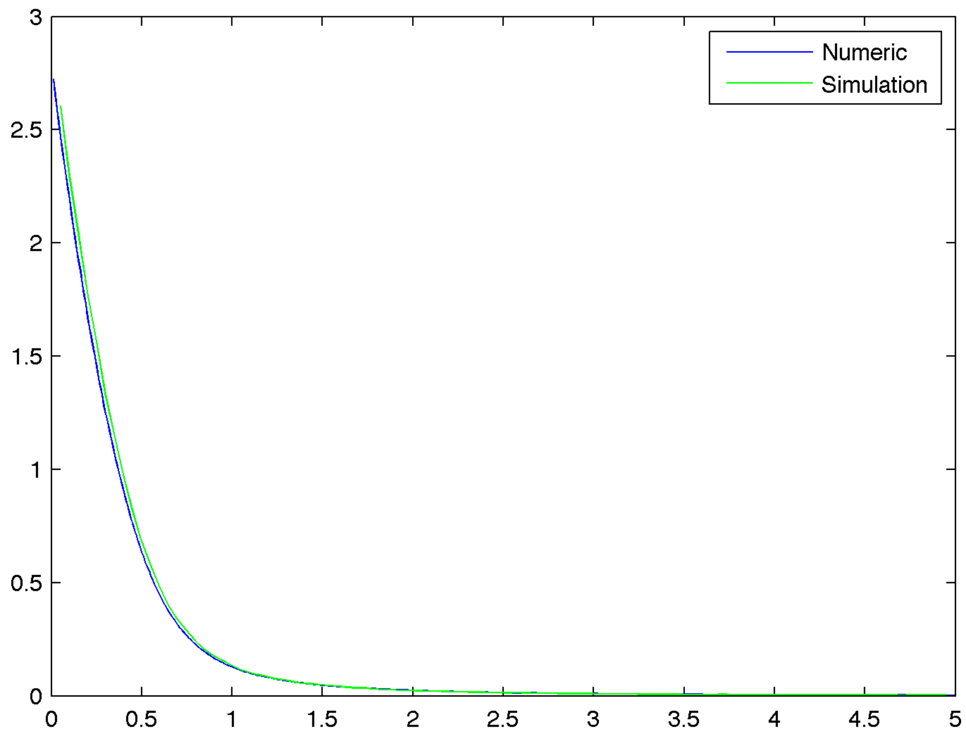

Figure 6: The distribution of simulated mean free paths with $\rho = 0.0001$ plotted for $0 < \xi \leq 5$, along with the numerical approximation of $f_{\mathcal{F}}(\xi)$.

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