Multiscale modelling of micromagnetic processes using the linearised Landau-Lifshitz equation

Fredrik Härlin
Abstract

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The problem with introducing multiscale domains when modelling micromagnetic processes is that some phenomena cannot be resolved in the coarser mesh. When using a structured grid, with an interface between the meshes, these phenomena reflect back into the fine mesh as a non-physical result of the discretization. Landau-Lifshitz is solved by using a new, higher order, energy conserving, finite difference, method. The particular issue of reflections is handled by weakly damping the high frequencies of spin waves, travelling across an interface, using penalty terms close to the interface. A couple of ways of imposing these penalty terms are implemented, which results in the damping of high frequencies, and creates unwanted reflections from the low frequencies that travel across the interface. The result is presented and other ways of implementing the penalty terms at the interface is discussed.
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1 Introduction

The aim of this report is to investigate the behaviour of a numerical solution to the Landau-Lifshitz equation, travelling across a multiscale interface. One of the main issues when working with several domains, with different coarseness of the mesh, is how to treat the high frequencies that travel from a fine mesh domain to a domain with a coarser mesh. An example of this phenomenon is presented in Figure 1. In this figure a spin wave of a certain frequency, travelling from left to right, is reflected back into the left domain where it can be represented, instead of transmitted into the coarser domain to the right.

![Figure 1: Illustration of a spin wave generated at x = 0 and travelling from left (fine mesh) to the right (coarse mesh), (blue line). Wavelength of the spin wave λ = 0.5µm, which cannot be represented in the coarse mesh and therefore creates a reflection back into the left domain. The dashed red line is a reference solution with no difference in mesh size and therefore no reflections.](image)

The Landau-Lifshitz equation is used to model the volume average magnetization in ferromagnets, and is used in this report to model spin waves. The Landau-Lifshitz equation and the physical setting of the model is further described in Section 2.

There are two desired outcomes of this report:

- The first aim is to present an investigation of the behaviour of micromagnetic spin waves travelling across a multiscale interface. The results of this investigation in one dimension is presented in Section 3. The investigation is then further expanded into two dimensions in Section 4, even though the main investigation is focused on the one dimensional case.

- The second aim is to develop a numerical, higher order, energy conserving method that is able to model micromagnetic phenomena in a physically realistic way. This model should be able to treat interfaces and domains with different coarseness. A suggestion how to do this, in one dimension, by weakly imposing the solution near the interface to equal an average, is presented in Section 3.7 together with the results of an implementation. Since the results do not agree with the desired outcomes a discussion on further development of the model is presented in Section 5.
2 Background

This section contains short descriptions of several background topics. It consists of six subsections and starts with a brief description of the physical context and the delimitations made to the continuum equation, Landau-Lifshitz equation, which is further considered in the second subsection. The third subsection contains a small discussion on the physical units within the current topic while the fourth describes the linearisation of Landau-Lifshitz equation. How the equation is discretized in space and the complications with multiscale interface, are questions addressed in Subsection 2.5 and 2.6 respectively.

2.1 Ferromagnetism and Micromagnetics

All materials can be divided into three categories with different basic magnetic properties. The first is paramagnets, where the magnetic dipoles of the material align with the external magnetic \( B \)-field. The second is diamagnets, where the dipoles align opposite to the \( B \)-field. For both these categories the magnetic dipoles become randomly aligned when the external \( B \)-field is removed. The third category is ferromagnets where the magnetic dipoles of the material remains aligned in a certain way even when the external \( B \)-field is removed. The Landau-Lifshitz equation is used to model magnetic processes in ferromagnetic materials. The most common ferromagnetic materials are iron, cobalt and nickel [1].

The magnetic dipole moment per unit volume is defined as the magnetization. The magnetization can be represented with a vector field within the material which is denoted by \( M \). When modelling micromagnetic processes it is often the magnetization as a function of space and time that is considered. The magnetization is a volume average of magnetic dipoles, which are in fact a macroscopic description of small currents: electrons spinning and orbiting around nuclei. A numerical model that treat each individual magnetic dipole moment in a sample, at an atomistic scale, would be computationally heavy. In this report larger scales are used as computational domains and therefore the continuous magnetization is modelled. There are models that consider an atomistic description in some domains and a continuum description in connected domains. This approach is further discussed in Section 2.6.

The dipole moments within a ferromagnetic material are inclined to point in the same direction as the neighbouring dipole moments, this is due to something called an exchange force between magnetic dipoles. There are other interactions contributing to the energy of the system, but since the exchange energy provides the largest energy contribution, the others are neglected [2]. The energy in the system due to the exchange energy is described in its integral form by Equation (2.1),

\[
E_{exc} = \int A_{exc} |\nabla m|^2.
\]

A more thorough discussion of ferromagnetism, and the connection to the continuous Landau-Lifshitz equation is further explained in the next section and in [3].
2.2 The Landau-Lifshitz equation

The Landau-Lifshitz-equation is a partial differential equation (2.2) [2], that is used to describe the magnetization, $M$. In this context the magnetization is described as a continuous volume averaged quantity of magnetic dipole moments, even though the magnetic dipole moment is actually a discrete property of single molecules. This equation is used to describe the continuous magnetization in ferromagnets, and a linearised version of it is also used throughout this report to model the magnetization.

\[ M_t = -\gamma' M \times H_{eff} - \lambda M \times (M \times H_{eff}), \quad (2.2) \]

where $M$ is the magnetization vector, $H_{eff}$ is the effective magnetization field, $\gamma'$ is a parameter that depends on the gyromagnetic ratio of the material ($\gamma$) and is defined by $\gamma' = \frac{1}{1+\gamma^2\eta^2M_s^2}$, $\lambda = \frac{\gamma'^2}{1+\gamma^2\eta^2M_s^2}$ and $\eta$ the phenomenological damping parameter [3].

To simplify the modelling of Equation (2.2) three modifications are used. First of the equation is normalized and rewritten in its dimensionless form. To be able to make Equation (2.2) dimensionless an analysis of the values and dimensions of the parameters involved has to be made. The normalised version is presented in Equation (2.4), and a further discussion on the dimensions and
units can be found in Section 2.3. The second modification is regarding the effective magnetization field, $H_{\text{eff}}$, which is considered to consist of the exchange energy in Equation 2.1 only. The other contributions are neglected in this model. The force arising from the exchange energy is calculated as a function of the Laplacian of the magnetization $\Delta m$. These two modifications lead to Equation (2.4). The third modification of the equation is the linearisation of the equation. This modification is further described in Section 2.4.

The normalised equation with the simplification for $H_{\text{eff}}$ is presented in Eq. (2.3):

$$m_t = -\gamma \left( m \times \frac{2A}{\mu_0 M_s} \Delta m + \alpha m \times \left( m \times \frac{2A}{\mu_0 M_s} \Delta m \right) \right),$$  

(2.3)

$$\Rightarrow m_t = -\gamma A_{\text{exc}} m \times \Delta m - \alpha \gamma A_{\text{exc}} m \times (m \times \Delta m),$$

$$\Rightarrow m_t = -\gamma_D m \times \Delta m - \alpha_D m \times (m \times \Delta m),$$  

(2.4)

where $\alpha$ is a dimensionless damping parameter and $m$ is normalized so that $||m|| = 1$. Further description of the parameters used above can be found in Appendix A.

2.3 Units

Traditionally literature on magnetism used the Gaussian cgs (centimetre, gram and second) system of units instead of the now mainly used SI-system of units. The Landau-Lifshitz equation is usually written in cgs-units in older articles, and it occurs in these units even in recent papers on magnetism. In this report the standard SI-system will be used, but it is still necessary to be aware of the differences, and take care when reading articles so that no unnecessary mistakes are made. Newer papers as [4] and [5] use SI-units in their simulations and there are several articles written that aim to explain the conversion procedure, see for example [6] or [7]. To cite [3] would be appropriate when it comes to units:

“For some people, converting to SI units has become an obsession, bordering on a religious conviction to abolish heresy and make everybody use the ‘true’ units. However, there is no way of ignoring the fact that there are many researchers who have not been converted ...”

2.4 Linearisation

As a starting point of the model a ferromagnetic material with an initial magnetization is considered. The ferromagnetic material is uniformly magnetized, with the constant magnetization $m_0 = (0,0,1)$. Exciting a spin wave in the material creates a small disturbance $m_s$ of the initial constant magnetization in each point. Figure 3 illustrates this initial setting, while Figure 4 describes the precession of a spin wave. The magnitude of $|m_s(x,t)|$ is assumed to be
such that the angle $\varphi$ in Figure 3 is considered small, and hence $\sin \varphi$ can be approximated by $\varphi$. The total magnetization as a function of position and time $\mathbf{m}(r, t) = m_0 + m_s(r, t) = [m^{(1)} \quad m^{(2)} \quad m^{(3)}]^T$ is illustrated in Figure 3.

Figure 3: Illustration of the magnetization $\mathbf{m}(r, t)$ slightly shifted from the initial magnetization $\mathbf{m}_0$ with $\mathbf{m}_s(r, t)$.

Landau-Lifshitz-equation (2.4) with the assumption that the angle $\varphi$ is small

Figure 4: Illustration of a spin wave. The $x$-component of $\mathbf{m}_s$ corresponds to $m^{(1)}$ and the $y$-component to $m^{(2)}$ in this model.
yields the linearisation in Eq. (2.5),

\[
\frac{\partial}{\partial t} (m_0 + m_s) = -\gamma_D (m_0 + m_s) \times \Delta (m_0 + m_s)
- \alpha_D (m_0 + m_s) \times \left( (m_0 + m_s) \times \Delta (m_0 + m_s) \right),
\]

\[
\Rightarrow \frac{\partial}{\partial t} m_0 + \frac{\partial}{\partial t} m_s = -\gamma_D \left( m_0 \times \Delta m_0 + m_0 \times \Delta m_s + m_s \times \Delta m_0 + m_s \times \Delta m_s \right)
- \alpha_D (m_0 + m_s) \times \left( (m_0 + m_s) \times \Delta (m_0 + m_s) \right),
\]

\[
\Rightarrow \frac{\partial}{\partial t} m_0 \approx -\gamma_D m_0 \times \Delta m_0 - \alpha_D (m_0 \times (m_0 \times \Delta m_s) + m_s \times (m_0 \times \Delta m_s))
\approx 0 \text{ small}
\]

\[
\Rightarrow \frac{\partial}{\partial t} m_s \approx -\gamma_D m_0 \times \Delta m_s - \alpha_D m_0 \times (m_0 \times \Delta m_s),
\tag{2.5}
\]

which is the linearized version of the normalized Landau-Lifshitz-equation (2.4).

The vector Laplacian in Cartesian coordinates and the two terms on the right side are defined as

\[
\Delta m_s = \begin{pmatrix}
\Delta (m_s \cdot \hat{i}) \\
\Delta (m_s \cdot \hat{j}) \\
\Delta (m_s \cdot \hat{k})
\end{pmatrix}
= \begin{pmatrix}
\Delta m_s^{(1)} \\
\Delta m_s^{(2)} \\
\Delta m_s^{(3)}
\end{pmatrix},
\]

\[
m_0 \times \Delta m_s = \begin{pmatrix}
-\Delta m_s^{(1)} \\
-\Delta m_s^{(2)} \\
0
\end{pmatrix}.
\]

Together with Eq. (2.5) this lead to the system of equations:

\[
\begin{pmatrix}
m_s^{(1)} \\
m_s^{(2)} \\
m_s^{(3)}
\end{pmatrix}_t =
\begin{pmatrix}
\alpha_D & \gamma_D & 0 \\
-\gamma_D & \alpha_D & 0 \\
0 & 0 & 0
\end{pmatrix} \Delta
\begin{pmatrix}
m_s^{(1)} \\
m_s^{(2)} \\
m_s^{(3)}
\end{pmatrix},
\tag{2.6}
\]

where third component is constant due to the fact that \(\varphi\) is considered small. Therefore this component can be left out of the equation, resulting in a system of equations using only two unknowns,

\[
\begin{pmatrix}
m_s^{(1)} \\
m_s^{(2)}
\end{pmatrix}_t =
\begin{pmatrix}
\alpha_D & \gamma_D \\
-\gamma_D & \alpha_D
\end{pmatrix} \Delta
\begin{pmatrix}
m_s^{(1)} \\
m_s^{(2)}
\end{pmatrix}.
\tag{2.7}
\]

Note here that \(m_s^{(1)} = m^{(1)}\) and \(m_s^{(2)} = m^{(2)}\) which can be observed in Figure 3. Therefore the subscript \(s\) is not used in the rest of this report.
2.5 Discretization

For the spatial discretization the finite difference scheme **Summation-by-Parts–Simultaneous Approximation Term** (SBP-SAT) is used. The method of SBP-SAT is divided into two parts. The first part is using a specially developed SBP-operator, which, when applied to the discrete solution returns an estimate of the derivative of the solution. See Definition 3.1 for further details. SBP-operators have been developed for approximating different types of derivatives, and for several orders of accuracy. In this report the fourth and sixth order accuracy operators, approximating the second order derivative in space, are used. The full definition of these higher order, energy conserving, finite difference-operators can be found in for example [8]. The second part of SBP-SAT is the SAT. The Simultaneous Approximation Terms are terms that operate at the boundaries of the domain in order to weakly impose the given boundary conditions. A further discussion of the implementation and the stability of the method, for this particular problem, can be found for both one and two dimensions in Section 3.3 and 4.3 respectively.

There are several reasons to the choice of discretization-method, here two main reasons are explained:

- One reason that SBP-SAT is chosen is that it can be implemented with fourth and sixth, and even higher, order accuracy. This means that the amount of discretization points needed in the domain, in order to for the solution to achieve high accuracy, is less than when using an ordinary first order difference method like the Euler method. Proof of order of convergence of the SBP-SAT method, in this particular context, can be found in Section 3.5 and 4.5 for the one and two dimensional implementation respectively.

- Another reason to use SBP-SAT is the possibility to show that the method is stable by using an energy estimate. The discrete energy estimate coming from the SBP-operators and SATs is created in such a way so that they mimic the continuous energy estimate, and the proof of stability of the method is presented in Section 3.3 and 4.3 for one and two dimensions. One of the benefits of this method is that it is straightforward to extend the model to multi-block geometries, and still obtain an energy estimate and thereby proving the method to be stable.

Examples of how the SBP-SAT method is used can be studied in [9], [10] and the review [11]. Interface treatment with SBP-SAT can be further studied in for example [12], [13] and [14].

A fourth order Runge-Kutta method is used for time-integration for all simulations performed in the scope of this report. An explicit method like this is chosen to keep the computational efficiency and due to its sufficient accuracy. The CFL-conditions is numerically analysed and fulfilled in all simulations, but since the spatial discretization is the focus of investigation in this report, this analysis is omitted from the report.
2.6 Multiscaling

The challenge in connecting two computational domains with different coarseness of the grids is a common problem in numerical modelling. In this report a structured grid is used, which requires some kind of interpolation between domains when solving for two or more dimensions. When using unstructured grids and solving with for example the finite element method (FEM), the interpolation issue do not occur in the same way since the grid can be gradually decreased or increased. On the other hand the issue of reflected phenomena still occur in unstructured meshes.

In [15] the authors have made extensive comparative experiments on different multiscaling methods for a benchmarking problem within mechanics. A Lomer dislocation dipole in face-centred cubic aluminium is modelled with different methods. In these methods the transition between a fine scale atomistic description and a coarse grid continuum model is considered. Furthermore most of these implementations require an unstructured grid and the PDE is solved using FEM. Some benefits of using the SBP-SAT method with a structured grid instead of using an unstructured grid are mentioned in previous section. Another reason for using a finite difference method is the fact that using a structured grid is computationally more efficient. There are of course also disadvantages in using a finite difference method, such as the inability for the mesh to conform to the domain boundary in complex geometries.

There are a lot of implementations with multiscale modelling of the Landau-Lifshitz-equation, examples can be found in [4] and in the PhD-thesis [5]. In these implementations unstructured grids and FEM have been used, and the authors have used the continuous Landau-Lifshitz-equation at the large scale, while they have used the Heisenberg formulation for the atomistic regions. In this report two domains with different scales will be connected and the continuous model (2.4) will be modelled in both domains. The same type of interactions at the interface would occur using an atomistic model in one of the domains, but since the focus of interest is within the behaviour of the numerical model, the continuous model is used in both domains. The implementation in this report only discuss two grids connected together with one interface, but the idea of how to treat the interfaces is the same in a multiscale setting with several interfaces.

3 The 1-D Problem

Assuming one dimension means that the only non-zero contribution from the Laplacian will be $m_{xx}$. When later assuming two dimensions the term $m_{yy}$ is simply added to the equation.

$$\begin{pmatrix} m^{(1)}_x \\ m^{(2)}_x \end{pmatrix}_t = \begin{pmatrix} \alpha_D & \gamma_D \\ -\gamma_D & \alpha_D \end{pmatrix} \begin{pmatrix} \frac{\partial^2 m^{(1)}_x}{\partial x^2} \\ \frac{\partial^2 m^{(2)}_x}{\partial x^2} \end{pmatrix} \Rightarrow m_t = A m_{xx} \quad (3.1)$$
3.1 Definitions

Firstly the modelling domain in one dimension is defined to be $x \in [x_l, x_r]$. Secondly, let $u, w \in L^2[x_l, x_r]$, $u = [u^{(1)}, u^{(2)}]^T$. $u^T$ denotes the Hermitian transpose of $u$, see Remark 3.1. Let the inner product be defined by $(u, w) = \int_{x_l}^{x_r} u^T w \, dx$ and let the corresponding norm be $||u||^2 = (u, u)$.

Remark 3.1. In this report the magnetization $m$ will always be real, therefore the Hermitian transpose (also called conjugate transpose) will be equivalent to the ordinary transpose, hence $m^* = m^T$. For consistency the notation $m^T$ will be used throughout this report.

The domain is discretized using $N + 1$ equidistant grid points $x_k = x_l + kh$, $k = 0, 1, ..., N$, $h = \frac{x_r - x_l}{N + 1}$, and the approximate solution at grid point $x_i$ is a $2 \times 1$ vector $v_k = [v_k^{(1)}, v_k^{(2)}]^T$ and the discrete solution vector corresponding to the continuous solution $u$ will be

$\mathbf{v} = [v_0^{(1)}, v_1^{(1)}, ..., v_N^{(1)}, v_0^{(2)}, ..., v_N^{(2)}]^T$.

Mimicking the inner product for continuous functions, the inner product for discrete vector functions $\mathbf{v}, \mathbf{q} \in \mathbb{R}^{2(N+1)}$ is defined by $(\mathbf{v}, \mathbf{q})_H = \mathbf{v}^T I_2 \otimes H \mathbf{q}$, where $H_{N+1 \times N+1}$ is Hermitian and positive definite. The corresponding norm is defined as $||\mathbf{v}||_H^2 = \mathbf{v}^T I_2 \otimes H \mathbf{v}$.

The vectors $\mathbf{e}_0$ and $\mathbf{e}_N$ are used throughout this report and they are defined as,

$\mathbf{e}_0 = [1, 0, ..., 0]^T_{(N+1) \times 1}$, $\mathbf{e}_N = [0, ..., 0, 1]^T_{(N+1) \times 1}$

and their use is further explained in Definition 3.1 where the 2nd order SBP-operator is defined.

Definition 3.1. SBP - second derivative SBP-operator

Let $D_2 = H^{-1}(-M - \mathbf{e}_0 d_0 + \mathbf{e}_N d_N)$ approximate $\frac{\partial^2}{\partial x^2}$ such that $H = H^T > 0$, $M = M^T \geq 0$ where $d_0$ and $d_N$ are first derivative difference operators at the boundaries, i.e., $d_0 \mathbf{v} \approx \mathbf{u}_x(x = x_l)$, $d_N \mathbf{v} \approx \mathbf{u}_x(x = x_r)$ and $\mathbf{e}_0$ and $\mathbf{e}_N$ are operators that extracts the values at the boundaries, i.e., $(\mathbf{e}_0)^T \mathbf{v} \approx \mathbf{u}(x = x_l)$, $(\mathbf{e}_N)^T \mathbf{v} \approx \mathbf{u}(x = x_r)$. Where $\mathbf{v}$ is the discrete approximation of the continuous solution $\mathbf{u}$.

The second derivative SBP-operators for implementation are, as mentioned earlier completely defined in [8].

The observant reader noticed the $\otimes$-symbol in the definition of the inner product. This symbol denotes the Kronecker product which is used mainly for book-keeping, and more extensively used when extending the problem to two dimensions. The Kronecker product is defined in Definition 3.2 below.
Definition 3.2. **Kronecker product** $\otimes$

$$A_{m \times n} \otimes B_{p \times q} = \begin{pmatrix} a_{11}B & a_{12}B & \cdots \\ a_{12}B & a_{22}B & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}_{mp \times nq},$$

assuming $A$ is compatible with $C$ and $B$ with $D$. $(A \otimes B)^T = (A^T \otimes B^T)$.

### 3.2 Continuous analysis

To be able to numerically simulate the problem it is necessary to first analyse if the the continuous problem is well-posed, and thereby also finding how the boundary conditions can be defined in order to keep this well-posedness.

Definition 3.3. **Well-posed problem**

The initial-boundary value problem 3.1 is well-posed if the problem has a unique solution, and that there is an energy estimate such that

$$||m|| \leq K(||m_0|| + ||g||),$$

where $K$ is a constant independent of the data and $g$ is the boundary data. Further explanation of the definition can be found in [16].

Usually the energy method is used to analyse the well-posedness of the problem. This is done by multiplying Eq. (3.1) by $m^T$, integrating by parts over the domain and by adding the transpose.

Remark 3.2. According to the definition $(m, m_t) + (m_t, m) = \frac{d}{dt}||m||^2$. This fact is used in the energy method to show that a problem is well-posed.

Using the energy method an energy estimate can be derived,

$$(m, m_t) = (m, Am_{xx}) = m^T Am_x|_{x_l} - (m_x, Am_x),$$

and the transpose

$$(m_t, m) = (Am_{xx}, m) = m_x^T A^T m|_{x_l} - (Am_x, m_x),$$

$$\Rightarrow \frac{d}{dt}||m||^2 = -(Am_{xx}, m_x) - (m_x, Am_x) + m^T Am_x|_{x_l} + m_x^T A^T m|_{x_l}. $$

From this it can be seen that well-posedness requires $A^T + A = 2\alpha_D I_2$ to be positive semi-definite. This is fulfilled if $\alpha_D \geq 0$. Another condition for the problem to be well posed is to define the correct amount of boundary conditions to bind the energy estimate in terms of known data. The boundary terms can be written as

$$BT = \begin{bmatrix} m^{(1)}_x \\ m^{(2)}_x \\ m^{(1)}_r \\ m^{(2)}_r \\ A \\ 0 \\ 0 \\ A^T \end{bmatrix}_{=B} = \begin{bmatrix} m^{(1)} \\ m^{(2)} \\ m^{(1)}_x \\ m^{(2)}_x \\ m^{(1)}_r \\ m^{(2)}_r \\ A \end{bmatrix}_{w}^T = w^T Bw,$$
and can also be represented in the following way,
\[
BT = 2 \left( \alpha_D m_1^{(1)} m_1^{(1)} + \gamma_D m_1^{(1)} m_1^{(2)} - \gamma_D m_2^{(2)} m_1^{(1)} + \alpha_D m_2^{(2)} m_2^{(2)} \right) \bigg|_{x_l}^{x_r}.
\]

### Boundary conditions

There are several ways to choose the boundary conditions and still have a well-posed problem. In this report two ways are described. The first and easiest way is to set \(m_x\) to zero at both boundaries, corresponding to homogeneous Neumann boundary conditions. With this assumption the boundary terms, \(BT\) vanish and the hence the problem fulfills the inequality in Equation (3.2), and the problem is proved to be well-posed. The final energy estimate will be
\[
||m(x, t)||^2 = ||m_0(x, 0)||^2 - \alpha_D \int_0^t ||m_x(x, \tau)||^2 d\tau,
\]
which means that the energy of the system is bound over time, and therefore the Problem (3.1) is well posed with the proper boundary conditions and initial condition. Note that the energy of the system will decrease only when the damping parameter \(\alpha_D \neq 0\) and that the equality sign holds when \(\alpha_D = 0\).

The above discussed boundary condition added to Eq. (3.1) leads to the problem setting in Eq. (3.4) with the proper boundary conditions and initial conditions defined
\[
\begin{align*}
    m_t &= A m_{xx}, \quad x \in [x_l, x_r], \quad t > 0, \\
    m_x &= 0, \quad x = x_l, \quad t > 0, \\
    m_x &= 0, \quad x = x_r, \quad t > 0, \\
    m &= m_0, \quad x \in [x_l, x_r], \quad t = 0.
\end{align*}
\]

The second way to define boundary conditions that lead to a well-posed problem is to define appropriate well-posed boundary conditions in a more general way. This is done by finding the eigenvalues of the boundary matrix \(B\) and specifying boundary conditions for \(\tilde{w} = T^{-1} w\) where \(T\) diagonalizes \(B\):

\[
A = T^T BT, \quad A = \begin{pmatrix}
-\sqrt{\alpha_D^2 + \gamma_D^2} & 0 & 0 & 0 \\
0 & -\sqrt{\alpha_D^2 + \gamma_D^2} & 0 & 0 \\
0 & 0 & \sqrt{\alpha_D^2 + \gamma_D^2} & 0 \\
0 & 0 & 0 & \sqrt{\alpha_D^2 + \gamma_D^2}
\end{pmatrix},
\]

\[
T = \frac{1}{\sqrt{\alpha_D^2 + \gamma_D^2}} \begin{pmatrix}
-\gamma_D & -\alpha_D & \gamma_D & \alpha_D \\
-\alpha_D & \gamma_D & \alpha_D & -\gamma_D \\
\gamma_D & -\alpha_D & -\gamma_D & \alpha_D \\
\alpha_D & \gamma_D & \alpha_D & \gamma_D
\end{pmatrix},
\]

where the entries in the diagonal of \(A\) correspond to the eigenvalues of \(B\). With this diagonalization the boundary terms can be expressed as,
\[
BT = w^T B w \bigg|_{x_l}^{x_r} = \tilde{w}^T T^T BT \tilde{w} \bigg|_{x_l}^{x_r} = \tilde{w}^T A \tilde{w} \bigg|_{x_l}^{x_r} = \sum_{j=1}^4 \lambda_j (\tilde{w}^{(j)})^2 \bigg|_{x_l}^{x_r},
\]
which means that to create a bound energy estimate in terms of the known data, the \( \tilde{w}^{(j)} \) corresponding to the positive eigenvalues have to be defined at the right boundary, and the components of \( \tilde{w} \) corresponding to the negative eigenvalues have to be defined at the left boundary. This corresponds to prescribing ingoing waves, and in analogy with hyperbolic wave propagation the term characteristic boundary conditions is used throughout this report.

The characteristic boundary conditions will be to define \( \tilde{w} \),

\[
\tilde{w} = T^T w = \frac{1}{\sqrt{\alpha_D^2 + \gamma_D^2}} \begin{pmatrix}
-\gamma_D w^{(1)} - \alpha_D w^{(2)} + \sqrt{\alpha_D^2 + \gamma_D^2} w^{(4)} \\
-\alpha_D w^{(1)} + \gamma_D w^{(2)} + \sqrt{\alpha_D^2 + \gamma_D^2} w^{(4)} \\
\gamma_D w^{(1)} + \alpha_D w^{(2)} + \sqrt{\alpha_D^2 + \gamma_D^2} w^{(4)} \\
\alpha_D w^{(1)} - \gamma_D w^{(2)} + \sqrt{\alpha_D^2 + \gamma_D^2} w^{(4)}
\end{pmatrix},
\]

(3.6)

When setting \( \lambda = \sqrt{\alpha_D^2 + \gamma_D^2} \), the characteristic boundary conditions will be

\[
\begin{align*}
-A^T m + \lambda m_x &= g_l(t), & x &= x_l, \\
A^T m + \lambda m_x &= g_r(t), & x &= x_r,
\end{align*}
\]

(3.7)

which, if \( g_l(t) = g_r(t) = 0 \), will result in the energy estimate:

\[
\frac{d}{dt} |m|^2 = -\alpha_D |m_x|^2 - \lambda m^2 \leq 0.
\]

(3.8)

This energy estimate fulfills the conditions for a well-posed problem and therefore the system of equations in Eq. (3.9)

\[
\begin{cases}
-\tau_0 H^{-1} \tau_1 = 0, \\
-A^T m + \lambda m_x = g_l(t), & x = x_l, & t > 0, \\
A^T m + \lambda m_x = g_r(t), & x = x_r, & t > 0, \\
m = m_0, & x \in [x_l, x_r], & t = 0,
\end{cases}
\]

(3.9)

also represents a well-posed problem-setting.

### 3.3 Semi-discretization

The semi-discretization of Eq. (3.4), complete with SBP-operators and SATs is presented in Eq. (3.10). The two SATs in Eq. (3.10) corresponds to the two boundary conditions in Eq. (3.4) for the left and right boundary respectively,

\[
\begin{align*}
\tau_0 \partial_0 H^{-1} \tau_1 &= 0, \\
(A \otimes D_2) v + \tau_0 H^{-1} \epsilon_0 ((A \otimes d_0) v) + \tau_1 H^{-1} \epsilon_N ((A \otimes d_N) v),
\end{align*}
\]

(3.10)

where \( \tau_0, \tau_1 \) are \( 2 \times 2 \) penalty parameters. The SBP-operator and SATs are used to mimic the integration by parts in the continuous case in order to show stability with a discrete energy estimate.

**Lemma 3.1.** The semi-discretization in Eq. (3.10) is stable if \( \tau_0 = I_2 \) and \( \tau_1 = -I_2 \).
Proof. Multiplying Eq. (3.10) with $v^T I_2 \otimes H$ and then adding the transpose yields the following relation for the energy estimate

$$\frac{d}{dt} \|v\|_H^2 = v^T I_2 \otimes H v + v^T I_2 \otimes H v^{\text{transpose}}$$

$$\#1 = v^T (I_2 \otimes H)(A \otimes D_2) v + v^T (I_2 \otimes H)(\tau_0 \otimes H^{-1} e_0)(A \otimes d_0) v$$

$$= A \otimes H D_2 v$$

$$+ v^T (I_2 \otimes H)(\tau_1 \otimes H^{-1} e_N)(A \otimes d_N) v$$

$$= -v^T A \otimes H D_2 v + v^T (\tau_0 \otimes e_0)(A \otimes d_0) v + v^T (\tau_1 \otimes e_N)(A \otimes d_N) v$$

$$= -v^T (A \otimes M) v - v^T (A \otimes e_0 d_0) v + v^T (A \otimes e_N d_N) v$$

$$= -v^T (A \otimes M) v$$

$$\Rightarrow \frac{d}{dt} \|v\|_H^2 = -v^T (A \otimes M) v - v^T (A T \otimes M) v$$

$$\leq 0$$

which leads to energy conservation in $H$-norm since both $A^T + A = \alpha_D I_2$ and $M$ is positive semi-definite. \qed

Remark 3.3. Exactly as in the continuous case the discrete energy estimate will be constant in time when $\alpha_D = 0$. This means that the energy estimate will decrease with time if $\alpha_D > 0$.

Characteristic boundary condition

Specifying the characteristic boundary conditions, taken from the problem setting presented in Eq. (3.9), will give rise to the following boundary conditions in the spatial semi-discretization:

$$\begin{cases} - (A^T \otimes e_0^T) v + \lambda (I_2 \otimes d_0) v - g_l = 0 \\ (A^T \otimes e_N^T) v + \lambda (I_2 \otimes d_N) v - g_r = 0 \end{cases}, \quad (3.11)$$

resulting in the complete semi-discretization for the characteristic boundary conditions,

$$v_t = A \otimes D_2 v + \tau_0 \otimes H^{-1} e_0 \left( - (A^T \otimes e_0^T) v + \lambda (I_2 \otimes d_0) v - g_l \right) + \tau_1 \otimes H^{-1} e_N \left( (A^T \otimes e_N^T) v + \lambda (I_2 \otimes d_N) v - g_r \right), \quad (3.12)$$

The characteristic boundary conditions are used in the implementation at the left boundary in order to impose a spin wave. The implementation is further explained in Section 3.7.

Lemma 3.2. The semi-discretization in Eq. (3.12) is stable if $\tau_0 = A/\lambda$ and $\tau_N = -A/\lambda$.

Proof. Assuming $g_l = g_r = 0$, an energy estimate can be derived,

$$\frac{d}{dt} \|v\|_H^2 = -v^T \left( (A + A^T) \otimes M \right) v - \lambda \left( v_0^2 + v_N^2 \right). \quad (3.13)$$
Since the discrete energy in the $H$-norm is bound the conclusion is that the semi-discretization is stable.

### 3.4 Interface treatment

Splitting the domain and thereby also adding an interface, the domain will be $x_l < x_i < x_r$ where $x_i$ is the position of the interface. This introduction of an interface turns the 1D-problem in Eq. (3.4) into,

\[
\begin{cases}
  m^L_x = A m^L_{xx} & x \in [x_l, x_i], \ t > 0, \\
  m^R_x = A m^R_{xx} & x \in [x_i, x_r], \ t > 0, \\
  m^L = m^L_0 & x \in [x_l, x_i], \ t = 0, \\
  m^R = m^R_0 & x \in [x_i, x_r], \ t = 0,
\end{cases}
\]

which is illustrated in Figure 5.

\[
\vec{m}^L(x) \in L^2(x_r, x_i) \quad \vec{m}^R(x) \in L^2(x_i, x_l)
\]

Figure 5: Illustration of the one-dimensional domain with an interface. The left and the right domain can be discretized differently, resulting in different amounts of discretization intervals $N_L$ and $N_R$ respectively. One of the main issues in this report is to investigate the behaviour of frequencies that travel through the interface.

Again, as in the case with no interface, the energy method for the continuous problem in Eq. (3.14) is used in order to investigate what necessary conditions that has to be set for the problem to be well-posed. Recall Remark 3.2 and use
this to make an energy estimate for the case with an interface,
\[
\frac{d}{dt} (||m^L||^2 + ||m^R||^2) = \int_{x_l}^{x_i} (m^L)^T m^L_t \, dx + \int_{x_i}^{x_r} (m^R)^T m^R_t \, dx \\
+ \int_{x_l}^{x_i} (m^L_x)^T m^L_x \, dx + \int_{x_i}^{x_r} (m^R_x)^T m^R_x \, dx \\
= (m^L)^T A m^L_x \bigg|_{x_l}^{x_i} - \int_{x_l}^{x_i} (m^L_x)^T A m^L_t \, dx \\
+ (m^R)^T A m^R_x \bigg|_{x_i}^{x_r} - \int_{x_i}^{x_r} (m^R_x)^T A m^R_t \, dx \\
+ (m^L_x)^T A^T m^L_t \bigg|_{x_l}^{x_i} - \int_{x_l}^{x_i} (m^L)^T A^T m^L_t \, dx \\
+ (m^R_x)^T A^T m^R_t \bigg|_{x_i}^{x_r} - \int_{x_i}^{x_r} (m^R)^T A^T m^R_t \, dx, \\
= - \int_{x_l}^{x_i} (m^L_x)^T (A + A^T) m^L_t \, dx \\
- \int_{x_i}^{x_r} (m^R_x)^T (A + A^T) m^R_t \, dx + BT.
\]

Assuming homogeneous Neumann B.C. at \( x_l \) and \( x_r \), \( m^L_x \bigg|_{x_l}^{x_i} = 0 \) and \( m^R_x \bigg|_{x_i}^{x_r} = 0 \)
this leaves the boundary terms to be,
\[
BT = (m^L)^T A m^L_x \bigg|_{x_l}^{x_i} - (m^R)^T A m^R_x \bigg|_{x_i}^{x_r} + (m^L_x)^T A^T m^L_t \bigg|_{x_l}^{x_i} - (m^R_x)^T A^T m^R_t \bigg|_{x_i}^{x_r} \\
= 2\alpha D \left( (m^{(1)})^L (m^{(1)}_x)^L - (m^{(1)})^R (m^{(1)}_x)^R + (m^{(2)})^L (m^{(2)}_x)^L - (m^{(2)})^R (m^{(2)}_x)^R \right) \\
+ 2\gamma D \left( (m^{(1)})^L (m^{(2)}_x)^L - (m^{(1)})^R (m^{(2)}_x)^R + (m^{(2)})^L (m^{(1)}_x)^L - (m^{(2)})^R (m^{(1)}_x)^R \right),
\]

which require both \( m^L = m^R \) and \( m^L_x = m^R_x \) at \( x_i \) for the boundary terms at the interface to vanish. The conditions for the solution to have a bound energy estimate and to be well-posed, is that both the solution and the first derivative of the solution have to be continuous at the interface.

The problem setting presented in Eq. (3.15) is a well-posed problem.

\[
\begin{align*}
    m^L_t &= A m^L_x, \quad x \in [x_l, x_i], \ t > 0 \\
    m^R_t &= A m^R_x, \quad x \in [x_i, x_r], \ t > 0 \\
    m^L_t &= m^L_x, \quad x \in [x_l, x_i], \ t = 0 \\
    m^R_t &= m^R_x, \quad x \in [x_i, x_r], \ t = 0 \\
    m^L_x &= m^L, \quad x = x_l, \ t > 0 \\
    m^R_x &= m^R, \quad x = x_r, \ t > 0 \\
    m^L = 0 &\quad x = x_l, \ t > 0 \\
    m^R = 0 &\quad x = x_r, \ t > 0
\end{align*}
\]

(3.15)

There are several ways of making semi-discretizations of Eq. (3.15). One energy conserving way is the following:
presented in Eq. (3.15) is used. The results are presented in Table 2, and a Gaussian wave is used as an initial condition, and the boundary conditions of discretization points are used in both the left and the right domain. Again, the table the expected convergence rate is achieved in both the case with a 6th order SBP-operator, and in the case with a 4th order SBP-operator. The reference solution used in this case is the solution generated in the same norm of the error.

3.5 Convergence study

The convergence study is conducted using a test-case with a Gaussian wave, illustrated in Figure 6a as an initial condition. The boundary condition that is used for the convergence study is the reflecting, homogeneous Neumann boundary conditions given in Eq. (3.4). The result of a convergence study using one domain without interface is presented in Table 1. A fine grid, numerical solution, with $N_{ref} = 1600$, is used as the reference solution. As can be seen in the table the expected convergence rate is achieved in both the case with a 6th order SBP-operator, and in the case with a 4th order SBP-operator.

The convergence is measured with the parameter $q$ defined by,

$$q = \frac{\log_{10} \frac{||v_{ref} - v^{(N)}||_h}{||v_{ref} - v^{(N-2)}||_h}}{\log_{10} \left( \frac{N_1}{N_2} \right)^{1/d}},$$

where $d$ is the dimension, $v_{ref}$ is the reference solution, $v^{(N)}$ is the numerical solution corresponding to $N$ grid points and $||v_{ref} - v^{(N)}||_h$ is the discrete $l^2$ norm of the error.

A second convergence study were conducted where an interface is introduced. The reference solution used in this case is the solution generated in the same domain, but without an interface and $N_{ref} = 800$. In this case the same number of discretization points are used in both the left and the right domain. Again a Gaussian wave is used as an initial conditions, and the boundary conditions presented in Eq. (3.15) is used. The results are presented in Table 2, and
Table 1: $l^2$-norm of the error and the convergence rate according to Equation (3.18) for a fourth and sixth order SBP-operator. This convergence is for a single domain without an interface.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$l^2$-norm of the error</th>
<th>$q$-4th order</th>
<th>$l^2$-norm of error</th>
<th>$q$-6th order</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>$1.74 \cdot 10^{-4}$</td>
<td>6.45 $\cdot 10^{-9}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>$1.31 \cdot 10^{-5}$</td>
<td>1.51 $\cdot 10^{-6}$</td>
<td>5.42</td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>$8.58 \cdot 10^{-7}$</td>
<td>2.62 $\cdot 10^{-8}$</td>
<td>5.84</td>
<td></td>
</tr>
<tr>
<td>400</td>
<td>$5.41 \cdot 10^{-8}$</td>
<td>4.21 $\cdot 10^{-10}$</td>
<td>5.96</td>
<td></td>
</tr>
<tr>
<td>800</td>
<td>$3.19 \cdot 10^{-9}$</td>
<td>6.52 $\cdot 10^{-12}$</td>
<td>6.01</td>
<td></td>
</tr>
</tbody>
</table>

as can be seen in the table the convergence is not affected remarkably by the introduction of an interface in one dimension. Note here that the initial condition is a Gaussian wave that contains frequencies that can be represented with the lowest number of discretization points used in the implementation.

Table 2: $l^2$-norm of the error and the convergence rate according to Equation (3.18) for a fourth and sixth order SBP-operator. This convergence study is conducted with a numerical interface in the center of the domain.

<table>
<thead>
<tr>
<th>$N^L, N^R$</th>
<th>$l^2$-norm of the error</th>
<th>$q$-4th order</th>
<th>$l^2$-norm of error</th>
<th>$q$-6th order</th>
</tr>
</thead>
<tbody>
<tr>
<td>50, 50</td>
<td>$8.84 \cdot 10^{-7}$</td>
<td>4.03 $\cdot 10^{-7}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100, 100</td>
<td>$6.13 \cdot 10^{-8}$</td>
<td>5.48 $\cdot 10^{-9}$</td>
<td>6.20</td>
<td></td>
</tr>
<tr>
<td>200, 200</td>
<td>$3.61 \cdot 10^{-9}$</td>
<td>4.42 $\cdot 10^{-11}$</td>
<td>6.96</td>
<td></td>
</tr>
</tbody>
</table>

3.6 Simulations of micromagnetic processes

Results of simulations in two domains with different scales are presented in Figure 6. The purpose of this simulation is to illustrate the issue of reflection when modelling micromagnetic processes. The initial condition in this case is a Gaussian wave that could represent a magnetic pulse invoked by a laser in a ferromagnetic material. In this particular case some of the frequencies cannot be represented in the coarser grid on the right side, and they are therefore reflected back into the left domain. However some frequencies can be represented in the right domain, and therefore they are transmitted through the interface. In the next section the behaviour of individual frequencies travelling across an interface is investigated.
Figure 6: Illustration of the simulation of an initial Gaussian wave in (a) evolving in time according to the Equation (2.4). The solid blue line is modelled with two differently meshed domains, while the dotted red line is modelled with a single domain for comparison. A fourth order SBP operator is used for the semi-discretization.
3.7 Spin wave frequency behaviour at interface

There are several ways of investigating how separate frequencies behave when travelling through an interface. One way is to impose a spin wave of a certain amplitude and frequency to the left of the interface, and then measure the amplitude of the transmitted wave. Depending on the frequency of the spin wave, the difference in mesh-size, and the damping at the interface the spin wave will behave differently when reaching the interface. This phenomena is investigated in this section.

In order to study the behaviour of individual frequencies at the interface the spin wave has been imposed at the left boundary. Below one way creating the spin wave is presented.

At the left boundary the following characteristic boundary conditions should be set:

\[-A^T m + \lambda m_x = g_l, \quad \text{where} \quad g_l = -A^T m + \lambda (m)_x^{x_l} \text{ and } m|^{x_l} \text{ is the imposed value of the normalized magnetization at the left boundary presented in Equation (3.19)}\]

\[m|^{x_l} = \left[\begin{array}{c} C_1 \cos(\omega t - kx) \\ C_2 \sin(\omega t - kx) \end{array}\right]^{x_l}. \quad (3.19)\]

The general challenge in imposing a frequency at the left boundary is to fulfil the boundary conditions. Since the boundary conditions are weakly imposed at the boundaries via Simultaneous Approximations Terms the approximate numerical solution will oscillate around the imposed boundary conditions until convergence. This phenomenon can be seen in Figure 7.

**Remark 3.4.** When creating a spin wave the parameter \(\alpha_D\) will be set to 0. \(\alpha_D\) introduces overall damping within the system. In this particular case the behaviour of the solution at the interface is the subject of investigation and therefore this damping is removed.

The characteristic boundary conditions at the left boundary

\[g_l = -A^T m + \lambda m_x|^{x_l} = (\gamma D + \lambda k) \left[\begin{array}{c} C_1 \sin(\omega t - kx) \\ C_2 \cos(\omega t - kx) \end{array}\right]|^{x_l}. \quad (3.20)\]

The equation \(m = Am_{xx}\) supports waves (3.19),

\[
\begin{pmatrix}
-C_1 \omega \sin(\omega t - kx) \\
C_2 \omega \cos(\omega t - kx)
\end{pmatrix} = \gamma D \begin{pmatrix}
-C_2 k^2 \sin(\omega t - kx) \\
C_1 k^2 \cos(\omega t - kx)
\end{pmatrix}
\quad (3.21)
\]

if \(C_1 = C_2 = C\) and \(k = \sqrt{\omega/\gamma D}\). The corresponding characteristic boundary conditions at \(x = x_l\) is

\[m|^{x_l} = C \left[\begin{array}{c} \cos(\omega t - kx) \\ \sin(\omega t - kx) \end{array}\right]|^{x_l} \Rightarrow g_l = C(\gamma D + \sqrt{\omega \gamma D}) \left[\begin{array}{c} \sin(\omega t - kx) \\ \cos(\omega t - kx) \end{array}\right]|^{x_l}. \quad (3.22)\]
Figure 7: The evaluation of the numerical (blue and green) and imposed analytical (read and turquoise) boundary values at the left boundary for the components of magnetization and its spatial derivative.

Figure 8: Illustration of an initial condition to a simulation with a spin wave.

**Initial conditions**

For the boundary conditions at the left boundary to be fulfilled at all times, an initial condition is introduced. This initial condition has to be consistent with the numerical approximation of the solution and the derivative of the solution at the left and right boundary. Therefore an exponential envelope is introduced. An example of how this initial condition looks like is presented in Figure 8.

**Damping of high frequencies**

In order to damp the high frequencies at the interface an extra SAT is introduced in a way so that the energy is decreased at the interface. The idea is for this method is the same as to add a low pass filter for frequencies that can be represented in the right domain. The extra SAT does this by forcing the solution at the interface closer to an average, and thereby smoothing the solution and damping the high frequencies. The solution is pushed towards an average not only in one point but in a larger region in order to avoid reflections back into the left domain. The forcing increases quadratically towards the interface where it reaches its peak. Figure 9 illustrates the idea of the damping graphically.
Let this extra SAT be constructed as in Eq. (3.23),
\[ SAT_x = \sigma(x) \otimes H^{-1}(v^L - (I_2 \otimes Q)v^L), \]  
(3.23)
where \( Q \) is constructed in such a way that it gives the average value in the region directly to the left of the interface, marked with Damping width in Figure 9. This results in the complete spatial discretization in Eq. (3.24),
\[ v^L_t = A \otimes D_2 v^L + A/\lambda \otimes H^{-1} \left( e_0 - (A^T \otimes e_0^T)v + \lambda(I_2 \otimes d_0)v - g_l \right) \]
\[ + (\tau^L_1 \otimes H^{-1}(d^L_N)^T)(v^L_N - v^R_0) \]
\[ + (\tau^L_2 \otimes H^{-1}e^L_N) \left( (I_2 \otimes d^L_N)v^L - (I_2 \otimes d^R_N)v^R \right) \]
\[ + \sigma \otimes H^{-1}(v^L - (I_2 \otimes Q)v^L) \]
\[ v^R_t = A \otimes D_2 v^R + (\tau^R_0 \otimes H^{-1}e^R_Nd^R_N)v^R + (\tau^R_1 \otimes H^{-1}(d^R_0)^T)(v^R_0 - v^L_0) \]
\[ + (\tau^R_2 \otimes H^{-1}e^R_0) \left( (I_2 \otimes d^R_0)v^R - (I_2 \otimes d^L_N)v^L \right) \]
(3.24)
where the penalty terms is the same as in Lemma 3.3. The contribution to the energy estimate from introducing this extra term is
\[ 2(v^L)^T (\sigma \otimes (I_N - Q)) v^L. \]  
(3.25)

The requirement to keep the semi-discretization stable is that \( \sigma \otimes (I_N - Q) \) is negative semi-definite. If \( \sigma(x = x_i) \) is set to be \(-\sigma_N \cdot I_2, \sigma_N \in \mathbb{R}_{>0}\) this requires \( I_N - Q \) to be positive semi-definite. This will be true if \( Q \) is constructed as in Eq. (3.26)
\[ Q = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \\
0 & 0 & 0 & 0 & \frac{1}{n} & \frac{1}{n} & \frac{1}{n} \\
0 & 0 & 0 & 0 & \frac{1}{n} & \frac{1}{n} & \frac{1}{n}
\end{pmatrix}_{N \times N} \]  
(3.26)

There are other ways of constructing the matrix \( Q \). However the conditions on \( Q \) is not easily fulfilled. This because the requirement that \( I_N - Q \) should be positive semi-definite. Some ideas of how to get around this problem are further developed in the Section 5.

In this case there are two degrees of freedom in creating the SAT at the interface. The first can be represented as the parameter \( \sigma_N \), the strength of damping and the second degree of freedom is the width of the averaging region, \( w_A \) used in the averaging matrix \( Q \). The latter will affect how many points that is considered in calculating the average, and thereby the value of \( 1/n \).

In order to minimize the reflection the parameter \( \sigma \) is quadratically increased within the region of damping to the maximum value of \( \sigma_N \) at \( x = x_i \). Figure 9
Figure 9: Illustration of the width and strength, $w_A$ and $\sigma_N$ respectively, of the damping to the left of the interface. Parameters which will affect the construction of the matrix $Q$. At the right boundary the absorbing boundary is illustrated.

This method is implemented for wavelengths between $0.2\mu m$ and $2\mu m$, where the shorter wavelength corresponds to the high frequency that should be completely damped, and the longer wavelength should be completely transmitted. To study the effect from the damping interface treatment only, the same mesh size is used for both domains and all frequencies. The mesh size used is chosen such that the highest frequency can be well represented with at least 10 discretization points per wavelength everywhere. The discretization is defined by $x_l = 0$, $x_{int} = 5\mu m$, $x_r = 20\mu m$, $N^L = 250$ and $N^R = 750$.

The result, when setting the width of the damping region, $w_A$ to $2.5\mu m$ and the damping strength to 100, is presented in Figure 10 for wavelengths of size $0.2\mu m$ and $1.8\mu m$. For the short wavelength the bandpass filter works as intended. The wave is completely damped before reaching the interface and no reflections arises. In the case of the longer wavelength the spin wave is partially transmitted and partially reflected. The amplitude of the reflections is marked in the figure as $E_r$.

In the model an absorbing boundary is added at the right boundary in order to avoid reflections from it. The reflected error of this absorbing boundary layer is less than 2% of the amplitude for all modelled wavelengths. This percentage is experimentally confirmed by modelling all wavelengths with $\sigma_N$ set to 0.

In order to find the optimal values of the damping width, $w_A$, and the strength of damping, $\sigma$ the two parameters should fulfil three conditions. The first and second condition is that the introduction of the extra SAT should damp high frequencies that can not be represented to the right of the interface, and at the same time allow transmission of low frequencies that can be represented to the right of the interface. The third condition is that the introduction of a SAT should create small or no reflections back into the left domain. The result
Figure 10: Spin waves of different frequencies travelling towards an interface with added damping to the left of the interface according to the discretization in Eq. (3.24). The high frequency spin wave becomes completely damped while the low frequency is partially transmitted and creates reflections with amplitude $E_r$. 

(a) Wavelength 0.2µm.

(b) Wavelength 1.8µm.
in terms of amplitude of reflection and amplitude of transmission as the parameters $\sigma_N$ and the width of averaging region is varied is presented in Figure 11.

The results in Figure 11 show that no parameter values fulfills all three conditions. For the small values of damping strength, $\sigma_N = 0.1, 1$, there is less than 10% damping for the high frequencies, which is far from the desired 100%. The fact that the transmitted and reflected amplitude sometimes reach above 100% is due to the reflections that becomes trapped between the interface and the left boundary. These reflection accumulate energy which in the cases of longer wavelengths also spill over to the right side. A couple of suggested improvements is to either use a moving average instead of imposing the same average in the entire averaging region or to make $\sigma_N$ dependant on the time derivative of the solution. Results from the latter of these suggestions is implemented below. In general the results show that the largest damping width produces the best results according to the conditions.
Figure 11: Transmitted and reflected amplitude for different magnitude of $\sigma_N$ and three different widths of the averaging region.
Variable damping strength

The results in Figure 11 show that a variable damping strength, that changes with the frequency, would give results that fulfills the conditions on the low pass filter. Since it can not be assumed that the frequency is known, the dispersive property of the spin wave is used to establish the behaviour of the spin wave. The numerical time derivative, $v_t$ is used to estimate the frequency of the spin wave.

The same approach as in the Discretization (3.24) is used except that $\sigma$ instead changes with $x$ and as a scalar function of the time derivative of the solution, according to Equation (3.27):

\[
\text{SAT}_\sigma = \sigma(x) \otimes H^{-1} \left( v^L - (I_2 \otimes Q) v^L \right)
\]

\[
\sigma(x) = \begin{cases} 
(\sigma_0 |v_t(x)|)^2 \left( \frac{x-x_i+w_A}{w_A} \right)^2 \cdot I_2, & x_i - w_A < x < x_i \\
0, & \text{otherwise}
\end{cases}
\]

The results of implementation, using the approach in Eq. (3.27) is presented in terms of transmission and reflections for different frequencies in Figure 13. The results are closer to the desired low pass filter than when using a constant value of $\sigma$. Also the reflections are considerably smaller, and the maximum reflections is 30%, compared to the previous method when the reflections for low frequencies was about 100% or more.

The behaviour of a high frequency and a low frequency spin wave is presented in Figure 14. The high frequency wave is damped so that 20% of the original amplitude is transmitted, and no reflections occur. For the low frequency spin wave the transmission is almost 100% while the reflections are about 20%. Reflections which optimally would be 0% but still are considerably less than when using a constant damping constant, compare to Figure 10.
Figure 12: Averaging width, \( w_A = 2.5 \, \mu m \).

Figure 13: Transmitted and reflected amplitude for damping strength according to Eq. (3.27) and damping width, \( w_A = 2.5 \mu m \).

Figure 14: Spin waves of different frequencies travelling towards an interface with added damping to the left of the interface according to Eq. (3.27).
4 The 2-D Problem

As mentioned in Section 3 the extension to two dimensions is made by adding the term $m_{yy}$ to the equation, resulting in the equation,

$$ m_t = A(m_{xx} + m_{yy}) . \quad (4.1) $$

The idea behind the implementation and interface treatment in two dimensions is the same as in the one dimensional case. This section provides an explanation to how a multiscale interface in two dimensions could be treated and an example is implemented. However the actual analysis of the results is outside the scope of this report.

4.1 Definitions

Considering the domain to be the unit square

$$ \Omega_{w,n}^y = \{ (x,y) \in \mathbb{R}^2 : w \leq x \leq e ; s \leq y \leq n \} $$

Let $u, w \in L^2[\Omega_{w,n}^y]$ where $u = [u^{(1)}, u^{(2)}]^T$ and $w = [w^{(1)}, w^{(2)}]^T$ are real-valued vector functions with 2 components. Let the inner product be defined by $(u, w) = \int_{\Omega} u^T w \, dx dy$, and let the corresponding norm be $||u||^2 = (u, u)$.

Let $r = (x, y)$ denote the position vector in two dimensions and consider the computational domain $\Omega_{w,s}^y$. This domain is discretized using an $(N_x + 1) \times (N_y + 1)$-point equidistant grid defined as:

$$ x_i = w + ih_x, \ i = 0, 1, \ldots, N_x, \ h_x = \frac{e - w}{N_x}, $$
$$ y_j = s + jh_y, \ i = 0, 1, \ldots, N_y, \ h_y = \frac{n - s}{N_y}. $$

The approximate solution at the grid point $(x_i, y_j)$ is $v_{i,j} = [v_{i,j}^{(1)}, v_{i,j}^{(2)}]^T$ and, for computational purposes, the entire solution vector should be defined as

$$ v = \left[ v_{0,0}^{(1)}, \ldots, v_{0,N_y}^{(1)}, v_{1,0}^{(1)}, \ldots, v_{1,N_y}^{(1)}, \ldots, v_{N_x,0}^{(1)}, \ldots, v_{N_x,N_y}^{(1)}, v_{0,0}^{(2)}, \ldots, v_{0,N_y}^{(2)}, \ldots, v_{N_x,0}^{(2)}, \ldots, v_{N_x,N_y}^{(2)} \right]_{1 \times 2N^2}. $$

To distinguish whether a 2-D difference operator $R$ is operating in the $x$- or the $y$-direction, we use the notations $R_x$ and $R_y$, respectively. The following 2-D operators will be used:

$$ D_{2x} = I_2 \otimes D_2 \otimes I_{N_y}, \quad D_{2y} = I_2 \otimes I_{N_x} \otimes D_2, $$
$$ H_x = I_2 \otimes H \otimes I_{N_y}, \quad H_y = I_2 \otimes I_{N_x} \otimes H, $$
$$ E_W = I_2 \otimes e_0 \otimes I_{N_y}, \quad E_S = I_2 \otimes I_{N_x} \otimes e_0, $$
$$ E_E = I_2 \otimes e_N \otimes I_{N_y}, \quad E_N = I_2 \otimes I_{N_x} \otimes e_N, $$
$$ H = H_x H_y = I_2 \otimes H \otimes H, $$

where we refer to the 4 boundaries of the domain as $W$ (West), $E$ (East), $S$ (South) and $N$ (North) boundaries, $D_2$ and $H$ are the 1-D operators, $e_0, e_N, b_0$ and $b_N$ are the 1-D "boundary" vectors introduced in Definition 3.1. Note that the matrix $E_W$ is defined so that $v_{W} = E_W^T v$ is a vector that contains only the elements of $v$ that correspond to the west boundary. The matrices $E_E, E_S$ and $E_N$, and respective vectors $v_E, v_S$ and $v_N$, are defined in a similar way for each of the other boundaries.
4.2 Continuous analysis

Considering the case with reflecting, homogeneous Neumann, boundary conditions at all four boundaries and an initial condition, \( m_0 \) fulfilling these boundary conditions, the problem can be described by Eq. (4.2).

\[
\begin{cases}
  m_t = A(m_{xx} + m_{yy}), & x \in \Omega_{w,s}, \ t > 0, \\
  m_x = 0, & x \in W, E, \ t \geq 0, \\
  m_y = 0, & x \in S, N, \ t \geq 0, \\
  m(x, 0) = m_0, & x \in \Omega_{w,s}, \ t = 0.
\end{cases}
\]  

(4.2)

Lemma 4.1. Problem (4.2) is well-posed.

Proof. The proof is conducted using the energy method mentioned in Remark 3.2. The method is similar for this two dimensional problem. Multiply the equation with \( m^T \) and integrate over the entire domain and add the transpose:

\[
\frac{d}{dt} ||m||^2 = \int_{\Omega} m^T m_t + m^T m_t^T m d\Omega
\]

\[
= \int_{\Omega} m^T A m_{xx} + m^T A m_{yy} + m^T A_x m + m^T A_y m d\Omega
\]

\[
= \int_{s} m^T A_s m_{xs} dy + \int_{w} m^T A_s m_{xs} d\Omega + \int_{s} m^T A_y m_{xs} d\Omega + \int_{w} m^T A_y m_{xs} d\Omega + \int_{s} m^T A_y m_{xs} d\Omega
\]

\[
= - \int_{\Omega} m^T (A + A^T) m_x + m^T (A + A^T) m_y d\Omega + BT
\]

\[
= - \alpha D (||m_x||^2 + ||m_y||^2) + BT
\]

For the energy to be bound the boundary terms, \( BT \), has to be less or equal to zero. The terms are divided into each boundary so that \( BT = BT_W + BT_E + BT_S + BT_N \) where the boundary terms all look similar:

\[
BT_W = \int_{W} m^T A m_{xs} + m^T A_x m_{xs} d\Omega
\]

\[
BT_E = - \int_{E} m^T A m_{xs} + m^T A_x m_{xs} d\Omega
\]

\[
BT_S = \int_{S} m^T A m_{ys} + m^T A_y m_{ys} d\Omega
\]

\[
BT_N = - \int_{N} m^T A m_{ys} + m^T A_y m_{ys} d\Omega
\]

Note that all boundary terms vanish if the boundary conditions defined in Eq. (4.2) are used. This leads to the final energy estimate

\[
||m(r, t)||^2 = ||m(r, 0)||^2 - \int_{0}^{t} \alpha D (||m_x||^2 + ||m_y||^2) \ d\tau
\]

(4.3)
Which leads to the fact that the energy of the system is bound by the initial energy, that is \( \| m(r, t) \| \leq \| m(r, 0) \| \), and that the Problem (4.2) is well-posed.

4.3 Semi-discretization

Discretizing Eq. (4.2) in space with the SBP-SAT method leads to the following semi-discrete problem:

\[
v_t = AD_{2x}v + AD_{2y}v + SAT \tag{4.4}
\]

where \( SAT = SAT_W + SAT_E + SAT_S + SAT_N \) are the specific SATs for each of the four boundaries and are defined as:

\[
SAT_W = (I_2 \otimes H^{-1}e_0 \otimes I_{N_y}) (A \otimes d_0 \otimes I_{N_y})v = (A \otimes H^{-1}e_0d_0 \otimes I_{N_y})v
\]

\[
SAT_E = -(A \otimes H^{-1}e_N d_N \otimes I_{N_y})v
\]

\[
SAT_S = (A \otimes I_{N_x} \otimes H^{-1}e_0d_0)v
\]

\[
SAT_N = -(A \otimes I_{N_x} \otimes H^{-1}e_N d_N)v
\]

**Lemma 4.2.** The semi-discretization in Eq. (4.4) is stable.

**Proof.** Mimic the continuous energy estimate by multiplying the semi-discretization with \( v^T\tilde{H} \) and add the transpose, obtaining

\[
\frac{d}{dt} \| v \|_{\tilde{H}}^2 = v^T\tilde{H}v_t + v_t^T\tilde{H}v
\]

\[
= v^T\tilde{H}(AD_{2x}v + AD_{2y}v + v^T\tilde{H}(SAT_W) + v^T\tilde{H}(SAT_E) + v^T\tilde{H}(SAT_S) + v^T\tilde{H}(SAT_N) + v^T(AD_{2x}v + AD_{2y})^T\tilde{H}v + (SAT_W)^T\tilde{H}v + (SAT_E)^T\tilde{H}v
\]

\[
+ (SAT_S)^T\tilde{H}v + (SAT_N)^T\tilde{H}v
\]

\[
= -v^T((A + A^T) \otimes M \otimes H)v - v^T((A + A^T) \otimes H \otimes M)v + BT,
\]

where the first two terms are positive semidefinite since \( A + A^T = \alpha_D I_2 \) and \( M = M^T \) is defined to be positive semi-definite and the boundary terms, \( BT = BT_W + BT_E + BT_S + BT_N \) will be

\[
BT_W = -v^T(A \otimes e_0d_0 \otimes H)v + v^T(A \otimes e_0d_0 \otimes H)v
\]

\[
- v^T(A^T \otimes (d_0)^T(e_0) \otimes H)v + v^T(A^T \otimes (d_0)^T(e_0) \otimes H)v = 0,
\]

the other boundary terms look similar to this and they will also vanish. This leaves the concluding discrete energy estimate

\[
\frac{d}{dt} \| v \|_{\tilde{H}}^2 = -\alpha_D v^T(I_2 \otimes M \otimes H)v + v^T(I_2 \otimes H \otimes M)v \leq 0 \tag{4.5}
\]

which proves that the discrete energy estimate is bound over time and that the semi-discretization of the system is stable.
Figure 15: Illustration of the two-dimensional domain with an interface. The left and the right domain can be discretized differently.

4.4 Interface treatment

As in the one dimensional case an interface is introduced, creating two domains with potentially different discretization. For the problem to be well-posed both the solution and the derivative of the solution have to be continuous, see analysis in Section 3.4. With reflecting boundary conditions the problem set up in two dimensions with an interface is described in mathematical terms in Eq. (4.6) and the domain is illustrated in Figure 15.

\[
\begin{align*}
  m^L_t &= A m^L_{xx} + A m^L_{yy}, \quad r \in \Omega^L, \quad t > 0, \\
  m^R_t &= A m^R_{xx} + A m^R_{yy}, \quad r \in \Omega^R, \quad t > 0, \\
  m^L_x &= 0, \quad r \in W, \quad t \geq 0, \\
  m^R_x &= 0, \quad r \in S, N, \quad t \geq 0, \\
  m^L_y &= 0, \quad r \in W, \quad t \geq 0, \\
  m^R_y &= 0, \quad r \in S, N, \quad t \geq 0, \\
  m^L_{\Omega^L} &= m^R_{\Omega^L}, \quad r \in I, \quad t = 0, \\
  m^R_{\Omega^R} &= m^R_{\Omega^R}, \quad r \in \Omega^R, \quad t = 0,
\end{align*}
\]

(4.6)

Discretizing 4.6 in space using SBP-SAT leads to the semi-discretization in Eq. (4.7),

\[
\begin{align*}
  v^L_t &= AD_{2x}v^L + AD_{2y}v^L + SAT^L \\
  v^R_t &= AD_{2x}v^R + AD_{2y}v^R + SAT^R
\end{align*}
\]

(4.7)

where \( SAT^L = SAT^L_W + SAT^L_S + SAT^L_N + SAT^L_I \) and \( SAT^R = SAT^R_E + SAT^R_S + SAT^R_N + SAT^R_I \) which in turn are defined as in Equation 4.4 and the interface terms defined as
\begin{align*}
SAT_{L1} &= (\tau_L^{r} \otimes H^{-1}(\delta_N^{l})^T \otimes I_{N_y}) ((I_2 \otimes (e_N^{y})^T \otimes I_{N_y})v^L - (I_2 \otimes (e_N^{y})^T \otimes I_{N_y})v^R) \\
SAT_{L2} &= (\tau_L^{2} \otimes H^{-1}(\delta_N^{l})^T \otimes I_{N_y}) ((I_2 \otimes (e_N^{y})^T \otimes I_{N_y})v^L - (I_2 \otimes (e_N^{y})^T \otimes I_{N_y})v^R) \\
SAT_{R1} &= (\tau_R^{l} \otimes H^{-1}(\delta_0^{r})^T \otimes I_{N_y}) ((I_2 \otimes (e_0^{y})^T \otimes I_{N_y})v^R - (I_2 \otimes (e_0^{y})^T \otimes I_{N_y})v^L) \\
SAT_{R2} &= (\tau_R^{2} \otimes H^{-1}(\delta_0^{r})^T \otimes I_{N_y}) ((I_2 \otimes (e_0^{y})^T \otimes I_{N_y})v^R - (I_2 \otimes (e_0^{y})^T \otimes I_{N_y})v^L)
\end{align*}

Lemma 4.3. The semi-discretization 4.7 is stable if the penalty parameters are chosen as in Lemma 3.3.

The proof of the lemma is straightforward to extend from the one dimensional interface proof in Appendix B.

Remark 4.1. In order for the semi-discretization in Eq. (4.7) to work when the two domains have different mesh size an interpolation has to be performed at the interface. A further discussion on the implementation of this interpolation and definitions of appropriate energy conserving interpolation operators can be found in [17].

4.5 Convergence study

The convergence study were carried out in the same way as in the one dimensional case, described in Section 3.5. As a test problem a square domain with an \(N \times N\) discretization without interface is used with a Gaussian wave as an initial condition at the west boundary. Both the fourth and sixth order convergence rates are investigated with a reference discretization of \(N_{ref} \times N_{ref} = 400 \times 400\). The result of the convergence study is as expected for both the sixth order and the fourth order operators. The magnitude of the error and the convergence rates are presented in Table 3.

<table>
<thead>
<tr>
<th>(N_x \times N_y)</th>
<th>(l^2)-norm of the error</th>
<th>(q)-4th order</th>
<th>(l^2)-norm of error</th>
<th>(q)-6th order</th>
</tr>
</thead>
<tbody>
<tr>
<td>25 \times 25</td>
<td>1.33 \cdot 10^{-9}</td>
<td>6.10</td>
<td>1.46 \cdot 10^{-9}</td>
<td>6.28</td>
</tr>
<tr>
<td>50 \times 50</td>
<td>1.93 \cdot 10^{-11}</td>
<td>3.89</td>
<td>1.89 \cdot 10^{-11}</td>
<td>6.72</td>
</tr>
<tr>
<td>100 \times 100</td>
<td>1.30 \cdot 10^{-12}</td>
<td>4.21</td>
<td>1.78 \cdot 10^{-13}</td>
<td>6.72</td>
</tr>
<tr>
<td>200 \times 200</td>
<td>7.04 \cdot 10^{-14}</td>
<td></td>
<td>2.18 \cdot 10^{-15}</td>
<td>6.36</td>
</tr>
</tbody>
</table>

4.6 Simulations of micromagnetic processes

When modelling magnetic processes in a two dimensional domain a Gaussian wave is used as an initial condition, see Figure 16a. Since the propagation of this Gaussian is in two dimensions and the boundary conditions are reflecting the resulting solution interferes with itself and the resulting simulations becomes
difficult to predict. In Figure 16 the initial condition and the evolution after some time is presented.

In this simulation the difference in mesh size in the two domains are two to one which means that an interpolation has to be made at the interface. This interpolation has to be energy conserving, which is why special interpolation operators discussed in Remark 4.1, are used. The simulations illustrated in Figure 16 show an evolving vector field that behaves smoothly. The smoothness of the solution is due to the fact that the solutions is well resolved in both domains. The SBP-operators used in the simulation is fourth order accurate and the mesh size is $100 \times 100$ and $50 \times 50$ in the left and right domain respectively. Vectors close to each other point in roughly the same direction and has about the same magnitude. The solution can also be seen to be continuous across the interface and no visible reflections from the interface can be observed from the simulation.

Figure 16: Propagation of a Gaussian wave in a two-dimensional domain with an multiscale interface and reflecting boundary conditions.
5 Conclusion

The problem when introducing multiscale domains is that the high frequency phenomena that cannot be represented in the coarser of the meshes is partially reflected back into the finer mesh as a non-physical result of the discretization.

A higher order, energy conserving, finite-difference-method with the purpose of solving the previously mentioned problem is developed in this report. The idea behind this developed method is to weakly impose an average to the incoming phenomena and thereby smoothing the solution close to the interface. The approach is evaluated for spin waves of different wave-lengths. The method is shown to damp the high frequency spin waves as desired but also to generate unwanted reflections for low frequency spin waves that should be transmitted without being influenced at the interface. From the observed results in Figure 11 the largest damping width gives the best results when it comes to minimizing reflections, and transmitting low frequencies.

The method could probably be improved by changing the approach of smoothing the solution. For example the averaging could be done differently. Instead of imposing the same average over the entire region a moving average could be used. This would create challenges within making a proper energy estimate and thereby guarantee the method to be stable but on the other hand it would minimize the reflections created for the low frequencies. Another suggestions for improvement is to use a value of the damping strength, $\sigma_N$, that is directly dependent on the value of the time-derivative of the solution. This approach is implemented with results presented in Figure 13 and 14. These implementation improve the method, but the method can be further developed in order to minimize reflections.

6 Acknowledgements

Lastly, I want to thank the people who made this work possible, and supported me throughout the process.

- Special thanks to my supervisor, Mikhail Pulektov, who always supported me during my work, and always expected more of me than I did of myself.
- Thanks also to my reviewer, Gunilla Kreiss, who always is a source of inspiration and inspired the entire project.
- My gratitude also includes, but is not limited to my classmates, Marcus Näslund, who on several occasions proof read my report, and Daniel Salvador, who had several good questions regarding my work.
- Lastly I want to thank the staff of the department of Information Technology who always were there and answered questions and made it possible for me to develop within the field of Computational Science.
## A Definitions of parameters

In this appendix parameters defined or used in this report are collected and shortly described.

Table 4: The physical quantities used in this report and the corresponding units, both in cgs-units and in SI-units, which is used in this report and the mainly used system. Further discussions of conversion factors are can be found in [3, ch. 6.4], [7] and [6].

<table>
<thead>
<tr>
<th>Notion</th>
<th>Quantity</th>
<th>cgs-units</th>
<th>Corresponding in SI-units</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>Magnetization - dipole moment per unit volume</td>
<td>$1\text{emu/cm}^3$</td>
<td>$10^{-3}\text{A/m}$</td>
</tr>
<tr>
<td>$M_s$</td>
<td>Magnetization saturation ($\approx 1.8 \cdot 10^6[A/m]$ for Iron in room temp [18])</td>
<td>$1\text{emu/cm}^3$</td>
<td>$10^{-3}\text{A/m}$</td>
</tr>
<tr>
<td>$m$</td>
<td>$M/M_s$, unit length</td>
<td>$1[-]$</td>
<td>$1[-]$</td>
</tr>
<tr>
<td>$m_s$</td>
<td>The perturbation of continuous magnetization $m - m_0, m_0 = (0, 0, 1)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H$</td>
<td>Magnetic-field, $H$-field</td>
<td>$1\text{Oe}$</td>
<td>$\frac{1}{4\pi} \cdot 10^4\text{A/m}$</td>
</tr>
<tr>
<td>$B$</td>
<td>Magnetic-field, $B$-field</td>
<td>$1\text{G}$</td>
<td>$10^{-4}\text{T}$</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Gyromagnetic ratio ($\approx 1.00e/m, 2.3 \cdot 10^5[m/As]$ for Iron [3], [19])</td>
<td>$1\text{rad/sG}$</td>
<td>$10^4\text{rad/sT}$</td>
</tr>
<tr>
<td>$A_{exc}$</td>
<td>Material exchange constant $\approx 2 \cdot 10^{-11}$</td>
<td></td>
<td>$[J/m]$</td>
</tr>
<tr>
<td>$\gamma_D$</td>
<td>$\gamma A_{exc}$</td>
<td></td>
<td>$1\text{m}^2/\text{s}$</td>
</tr>
<tr>
<td>$\mu_0$</td>
<td>Permeability in vacuum $= 4\pi \cdot 10^{-7}$</td>
<td></td>
<td>$[kgm/A^2s^2]$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Dimensionless damping parameter</td>
<td>$1[-]$</td>
<td>$1[-]$</td>
</tr>
<tr>
<td>$\alpha_D$</td>
<td>$= \alpha \gamma A_{exc} = \alpha \gamma_D$</td>
<td></td>
<td>$1\text{m}^2/\text{s}$</td>
</tr>
<tr>
<td>$x_l, x_r$</td>
<td>Left and right boundary in x-direction respectively</td>
<td></td>
<td>$1[m]$</td>
</tr>
<tr>
<td>$x_i$</td>
<td>Position of the interface in x-direction</td>
<td></td>
<td>$1[m]$</td>
</tr>
</tbody>
</table>

The definitions of the discrete parameters are collected in separate chapters divided into one dimensional and two dimensional definitions in Section 3.1 and Section 4.1 respectively. In Section 3.1 the second order SBP-operator is defined.
B Proof of stability - 1D interface

To be able to prove stability of the Scheme 3.15 it is necessary for the energy estimate to be non increasing. Therefore the energy method is used to prove this.

First the semi-discretization in Eq. (3.16) is written in a more explicit form as:

\[
v^L_t = A \otimes D_2 v^L + \left( \tau^L_1 \otimes H^{-1} e_0^L d_0^L \right) v^L + \left( \tau^L_2 \otimes H^{-1} e_N^L \right) \left( (I_2 \otimes (e_N^L)^T) v^L - (I_2 \otimes (e_R^L)^T) v^R \right)
\]

\[
v^R_t = A \otimes D_2 v^R + \left( \tau^R_1 \otimes H^{-1} e_N^R d_N^R \right) v^R + \left( \tau^R_2 \otimes H^{-1} e_0^R \right) \left( (I_2 \otimes (e_N^R)^T) v^R - (I_2 \otimes (e_L^R)^T) v^L \right)
\]

then the energy method is used by multiplying with

\[
\frac{d}{dt} \left( ||v^L||_H + ||v^R||_H \right) = (v^L, v^L)_H + (v^L, v^L^L)_H + (v^R, v^R)_H + (v^L^R, v^R^R)_H
\]

and investigate the behaviour of every term individually,

\[
\#1 = (v^L)^T (I_2 \otimes H) (A \otimes H^{-1} (-M - e_0^L d_0^L + e_N^L d_N^L)) v^L + (v^L)^T (I_2 \otimes H) (\tau^L_1 \otimes H^{-1} e_0^L d_0^L) v^L
\]

\[
+ (v^L)^T (I_2 \otimes H) (\tau^L_2 \otimes H^{-1} d_N^L) \left( (I_2 \otimes (e_N^L)^T) v^L - (I_2 \otimes (e_R^L)^T) v^R \right)
\]

\[
+ (v^L)^T (I_2 \otimes H) (\tau^R_1 \otimes H^{-1} e_N^R d_N^R) \left( (I_2 \otimes d_N^L) v^L - (I_2 \otimes d_0^L) v^R \right)
\]

\[
= -(v^L)^T A \otimes M v^L + (v^L)^T (A \otimes e_0^L d_0^L) v^L + (v^L)^T A \otimes e_N^L d_N^L v^L + (v^L)^T \tau^L_1 \otimes e_0^L d_0^L v^L
\]

\[
+ (v^L)^T \tau^L_2 \otimes (d_N^L)^T (e_N^R)^T v^L - (v^L)^T \tau^L_2 \otimes (d_N^L)^T (e_0^R)^T v^R
\]

\[
+ (v^L)^T \tau^L_1 \otimes e_N^L d_N^L v^L - (v^L)^T \tau^L_2 \otimes e_N^L d_N^L v^R
\]

\[
= -(v^L)^T A \otimes M v^L + (v^L)^T \left( -A + \tau^L_0 \right) \otimes e_0^L d_0^L v^L + (v^L)^T \left( A + \tau^L_2 \right) \otimes e_N^L d_N^L v^L
\]

\[
+ (v^L)^T \tau^L_1 \otimes (d_N^L)^T (e_N^R)^T v^L - (v^L)^T \tau^L_2 \otimes (d_N^L)^T (e_0^R)^T v^R
\]

\[
+ (v^L)^T \tau^L_2 \otimes e_N^L d_N^L v^L - (v^L)^T \tau^L_2 \otimes e_N^L d_N^L v^R
\]

\[
\#2 = -(v^L)^T A^T \otimes M^T v^L + (v^L)^T (A^T + \tau^R_1)^T \otimes (d_N^R)^T (e_N^L)^T v^L
\]

\[
+ (v^L)^T (\tau^R_2)^T \otimes e_N^R d_N^R v^L - (v^L)^T (\tau^R_2)^T \otimes e_N^R d_N^R v^R
\]
#3 = (v^R)^T (I_2 \otimes H) (A \otimes H^{-1}(-M - e_0^R d_0^R + e_N^R d_N^R)) v^R + (v^R)^T (I_2 \otimes H)(\tau^R \otimes H^{-1} e_N^R d_N^R) v^R
+ (v^R)^T (I_2 \otimes H)(\tau^R \otimes H^{-1} (d_0^R)^T) ((I_2 \otimes (e_N^R)^T) v^R - (I_2 \otimes (e_N^R)^T) \nu^L)
+ (v^R)^T (I_2 \otimes H)(\tau^R \otimes H^{-1} e_0^R) ((I_2 \otimes d_0^R) v^R - (I_2 \otimes d_0^R) \nu^L)

= - (v^R)^T A \otimes M \nu^R - (v^R)^T A \otimes e_0^R d_0^R v^R + (v^R)^T A \otimes e_N^R d_N^R v^R + (v^R)^T \tau^R \otimes e_0^R d_0^R v^R
+ (v^R)^T \tau^R \otimes (d_0^R)^T (e_0^R)^T v^R - (v^R)^T \tau^R \otimes (d_0^R)^T (e_N^R)^T \nu^L
+ (v^R)^T \tau^R A \otimes e_0^R d_0^R v^R - (v^R)^T \tau^R A \otimes e_N^R d_N^R \nu^L.

#4 = - (v^R)^T A^T \otimes M \ast \nu^R + (v^R)^T (-A^T + (\tau^R)^T) \otimes (d_0^R)^T (e_0^R)^T
+ (v^R)^T ((\tau_1^R)^T \otimes d_0^R) \nu^L - (v^L)^T (\tau_1^R)^T \otimes e_N^R d_N^R \nu^R - (v^R)^T (\tau_2^R)^T \otimes (d_N^R)^T (e_0^R)^T v^R.

In total the energy estimate is (recall that by definition $M^T = M$)

$$\frac{d}{dt} (||v^L||_H + ||v^R||_H) = - (v^L)^T (A + A^T) \otimes M^T v^L - (v^R)^T (A + A^T) \otimes M v^R$$

$$+ (v^L)^T \left( A + A^T + (\tau_2^R)^T \otimes e_N^R d_N^R \right) v^L + (v^L)^T \left( (\tau_2^R)^T + \tau_1^R \otimes (d_N^R)^T (e_N^R)^T \right) v^L
+ (v^R)^T \left( A + A^T + (\tau_2^R)^T \otimes e_0^R d_0^R \right) v^R + (v^R)^T \left( (\tau_2^R)^T + \tau_1^R \otimes (d_0^R)^T (e_0^R)^T \right) v^R
- (v^R)^T \left( (\tau_2^R)^T \otimes (d_N^R)^T (e_N^R)^T \right) v^R - (v^R)^T \left( (\tau_2^R)^T + \tau_1^R \otimes e_N^R d_N^R \right) \nu^L
- (v^R)^T \left( (\tau_2^R)^T \otimes (d_0^R)^T (e_0^R)^T \right) \nu^L - (v^L)^T \left( (\tau_2^R)^T + \tau_1^R \otimes e_0^R d_0^R \right) \nu^L.$$

The penalty parameters are set to $\tau_0^L = A$, $\tau_0^R = -A$, $\tau_1^L = -\alpha_D I_2$, $\tau_2^R = \alpha_D I_2$ and $\tau_1^R = \tau_2^L = \begin{pmatrix} 0 & \gamma_D \\ -\gamma_D & 0 \end{pmatrix}$. Recall also that $A + A^T = \alpha_D + I_2$. This leads to the energy estimate found in Eq. 3.17:

$$\frac{d}{dt} (||v^L||_H + ||v^R||_H) = -\alpha_D \left( (v^L)^T I_2 \otimes M v^L + (v^R)^T I_2 \otimes M v^R \right) \leq 0,$$

which is $\leq 0$ due to the fact that $\alpha_D \geq 0$ and $M$ by definition is positive semi-definite. Hence the Scheme (3.15) is stable.
References


