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Dynamical modelling

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1 Introduction

My point of departure was the article *Foraging at the Edge of Chaos: Internal Clock versus External Forcing* [1]. Here nonlinear differential equations were used to model the behavior of foraging ants. Both external factors and the ants' interaction were incorporated in the model. An ant could be in one of two states: *active* of *inactive* and the transition rates between the two depend on the temperature (external factor) and on the proportion of active ants.

[1] In Physical Review Letters 110, 268104 (2013)

Instead of modeling behavior of ants, D. Helbings [2] focuses on human behavior.

My article is divided into two main parts. The first part is an exploration of quantitative sociodynamics, based on the first chapters of part 1 and part 2 of D. Helbings *Quantitative Sociodynamics.* As the ants, people are seen as individual participants that display a particular behavior and in interplay with each other constitute global processes. The second part consists of an analysis of a particular dynamical model. As in the model of foraging ants, the model analyzed also makes use of a division into active(, less active) and non-active participants. In addition to this, numerics are used to confirm the analytic results. This is done with Python. Before going on to the two main parts, I will first cover some basic theory of nonlinear dynamics and chaos.

2 Basic theory of nonlinear dynamics and chaos

A two-dimensional non-linear dynamical system consists of two non linear differential equations, one describing the growth (or decline) of a first variable and one equation for a second:

$$\begin{cases} \dot{x_1} &= f_1(x_1, x_2) \\ \dot{x_2} &= f_2(x_1, x_2) \end{cases}$$

A dynamical system can be visualized by a *phase plane*, which is a two-dimensional representation, where the first variable is put on the horizontal axis and the second on the vertical. Arrows indicate the direction of "movement" starting from the initial conditions indicated by the coordinates of the arrow. In this way, starting from a particular initial condition and "following" the arrows, results in an *trajectory.*

A *null cline* is a curve in the phase plane from which there is purely horizontal or purely vertical movement. This resembles the situation where $\dot{x_1} = 0$ or $\dot{x_2} = 0$. A null cline with no movement in the x_1 direction is typically called a x_1*-null cline* and, analogously, a x_2*-null cline* for a null cline with no movement in the x_2-direction.

Where a x_1-null cline intersects with a x_2-null cline, there is a *fixed point.* This means that starting from this point, there is neither movement in the x_1 direction, nor in the x_2-direction, so the trajectory starting from this

[2] D. Helbing, *Quantitative Sociodynamics*, 2nd ed., Springer-Verlag Berlin Heidelberg

point is just the point itself (i.e. it doesn't move at all).

Fixed points are described in terms of their *stability* and shape. With respect to stability, a fixed point is called *stable* if trajectories started in the vicinity of the point move *towards* it and *unstable* if trajectories move away from it.

The type of the fixed point can be obtained by calculating the jabobi matrix of the system and evaluating it for the particular fixed point. A jacobi matrix is in case of a two-dimensional dynamical system a 2x2 matrix with a set of derivatives in the following way:

$$\begin{pmatrix} \frac{\partial \dot{x_1}}{\partial x_1} & \frac{\partial \dot{x_1}}{\partial x_2} \\ \frac{\partial \dot{x_2}}{\partial x_1} & \frac{\partial \dot{x_2}}{\partial x_2} \end{pmatrix}$$

We use this Jacobian to turn a *non linear* problem into a *linear* one. This process is called *linearization.* (maybe refer to Taylor formula)

The *eigenvalues*, λ_1 and λ_2, of the matrix that results from this evaluation tell us about the behavior of the system near the fixed point in question. The two properties related to this are the *determinant* (Δ) of the matrix and the *trace* (τ). Their relations are described by the following formulas:[3]

$$\lambda_{1,2} = \frac{\tau^2 \pm \sqrt{\tau - 4\Delta}}{2} \qquad \Delta = \lambda_1 \lambda_2 \qquad \tau = \lambda_1 + \lambda_2$$

If the determinant is negative, the eigenvalues have to be real and of opposite sign. In that case we have a *saddle point*, i.e. approaching from one direction and leaving in one other direction. If the determinant is positive, we have a few different cases: if both eigenvalues are real and negative, we have a *sink* (stable node) and if they are real and positive, we have a *source* (unstable node). In case of complex conjugates with negative real part, we have a spiral sink and with positive real part a spiral source. Purely imaginary eigenvalues result in a *center*, which means that trajectories in the vicinity of the fixed point orbit around it. [4]

3 Theory and examples of sociodynamics

This section is based on D. Helbings *Quantitative Sociodynamics* of 1994, which deals with the mathematics used to describe and analyse social pro-

[3]Steven H. Strogatz, Nonlinear Dynamics and Chaos, chapter 5.2

[4]https://www.math.stonybrook.edu/ scott/mat308.spr11/TraceDet.pdf

cesses and phenomena.

In his introduction he stresses the growing importance of quantitative sociodynamics in the fields of social and economical sciences: in comparison to qualitative sociodynamics, a quantitative approach is more precise and makes more rigid mathematical formulations possible and therefore, more reliable conclusions and better forecasts can be made and hypotheses can be tested more easily. Also the quite recently developed sophisticated methods to describe more complex systems which include several elements interacting with each other is accentuated.

3.1 Theory

Essential models that are covered in the book are the logistic, diffusion, gravity and decision model(s) and also game theory.

Initially, exponential functions have been used to describe growth processes, but this proved only adequate in the very beginning of the growing process. The fact is, most of the phenomena don't increase unlimitedly, but slow down and converge to some state. The exponential model was therefore replaced by the logistic model, that after an initial period of excessive growth accordingly turned into a so called saturation phase.

The diffusion model is mostly about distribution processes (i.e.: the spreading of *something*) and the gravity model is about exchange processes which for instance has applications in migration.

In game theory one tries to model more complex phenomena such as social competition and cooperation, where the behavior of an individual is explained or predicted in terms of how beneficial a specific outcome for that individual is. In time there has been a shift from seeing this kind of social processes as an iteration of single games, to seeing them, with help of differential equations, as a *dynamical game process*. An interesting example of competing actors were the video systems *VHS* and *BETA MAX*. Game theory then focuses on what factors play a role in whether or not two competing actors can coexist and if not, what makes one of them *win*. A similar example is that of the *behavior of pedestrians* in crowded areas, where a preference for either the left or right side of the side walk develops when meeting other pedestrians.

As mentioned, it is also possible that a system never does result in a *winning* option. One cause can be so called spontaneous strategy changes, which can be understood as individuals - with a certain rate - not complying to the present majority but choosing another option. This aspect can be taken into account in mathematical models too. In case of a small number of these spontaneous strategy changes, the system yields a preferred option anyway (i.e. the few disturbances cannot prevent the development of a dominant option), with the other option(s) coexisting in a certain (lower) frequency. But when this critical value is surpassed, both (or more) options are equally probable and the system never results in a dominant alternative.

The decision models model making decisions and try to estimate the probability of specific outcomes. Here decision making is seen as a process of counting arguments in favor of the respective alternatives and opting for the one having the most. Hence: the more arguments in favor of a certain alternative, the higher its probability. How beneficial a certain outcome is, is coined as its *utility*.

I would argue that it is quite strange that only arguments in favor are being taken into account and not arguments against. Aren't cons of importance too? One alternative with more pros than another one, but with more negative effects, isn't the best candidate, is it? But maybe this is already incorporated in the model by seeing the disadvantages of a specific outcome as arguments in favor of alternatives without those disadvantages.

The multinomial logit model [5] is used for situations where there are more than two nominal outcomes that are not ordered.

Sometimes not all relevant information is accessible and this can result in quite some variation in the expected utility of a certain decision, thereby broadening the probability estimates. Also, decisions can depend on other factors, like personality, which too give rise to broader estimates.

Wanting to take previous decisions into account for making predictions on upcoming behavior gave rise to the development of *dynamical* decision models, which instead of probabilities for specific decisions, describe things in terms of probabilities of going from one alternative to another. These are called *transitional* probabilities.

Helbing claims that he will expound a general, mathematical and dynamical model model that treats all of the above mentioned models as special cases. I unfortunately cannot discuss all of them in detail.

3.2 Examples

Over to part 2 of the book about quantitative models of social processes, where the theory discussed in the first part will be applied. Helbing stresses the strong analogy between social en physical processes, which is, according to him, often just a matter of different terminology. He speaks of a *system* in the meaning of a *social* system, where its *elements* consist of the individuals taking part in the system and where each individual has certain properties. All these properties are summarized by the *state* x that will result in some kind of *behavior*. Usually some (nearly) invariable characteristics of individuals (like age, gender, socio-economic status and so forth) are used to partition the population into subgroups. Of course all subpopulations add up to the total number of the individuals involved. One assumes that persons within the same subpopulation have the same behavioral probabilities

[5] also known as multinomial logistic regression, http://en.wikipedia.org/wiki/Multinomial_logistic_regression

(but **not** necessarily the same *behavior*), because people in the same (social) environment tend to influence each other.

The following model makes this clear:

$$U_a(x,t) = B_a(x,t) + S(x,t)$$

In other words: the utility of behavior x for someone in group a at time t, is the sum of the own *attitude* towards behavior x plus the *social reinforcement* (group/peer pressure). A few things can happen to maximize the utility: one can choose the attitude one values the most, or one can move to an environment more in line with one's own beliefs and by that reducing the negative social pressure. Mental dissonance occurs when one chooses a behavior x' over a behavior x, despite the more positive attitude towards x. This eventually leads to re-evaluating behavior x' (i.e. more favorable), to reduce this dissonance. These phenomena explain why behavior in some social environment is more homogeneous. Helbing includes a clarifying figure which simply conveys that two befriended persons having two different opinions can result in two equal opinions, and two persons on bad terms having the same opinion can result in two different opinions.

The so called *socioconfiguration* is a vector:

$$n := (..., n_x^a, ...)^{tr}$$

where every n_x^a stands for the number of individuals in sample a exhibiting behavior x. This terminology will be used later. Typical to non-linear interactions is the phenomenon that actions of one individual can influence the system as a whole and in effect can influence the same individual. This is called the feedback effect, which eventually may result in *self-organization.*

The problems with modeling social processes stem from the fact that human behavior is determined by the brain and that brain processes are extremely complex. Fortunately a lot of variables can be neglected or grouped together, so that one ends up with just a few slowly changing variables that satisfactorily prescribe the behavior. [6] All the variables are divided into external influences Q_a and internal influences which are subdivided into slowly changing variables A_α., variables that describe the (macroscopic) behavior X_α. and the rest O_α. But, because of the faster time scale on which O_α operate, they are determined (*slaved*) by Q_α. This reduces the data load and comprises the complexity of the system considerably.

The transition rate from one behavior to another then boils down to:

[6] In technical terms it is said that these variables are *non-linearly coupled* and can be *adiabatically eliminated.* Helbing refers to the *slaving principle* in Haken H (1983), Advanced synergetics. Springer, Berlin

$w_\alpha(Q'_\alpha, X'_\alpha | Q_\alpha, X_\alpha; t)$

where A_α. and O_α are neglected.

In the case of opinion formation, external influences Q_α will consist of the behavior of other individuals:

$Q_\alpha = (X_1, ..., X_{\alpha-1}, X_{\alpha+1}, ..., x_N)^{tr}$

Where N is the total number of the population and where X_α is left out because that's the behavior of the individual in question.

The transition rate then translates into:

$$w_\alpha(Q'_\alpha, X'_\alpha | Q_\alpha, X_\alpha; t) = w_\alpha(X'_1, ..., X'_\alpha, ..., X'_N | X_1, ..., X_\alpha, ..., X_N; t)$$

Another problem is that every human being is unique and therefore no two persons show exactly the same behavior. But in order to still be able to reproduce studies and re-test models, one needs to use *simple models*. This involves the restriction to only a few situative variables and a few behavior options, ignoring the behavior in all other situations, as long as this doesn't have a systematic effect on the behavior in question [7]. This leads to a limited number of necessary behavioral types. One can say that the model is made more discrete: one can for example only vote for party A or B and not just any politician. This also leads to the situation that despite the fact that no two persons act alike, in the sense of the simple model, there are way more people acting the same. That's also why the transition rates now can be simplified to:

$w_\alpha(q', x' | q, x; t)$

Due to the fact that the internal variables (which we left out in the model) lead to fluctuations, we cannot predict behavior exactly, but we can construct stochastic models and probabilities for particular behaviors. This also takes into account the problem of disturbances in external influences and small individual variations.

Helbing describes Feger's model of decision making [8] where decision making is seen as a conflict situation, where some alternative of a series of excluding alternatives has to be chosen. The (estimated) importance of a decision influences the time willing to spend. Then, for every option the consequences are anticipated and valued. If k (where k depends on the im-

[7] As is the case in the later Facebook user model in the fourth section

[8] Feger H (ed) (1977) Studie zur intraindividuellen Konfliktforschung. Huber, Bern. Feger H (1978) Konflikterleben und Konfliktverhalten. Huber, Bern

portance of the decision) arguments are valued in favor of some decision, the decision is made. I have to say that I don't understand why the arguments in favor should be *successive*. Maybe this is some arbitrary way of structuring the decision process.

Let $U_\alpha(x|y,t)$ denote the preference for some individual α to choose alternative x at time t, given decision y. The probability of the occurrence of argument in favor of x is defined by:

$$\widehat{p}_\alpha(x|y;t) := \frac{e^{U_\alpha(x|y;t)}}{\sum\limits_{x'} e^{U_\alpha(x'|y;t)}} := e^{\widehat{U}_\alpha(x|y,t)}$$

The expression in the middle can be interpreted as the preference for alternative x, divided by the sum of all preferences for all alternatives. $\widehat{U}_\alpha$ now represents scaled *scaled* preferences. The use of the natural logarithm transfers the property of being monotonically increasing to the model.

The probability that the next k arguments are in favor of behavior x then becomes:
$p'_\alpha(x|y;t) = [\widehat{p}_\alpha(x|y;t)]^k = e^{k\widehat{U}_\alpha(x|y,t)}$

The probability of an option x, given behavior y, at time t, then becomes:

$$p_\alpha(x|y,t) := \frac{e^{k\widehat{U}_a(x|y,t)}}{\sum\limits_{x'} e^{k\widehat{U}_a(x'|y,t)}} = e^{U'_\alpha(x|y;t)}$$

Which can be interpreted as: the probability of the next k arguments being in favor of x, divided by the sum of all probabilities of the next k arguments being in favor of all other alternatives (which is represented with x' below the summation). $U'_\alpha(x|y;t)$ is the *attractiveness* of decision x for person α at time t, with previous decision y.

In this section I gave a quick introduction to dynamical models in the realm of sociology. I covered the most important models and illustrated some of these with some examples. In the next section I will treat my own example of a sociodynamic model following the same principles albeit a little less complex.

4 Analysis of a model: Facebook users

4.1 Introduction

The example of a dynamical model is a model used to describe the behavior and interaction of different user types on the social medium Facebook.

Facebook users in this model are one of three types:

x_1	:	readers (only read posts of others but don't post something themselves)
x_2	:	posters (the ones who actually write/share stuff)
s	:	sleepers (the ones not logged in)

The model uses the following three processes:

$$\begin{cases} x_1 + x_2 & \stackrel{k_1}{\rightarrow} 2x_2 \\ x_1 + s & \stackrel{k_2}{\rightarrow} 2x_1 \\ x_2 + s & \stackrel{k_3}{\rightarrow} 2s \end{cases}$$

Where the first equations means that with a factor k_1 one reader and one poster turn into two posters (i.e.: the reader gets activated to post too, because of having read a post of the poster). The second one represents the event of one sleeper and one reader with factor k_2 turning into two readers (i.e.: the sleeper logs in). The third one represents the event of one poster and one sleeper with factor k_3 resulting in two sleepers (i.e.: the poster logs out)

The associated differential equation system is:

$$\begin{cases} \dot{x_1} & = -k_1 x_1 x_2 + k_2 x_1 s \\ \dot{x_2} & = k_1 x_1 x_2 - k_3 x_2 s \\ \dot{s} & = -k_2 x_1 s + k_3 x_2 s \end{cases}$$

For the simple reason that a sleeper cannot have an encounter with a reader, it is maybe better to first change the second equation into $s \stackrel{k_3}{\rightarrow} x_1$ (i.e.: with a rate k_2 the sleeper turns into a reader.) In this way a sleeper logs in without interference of another person. The third equation isn't affected since a poster who wants to talk to someone and finds this person off line, can because of this decide to go off line too. I will first discuss this simpler model.

4.2 The first model

The discussion in the previous paragraph yields the following model:

$$\begin{cases} x_1 + x_2 & \stackrel{k_1}{\rightarrow} 2x_2 \\ s & \stackrel{k_2}{\rightarrow} x_1 \\ x_2 + s & \stackrel{k_3}{\rightarrow} 2s \end{cases}$$

The differential equation system then gets:

$$\begin{cases} \dot{x_1} &= -k_1x_1x_2 + k_2s \\ \dot{x_2} &= k_1x_1x_2 - k_3x_2s \\ \dot{s} &= -k_2s + k_3x_2s \end{cases}$$

When viewing the values of x_1 , x_2 and s as the proportions of the total number of facebook users, it is apparent that $s + x_1 + x_2 = 1$.

Since we in this model have a fixed number of facebook users who alternate between being a reader, poster or sleeper, we must have that the sum of the growth of x_1, x_2 and s has to be zero since in our model. So:

$$\dot{x_1} + \dot{x_2} + \dot{s} = 0$$

$$\begin{cases} \dot{x_1} &= -k_1x_1x_2 + k_2(1 - x_1 - x_2) \\ \dot{x_2} &= k_1x_1x_2 - k_3x_2(1 - x_1 - x_2) \end{cases}$$

We see that $(1, 0)$ is an answer to $\begin{cases} \dot{x_1} = 0 \\ \dot{x_2} = 0 \end{cases}$

(because when we put $x_2 = 0$ the system gets:)

$$\begin{cases} \dot{x_1} = k_2(1 - x_1) \\ \dot{x_2} = 0 \end{cases} \Leftrightarrow k_2 - k_2x_1 = 0 \Leftrightarrow k_2 = k_2x_1 \Leftrightarrow x_1 = 1$$

This describes the situation where there are only readers. Since there are no posters left, readers cannot turn into posters (in line with the first arrow equation) and since there are no sleepers left either, a poster cannot turn into a sleepers (in line with the third arrow equation)

The analogue holds for (0,1): Put $x_1 = 0$:

$$\begin{cases} \dot{x_1} = k_2(1 - x_2) \\ \dot{x_2} = k_3x_2(1 - x_2) \end{cases} \Leftrightarrow x_2 = 1$$

Which describes the situation of only posters, who, since there are no sleepers left, cannot turn into a sleeper themselves.

4.2.1 Calculation of null clines and fixed points

To give a qualitative description of the behaviour, we now want to calculate the null clines:

$\dot{x_1} = -k_1x_1x_2 + k_2(1 - x_1 - x_2) = 0$
$\Leftrightarrow k_2(1 - x_1 - x_2) = k_1x_1x_2$
$\Leftrightarrow k_2 - k_2x_1 - k_2x_2 = k_1x_1x_2$
$\Leftrightarrow k_2 - k_2x_1 = (k_1x_1 + k_2)x_2$
$\Leftrightarrow \frac{k_2 - k_2x_1}{k_1x_1 + k_2} = x_2$

$\Leftrightarrow \frac{k_2(1-x_1}{k_1x_1+k_2} = x_2$
$\Leftrightarrow x_2 = \frac{1-x_1}{\frac{k_1}{k_2}x_1+1}$
Which is a hyperbolic function that goes through the points $(0,1)$ and $(1,0)$

$\dot{x_2} = k_1x_1x_2 - k_3x_2(1-x_1-x_2) = 0$
Has a solution for
$x_2 = 0$
Which then gives us:
$k_1x_1 - k_3(1-x_1-x_2) = 0$
$\Leftrightarrow \frac{k_1x_1}{k_3} = 1 - x_1 - x_2$
$\Leftrightarrow x_2 = -\frac{k_1x_1}{k_3} - x_1 + 1 = -\frac{k_1x_1+k_3(-x_1+1)}{k_3}$
$\Leftrightarrow x_2 = -\frac{k_1}{k_3}x_1 + (-x_1+1) = (-\frac{k_1}{k_3} - 1)x_1 + 1$
Which is a straight line that always goes through $(0,1)$ and has a slope in the interval $(-\infty, -1)$ which is closer to -1 if $k_1 << k_3$ and closer to $-\infty$ if $k_1 >> k_3$.

The hyperbolic curve (where $\dot{x_1} = 0$) and the straight line (where $\dot{x_2} = 0$) clearly intersect at $(0,1)$. The hyperbolic curve clearly also intersects with $x_2 = 0$ (where $\dot{x_2} = 0$). These are two equilibrium points.

In order to see if there are more equilibrium points, we analyze the situation where $\dot{x_1} = \dot{x_2} = 0$:

$\dot{x_1} = -k_1x_1x_2 + k_2(1-x_1-x_2) = 0$
$\Leftrightarrow k_1x_1x_2 = k_2(1-x_1-x_2)$
$\Rightarrow \dot{x_2} = k_2(1-x_1-x_2) - k_3x_2(1-x_1-x_2) = 0$
$\Rightarrow k_2 - k_3x_2 = 0$
$\Rightarrow x_2 = \frac{k_2}{k_3}$
$\dot{x_1} = -\frac{k_1k_2}{k_3}x_1 + k_2(1-x_1-\frac{k_2}{k_3}) = 0$
$\Leftrightarrow (\frac{k_1k_2}{k_3} + k_2)x_1 = k_2 - \frac{k_2^2}{k_3}$
$\Leftrightarrow (\frac{k_1}{k_3} + 1)x_1 = 1 - \frac{k_2}{k_3}$
$\Leftrightarrow x_1 = \frac{(1-\frac{k_2}{k_3})}{(\frac{k_1}{k_3}+1)} = \frac{k_3-k_2}{k1+k_3}$

So according to the previous calculations we have an equilibrium point at $(\frac{k_3-k_2}{k1+k_3}, \frac{k_2}{k_3})$. This makes sense since the straight line and the hyperbolic function could intersect.

This means that we have three fixed points when $k_2 < k_3$ and two when $k_2 > k_3$.

4.2.2 Analysis of fixed points

To study the direction of the phase plane in the fixed points, we calculate the Jacobian matrix and examine it in the points in question: $(0,1)$, $(1,0)$ and $(\frac{k_3-k_2}{k1+k_3}, \frac{k_2}{k_3})$

The Jacobian is:

$$\begin{pmatrix} -k_1x_2 - k_2 & -k_1x_1 - k_2 \\ k_1x_2 + k_3x_2 & k_1x_1 - k_3 + k_3x_1 + 2k_3x_2 \end{pmatrix}$$

The point (0,1)

At $(0,1)$ the Jacobian turns into:

$$\begin{pmatrix} -k_1 - k_2 & -k_2 \\ k_1 + k_3 & k_3 \end{pmatrix}$$

Which has determinant

$$\Delta = (-k_1 - k_2)k_3 + (k_1 + k_3)k_2$$
$$= -k_1k_3 - k_2k_3 + k_1k_2 + k_3k_2$$
$$= -k_1k_3 + k_1k_2$$
$$= k_1(k_2 - k_3)$$
$$\Rightarrow \begin{cases} \Delta > 0 \text{ if } k_2 > k_3 \\ \Delta < 0 \text{ if } k_2 < k_3 \end{cases}$$

and trace

$$\tau = -k_1 - k_2 + k_3$$
$$\Rightarrow \begin{cases} \tau > 0 \text{ if } k_3 > k_1 + k_2 \\ \tau < 0 \text{ if } k_3 < k_1 + k_2 \end{cases}$$

We know that the third point $(\frac{k_3-k_2}{k1+k_3}, \frac{k_2}{k_3})$ only exists if $k_2 < k_3$ (for then the point is in the first quadrant). It then follows that in the case of three fixed points, where $k_2 < k_3$, the determinant is negative. The trace is positive or negative, depending on the conditions above, but if the determinant is negative, we can already be sure we have a saddle point at $(0,1)$. So for $k_2 < k_3$ $(0,1)$ is a saddle point.

If we are in the second case of only two fixed points, where $k_2 > k_3$, the determinant is positive and the trace is negative. A positive determinant and negative trace gives rise to a stable spiral or a stable node. Since:

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

With a positive Δ and negative τ the two eigen values are either both real and negative (giving rise to a node), or complex conjugates with negative real parts (giving rise to a spiral). In order to determine which one, we look at the behavior of $\tau^2 - 4\Delta$. If this is positive we are in the first case, if negative we are in the second case:

$$
\begin{aligned}
&\tau^2 - 4\Delta = (-k_1 - k_2 + k_3)^2 - 4(k_1(k_2 - k_3)) \\
&= k_1^2 + k_2^2 + k_3^2 + 2k_1k_2 - 2k_1k_3 - 2k_2k_3 - 4k_1k_2 + 4k_1k_3 \\
&= k_1^2 + k_2^2 + k_3^2 - 2k_1k_2 + 2k_1k_3 - 2k_2k_3
\end{aligned}
$$

Fixing k_1 and k_3, we can solve the quadratic equation in terms of k_2:

$$
\begin{aligned}
&k_2^2 - 2(k_1 + k_3)k_2 + k_1^2 + k_2^2 + k_3^2 = 0 \\
&\Leftrightarrow k_2 = \tfrac{2(k_1+k_3)\pm\sqrt{4(k_1+k_3)^2-4k_1^2-4k_3^2-8k_1k_3}}{2} = k_1 + k_3
\end{aligned}
$$

This means that if $k_2 > k_1 + k_3$, $(0, 1)$ is a stable node and if $k_2 < k_1 + k_3$, it is a stable spiral.

Finally we check the behavior on the x_2-axis: $\begin{cases} \dot{x_1} &= k_2(1 - x_2) = 0 \\ \dot{x_2} &= -k_3x_2(1 - x_2)0 \end{cases}$

$\Leftrightarrow x_2 = 1$ is the turning point so:

If $x_2 > 1$, $\begin{cases} \dot{x_1} &= k_2(1 - x_2) < 0 \\ \dot{x_2} &= -k_3x_2(1 - x_2) > 0 \end{cases}$

And if $x_2 < 1$, $\begin{cases} \dot{x_1} &= k_2(1 - x_2) > 0 \\ \dot{x_2} &= -k_3x_2(1 - x_2) < 0 \end{cases}$

So just above $(0, 1)$ the direction is to the left and up and just below $(0, 1)$ this is to the right and down.

The point (1,0)

At $(1, 0)$ the Jacobi matrix turns into:

$$\begin{pmatrix} -k_2 & -k_1 - k_2 \\ 0 & k_1 \end{pmatrix}$$

Which has eigen values $\lambda_1 = -k_2 < 0$ and $\lambda_2 = k_1 > 0$ and therefore $(1, 0)$ is a saddle point (since the determinant is negative), both for the case $k_2 < k_3$ and $k_2 > k_3$

For this saddle point, we want to determine the direction of the arrows. First we study the behavior at the x_1-axis, where $x_2 = 0$. We then have:

$$\begin{cases} \dot{x_1} &= k_2(1 - x_1) \\ \dot{x_2} &= 0 \end{cases}$$

So that to the right of $(1,0)$ $\dot{x_1} < 0$ and to the left $\dot{x_1} > 0$. So on the x_1-axis, movement is *towards* the point $(1,0)$.

In the vertical direction (the direction of x_2), we have that $x_1 = 1$, so that:

$$\begin{cases} \dot{x_1} &= -k_1x_2 + k_2(-x_2) = (-k_1 - k_2)x_2 \\ \dot{x_2} &= k_1x_2 - k_3x_2(-x_2) = k_1x_2 + k_3x_2^2 \end{cases}$$

From this we can conclude that above $(1,0)$, $\dot{x_2} > 0$, but beneath $(1,0)$ this depends on the values of k_1 and k_3, if k_1 is not too small compared to k_3, $\dot{x_2} < 0$, so that in the vertical direction the arrows move *away* from $(1,0)$. We can also conclude that $\dot{x_1} < 0$ above $(1,0)$ and $\dot{x_1} > 0$ beneath $(1,0)$.

The point $(\frac{k_3-k_2}{k1+k_3}, \frac{k_2}{k_3})$

At $(\frac{k_3-k_2}{k1+k_3}, \frac{k_2}{k_3})$ the Jacobi matrix turns into:

$$\begin{pmatrix} \frac{-k_1k_2}{k_3} - k_2 & \frac{-k_1(k_3-k_2)}{k_1+k_3} - k_2 \\ \frac{k_1k_2}{k_3} + \frac{k_3k_2}{k_3} & \frac{k_1(k_3-k_2)}{k_1+k_3} - k_3 + \frac{k_3(k_3-k_2)}{k_1+k_3} + \frac{2k_3k_2}{k_3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-k_1k_2-k_3k_2}{k_3} & \frac{-k_1(k_3-k_2)-k_2(k_1+k_3)}{k_1+k_3} \\ \frac{k_1k_2+k_3k_2}{k_3} & \frac{(k_1+k_3)(k_3-k_2)}{k_1+k_3} - k_3 + 2k_2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-k_1k_2-k_3k_2}{k_3} & \frac{-k_1k_3-k_2k_3}{k_1+k_3} \\ \frac{k_2(k_1+k_3)}{k_3} & k_2 \end{pmatrix}$$

The trace is:

$\tau = \frac{-k_1k_2-k_3k_2}{k_3} + \frac{k_2k_3}{k_3} = \frac{-k_1k_2}{k_3} < 0$

So the trace is negative.

The determinant is:

$\Delta = \frac{-k_1k_2^2-k_3k_2^2}{k_3} + (\frac{k_1k_3+k_2k_3}{k_1+k_3})(\frac{k_2(k_1+k_3)}{k_3})$

$= \frac{-k_1k_2^2-k_3k_2^2}{k_3} + k_2(k_1 + k_2)$

$= \frac{-k_1k_2^2-k_3k_2^2+k_3k_2^2+k_1k_2k_3}{k_3}$

$= \frac{-k_1k_2^2+k_1k_2k_3}{k_3}$

$= \frac{k_1k_2(k_3-k_2)}{k_3}$

$$\Rightarrow \begin{cases} \Delta > 0 \text{ if } k_3 > k_2 \\ \Delta < 0 \text{ if } k_3 < k_2 \end{cases}$$

Since we know from earlier that $k_2 < k_3$, the determinant has to be positive.

A positive determinant and negative trace gives rise to a stable spiral or a stable node. Since:

$$\lambda_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}$$

With a positive Δ and negative τ the two eigen values are either both real and negative (giving rise to a node), or complex conjugates with negative real parts (giving rise to a spiral). In order to determine which one, we look at the behavior of $\tau^2 - 4\Delta$. If this is positive we are in the first case, if negative we are in the second case:

$$\tau^2 - 4\Delta = (\frac{-k_1k_2}{k_3})^2 - 4\frac{k_1k_2(k_3-k_2)}{k_3}$$
$$= \frac{k_1^2k_2^2}{k_3^2} - 4\frac{k_1k_2k_3 - k_1k_2^2}{k_3}$$
$$= \frac{k_1^2k_2^2 - 4k_1k_2k_3^2 - 4k_1k_2^2k_3}{k_3}$$
$$= \frac{k_1k_2 - 4k_3^2 - 4k_2k_3}{k_1k_2k_3}$$

which is positive or negative if the enumerator of the fraction is:

$$k_1k_2 - 4k_3^2 - 4k_2k_3 > 0$$
$$\Leftrightarrow k_1k_2 - 4k_3(k_3 + k_2) > 0$$
$$\Leftrightarrow k_1k_2 > 4k_3(k_3 + k_2)$$
$$\Leftrightarrow k_1 > \frac{4k_3(k_3+k_2)}{k_2}$$

Or, in terms of k_2:

$$\Leftrightarrow k_1k_2 - 4k_3^2 - 4k_2k_3 > 0$$
$$\Leftrightarrow k_1k_2 - 4k_2k_3 > 4k_3^2$$
$$\Leftrightarrow k_2(k_1 - 4k_3) > 4k_3^2$$
$$\Leftrightarrow k_2 > \frac{4k_3^2}{k_1 - 4k_3}$$

So if $k_2 > \frac{4k_3^2}{k_1-4k_3}$ the eigenvalues are real and negative and we have a stable node, and if $k_2 < \frac{4k_3^2}{k_1-4k_3}$ the eigenvalues are complex conjugates with negative real part and then we have a stable spiral.

Finally we look at the behavior of the model on the x_1-axis to determine the directions of the arrows in the saddle point.

On the x_1-axis, where $x_2 = 0$:

$$\begin{cases} \dot{x_1} &= k_2(1 - x_1) = 0 \\ \dot{x_2} &= 0 \end{cases}$$

$k_2(1 - x_1) = 0$

$\Leftrightarrow$ $x_1 = 1$ is the turning point. Assuming that all parameters k_1, k_2 and $k_3 > 0$, we get that:
if $x_1 > 1$, $\dot{x_1} = k_2(1 - x_1) < 0$ and
if $x_1 < 1$, $\dot{x_1} = k_2(1 - x_1) > 0$

So on the x_1-axis, movement is *towards* the saddle point and therefore move in the vertical direction is *away* from the point.

In summary:
for $k_2 < k_3$ the system has *three* fixed points: $(1, 0)$ (saddle), $(0, 1)$ (saddle) and $(\frac{k_3 - k_2}{k1 + k_3}, \frac{k_2}{k_3})$ (stable node if $k_2 > \frac{4k_3^2}{k_1 - 4k_3}$ and stable spiral if $k_2 < \frac{4k_3^2}{k_1 - 4k_3}$). In intuitive terms this means that except for the situation with no posters ($x_2 = 0$), which results in all readers, the situation with all posters ($x_2 = 1$), which stays the same, every valid proportion of readers, posters and sleepers results in $\frac{k_3 - k_2}{k1 + k_3}$ readers, $\frac{k_2}{k_3}$ posters and $1 - \frac{k_3 - k_2}{k1 + k_3} - \frac{k_2}{k_3}$ sleepers.

For $k_2 > k_3$, the system has *two* fixed points: (1,0) (saddle), (0,1) (stable node if $k_2 > k_1 + k_3$ and stable spiral if $k_2 < k_1 + k_3$). In intuitive terms this means that except for the situation with no posters ($x_2 = 0$), which results in all readers, every valid proportion of readers, posters and sleepers results in all posters.

4.2.3 Graphs

Combining all the information above, this yields the following phase plane in Figure 1. Note that the arrows are not on scale (they don't resemble the speed of the movement). The dark dot indicates stable point and the white dots indicate the saddle point. The null clines are also drawn. The thicker line resembles the null cline for x_1.

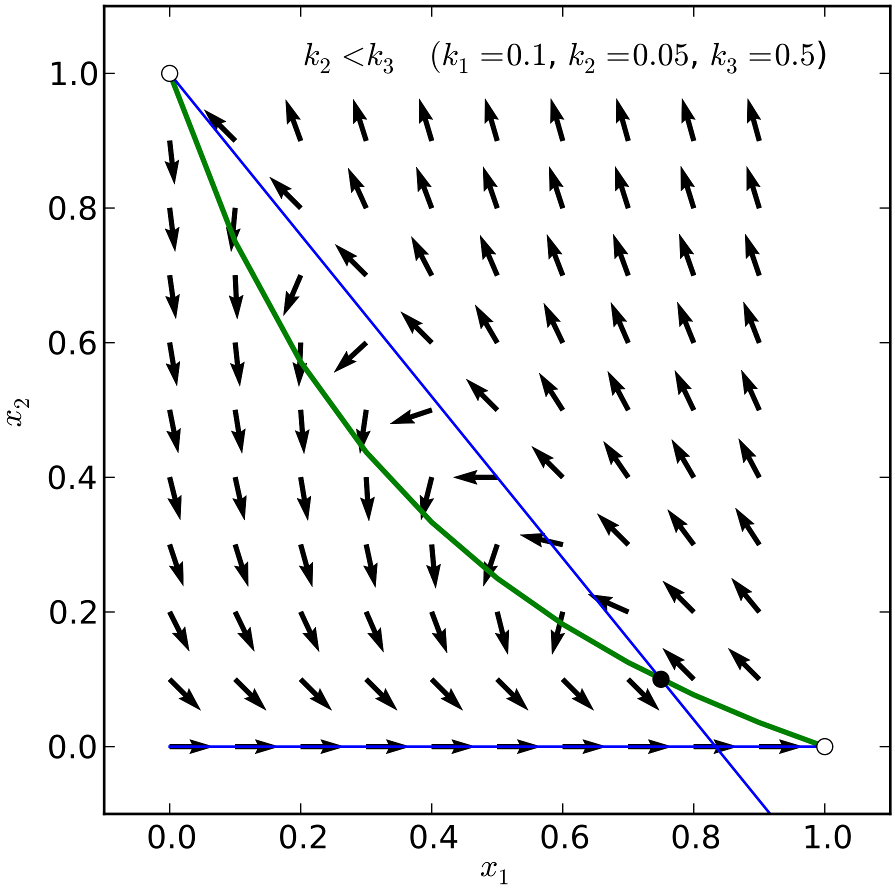

Figure 1: phase plane for the first case of the first model, where $k_2 < k_3$: $k_1 = 0.1$, $k_2 = 0.05$ and $k_3 = 0.5$.

Now we turn to the case where $k_2 > k_3$. In this case we only have the two fixed points at $(0, 1)$ and $(1, 0)$. The eigen values of the Jacobi matrix at $(1, 0)$ are still of different sign, independent of whether $k_2 < k_3$ or not, so this point is still a saddle point and has the same behavior as in the previous case. At $(0, 1)$ however, since:

$$\begin{cases} \Delta > 0 \text{ if } k_2 > k_3 \\ \Delta < 0 \text{ if } k_2 < k_3 \end{cases}$$
$$\begin{cases} \tau > 0 \text{ if } k_3 > k_1 + k_2 \\ \tau < 0 \text{ if } k_3 < k_1 + k_2 \end{cases}$$
we now have to do with a stable point, since $k_2 > k_3$ implies that $\Delta > 0$ and $\tau < 0$.

The information above is combined to find the following phase plane in Figure 2 (the behavior on the null clines themselves and on the axis is the same as in the previous situation).

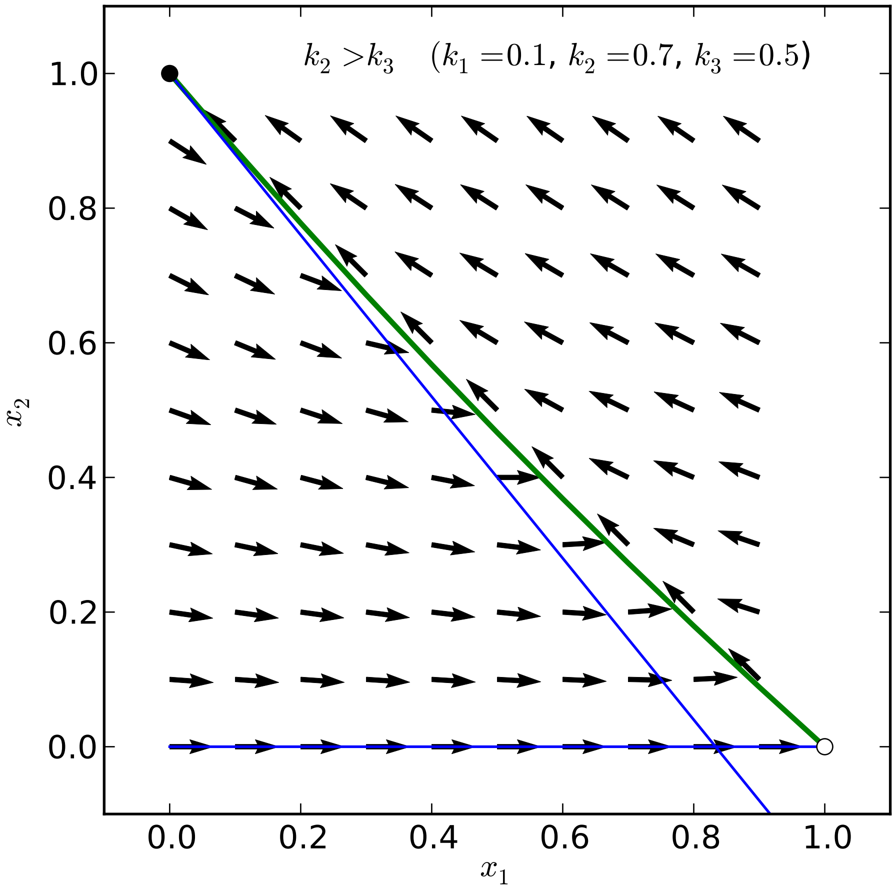

Figure 2: phase plane for the second case of the first model, where $k_2 > k_3$: $k_1 = 0.1$, $k_2 = 0.7$ and $k_3 = 0.5$.

The different phase planes depending on if $k_2 < k_3$ or $k_3 > k_2$ are summarized in the bifurcation diagram in Figure 3, where a dashed line represents an unstable fixed point and a continuous line a stable one.

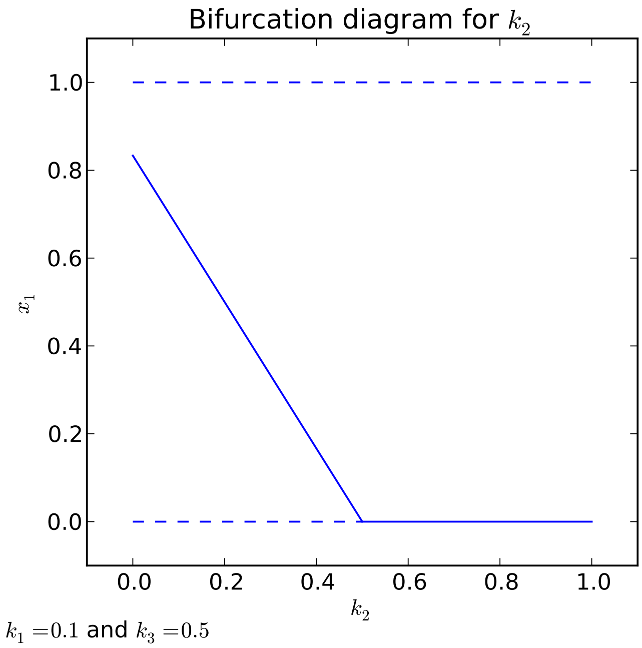

Figure 3: Bifurcation diagram for k_2 in the first model. $k = 0.1$, $k_3 = 0.5$.

This is verified with the ode-solver of Python, as shown in Figure 4. In the case of $k_2 > k_3$, initial values for x_1 and x_2 converge to the point $(0, 1)$, except when $x_2 = 0$, then it converges to $(1, 0)$, our saddle point. If the initial values are in the upper right half of the square $(0, 1), (1, 0)$, i.e. if $x_1 + x_2 > 1$, this still converges to $(0, 1)$, but in this case x_2 trespasses the maximum value of 1. Note that these initial values are not of interest to us since we had the condition that $x_1 + x_2 + s = 1$ with all values non-negative, so $x_1 + x_2 > 1$ wouldn't occur.

The same verification is done for the case where $k_2 < k_3$ as shown in Figure 5. Here, an initial condition starting at $x_2 = 0$ will once again converge to the saddle point $(1, 0)$. But now, all the other initial conditions will converge to the stable point at $(\frac{k_3-k_2}{k_1+k_3}, \frac{k_2}{k_3})$, which in the run case $k_1 = 0.5$, $k_2 = 0.05$, $k_3 = 0.1$, comes down to $x_1 = \frac{k_3-k_2}{k_1+k_3} = \frac{0.05}{0.6} = 0.0833$ and $x_2 = \frac{k_2}{k_3} = \frac{0.05}{0.1} = 0.5$. Again, for initial conditions in the upper right half, x_2 will trespass 1, but this is not of interest to us because of the beforementioned condition.

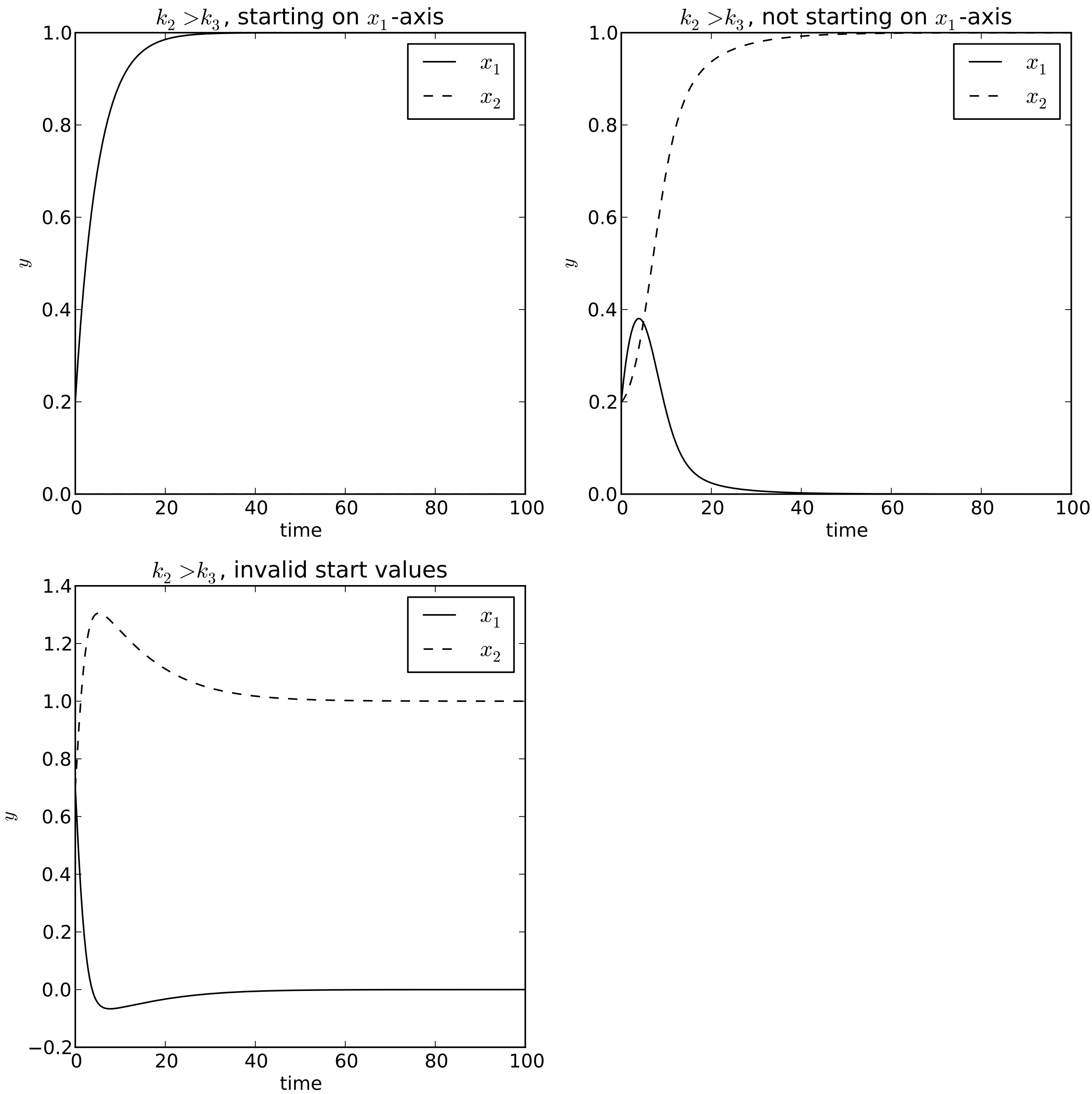

Figure 4: Different plots of Python's ode-solver. First main case: $k_2 > k_3$ Parameters: $k_1 = 0.5$, $k_2 = 0.2$, $k_3 = 0.1$. Start values: upper left: $(0.2, 0.0)$, upper right: $(0.2, 0.2)$, down: $(0.7, 0.7)$

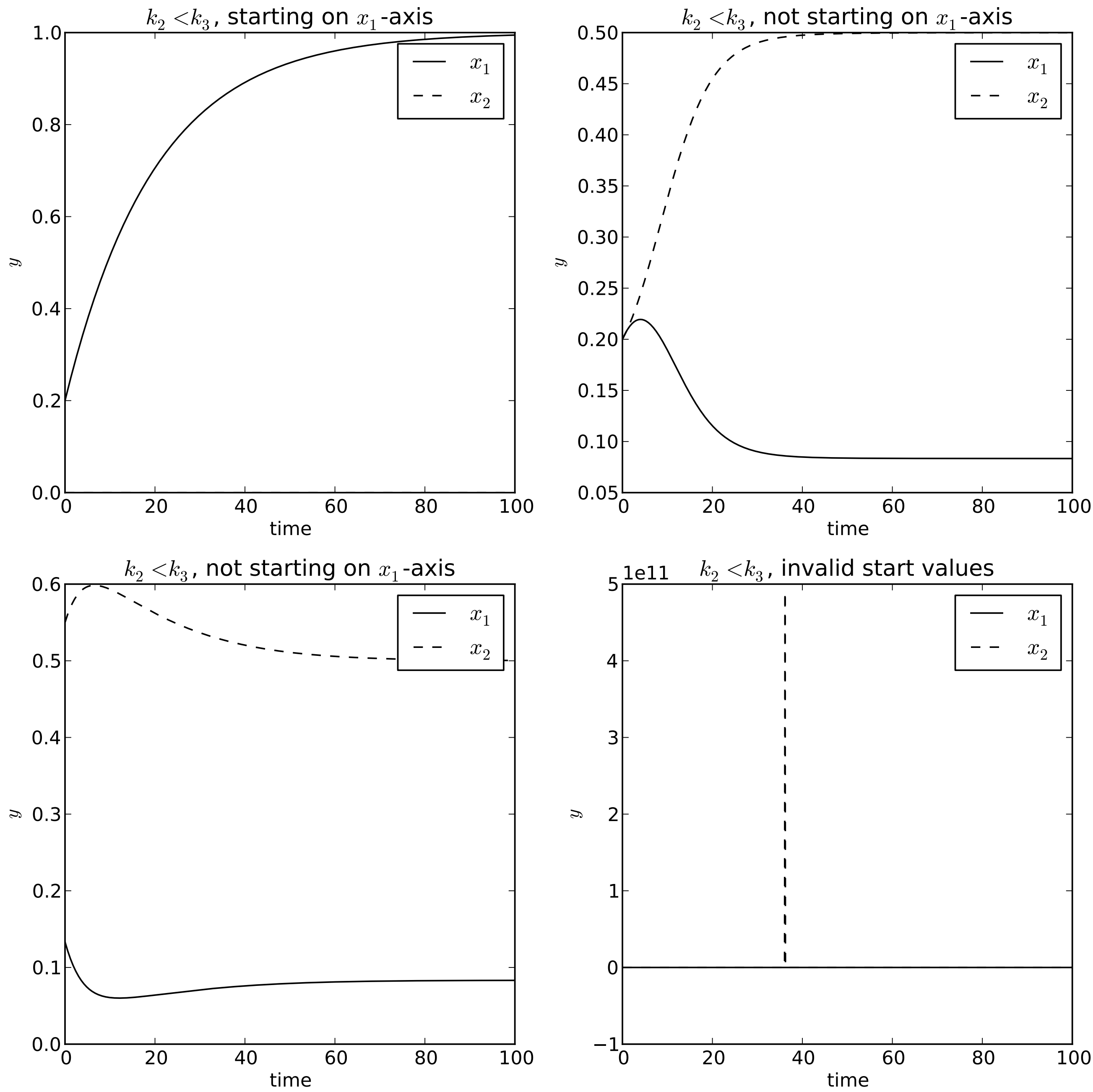

Figure 5: Different plots of Python's ode-solver. Second main case: $k_2 < k_3$ Parameters: $k_1 = 0.5$, $k_2 = 0.05$, $k_3 = 0.1$. Start values: upper left: $(0.2, 0.0)$, upper right: $(0.2, 0.2)$, down left: $(0.1333, 0.55)$, down right: $(0.3, 0.8)$

4.3 The second model

Let's return to the (more complex) model we started with:

$$\begin{cases} \dot{x_1} &= -k_1x_1x_2 + k_2x_1(1-x_1-x_2) \\ \dot{x_2} &= k_1x_1x_2 - k_3x_2(1-x_1-x_2) \end{cases}$$

We see immediately that this has solutions $(0,0)$, $(1,0)$ and $(0,1)$.

4.3.1 Calculation of null clines and fixed points

In determining the null clines it is obvious that $\dot{x_2} = 0$ has the same solutions, because nothing has changed:

$x_2 = 0$, and
$x_2 = (-\frac{k_1}{k_3} - 1)x_1 + 1$

For $\dot{x_1} = 0$ we get:
$\dot{x_1} = -k_1x_1x_2 + k_2x_1(1-x_1-x_2)$
Which obviously had the solution $x_1 = 0$.

Furthermore:
$k_1x_1x_2 = k_2x_1(1-x_1-x_2)$
$\Leftrightarrow k_1x_2 = k_2(1-x_1-x_2)$
$\Leftrightarrow (k_1+k_2)x_2 = k_2(1-x_1)$
$\Leftrightarrow x_2 = \frac{k_2}{k_1+k_2}(1-x_1)$

Because of the possibles slopes (because of the properties of the parameters) of the null clines that are not the axes, they have to intersect and we have a fourth fixed point. We find this by equating the first equation to zero and substituting some convenient term into the second equation:

$\dot{x_1} = 0$ gives:
$k_1x_1x_2 = k_2x_1(1-x_1-x_2)$
$\Leftrightarrow k_1x_2 = k_2(1-x_1-x_2)$
$\Leftrightarrow -k_2 + (k_1+k_2)x_2 = -k_2x_1$
$\Leftrightarrow x_1 = (-\frac{k_1}{k_2} - 1)x_2 + 1$

$\dot{x_2}$
$= k_1[(-\frac{k_1}{k_2} - 1)x_2 + 1]x_2 - k_3x_2(1 - [(-\frac{k_1}{k_2} - 1)x_2 + 1] - x_2)$
$= k_1(-\frac{k_1}{k_2} - 1)x_2^2 + k_1x_2 - k_3x_2 + k_3x_2[(-\frac{k_1}{k_2} - 1)x_2 + 1] + k_3x_2^2$
$= k_1(-\frac{k_1}{k_2} - 1)x_2^2 + (k_1 - k_3)x_2 + k_3x_2 + k_3(-\frac{k_1}{k_2} - 1)x_2^2 + k_3x_2^2$
$= k_1(-\frac{k_1}{k_2} - 1)x_2^2 + k_1k_2 + (-\frac{k_1k_3}{k_2} - k_3)x_2^2 + k_3x_2^2$

$= (-\frac{k_1^2}{k_2} - 1)x_2^2 + k_1k_2 + (-\frac{k_1k_3}{k_2})x_2^2$
$= (\frac{-k_1^2 - k_1k_3 - k_2}{k_2})x_2^2 + k_1x_2 = 0$
$\Leftrightarrow (\frac{-k_1 - k_3}{k_2} - 1)x_2^2 + x_2 = 0$

$x_2 = 0$ and
$x_2 = \frac{-1-\sqrt{1}}{\frac{-2k_1-2k_3}{k_2} - 2)} = \frac{-2}{\frac{-2k_1-2k_3}{k_2} - 2)} = \frac{-2}{\frac{-2k_1-2k_3-2k_2}{k_2})} = \frac{-2k_2}{-2k_1-2k_3-2k_2} = \frac{k_2}{k_1+k_2+k_3}$

$x_1 = (\frac{-k_1}{k_2} - 1)x_2 + 1$
$= (\frac{-k_1}{k_2} - 1)\frac{k_2}{k_1+k_2+k_3} + 1$
$= (\frac{-k_1-k_2}{k_2})\frac{k_2}{k_1+k_2+k_3} + 1$
$= \frac{-k_1-k_2}{k_1+k_2+k_3} + \frac{k_1+k_2+k_3}{k_1+k_2+k_3}$
$= \frac{k_3}{k_1+k_2+k_3}$

Which yields the point for the intersection to be $(\frac{k_3}{k_1+k_2+k_3}, \frac{k_2}{k_1+k_2+k_3})$

4.3.2 Analysis of fixed points

We study the behavior in the fixed points by again acquiring the Jacobian and evaluating at the fixed points:

$$\begin{pmatrix} -k_1x_2 + k_2 - 2k_2x_1 - k_2x_2 & -k_1x_1 - k_2x_1 \\ k_1x_2 + k_3x_2 & k_1x_1 - k_3 + k_3x_1 + 2k_3x_2 \end{pmatrix}$$

(The second row being identical to the previous Jacobi matrix.)
$$= \begin{pmatrix} (-k_1 - k_2)x_2 + k_2 - 2k_2x_1 & (-k_1 - k_2)x_1 \\ (k_1 + k_3)x_2 & (k_1 + k_3)x_1 - k_3 + 2k_3x_2 \end{pmatrix}$$

Evaluating at $(0, 0)$ yields:
$\begin{pmatrix} k_2 & 0 \\ 0 & -k_3 \end{pmatrix}$
Where the eigenvalues are of opposite sign, so at $(0, 0)$ we have a saddle point.

Evaluating at $(1, 0)$ yields:
$\begin{pmatrix} k_2 - 2k_2 & -k_1 - k_2 \\ 0 & k_1 - k_3 + k_3 \end{pmatrix} = \begin{pmatrix} -k_2 & -k_1 - k_2 \\ 0 & k_1 \end{pmatrix}$
With eigenvalues $\lambda_1 = -k_2 < 0$ and $\lambda_2 = k_1 > 0$
The determinant is negative: $\Delta = -k_2k_1$ which leads to the conclusion that $(1, 0)$ is a saddle point.

Evaluating at $(0, 1)$ yields:

$$\begin{pmatrix} -k_1+k_2-k_2 & 0 \\ k_1+k_3 & -k_3+2k_3 \end{pmatrix} = \begin{pmatrix} -k_1 & 0 \\ k_1+k_3 & k_3 \end{pmatrix}$$

Which has eigen values:
$\lambda_1 = -k_1 < 0$
$\lambda_2 = k_3 > 0$
So it can be concluded that this also is a saddle point.

Evaluating at $(\frac{k_3}{k_1+k_2+k_3}, \frac{k_2}{k_1+k_2+k_3})$ gives the following matrix:

$$\begin{pmatrix} (-k_1-k_2)[\frac{k_2}{k_1+k_2+k_3}]+k_2-2k_2[\frac{k_3}{k_1+k_2+k_3}] & (-k_1-k_2)[\frac{k_3}{k_1+k_2+k_3}] \\ (k_1+k_3)[\frac{k_2}{k_1+k_2+k_3}] & (k_1+k_3)[\frac{k_3}{k_1+k_2+k_3}]-k_3+2k_3[\frac{k_2}{k_1+k_2+k_3}] \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-k_1k_2-k_2^2}{k_1+k_2+k_3}+\frac{k_2(k_1+k_2+k_3)}{k_1+k_2+k_3}-\frac{2k_2k_3}{k_1+k_2+k_3} & \frac{(-k_1-k_2)k_3}{k_1+k_2+k_3} \\ \frac{(k_1+k_3)k_2}{k_1+k_2+k_3} & \frac{(k_1+k_3)k_3}{k_1+k_2+k_3}-\frac{k_3(k_1+k_2+k_3)}{k_1+k_2+k_3}+\frac{2k_3k_2}{k_1+k_2+k_3} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-k_2k_3}{k_1+k_2+k_3} & \frac{(-k_1-k_2)k_3}{k_1+k_2+k_3} \\ \frac{(k_1+k_3)k_2}{k_1+k_2+k_3} & \frac{k_3k_2}{k_1+k_2+k_3} \end{pmatrix}$$

Which is equivalent to:

$$= \begin{pmatrix} -k_2k_3 & -(k_1+k_2)k_3 \\ (k_1+k_3)k_2 & k_3k_2 \end{pmatrix}$$

We see immediately that the trace is zero:
$\tau = -k_2k_3 + k_3k_2 = 0$

This leads to the following eigenvalues:
$\lambda_{1,2} = \frac{\tau^2 \pm \sqrt{\tau - 4\Delta}}{2} = \pm\sqrt{-\Delta}$

This means that if $\Delta < 0$, we have two real eigenvalues, one negative and one positive. In that case we have another saddle point. If $\Delta > 0$ we have two complex conjugate eigenvalues. If they both have negative real parts, we have a stable spiral, if they have both positive real parts, we have an unstable spiral. If the real part is zero, we have a center.

So is Δ positive or negative?

$\Delta = -k_2^2k_3^2 + (k_1+k_3)k_2k_3(k_1+k_2)$
$= -k_2^2k_3^2 + k_2^2k_3^2 + k_1^2k_2k_3 + k_1k_2^2k_3 + k_1k_2k_3^2$
$= k_1^2k_2k_3 + k_1k_2^2k_3 + k_1k_2k_3^2$
$= k_1k_2k_3(k_1+k_2+k_3) > 0$

Which means we have two eigenvalues that are complex conjugates,

$$\lambda_{1,2} = \pm\sqrt{-k_1k_2k_3(k_1+k_2+k_3)} = \pm\sqrt{k_1k_2k_3(k_1+k_2+k_3)}i$$

and since the real part is zero, we have a center. This is reflected in the figure resulting from Python's ode-solver.

4.3.3 Graphs

When plotting the phase portrait with Python, we get Figure 6:

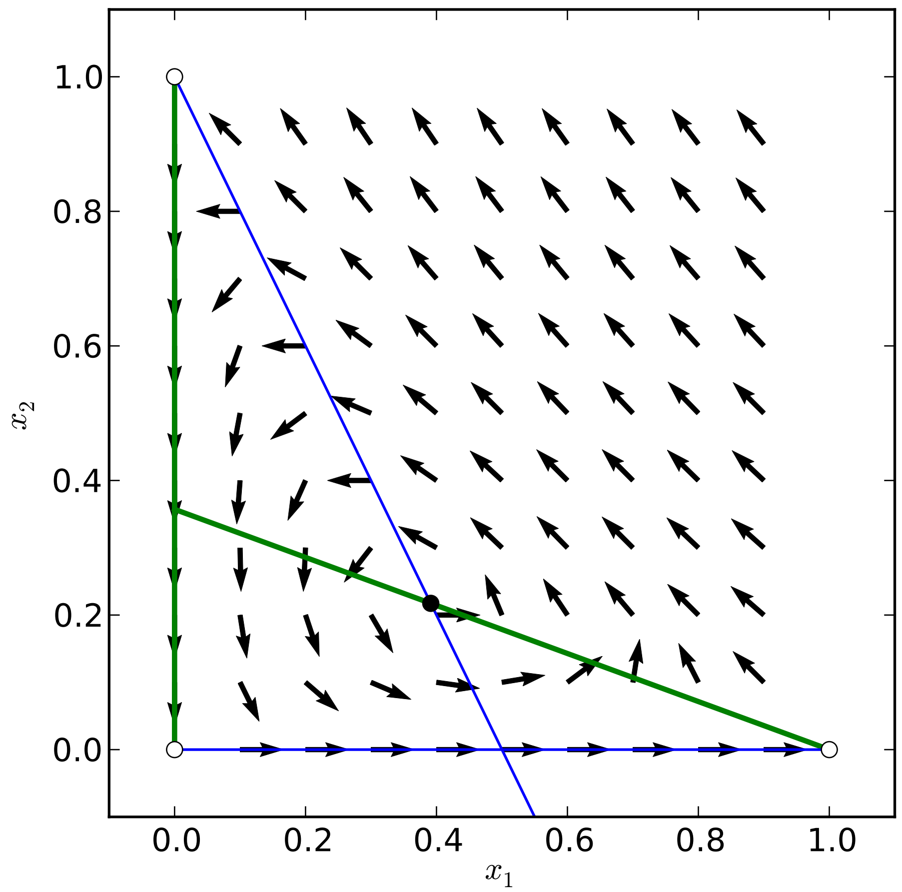

Figure 6: phase plane for the second model. $k_1 = 0.1$, $k_2 = 0.5$ and $k_3 = 0.3$.

In intuitive terms this means that:

- The situation of all sleepers, all readers, all posters, or $[\frac{k_3}{k_1+k_2+k_3}$ readers, $\frac{k_2}{k_1+k_2+k_3}$ posters and $1 - \frac{k_3}{k_1+k_2+k_3} - \frac{k_2}{k_1+k_2+k_3}$ sleepers$]$ does not change.
- Excluding the situation with all posters, no readers ends in all sleepers.
- Excluding the situation with all sleepers, no posters ends in all readers.
- All other valid proportions will eventually show oscillatory behavior and center around the point $(\frac{k_3}{k_1+k_2+k_3}, \frac{k_2}{k_1+k_2+k_3})$, which means that the proportions behave periodically.

Running Python's ode-solver for $k1 = 0.1$, $k2 = 0.2$ and $k3 = 0.5$, for the initial values $(0.4, 0.3)$, gives the following oscillation behavior in figure 7:

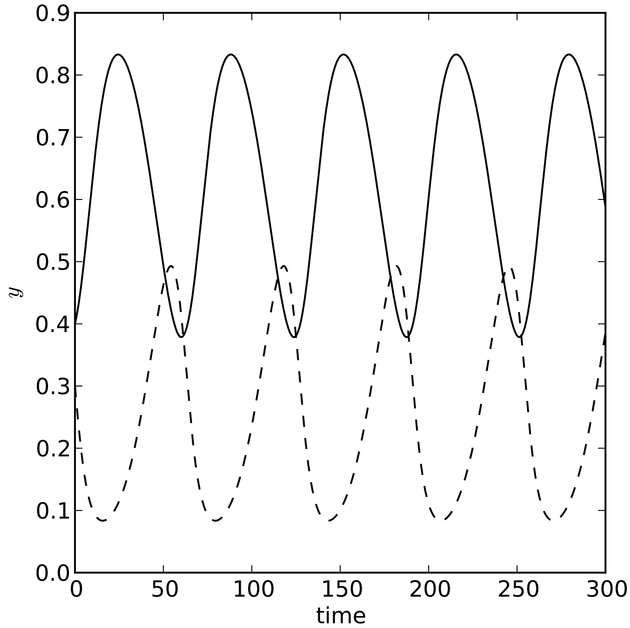

Figure 7: plot of Python's ode-solver. $k_1 = 0.1$, $k_2 = 0.2$, $k_3 = 0.5$. Initial values: $(0.4, 0.3)$

5 Conclusion

Dynamical modeling is used in a variety of disciplines. It can be used to model behavior in physics, behavior of animals and in a lot of cases even behavior of humans, the latter being known as sociodynamics. Inspired by an article about a dynamical model of foraging ants, I made an exploration into sociodynamics.

Subsequently I analyzed two dynamical systems modeling facebook users. Three types of users were distinguished: readers, posters and sleepers. Three transition processes were described and two differential equations were inferred. Firstly the fixed points of the models were determined with the help of the null clines and secondly these fixed points were classified using linearization with the Jacobi matrix. The first model broke down into two cases. The first one having two saddle points and either a stable node or stable spiral, depending on the parameters, and the second one having one saddle point and a stable node or spiral, depending on the parameters.

The second model only had one case. This case had two saddle points and a center.

The above was verified by plotting the phase planes with Python and using its ode-solver.