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Interface waves in almost incompressible elastic materials

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Abstract

We study the problem of two elastic half-planes in contact and the Stoneley interface wave that may exist at the interface between two different elastic materials, emphasis is put on the case when the half-planes are almost incompressible. We show that numerical simulations involving interface waves requires an unexpectedly high number of grid points per wavelength as the materials become more incompressible. Let λ, μ, ρ and λ', μ', ρ' be the Lamé parameters and densities of the first and second half-plane, respectively. A theoretical study shows that if K is a real constant, $\lambda' = K\lambda$, $\mu' = K\mu$, $\rho' = K\rho$ and $\mu \rightarrow 0$, then for an accurate solution the required number of grid points per wavelength scales as $(\mu/\lambda)^{-1/p}$, where p is the order of accuracy of the numerical method. This requirement becomes very restrictive close to the incompressible limit $\mu \ll \lambda$, especially for lower order methods i.e., a small p . The theoretical findings are supported by numerical experiments that illustrate the demanding resolution requirement as well as the superiority of higher order methods. The scaling is also seen to hold for a more general choice of Lamé parameters. Numerical experiments when one of the half-planes is a vacuum are also presented, where the higher resolution requirement is illustrated in a numerical solution of Lamb's problem.

Keywords: elastic waves, interface waves, surface waves, high order accuracy, SBP–SAT.

1. Introduction

Consider two homogeneous elastic isotropic half - planes in contact. Let $y = 0$ be the line of contact and denote the displacement fields (u, v) and (u', v') in the half planes $y > 0$ and $y < 0$, respectively. Elastic wave motion in the half-planes is governed by the elastic wave equation,

$$\begin{aligned} \rho u_{tt} &= \mu \Delta u + (\lambda + \mu)(u_x + v_y)_x, & (x, y) \in (-\infty, \infty) \times [0, \infty), t \geq 0, \\ \rho v_{tt} &= \mu \Delta v + (\lambda + \mu)(u_x + v_y)_y, \\ \rho' u'_{tt} &= \mu' \Delta u' + (\lambda' + \mu')(u'_x + v'_y)_x, & (x, y) \in (-\infty, \infty) \times (-\infty, 0], t \geq 0. \\ \rho' v'_{tt} &= \mu' \Delta v' + (\lambda' + \mu')(u'_x + v'_y)_y, \end{aligned} \tag{1}$$

Here $\rho, \mu, \lambda > 0$ and $\rho', \mu', \lambda' > 0$ are the density and the first and second Lamé parameters of the upper and lower half - planes, respectively. We assume that both the densities and Lamé parameters are constant. Initial data for (u, v) , (u', v') and (u_t, v_t) , (u'_t, v'_t) is given at $t = 0$. At the interface $y = 0$ we require continuity of both displacements,

$$\begin{aligned} u - u' &= 0, \\ v - v' &= 0, \end{aligned} \quad y = 0, t > 0, \tag{2}$$

and tangential and normal tractions

$$\begin{aligned} u_y + v_x - \frac{\mu'}{\mu}(u'_y + v'_x) &= 0, \\ v_y + \frac{\lambda}{\lambda + 2\mu}u_x - \frac{\lambda' + 2\mu'}{\lambda + 2\mu}v'_y - \frac{\lambda'}{\lambda + 2\mu}u'_x &= 0, \end{aligned} \quad y = 0, t > 0. \tag{3}$$

It is well known that (1) admits compressional and shear waves. This becomes transparent when considering the simpler set of equations equivalent to (1),

$$\begin{aligned} \rho \phi_{tt} &= (\lambda + 2\mu) \Delta \phi, & (x, y) \in (-\infty, \infty) \times [0, \infty), t \geq 0, \\ \rho H_{tt} &= \mu \Delta H, \\ \rho' \phi'_{tt} &= (\lambda' + 2\mu') \Delta \phi', & (x, y) \in (-\infty, \infty) \times (-\infty, 0], t \geq 0. \\ \rho' H'_{tt} &= \mu' \Delta H', \end{aligned} \tag{4}$$

Here the equations for ϕ , H , governs the propagation of compressional and shear waves in the upper half - plane with phase velocities $c_p = \sqrt{\frac{\lambda+2\mu}{\rho}}$ and $c_s = \sqrt{\frac{\mu}{\rho}}$, respectively. Similarly the equations for ϕ' , H' governs compressional and shear waves with phase velocities c'_p and c'_s in the lower half - plane. The displacement fields (u, v) and (u', v') are obtained via

$$\begin{aligned} u &= \phi_x + H_y, & u' &= \phi'_x + H'_y, \\ v &= \phi_y - H_x, & v' &= \phi'_y - H'_x. \end{aligned} \quad (5)$$

The conditions (2) in terms of ϕ, ϕ', H and H' become,

$$\begin{aligned} \phi_x + H_y - \phi'_x - H'_y &= 0, \\ \phi_y - H_x - \phi'_y + H'_x &= 0, \quad y = 0, t > 0, \end{aligned} \quad (6)$$

and the conditions (3) become

$$\begin{aligned} 2\phi_{xy} + H_{yy} - H_{xx} - \frac{\mu'}{\mu} (2\phi_{xy} + H_{yy} - H_{xx}) &= 0, \\ \phi_{xx} + \phi_{yy} - \frac{2\mu}{\lambda+2\mu} (\phi_{xx} + H_{xy}) - & \quad y = 0, t > 0. \\ \frac{\lambda'+2\mu'}{\lambda+2\mu} (\phi'_{xx} + \phi'_{yy}) + \frac{2\mu'}{\lambda+2\mu} (\phi'_{xx} + H'_{xy}) &= 0, \end{aligned} \quad (7)$$

For a discussion on how to arrive at (4) - (7) from (1) - (3) see [9, p. 273 - 278].

For some choices of material parameters in the half - planes the elastic wave equation with the interface conditions (2) and (3) admits Stoneley interface waves [18]. These waves travel harmonically along the interface between the half - planes, whereas the amplitude decay exponentially away from the interface. The phase velocity c of the waves is proportional to $\min\{c_s, c'_s\}$ and satisfies $c/\min\{c_s, c'_s\} < 1$. Hence, the Stoneley interface waves always travel slower than both the compressional and shear waves.

In many applications the period of the solution is given through boundary and internal forcing and can be considered as known. Then, as the phase velocity of a wave is defined by the ratio of its length to its period, the shortest present wavelength becomes proportional to $\min\{c_s, c'_s\}$. According to the classical theory in [10] concerning the resolution requirements for numerical simulations of linear hyperbolic wave propagation, an accurate numerical solution is obtained if the shortest wave length is resolved by a constant number of grid points. Here the constant depends on the order of accuracy of the numerical method. This predicts that the number of grid points used in the computational domain should be proportional to $1/\min\{c_s, c'_s\}$ in a simulation where interface waves occur.

In a recent paper by H - O. Kreiss and N.A. Petersson [11] Rayleigh surface waves [16] in elastic half-planes with $\mu \ll \lambda$ are studied. There they show that the classical theory in [10] is inadequate when simulating surface waves. Instead it is proved that the grid size must be proportional to $(\mu/\lambda)^{-1/p}$ in order to achieve an accurate solution. Here p is the order of accuracy of the numerical method. This requirement becomes very restrictive close to the incompressible limit $\mu/\lambda \ll 1$, especially for low order methods. The theory in [11] was supported by numerical experiments using 2nd and 4th order discretizations of the elastic wave equation in a half-plane with μ/λ as small as 10^{-3} .

The Stoneley interface wave can be viewed as a generalization of the Rayleigh surface wave, which appears at the traction free surface of a single half - plane, or equivalently at the interface between an elastic half-plane and vacuum. One might now inquire, due to the similarity between interface and surface waves, whether a numerical simulation of the Stoneley interface wave suffers from a similar resolution requirement. As an example motivating a positive answer to this inquiry we propagate an interface wave using a second order accurate finite difference method. The closed form expression for the solution is available in Appendix B and the numerical method is described in [4]. In the numerical example we make the x-direction 2π -periodic and take the wavelength to be 2π . The interface conditions (2)-(3) are imposed at $y=0$. The interface wave propagates harmonically in the x-direction and decays exponentially with $|y|$ away from $y = 0$, see the left of Figure 1. We take $\lambda = 1, \lambda' = 3, \rho = 1, \rho' = 1.9999, \mu' = 2\mu$ and decrease μ , which gives the interface wave a phase velocity proportional to $\sqrt{\mu}$. We discretize the elastic wave equation on a grid with P_S points per wave length. In the right of Figure 1 we report the error in the numerical solution after five periods in time. This experiment indicates that numerical simulations involving the Stoneley interface wave also suffers from a stricter than expected resolution requirement. For $\mu = 1$, 21 grid points per wavelength is needed to give a relative maximum error of less than 10% whereas for $\mu = 10^{-3}$, 321 grid points are required for the same error tolerance. The gradient of the exact solution is of the order $\mathcal{O}(1/2\pi)$ for all values of $\mu > 0$. That is, the loss of accuracy is not due to poor resolution in space.

The first goal of this paper is to explain the loss of accuracy of the simulation of interface waves as the half-planes becomes more and more incompressible. The analysis is influenced by [11] in that we view the discretized interface conditions as perturbations of the exact interface conditions. The analysis then highlights how the perturbations influence the phase velocity of the solution and how the material parameters enter in the relation.

In the concluding section of [11] remarks are made on the use of methods of higher order than four to simulate surface waves in a half - plane. It is there concluded that numerical experiments must be performed to evaluate how small μ/λ has to be to compensate for the higher complexity of higher order methods. The second goal of this paper aims at performing such experiments using methods of order of accuracy higher than four. These experiments show a distinct advantage of methods of order higher than four when materials in which $\mu/\lambda < 10^{-3}$ are considered.

Moreover, the numerical method used in [11] uses ghost points to implement the traction free boundary condition [17]. In our experiments we use a weak enforcing of the boundary conditions [1]. As a consequence our experiments on the Rayleigh surface wave also show that the problem is generic and not a consequence of the particular implementation of boundary conditions.

The rest of the paper is organized as follows. Section 2 describes the construction of and conditions for existence of the Stoneley interface wave. This material follows the original work by Stoneley [18] and is included for completeness. New results then follow in Section 3, where the above noticed loss of accuracy is explained. Section 4 briefly describes the numerical method used in the experimental work that is made in Section 5, where the loss of accuracy is confirmed by experiments that illustrate the importance of higher order methods. Conclusions are given in Section 6.

Figure 1: Left: Plot of the horizontal and the vertical components of the Stoneley interface wave. The u – component is shown to the left and the v – component to the right. Right: Maximum error in the numerical solution after having propagated the interface wave for five periods in time with a second order accurate method. The error is shown as a function of the number of grid points per wavelength.

2. Interface waves

Consider the equations (4) with the interface conditions (6) and (7). We examine the existence of Stoneley interface wave solutions of the type

$$\begin{aligned} \phi &= f(y)e^{i\xi(x-ct)}, & |f|_\infty < \infty, \\ H &= h(y)e^{i\xi(x-ct)}, & |H|_\infty < \infty, \end{aligned} \quad , y \geq 0, -\infty < x < \infty, \xi \in \mathbb{R} \setminus \{0\} \quad (8)$$

and

$$\begin{aligned} \phi' &= f'(y)e^{i\xi(x-ct)}, & |f'|_\infty < \infty, \\ H' &= h'(y)e^{i\xi(x-ct)}, & |H'|_\infty < \infty, \end{aligned} \quad y \leq 0, -\infty < x < \infty, \xi \in \mathbb{R} \setminus \{0\}. \quad (9)$$

To define a wave that clings to the interface and travels harmonically in the tangential direction we only look for solutions where $f(y)$ and $f'(y)$ are decaying into the half-planes. Inserting (8) and (9) into (4) we get

$$\begin{aligned}\frac{d^2}{dy^2}f - r^2f &= 0, r^2 = \xi^2 \left(1 - \frac{c^2}{c_p^2}\right), \\ \frac{d^2}{dy^2}h - s^2h &= 0, s^2 = \xi^2 \left(1 - \frac{c^2}{c_s^2}\right)\end{aligned}\tag{10}$$

and

$$\begin{aligned}\frac{d^2}{dy^2}f' - r'^2f' &= 0, r'^2 = \xi^2 \left(1 - \frac{c'^2}{c_p'^2}\right), \\ \frac{d^2}{dy^2}h' - s'^2h' &= 0, s'^2 = \xi^2 \left(1 - \frac{c'^2}{c_s'^2}\right).\end{aligned}\tag{11}$$

The solutions to (10) and (11) are

$$\begin{aligned}f &= A_1 e^{ry} + A_2 e^{-ry}, \\ h &= B_1 e^{sy} + B_2 e^{-sy}, \\ f' &= A'_1 e^{r'y} + A'_2 e^{-r'y}, \\ h' &= B'_1 e^{s'y} + B'_2 e^{-s'y}.\end{aligned}\tag{12}$$

The functions ϕ, H of (8) and ϕ', H' of (9) then become

$$\begin{aligned}\phi &= A_1 e^{ry} e^{i\xi(x-ct)} + A_2 e^{-ry} e^{i\xi(x-ct)}, \\ H &= B_1 e^{sy} e^{i\xi(x-ct)} + B_2 e^{-sy} e^{i\xi(x-ct)}, \\ \phi' &= A'_1 e^{r'y} e^{i\xi(x-ct)} + A'_2 e^{-r'y} e^{i\xi(x-ct)}, \\ H' &= B'_1 e^{s'y} e^{i\xi(x-ct)} + B'_2 e^{-s'y} e^{i\xi(x-ct)}.\end{aligned}\tag{13}$$

If $c > c_s$, then s becomes imaginary and H of (13) defines a plane wave propagating in the entire upper half-plane. Similarly, if $c > c'_s$, $c > c_p$ or $c > c'_p$ then H', ϕ or ϕ' does not define an interface wave localized at the interface. For this reason we only look for a phase velocity c such that all of r^2, r'^2, s^2, s'^2 are positive. By the requirement that the functions of (12) are bounded we necessarily have $A_1 = B_1 = A'_2 = B'_2 = 0$. Inserting the expressions (13) into the interface conditions (6) - (7) and solving for $e^{i\xi(x-ct)}iA_2$, $e^{i\xi(x-ct)}iA'_1$, $-e^{i\xi(x-ct)}B_2$ and $e^{i\xi(x-ct)}B'_1$ we get a system of equations with homogeneous right hand side. This system has a solution iff its determinant

$$\det \begin{vmatrix} 2\left(1 - \frac{c^2}{c_p^2}\right)^{1/2} & \frac{2\mu'}{\mu}\left(1 - \frac{c^2}{c_p'^2}\right)^{1/2} & \left(2 - \frac{c^2}{c_s^2}\right) & \frac{\mu'}{\mu}\left(2 - \frac{c^2}{c_s'^2}\right) \\ \left(\frac{c^2}{c_s^2} - 2\right) & \frac{\mu'}{\mu}\left(2 - \frac{c^2}{c_s'^2}\right) & -2\left(1 - \frac{c^2}{c_s^2}\right)^{1/2} & \frac{2\mu'}{\mu}\left(1 - \frac{c^2}{c_s'^2}\right)^{1/2} \\ 1 & -1 & \left(1 - \frac{c^2}{c_s^2}\right)^{1/2} & -\left(1 - \frac{c^2}{c_s'^2}\right)^{1/2} \\ \left(1 - \frac{c^2}{c_p^2}\right)^{1/2} & \left(1 - \frac{c^2}{c_p'^2}\right)^{1/2} & 1 & 1 \end{vmatrix}\tag{14}$$

vanishes. The existence of a phase velocity $c = c_0$ such that the determinant is zero is not always guaranteed, the range of values of the density and Lamé parameters such that a zero is possible has been discussed in [2].

In the following we investigate the special case

$$c_p = c'_p, c_s = c'_s.\tag{15}$$

Define $\gamma = c_s/c_p$, $\tilde{c} = c/c_s$ and $K = \rho'/\rho$. Without loss of generality we may assume $\rho > \rho'$ so that $0 < K < 1$. The condition for the existence of a Stoneley interface wave becomes

$$\begin{aligned}\psi(\tilde{c}) &= \frac{1}{\rho^2} \left(\tilde{c}^2 \left((1-K)^2 - (1+K)^2 (1-\tilde{c})^{1/2} (1-\gamma\tilde{c})^{1/2} \right) \right. \\ &\quad \left. + 4(1-K)^2 \left((1-\tilde{c})(2-\gamma\tilde{c}) + (1-\gamma\tilde{c})^{1/2} (1-\tilde{c})^{1/2} (\tilde{c}-2) \right) \right) = 0.\end{aligned}\tag{16}$$

Figure 2: The root $\tilde{c}_0$ for $0 < K < 1$, $\lambda = 1$, $c_s = c'_s$ and $c_p = c'_p$.

We see immediately that $\psi(1) = (1 - K)^2/\rho^2 > 0$ and $\psi(0) = 0$. Assume that $0 < \tilde{c} < 1$ such that the square roots may be expanded as a power series in $\tilde{c}$, then

$$\frac{\rho^2}{\tilde{c}^2} \psi(\tilde{c}) = -(1 + K)^2 + \gamma^2 (1 - K)^2 + \mathcal{O}(\tilde{c}).$$

Since $0 < \gamma < 1$ and $1 + K > 1 - K$ we have $-(1 + K)^2 + \gamma^2 (1 - K)^2 < 0$ and therefore

$$\lim_{\tilde{c} \rightarrow 0^+} \frac{\rho^2}{\tilde{c}^2} \psi(\tilde{c}) < 0,$$

It follows that for some $\epsilon > 0$, $\psi(\tilde{c}) < 0$ for $0 < \tilde{c} < \epsilon$. Since $\psi(1) > 0$ a root $\tilde{c}_0$ always exists in $(0, 1)$ and therefore interface waves with phase velocity $c = c_s \tilde{c}_0 < c_s$ always exists for the case $c_p = c'_p$, $c_s = c'_s$.

The root $\tilde{c}_0$ depends on the quotients $K = \rho'/\rho$ and $\gamma = c_s/c_p = \sqrt{\mu/(\lambda + 2\mu)} = \sqrt{\mu'/(\lambda' + 2\mu')}$. In the present study we are interested in the case $\mu/\lambda \rightarrow 0$ for this reason we scale time so that $\lambda = 1$, then by (15) $\rho' = K\rho$, $\lambda' = K$, $\mu' = K\mu$. In Figure 2 we have plotted the root $\tilde{c}_0$ for $0 < K < 1$ and some values of μ . The figure show that the phase velocity of the interface wave approaches that of the shear wave velocity as $K \rightarrow 1$.

3. Sensitivity of interface waves to boundary truncation errors

In this section we explain the stricter resolution requirement that appears when letting $\mu/\lambda \rightarrow 0$ and $\mu'/\lambda' \rightarrow 0$. To guarantee the existence of an interface wave we assume that (15) holds, we scale time so that $\lambda = 1$ and with out loss of generality we assume that $\rho > \rho'$. That is, in this section for $0 < K < 1$

$$\rho' = K\rho, \lambda' = K, \mu' = K\mu.$$

We use the modified equation approach to investigate how the phase velocity of the Stoneley interface wave will be perturbed in a numerical simulation. The modified equation approach involves modeling the truncation errors arising from a discretization of the interface conditions by introducing non-zero interface forcing functions,

$$g_1 e^{i\xi(x-ct)} \text{ and } g_2 e^{i\xi(x-ct)}$$

to the right hand sides of (3) and (7). The forcing functions have a specific form, which is given by the discretization. Typically the terms depend on spatial derivatives of the continuous solution and the grid size. Consider solutions of the type

$$\begin{aligned} \phi &= A_2 e^{-ry} e^{i\xi(x-ct)}, \\ H &= B_2 e^{-sy} e^{i\xi(x-ct)}, \\ \phi' &= A'_1 e^{r'y} e^{i\xi(x-ct)}, \\ H' &= B'_1 e^{s'y} e^{i\xi(x-ct)}. \end{aligned} \tag{17}$$

Define

$$X = \left(1 - \frac{\tilde{c}^2 c_s^2}{c_p^2}\right)^{1/2} = r/\xi, Y = (1 - \tilde{c}^2)^{1/2} = s/\xi. \quad (18)$$

Inserting the solutions (17) into the now inhomogeneous interface conditions we get after some algebra

$$2XiA_2 + 2KXiA'_1 - (1 + Y^2) B_2 + K(1 + Y^2) B'_1 = \frac{g_1}{\xi^2}, \quad (19)$$

$$- (1 + Y^2) iA_2 + K(1 + Y^2) iA'_1 + 2YB_2 + 2KYB'_1 = -i \frac{g_2(\lambda + 2\mu)}{\xi^2 \mu}, \quad (20)$$

$$iA_2 + -iA'_1 - YB_2 - YB'_1 = 0, \quad (21)$$

$$XiA_2 + XiA'_1 - B_2 + B'_1 = 0, \quad (22)$$

where $K \in (0, 1)$ is the quotient of the densities between the upper and lower half-plane. The determinant of the system (19)-(22) is $\psi(\tilde{c})$ defined by (16). Hence, the system becomes singular exactly at the roots of $\psi(\tilde{c})$. We solve for iA'_1 using Cramer's rule,

$$\begin{aligned} \psi(\tilde{c})iA'_1 = & -\frac{g_1}{\xi^2} \det \begin{vmatrix} -(1 + Y^2) & -2Y & 2KY \\ 1 & Y & -Y \\ X & 1 & 1 \end{vmatrix} \\ & -i \frac{g_2(\lambda + 2\mu)}{\xi^2 \mu} \det \begin{vmatrix} 2X & 1 + Y^2 & K(1 + Y^2) \\ 1 & Y & -Y \\ X & 1 & 1 \end{vmatrix}. \end{aligned} \quad (23)$$

Discretizing the conditions (3) with a second order accurate method the principal parts of the truncation errors become

$$\begin{aligned} g_1 e^{i\xi(x-ct)} &= \tau_{11} h^2 v_{xxx} + \tau_{12} h^2 u_{yyy} + \tau_{13} h^2 v'_{xxx} + \tau_{14} h^2 u'_{yyy}, \\ g_2 e^{i\xi(x-ct)} &= \tau_{21} h^2 u_{xxx} + \tau_{22} h^2 v_{yyy} + \tau_{23} h^2 u'_{xxx} + \tau_{24} h^2 v'_{yyy}. \end{aligned} \quad (24)$$

Here τ_{ij} are constants which can be determined by a standard Taylor series approach. These constants depend on the numerical scheme, but also on the material parameters in the same way as the coefficients in (3). Note that for the case under consideration here the coefficients in (3) all remain bounded as $\mu, \mu' \rightarrow 0$. By (5),

$$\begin{aligned} g_1 e^{i\xi(x-ct)} &= \tau_{11} h^2 (\phi_{xxxy} - H_{xxx}) + \tau_{12} h^2 (\phi_{yyyy} + H_{yyyy}) \\ &\quad + \tau_{13} h^2 (\phi'_{xxxy} - H'_{xxx}) + \tau_{14} h^2 (\phi'_{yyyy} + H'_{yyyy}) \\ g_2 e^{i\xi(x-ct)} &= \tau_{21} h^2 (\phi_{xxx} + H_{xxxy}) + \tau_{22} h^2 (\phi_{yyyy} - H_{xyyy}) \\ &\quad + \tau_{23} h^2 (\phi'_{xxx} + H'_{xxxy}) + \tau_{24} h^2 (\phi'_{yyyy} - H'_{xyyy}). \end{aligned}$$

Using (17) the functions g_1 and g_2 become

$$\begin{aligned} g_1 &= \tau_{11} h^2 (i\xi^3 r A_2 - \xi^4 B_2) + \tau_{12} h^2 (-i\xi r^3 A_2 + s^4 B_2) \\ &\quad + \tau_{13} h^2 (-i\xi^3 r' A'_1 - \xi^4 B'_1) + \tau_{14} h^2 (i\xi r'^3 A'_1 + s'^4 B'_1) \\ g_2 &= \tau_{21} h^2 (\xi^4 A_2 + i\xi^3 s B_2) + \tau_{22} h^2 (r^4 A_2 + i\xi s^3 B_2) \\ &\quad + \tau_{23} h^2 (\xi^4 A'_1 - i\xi^3 s' B'_1) + \tau_{24} h^2 (r'^4 A'_1 - i\xi s'^3 B'_1). \end{aligned} \quad (25)$$

Since the right hand side of (20) is proportional to g_2/μ while the right hand side of (19), (21) and (22) does not contain a factor $1/\mu$ we argue that the main effect comes from g_2 as $\mu \rightarrow 0$. To simplify we therefore assume that $g_1 = 0$. Using (19), (21) and (22) with $g_1 = 0$ we express $iA_2, -B_2$ and B'_1 in terms of iA'_1 .

$$\begin{aligned} iA_2 &= \frac{Y^2 - K + 1 - XY - KY^2 - XY^3 + 3KXY - KXY^3}{XY^3 - 3XY - KY^2 - K + Y^2 + KXY + KXY^3 + 1} iA'_1 \\ -B_2 &= 2 \frac{X^2 Y - X + KXY^2 - KX^2 Y}{XY^3 - 3XY - KY^2 - K + Y^2 + KXY + KXY^3 + 1} iA'_1 \\ B'_1 &= 2 \frac{KX - XY^2 + X^2 Y - KX^2 Y}{XY^3 - 3XY - KY^2 - K + Y^2 + KXY + KXY^3 + 1} iA'_1 \end{aligned} \quad (26)$$

Figure 3: $\frac{d}{d\tilde{\epsilon}}\psi(\tilde{\epsilon})$ at $\tilde{\epsilon}_0$ for $0 < K < 1$.

Substituting the expressions (26) for iA_2 , $-B_2$ and B'_1 into (25) and using the solution formula (23) with $g_1 = 0$ we get the scalar eigenvalue problem

$$\psi(\tilde{\epsilon})A'_1 = \theta(\tilde{\epsilon})A'_1. \quad (27)$$

An eigenvalue $\tilde{\epsilon}_1$ of this problem determines the numerical phase velocity of the interface waves in a numerical approximation of (1) - (3), while the phase velocity in the continuous case is determined by the root $\tilde{\epsilon}_0$ of $\psi(\tilde{\epsilon})$. We describe the derivation of $\theta(\tilde{\epsilon})$ in Appendix A. where we also show that we can approximate $\theta(\tilde{\epsilon}_0)$ for $K \in (0.01, 0.99)$ by

$$\begin{aligned} \theta(\tilde{\epsilon}_0) &\approx \frac{-\lambda h^2 \xi^2}{\mu} (\kappa_1 \tau_{21} + \kappa_2 \tau_{22} + \kappa_3 \tau_{23} + \kappa_4 \tau_{24}), \\ \kappa_1 &\in (1.02, 118.54), \\ \kappa_2 &\in (1.01, 77.78), \\ \kappa_3 &\in (1.01, 21.32), \\ \kappa_4 &\in (-1.86, 1.17). \end{aligned} \quad (28)$$

Here $\tau_{21}, \tau_{22}, \tau_{23}, \tau_{24}$ are the same constants as in (24). The numerical values involved in the approximation (28) of $\theta(\tilde{\epsilon}_0)$ hold for all $K \in (0.01, 0.99)$. Taking $K = 0$ would imply that one of the half-planes is a vacuum and this case would reduce to the case of a Rayleigh surface wave, then the theory of [11] would apply. Letting $K = 1$ implies that the half-planes are of the same material and no interface wave could exist. The authors believe that the range $K \in (0.01, 0.99)$ captures most interesting materials.

To estimate the difference $\tilde{\epsilon}_1 - \tilde{\epsilon}_0$ between the phase velocity of the discrete and continuous interface wave we Taylor expand ψ of (27) about $\tilde{\epsilon}_0$ and to first order we have

$$(\tilde{\epsilon}_1 - \tilde{\epsilon}_0) \psi'(\tilde{\epsilon}_0) \approx \theta(\tilde{\epsilon}_0).$$

Figure 3 shows $\psi'(\tilde{\epsilon}_0)$ for $K \in (0.01, 0.99)$ for some values of μ as well as the limit of $\psi'(\tilde{\epsilon}_0)$ as $\mu \rightarrow 0$. There we see that $\psi'(\tilde{\epsilon}_0)$ is bounded away from zero as $\mu \rightarrow 0$. Hence,

$$\tilde{\epsilon}_1 - \tilde{\epsilon}_0 \approx \frac{\theta(\tilde{\epsilon}_0)}{\psi'(\tilde{\epsilon}_0)} \approx \frac{-\lambda h^2 \xi^2}{\mu} \frac{(\kappa_1 \tau_{21} + \kappa_2 \tau_{22} + \kappa_3 \tau_{23} + \kappa_4 \tau_{24})}{\psi'(\tilde{\epsilon}_0)}. \quad (29)$$

To achieve an error in the phase velocity of size ϵ with $0 < \epsilon \ll 1$ we must choose the grid size h such that

$$\frac{\lambda h^2 \xi^2 |\tau|}{\mu} = \epsilon, \tau = \frac{(\kappa_1 \tau_{21} + \kappa_2 \tau_{22} + \kappa_3 \tau_{23} + \kappa_4 \tau_{24})}{\psi'(\tilde{\epsilon}_0)}. \quad (30)$$

If the computational grid has P_S point per interface wavelength $L_S = 2\pi/|\xi|$ we get

$$h = \frac{L_S}{P_S}, h|\xi| = \frac{2\pi}{P_S}.$$

If we require an error in the phase velocity of size ϵ then, by (30) we get

$$P_S = 2\pi \left(\frac{|\tau| \lambda}{\epsilon \mu} \right)^{1/2}.$$

That is, as $\mu/\lambda \rightarrow 0$ the number of points per surface wavelength for a second order method must be proportional to $(\lambda/\mu)^{1/2}$ to maintain a relative error in the phase velocity of size ϵ .

For a p -th order method the leading order truncation error terms are

$$g_2 e^{i\xi(x-ct)} = \bar{\tau}_{21} h^p \frac{\partial^{p+1} u}{\partial x^{p+1}} + \bar{\tau}_{22} h^p \frac{\partial^{p+1} v}{\partial y^{p+1}} + \bar{\tau}_{23} h^p \frac{\partial^{p+1} u'}{\partial x^{p+1}} + \bar{\tau}_{24} h^p \frac{\partial^{p+1} v'}{\partial y^{p+1}} \quad (31)$$

and (30) is replaced by

$$\frac{\lambda h^p \xi^2 |\bar{\tau}|}{\mu} = \epsilon. \quad (32)$$

The number of grid points required to maintain an error ϵ in the phase velocity now becomes

$$P_S = 2\pi \left(\frac{|\bar{\tau}| \lambda}{\epsilon \mu} \right)^{1/p}.$$

Hence, as $\mu/\lambda \rightarrow 0$ the number of grid points per surface wave length grows much slower for larger p .

We have established that the error in the numerical phase velocity of interface waves is governed by $(\lambda/\mu)^p$. In the introductory numerical example as well as in the numerical experiments in Section 5.1 we have seen that although the numerical solution is extremely well resolved in the spatial dimensions large errors are still present. We therefore assume that phase errors dominate the numerical solution. For a spatially well resolved numerical solution the amplitude of the errors as a function of time are then governed to first order in time by

$$e(t) = C_1 \xi c_s (\tilde{c}_1 - \tilde{c}_0) t = C_1 \xi c_s \epsilon t, \quad C_1 = \text{constant}.$$

The period of the interface wave is given by

$$q = \frac{2\pi}{\xi \tilde{c}_0 c_s}.$$

For a grid with grid size $h = L_S/P_S$ we get after n periods in time at $t = nq$

$$e(nq) = C_1 (2\pi)^{p+1} |\bar{\tau}| \frac{n\lambda}{\tilde{c}_0 P_S^p \mu}. \quad (33)$$

For a constant λ and a fixed number of periods n we therefore need to choose $P_S \approx \left(\frac{n}{\mu}\right)^{1/p}$ to achieve a constant error as μ decreases. The classical result of [10] states that for a fixed number of periods we need only take $P_S \approx Cn^{1/p}$ for a constant error, where C depends on the numerical method. For μ sufficiently small we have

$$Cn^{1/p} \ll \left(\frac{n}{\mu}\right)^{1/p},$$

and for such values of μ the classical theory fails to give a good estimate of a sufficient number of grid points per wavelength of the interface wave.

4. The numerical method

The numerical method used in the experiments of Section 5 is based on discretizing the second order system (1) and the interface conditions (2) - (3) in space using high order finite difference operators that satisfies a summation-by-parts (SBP) rule [19, 13]. The interface conditions are imposed weakly via the simultaneous-approximation-term (SAT) method [1] by adding to the discretized system a forcing term proportional to the difference between the correct and discrete value at the interface. The proof of energy stability, i.e., existence of a discrete semi-norm $\|\cdot\|_H$ in which the numerical solution is proved to be bounded for all times, guaranties that the error (measured in the same semi-norm) is at least of the order of the truncation error. In particular we expect the

interface conditions to be satisfied to the same order. However, in many cases when the dominating truncation error is localized near boundaries the energy technique yields a pessimistic error bound. The numerical method and the SBP-SAT approach are described in [4]. In the numerical experiments we also present computations involving surface waves at a traction free surface. These computations were made with the method using a SBP-SAT approach described in [5] combined with a perfectly matched layer (PML) to absorb outgoing waves [3]. The capabilities of the methods are schematically described in Figure 4.

The estimate (33) describes the error as a function of time. It suggests that if $P_S \mu^{1/p}$ is kept constant as μ decreases the error will grow linearly in time. This is investigated numerically in Experiment 5.1. The formula involves the order of accuracy p of the discretization of the condition that normal tractions are continuous across the interface. With the SBP-SAT method the tangential and normal derivatives appearing in the expression for the normal tractions have to be, for stability reasons, approximated to different orders of accuracy. In particular, the truncation error term corresponding to g_2 of (31) becomes

$$\bar{g}_2 e^{i\xi(x-ct)} = \bar{\tau}_{21} h^{2m} \frac{\partial^{2m+1} u}{\partial x^{2m+1}} + \bar{\tau}_{22} h^{m+1} \frac{\partial^{m+2} v}{\partial y^{m+2}} + \bar{\tau}_{23} h^{2m} \frac{\partial^{2m+1} u'}{\partial x^{2m+1}} + \bar{\tau}_{24} h^{m+1} \frac{\partial^{m+2} v'}{\partial y^{m+2}}. \quad (34)$$

Choosing $p = m + 1$ as the leading order of the truncation term in (31) would give the estimate (33) with $p = m + 1$. However, the numerical experiments of Section 5 show that the asymptotic behavior of (33) is consistent with $p = 2m$, i.e., a higher than expected order of accuracy is obtained in practice.

In general, SBP operators approximate derivatives appearing in the partial differential equation of order of accuracy $2m$ in the interior of the computational domain and order m in the vicinity of boundaries. Discretizing second order partial differential equations using SBP operators a global order of accuracy of $\min\{2m, m + 2\}$ has previously been observed in [4, 5, 8, 14, 15]. In these cases the observed convergence order is faster than the order predicted by the energy estimate. In the computations involving interface waves use SBP-SAT methods with $2m = 2, 4, 6$, in computations involving surface waves we also use an SBP-SAT method with $2m = 8$. We will refer to these as $2m$ th order SBP-SAT methods.

The discretization of spatial derivatives with SBP operators, enforcing of a traction free boundary condition, interface conditions and a homogeneous Dirichlet condition with the SAT technique results in a semi - discrete system of the type

$$\mathbf{U}_{tt} = \frac{1}{h^2} \mathbf{Q} \mathbf{U}. \quad (35)$$

Here $\mathbf{U}$ is a vector with approximative values of the displacement components at the grid points of the computational domain and h is the grid size. $\mathbf{Q}$ is a matrix with elements independent of h . To discretize (8) in time we use the time stepping schemes of [7]. These schemes were designed for a system of the type (8) in that it is not rewritten as a system first order in time. In [7] it is shown that if a time step k is chosen as

$$k = h \frac{\alpha_n^{1/2}}{\|\mathbf{Q}\|_H^{1/2}} \quad (36)$$

stability is guaranteed provided that the semi-discrete solution $\mathbf{U}$ of (35) can be proved to be bounded in the semi-norm $\|\cdot\|_H$. Here $\|\mathbf{Q}\|_H$ is proportional to the fastest wave speed, $\max\left\{\sqrt{(2\mu + \lambda)/\rho}, \sqrt{(2\mu' + \lambda')/\rho'}\right\}$. The constants α_n depends on the order of accuracy of the temporal discretization and can be found in [7].

Using the PML results in a system of the type

$$\begin{bmatrix} \mathbf{U}_{tt} \\ \mathbf{W}_t \end{bmatrix} = \mathbf{P} \begin{bmatrix} \mathbf{U} \\ \mathbf{W} \end{bmatrix}. \quad (37)$$

Here $\mathbf{W}$ contains auxiliary variables arising from the addition of the PML. The system (37) is written as a first order system in time and the classical Runge - Kutta 4 scheme is used for time integration.

5. Numerical experiments

In this section we first present experiments involving the Stoneley interface wave where we see the theoretically expected need to increase the number of grid points for a fixed accuracy as μ tends to zero. In a second study we investigate the mode conversion phenomena that occurs as waves of a certain type, pressure or shear,

Figure 4: The capabilities of the method includes traction free and Dirichlet boundary conditions, interface conditions and a PML to absorb outgoing waves.

impinges on an interface. The refractions and reflections are known to be compounded of both types of waves. In particular we see that this phenomena does not suffer from a stricter resolution requirement as the materials involved becomes more incompressible. In a third set of experiments we consider the use of high order accurate methods in computations involving surface waves, here we also solve numerically a version of Lamb's problem.

Let $\mathbf{U}_{i,j}^{\text{num}}$ and $\mathbf{U}_{i,j}^{\text{exact}}$ be the numerical and the exact solution at a grid point (x_i, y_j) , respectively. On an equidistant grid with $N \times N$ grid points and grid size h , we compute the maximum error e_{max} and the l_2 error, e_{l_2} as

$$e_{\text{max}} = \frac{\max_{i,j} \{|\mathbf{U}_{i,j}^{\text{num}} - \mathbf{U}_{i,j}^{\text{exact}}|\}}{\max_{i,j} \{|\mathbf{U}_{i,j}^{\text{exact}}|\}}, \quad (38)$$

$$e_{l_2} = \left(\sum_{i=0}^N \sum_{j=0}^N |\mathbf{U}_{i,j}^{\text{num}} - \mathbf{U}_{i,j}^{\text{exact}}|^2 h^2 \right)^{1/2}.$$

5.1. Interface waves

The parts of an interface wave (u, v) and (u', v') in the upper and lower half-plane, respectively, can be written by inserting the expressions (13) for ϕ and H into (5),

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} i\xi A_2 e^{-ry} - sB_2 e^{-sy} \\ -rA_2 e^{-ry} - i\xi B_2 e^{-sy} \end{pmatrix} e^{i\xi(x-ct)}, y > 0, \\ \begin{pmatrix} u' \\ v' \end{pmatrix} &= \begin{pmatrix} i\xi A'_1 e^{r'y} + s'B'_1 e^{s'y} \\ r'A'_1 e^{r'y} - i\xi B'_1 e^{s'y} \end{pmatrix} e^{i\xi(x-ct)}, y < 0. \end{aligned} \quad (39)$$

The values of c , A_2 , B_2 , A'_1 and B'_1 used in the following two experiments are presented in Appendix A. The theory in Section 3 was developed for the special case $c_p = c'_p$, $c_s = c'_s$. This was done to guarantee the existence of an interface wave and to minimize the parameter space. In the numerical experiments involving interface waves we consider the more general case $c_p \neq c'_p$ and $c_s \neq c'_s$. We vary μ take $\rho = \lambda = 1$, $\rho' = 1.9999$, $\lambda' = 3$ and $\mu' = 2\mu$. The phase velocity of the interface wave is computed by solving (14) which has a solution c for all values of μ we consider.

In the computations of this section we scale the wavelength and the temporal period of the interface wave to be 2π and $2\pi/c$, respectively. We use a computational domain with grid size $h = 2\pi/(P_S - 1)$ periodic in the x -direction. With this grid size P_S grid points per wavelength of the interface wave are used. The computational domain is truncated by imposing exact Dirichlet data at $y = \pm 4\pi$. We compute in real arithmetic, for this reason the real part of (39) is used as initial data at time $t = 0$ and $t = -k$, where k is the time step.

In a first experiment we establish the order of convergence for computations involving interface waves. To achieve this we fix $\mu = 0.1$ and choose all other material parameters as described above. A grid refinement study

Second order SBP-SAT, second order in time				
P_S	e_{l_2}	pl_2	$e_{\max}$	$p_{\max}$
21	1.0×10^{-1}	-	1.0×10^{-1}	-
41	3.3×10^{-2}	1.6	3.1×10^{-2}	1.8
81	8.8×10^{-3}	1.9	8.3×10^{-3}	1.9
161	2.2×10^{-3}	2.0	2.1×10^{-3}	2.0
321	5.6×10^{-4}	2.0	5.3×10^{-4}	2.0
Fourth order SBP-SAT, fourth order in time				
P_S	e_{l_2}	pl_2	$e_{\max}$	$p_{\max}$
21	2.0×10^{-2}	-	2.1×10^{-2}	-
41	2.0×10^{-3}	3.4	2.4×10^{-3}	3.1
81	1.1×10^{-4}	4.0	2.2×10^{-4}	3.4
161	6.2×10^{-6}	4.2	1.7×10^{-5}	3.7
321	3.5×10^{-7}	4.2	1.2×10^{-6}	3.9
Sixth order SBP-SAT, fourth order in time				
N	e_{l_2}	pl_2	$e_{\max}$	$p_{\max}$
21	3.2×10^{-4}	-	8.3×10^{-4}	-
41	1.1×10^{-5}	4.9	4.0×10^{-5}	4.4
81	3.0×10^{-7}	5.1	1.6×10^{-6}	4.7
161	8.0×10^{-9}	5.2	5.5×10^{-8}	4.9
321	3.2×10^{-10}	4.6	1.8×10^{-9}	4.9
Sixth order SBP-SAT, sixth order in time				
N	e_{l_2}	pl_2	$e_{\max}$	$p_{\max}$
21	3.2×10^{-4}	-	8.3×10^{-4}	-
41	1.1×10^{-5}	4.9	4.0×10^{-5}	4.4
81	3.0×10^{-7}	5.1	1.6×10^{-6}	4.7
161	8.0×10^{-9}	5.2	5.5×10^{-8}	4.9
321	3.0×10^{-10}	4.7	1.8×10^{-9}	4.9

Table 1: Maximum errors and l_2 errors after 5 temporal periods and the corresponding rates of convergence for 2,4 and 6 th order SBP-SAT methods.

is performed by successively increasing the number of grid points per wavelength used, P_S , and computing for 5 temporal periods. Maximum and l_2 errors are computed according to (38) from which the rate of convergence of the maximum error, $p_{\max}$ and the rate of convergence of the l_2 error, pl_2 , are computed. The results for the second, fourth and sixth order SBP-SAT methods are displayed in Table 1. Here the expected rate of convergence $\min\{2m, m+2\}$ is observed for $2m$ th order SBP-SAT methods, where $2m = 2, 4, 6$. For the sixth order SPB-SAT method we have performed the experiment using both a fourth and a sixth order method for the temporal discretization. The results are identical, indicating that the spatial discretization error is dominating. For this reason, in the following experiments we use a fourth order time-stepping method.

In our second experiment we take $\mu = 0.01$ and evolve the numerical solution for 5 periods in time. Figure 5 displays the maximum error as a function of time for a sequence of finer grids. Results for the second and fourth order SBP-SAT methods are shown in the left and right figures, respectively. Note that the grids used with the second order SBP-SAT method are much finer than those used with the fourth order SBP-SAT method. The stricter resolution requirement is exemplified by noting that for a maximum error of about 5% 101 grid points per interface wave length are needed using the second order SBP-SAT method. The corresponding quantity is 21 for the fourth order SBP-SAT method.

In our third experiment we use the fourth and sixth order SBP-SAT methods to propagate an interface wave in time. We choose the integer P_S such that $\mu^{1/p}P_S$ is approximately constant as μ decreases. Here we vary p depending on the method we use. The order of accuracy of the discretization of the normal derivatives appearing in the discretization of the tractions is 3 or 4 corresponding to the fourth and sixth order SBP-SAT method, respectively. The order of accuracy of the derivatives in the tangential direction is 4 and 6, respectively. At this point it is unknown if the truncation error from the discretization of derivatives in the normal direction or in the tangential direction is dominating. For this reason we first investigate how many grid points P_S per wavelength are needed to achieve an error of 0.1% after one period in time, as μ decreases. These values are then used to approximate p in the expression $P_S\mu^{1/p} = \text{constant}$ for each method. The recorded values of P_S are displayed in Table 2. For the fourth order SBP-SAT method we get $p \approx 4.53$. For the sixth order SBP-SAT method we get $p \approx 7.83$. Both values are larger than the order of the approximation of the tangential derivative for each method. For this reason we conjecture that the dominating errors are due to discretization of the tangential

Figure 5: Maximum errors as function of time

Fourth order SBP-SAT		Sixth order SBP-SAT	
μ	P_S	μ	P_S
0.1	26	0.1	20
0.01	45	0.01	29
0.001	71	0.001	39
0.0001	119	0.0001	49
0.00001	203	0.00001	67

Table 2: The required number of grid points P_S per wave length needed to achieve an maximum error of 0.1% after one temporal period for each method considered

derivatives and choose $p = 4$ and $p = 6$ for the fourth and sixth order SBP-SAT method, respectively. Note that the interface wave travels in the tangential direction and keeps a constant shape in the normal direction.

The maximum errors as functions of time are displayed in Figure 6(a) for the choice $p = 4$ and the fourth order SBP-SAT method and in Figure 6(b) for the choice $p = 6$ and the sixth order SBP-SAT method. The values of μ are 0.1, 0.01, 0.001, 0.0001, 0.00001. In both figures the magnitude of the error remains close together for all μ even though the spatial resolution increases as μ decreases. The figures also show that the errors grows linearly with the number of temporal periods. These observations corroborates equation (33) and indicates a correct choice of p for both methods.

Figure 6: Maximum errors measured as μ decreases and $P_S \mu^{1/p} \approx \text{constant}$

5.2. Mode conversion

Consider a compressional plane wave of unit amplitude traveling in the negative y -direction making the angle θ , $0 < \theta < \frac{\pi}{2}$, with the x -axis in a homogeneous elastic material. The displacement field is described by

$$\mathbf{u}^{(\text{in})} = \begin{pmatrix} \sin \theta \\ -\cos \theta \end{pmatrix} e^{i\gamma(\sin \theta x - \cos \theta y - c_p t)}. \quad (40)$$

Here $\gamma = \omega/c_p$ where ω is the temporal frequency. If this wave impinges on an interface to another half-plane it will be split into a reflected part traveling in the positive y -direction and a refracted part, traveling in the negative y -direction. Each part consisting of both pressure and shear waves. The reflected waves may be written

$$\begin{aligned}\mathbf{u}^{(\text{refl})} &= \mathbf{u}^{(\text{Prefl})} + \mathbf{u}^{(\text{Srefl})}, \\ \mathbf{u}^{(\text{Prefl})} &= A_2 \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix} e^{i\gamma(\sin \theta x + \cos \theta y - c_p t)}, \\ \mathbf{u}^{(\text{Srefl})} &= B_2 \begin{pmatrix} -\beta \cos \eta \\ \gamma \sin \theta \end{pmatrix} e^{i\beta \left(\frac{c_s}{c_p} \sin \theta x + \cos \eta y - c_s t \right)}.\end{aligned}\tag{41}$$

Here $\beta = \frac{\omega}{c_s}$ and $\sin \eta = \frac{c_s}{c_p} \sin \theta$. The refracted waves may be written

$$\begin{aligned}\mathbf{u}^{(\text{refr})} &= \mathbf{u}^{(\text{Prefr})} + \mathbf{u}^{(\text{Srefr})}, \\ \mathbf{u}^{(\text{Prefr})} &= A'_1 \begin{pmatrix} \gamma \sin \theta \\ \gamma' \cos \theta' \end{pmatrix} e^{i\gamma' \left(\frac{c'_p}{c_p} \sin \theta x - \cos \theta' y - c'_p t \right)}, \\ \mathbf{u}^{(\text{Srefr})} &= B'_1 \begin{pmatrix} -\beta' \eta' \\ \gamma \sin \theta \end{pmatrix} e^{i\beta' \left(\frac{c'_s}{c_p} \sin \theta x - \cos \eta' y - c'_s t \right)}.\end{aligned}\tag{42}$$

Here $\gamma' = \frac{\omega}{c'_p}$, $\beta' = \frac{\omega}{c'_s}$ and $\sin \eta' = \frac{c'_s}{c_p} \sin \theta$. The amplitudes B_2, A'_1 and B'_1 are functions of c_p, c_s, c'_p, c'_s and θ and are uniquely determined by the interface conditions (2) - (3), for a detailed discussion we refer to [9, p. 377-380].

The wavelengths of the reflected compressional and shear waves are given by

$$\begin{aligned}L_{\text{Prefl}} &= \frac{2\pi}{\sqrt{\sin^2 \theta + \cos^2 \theta}} = 2\pi, \\ L_{\text{Srefl}} &= \frac{2\pi}{\sqrt{\beta^2 \cos^2 \eta + \gamma^2 \sin^2 \theta}} = \frac{2\pi c_s}{\omega} = \frac{2\pi}{\omega} \sqrt{\frac{\mu}{\rho}}.\end{aligned}\tag{43}$$

The wavelengths of the refracted compressional and shear waves are given by

$$\begin{aligned}L_{\text{Prefr}} &= \frac{2\pi}{\sqrt{\sin^2 \theta + \cos^2 \theta}} = \frac{2\pi c'_p}{\omega} = \frac{2\pi}{\omega} \sqrt{\frac{(\lambda' + 2\mu')}{\rho'}}, \\ L_{\text{Srefr}} &= \frac{2\pi}{\beta'^2 \cos^2 \eta' + \gamma^2 \sin^2 \theta} = \frac{2\pi c'_s}{\omega} = \frac{2\pi}{\omega} \sqrt{\frac{\mu'}{\rho'}}.\end{aligned}\tag{44}$$

Note that for fixed a ω, ρ, ρ' and λ, λ' the wave length of the reflected and refracted shear waves are proportional to $\sqrt{\mu}$ and $\sqrt{\mu'}$, respectively.

In this experiment we chose the same values of the densities and Lamé parameters as in the previous experiment as well as the case $\mu = 1$. The reflected and refracted shear waves for $\mu = 1, 0.1, 0.01$ are displayed in Figure 7. We use the fourth order SBP-SAT method and fix the number of grid points per smallest wavelength as μ decreases at 11 points per wavelength. The solutions are propagated for 5 periods in time. In Figure 8 the maximum errors are reported as functions of time for the values of $\mu = 1, 0.1, 0.01, 0.001$. As the magnitude of the errors are of the same order we conclude that the mode conversion phenomena does not suffer from a similar stricter resolution requirement as do the interface waves. Instead the main difficulty of mode conversion in almost incompressible materials is that the reflected and refracted shear waves has a wavelength several magnitudes shorter than that of the incoming pressure wave.

5.3. Surface waves

The Rayleigh surface wave was investigated theoretically and numerically in [11]. There it was shown that in almost incompressible elastic materials the Rayleigh surface wave also requires a higher resolution of the type $P_R \approx \text{constant} (\mu/\lambda)^{-1/p}$ for accuracy. Here P_R is the number of grid points per wavelength of the surface wave. Numerical experiments that supported this requirement were done with a fourth order method and

Figure 7: u -component of the reflected and refracted shear waves as a function of (x, y) as $t = 0$. The angle of the incoming pressure wave is $\theta = \pi/4$. The snapshots correspond to $\mu = 1$, $\mu = 0.1$ and $\mu = 0.01$.

Figure 8: Maximum errors as μ decreases and the number of grid points per smallest wavelength is constant.

μ/λ as small as 10^{-3} . In the concluding remarks of their work [11] the authors suggests an experiment that investigates how small this quotient can be, using a numerical method of order higher than four.

In this experiment we perform computations that involve surface waves in elastic materials with $10^{-6} \leq \mu/\lambda \leq 10^{-2}$ using schemes of order of accuracy higher than four.

A first experiment is done as follows, for a given value of μ/λ we determine the number of grid points, P_R , per surface wavelength needed to achieve a maximum error of at most 5% after having propagated for 10 periods in time. As the number of grid points per surface wavelength needed for accuracy is proportional to $(\mu/\lambda)^{-1/p}$ where p is the order of accuracy of the method, high order of accuracy is expected to become more important as μ/λ decreases. A high order scheme requires more computational effort per grid point than a low order scheme. We have recorded the execution times in seconds, and used execution time to determine how small μ/λ must be for a higher order scheme to be more efficient.

We use an analytic expression of a Rayleigh surface wave

$$\begin{pmatrix} u \\ v \end{pmatrix} = A_2 \begin{pmatrix} \left(-\xi e^{-ry} + \frac{s^2 + \xi^2}{2\xi} e^{-sy} \right) \sin(\xi(x - c_R t)) \\ \left(-r e^{-ry} + \frac{s^2 + \xi^2}{2s} e^{-sy} \right) \cos(\xi(x - c_R t)) \end{pmatrix}, A_2 \in \mathbb{R}. \quad (45)$$

We keep the wavelength fixed at $L_R = 1$ by choosing $\xi = 2\pi$. The values of r and s then follow from (10). The constant A_2 is arbitrary, here we take $A_2 = -\frac{1}{2\pi}$ and keep $\rho = \lambda = 1$ fixed. The phase velocity c_R is proportional to $\sqrt{\mu}$ and thus decreases as μ decreases. The period, T , of the solution is proportional to $1/\sqrt{\mu}$. We consider a domain which is periodic in the x - direction. The computational domain is chosen to contain exactly one wavelength of the solution. At the boundary $y = 0$ a traction free boundary condition is imposed. The computational domain is truncated at $y = 10$ by imposing exact boundary data given by the exact solution (45). The numerical computations were made on a single Intel Xenon W3680 3.33 GHz processor.

Results for different values of μ/λ and different grid sizes are reported in Table 3. We see that for $10^{-3} \leq \mu/\lambda \leq 10^{-2}$ the sixth and eighth order SBP-SAT methods perform similarly. For $10^{-5} \leq \mu/\lambda \leq 10^{-4}$ the sixth order SBP-SAT method is clearly disadvantageous compared to the eight order SBP-SAT method. It uses more than 4 times the amount of time to obtain a relative maximum error of at most 5% for μ/λ in this interval. For $\mu/\lambda \leq 10^{-6}$ the required number of points per surface wavelength used by the 8th order scheme has increased very much above the value predicted by the classical theory and for such materials even higher order methods would be needed for an efficient numerical method.

Case	P_R	e_6	ET_6	e_8	ET_8
$\mu = 10^{-2}$	13	1.3×10^{-1}	4	1.9×10^{-2}	7
$T = 10.474$	25	9.0×10^{-3}	31	-	-
$\mu = 10^{-3}$	13	6.6×10^{-1}	11	7.5×10^{-2}	19
$T = 33.104$	25	2.7×10^{-2}	96	4.3×10^{-3}	173
$\mu = 10^{-4}$	13	3.5×10^0	36	3.0×10^{-1}	71
$T = 104.678$	25	1.4×10^{-1}	318	6.6×10^{-3}	545
	49	5.2×10^{-3}	2936	-	-
$\mu = 10^{-5}$	13	4.2×10^0	114	2.4×10^{-0}	196
$T = 331.020$	25	8.8×10^{-1}	990	1.8×10^{-2}	1746
	49	2.4×10^{-2}	7397	-	-
$\mu = 10^{-6}$	13	5.2×10^0	513	3.7×10^0	1043
$T = 1046.778$	25	4.5×10^0	4288	1.3×10^{-1}	9993
	49	1.8×10^{-1}	39233	1.9×10^{-3}	78775
	97	4.4×10^{-3}	322546	-	-

Table 3: The leftmost column displays the value of μ/λ and the period, T , of the solution. The rest of the columns report the number of grid points, P_R , per surface wavelength, the corresponding relative maximum errors, e_6 , e_8 recorded at time $t = 10T$ and execution times, ET_6 , ET_8 in seconds for schemes using 6-th and 8-th order SBP operators, respectively.

As an application we use the constructed table to gain a priori information on the required number of grid points when solving a version of Lamb's problem [12]. The difference between Lamb's original problem and our version is that where Lamb considered a single line source with loading normal to a plane surface we subject the surface of a half - space to a periodic array of line sources with loading normal to the surface. Under these conditions compressional, shear and Rayleigh waves are generated. The stress boundary conditions are

$$\begin{aligned} v_y + \frac{\lambda}{\lambda + 2\mu} u_x &= f(t)\delta(x - kM), M > 0, k = 0, \pm 1, \dots, \quad -\infty < x < \infty, y = 0. \\ u_y + v_x &= 0, \end{aligned}$$

where M is the distance between the sources and δ the Dirac delta function. A discussion on how to construct high-order approximations of the Dirac delta function is given in [20]. We let f be the wavelet given by

$$f(t) = \begin{cases} \sin(2\pi\omega t) - \frac{1}{2}\sin(4\pi\omega t), & 0 \leq t \leq \frac{1}{\omega} \\ 0, & \text{else.} \end{cases}$$

f is shown as an inset in Fig 9(a) with $\omega = 1$. With $\lambda = 1, \mu = 10^{-3}$ the Rayleigh phase velocity becomes

Figure 9: The numerical solution (a) and the relative maximum error (b) at time $t = 3.2$

$c_R \approx 0.954\sqrt{\mu} = 0.0302$. The amplitude of the Fourier transform of the time function f with $\omega = 1$ has local maxima in the vicinity of the points ± 1 and ± 2 and decays quadratically for larger frequencies. For this reason we estimate the highest significant frequency in f to be 2, then the corresponding shortest wavelength of the Rayleigh wave is approximately $L_{\min} = \frac{c_R}{2} = 0.0151$. We choose the domain $[-2L_{\min}, 2L_{\min}] \times [0, 6L_{\min}]$, $M = 4L_{\min}$ and solve numerically until time $t = 3.2$. Figure 9(a) shows the magnitude of the displacement field. Periodic boundary conditions are applied at the vertical boundaries and the domain is truncated above with a perfectly matched layer [3]. By consulting Table 3 we see that since $\mu/\lambda = 10^{-3}$ the sixth order SBP-SAT method should be used.

To compute the error in our solution we construct a reference solution that uses 400 points per shortest wavelength, which corresponds to using a grid with 1600×1600 grid points. We verify that the reference solution is sufficiently converged by using it to compute the l_2 error on successively refined grids of size $N \times N$ and plotting the error versus N . Figure 10 shows that the asymptotic rate of convergence of the l_2 error approaches the expected value 5, which indicates that the reference solution is computed on a sufficiently fine grid.

To estimate the required number of points per wavelength to achieve a relative maximum error of at most 5% using the sixth order SBP-SAT method we consult Table 2 to conclude that 25 points per wavelength should

P_R	e_6
10	7.5×10^{-1}
25	2.7×10^{-2}

Table 4: Points per shortest wavelength P_R and corresponding maximum error e_6 .

Figure 10: The l_2 errors in each of the horizontal and vertical displacements as functions of number of grid points. The line corresponding to order of convergence 5 is plotted for reference.

suffice. As a comparison a solution using 10 points per shortest wavelength, a quantity predicted by the classical theory to yield a relative error lower than 5%, is also computed. The results presented in Table 4 verifies the claim for this application. Figure (9(b)) shows the relative error in the magnitude of the displacement field at time $t = 3.2$. It is interesting to see that the main bulk of the error is seen to be located in the vicinity of the surface. This is in accordance with the theory presented in [11], which predicts that the Rayleigh waves are much more sensitive to discretization errors than the shear and pressure waves.

6. Conclusions

For two elastic materials in contact there may exist a Stoneley interface wave at the surface of contact. We have shown that when these interface waves appears in numerical simulations they suffer from a higher than expected resolution requirement in terms of grid points per wavelength when the present elastic materials become almost incompressible. The theoretical findings show that the required number of points per wavelength scales as $(\mu/\lambda)^{-1/p}$, where λ and μ are the first and second Lamé parameters, respectively, and p is the order of accuracy of the numerical method. The theoretical study was exemplified with numerical experiments that uses finite-difference methods to approximate derivatives. The methods used were of high order, in particular of order higher than 2.

The Stoneley interface wave propagates in the direction tangential to the surface of contact and remains a constant shape in the normal direction. In the numerical experiments all spatial derivatives of the wave are of the same magnitude regardless of the quotient μ/λ . For stability reasons normal and tangential derivatives are approximated to different orders of accuracy in the vicinity of the boundary, the order of approximation of tangential derivatives being in general the largest. Quantitative results from experiments show that it is the higher order of accuracy of the discretization of tangential derivatives that enter in to the estimates. That is, a lower order of accuracy in the normal direction does not seem to affect the overall behavior of the interface wave.

Another feature of elastic wave propagation in heterogeneous materials is that waves of a certain type, shear or pressure, reflects and refracts as both shear and pressure upon impinging on an interface to another material. We show numerically that these refracted and reflected waves does not require an unexpected number of points per wavelength for accuracy. But instead the classical results on numerical simulation of hyperbolic wave propagation [10] holds in this case.

An analogous estimate on the number of grid points per wavelength was already known for the Rayleigh surface wave [11]. We have also performed numerical experiments involving surface waves at a traction free

surface. These experiments was done with methods that discretize the tangential derivatives at the surface of order of accuracy six and eight, respectively, and were included to study how small μ/λ must be to compensate for the higher complexity of higher order methods. The results were then used in the numerical solution of a version of Lamb's problem.

Appendix A. The function $\theta(\tilde{c})$

Define

$$D = \det \begin{vmatrix} 2X & 1+Y^2 & K(1+Y^2) \\ 1 & Y & -Y \\ X & 1 & 1 \end{vmatrix},$$

where X and Y are defined by (18). and $K \in (0.01, 0.99)$. Inserting the expression (25) for g_2 in the solution formula (23) with $g_1 = 0$ we get

$$\begin{aligned} \psi(\tilde{c})iA'_1 = & -\frac{\lambda+2\mu}{\mu}h^2\xi^2D((iA_2-B_2Y)\tau_{21}+(iA_2X^4-B_2Y^3)\tau_{22} \\ & + (iA'_1+B'_1Y)\tau_{23}+(iA'_1X^4-B'_1Y^3)\tau_{24}) \end{aligned} \quad (\text{A.1})$$

The function $\theta(\tilde{c})$ is obtained by using the expressions (26) for $iA_2, -B_2$ and B'_1 in the right hand side of (A.1) and factoring out the term A'_1 . For $\mu \rightarrow 0$ and $K \in (0.01, 0.99)$ we have

$$\begin{aligned} \tilde{c}_0 & \in (0.92, 0.99), \\ X & = \left(1 - \frac{\mu\tilde{c}_0^2}{\lambda+2\mu}\right)^{1/2} \approx 1, \\ Y & = (1 - \tilde{c}_0^2)^{1/2} \in (3.57 \times 10^{-5}, 0.40). \end{aligned} \quad (\text{A.2})$$

Using (A.2) we may approximate $\theta(\tilde{c}_0)$ as $\mu \rightarrow 0$ by

$$\begin{aligned} \theta(\tilde{c}_0) & \approx \frac{-\lambda h^2 \xi^2}{\mu} (\kappa_1 \tau_{21} + \kappa_2 \tau_{22} + \kappa_3 \tau_{23} + \kappa_4 \tau_{24}), \\ \kappa_1 & \in (1.02, 118.54), \kappa_2 \in (1.01, 77.78), \\ \kappa_3 & \in (1.01, 21.32), \kappa_4 \in (-1.86, 1.17). \end{aligned}$$

Appendix B. Parameter values for the Stoneley interface wave

In the numerical experiments the upper elastic half-plane was defined by $\lambda = 1, \rho = 1$ and $\mu = 0.1, 0.01, 0.001, 0.0001, 0.00001$. The lower elastic half-plane was defined by $\lambda' = 3, \rho' = 1.9999, \mu' = 2\mu$. The vertical and horizontal components of a Stoneley interface wave with wavelength one is then described by

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix} & = \begin{pmatrix} iA_2 e^{-ry} - sB_2 e^{-sy} \\ -rA_2 e^{-ry} - iB_2 e^{-sy} \end{pmatrix} e^{i(x-ct)}, y > 0, \\ \begin{pmatrix} u' \\ v' \end{pmatrix} & = \begin{pmatrix} iA'_1 e^{r'y} + s'B'_1 e^{s'y} \\ r'A'_1 e^{r'y} - iB'_1 e^{s'y} \end{pmatrix} e^{i(x-ct)}, y < 0, \end{aligned}$$

where

$$r = \sqrt{1 - c^2 \frac{\rho}{\lambda + 2\mu}}, r' = \sqrt{1 - c^2 \frac{\rho'}{\lambda' + 2\mu'}}, s = \sqrt{1 - c^2 \frac{\rho}{\mu}}, s' = \sqrt{1 - c^2 \frac{\rho'}{\mu'}}.$$

We fix $A_2 = -i$ the values of c, B_2, A'_1 and B'_1 for each μ considered are then computed in Table B.5

μ	c	B_2	A'_1	B'_1
0.1	0.315138096754869	-3.417353225815823	$-i1.792431528663013$	-6.114370209408633
0.01	0.099675390917174	-3.511113557139498	$-i1.791132691635746$	-6.291501145397127
0.001	0.031520802631186	-3.521549959800072	$-i1.790982715627473$	-6.311444334798907
0.0001	0.009967774542550	-3.522605427057088	$-i1.790967482603149$	-6.313463935215666
0.00001	0.003152087757500	-3.522711093663645	$-i1.790965957030506$	-6.313666151843764

Table B.5: The computed values of c , B_2 , A'_1 and B'_1 for all values of μ used in the experiments.

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