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Numerics of Elastic and Acoustic Wave Motion

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Abstract
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The elastic wave equation describes the propagation of elastic disturbances produced by seismic events in the Earth or vibrations in plates and beams. The acoustic wave equation governs the propagation of sound. The description of the wave fields resulting from an initial configuration or time dependent forces is a valuable tool when gaining insight into the effects of the layering of the Earth, the propagation of earthquakes or the behavior of underwater sound. In the most general case exact solutions to both the elastic wave equation and the acoustic wave equation are impossible to construct. Numerical methods that produce approximative solutions to the underlaying equations now become valuable tools. In this thesis we construct numerical solvers for the elastic and acoustic wave equations with focus on stability, high order of accuracy, boundary conditions and geometric flexibility. The numerical solvers are used to study wave boundary interactions and effects of curved geometries. We also compare the methods that we have constructed to other methods for the simulation of elastic and acoustic wave motion.

Keywords: finite differences, stability, high order accuracy, elastic wave equation, acoustic wave equation

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Dedicated to all my past teachers,
from elementary school to university

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I K. Duru and K. Virta. Stable and High Order Accurate Difference Methods for the Elastic Wave Equation in Discontinuous Media. In *Journal of Computational Physics*, 2014, Volume 279, pp. 37-62.
Contributions: The author of this thesis initiated the project and developed the ideas together with the first author of the paper. The author of this thesis implemented and designed all numerical experiments except Experiment 5.2 and wrote parts of the manuscript.

II K. Virta and G. Kreiss. Interface Waves in Almost Incompressible Elastic Materials. In *Journal of Computational Physics*, 2015, Volume 303, pp. 313-330.
Contributions: The author of this thesis developed the analysis, designed and performed the numerical experiments. The manuscript was written in discussion with the second author.

III K. Virta and D. Appelö. Formulae and Software for Particular Solutions to the Elastic Wave Equation in Curved Geometries. Submitted to Journal of Computational Physics, (under review).
Contributions: The author of this thesis developed the analysis in this paper. The manuscript was written and the numerical experiments were performed in close collaboration with the second author.

IV K. Virta, C. Juhlin and G.Kreiss. Elastic wave propagation in complex geometries: A qualitative comparison between two high order finite difference methods. Technical report, arXiv:1511.07596 [math.NA].
Contributions: The author of this thesis designed and performed the numerical experiments in close collaboration with the second author. The manuscript was written in discussion with the second and third author.

V K. Virta and K. Mattsson. Acoustic Wave Propagation in Complicated Geometries and Heterogeneous Media. In *Journal of Scientific Computing*, 2014, Volume 61, pp. 90-118.
Contributions: The author of this thesis developed the analysis, designed and performed the numerical experiments. The manuscript was written in discussion with the second author.

VI S. Wang, K. Virta and G. Kreiss. High order finite difference methods for the wave equation with non-conforming grid interfaces. Submitted to Journal of Scientific Computing (under review).
Contributions: The author of this thesis initiated the project and developed the ideas together with the first author and third author of the paper. The author of this thesis implemented and designed the numerical experiments in close collaboration with the first author and wrote parts of the manuscript.

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Contents

1. Introduction

A disturbance, confined to a bounded part of a medium, which propagates with a finite speed to other parts of the medium forms the basis for the study of wave propagation. Wave propagation manifests in forms that are familiar in everyday life such as acoustic waves from musical instruments, water waves breaking on a coastline or elastic displacements in the Earth. Associated with a propagating wave apart from the underlying type of media and its propagation speed is its frequency and wavelength. A particular type of wave propagation arises when the speed of propagation is determined by the underlying media and independent of both frequency and wavelength. Such wave propagation is called hyperbolic. Acoustic and elastic waves are examples of hyperbolic waves, although we in what follows we will encounter particular waves which have a speed of propagation that depend on the geometry of the underlying domain. Water waves are traveling with a speed that also depend on their frequency, making them an example of dispersive non-hyperbolic waves. For a thorough discussion on dispersive wave propagation see the volume "Linear and nonlinear waves" [56] by G. B. Whitham. This thesis is about elastic and acoustic wave propagation.

In an elastic material at least two types of waves can propagate, pressure waves and shear waves, whereas in an acoustic material only pressure waves are propagating. Both shear waves and pressure waves are governed by the same equation, the wave equation. It is when the present waves strike a boundary that features of elastic wave propagation become distinguished from the acoustic counterpart. Now the wave equation governing the propagation of shear waves is coupled through boundary conditions to the wave equation governing pressure waves, the resulting system constitutes the elastic wave equation. The coupling of the equations at a boundary can for instance have as an effect that one type of wave, be it pressure or shear, will in general upon striking a boundary reflect as waves of both types. This phenomena is called mode conversion has no counterpart in acoustics. Another boundary phenomena supported by the elastic wave equation is the existence of a wave that clings to a boundary and is independent of incoming shear and pressure waves, this wave is called a Rayleigh surface wave after its finder Lord Rayleigh [45] and accounts for the significantly damage causing effects of aftershocks of earthquakes.

In general, the disturbance that generates propagating waves can either manifest in an initial state or as a time dependent forcing, the geometry of the underlying domain can have a complex structure with curved boundaries, and the media may have discontinuous parameters with the discontinuity taking place along a non-planar interface. The complexity of boundary conditions, transient time behavior, geometries and material properties make the description of the resulting wave field hard, if not impossible, to express using mathematical analysis. For this reason we abandon the ambition to seek exact solutions for the general case and instead consider numerical approximations of the solution at a finite number of points in space and time. Although this makes it possible to arrive at approximate descriptions using an electronic computer machine, questions arise concerning the particular method that was used to obtain the approximation. These questions concern convergence to the true solution as the level of refinement increases, time stability and order of accuracy. Technical details such as computational efficiency and ease of implementation also become important factors.

In this thesis we construct one numerical solver for the elastic wave equation and one for the acoustic wave equation. The focus when constructing the numerical methods is in particular on approximating the correct boundary conditions so that the numerical solution is able to capture all significant wave boundary interactions. Other key aspects of the numerical methods are high order of accuracy, time stability and geometric flexibility. The methods are then used to study wave interface interaction, wave fields in curved geometries and for comparison between other numerical methods that solve the same problem with another solution strategy. Before presenting the manuscripts that form this thesis there is an introductory part in which the author has selected a limited number of topics that he feels give a good start to an excursion in the numerics of elastic and acoustic wave motion.

2. Elastic and acoustic wave motion

The first problem of this thesis considers a heterogeneous elastic medium compounded of a number of homogeneous elastic bodies. Each elastic body is being either semi-infinite or bounded. The elastic bodies that constitute the entire elastic medium are in welded contact along internal interfaces of arbitrary shape. Boundaries that are not welded to a differing material are assumed to be adjacent to a vacuum. At an initial time the elastic medium is in an initial state and both internal and boundary forces are acting on the medium. We seek to describe the time evolution of the spatial disturbance in the medium arising from the initial state and the present forces, with particular emphasis being put on the effect of outer boundaries and internal interfaces. The disturbance is represented by a vector field of spatial displacements of the elastic material away from its equilibrium state.

The first problem is representative of seismic events in the Earth due to its layered structure, its curved surface and the appearance of internal inclusions and cavities. A seismic source may arise due to an earthquake or due to man-made signals placed at its surface to study their response and give a description of the internal structures of the Earth. Internal cavities and inclusions can arise due to deposits of gas or oil and ore bodies, respectively. An illustration is given in Figure 2.1 which shows a fictive cross-section of the upper part of the Earth's crust.

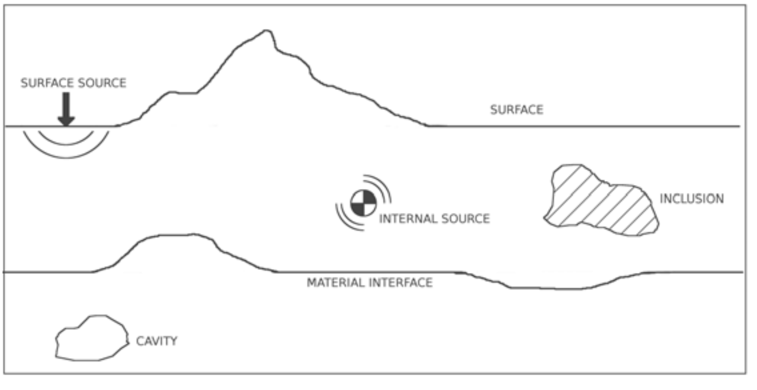

Figure 2.1. A fictive cross-section of the upper part of the crust of the Earth.

The second problem of this thesis considers the acoustic counterpart of the first problem. Now disturbances in the material manifests in a disturbance of the scalar valued pressure in the material.

The second problem is representative of sound propagation in the oceans and soft seabed sediments of the Earth. In this case an acoustic source may be a sonar used to locate underwater objects or to communicate in the ocean as electromagnetic radar and radio waves are highly attenuated in water.

2.1 The elastic wave equation

In this section we present the elastic wave equation in two spatial dimensions and in the Cartesian coordinate system x,y. We begin by introducing the equation in a homogeneous infinite medium. Boundaries and material heterogeneities are considered in the next section.

An elastic medium Ω is defined by its density $\rho > 0$ and the Lamé parameters $\lambda > 0$ and $\mu > 0$. Let $u(t,x,y), v(t,x,y)$ be the horizontal and vertical displacement components, respectively, at time t and the point (x,y). The displacements are resulting from an initial configuration $u(0,x,y)$, $v(0,x,y)$, $u_t(0,x,y)$, $v_t(0,x,y)$ and forces acting on the medium via $f(t,x,y), g(t,x,y)$. The evolution of u and v is governed by elastic wave equation

$$\begin{aligned} \rho u_{tt} &= \mu\,(u_{xx}+u_{yy}) + (\lambda+\mu)\,(u_x+v_y)_x + \rho f \\ \rho v_{tt} &= \mu\,(v_{xx}+v_{yy}) + (\lambda+\mu)\,(u_x+v_y)_y + \rho g, \end{aligned}, (x,y) \in \Omega, t > 0. \tag{2.1}$$

An alternative formulation of the elastic wave equation is obtained by introducing functions $\phi(t,x,y), \psi(t,x,y), \alpha(t,x,y), \beta(t,x,y)$ and assuming that

$$\begin{aligned} u &= \phi_x + \psi_y, \\ v &= \phi_y - \psi_x, \\ f &= \alpha_x + \beta_y, \\ g &= \alpha_y - \beta_x. \end{aligned} \tag{2.2}$$

Inserting (2.2) into (2.1) we get

$$\begin{aligned} \left(\phi_{tt} - c_p^2 \nabla^2 \phi - \alpha\right)_x + \left(\psi_{tt} - c_s^2 \nabla^2 \psi - \beta\right)_y &= 0, \\ \left(\phi_{tt} - c_p^2 \nabla^2 \phi - \alpha\right)_y - \left(\psi_{tt} - c_s^2 \nabla^2 \psi - \beta\right)_x &= 0. \end{aligned} \tag{2.3}$$

where $c_p = \sqrt{(\lambda+2\mu)/\rho}$ and $c_s = \sqrt{\mu/\rho}$. That is, if ϕ satisfies the wave equation

$$\phi_{tt} = c_p^2 \nabla^2 \phi + \alpha \tag{2.4}$$

and if ψ satisfies the wave equation

$$\psi_{tt} = c_s^2 \nabla^2 \psi + \beta, \tag{2.5}$$

then (2.3) holds. The possibility of solutions to (2.3) not satisfying (2.4) and (2.5) have been ruled out by Sternberg [50]. The complete solution to (2.3), and via (2.2) the solution to (2.1), are now given by the solutions to (2.4) - (2.5). We have stated the elastic wave equation (2.1) in terms of the wave equation (2.4) which governs waves propagating with wave speed c_p, and the wave equation (2.5) which governs waves propagating with wave speed c_s. It is now transparent that in general at least two different types waves are propagating, independent of each other, in an elastic body. In fact, it can be shown [18] that the solution ϕ represents traveling pressure waves (P-waves), and that the solution ψ represents traveling shear waves (S-waves).

Example 1: P-waves

In the introduction to his volume "Reflection and Refraction of Progressive Seismic Waves" [9] L. Cagniard suggests that the fundamental problem of seismology may be:

two homogeneous, isotropic, perfectly elastic, semi-infinite media are in welded contact at a plane surface. In one of the media there is set up an initially spherical, progressive, isotropic, compressional elastic wave. We seek to describe the displacement exactly at any point in either medium as a function of time, especially the phenomena which arise from the influence of the plane boundary on the incident spherical wave.

That is, Cagniard suggests choosing initial data $u(0,x,y)$ and $v(0,x,y)$ or forcing functions f and g to the elastic wave equation (2.1) such that initially only P-waves (compressional waves) are present. Looking at (2.1) it is not immediately clear how such a choice should be done. Using the alternative formulation (2.2), (2.4) and (2.5) it becomes clear that if $\psi(0,x,y) \equiv 0$ and $\beta(t,x,y) \equiv 0$, then only P-waves are present initially. Proper choices for $u(0,x,y)$,$v(0,x,y)$,$u_t(0,x,y)$,$v_t(0,x,y)$, f and g are now obtained via (2.2). In Paper I we use this observation for the construction of initial data to study a tapered sinusoidal plane P-wave wave that impinges on an irregular shaped cavity. In Paper IV we use the observation to chose forcing functions representing a point source emitting only P-waves $\square$

We define the tractions in the elastic medium as

$$\begin{aligned} \tau^{xx} &= (\lambda + 2\mu)\, u_x + \lambda v_y, \\ \tau^{xy} &= \mu\,(u_y + v_x) = \tau^{yx}, \\ \tau^{yy} &= (\lambda + 2\mu)\, v_y + \lambda u_x. \end{aligned} \tag{2.6}$$

Let

$$W = \begin{bmatrix} \rho \dot{u} \\ \rho \dot{v} \\ \tau^{xx} \\ \tau^{xy} \\ \tau^{yy} \end{bmatrix}, F = \begin{bmatrix} \rho f \\ \rho g \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

where $\dot{u}$ and $\dot{v}$ denotes the velocities of the horizontal and vertical displacements, respectively. We can now write the elastic wave equation in the first order velocity-stress formulation,

$$W_t = AW_x + BW_y + F. \tag{2.7}$$

Here

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ (\lambda+2\mu) & 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \lambda & 0 & 0 & 0 \\ \mu & 0 & 0 & 0 & 0 \\ 0 & (\lambda+2\mu) & 0 & 0 & 0 \end{bmatrix}.$$

Let (ξ, η) be a unit vector. The eigenvalues of $\xi A + \eta B$ are

$$e_1 = 0, e_{2,3} = \pm\sqrt{(\lambda+2\mu)}, e_{4,5} = \pm\sqrt{\mu},$$

with the corresponding eigenvectors

$$l_1 = \begin{bmatrix} 0 \\ 0 \\ \eta^2/\xi^2 \\ -\eta/\xi \\ 1 \end{bmatrix}, l_{2,3} = \begin{bmatrix} \pm\frac{\xi\sqrt{\lambda+2\mu}}{2\eta^2\mu+\lambda} \\ \pm\frac{\eta\sqrt{\lambda+2\mu}}{2\eta^2\mu+\lambda} \\ \frac{\lambda+2\mu\xi^2}{\lambda+2\eta^2\mu} \\ \frac{2\eta\mu\xi}{\lambda+2\eta^2\mu} \\ 1 \end{bmatrix}, l_{4,5} = \begin{bmatrix} \mp\frac{\sqrt{\mu}}{2\mu\xi} \\ \pm\frac{\sqrt{\mu}}{2\mu\eta} \\ -1 \\ -\frac{\eta^2-\xi^2}{2\eta\xi} \\ 1 \end{bmatrix}.$$

The eigenvalues distinct and real and the eigenvectors are linearly independent. Hence, the Cauchy problem represented by the equation (2.7) is well-posed and hyperbolic [20]. This indicates that also the Cauchy problem represented by equation (2.1) is well-posed.

2.1.1 Boundary and interface conditions

At a boundary $\partial\Omega$ of the elastic medium Ω adjacent to a vacuum the tractions are subjected to forces represented by the functions r and s. Let $n = (n^x, n^y)$ be one normal of $\partial\Omega$, the boundary conditions are

$$\begin{aligned} \tau^{xx}n^x + \tau^{xy}n^y &= r, \\ \tau^{yx}n^x + \tau^{yy}n^y &= s, \end{aligned} \quad (x,y) \in \partial\Omega, t \geq 0. \tag{2.8}$$

When $r \equiv s \equiv 0$ we say that the boundary condition is traction free. Let I be an internal interface of Ω where ρ, μ and λ have discontinuities. The discontinuities represent two differing materials, we assume that these materials are in welded contact. Let $[\rho]_I$ e.t.c., denote the jump in ρ e.t.c., at I. At I the welded contact requires that continuity of both displacements u, v and tractions is imposed,

$$\begin{aligned} &[u]_I = 0, \\ &[v]_I = 0, \\ &[\tau^{xx}n^x + \tau^{xy}n^y]_I = 0, \\ &[\tau^{yy}n^x + \tau^{yx}n^y]_I = 0, \end{aligned}, (x,y) \in I, t \geq 0. \tag{2.9}$$

Here n^x, n^y are the components of one normal of I.

Example 2: Surface and interface phenomena

Returning to Cagniard's formulation of the fundamental problem of seismology stated in Example 2.1 we note that Cagniard suggests that particular emphasis is put on the phenomena that arise from the influence of boundaries and internal interfaces in the elastic medium. We have seen that in the interior of an elastic body P-waves and S-waves travel independent of each other. The corresponding wave equations (2.4) and (2.5) are coupled only via the boundary conditions (2.8) and the interface conditions (2.9). A consequence is that the incidence of say a P-wave on a boundary or interface generates scattered S-waves. This phenomenon is called mode conversion. Numerical experiments and analytic expressions involving the mode conversion phenomenon are described in papers I,II and III of this thesis.

Boundary and interface phenomena are closely related to seismology, as it was early observed that earthquakes consisted of two early tremors corresponding to the arrival of P and S waves, followed by a third more damage-causing tremor. Such a third arrival was first investigated by Lord Rayleigh [45]. Lord Rayleigh showed that a third type of wave could exist and that it is dependent on the presence of a traction free surface but independent of any incoming waves. This wave has its energy concentrated close to the surface, accounting for its damage causing effects and is called a Rayleigh surface wave. A natural question is whether a similar type of wave can exist at an interface where two elastic bodies are welded together. Stoneley investigated the existence of such a wave [51] and showed that they can appear, for example at internal layers in the crust of the Earth. This fourth type of wave is called a Stoneley wave and has properties similar to the Rayleigh surface wave. Paper II is devoted to a study of the Stoneley interface wave.

A fifth type of wave can be observed as a shear wave polarized in the transverse (out-of-plane) direction. Love explained that this type of wave is a consequence of a layered structure of the Earth [34], in particular when a layer of finite thickness is overlaying a much wider layer of a differing material. Under these circumstances waves can be trapped in the thinner layer and generate an interface wave in the thicker layer, which is dependent on the trapped waves. Such a wave is called a Love wave. Figure 2.2 displays Rayleigh, Stoneley and Love waves clinging to a plane boundary or interface.

It is interesting to note that surface and interface waves are dispersive and that the dispersion enter through geometrical features of the domain [48, 34, 15] such as the local radius of curvature of the surface or the thickness of the present layers. In Paper III we construct closed form expressions that represent surface waves on curved surfaces and see that the corresponding phase velocities depend on the radius of curvature of the surface □

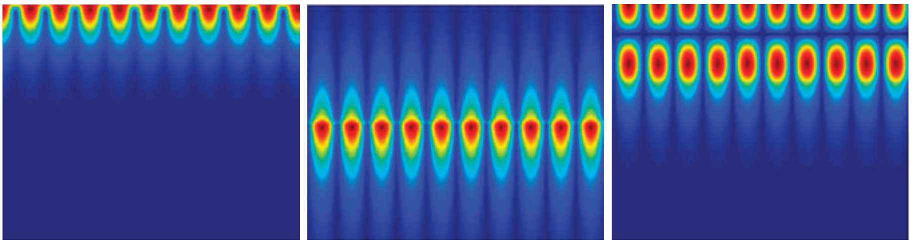

Figure 2.2. Left: Rayleigh surface waves, Middle: Stoneley interface waves, Right: Love waves

2.1.2 The elastic wave equation in curvilinear coordinates

To model elastic wave motion in bodies with specific geometries it is more convenient, albeit more technical, to formulate the elastic wave equation in curvilinear coordinates. Let the domain of interest Ω be such that there is a one-to-one and differentiable mapping from the unit square $S = [0,1] \times [0,1]$ to Ω,

$$x = x(\xi,\eta), y = y(\xi,\eta), (\xi,\eta) \in S. \tag{2.10}$$

Since the mapping is one-to-one and differentiable its inverse exists and is differentiable,

$$\xi = \xi(x,y), \eta = \eta(x,y), (x,y) \in \Omega.$$

By the chain rule, in S we have

$$\frac{\partial}{\partial x} = \xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta}, \frac{\partial}{\partial y} = \xi_y \frac{\partial}{\partial \xi} + \eta_y \frac{\partial}{\partial \eta}, \tag{2.11}$$

and in Ω we have

$$\frac{\partial}{\partial \xi} = x_\xi \frac{\partial}{\partial x} + y_\xi \frac{\partial}{\partial y}, \frac{\partial}{\partial \eta} = x_\eta \frac{\partial}{\partial x} + y_\eta \frac{\partial}{\partial y}. \tag{2.12}$$

From (2.11) and (2.12) we get

$$\begin{bmatrix} \xi_x & \eta_x \\ \xi_y & \eta_y \end{bmatrix} = \frac{1}{J} \begin{bmatrix} y_\eta & -y_\xi \\ -x_\eta & x_\xi \end{bmatrix},$$

where $J = x_\xi y_\eta - y_\xi x_\eta$ is the Jacobian of the mapping from S to Ω. Since the mapping is one-to-one J is always non-zero. Let $w(x,y)$ be a two times differentiable function in Ω. We get

$$w_x = \frac{(y_\eta w)_\xi - (y_\xi w)_\eta}{J}, w_y = \frac{-(x_\eta w)_\xi + (x_\xi w)_\eta}{J}. \tag{2.13}$$

By using (2.13) repeatedly (for details see [26]) and setting

$$\begin{aligned}
a_{11} &= \frac{(\lambda+\mu)y_\eta^2 + \mu(x_\eta^2 + y_\eta^2)}{J},\\
a_{12} = a_{21} &= -\frac{(\mu+\lambda)y_\xi y_\eta + \mu(x_\xi x_\eta + y_\xi y_\eta)}{J},\\
a_{22} &= \frac{(\lambda+\mu)y_\xi^2 + \mu(x_\xi^2 + y_\xi^2)}{J}\\
b_{11} = c_{11} &= -\frac{(\lambda+\mu)y_\eta x_\eta}{J},\\
b_{12} = c_{21} &= \frac{\lambda x_\xi y_\eta + \mu y_\xi x_\eta}{J},\\
b_{21} = c_{12} &= \frac{\lambda x_\eta y_\xi + \mu y_\eta x_\xi}{J},\\
b_{22} = c_{22} &= -\frac{(\lambda+\mu)y_\xi x_\xi}{J}\\
d_{11} &= \frac{(\lambda+\mu)x_\eta^2 + \mu(x_\eta^2 + y_\eta^2)}{J},\\
d_{22} &= \frac{(\lambda+\mu)x_\xi^2 + \mu(x_\xi^2 + y_\xi^2)}{J},\\
d_{12} = d_{21} &= -\frac{(\lambda+\mu)x_\xi x_\eta + \mu(y_\xi y_\eta)}{J}
\end{aligned}$$

We can write the elastic wave equation in the curvilinear coordinates ξ, η,

$$\begin{aligned}
\rho J u_{tt} &= (a_{11}u_\xi)_\xi + (a_{12}u_\xi)_\eta + (a_{21}u_\eta)_\xi + (a_{22}u_\eta)_\eta\\
&\quad + (b_{11}v_\xi)_\xi + (b_{12}v_\xi)_\eta + (b_{21}v_\eta)_\xi + (b_{22}v_\eta)_\eta + Jf,\\
\rho J v_{tt} &= (c_{11}u_\xi)_\xi + (c_{12}u_\xi)_\eta + (c_{21}u_\eta)_\xi + (c_{22}u_\eta)_\eta\\
&\quad + (d_{11}v_\xi)_\xi + (d_{12}v_\xi)_\eta + (d_{21}v_\eta)_\xi + (d_{22}v_\eta)_\eta + Jg.
\end{aligned} \tag{2.14}$$

The wave equation in curvilinear coordinates is now solved on the unit square S. The effects of the geometry of Ω are now seen through anisotropic wave motion in the solution to (2.14) on S. That is, the waves move in preferred directions as a result of the geometry of Ω. Anisotropic wave motion also appears in materials such as crystals, wood and certain rocks. Elastic wave motion in an anisotropic material that can be represented by parameters a_{11}

e.t.c., can also be described by (2.14). In a general application the parameters a_{11} e.t.c. result from both anisotropic media and a transformation to curvilinear coordinates.

One limitation of the treatment of geometries by a transformation to curvilinear coordinates is that it requires a suitable mapping (2.10) from the unit square. Consider a cavity embedded in a square as in Figure 2.3, the author knows of no suitable mapping of the resulting domain from the unit square. By cutting the domain along the line segments ab, cd, ef and gh we decompose the domain into four blocks where each block has a suitable mapping $x_i = x_i(\xi,\eta), y_i = y_i(\xi,\eta), i = 1,2,3,4$ from the unit square. The blocks are then patched together at the artificial interfaces ab, cd, ef and gh by imposing the conditions (2.9). In this way wave motion in much more general domains may be modeled by the elastic wave equation in curvilinear coordinates. This strategy is used for the elastic wave equaiton in papers I, III and IV of this thesis. The analogous treatment of geometries for acoustic wave propagation is used in papers V and VI of this thesis.

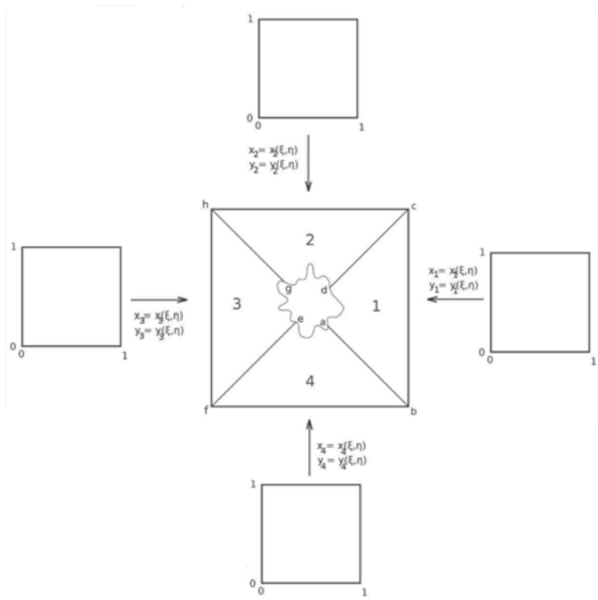

Figure 2.3. Illustration of representing a cavity in a square as a collection of four sub domains.

2.2 The acoustic wave equation

In an acoustic medium with wave speed $c > 0$ and density $\rho > 0$ let the scalar u be the disturbance in pressure arising from an initial state and an interior source described by f. The evolution of u is governed by the acoustic wave equation

$$\frac{1}{c^2}u_{tt} - \frac{1}{\rho}\nabla\cdot(\rho\nabla u) = f. \tag{2.15}$$

The acoustic wave equation finds application in for example underwater acoustics, where the domain of interest is a column of water bounded above by the sea surface and below by layers of acoustic bottom sediments, which in turn end with the totally reflecting bedrock or a damping elastic bottom. At the vertical sides of the column of water the acoustic waves are assumed to radiate out to infinity. In this setting a number of boundary conditions appears. Let Ω denote the total acoustic medium and $\partial\Omega_{SS}$, $\partial\Omega_{BS}$, $\partial\Omega_{RB}$ and $\partial\Omega_{EB}$ denote the sea surface, a surface interface between bottom sediments, the surface of the totally reflecting bedrock and the surface of a damping elastic bottom, respectively. Boundary conditions that arise in applications are

$$u = 0, x \in \partial\Omega_{SS} \tag{2.16}$$

$$\left\{\begin{array}{l} [u]_{\partial\Omega_{BS}} = 0, \\ [\rho\partial_n u]_{\partial\Omega_{BS}} = 0, \end{array}\right. x \in \partial\Omega_{BS} \tag{2.17}$$

$$\partial_n u = 0, x \in \Omega_{RB}, \tag{2.18}$$

$$\partial_n u + \alpha u + \beta u_t = 0, x \in \partial\Omega_{EB}. \tag{2.19}$$

Here ∂_n denotes the normal derivative at the surface. The parameters α and β are determined by the material properties of the damping elastic bottom.

With the boundary conditions (2.16)-(2.19) the sum of the kinetic and potential energy of solutions to the acoustic wave equation is bounded in terms of initial data and interior forcing. That is,

$$\int_\Omega \left(\frac{\rho}{c^2}u_t^2 + \rho\nabla u\cdot\nabla u\right) d\Omega + \int_{\partial\Omega_{EB}} \alpha u^2 dS \le \int_0^t \int_\Omega \rho c^2 f^2 d\Omega d\tau + C, \tag{2.20}$$

where the constant C is given by initial data, a derivation of the estimate (2.20) is given in Paper V. We have seen that the elastic wave equation can be written as a system of scalar wave equations. For this reason a similar estimate also holds for the elastic wave equation with the boundary conditions (2.8) and interface conditions (2.9). One key property of the numerical methods constructed in this thesis is that the numerical solutions they produce all satisfy analogous estimates i.e., the numerical methods does not allow for any unphysical growth in the numerical solution.

3. Numerical solution of the governing equations

In general, the domain where we seek a solution to the problem of elastic and acoustic wave propagation has a complicated structure. Layers, inclusions and cavities can have complex geometrical features and the material may be anisotropic i.e., waves may travel in preferred directions. Anisotropy in the most general case is represented by variable material parameters that may also be discontinuous. Present sources are usually transient so that the time dependence of the solution is non trivial. Due to the complexity of the general problem we do not expect to find an exact solution. For this reason the solution strategy used in this thesis is to seek an approximate numerical solution at a discrete set of points in the domain of interest.

In the previous chapter we have seen that the equations of motion may be written as second order systems, the displacement formulation of the elastic wave equation (2.1) and the scalar acoustic wave equation (2.15). The equations may also be formulated as first order systems, the velocity-stress formulation of the elastic wave equation (2.7) and an analogous first order formulation of the acoustic wave equation. Numerical methods that computes approximate solutions can use both the first and second order formulations. We also presented the elastic wave equation as a system of two scalar wave equations. This formulation is invaluable when constructing exact solutions or studying boundary and interface phenomena. When designing numerical methods this formulation is avoided due to the rather complicated formulation of the boundary and interface conditions the scalar wave equations must satisfy (e.g., insert (2.2) into (2.6) to formulate the boundary conditions (2.8) in terms of the functions ϕ and ψ).

Early finite difference methods approximated the second order displacement formulation of the elastic wave equation [3, 4]. These methods suffered from instabilities when approximating a traction free boundary condition or the conditions at an interface between two materials in bonded contact [24]. The instabilities depended on the present material parameters and was particularly severe for materials with a high ratios between P-wave speeds and S-wave speeds.

A stable numerical method using second order accurate finite-difference stencils on a staggered grid that approximated the first order system formulation of the wave equation and the velocity-stress formulation of the elastic wave equation in two spatial dimensions was developed by Virieux in [54] and [55], respectively. This method was later extended to fourth order accuracy in space by Levander [33]. An extension to three spatial dimensions has been implemented by Graves [19]. The resulting method is of high order in space (higher than 2), stable and can handle variety of boundary conditions, including the traction free and radiation boundary conditions. Implementations based on these methods are available in the Seismic Unix software package [25, 47]. The underlying geometry is limited to the Cartesian case and no explicit treatment of interfaces where material parameters have discontinuities is done. The drawback of this approach to treat material interfaces, albeit easy to implement and stable, is that the order of accuracy is reduced [8]. One way to treat geometrical features with this method is to embed the elastic body in a surrounding material e.g., air and treat the geometry as a material discontinuity. In Paper IV we make a qualitative comparison between using this approach and a method that treats geometrical features explicitly to study wave boundary interaction at a circular cavity embedded in a surrounding elastic material.

The numerical methods developed in this thesis are based on the second order displacement formulations of the elastic and acoustic wave equations. We use the second order formulation to reduce the number of unknowns and to avoid spurious occilations that can appear if a staggered grid is not used for the first order formulation. The approximation of derivatives employs a class of finite-difference operators satisfying a Summation-By-Parts (SBP) property [52] and imposes boundary conditions explicitly via the Simultaneous-Approximation-Term (SAT) [10] methodology.

Example 3: SBP-SAT finite difference solution of the 1D wave equation
Consider the 1D wave equation on the unit interval $I = [0,1]$ with homogeneous Neumann conditions $u_x = 0$ at both $x = 0$ and $x = 1$,

$$u_{tt} = u_{xx},\ x \in I,\ t > 0. \tag{3.1}$$

Initial data is given by $u(x,0) = u_0(x)$ and $u_t(x,0) = u_1(x)$.

Introduce the grid points $x_i = ih, i = 0 \dots N$, where $h = \frac{1}{N}$. A second order accurate approximation of u_{xx} at the point $x_i, 1 \leq i \leq N-1$ is

$$u_{xx}(x_i,t) \approx \frac{u(x_{i+1},t) - 2u(x_i,t) + u(x_{i-1},t)}{h^2} = u_{xx}(x_i,t) + \mathscr{O}(h^2). \tag{3.2}$$

First order accurate approximations of u_{xx} at the points $x_0 = 0$ and $x_N = 1$ are

$$\begin{aligned} u_{xx}(x_0,t) &\approx \frac{u(x_0,t) - 2u(x_1,t) + u(x_2,t)}{h^2} = u_{xx}(x_0,t) + \mathscr{O}(h), \\ u_{xx}(x_N,t) &\approx \frac{u(x_{N-2},t) - 2u(x_{N-1},t) + u(x_N,t)}{h^2} = u_{xx}(x_N,t) + \mathscr{O}(h), \end{aligned} \tag{3.3}$$

and second order accurate approximations of u_x at the points x_0 and x_N are

$$\begin{aligned} u_x(x_0,t) &\approx \frac{-\frac{3}{2}u(x_0,t)+2u(x_1,t)-\frac{1}{2}u(x_2,t)}{h} = u_x(x_0,t)+\mathscr{O}(h^2), \\ u_x(x_N,t) &\approx \frac{\frac{1}{2}u(x_{N-2},t)-2u(x_{N-1},t)+\frac{3}{2}u(x_N,t)}{h} = u_x(x_N,t)+\mathscr{O}(h^2). \end{aligned} \tag{3.4}$$

We seek approximate solutions $\mathbf{u}_i(t)$ of (3.1) at the points x_i. Consider the set of ordinary differential equations

$$\frac{d^2}{dt^2}\mathbf{u}_0 = \frac{\mathbf{u}_0-2\mathbf{u}_1+\mathbf{u}_2}{h^2}+SAT_0, \tag{3.5}$$

$$\frac{d^2}{dt^2}\mathbf{u}_i = \frac{\mathbf{u}_{i+1}-2\mathbf{u}_i+\mathbf{u}_{i-1}}{h^2},\ 1\le i\le N-1, \tag{3.6}$$

$$\frac{d^2}{dt^2}\mathbf{u}_N = \frac{\mathbf{u}_{N-2}-2\mathbf{u}_{N-1}+\mathbf{u}_N}{h^2}+SAT_N, \tag{3.7}$$

with initial data $\mathbf{u}_i(0)=u_0(x_i)$ and $\frac{d}{dt}\mathbf{u}_i(0)=u_1(x_i)$. Here

$$\begin{aligned} SAT_0 &= \frac{\sigma_0}{H_0 h}\left(\frac{-\frac{3}{2}\mathbf{u}_0+2\mathbf{u}_1-\frac{1}{2}\mathbf{u}_2}{h}-0\right), \\ SAT_N &= \frac{\sigma_N}{H_N h}\left(\frac{\frac{1}{2}\mathbf{u}_{N-2}-2\mathbf{u}_{N-1}+\frac{3}{2}\mathbf{u}_N}{h}-0\right), \end{aligned} \tag{3.8}$$

where H_0, H_N, σ_0 and σ_N are constants independent of h to be determined. The terms SAT_0 and SAT_N are consistent approximations of the difference between u_x and the imposed values $u_x=0$ at $x=0$ and $x=1$. The resulting set of equations (3.5)-(3.7) is a consistent semi-discretization of (3.1). To see this let u be the solution to (3.1) and set $\mathbf{u}_i(t)=u(x_i,t)$ we get,

$$\begin{aligned} u_{tt}(x_0,t) &- \frac{u(x_0,t)-2u(x_1,t)+u(x_2,t)}{h^2} \\ &- \frac{\sigma_0}{H_0 h}\left(\frac{-\frac{3}{2}u(x_0,t)+2u(x_1,1)-\frac{1}{2}u(x_2,t)}{h}\right) \\ &= u_{tt}(x_i,t)-u_{xx}(x_i,t)+\mathscr{O}(h) \\ &- \frac{\sigma_0}{H_0 h}\left(u_x(x_0,t)+\mathscr{O}(h^2)\right) = \mathscr{O}(h). \end{aligned}$$

Similarly, u satisfies the equation (3.7) to order $\mathscr{O}(h)$. For $1\le i\le N-1$

$$\begin{aligned} u_{tt}(x_i,t) &- \frac{u(x_{i+1},t)-2u(x_i,t)+u(x_{i-1},t)}{h^2} \\ &= u_{tt}(x_i,t)-u_{xx}(x_i,t)+\mathscr{O}(h^2) = \mathscr{O}(h^2), \end{aligned}$$

The solution u of (3.1) satisfies the equations (3.5)-(3.7) to order $\mathscr{O}(h^2)$ for $1 \leq i \leq N-1$ and to order $\mathscr{O}(h)$ for $i = 0$ and $i = N$.

If the semi-discretization is also stable its solutions $\mathbf{u}_i(t)$ converges to the solution $u(x_i, t)$ as $h \to 0$ by the Lax-Richtmyer theorem [32]. Convergence implies that any departure from the imposed boundary conditions in the numerical solution will be be penalized by the terms SAT_0 and SAT_N and the numerical solution forced to satisfy the boundary conditions to order of accuracy. The boundary forcing terms are in the SBP-SAT folklore called penalty terms and the corresponding σ_0, σ_N are called penalty parameters.

We now prove stability. The right hand side of (3.5) is

$$2\frac{-\mathbf{u}_0 + \frac{3}{2}\mathbf{u}_0 + \mathbf{u}_1 - 2\mathbf{u}_1 + \frac{1}{2}\mathbf{u}_2}{h^2} + \frac{\sigma_0}{H_0 h}\left(\frac{-\frac{3}{2}\mathbf{u}_0 + 2\mathbf{u}_1 - \frac{1}{2}\mathbf{u}_2}{h}\right).$$

Hence, by setting $\sigma_0 = 1$ and $H_0 = \frac{1}{2}$ in (3.5) we get

$$\frac{d^2}{dt^2}\mathbf{u}_0 = 2\frac{-\mathbf{u}_0 + \mathbf{u}_1}{h^2}.$$

Similarly, by setting $\sigma_N = -1$ and $H_N = 1/2$ in (3.7) we get,

$$\frac{d^2}{dt^2}\mathbf{u}_N = 2\frac{\mathbf{u}_{N-1} - \mathbf{u}_N}{h^2}. \tag{3.9}$$

By multiplying (3.6) by $\frac{d}{dt}\mathbf{u}_i h$ and summing over $1 \leq i \leq N-1$ we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{N-1}\left(\frac{d}{dt}\mathbf{u}_i\right)^2 h = &-\frac{1}{2}\frac{d}{dt}\sum_{i=1}^{N-1}\left(\frac{\mathbf{u}_i - \mathbf{u}_{i+1}}{h}\right)^2 h \\ &- h\frac{\frac{d}{dt}\mathbf{u}_1\mathbf{u}_1}{h^2} + h\frac{\frac{d}{dt}\mathbf{u}_1\mathbf{u}_0}{h^2} \\ &- h\frac{\frac{d}{dt}\mathbf{u}_{N-1}\mathbf{u}_{N-1}}{h^2} + h\frac{\frac{d}{dt}\mathbf{u}_{N-1}\mathbf{u}_N}{h^2}.\end{aligned} \tag{3.10}$$

Multiplying (3) and (3.9) by $\frac{d}{dt}\mathbf{u}_0 h H_0$ and $\frac{d}{dt}\mathbf{u}_N h H_N$, respectively, and adding the result to (3.10) gives

$$\frac{1}{2}\frac{d}{dt}E(t) = 0, \tag{3.11}$$

where

$$E(t) = \frac{1}{2}\left(\frac{d}{dt}\mathbf{u}_0\right)^2 h + \sum_{i=1}^{N-1}\left(\frac{d}{dt}\mathbf{u}_i\right)^2 h + \frac{1}{2}\left(\frac{d}{dt}\mathbf{u}_N\right)^2 h + \sum_{i=0}^{N-1}\left(\frac{\mathbf{u}_i - \mathbf{u}_{i+1}}{h}\right)^2 h.$$

The quadratic form $E(t)$ is positive semi-definite (e.g., $E(t) = 0$ if all $\mathbf{u}_i = constant$) and by (3.11) $E(t) = constant = E(0)$. Hence, $\mathbf{u}_i$ are bounded for all t. The semi-discretization (3.5)-(3.7) is stable and the system can be propagated in time with a numerical method for ordinary differential equations.

That $E(t)$ is positive semi-definite is completely dependent on the possibility to write (3.5) as (3) and (3.7) as (3.9), which in turn is a result of the particular choices (3.3) and (3.4) for the approximation at the boundary points of u_{xx} and u_x, respectively. The property of the chosen approximations in this example that makes it possible to prove stability is called the Summation-By-Parts property □

Example 3 illustrates the general procedure for using the SBP-SAT method: Discretize spatial derivatives with finite-difference operators having a SBP property, add SAT penalty terms to enforce the boundary conditions, use the SBP property to prove stability and propagate the resulting system of ordinary differential equations in time which may be done with a Runge-Kutta method or by a method specialized for second order systems [16]. Proving stability often amounts to tuning the penalty parameters corresponding to the SAT penalty terms, which may turn out to be more or less involved. In [38] SBP operators approximating second derivatives with constant coefficients to different orders of accuracy where developed and latter extended to variable coefficients [36]. First derivative SBP approximations can be found in [52]. SBP operators are non-unique and their construction involves determining unknown parameters, often using a symbolic software package [1]. Apart from the elastic and acoustic wave equation recent use of the SBP-SAT methodology for other wave propagation problems include, the Schrödinger equation [40] the Dirac equation [2] and the dynamic beam equation [37].

Another approach to enforce boundary conditions while using SBP finite-difference operators to approximate spatial derivatives is to introduce so called ghost-points outside the boundary of the computational grid. The elastic wave equation in isotropic materials has been solved numerically using fourth order accurate SBP finite-difference operators and ghost points to enforce boundary conditions [49]. This method was latter extended to treat anisotropic materials and curvilinear coordinates [41]. These numerical schemes have support for second-order accurate treatment of material discontinuities and non-conforming grids via interpolation [42] and absorbing sponge layers [43]. The resulting numerical method has been implemented i three spatial dimensions and is available in the software package SW4 [44]. The boundary treatment using ghost-points has been validated for stability and accuracy using analytic solutions representing surface phenomena such as mode conversion and surface waves [29]. The reliability of the method has been further strengthened by producing convincing solutions to Lambs's problem [31] that agree with analytic solutions [39].

Discretizing the underlying equation using finite-differences results in a structured grid and geometric flexibility can be given by solving the equations in curvilinear coordinates. To handle more complicated geometries while still using a structured grid a multi-block strategy can be useful as we saw in Section 2.1.2. Now the domain is decomposed into sub-domains each having a suitable mapping from the unit square. The sub-domains are then patched together with artificially imposed interface conditions. This patching can be done numerically by explicitly treating the interface conditions using for example the SAT or the ghost point technique.

An alternative approach where overlapping grids is used to represent geometries of the underlying domain has been constructed in [5]. Now function values are interpolated to coinciding points where the used grids overlap. This approach allows for a flexible geometry treatment and has been implemented for the elastic wave equation written as both a first and second order system, respectively, in the software package Overture [22].

The different strategies to treat geometries we have discussed are illustrated in Figure 3.1 for the treatment of a circular geometry.

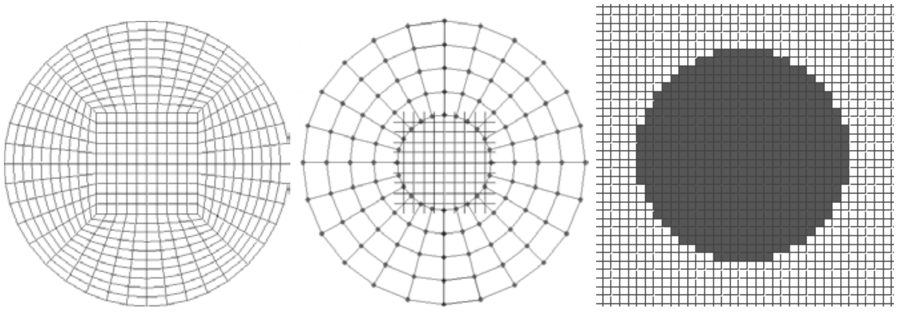

Figure 3.1. A circular domain represented by a collection of curvilinear grids (left), overlapping grids (middle), embedded in a surrounding medium (right).

When discretizing the underlying equations we seek approximate solutions at a finite number of grid points in the domain if interest. As the number of grid points increases, the error in the numerical solution gets smaller. For hyperbolic wave propagation problems such as the elastic and acoustic wave equation there has been established a rule of thumb that relates the discretization error to how many grid points are used per shortest wavelength present in the solution [28]. The wavelengths in the solution are in general the quotients between the present wave speeds and the frequencies given by initial data and forcing functions. As wave speeds and frequencies are typically known, so are the shortest wavelengths. Table 3.1 is taken from [28] and lists the number of grid points P_n per wavelength needed for a certain maximum error using a method of order of accuracy $n = 2,4,6$ after having propagated the waves for q periods in time. In heterogeneous materials a wave front can enter into a region of the material where the wave speed is reduced, which decreases its wave length. To keep the size of the discretization errors uniform through out the domain we must either use a grid size that is uniform through out the domain and chosen according to the smallest wavelength, or adjust the grid size in in each region of differing wave speeds and wave lengths. The first option results in over-resolving waves in regions of a faster wave speed, which gives higher computational costs. Using finite differences the second option can be realized by a multi-block strategy, as when representing complex geometries. Now each block has a grid size adjusted according to the wave speed in that block, the effect is that at block interfaces the grid sizes are non-uniform and hanging nodes appear. In the SBP-SAT framework hanging nodes are treated by interpolating solution values to missing nodes in a stable and accurate manner [35, 27]. In Paper VI we extend the method constructed in Paper V to treat hanging nodes and discuss resulting efficiency, accuracy and stability properties.

The analysis resulting in Table 3.1 was done by studying plane waves traveling in an infinite medium and did not take into account wave boundary interactions.

Max error	P_2	P_4	P_6
10 %	$20q^{1/2}$	$7q^{1/4}$	$5q^{1/6}$
1 %	$64q^{1/2}$	$13q^{1/4}$	$8q^{1/6}$

Table 3.1. *Number of grid points P_n needed for a maximum error of 10 % and 1 % after q periods in time.*

Example 4: Discretization errors in surface waves
Poisson's ratio in an elastic material is $\nu = \frac{\lambda}{2(\lambda+\mu)}$. When $\nu \to 0.5$ or equivalently $\lambda/\mu \to \infty$ we say that the material is becoming nearly incompressible. Rubber, soft organic tissues [53] and in some cases crusts at the ocean floor [12] are examples of the appearance of nearly incompressible materials.

We take $\rho = 1, \mu = 0.001$ and $\lambda = 1$ in an elastic half-plane i.e., $\nu \approx 0.4995$, and for a numerical experiment we propagate a periodic train of Rayleigh surface waves for one period in time using a second order finite-difference method based on the SBP-SAT methodology [14]. Guided by Table 3.1 we expect a maximum error of 1% by choosing to resolve the waves by 64 grid points per wavelength. Figure 3.2 visualizes the numerical solution compared to the exact solution to this problem. The numerical solution displays dispersion errors and has lost its correct shape. The measured error in the numerical solution is about 50% i.e., 50 time larger than expected. In [29] it is explained that when using finite differences to propagate Rayleigh surface waves in nearly incompressible materials the number of grid points must be proportional to $(\lambda/\mu)^{1/p}$, where p is the order of accuracy of the numerical method. In Paper II we show that also the Stoneley interface wave suffers from an analogous resolution requirement as the involved materials become nearly incompressible.

Similar findings have been reported when using finite element methods to solve elasticity problems in nearly incompressible elastic materials [7]. This numerical difficulty is called volume locking and manifests in excessive non-realistic numerical stiffness as $\nu \to 0.5$ □

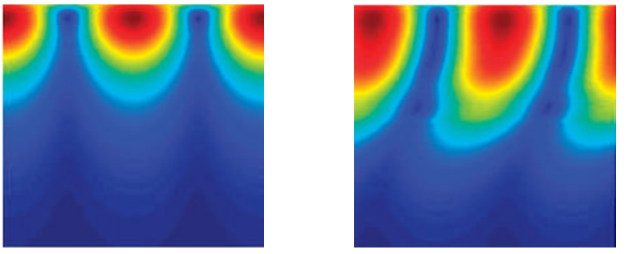

Figure 3.2. Left: Exact solution after one temporal period. Right: numerical solution after one temporal period.

Example 3 provides another motivation for using of a variable grid size. In a nearly incompressible material it is beneficial to have a smaller grid size close to a traction free surface even though the entire material is homogeneous.

There are of course many numerical methods for hyperbolic wave propagation problems that are not using finite-differences. Methods based on Hermite interpolation and the use of a staggered grid are promising arbitrary order of accuracy and a unit CFL number regardless of the order of accuracy [17].

Example 5: A Hermite interpolation method for the elastic wave equation

In his definition of the fundamental problem of seismology, quoted earlier in Example 1, Cagniard states that a description of the present waves should be described exactly at any point in the domain. When we use a numerical method we abandon the ambition to seek an exact description of the solution but instead look for a solution which is correct to order of accuracy. The claim that Hermite methods can be constructed to arbitrary order of accuracy now becomes interesting, as higher order accuracy indicates a smaller discretization error for a fixed grid size and hence, getting closer to a solution of Cagniard's fundamental problem.

Guided by the description in [17, 21] we implement the Hermite method for the elastic wave equation in an elastic strip $(x,y) \in [0,1] \times (-\infty,\infty)$. To test the claim of arbitrary order of accuracy we take $c_p = \sqrt{3}$, $c_s = 1$ and propagate initial data given by an exact solution at time $t = 0$ until time $t = 1$. At the boundaries of the strip the exact solution is injected. The exact solution is

$$\phi = \cos(\xi(xa + yb - c_p t)),$$
$$\psi = \cos(\eta(xc + yd - c_s t)),$$

where $a = b = \sin\pi/5, c = d = \sin\pi/3, \xi = 2\pi/a$ and $\eta = 2\pi/c$. From which $\dot{u}, \dot{v}, \tau^{xx}, \tau^{xy}, \tau^{yy}$ can be computed via (2.2) and (2.6). A plot of the magnitude of the velocity field $(\dot{u}, \dot{v})$ is displayed in the left of Figure 3.3. In the right of

Figure 3.3 the relative maximum errors obtained with third, fifth, seventh and ninth order Hermite methods are plotted as functions of number of degrees of freedom. The errors decay to correct order of accuracy □

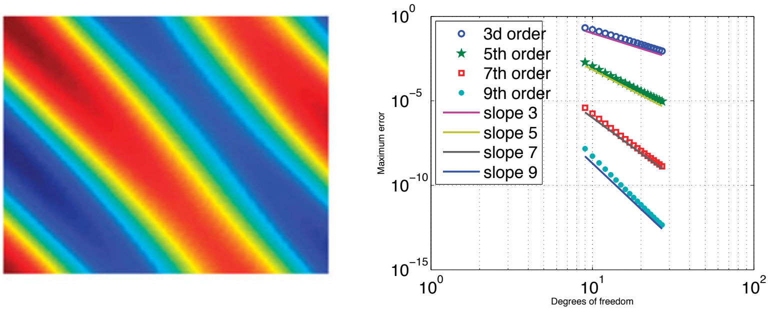

Figure 3.3. Left: magnitude of velocity field. Right: relative maximum errors as functions of degrees of freedom.

The usage of variable coefficients, and hence curvilinear coordinates, with the Hermite methods is discussed in [17], but the extension of stable boundary treatment in higher dimensions than one is to the author of this thesis knowledge an open problem. One approach studied in [11] treats boundaries and geometries by coupling a Cartesian grid on which a Hermite method is used to an unstructured grid in the vicinity of boundaries on which a discontinuous Galerkin (dG) method is used. The dG method [23] uses an unstructured mesh to represent the physical domain and thus simplifies the treatment of geometries. For complicated geometries, it is however, a nontrivial task to generate a high quality mesh and the connectivity of the mesh needs to be stored requiring additional memory usage. Benefits with the dG method apart from geometric flexibility is that arbitrary high order accuracy can be obtained, and boundary and interface conditions can be easy to implement and do not require tuning of parameters as with the SAT technique [6]. In Paper III we report that contrary to a finite difference method, computations involving surface waves with the dG method does not seem to suffer from the stricter resolution requirement seen in Example 3, although this is not proved. Examples of the dG method in the context of elastic wave propagation can be found in [30, 13]. In [57] the dG method is used to couple acoustic and elastic wave propagation and [46] uses the dG method for both the elastic and acoustic wave equation.

4. Summary in Swedish

En störning lokaliserad till en avgränsad del av ett medium som sprider sig med ändlig hastighet till andra delar av mediet utgör basen för läran om vågutbredning. Vågutbredning uppträder i vårt dagliga liv i form av till exempel ljudvågor från musikinstrument, vattenvågor som bryts vid en strand eller elastisk deformation av jordskorpan vid ett jordskalv. Frekvens och våglängd är förutom våghastighet och i vilket medium vågrörelsen sker två egenskaper hos en propagerande våg. I det fall då våghastigheten bestäms av mediet och är oberoende av frekvens och våglängd kallas vågrörelsen hyperbolisk. Både elastisk och akustisk vågutbredning är exempel på hyperbolisk vågrörelse. Vattenvågor är så kallat dispersiva och inte hyperboliska, deras hastighet beror på deras frekvens. Denna avhandling handlar om simulering av elastisk och akustisk vågutbredning.

I ett elastiskt medium, till exempel vår jordskorpa, kan minst två typer av vågor sprida sig, tryckvågor och skjuvvågor. I ett akustiskt material, till exempel vår atmosfär, propagerar inga skjuvvågor. Både tryckvågor och skjuvvågor beskrivs av samma ekvation, vågekvationen, dock färdas de med olika hastighet. Det är när en våg i ett elastiskt material interagerar med en rand som en tydlig distinktion uppstår mellan elastisk och akustisk vågutbredning, till exempel då en tryckvåg i jordens inre stöter på jordytan. Nu kopplas vågekvationen som beskriver tryckvågor ihop med vågekvationen som beskriver skjuvvågor genom villkor som måste uppfyllas vid randen. Det resulterande systemet av ekvationer utgör den ekvation som kallas elastiska vågekvationen. Kopplingen vid randen resulterar i en del intressanta fenomen. Då en typ av våg, antingen tryck eller skjuvvåg träffar en rand kommer den att reflekteras som både en tryckvåg och en skjuvvåg. Detta fenomen har ingen motsvarighet inom akustik. Ett annat randfenomen som svarar för distinktionen mellan elastisk och akustisk vågutbredning är den ytvåg som kan existera vid randen av en domän. Denna ytvåg kallas för en Rayleighvåg efter sin uptäckare Lord Rayleigh och uppträder i naturen som de kraftfulla efterskalv som uppträder i samband med jordbävningar.

Den generella problemställningen innefattar en störning som genererar vågrörelse, denna störning kan ha ett godtyckligt tidsberoende. Materialegenskaper och geometri hos det underliggande mediet kan ha en komplicerad struktur och vid ränder uppstår en rad olika randvillkor. Komplexiteten av det generella problemet gör att en exakt matematisk beskrivning av det slutgiltiga vågfältet är mycket svårt, om inte omöjligt att konstruera. Istället söker vi i denna avhandling en approximativ beskrivning av lösningen till den elastiska och akustiska vågekvationen i ett ändligt antal punkter i tid och rum. Även om vi nu kan hitta en approximativ lösning med hjälp av en elektronisk datormaskin uppstår frågor kring den metod vi använde för att räkna ut lösningen. Frågor som huruvida vi faktiskt hittar den rätta lösningen, om approximationen möjliggör icke-fysikalisk tillväxt i den approximativa lösningen och med vilken noggranhetsordning vi räknar ut vår lösning. Tekniska detaljer såsom beräkningseffektivitet och hur den underliggande beräkningsmetoden implementeras blir också viktiga faktorer att ta hänsyn till. I denna avhandling konstruerar vi en lösningsmetod för den elastiska vågekvationen och en för den akustiska vågekvationen. I vår konstruktion lägger vi fokus på hög noggrannhet, eliminerande av icke-fysikalisk tillväxt, hantering av randvillkor och geometrisk flexibilitet. Dessa metoder används sedan för att studera vågfronters interaktion med ränder, geometrisk påverkan av reflektion och ytvågor. Vi jämför även våra metoder med andra lösningsstrategier. Innan de manuskript som utgör denna avhandling presenteras finns ett introduktionsavsnitt där författaren har valt ut en del material som han tror kan ge en bra start till en exkursion i numerisk behandling av elastisk och akustisk vågrörelse.

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