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Calculation of Wave Propagation for Statistical Energy Analysis Models

Hussam Bashir

UPPSALA
UNIVERSITET

Teknisk- naturvetenskaplig fakultet
UTH-enheten

Besöksadress:
Ångströmlaboratoriet
Lägerhyddsvägen 1
Hus 4, Plan 0

Postadress:
Box 536
751 21 Uppsala

Telefon:
018 – 471 30 03

Telefax:
018 – 471 30 00

Hemsida:
http://www.teknat.uu.se/student

Abstract

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This thesis investigates the problems of applying Statistical Energy Analysis (SEA) to models that include solid volumes. Three wave types (Rayleigh waves, Pressure waves and Shear waves) are important to SEA and the mathematics behind them is explained here. The transmission coefficients between the wave types are needed for energy transfer in SEA analysis and different approaches to solving the properties of wave propagation on a solid volume are discussed. For one of the propagation problems, a solution, found in Momoi [6] is discussed, while the other problem remains unsolved due to the analytical difficulties involved.

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Ämnesgranskare: Per Isaksson
Examinator: Tomas Nyberg
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Populärvetenskaplig Sammanfattning

När man konstruerar industrimaskiner använder man ofta ett beräkningsverktyg för att förutsäga ljudvolymen som uppstår. Dessa ljudvågor kan färdas långt och man bör kostruera utrustningen på sådant sätt att påverkan på människor från oväsen blir minimal. Ett vanligt sådant beräkningsverktyg är Statistisk Energianalys (SEA) som använder ett antal tidigare uträknade värden för att förutsäga ljudvolymen i en situation. I dagsläget används SEA endast för system som består av tunna skivor, smala stänger samt luftutrymmen då dessa värden är relativt enkla att räkna ut. Det finns dock en önskan inom industrin att även kunna använda SEA till system som inkuderar solida objekt som ej kan approximeras som stänger eller skivor. Detta skulle göra SEA användbart även för att forutsäga hur ljud sprider sig genom t.ex. marken.

Målet med detta arbete var att bidra till utvecklingen av SEA genom att räkna ut hur ytvågor på en solid massa rör sig över hörn, samt hur de omvandlas till tryck- och skjuvvågor i materialet. Om tid fanns skulle även det omvända problemet, hur tryck- och skjuvvågor omvandlas till ytvågor vid ett hörn, även tas i beaktande. Den relevanta bakgrunden för uträkningen samt en lösningsansats är beskriven med mer detalj i arbetet. En lösning för det första problemet hittades i Momoi [1] vid antagande att hörnet består av en rät vinkel. Denna lösning är materialberoende och stämmer väl överens med verkligheten för t.ex. aluminium. Komplexiteten bakom att lösa det andra problemet samt att lösa det första med en annan vinkel på hörnet är stor och kommer kräva mycket mer arbete framöver.

Dessa är matematiskt avancerade problem som kräver bättre analytiskt kunnande för att lösa inom den tid som ett examensarbete rymmer. Därför kanske man bör ge detta arbete som uppdrag åt en renlärd matematikstudent inom Partiella Differentialekvationer (PDE) eller annan matematisk expert. Det arbete som har gjort hittills kan förhoppningsvis fungera some en startpunkt för ett mer lyckat försök i framtiden.

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1 Introduction

In the process of designing equipment for use in industrial applications, special care must be taken to make sure that the sound produced does not exceed levels appropriate for workplaces. For drilling and mining applications especially, there is a huge incentive to make sure vibrations from the drilling do not propagate needlessly through the machines. Furthermore, these vibrations can decrease the lifespan of parts as well as affect the well-being of staff negatively. In order to find the best design there should therefore be a way to try and estimate the sound properties without having to build a prototype for testing. For a long time this method has been Statistical Energy Analysis (SEA) [2], which is limited to solving problems involving 1D and 2D solids.

Figure 1: Surface waves in the rock are excited by the drilling process and propagate through the ground. It is of interest to predict the behaviour of sound in these kinds of situations.

The purpose of this thesis is to attempt general solutions to wave propagation on the surface of a 3D solid, so that the transmission coefficients can be used for energy flow parameters in SEA simulation models. For this a right angle corner is chosen as the discontinuity from which the waves scatter. In order to know boundary conditions, we also need to consider the excitation of waves on a solid. At the moment general solutions to these problems are hard to find, and require a heavy mathematical background to solve.

The rest of this thesis is structured as follows: An introduction of SEA is given in Section 2.1 followed by a theoretical background of wave equations in Section 2.4. The different wave types are explained in Sections 2.5 through Section 2.7 and a discussion of the point excitation of waves in a solid is given in Section 2.8. The Problems to be solved are presented in Section 3 and the results are presented in Section 4, paired with a with an explanation of the results in Momoi [1]. This is followed by a discussion of the thesis and of future work in Sections 5 and 7 respectively.

2 Theory

Using SEA requires the transmission coefficients between wave types at different surface discontinuities to be known, and the mathematics involved can be quite messy. First the properties of the different waves must be well understood, and the boundary conditions must be well defined. That is the point at which a solution can be attempted.

2.1 Statistical Energy Analysis

A common technique for solving sound propagation problems is SEA which uses known wave propagation properties and energy distributions to calculate the sound properties of a construction or workplace. In essence, SEA reduces the complexity of the problem to a linear system of equations instead of a multidimensional non-linear problem by assuming that the wavelengths are much smaller than the size of the model. At the present time, commercial SEA can only be used for simulation containing "thin rods", "thin plates" and air cavities [2]. This is because solving these 1D and 2D problems generally is relatively simple and results in solutions with a few wave-types. In comparison a solid volume has a 2D surface wave on each surface and two types of 3D body waves, while a "thin plate" has only a 1D surface wave and 2D body waves. Being able to use SEA on solid volume elements would be a great benefit, but at the moment the general solutions from which SEA is approximated do not exist.

Most of the information in this section can be found in greater detail in the thesis [3] by Daniel Norgren.

2.1.1 Sub-systems

In SEA the model is divided into different sub-systems, each comprising one part of the system. In effect, each plate, rod or volume of air becomes a sub-system. At that point the behaviour of the waves can be calculated since there are known solutions to these cases [2]. For each sub-system there are different parameters that describe the energy loss or transfer to or from the sub-system. For a 3D solid, the sub-systems would be one for each face of the volume and each body wave-type. For a simple cylinder, this comes to 5 connected sub-systems as illustrated in Figure 2.

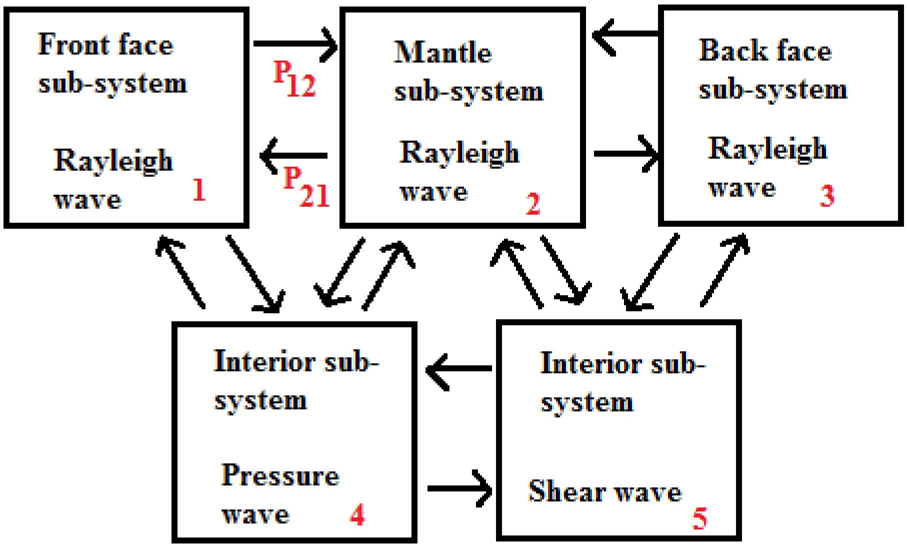

Figure 2: These are the sub-systems for a simple cylinder. The arrows, P_{ij}, represent the energy flow between sub-systems which require the transmission coefficients for the waves. P

2.1.2 Parameters for SEA

The following parameters are the parameters that decide the energy flow between our sub-systems. They are all calculated from the behaviours of waves in that sub-system. To complete the model there are four main parameters needed. Input power, Energy damping, Modal density and Power flow are all needed to characterize an SEA model. There are of course complications where additional information is required, but in general these four are needed.

Energy input

The external energy input to each wave-type is calculated from whichever disturbance causes the vibrations. This energy input is specified for each sub-system, and must be calculated beforehand from a known excitation. An example of this is described in Section 2.8.

Energy loss

The energy loss from each sub-system is calculated from material data sheets or measured separately. This is a form of damping and is proportional to the frequency and to the vibrational energy accumulated in the sub-system.

Modal density

The modal density of a sub-system is quantified as the amount of vibrational modes, N, there are per unit of frequency, f, as according to (1). This value must be calculated from the geometry and phase velocity of the waves in a sub-system. These equations exist for rods, plates and volumes [3], and can easily be calculated for solid volumes and surfaces since the analytical expressions only depend on the geometry, the frequency and the wave velocities.

$$n = \frac{\Delta N}{\Delta f} \tag{1}$$

Power flow

The flow of energy between the different sub-systems in the SEA model is dependent on the calculated wave transmission between the different wave types and is proportional to the angular frequency, ω, and energy accumulated in the two sub-systems according to (2)

$$P_{ij} = \omega \eta_{ij} (n_j E_i - n_i E_j) \tag{2}$$

where η_{ij} is the Coupling loss factor for the transmission of the wave types between the different sub-systems and E_i is the accumulated vibrational energy in the i-th sub-system. n_i is the modal density as described in (1). It is for these coupling loss factors that the transmission coefficients are needed according to (3) [3].

$$\eta_{ij} = \bar{T_{ij}} \frac{v}{\omega} \frac{l_c}{\pi S_v} \tag{3}$$

where the T_{ij} is the mean transmission coefficient, v is the phase velocity, ω is the angular frequency, l_c is the length of the contact line and S_v is the area of the sub-system. All of these are known except for the transmission coefficients.

2.2 Use of parameters

Table 1: Example of sub-system coefficients. These characterize the energy flow between the sub-systems which are needed to find the solution.

	Sub-system 1	Sub-system 2	Sub-system 3
To sub-system 1	-	P_{21}	P_{31}
To sub-system 2	P_{12}	-	P_{32}
To sub-system 3	P_{13}	P_{23}	-
Energy loss	$P_{1,loss}$	$P_{2,loss}$	$P_{3,loss}$
Energy input	$P_{1,in}$	$P_{2,in}$	$P_{3,in}$

In principle, the coefficients in Table 1 can all be calculated from general solutions of wave propagation and dispersion. However on a solid volume any sub-systems would interact with each other in very complicated ways due to the fact that all types of waves can exist within the same volume elements and on the surfaces. In our case we need to find the transmission coefficients for a surface wave over an edge, seen in Figure 3, and if possible the transmission coefficients from longitudinal and transversal waves to surface waves at an edge.

2.3 Measurement of waves

In any problem simulation, it is important to validate the models used. Making sure that a solution is physical is critical, but on a solid volume it becomes complicated to measure the vibrations for validation of the SEA model. This is because of the fact that both body waves and surface waves cause displacements on the surface. Considering that the surface waves and body waves all have the same frequency we encounter a problem that is simpler in conventional 2D SEA where sensors only need to discern between the pressure and shear waves. Therefore validating any solution is a difficult

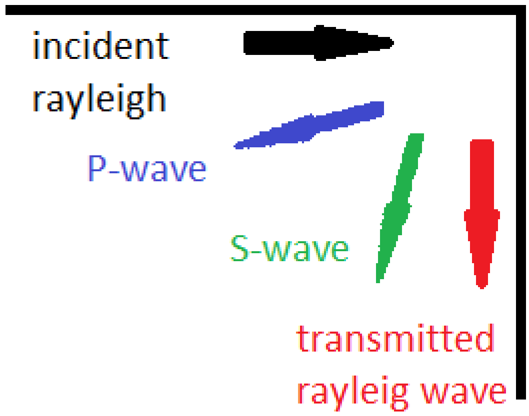

Figure 3: The incoming surface wave (black) is reflected and transmitted (red). Some energy is also transformed into pressure (blue) and shear (green) body waves which travel into the solid.

problem in itself since the effect of the body waves must be separated from the surface measurements. Without a reliable validation of the energies in each sub-system it is difficult to understand any errors in the model predictions. However it is possible that a combination of many sensors could distinguish the elliptical movement of Rayleigh waves with some accuracy as discussed in [4].

2.4 Elastic wave equations

To solve the wave equations, we need an understanding of the potentials and the boundary conditions as they apply in this situation. To gain an understanding of the role of stress and displacements, we will derive the potential wave equations. When deriving the elastic wave equation we begin with Newtons second law of motion on a fluid element dV

$$\int \vec{F} dV = \int \rho \vec{a} dV \tag{4}$$

and separate it between body forces and contact forces

$$\int \rho \vec{g} dV + \int \vec{T} dS = \int \rho \vec{a} dV \tag{5}$$

so that we can ignore body forces. We can do this since the effect of gravity is small at the assumption that the wavelength is small. We know that

$$\vec{T} = \sigma \cdot \hat{n} \tag{6}$$

where $\vec{T}$ is the surface tension and $\hat{n}$ is the stress tensor, and

$$\vec{a} = \frac{\partial^2 \vec{u}}{\partial t^2} \tag{7}$$

where $\vec{u}$ is the displacement of the solid element. Combining (5), (6), (7) and Gauss' theorem gives us

$$\int \nabla \cdot \sigma dV = \int \rho \frac{\partial^2 \vec{u}}{\partial t^2} dV \tag{8}$$

which finally starts to take the form of a wave equation. We can demand that the equality is satisfied for the integrands as well for each direction. In one direction this is written out as

$$\frac{\partial \sigma_{ii}}{\partial x_i} + \frac{\partial \sigma_{ij}}{\partial x_j} + \frac{\partial \sigma_{ik}}{\partial x_k} = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{9}$$

since the stress tensor, σ is symmetric. Substituting the stress for displacements according to the linear displacement-stress relationship yields

$$2\mu u_{i,ii} + \lambda \left(u_{i,ii} + u_{j,ji} + u_{k,ki}\right) + \mu \left(u_{i,jj} + u_{j,ij}\right) + \mu \left(u_{i,kk} + u_{k,ik}\right) = \rho \ddot{u}_i \tag{10}$$

which simplifies to the following in Einstein notation

$$\mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} + (\lambda+\mu)\frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j} = \rho\ddot{u}_i \tag{11}$$

From now on we will only consider a 2D case, which greatly simplifies our work. Now we put this constraint into (11) and get

$$(2\mu+\lambda)u_{xx} + (\lambda+\mu)v_{yx} + \mu u_{yy} = \rho\ddot{u} \tag{12}$$

$$(2\mu+\lambda)v_{yy} + (\lambda+\mu)u_{yx} + \mu v_{xx} = \rho\ddot{v} \tag{13}$$

where u and v are the horisontal and vertical velocities. We know that the longitudinal and shear waves in 2D can be described by their potentials

$$u = \frac{\partial\Phi}{\partial x} + \frac{\partial\Psi}{\partial y}, \qquad v = \frac{\partial\Phi}{\partial y} - \frac{\partial\Psi}{\partial x} \tag{14}$$

which can be plugged into (12) and (13). Doing that yields

$$(2\mu+\lambda)(\Phi_{xxx} + \Phi_{yyx}) + \mu(\Psi_{yxx} + \Psi_{yyy}) = \rho(\Phi_{xtt} + \Psi_{ytt}) \tag{15}$$

$$(2\mu+\lambda)(\Phi_{yyy} + \Phi_{yxx}) - \mu(\Psi_{yyx} + \Psi_{xxx}) = \rho(\Phi_{ytt} - \Psi_{xtt}) \tag{16}$$

which we can add and subtract from each other to uncouple our potentials.

$$(2\mu+\lambda)(\nabla^2(\Phi_y + \Phi_x) = \rho(\Phi_x + \Phi_y)_{tt} \tag{17}$$

$$\mu(\nabla^2(\Psi_y + \Psi_x) = \rho(\Psi_y + \Psi_x)_{tt} \tag{18}$$

and see that if we demand that these equations are satisfied by the two derivations separately, then it is also satisfied by the potentials themselves. Then we finally have the elastic wave equations for the potentials of the pressure and shear waves.

$$\frac{2\mu + \lambda}{\rho}\nabla^2\Phi = \ddot{\Phi}, \qquad \frac{\mu}{\rho}\nabla^2\Psi = \ddot{\Psi} \tag{19}$$

These equations are very important since the solutions define the wave-types we are interested in and the wave constants define our wave velocities. That they take the form of these regular wave equations also tells us that there exists a unique solution to our problems, assuming correctly defined boundaries.

2.5 Longitudinal waves

In a situation where we only have plane normal stresses, the solutions to (19) become a wave with a displacement in only one direction, the direction of wave propagation. This is called a longitudinal wave and is described by

$$\vec{u} = U_0 e^{i\omega\left(\frac{x_i}{v_l} \pm t\right)}\hat{e}_i \tag{20}$$

where v_l is the longitudinal phase velocity and ω is the angular frequency. This phase velocity is different for each wave type and for longitudinal waves in an elastic isotropic solid

$$v_l = \sqrt{\frac{\lambda + 2\mu}{\rho}}. \tag{21}$$

This longitudinal wave is in elastic solids the fastest wave type and therefore has the longest wavelengths. This is the wave type that limits the SEA analysis in the size of the sub-systems since the wavelength must be assumed small. If the wavelength is comparable to the size of the solid, the waves will exhibit standing wave phenomena where the transmission and reflection becomes heavily dependent on wavelength. This is not compatible with SEA which solves problems linearly. In a true 3D problem, this wave can move in any three directions and therefore has three degrees of freedom.

2.6 Shear waves

When we only have plane shear stresses, the solutions to 19 is a wave with displacement in a direction perpendicular to the propagation. The description follows as

$$\vec{u} = U_0 e^{i\omega\left(\frac{x_i}{v_t} \pm t\right)} \hat{e_j} \tag{22}$$

where v_t is the transversal phase velocity which is

$$v_t = \sqrt{\frac{\mu}{\rho}} \tag{23}$$

in an elastic isotropic media. Taking the ratio of the phase velocities gives us

$$\frac{v_l}{v_t} = \sqrt{\frac{\lambda + 2\mu}{\mu}} \tag{24}$$

which most often has a value of approximately 2 (non-dimensional) or higher. In the case that the longitudinal waves can be considered small, this will also apply to the transversal waves and we have no additional constraint. This wave type has two components for every propagation direction, which means it has six degrees of freedom. Together with the longitudinal waves this means that a wave problem will have 9 different dimensions that together describe the total displacement in the medium.

2.7 Rayleigh waves

If the medium does not extend infinitely in all directions, but is limited by the xz-plane, there will not only be shear waves and longitudinal waves, but also Rayleigh waves. These propagate on the surface of the medium and exist as a combination of both longitudinal and shear components. To find the description wave on the surface of the solid we have a condition of no normal or shear stress

$$\sigma_{yy}|_{y=0} = \sigma_{xy}|_{y=0} = 0 \tag{25}$$

since it moves freely. We will also only consider a wave travelling in the positive x-direction and ignore the z-direction entirely. In this case we know that the solutions to (19) have the form

$$\Phi = A(y)e^{iw(\frac{x}{v_r}-t)}, \qquad \Psi = B(y)e^{iw(\frac{x}{v_r}-t)} \tag{26}$$

which we can insert into (19) to get

$$A_{yy}(y) + (\frac{\omega^2}{v_l^2} - \frac{\omega^2}{v_r^2})A(y) = 0, \qquad B_{yy}(y) + (\frac{\omega^2}{v_t^2} - \frac{\omega^2}{v_r^2})B(y) = 0 \tag{27}$$

whose solutions are

$$A(y) = A_1 e^{y\omega\sqrt{\frac{1}{v_r^2}-\frac{1}{v_l^2}}} + A_2 e^{-y\omega\sqrt{\frac{1}{v_r^2}-\frac{1}{v_l^2}}} \tag{28}$$

$$B(y) = B_1 e^{y\omega\sqrt{\frac{1}{v_r^2}-\frac{1}{v_t^2}}} + B_2 e^{-y\omega\sqrt{\frac{1}{v_r^2}-\frac{1}{v_t^2}}} \tag{29}$$

where we can see that a solution is ensured by $v_r > v_t > v_l$ and the negative exponent since we have no boundary at negative y-values. These can be inserted into the stresses

$$\sigma_{yy} = \lambda(u_x + v_y) + 2\mu v_y = \lambda(\Phi_{xx} + \Phi_{yy}) + 2\mu(\Phi_{yy} - \Psi_{yx}) = 0 \tag{30}$$

$$\sigma_{xy} = \mu(v_x + u_y) = \mu(2\Phi_{yx} - \Psi_{xx} + \Psi_{yy}) = 0 \tag{31}$$

at the surface $y = 0$. These expressions will allow us to use the potentials even for finding the stresses and therefore the boundary conditions, which means we can solve our problem with the uncoupled equations 19. This together with (21) and (23) gives us

$$\Phi = Ae^{\omega\left(y\sqrt{\frac{1}{v_r^2}-\frac{1}{v_l^2}}+i(\frac{x}{v_r}-t)\right)} \tag{32}$$

$$\Psi = A\frac{2iv_r v_t^2\sqrt{\frac{1}{v_r^2}-\frac{1}{v_l^2}}}{v_r^2-2v_t^2}e^{\omega\left(y\sqrt{\frac{1}{v_r^2}-\frac{1}{v_t^2}}+i(\frac{x}{v_r}-t)\right)} \tag{33}$$

with v_r defined as the solution to the following Rayleigh Equation:

$$(\lambda+2\mu)\left((\lambda+2\mu)\mathbf{v_r^6}\rho^3-(8\lambda+16\mu)\mathbf{v_r^4}\rho^2\mu+(24\lambda+32\mu)\mathbf{v_r^2}\rho\mu^2-(16\lambda+16\mu)\mu^3\right)=0 \tag{34}$$

This relation shows that Rayleigh waves are non-dispersive, since there is no ω dependence, which means that the speed is the same for all wavelengths. This is very important for our solution since we can treat v_r as a constant. We can also see that the two potential functions are phase shifted with the imaginary unit, i, which means each point in the medium will move in an elliptical path.

2.8 Wave excitation

When modelling in SEA there must be some source from which the vibrational energy comes. Dividing that source energy correctly between the sub-systems is crucial, since this is the source of all wave propagation. We will consider a simple case of a small circular disk with radius, a, which is subjected to a stress according to

$$P = P_0 e^{i\omega t}. \tag{35}$$

This problem was solved in [5] where the excitations were calculated numerically from an integral and found to be

$$W_l = 0.333\frac{\pi\omega^2 a^4 P_0^2}{4\rho v_l^3}, \quad W_t = 1.246\frac{\pi\omega^2 a^4 P_0^2}{4\rho v_l^3}, \quad W_r = 3.257\frac{\pi\omega^2 a^4 P_0^2}{4\rho v_l^3} \tag{36}$$

for a material where $\lambda, \mu = \sqrt{3}$. This implies heavily that small excitations can not be put only into the surface sub-system of the SEA model. About a third of the energy flows into the body waves and is divided between longitudinal and shear waves.

3 Method

To find the energy flow between sub-systems, there are two distinct problems that need to be solved. They are different only in that the incident wave types differ, and therefore the wave speed differs. In this section the different possible approaches to finding transmission coefficients are discussed.

3.1 Incident rayleigh wave

The incident rayleigh wave problem deals with the transmission and scattering of a rayleigh wave that meets a discontinuity of the surface. The simplest case, the one we are considering, is a corner with an angle of 90° as depicted in Figure 4.

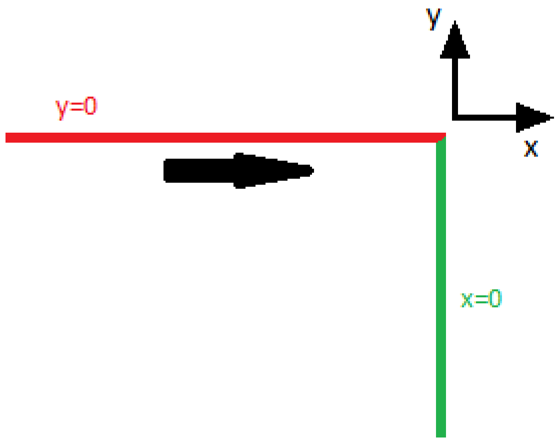

Figure 4: The incoming surface wave (black) is reflected and travels along the $y = 0$ face (red) and scattered by the $x = 0$ edge (green).

Solving this problem requires the assumption that both of the faces are free to move such that we have a condition of no normal or shear stresses on the surfaces. This is because the surrounding air can be considered light enough for the small section of the volume, the corner, does not radiate compared to the rest of the solid surface. We know this from the requirement that the wavelengths are small. This leads to the following

$$\sigma_{yy} = \lambda(\Phi_{xx} + \Phi_{yy}) + 2\mu(\Phi_{yy} - \Psi_{yx}) = 0 \quad \Big|_{y=0} \tag{37}$$

$$\sigma_{xy} = \mu(2\Phi_{yx} - \Psi_{xx} + \Psi_{yy}) = 0 \quad \Big|_{y=0} \tag{38}$$

$$\sigma_{xx} = \lambda(\Phi_{yy} + \Phi_{xx}) + 2\mu(\Phi_{xx} - \Psi_{yx}) = 0 \quad \Big|_{x=0} \tag{39}$$

$$\sigma_{xy} = \mu(2\Phi_{yx} - \Psi_{yy} + \Psi_{xx}) = 0 \quad \Big|_{x=0} \tag{40}$$

and the equations of motion for this is the same as (19) in Section 2.7. The difficulty with this problem lies in finding a general solution where the transmission to all wave types is taken into consideration. For example, [6] uses a constraint that all energy is transmitted or reflected but not transformed into body waves. This causes the solution to lose validity, since we know that some energy does transfer. It is possible that that energy can be ignored in some cases, but that should be motivated by previous measurements for validation. Measurement in a lab setting could give a credible result, like in [7] where transmission of surface waves on the edge of a thin plate were measured, but those results cannot be generalised and must be repeated for many frequencies and different values of the two elastic constants.

The method we will attempt is a common one of fourier transforming our equations and solving the resulting algebraic problems. This should hopefully result in a solution that can be reverse transformed back and give us the relations between the amplitudes of the resulting waves. Since the fourier transform is an integral operation, there is no guarantee that the solution can be resolved analytically and one should prepare for solutions that require integral solutions.

3.2 Incident body wave

The incident body wave problem deals with the same geometry and wave types, but with a pressure or shear wave as the incident wave instead. These incident waves are presented in Figure 5. In this problem we have the same boundary conditions (37) - (40) and the same resulting wave types as in the incident rayleigh wave problem. However the problem is complicated by several factors that do not apply in the rayleigh wave problem. Firstly the incoming wave can have a direction with any angle incident on the corner. This effectively means that we need a solution which has the angle as a variable as well as amplitude and frequency. There are also two types of body waves, pressure waves and shear waves, and any incoming wave is a superposition of these waves. This means that the problem has two types of initial conditions, each with an infinite number of angles compared to the rayleigh wave problem. It would be unexpected that this problem can be solved if we cannot solve that one first.

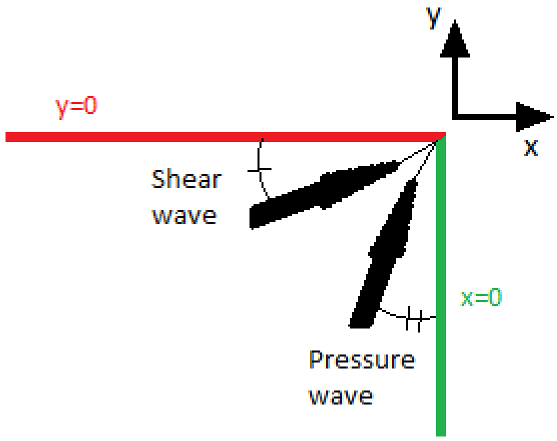

Figure 5: The incoming body waves (black) are diffracted by the corner and reflected by the faces. The waves can approach with any angle between the faces.

There have been measurements made for similar problems, but most seem to deal with an inverted corner as a wedge in to the material as in [8]. There they have varied the incident angle and shown results which are highly variable, which confirms the expectation that the incident angle plays a large role.

4 Results

After subjecting the real parts of the reflected and transmitted rayleigh potential equations to a fourier transform the following general expressions,

$$\bar{\Phi_1} = \sqrt{\frac{2}{\pi}} \int A_1 e^{\omega y \sqrt{\frac{1}{v_r^2} - \frac{1}{v_l^2}}} cos(\omega i \frac{x}{v_r}) d\omega \tag{41}$$

$$\bar{\Psi_1} = \sqrt{\frac{2}{\pi}} \int B_1 e^{\omega y \sqrt{\frac{1}{v_r^2} - \frac{1}{v_t^2}}} sin(\omega i \frac{x}{v_r}) d\omega \tag{42}$$

$$\bar{\Phi_2} = \sqrt{\frac{2}{\pi}} \int A_2 e^{\omega x \sqrt{\frac{1}{v_r^2} - \frac{1}{v_l^2}}} cos(\omega i \frac{y}{v_r}) d\omega \tag{43}$$

$$\bar{\Psi_2} = -\sqrt{\frac{2}{\pi}} \int B_2 e^{\omega x \sqrt{\frac{1}{v_r^2} - \frac{1}{v_t^2}}} sin(\omega i \frac{y}{v_r}) d\omega \tag{44}$$

were derived and inserted into the boundary conditions as defined by (37) - (40). We expect these, together with the stresses from the incoming equations, to cancel in order to satisfy the no stress boundary conditions. At the surface where $y = 0$ this yields

$$\bar{\sigma_{yy1}} = \sqrt{\frac{2}{\pi}} \int \left(A_1 \left((\lambda + 2\mu)(\frac{1}{v_r^2} - \frac{1}{v_l^2}) - \lambda \frac{1}{v_r^2} \right) - B_2 2\mu \sqrt{\frac{1}{v_r^2} - \frac{1}{v_t^2}} \frac{1}{v_r} \right) \omega^2 cos(\omega i \frac{x}{v_r}) d\omega \tag{45}$$

$$\bar{\sigma_{yy2}} = \sqrt{\frac{2}{\pi}} \int \left(A_1 \left(\lambda(\frac{1}{v_r^2} - \frac{1}{v_l^2}) - (\lambda + 2\mu) \frac{1}{v_r^2} \right) e^{\omega x \sqrt{\frac{1}{v_r^2} - \frac{1}{v_l^2}}} + B_1 2\mu \sqrt{\frac{1}{v_r^2} - \frac{1}{v_t^2}} \frac{1}{v_r} \right) \omega^2 e^{\omega x \sqrt{\frac{1}{v_r^2} - \frac{1}{v_t^2}}} d\omega \tag{46}$$

$$\bar{\sigma_{xy1}} + \bar{\sigma_{xy2}} = \sqrt{\frac{2}{\pi}} \int \left(2A_1 \frac{1}{v_r} \sqrt{\frac{1}{v_r^2} - \frac{1}{v_l^2}} + B_1(\frac{1}{v_r^2} - \frac{1}{v_t^2} + \frac{1}{v_r^2}) \right) \omega^2 sin(\omega i \frac{x}{v_r}) + 0 d\omega \tag{47}$$

and applying fourier transform to the boundary condition with these expressions, simplifies the boundary conditions to

$$A_1 \frac{2}{v_r} \sqrt{\frac{1}{v_r^2} - \frac{1}{v_l^2}} + B_1(\frac{2}{v_r^2} - \frac{1}{v_t^2}) = 0 \tag{48}$$

$$B_1 2\mu \frac{2}{v_r} \sqrt{\frac{1}{v_r^2} - \frac{1}{v_t^2}} + A_1 \mu(\frac{2}{v_r^2} - \frac{1}{v_t^2}) = -\sqrt{\frac{2}{\pi}} \int \bar{\sigma_{yy2}} cos(\frac{x}{v_r}) dx \tag{49}$$

which allows us to solve for the reflected amplitudes. Doing the same for the $x = 0$ edge gives us a similar but more complicated result for the transmitted amplitudes. It is here however that we run into problems.

There is a need of reduction to solve the integral in the right hand side of (49) and similar integrals for the $x = 0$ edge, and the best method for this was not realised during this thesis. While searching for an appropriate method, a solution in Momoi [1] was found, not only to the integrals but to the whole incident rayleigh wave problem. This was unfortunately not found previously during literature review and was the mathematical methods were considerably more complicated than expected. A brief explanation of the solution is given here.

In Momoi [1] the integral expressions that were left were separated into functions with differing properties and after a long reduction process yielded integral equations connecting the amplitudes of the reflected, transmitted and incoming waves. These integral equations become so complicated that they could not be solved analytically and are instead computed numerically. First the integral path is chosen and then an infinite system of simultaneous equations is formulated so that the equations can be normalised. Then the expressions are modified so that they can be expressed numerically near the corner and so that they will also give a result for the scattered body waves. This is followed by finding expressions for the energy fluxes and finally use of Simpson's formula for numerical integration. The results of the calculated energy flows are presented in Figure 6. There, some measurements from other works have been included and show remarkable agreement. These results also agree with measurements from [7].

Considering that the complications in solving the incident rayleigh wave problem, the incident body wave problem was deemed too difficult to solve within the scope of this thesis.

5 Discussion

The objective of this thesis was to solve the for the transmission coefficients of waves on a right angled corner. During the process a solution was found in in [1] by Momoi and the results are very general and useful. Unfortunately no work was found that looks at both scattered body waves and a varying angle of the corner, since that would add another variable to the already complicated mathematics. The varying angle problem was solved in [6] but there the assumption was made that no body waves are emitted, which results in an un-physical solution, but could be used to estimate results for angles not accounted for in [1].

6 Conclusions

The complexity of this problem exceeded expectations and it seems that a much more mathematical background is needed to solve these kinds of problems. In addition to the incident body wave problem, there is a need to solve these for a variation of the corner angle. Before a general solution for a variation of the corner angle is found, the SEA model cannot be applied in a general situation. However it is not possible within the scope of this thesis to predict the possibility of finding solutions to these problems.

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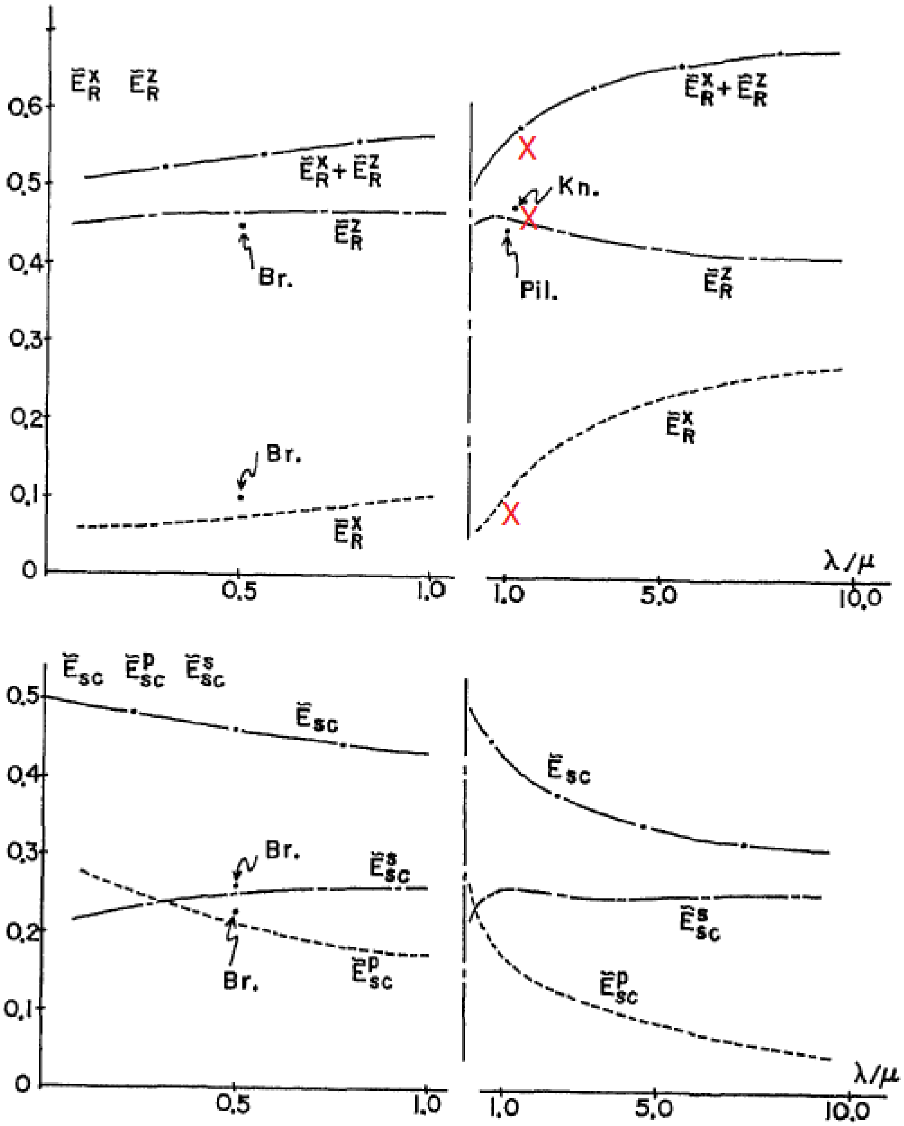

Fig. 2. Variations of energy fluxes $\breve{E}^x_R$, $\breve{E}^z_R$, $\breve{E}^p_{sc}$ and $\breve{E}^s_{sc}$ versus λ/μ. **Br.**, **Kn.** and **Pil.** in the figure denote de Bremaecker's, Knopoff *et al.*'s and Pilant *et al.*'s experimental data respectively.

Figure 6: This is an image from [1] by Momoi. The curves represent the transmission coefficients for the rayleigh waves (top) and the body waves (bottom) for variations in the ratio $\frac{\lambda}{\mu}$. Image taken from [1]. Mesurements from [7] are marked with an X (red)

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7 Future work

There are at least two large problems left to solve for the 3D solid SEA model. Firstly the incident body wave problem ,described in Section 3.2, where the incident waves have a varying angle should be solved. Secondly both of the problems in Section 3 should be solved for a varying corner angle, however this increases the complexity greatly. There is no possibility in the scope of this thesis to predict the success or failure of such a project, however a student with greater knowledge of Partial Differential Equations (PDE) could hopefully attack problems like these with a stronger set of mathematical tools.

Due to the large number of variables a comprehensive solution would be very time consuming but may ultimately be the least difficult way to approach the problem. In that case there would be a need for a reliable way to extract the wave type amplitudes from numerical solutions. There are also different computation methods that could be useful. It is possible that a search through a solution space consisting of the amplitudes of the wave types could be orders of magnitude faster than a grid based simulation. This is however still an area of active research within computer science.

8 References

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