

Expander Graphs and Explicit Constructions

Aggeliki Lountzi

A thesis submitted in partial fulfillment for the
degree of Master of Science

in the

Department of Mathematics
Uppsala University

Supervisor: Vera Koponen

December, 2015

Contents

Introduction	5
Acknowledgement	7
1 Prerequisites and Notation	9
1.1 General Definitions	9
1.2 Basic Facts from Linear Algebra	14
1.3 Matrices of Graphs	15
1.3.1 Adjacency Matrix	15
1.3.2 Examples of Graphs	16
1.3.3 Laplacian Matrix	17
1.3.4 Eigenvalue Invariance under Vertex Permutation	19
1.3.5 Spectrum	19
1.3.6 What a Graph's Spectrum can tell us	20
2 Expander Graphs	21
2.1 Expansion in Graphs	22
2.1.1 Examples	23
2.1.2 Further	24
2.2 Cheeger Inequality	26
2.2.1 Examples - Edge Expansion	30
2.3 Definitions and Properties	32
2.3.1 Combinatorial Definition	32
2.3.2 Algebraic Definition	33
2.3.3 Equivalence of the Definitions	33
2.3.4 Important Facts about Expanders and Further Results	34
2.4 Diameter in Expanders	36
3 Existence and Construction of Expanders	41
3.1 Applications to Expanders	41
3.2 Existence of Expanders	42
3.3 Two levels of Explicitness	44
3.4 Construction of Expander Families	44
3.4.1 The Margulis - Gabber - Galil Expander	45
3.4.2 Construction of Constant Degree Products	46
3.5 Cayley Graphs	48
3.6 Some Negative Results for Cayley Expansion	50

4	Ramanujan Graphs	53
4.1	Motivation of the Definition	53
4.2	Existence of Bipartite Ramanujan Graphs	54
4.2.1	2-Lifts	54
5	Random Walks and Spectral Expansion	57
5.1	Random Walks	57
5.1.1	Random Graphs	57
5.1.2	Random Walks	58
5.1.3	Expanders are Sparse Approximations of Complete Graphs	59
5.2	Spectral Expansion	61
5.3	Spectral Expansion Properties	66
5.3.1	Eigenvalue Bounds on Expansion	68
5.4	Other Measures of Expansion	69
5.4.1	Expansion and Poincare Inequality	69
5.4.2	Connection with Edge Expansion	70
5.4.3	Weak Discrete Cheeger Inequality	70
5.4.4	Expander Mixing Lemma	70
5.4.5	Expansion as Pseudorandomness	72
5.5	Further Results	72
5.5.1	Spectral Construction	73
6	Kazhdan's Property (T)	75
6.1	Unitary Representation of Locally Compact Groups	75
6.2	Property (T)	79
6.3	Kazhdan Expanders	80
6.3.1	Margulis's Construction of Expanders	83
	Bibliography	84

Introduction

Expander graphs are highly connected sparse finite graphs, which play an important role in Pure(number theory, group theory, geometry, and many more) and Applied Mathematics. They are also very important in computer science as basic building blocks for network constructions, error correcting codes, algorithms and more.

Their existence follows easily by random considerations, but explicit constructions, which are very desirable for applications, are much more difficult. Various deep mathematical theories have been used to give explicit constructions, e.g, the Kazhdan property (T) from representation theory of semisimple Lie groups and their discrete subgroups, the Ramanujan conjecture from the theory of automorphic forms, and more. We will start with general definitions of Graphs and basic definitions of expander graphs, their properties, their eigenvalues and random walks on them. We will give various examples, such as Cayley graphs, which are expanders.

An expander graph is a sparse graph that has strong connectivity properties, quantified using vertex, edge or spectral expansion.

Intuitively, an expander is a finite, undirected multigraph in which every subset of the vertices that is not “too large” has a “large” boundary. For instance, a graph for which any “small” subset of vertices has a relatively “large” neighborhood. Conceivably, that means that removing random edges (local connection failures) does not reduce the property of an expander by much. Hence, a disconnected graph is not an expander, since the boundary of a connected component is empty.

Every connected graph is an expander. However, different connected graphs have different expansion parameters. For example, the complete graph has the best expansion property, but it has the largest possible degree. So informally, a graph is a good expander if it has low degree and high expansion parameters.

A very interesting prospective of expander graphs, which also shows their importance, is that they can be considered into the human brain : Viewing neurons as vertices and axons as edges, the brain is -as a rough first approximation- also a graph, and this graph actually appears as one of the first motivations that led to expander graphs.

There are mainly three definitions but they are all equivalent.

Roughly, Expander graphs are certain families of graphs, becoming larger and larger, which have the following two competing properties :

1. they are fairly sparse(in terms of number of edges, relative to the number of vertices)
2. yet they are highly connected, and in fact highly robust in some sense.

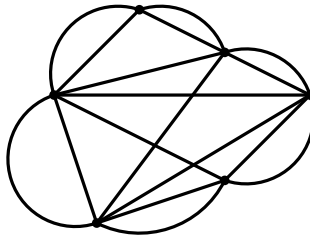
The property of connectedness would simply mean that one can go from any vertex to any other in the graph. One natural strengthening is to ask that such a path is always rather short, which means that the maximal distance in the graph between two points is much smaller than the number of vertices.

Any subset $V \subset V_n$, where n is the number of vertices, should have many connections with its complement $W = V_n \setminus V$, i.e. , there should be many edges linking vertices v and w , with $v \in V$ and $w \in W$.

More precisely, expanders are determined by the condition that, for some constant $c > 0$, independent of n , the number of edges should be at least

$$c \min(|V|, |W|)$$

for all non-empty subsets $V \subset V_n$, for all n .



This definition of sparse, highly connected, robust families of graphs is obviously quite strong.

But do they exist?

We will show that expander families do exist, and in fact exist in great abundance (this was first shown using probabilistic methods). Moreover, it is maybe even more surprising that they turn out to appear in so many different areas of Mathematics, and lead to extremely remarkable results in unexpected directions, such as, embedding graphs in space, knot distortion and many more.

We will now begin by defining the concept of expander graphs formally. As we have already mentioned, with many fundamental important concepts in Mathematics, there are a number of equivalent definitions of this concept.

We will adopt a “spectral” prospective towards expander graphs, defining them in terms of a certain gap, but will relate this formulation of expansion to the more classical notion of edge expansion later.

We recall the notions of basic graph theory and we note that we will work with undirected, loop free multiplicity-free graphs.

Acknowledgement

I would like to express my sincere gratitude to my supervisor Vera Koponen for the useful comments, remarks and engagement through the learning process of this master thesis.

Also, I want to thank my family for supporting me spiritually throughout my life.

Chapter 1

Prerequisites and Notation

1.1 General Definitions

The material of pages 9-13 is found in [A], [B].

Definition 1.1.1 (Graph). A graph is a pair $G = (V, E)$, where V is a set called the vertex set of G , and $E \subset \binom{V}{2}$ is a collection of unordered pairs $\{v, w\}$ of distinct elements v, w of V , known as the edge set of E . A graph with the edges being ordered pairs of vertices is directed. If we instead consider the edges to be unordered pairs of elements of V , then it is undirected. Edges of the form (v, v) are called loops. If a graph is undirected and has no loops, we call it simple. If there is an edge (v, w) , we say that the vertices v and w are adjacent and that the edge (v, w) is incident to the vertices v and w .

We often sketch graphs like polygons, or perhaps with curved rather than straight edges. However, it is important to remember that the geometry of a particular sketch is not important, at least not in general.

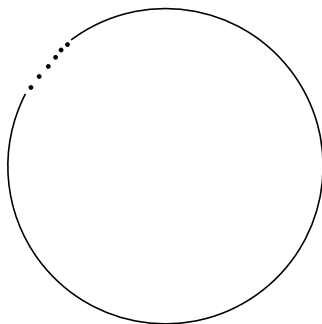
Another very useful way of defining a graph, that we usually come across in most books is the following.

Definition 1.1.2. A graph Γ is given by a triple (V, E, ep) where V and E are arbitrary sets called respectively, the set of vertices of Γ and the set of edges of Γ and also

$$ep : E \rightarrow V^{(2)}$$

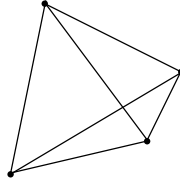
is an arbitrary map, called the endpoint map, where $V^{(2)}$ denotes the set of subsets $e \subset V$ of cardinality either 1 or 2.

Example 1.1.3. 1. The cycle graph on $n \geq 3$ vertices, denoted as C_n , is a graph with vertices $V = \mathbb{Z}/n\mathbb{Z} = \{0, 1, 2, \dots, n-1\}$ and edges joining each i to $i+1$.

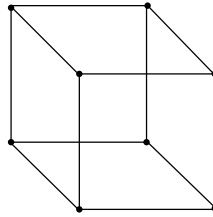


2. The complete graph is a simple undirected graph on n vertices, denoted K_n , has an edge joining every pair of distinct vertices. It has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges, which is maximal for an undirected graph.

For $n = 4$, we get :



3. The n^{th} hypercube graph, denoted Q_n , on 2^n vertices, has vertex set $\{0,1\}^n$, and edges joining those n -tuples that differ in exactly one coordinate. It can be identified with the 1-skeleton, that is the vertices and edges of the n - dimensional cube, so in particular the graph Q_3 can be indentified with the 1-skeleton of the familiar cube, which has 8 vertices and 12 edges in both the geometric and graph-theoretic senses. That is :



It will be necessary sometimes to loosen our definition of a graph to the more general notion of a multigraph, where two vertices can be connected by multiple edges. We will indicate when this generalisation is required. A *multiset* is a generalized set where the elements are allowed to appear more than once, that is, they can have arbitrary positive integer multiplicity.

Definition 1.1.4 (Multigraph). A multigraph $G = (V, E)$ is a pair consisting of a set V and a multiset E such that the elements of E are all members of $V \times V$.

Note that, if a graph is simple, it is implicit that it is not a multigraph.

Remark 1.1.5. Expanders are “sparse graphs”, in a sense that is made precise by the definition of degree. There is no precise, universally agreed-upon definition of “sparseness” for graphs. It means roughly speaking that the total number of edges is much less than the number of unordered pairs on n vertices, that is, $\binom{n}{2}$, which for a simple graph on n vertices is the maximum number of edges possible. To give one possible precise definition, for an infinite family of graphs (V_i, E_i) we could say that the total number of edges grows subquadratically, that is, $|E_i|/|V_i|^2 \rightarrow 0$ as $i \rightarrow \infty$.

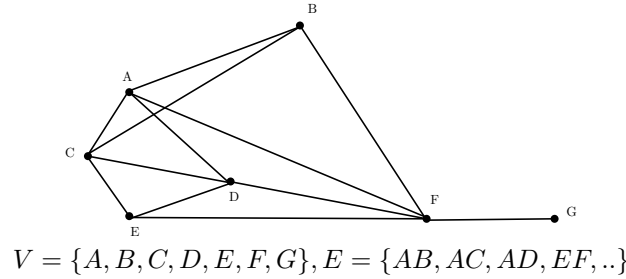
Remark 1.1.6. While the theory of infinite graphs is interesting, covering such topics as Cayley graphs for infinite groups and automorphism groups of trees are not included in this thesis. From now on we restrict attention almost exclusively to finite graphs, as these are the graphs for which expansion is defined.

Definition 1.1.7. Elements of V or E are called vertices and edges of E . A graph is finite if the vertex set (and hence the edge set) is finite.

Definition 1.1.8. If $\alpha \in E$ is an edge of Γ , the elements of $ep(\alpha)$ are called *extremities* of α . If $\alpha \neq \beta$ are distinct edges of Γ , they are called *adjacent* at a vertex $x \in V$ if $x \in ep(\alpha) \cap ep(\beta)$ is a common extremity.

Definition 1.1.9. Given a vertex $x \in V$, the number of edges α such that x is an extremity, i.e., such that $x \in ep(\alpha)$, is called the *valency*, $val(x)$ of x . If the valency is the same, say equal to $r \geq 0$, at all vertices, the graph is called *regular*, or *r-regular*.

In other words, If $k \geq 0$ is a natural number, we say that a graph $G = (V, E)$ is *k-regular* if each vertex of V is contained in exactly k edges in E . We refer to k as the *degree* of the regular graphs.



We will mostly be interested in constant-degree large finite regular graphs, in which k is fixed (e.g. $k = 4$), and the number $n = |V|$ of vertices is going off to infinity.

- Definition 1.1.10.**
1. For a set X , $|X| \in [0, \infty)$ denotes its cardinality with $|X| = \infty$ if X is infinite.
 2. For a group G , we denote by $[G, G]$ the commutator group of G , which is generated by all commutators $[gh] = ghg^{-1}h^{-1}$. Also, $[G, G] \triangleleft G$ and $G/[G, G]$ is abelian.
 3. F_p is the finite field $\mathbb{Z}/p\mathbb{Z}$, p prime and more generally by F_q a finite field with q elements, where $q = p^n$, $n \geq 1$, is a power of p .

Definition 1.1.11. A path is a non- empty graph $P = (V, E)$ of the form $V = \{x_0, x_1, \dots, x_k\}$ and $E = \{x_0x_1, x_1x_2, \dots, x_{k-1}x_k\}$, where x_i are all distinct. The vertices x_0 and x_k are linked by P and are called the *endvertices* or *ends*; the vertices x_1, \dots, x_{k-1} are the *inner vertices* of P . The number of edges of a path is its length, and the path of length k is denoted by P^k . Note that k is allowed to be zero, and thus $P^0 = K^1$.

Example 1.1.12. Here we will see a path $P = P^6$ in G :



Definition 1.1.13. If $P = x_0 \dots x_{k-1}$ is a path and $k \geq 3$, then the graph $C := P + x_{k-1}x_0$ is called a *cycle*. As with paths, we often denote a cycle by its (cyclic) sequence of vertices; the above cycle C might be written as $x_0 \dots x_{k-1}x_0$. The length of a cycle is its number of edges (or vertices); the cycle of length k is called a *k-cycle* and denoted by C^k .

Definition 1.1.14. *The distance $d_G(x, y)$ in G of two vertices x, y is the length of the shortest $x - y$ path in G ; if no such path exists, we set $d(x, y) := \infty$. The greatest distance between any two vertices in G is the diameter of G , denoted by $\text{diam}G$*

There are many different connectivity notions for graphs, which form an essential part of graph theory. The edges connecting vertices are the fundamental structure that graphs have beyond simply being sets, so in some sense the connectivity properties are the *raison d'être* of graphs. The most basic connectivity property of a graph is simply called connectedness.

Definition 1.1.15. *Two vertices v and w in a graph $G = (V, E)$ are said to be connected if there is a path joining them. If all pairs of vertices $v, w \in V$ are connected, we say that the graph G is connected.*

If a graph is not connected, it is often convenient to discuss its connected components. For the definition of connected components, we note that in an undirected graph, the relation of connectedness is an equivalence relation on the set V of vertices : symmetry holds by taking a path of length zero, reflexivity follows from taking a reversed path, and transitivity follows from concatenation of paths.

Definition 1.1.16. *A connected component of an undirected graph is an equivalence class of vertices under the equivalence relation of connectedness.*

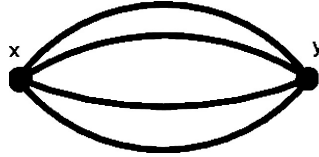
Connected graphs have a natural metric, usually referred to as distance. For unconnected graphs, we can consider two unconnected vertices to be at distance ∞ from each other, but this is not actually a metric because the distances in a metric space must be finite by definition (although with the natural addition in $\mathbb{R}_0^+ \cup \{\infty\}$ the axioms for a metric still holds).

Remark 1.1.17. :



a loop, which are edges where $\text{ep}(\alpha) = \{x\}$ is a singleton. The loop is then based at x .

Remark 1.1.18. Multiple edges with the same endpoints, say $\alpha_1 \neq \alpha_2$, with $ep(\alpha_1) = ep(\alpha_2) = \{x, y\}$:



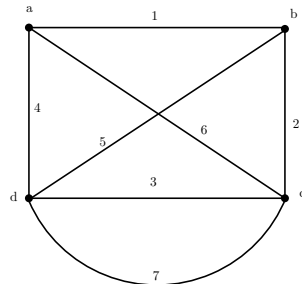
Conversely, to draw a graph Γ coded as a triple (V, E, ep) , we can draw the points, then for each $\alpha \in E$ we look at $ep(\alpha)$ and draw either a loop from x to x if $ep(\alpha) = \{x\}$ is a single element, or an arc from x to y if $ep(\alpha) = \{x, y\}$ with $x \neq y$.

Example 1.1.19. Consider the graph with $V = \{a, b, c, d\}$, $E = \{1, 2, 3, 4, 5, 6, 7\}$ and

$$ep(1) = \{a, b\}, ep(2) = \{b, c\}, ep(3) = \{c, d\},$$

$$ep(4) = \{a, d\}, ep(5) = \{a, c\}, ep(6) = \{b, d\}, ep(7) = \{c, d\}.$$

That is,



Remark 1.1.20. 1. We will denote $ep(\alpha) = \{x, y\}$ without specifying that $x \neq y$. A more geometric flavor is : $x \xrightarrow{\alpha} y$

2. In a simple graph, we actually have that every set of endpoint $ep(\alpha)$ contains two elements, and no multiple edges so that ep is an injection of E into the set of subsets of order 2 in V .

In that case, the set of edges can also be identified with a subset $R \subset V \times V$ such that $(x, y) \in R$ if and only if $(y, x) \in R$ and such that $(x, x) \notin R, \forall x \in V$. This is more common of “coding” simple graphs.

3. For a graph Γ , we write $|\Gamma| = |V|$: the “size” of Γ is identified with the number of vertices. We also sometimes write $x \in \Gamma$ to mean $x \in V$. Note that V_Γ is the set of vertices of a graph Γ .

1.2 Basic Facts from Linear Algebra

We will note several useful definitions and results from linear algebra. Let $M \in R^{n \times n}$ be a square symmetric matrix of n rows and n columns.

The material of pages 14-18 can be found in [A], [B].

Definition 1.2.1. *An eigenvalue of M is a scalar $\lambda \in R$ such that there exist a vector $x \in R^n$ for which $M \cdot x = \lambda \cdot x$. The vector x is called the eigenvector corresponding to the eigenvalue λ . The multiset of eigenvalues is called the spectrum.*

Facts about eigenvalues and eigenvectors of symmetric matrices over R :

1. M has n real eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$. The eigenvectors associated with these eigenvalues form an orthogonal basis for the vector space R^n . Actually, for any two such vectors the inner product is zero and all vertices are linearly independent.
2. The smallest eigenvalue satisfies :

$$\lambda_1 = \min_{x \in R^n, x \neq 0} \frac{x^T M x}{x^T x}.$$

Denote the eigenvector corresponding to λ_i as x_i . Denote the vector space of all vectors in R^n that are orthogonal to x_1 as $W(x_1) := R^n \text{ span}\{x_1\}$. Then the second smallest eigenvalues satisfies :

$$\lambda_2 = \min_{x \in W(x_1)} \frac{x^T M x}{x^T x}$$

If we denote $W(x_1, \dots, x_{k-1}) := R^n \text{ span}\{x_1, \dots, x_{k-1}\}$, then the k 'th smallest eigenvalue is

$$\lambda_k = \min_{x \in W(x_1, \dots, x_{k-1})} \frac{x^T M x}{x^T x}$$

This characterization of the spectrum is called the Courant Fisher Theorem.

3. Denote by $\text{Spec}(M)$ the spectrum of the matrix M , that is the multi-set of its eigenvalues. Then for a block diagonal matrix M , that is, a matrix of the form

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}.$$

The following holds : $\text{Spec}(M) = \text{Spec}(A) \cup \text{Spec}(B)$.

4. Eigenvalues of a matrix can be computed in polynomial time. In fact, eigenvalues are the roots of the characteristic polynomial of a matrix.
5. The Interlacing theorem : A matrix B is denoted a principal minor of matrix M if it can be obtained from M by deleting $k < n$ columns and k rows. Let $A \in R^{(n-1) \times (n-1)}$ be a principal minor of the matrix M .

Now let :

$$\text{Spec}(A) = \{\mu_1 \leq \dots \leq \mu_{n-1}\}$$

Then,

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \dots \leq \mu_{n-1} \leq \lambda_n$$

1.3 Matrices of Graphs

1.3.1 Adjacency Matrix

In order to encode more efficiently a finite graph, one can use its adjacency matrix. Actually, the most common matrix associated with graphs in literature is the adjacency matrix. Specifically, the adjacency matrix of a finite graph G on n vertices is the $n \times n$ matrix where the non-diagonal entry a_{ij} is the number of edges from vertex i to vertex j , and the diagonal entry a_{ii} , depending on the convention, is either once or twice the number of edges(loops) from vertex i to itself. For a simple graph with no self-loops, the adjacency matrices must have zeros on the diagonal.

Definition 1.3.1. For a graph $G = (V, E)$, the adjacency matrix $A = A_G$ is defined as :

$$A_{i,j} = \begin{cases} 1, & (i, j) \in E \\ 0, & \text{otherwise} \end{cases}$$

In other words, for a finite graph G , the adjacency matrix $A_G = (\alpha(x, y))$ is the matrix with rows and columns indexed by V_G and write $\alpha(x, y)$ equal to the number of edges with extremities (x, y) . Formally, that is,

$$\alpha(x, y) = |\{\alpha \in E_G : ep(\alpha) = \{x, y\}\}|.$$

Note the adjacency matrix is always symmetric, which reflects our use of unoriented edges.

It is easy to go in the opposite direction: Given any symmetric “matrix” $A = \alpha(x, y)$ with rows and columns indexed with a finite set V and non-negative integral coefficients $\alpha(x, y)$, one defines a finite graph with adjacency matrix A by taking V as a set of vertices and (for instance!)

$$E = \{(x, y, i) \in V \times V \times Z : \alpha(x, y) \neq 0 \text{ and } 1 \leq i \leq |\alpha(x, y)|\}$$

with $ep(x, y, i) = \{x, y\}$, for all (x, y, i) in E .

A simple but useful result of the previous definition, which can also be generalized to directed graphs, is given below :

Proposition 1.3.2. Let G be a graph on n vertices with associated adjacency matrix A . Then for any positive integer k , the $(i, j)^{th}$ entry of A^k is the number of k -walks from i to j . In particular, the entries along the main diagonal of A^k are the numbers of closed k -walks in G .

What the adjacency matrix fails to provide however, is a method of counting self-avoiding walks and cycles in G . We will not investigate this path though in this thesis.

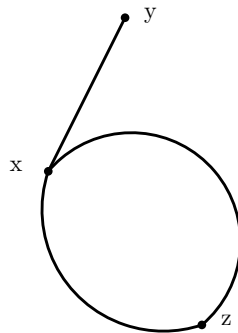
Remark 1.3.3. • At this point, someone may ask about the diagonal part of the matrix.

Are these cells always zero? And what happens when we have loops?

The answer to the first question is obviously negative.

If we find that the graph has some loop in some vertices, we can fill the diagonal element of the adjacency matrix with the number of loop.

For example,

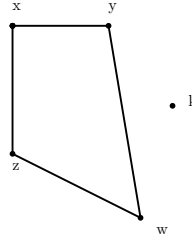


which has adjacency matrix :

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix}.$$

- If a graph has some vertex that is not connected to any other vertices, the adjacency matrix correspond to that single vertex is zero.

For example,



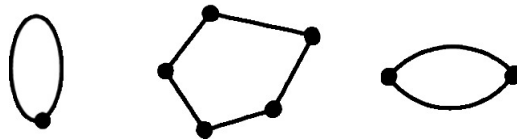
which its adjacency matrix is :

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

1.3.2 Examples of Graphs

Example 1.3.4. 1. Cycle : Let $m \geq 1$ be an integer. The m -cycle C_m is the graph with vertices $V_m = \mathbb{Z}/m\mathbb{Z}$, and endpoint map given by $ep(i) = \{i, i+1\}$ for $i \in \mathbb{Z}/m\mathbb{Z}$. In other words, except when $m = 1$ (in which the cycle is a single loop based at 0), there are two edges adjacent to any given $i \in V_m$, which are the edges coded by $i - 1$ and the one coded by i itself.

Here, we can see the graphs for $m = 1, m = 5$ and $m = 2$ respectively :



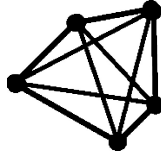
2. Path : Let $m \in \mathbb{Z}$. The path of length m , denoted by P_m , is the graph with $m + 1$ vertices, $V_m = \{0, 1, \dots, m\}$, m edges, $E_m = \{1, \dots, m\}$ and $ep(i) = \{i - 1, i\}$ for $1 \leq i \leq m$. A path of length zero is a graph with a single vertex and no edges.

For example, the path of length 4 is :



3. Complete graph : Let $m \geq 1$ an integer. The complete graph K_m with m vertices has also $V_m = \{1, \dots, m\}$ but now $E_m = \{(x, y) \in V_m/x < y\}$, with $ep((x, y)) = \{x, y\}$. In other words, each pair of distinct vertices is joined by (exactly) one edge.

For example, the complete graph K_5 is :



All graphs in these first examples are simple graphs, except for the cycles C_1 and C_2 .

Most of them are regular, that is,

- C_1 is 1-regular
- C_m is 2-regular for $m \geq 2$
- P_0 is 1-regular
- P_1 is 1-regular, but P_k is not regular for $k \geq 2$
- K_m is $(m - 1)$ -regular $\forall m \geq 1$.

1.3.3 Laplacian Matrix

The Laplacian matrix is a matrix representation of a graph. We can find the material in [U]. Recall the definition of the diagonal matrix :

Definition 1.3.5. A diagonal matrix is a matrix in which the entries outside the main diagonal are all zero. The diagonal entries themselves may or may not be zero. Thus the matrix $D = (d_{i,j})$ with n columns and n rows is diagonal if $d_{i,j} = 0$ if $i \neq j$ for all i, j in $\{1, 2, 3, \dots, n\}$.

Example 1.3.6. The following is an example of a diagonal matrix.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Definition 1.3.7. Given a simple graph G with n vertices, its Laplacian matrix $L_{n \times n} = L$ is defined as $L = D - A$, where D is the degree matrix and A is the adjacency matrix of the graph.

The elements of L are :

$$L_{i,j} = \begin{cases} \deg(v_i), & \text{if } i = j \\ -1, & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise} \end{cases}$$

where $\deg(v_i)$ is the degree of the vertex i .

Notice that if the graph G is d -regular, then its matrices satisfy $A_G = d(I - L_G)$. Denote by $\{\lambda_i\}$ the eigenvalues of A and by $\{\mu_i\}$ the eigenvalues of L . Then the previous relation implies that : $\lambda_i = d(1 - \mu_i)$. In fact, for any graph, the Laplacian is a positive semi-definite matrix, that is, for any vector $y \in R^n$, we have that : for every $y \in R^n$, $y^T L y \geq 0$, or equivalently, all eigenvalues are nonnegative.

Example 1.3.8 (The cycle). Consider a cycle on n nodes. The laplacian is

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & -\frac{1}{2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & -\frac{1}{2} & \cdots & 1 \end{bmatrix}.$$

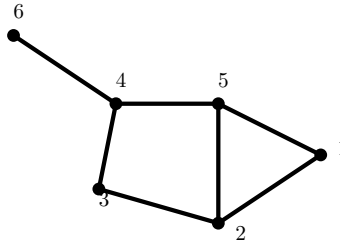
This matrix has 1's on the diagonal, and 0's or $-\frac{1}{2}$ elsewhere, depending on whether the indices correspond to an edge. Since a cycle is 2-regular, in each row and column there are exactly two entries with $-\frac{1}{2}$.

Definition 1.3.9. The symmetric normalized Laplacian matrix is defined as $L^{sym} := D^{-1/2}LD^{1/2} = I - D^{-1/2}AD^{-1/2}$.

The elements of L^{sym} is :

$$L_{i,j} = \begin{cases} 1, & \text{if } i = j \\ \frac{-1}{\sqrt{\deg(v_i)\deg(v_j)}}, & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \\ 0, & \text{otherwise} \end{cases}$$

Example 1.3.10. Here is another example of a Laplacian of a labeled graph.



The degree matrix of the graph is :

$$\begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The adjacency matrix is :

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Hence the Laplacian matrix is :

$$\begin{pmatrix} 2 & -1 & 0 & 0 & -1 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 3 & -1 & -1 \\ -1 & -1 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}.$$

Note that we usually only use these definitions for graphs without self-loops, multiple edges, or isolated vertices (L is not defined on graphs with a degree zero vertex). These matrices are useful due to how they act.

1.3.4 Eigenvalue Invariance under Vertex Permutation

We are going to concern ourselves with the eigenvalues of these matrices that we just defined. However, it is apparent in the definition of these matrices that each vertex had to be assigned a number. Thus it is fair to ask whether renaming the vertices will affect the eigenvalues that we will study. Of course, we will find exactly what is expected - that these values are intrinsic to the graph and not the choice of vertex naming. Let us sketch of how we can know this:

Suppose that M is some matrix defined in terms of a graph G and that a permutation $\sigma \in S_n$ is applied to the vertex names. Now, define M' the same way that M was defined - just applied to the newly vertices. Then, $M_{i,j} = M'_{\sigma(i)\sigma(j)}$. Let P_σ be the permutation matrix for σ . Then, $P_\sigma^T M P_\sigma = M'$. This is because the multiplication where M is on the right will swap the columns accordingly, and the other multiplication will swap the rows. It is easy to check that for permutation matrices, $P^T = P^{-1}$, meaning that M and M' are similar. Then we only need recall the linear algebra fact that similar matrices have the same characteristic polynomial and thus the same eigenvalues and multiplicities.

1.3.5 Spectrum

Now, let A_G be the adjacency matrix of G . Let $\lambda_1 \geq \dots \geq \lambda_n$ be the eigenvalues of A_G . Also define the Laplacian matrix of G as $L_G = D - A_G$, where D is the diagonal matrix where D_{vv} equals the degree of the vertex v . For d -regular graphs, $L_G = dI - A_G$, and hence the eigenvalues of L_G are $d - \lambda_1, d - \lambda_2, \dots, d - \lambda_n$.

We refer to [T], [X].

Lemma 1.3.11. • $\lambda_1 = d$

- $\lambda_2 = \lambda_3 = \dots = \lambda_k = d$ if and only if G has at least k connected components.

Proof. For the first part, verify that the vector 1 is an eigenvector with eigenvalue d .

For the second part, let C_1, \dots, C_r be all the connected components of G . Then $1_{C_1}, \dots, 1_{C_r}$ are all mutually orthogonal eigenvectors with eigenvalue d . Thus if G has at least k connected components, then $\lambda_2 = \dots = \lambda_r = d$.

Suppose there is an eigenvector f not in the span of $1_{C_1}, \dots, 1_{C_r}$ with eigenvalue d . Let C_i be a component on which that eigenvector is not constant. Let $v \in C_i$ be a vertex such that $f(v)$ is maximum. Since f has eigenvalue d , $df(v) = \sum_{u \in \Gamma(v)} f(u)$, and by maximality this implies that

$f(u) = f(v)$ for each $u \in \Gamma(v)$, where $\Gamma(v)$ is the neighborhood of v . Repeating this, we conclude that f must be constant on the component C_i , which is a contradiction. ■

In particular, connected graphs have $\lambda_2 < d$. We will actually study in the next chapter the notion of expansion, where λ_2 is less than d by a significant amount; this can be thought of as a strong form of connectedness.

1.3.6 What a Graph's Spectrum can tell us

In this section, we refer to $[[T], [U], [W], [S], [X]]$.

We start with a quick linear-algebraic lemma.

Lemma 1.3.12. *For square matrices A and B (note that the zero-blocks are not necessarily square),*

$$\det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = (\det A)(\det B).$$

Proof. Fix the dimensions of B to be $n \times n$. We induct on r , where the dimensions of A are $r \times r$. The base case is clear, and we suppose that the result holds for $r \leq n-1$. Using Laplace's formula for the determinant, we have (let M_{ij} (resp. A_{ij}) be the minor of the entire block matrix (resp A) formed by excluding the i th row and j th column) :

$$\det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \alpha_{11} \det M_{11} + \cdots + \alpha_{1n} \det M_{1n}.$$

By the inductive hypothesis, we have that

$$\det \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \alpha_{11} \det M_{11} + \cdots + \alpha_{1n} \det M_{1n} = \alpha_{11} \det A_{11} B + \cdots + \alpha_{1n} \det A_{1n} \det B = (\det A)(\det B).$$

■

Therefore, first of all, from the spectrum we can get information about the following :

- The connectedness of a graph as we can see from the following theorem :

Theorem 1.3.13. *$\max\{k/\lambda_k = 0\} = K_0$ if and only if G has precisely K_0 disjoint connected components, where λ_j is the $j+1^{\text{st}}$ smallest eigenvalue of the normalized Laplacian matrix.*

- Bipartiteness. We can see from the following results that all eigenvalues of the normalized Laplacian are at most 2. This leads to the fact that, this upper bound is attained if and only if G is bipartite.

Corollary 1.3.14. *$\lambda_{n-1} \leq 2$, where λ_{n-1} is the largest eigenvalue of a normalized Laplacian matrix.*

This Corollary leads to the equivalency between bipartiteness and $\lambda_{n-1} = 2$ from the below theorem. I will state the theorem without its proof.

Theorem 1.3.15. *G is bipartite if and only if $\lambda_{n-1} = 2$.*

- Cheeger inequality implies expansion. We actually want to show that λ_2 , the second smallest eigenvalue, is a measure of how well connected a graph is. This will be accomplished by defining a constant associated with each graph (one that clearly is related to how well connected a graph is) and then putting bounds on it in terms of λ_2 . This will be our main goal in the next chapter.

Chapter 2

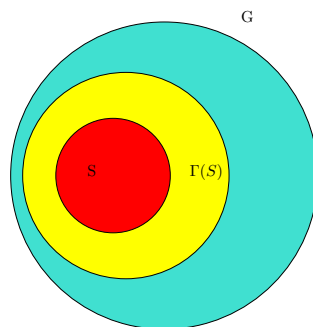
Expander Graphs

Let $G = (V, E)$ an undirected graph and S, T subsets of V , we have that

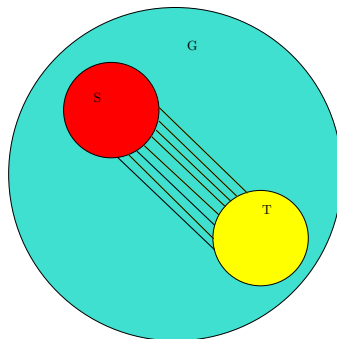
$$\Gamma(S) = \{v \in V \mid \exists u \in S : (u, v) \in E\}$$

is the neighbourhood of the set S

This can be seen from the figure below.

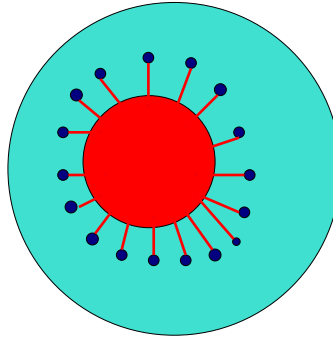


Also, we have that :



Then, $E(S, T) = \{(u, v) \in E \mid u \in S, v \in T \text{ or } u \in T, v \in S\}$.

Finally, for the figure following, we have that $\partial S = \{(u, v) \in E \mid u \in S, v \notin S\} = E(S, \bar{S})$.

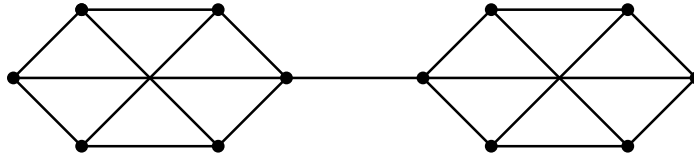


Also, we define $\partial_{\text{out}}(S) = \{v \in V \mid \exists u \in S : (u, v) \in E\}$ to be the neighbouring of S . Hence, the boundary of S is defined as $\partial(S) = \Gamma(S) \setminus S$. In other words, the boundary of S , ∂S , is the set of vertices in the complement of S which are adjacent to at least one vertex in S , that is,

$$\partial S = \{v \in V \mid d(v, S) = 1\}.$$

Of course, assuming the graph is connected, the diameter is the first invariant that comes to mind.

Recall that for a fixed number of vertices, a graph with smaller diameter is “better connected”. However, as we have already said, we also wish to be able to detect (using our invariant) that the graph is robust, that is, it cannot be disconnected too easily. For instance, consider the graph Γ_m given by taking the disjoint union of two copies Γ and Γ' of a complete graph K_m , for some $m \geq 2$ and adding a single edge between chosen vertices $x_1 \in \Gamma$ and $x_2 \in \Gamma'$:



We clearly have $\text{diam}(\Gamma_m) = 3$, for any m , which shows that Γ_m has very small diameter. However, if we remove the single additional edge between x_1 and x_2 , we obtain a disconnected graph. This behavior is not desirable in many applications, and leads to the definition of the “expansion constant” or Cheeger constant of a graph.

2.1 Expansion in Graphs

We begin the study of “expansion” properties in graphs. Then we will give the definition of an expander family of graphs and we will show the equivalence of different notions of expansion. Our goal is to find a quantitative invariant that can be used to measure a very high level of connectedness of a graph.

In this section, we use material from [[C], [T], [Y], [Z], [AB]].

Remark 2.1.1. Recall that, for any disjoint subsets of vertices $V_1, V_2 \subset V$, we denote by $\epsilon(V_1, V_2)$ or $\epsilon_\Gamma(V_1, V_2)$ the set of edges of Γ with one extremity in V_1 and one extremity in V_2 , that is

$$\epsilon(V_1, V_2) = \{\alpha \in E : \text{ep}(\alpha) \cap V_1 \neq \emptyset, \text{ep}(\alpha) \cap V_2 \neq \emptyset\}$$

and we denote by $\epsilon(V_1)$ or $\epsilon_\Gamma(V_1)$ the set $\epsilon(V_1, V - V_1)$ of edges with one extremity in V_1 and one outside V_1 .

Now, we can define the expansion constant.

Definition 2.1.2. Let $\Gamma = (V, E, ep)$ a finite graph.

The expansion constant $h(\Gamma)$ is defined by :

$$h(\Gamma) = \min\left\{\frac{|\epsilon(W)|}{|W|} : \emptyset \neq W \subset V, \text{ such that } |W| \leq \frac{1}{2}|\Gamma|\right\}.$$

In other words, $h(\Gamma)$ is the smallest possible ratio between the number of edges exiting from W and the size of W , when W is a non-empty, but not too big, subset of vertices.

This will provide a measure of robustness, in the following sense : The larger $h(\Gamma)$ is, the more difficult it is to disconnect a large subset of V from the rest of the graph.

We can see this in the following result.

Proposition 2.1.3. Let $\Gamma = (V, E, ep)$ be a finite graph.

1. We have $h(\Gamma) > 0$ if and only if Γ is connected.
2. If $W \subset V$ is a subset of vertices with $|W| = \delta|V|$, $\delta \leq \frac{1}{2}$, one must remove at least $\delta h(\Gamma)|V|$ edges from Γ to disconnect W from the rest of the graph.

Proof. 1. $h(\Gamma) = 0$, that means that there exist some $W \subset V$, non-empty, of size $\leq \frac{|\Gamma|}{2}$, such that $\epsilon(W)$ is empty. Now, if $x \in W$ and $y \notin W$ are any two vertices, there can not be a path in Γ between x and y , since it would have to cross from W to $V - W$ at some point.

Conversely, if Γ is not connected, there must be at least two connected components, and at least one of them must have size $|W| \leq \frac{|\Gamma|}{2}$. Then $\epsilon(W) = \emptyset$, and hence $h(\Gamma) = 0$.

2. Once, we explain the meaning of the sentence, it will become clear. So, we say that removing a set C of edges disconnects W from $V - W$ if $\epsilon(W) \subset C$, i.e. , all edges that go from W to "somewhere else" are contained in C . Then, since

$$|\epsilon(W)| \geq h(\Gamma)|W| = \delta h(\Gamma)|V|,$$

by the definition of $h(\Gamma)$, our statement is just a formulation. ■

2.1.1 Examples

1. The complete graph K_m with $m \geq 2$ vertices :

Any two subsets of the vertices with the same cardinality are equivalent, i.e. , there is an automorphism of the graph mapping one to the other, and hence

$$h(K_m) = \min_{1 \leq j \leq \frac{m}{2}} \frac{1}{j} |\epsilon(\{1, \dots, j\})| = \min_{1 \leq j \leq \frac{m}{2}} (m - j) = m - \lfloor \frac{m}{2} \rfloor.$$

Since there are $j(m - j)$ edges in K_m from $\{1, \dots, j\}$ to its complement $\{j + 1, \dots, m\}$.

Note that it is very visible that expansion "slows down" when sets of vertices larger than half of the graph are considered, so the condition $|V| \leq \frac{|\Gamma|}{2}$ is needed to obtain a good definition.

2. $\Gamma = C_m$, the cycle with $m \geq 2$ vertices :

The subsets of size $\leq \frac{m}{2}$ that expand least are given (this is very clear intuitively) by the images W of paths in C_m of length $\text{diam}(C) = \lfloor \frac{m}{2} \rfloor \leq \frac{m}{2}$. In this case, $\epsilon(W)$ has two elements (one edge from each end of the path), and so,

$$h(C_m) = \frac{2}{\lfloor \frac{m}{2} \rfloor} \leq \frac{4}{m-1}.$$

Note : The inequality $h(C_m) \leq \frac{4}{m-1}$ follows in this way, even if one does not know that paths are the last expanding subsets,

$$h(C_m) \leq \frac{|\epsilon(W)|}{|W|}$$

by definition of any subset W .

3. Let Γ be a graph like the one discussed at the beginning of this section, two copies of K_m joined by a single edge α . Then if we take W to be the first copy of K_m , we see that $\epsilon(W) = \{\alpha\}$, and hence $h(\Gamma) \leq \frac{1}{m}$.
4. Let $T = T_{d,k}$ be a finite tree with degree ≥ 3 and depth $k \geq 1$:

The expansion constant can be bounded from the above by taking as subset W one of the subtrees “below a neighbor of the root”, i.e. , if x_0 is the root and x_1 is a vertex indexed with a single letter of the alphabet (e.g. $x = 1$), we let

$$W = \cap_{2 \leq j \leq k} \{(1, s_1, \dots, s_j \in V_T)\}$$

which can be written equivalently as

$$W = \{y \in V_T : d_T(y, x_0) \geq d_T(y, 1)\}.$$

We then have, $|W| = \frac{|T|-1}{d} \leq \frac{|T|}{2}$ and so,

$$h(T) \leq \frac{|\epsilon(W)|}{|W|}.$$

These examples are already instructive.

In particular, they show that $h(\Gamma)$ in a way is consistent with our goal : the “super”- connected complete graphs have $h(\Gamma)$ very large, while large, easily - disconnected graphs, like C_m or example 3 have quite small expansion constants. Also, since the expansion constant is defined as a minimum, a single well-chosen subset W may lead to a good upper-bound, while we need to know which sets are the worst behaved in order to give a non-trivial lower-bound. This indicates that it can be easier to give an upper-bound for $h(\Gamma)$ than a lower bound. However, we will discuss bounds later on.

2.1.2 Further

Lemma 2.1.4 (Trivial bounds). *For any finite connected graph Γ such that V and E are non-empty, we have that*

$$\frac{2}{|\Gamma|} \leq h(\Gamma) \leq \min_{x \in V} \text{val}(x)$$

where $\text{val}(x)$ is the valency of a vertex x .

Proof. For the lower-bound, we just note that since Γ is connected, we must have

$$|\epsilon(W)| \geq 1$$

for any non-empty proper subset W in V . So,

$$\frac{|\epsilon(W)|}{|W|} \geq \frac{1}{|W|} \geq \frac{2}{|\Gamma|}$$

if $1 \leq |W| \leq \frac{|\Gamma|}{2}$.

On the other hand, for the upper-bound, take $W = \{x\}$ for some x . We have that $|\epsilon(W)| = \text{val}(x)$ and hence $h(\Gamma) \leq \text{val}(x)$. ■

We now come to a proper result. We will show that a large $h(\Gamma)$ implies that the diameter of a graph is relatively small. This actually means that the expansion constant does not control this more natural-looking invariant.

Proposition 2.1.5. *Let Γ be a finite non-empty connected graph. We have that*

$$\text{diam}(\Gamma) \leq 2 \frac{\log \frac{|\Gamma|}{2}}{\log(1 + \frac{h(\Gamma)}{u})} + 3$$

where $u = \max_{x \in V} \text{val}(x)$.

The intuitive idea is the following :

To “join” x to y with a short path, we look at how many elements there are at increasing distance from x to y . The definition of the expansion ratio gives a geometric lower-bound on the number of new elements when we increase the distance by one, and at some point the sets which can be reached in n -steps from both sides are so big that they have to intersect, giving a distance at most $2n$ by the triangle inequality.

Now, before we will give the proof of the above proposition, we will state and show a very important and useful lemma.

Lemma 2.1.6. *Let Γ be a finite non-empty connected graph and $x \in V$. For any $n \geq 0$, let*

$$B_x(n) = \{y \in V : d_\Gamma(x, y) \leq n\}.$$

Then, with u as before, we have that

$$|B_x(n)| \geq \min\left(\frac{|\Gamma|}{2}, \left(1 + \frac{h(\Gamma)}{u}\right)^n\right).$$

Proof. It is enough to show that if $n \geq 0$ is such that $|B_x(n)| \leq \frac{|\Gamma|}{2}$, we have that

$$|B_x(n+1)| \geq \left(1 + \frac{h(\Gamma)}{u}\right) |B_x(n)|$$

since $B_x(0) = \{x\}$. To prove this inequality, we observe simply that if $\alpha \in \epsilon(B_x(n))$ is an edge exiting from $B_x(n)$, its extremity which is not in $B_x(n)$ is in $B_x(n+1) - B_x(n)$, i.e. , is at distance $n+1$ from x . This is actually a new point.

It is possible that multiple edges α starting from $B_x(n)$ lead to the same y , but since all these edges share the extremity y , the maximal number of edges leading to y is $\text{val}(y) \leq u$, so that

$$|B_x(n+1) - B_x(n)| \geq \frac{|\epsilon(B_x(n))|}{u} \geq \frac{h(\Gamma)}{u} |B_x(n)|,$$

by definition of $h(\Gamma)$, using the assumption that $|B_x(n)| \leq \frac{|\Gamma|}{2}$. Then we get that,

$$|B_x(n+1)| = |B_x(n)| + |B_x(n+1) - B_x(n)| \geq (1 + \frac{h(\Gamma)}{u})|B_x(n)|.$$

■

Now we can prove the previous proposition.

Proof. Let $x, y \in V$ 3 arbitrary vertices. We are going to estimate $d_\Gamma(x, y)$ from above.

We denote $\beta = 1 + \frac{h(\Gamma)}{u}$, and we denote by $n \geq 1$ the smallest integer such that $\beta^n \geq \frac{|\Gamma|}{2}$, which is possible since $\beta > 1$, in view of the connectedness of Γ . Then, by previous lemma, applied to x and y , we find that

$$|B_x(n)| \geq \frac{|\Gamma|}{2}, |B_y(n)| \geq \frac{|\Gamma|}{2}.$$

In fact, we must have $|B_x(n+1)| > \frac{|\Gamma|}{2}$ because either this is true for $B_x(n)$, or else $|B_x(n)| = \frac{|\Gamma|}{2}$ and then there are some vertices at distance $n+1$, and so

$$B_x(n+1) \cap B_y(n) \neq \emptyset,$$

which means that $d_\Gamma(x, y) \leq 2n+1$ by passing through an intermediate point z lying in this intersection. Since now, x and y were arbitrary, we have that

$$\text{diam}(\Gamma) \leq 2n+1$$

and since $n = \lceil \frac{\log \frac{|\Gamma|}{2}}{\log \beta} \rceil \leq \frac{\log \frac{|\Gamma|}{2}}{\log \beta} + 1$, and so we obtain the diameter bound that we stated. ■

Example 2.1.7. For the complete graphs and the cycles, this translates to the following asymptotic upper bounds on the diameter. That is,

$$\text{diam}(K_m) \ll \log m, \text{diam}(C_m) \ll m \log m, \text{ for } m \geq 2.$$

Both are off by a factor of size $\log m = \log |K_m| = \log |C_m|$ from the actual values.

2.2 Cheeger Inequality

We want to show that λ_2 , the second smallest eigenvalue, is a measure of how well connected a graph is. This will be accomplished by defining a constant associated with each graph (one that clearly is related to how well connected a graph is) and then putting bounds on it in terms of λ_2 .

During this section, we refer to [W], [U], and [T].

Definition 2.2.1 (volume). We define the volume of a subset of the vertices $X \subset V$ to be :

$$\text{vol}(X) = \sum_{x \in X} \text{deg}(x).$$

Definition 2.2.2 (Cheeger constant on a vertex subset). We define :

$$h_G(X) = \frac{|E(X, \bar{X})|}{\min\{\text{vol}(X), \text{vol}(\bar{X})\}},$$

as the Cheeger constant on a vertex subset (which is actually the expansion constant that we have already defined).

For a subset of vertices $S \subseteq V$ denote by $E(S, \bar{S})$ the set of edges crossing the cut defined by S . Using the above the definition, we can now define the Cheeger constant in new terms.

Definition 2.2.3. *The cheeger constant of a graph G is :*

$$h_G := \min_{S \subseteq V} \frac{|E(S, \bar{S})|}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}} = \min_{S \subseteq V, \text{vol}(S) \leq |E|} \frac{|E(S, \bar{S})|}{\text{vol}(S)}.$$

Note that the volume of some subset of the vertices basically represents how important or large that subset is, taking into account differences in degree (since different vertices may have different degree, we do not simply take $\text{vol}(X) = |X|$). We can consider Cheeger constant, h_G , as representing how difficult it is to disconnect G . A small value of h_G means that there is a way to divide up the vertices into sets X and \bar{X} so that there are few edges leading between the sets, relative to the size of the sets.

Example 2.2.4. *If G is already disconnected, then h_G is zero (take X to be a connected component). If G is complete (without self-loops), then consider a partition X, \bar{X} , such that $|X| = n, |\bar{X}| = m$, and without loss of generality, $n \geq m$. Then,*

$$\begin{aligned} h_G &= \frac{|E(X, \bar{X})|}{\min\{\text{vol}(X), \text{vol}(\bar{X})\}} = \frac{mn}{\text{vol}(\bar{X})} = \frac{mn}{\sum_{x \in \bar{X}} \text{deg}(x)} \\ &= \frac{mn}{(n+m-1)\sum_{x \in \bar{X}} 1} = \frac{n}{n+m-1} = \frac{n}{|V|-1}. \end{aligned}$$

Since $n \geq m$, we know that $n \geq \lceil \frac{|V|}{2} \rceil$. Thus,

$$h_G = \min_X h_G(X) = \frac{\lceil |V|/2 \rceil}{|V|-1}$$

At this point we need to define the adjacency operator.

Recall that l^2 is the space of square - summable sequences, which is actually a Hilbert space.

Definition 2.2.5. *Given a finite graph $G = (V, E)$, one can define the adjacency operator*

$$A = l^2(V) \rightarrow l^2(V)$$

on functions $f : G \rightarrow \mathbb{C}$ by the formula

$$Af(v) := \sum_{w \in V, \{v,w\} \in E} f(w)$$

thus, $Af(v)$ is the sum of f over all of the neighbours of V . If one enumerates the vertices of V as v_1, \dots, v_n in some fashion, then one can associate A with an $n \times n$ matrix, known as the adjacency matrix of G .

Remark 2.2.6. *Recall that, a self-adjoint operator on a complex vector space V with inner product $\langle \cdot, \cdot \rangle$ is an operator (a linear map A from V to itself) that is its own adjoint :*

$$\langle Av, w \rangle = \langle v, Aw \rangle.$$

As our graphs are undirected, the adjacency operator A is clearly self-adjoint, and the adjacency matrix is real symmetric.

By the spectral theorem, A thus has n real eigenvalues

$$\lambda_1 \geq \dots \geq \lambda_n.$$

Theorem 2.2.7. $2h_G \geq \lambda$, where λ is the second smallest eigenvalue of the Laplacian of the graph G .

Proof. Since the Cheeger constant of G equals h_G , there exists a partition of the vertices $A, B (B = \bar{A})$ such that $h_G(A) = h_G$. Define a function of vertices :

$$f(v) = \begin{cases} \frac{1}{\text{vol}(A)} & \text{if } v \in A \\ \frac{-1}{\text{vol}(B)} & \text{otherwise} \end{cases}$$

Note that

$$\begin{aligned} \frac{\sum_{\{i,j\} \in E} (f(i) - f(j))^2}{\sum_{v \in V} \text{deg}(v) f(v)^2} &= \frac{\sum_{\{i,j\} \in E(A,B)} \left(\frac{1}{\text{vol}(A)} + \frac{1}{\text{vol}(B)} \right)^2}{\frac{1}{\text{vol}^2(A)} \sum_{v \in A} \text{deg}(v) + \frac{1}{\text{vol}^2(B)} \sum_{v \in B} \text{deg}(v)} \\ &= \frac{\left(\frac{1}{\text{vol}(A)} + \frac{1}{\text{vol}(B)} \right)^2 \sum_{\{i,j\} \in E(A,B)} 1}{\frac{1}{\text{vol}(A)} + \frac{1}{\text{vol}(B)}} \\ &= |E(A, B)| \left(\frac{1}{\text{vol}(A)} + \frac{1}{\text{vol}(B)} \right) \leq \frac{2|E(A, B)|}{\min\{\text{vol}(A), \text{vol}(B)\}} \\ &= 2h_G(A) = 2h_G. \end{aligned}$$

Now, it is important to realize that without loss of generality, we can assume that

$$\psi_0 = \frac{D^{1/2} \mathbb{1}}{\|D^{1/2} \mathbb{1}\|} \perp f.$$

Thus, we have that :

$$\lambda = \min_{h \perp \psi_0} \frac{h^T L h}{h^T h} \leq \frac{f^T L f}{f^T f} = 2h_G$$

■

Therefore, the upper bound can be stated in the below theorem :

Theorem 2.2.8. $\lambda \geq \frac{h_G^2}{2}$, where λ is the second smallest eigenvalue of the Laplacian of the graph G .

Before we proceed to the proof, we recall the Cauchy-Schwartz inequality :

Let $a_1, \dots, a_n \in \mathbb{R}$ and $b_1, \dots, b_n \in \mathbb{R}$. Then :

$$\sum_{i=1}^n a_i b_i \leq \left(\sum_i a_i^2 \right)^{\frac{1}{2}} \cdot \left(\sum_i b_i^2 \right)^{\frac{1}{2}}.$$

Proof. Let \vec{y} be the vector so that

$$\lambda = \frac{\sum_{(i,j) \in E} (y_i - y_j)^2}{\sum_i d_i y_i^2}.$$

Now define two vectors \vec{u}, \vec{v} with coordinates :

$$u_i = \begin{cases} -y_i & y_i < 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$v_i = \begin{cases} y_i & y_i > 0 \\ 0 & \text{otherwise} \end{cases}$$

Observe that $(y_i - y_j)^2 \geq (u_i - u_j)^2 + (v_i - v_j)^2$.

Hence,

$$\lambda \geq \frac{\sum_{(i,j) \in E} [(u_i - u_j)^2 + (v_i - v_j)^2]}{\sum_i d_i (u_i + v_i)^2}.$$

Since $\frac{a+b}{c+d} \geq \min \left\{ \frac{a}{c}, \frac{b}{d} \right\}$ it is enough to show that :

$$\frac{\sum_{(i,j) \in E} (u_i - u_j)^2}{\sum_i d_i u_i^2} \geq \frac{h_G^2}{2}.$$

Now we multiply and divide by the same quantity :

$$\begin{aligned} \frac{\sum_{(i,j) \in E} (u_i - u_j)^2}{\sum_i d_i u_i^2} &= \frac{\sum_{(i,j) \in E} (u_i - u_j)^2}{\sum_i d_i u_i^2} \times \frac{\sum_{(i,j) \in E} (u_i + u_j)^2}{\sum_{(i,j) \in E} (u_i + u_j)^2} \\ &\geq \frac{[\sum_{(i,j) \in E} (u_i - u_j)^2][\sum_{(i,j) \in E} (u_i + u_j)^2]}{\sum_i d_i u_i^2 \cdot 2 \sum_{(i,j) \in E} (u_i^2 + u_j^2)}, \end{aligned}$$

where the last inequality comes from $(a+b)^2 \leq 2(a^2 + b^2)$.

Now, by Cauchy-Schwartz inequality, we get :

$$\begin{aligned} &\geq \frac{[\sum_{(i,j) \in E} (u_i - u_j)(u_i + u_j)]^2}{2(\sum_i d_i u_i^2)^2} \\ &= \frac{[\sum_{(i,j) \in E} (u_i^2 - u_j^2)]^2}{2(\sum_i d_i u_i^2)^2}. \end{aligned}$$

At this point, we denote by $S_k = \{u_1, \dots, u_k\} \subseteq V$ the set of the first k vertices. Also, denote by C_k the size of the cut induced by S_k . Now, since $u_i^2 - u_j^2 = u_i^2 - u_{i+1}^2 + u_{i+1}^2 - u_{i+1}^2 + \dots + u_{j-1}^2 - u_j^2$, we can write that :

$$\frac{\sum_{(i,j) \in E} (u_i - u_j)^2}{\sum_i d_i u_i^2} \geq \frac{[\sum_{(i,j) \in E} (u_i^2 - u_j^2)]^2}{2(\sum_i d_i u_i^2)^2} = \frac{[\sum_k (u_k^2 - u_{k+1}^2) \cdot C_k]^2}{2(\sum_i d_i u_i^2)^2}.$$

According to the definition of h_G , we know that $C_k \geq h_G \cdot (\sum_{i \leq k} d_i)$, since h_G is the minimum of a set of expressions containing these.

Therefore :

$$\begin{aligned}
&\geq \frac{[\sum_k (u_k^2 - u_{k+1}^2) \cdot h_G \cdot (\sum_{i \leq k} d_i)]^2}{2(\sum_i d_i u_i^2)^2} \\
&= h_G^2 \cdot \frac{[\sum_k (u_k^2 - u_{k+1}^2) \cdot (\sum_{i \leq k} d_i)]^2}{2(\sum_i d_i u_i^2)^2} \\
&= h_G^2 \cdot \frac{(\sum_k d_k u_k^2)^2}{2(\sum_i d_i u_i^2)^2} = \frac{h_G^2}{2}.
\end{aligned}$$

And this concludes the proof. ■

Remark 2.2.9. Note that we actually showed that the previous result gives us a method to find a cut (S, \bar{S}) such that $\frac{E(S, \bar{S})}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}} \leq \sqrt{2\lambda}$.

Namely, take the eigenvector (y_1, \dots, y_n) corresponding to λ , and check all the n cuts of the type $S_k = \{i : x_i \leq x_k\}$.

Expansion can be defined both with respect to the number of the edges or vertices on the boundary of S . We will mostly stick with the edge expansion, which is more directly related to eigenvalues.

Definition 2.2.10 (Vertex expansion). We define the vertex expansion factor as :

$$h_{out}(G) = \min_{0 < |S| \leq \frac{n}{2}} \frac{|\partial_{out}(S)|}{|S|}$$

In the definition of edge expansion (cuts), instead of $N(S)$, we use the number of edges leaving S . Hence :

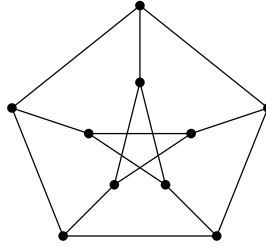
Definition 2.2.11. The edge expansion (or cheeger constant) of a graph G with n vertices is :

$$h(G) = \min_{|S| \leq \frac{n}{2}} \frac{e(S, \bar{S})}{|S|},$$

where $e(S, \bar{S})$ is the number of edges between S and its complement.

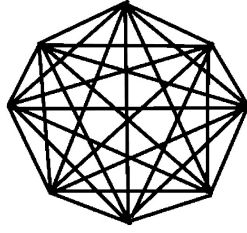
2.2.1 Examples - Edge Expansion

1. Petersen graph :



We have $h(G) = 1$.

2. $G = K_n$.



We have here that $h(G) = \frac{n}{2}$.

3. If G is a complete graph, then $h(G) = \frac{|V|}{2}$.
4. If G is not connected, then $h(G) = 0$.

The following proposition relates the different expansion constants of a single graph, which will give us the desired equivalence of definitions of expander families later on in this chapter.

Proposition 2.2.12. (*Bounds between expansion constants*)
 For a k -regular graph G ,

$$\frac{h(G)}{k} \leq \epsilon(G) \leq h(G).$$

Proof. Suppose $V = A \sqcup B$ is a partition of the vertices. Without loss of generality, we may assume that $|A| \leq |B|$ and thus $|A| \leq \frac{n}{2}$.

Since to each vertex $b \in \partial B \subseteq B$ there corresponds at least one edge $(a, b) \in E(A, B)$, we have $|\partial A| \leq |E(A, B)|$.

As G is k -regular, for each vertex $b \in \partial A$ there are at most k edges incident to b , and in particular there can be at most k edges of the form $(a, b) \in E(A, B)$. Thus, $|E(A, B)| \leq k|\partial A|$.

Putting these together, we have that

$$\frac{1}{k} \cdot \frac{|E(A, B)|}{\min(|A|, |B|)} \leq \frac{|\partial A|}{|A|} \leq \frac{|E(A, B)|}{\min(|A|, |B|)}.$$

As this holds for any partition with $|A| \leq \frac{n}{2}$, taking minima over all such partitions gives

$$\frac{1}{k}h(G) \leq \epsilon(G) \leq h(G).$$

■

Corollary 2.2.13 (Equivalence of vertex and edge expansion). *Vertex and edge expansion are equivalent, that is, a family of k -regular graphs is a family of vertex expanders if and only if it is a family of edge expanders (that is, the cheeger constants are bounded away from zero).*

Remark 2.2.14. *Note that the equivalence of vertex and edge expansion depends very essentially on the fact that the degree of vertices is bounded (actually a constant k). For example, the trees of diameter 2, which are also known as the star graphs, in which one central vertex is joined to all the other vertices, have good edge expansion but poor vertex expansion.*

We can now turn to the next section to define expander graphs, which encapsulate the idea of graphs which are both relatively sparse and highly, and robustly connected.

2.3 Definitions and Properties

Loosely speaking, expander graphs are regular graphs of small degree that exhibit various properties of cliques. Another useful intuition is that expander graphs exhibit various properties of random regular graphs on the same degree.

In particular, we refer to properties such as the relative sizes of cuts in the graph (i.e., relative to the number of edges), and the rate at which a random walk converges to the uniform distribution (relative to the algorithm of the graph size to the base of its degree).

In this section we give definitions of expander graphs and expander families, illustrated by some examples.

Throughout the rest of this section, we will assume that $G = (V, E)$ is a k -regular graph on n vertices. There are many different definitions of expander graphs, but they are all equivalent up to a change of constant (possibly after some transformation). We will present the two most commonly used definitions and prove they are equivalent. And then we will stick using the most fundamental one.

Recall that a sparse graph is the graph in which the number of edges is much less than the possible number of edges.

The material of this section can be found in [BB], [BC], [Y].

2.3.1 Combinatorial Definition

Our first and most important definition of expanders is as follows.

Definition 2.3.1. *Let $c > 0$. A finite k -regular graph $G = (V, E)$ on n vertices is called an (n, k, c) -expander if for every set of vertices $A \subseteq V$ with $|A| \leq \frac{n}{2}$,*

$$|\partial A| \geq c|A| \quad (1).$$

If n and k are understood, then we will simply call G a c -expander.

We consider only sets of size less than or half of all vertices because then no graph ever satisfies any c value greater than 0; take $A = V$.

This definition means that all small sets of vertices are connected somewhat robustly to the rest of the graph, thus higher c values correspond to better expansion. In other words, equation (1) means that sets A which are “not too large” cannot have boundaries that are very small relative to A , and the condition $|A| \leq \frac{n}{2}$ must be imposed because otherwise as A becomes a large proportion of V , ∂A can only be very small, and consequently c would need to be very small for large graphs (in fact, if we allowed $A = V$ this would force $c = 0$). All connected graphs are expanders for some $c \geq 0$, and non-connected graphs are not expanders as taking A to be the smallest connected component requires c to be zero.

Notice that the *Cheeger inequality* is closely related to this definition as we can see below :

$$\begin{aligned} h_G = c &\implies \text{for all } S, \text{ we have } h_G(S) \geq c \\ &\implies \text{for all } S, \text{ we have } \frac{|E(S, \bar{S})|}{\min\{\text{vol}(S), \text{vol}(\bar{S})\}} \geq c \\ &\implies \text{for all } S \text{ such that } |S| \leq \frac{|V|}{2}, \frac{|E(S, \bar{S})|}{\text{vol}(S)} \geq c \\ &\implies \frac{|E(S, \bar{S})|}{d|S|} \geq c \implies |E(S, \bar{S})| \geq cd|S|, \text{ where } d \text{ is the degree} \end{aligned}$$

$\implies G$ is an (n, k, c) – expander.

Then using the *Cheeger inequality* to relate h_G to λ_2 and the fact that $\lambda_i = d(1 - \mu_i)$, we can relate combinatorial expansion to the spectral quantities of G . Also, all of the above steps are reversible, allowing for if and only if statements that relate the spectral quantities to the combinatorial expansion coefficient.

2.3.2 Algebraic Definition

Recall that for any k -regular graph G , the eigenvalues of its adjacency matrix will lie in $[-k, k]$. Also, recall that μ_i are the eigenvalues of the Laplacian L .

Definition 2.3.2. *Such a graph is said to an (n, k, ϵ) -expander if it is connected and all of the non-trivial eigenvalues lie in $[-k + \epsilon, k - \epsilon]$*

Since k is always an eigenvalue of G (with multiplicity one, since G is assumed to be connected), k is considered a trivial eigenvalue. Sometimes $-k$ is also considered trivial since every bipartite graph has this eigenvalue. In what follows, we will assign an expansion coefficient, $\hat{\mu}(G)$ to each n -vertex, k -regular graph, and say that G is a spectral expander if $\hat{\mu} < k$ and an ϵ -expander if $\hat{\mu}(G) \leq k - \epsilon$. However, we will define $\hat{\mu}$ in two different ways, which will affect what graphs are considered to be expanders.

The two $\hat{\mu}$ definitions are :

Definition 2.3.3. 1. $\hat{\mu}(G) = \max\{|\mu_1|, |\mu_{n-1}|\}$

2. $\hat{\mu}(G) = \max_{|\mu_i| \neq d} |\mu_i|$, where d is the degree of the vertices of G .

The only discrepancy arises in the case of bipartite graphs. The first definition assigns bipartite graphs an expansion coefficient that is k (due to the $-k$ eigenvalue) and thus says that no bipartite graph is ever an expander. The second definition allows for bipartite graphs. Note than in the case of definition 1, $\hat{\mu}$ is the second largest absolute value of any eigenvalue of G , and in definition 2, $\hat{\mu}$ is the second largest absolute value except for bipartite graphs, in which case it is the third largest. We will specify which definition we use when needed.

2.3.3 Equivalence of the Definitions

The following theorem is referred to [BB], [BC].

Theorem 2.3.4 (equivalence of the two definitions). *Let G be a k -regular graph having a self-loop on each vertex.*

1. *The graph G is c -expanding for $c \geq (k - \lambda(G))/2d$.*
2. *If G is c -expanding then $k - \lambda(G) \geq c^2/(4 + 2c^2)$.*

Thus, any non-zero bound on the combinatorial expansion of a family of k -regular graphs yields a non-zero bound on its eigenvalue gap, and vice versa. Note however, that the back-and-forth translation between these measures is not tight.

Recall that in such a graph as in the previous theorem, we have that for $G = (V, E)$, it holds that $N_G(S) \supseteq S$ for every $S \subseteq V$, and thus $|N_G(S)| = |N_G(S) \setminus S| + |S|$. Furthermore, in such a graph all eigenvalues are greater than or equal to $-k + 1$, and thus if $k - \lambda(G) < 1$ then this is due to a positive eigenvalue of G .

2.3.4 Important Facts about Expanders and Further Results

Lemma 2.3.5. *Let G be a graph. Then G is a c -expander for some $c > 0$ if and only if G is connected.*

Proof. If a graph G is connected, then $|\partial A| > 0$ whenever $0 < |A| < n$, so since there are only finitely many $A \subseteq V$, G will be an expander for

$$c(G) := \min_{0 < |A| < \frac{n}{2}} \frac{|\partial A|}{|A|} > 0 \quad (2).$$

If on the other hand G is not connected, letting A be a connected component of minimal size in G we then have $\partial A = \emptyset$ and $|A| \leq \frac{n}{2}$, since there must be at least two connected components. ■

Remark 2.3.6. *Although the above lemma completely classifies which graphs are expanders when taken by themselves, we wish to find large sparse graphs with the expansion coefficient c bounded away from zero uniformly. It is applications of expanders that first motivated the following definition. However, we can also motivate it by the observation it allows us to move from the definition of single expander graphs, something which requires choosing an arbitrary c , to families of expanders which can be described more “qualitatively”, that is, without imposing some constant priori.*

At this point, we have to say that, we often talk of expander graphs while we actually mean an infinite collection of graphs such that each graph in this collection satisfies the same property. For example, when talking of a k -regular expander graph, we actually refer to an infinite collection of graphs such that each of these graphs is k -regular. Typically, such a collection (or family) contains a single N -vertex graph for every $N \in \mathbb{S}$, where \mathbb{S} is an infinite subset of \mathbb{N} .

Generally, let $d \in \mathbb{N}$. A sequence of d -regular graphs $\{G_i\}_{i \in \mathbb{N}}$ of size increasing with i is a family of expanders if there is $\epsilon > 0$ such that $h(G_i) \geq \epsilon$ for all i .

In other words,

Definition 2.3.7. *A family $(G_i)_{i \in I}$ of finite non-empty connected graphs $G_i := (V_i, E_i, ep)$ is an expander family, or a family of expanders, if there exist constants $u \geq 1$ and $h > 0$, independent of i , such that :*

- *The number of vertices $|V_i|$ “tends to infinity”, in the sense that for any $N \geq 1$, there are only finitely many $i \in I$ such that G_i has at most N vertices.*
- *For each $i \in I$, the expansion ratio satisfies $h(G_i) \geq h > 0$, i.e. , it is bounded away from 0 by a constant independent of i .*

We will say that a pair (h, u) for which the two above properties hold are expansion parameters of the family.

Remark 2.3.8. *Of course, most often the index set I is just the set of positive integers, so that we have a sequence of expander graphs. However, it is convenient to allow more general index sets.*

Let us now review these conditions :

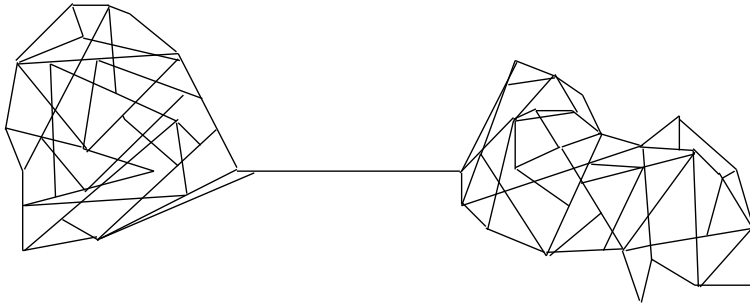
1. The first is a matter of convention. That is, if Γ is a fixed non-empty connected graph, it has bounded valency, of course, as well as positive expansion constant, and hence a 'constant' family with $\Gamma_i = \Gamma$, for all i , would qualify as expanders if the number of vertices was not allow to grow.

The idea is that a family of expanders should allow us to construct arbitrarily large graphs, measured with the number of vertices, which are "sparse" and "super-connected", and hence repeating a single graph infinitely many times is not interesting.

2. The second condition is our interpretation of sparsity. The point is that if the valency of vertices of a graph Γ is $\leq k$, the number of edges is controlled by the number of vertices, namely $|E_\Gamma| \leq k|V_\Gamma|$.
3. The last condition is connectedness and robustness assertion. It is natural in view of our examples and of previous proposition. It is the best to hope for here, since the trivial bound of previous lemma shows that one cannot do better than having $h(\Gamma)$ bounded from below for a family of graphs with bounded valency.

Lemma 2.3.9. *Any expander graph is a connected graph.*

The figure following is a graph G with poor expansion.



This figure presents a graph G which is a manifestly poor expander. If we take the set A in the combinatorial definition to be the 'cluster' of vertices on the left, then we have $\partial A = 1$, but A comprises half of the vertices. Since G is connected, it will be a c -expander for some sufficiently small positive c . However, the best possible c will be very small for a graph of this size, and we can imagine that if we had a sequence of graphs of increasing size with similar structure (only one edge connecting the two 'sides' of the vertex set) then their respective constants c would tend to 0. (Unsurprisingly, the technical term of an edge, like the one joining the two sides in the above figure, whose deletion disconnects the graph, is a bridge.)

Example 2.3.10. *The sequence (K_n) of complete graphs is not an expander family because their degree is not constant.*

Lemma 2.3.11. *The sequence of cycle graphs (C_n) with $n \geq 3$ is not an expander family.*

Proof. Take A in the combinatorial definition to be the vertices on a path of length $\lfloor \frac{n}{2} \rfloor$. Then if C_n is a c -expander, we must have

$$c \leq \frac{|\partial A|}{|A|} = \frac{2}{\lfloor \frac{n}{2} \rfloor} = O(1/n)$$

as $n \rightarrow \infty$.

■

Corollary 2.3.12. *A family of k -regular expanders must have $k \geq 3$.*

Proof. As expanders are connected, they must have degree at least 2, except for the case of graphs on 2 vertices (degree 1 graphs have the form of an even number of vertices connected in disjoint pairs). However, up to isomorphism, the only connected 2-regular graph on $n \geq 3$ vertices is the cycle graph C_n . ■

Remark 2.3.13. *As is common practice, we say that graphs themselves are expanders to mean that they form a family of expanders, and likewise when saying that graphs are not expanders (even though, as noted above, all connected graphs are automatically c -expanders for some sufficiently small c , and could in fact be realised as a member of an expander family by simply adding them to any existing expander family). For example, we say that cycle graphs are not expanders.*

Remark 2.3.14. *It turns out that there are in fact expanders of degree 3, and in certain models of random regular graphs, almost all 3-regular graphs are expanders [[C], Theorem 4.16]. As a constructive example, for each prime p we construct a graph whose vertex set is $F_p = \mathbb{Z}/p\mathbb{Z}$, with an edge joining each vertex x to $x - 1$, $x + 1$ and x^{-1} (taking 0 to be its own inverse).*

Remark 2.3.15. *An alternative way of defining expander families would be to require that*

$$\liminf_{i \rightarrow \infty} c(G_i) > 0$$

where c is as defined by equation (2). Naturally, it is possible that a sequence G_i of graphs is not an expander family for the reason that there is no uniform lower bound on the $c(G_i)$, even though the sequence of graphs (G_i) still has a subsequence that is an expander family (consider for example the sequence of constants $0, 1, 0, 1, \dots$).

There is a straightforward upper bound on the expansion constant c in the inequality (1). As $\partial A \subseteq V \setminus A$, we have that $|\partial A| \leq n - |A|$. Thus for any $A \subseteq V$ of an (n, k, c) -expander, if $|A| = \lfloor \frac{n}{2} \rfloor$ then we have that

$$c \leq \frac{|\partial A|}{|A|} \leq \frac{n - \lfloor n/2 \rfloor}{\lfloor n/2 \rfloor} = \begin{cases} 1 & \text{if } n \text{ even} \\ 1 + \frac{2}{n-1} & \text{if } n \text{ odd} \end{cases}$$

In a complete graph K_n , the inequalities become equalities, and for any smaller A , that is $|A| < \lfloor n/2 \rfloor$, we have $|\partial A|/|A| > c$. Thus the upper bound on c is tight.

2.4 Diameter in Expanders

We have already mentioned that good expansion guarantees that a graph is both ‘robust’ and ‘efficient’, which respectively means that it is not easily disconnected and that any pair of vertices has a small distance between them. In this section, we pursue the second of those notions in detail.

Recall that for any set $S \subseteq V$, let

$$N(S) = \{v \in V \mid \exists u \in S : (u, v) \in E\} = \Gamma(S)$$

be the neighbourhood set of S .

Let us first define balls and spheres in graphs more in detail than before, which will be used to study growth, and show that the boundary of a ball is a sphere.

Recall that a metric is a function that defines a distance between each pair of elements of a set. Formally, that is :

Definition 2.4.1. *A metric on a set X is a function (called distance) $d : X \times X \rightarrow [0, +\infty)$, where for all x, y, z in X , the following conditions are satisfied :*

1. $d(x, y) \geq 0$ (non - negativity).
2. $d(x, y) = 0$ if and only if $x = y$ (coincidence axiom).
3. $d(x, y) = d(y, x)$ (symmetry).
4. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Definition 2.4.2 (Balls and Spheres). *Let $G = (V, E)$ be a connected, undirected graph with metric d . Let $v \in V$ be a vertex of X , and r a non-negative integer. The ball of radius r centered at v is defined to be*

$$B_r(v) = \{w \in V | d(v, w) \leq r\}.$$

The sphere of radius r centered at v is defined to be

$$S_r(v) = \{w \in V | d(v, w) = r\}.$$

It is immediate from this definition that the balls partition into spheres :

$$B_r(v) = \bigcup_{r'=0}^r S_{r'}(v).$$

The following result, however, does require proof (for example, it is not true for arbitrary metric spaces whose distances are integral).

Proposition 2.4.3. *Let v be a vertex of an undirected graph G , and $r \in \mathbb{N}$. Then*

$$\partial B_r(v) = S_{r+1}(v).$$

Proof. First let $w \in \partial B_r(v)$. By the definition of boundary, $w \notin B_r(v)$, so $d(v, w) > r$, and there is a $w' \in B_r(v)$ such that $d(w', w) = 1$. Then by the triangle inequality, we have that

$$d(v, w) \leq d(v, w') + d(w', w) \leq r + 1.$$

Thus, $d(v, w) = r + 1$, and so $w \in S_{r+1}(v)$.

Now, let $w \in S_{r+1}(v)$. Then there is a path w, w', \dots, v in G of length $r + 1$ from w to v , and for this w' we have $d(w, w') = 1$. This path also gives a path of length r from w' to v , so that $w' \in B_r(v)$. As $B_r(v)$ and $S_{r+1}(v)$ are disjoint, $d(w, B_r(v)) > 0$. Thus, $d(w, B_r(v)) = 1$, that is, $w \in \partial B_r(v)$. ■

Proposition 2.4.4. *Let $G = (V, E)$ be an (n, k, c) -expander. Then*

$$\text{diam}(G) \leq \frac{2}{\log(1+c)} \log n.$$

Proof. [[D], p.97]

Let $v_1, v_2 \in V$ be two arbitrary vertices of G . Because G is an expander it is connected. So, any vertex $w \in V$ is joined to v_1 by a path, which must have length at most $n - 1$ (by definition, a path consists of distinct vertices). So, $d(v_1, w) \leq n - 1$. Hence $B_{n-1}(v_1) = V$ and thus

$$\{v_1\} = B_0(v_1) \subseteq B_1(v_1) \subseteq B_2(v_1) \subseteq \dots \subseteq B_{n-1}(v_1) = V.$$

Since $|V| > \frac{n}{2}$ we can define r_1 to be the least positive integer r such that

$$|B_r(v_1)| > \frac{n}{2}.$$

Define r_2 similarly, with respect to v_2 . For any $r < r_1$ we have that $|B_r(v_1)| \leq \frac{n}{2}$, so by previous lemma and expansion of G we have that

$$|B_{r+1}(v_1)| = |B_r(v_1)| + |S_{r+1}(v_1)| = |B_r(v_1)| + |\partial B_r(v_1)| \geq (1+c)|B_r(v_1)|.$$

As $|B_0(v_1)| = 1$, a trivial induction gives

$$|B_{r_1}(v_1)| \geq (1+c)^{r_1}.$$

Since $|B_{r_1}(v_1)| \leq n$, we now have that

$$r_1 \leq \log_{1+c} n = \frac{\log n}{\log(1+c)}$$

and similarly the same upper bound holds for r_2 .

Because $|B_{r_1}(v_1)| + |B_{r_2}(v_2)| > n$, these two balls cannot be disjoint, that is, there exists w such that $d(v_1, w) \leq r_1$ and $d(v_2, w) \leq r_2$. Thus,

$$d(v_1, v_2) \leq d(v_1, w) + d(w, v_2) \leq \frac{2}{\log(1+c)} \log n.$$

As v_1 and v_2 were arbitrary, we conclude that

$$\text{diam}(G) \leq \frac{2}{\log(1+c)} \log n.$$

■

Corollary 2.4.5 (Expanders have logarithmic diameter, [AG], [S]). *Let $(G_i) = (V_i, E_i)$ be a family of expanders. Then,*

$$\text{diam}(G_i) = O(\log |V_i|).$$

Remark 2.4.6. *This corollary is a very simple example of the kind of results that follow from the definition of an expander family. In this way, it validates the definition of expander families. It is only because we have a fixed lower bound on the expansion constants of all the graphs G_i that we can make such a statement : the big -oh notation hides a constant which depends on the constant of expansion c .*

Remark 2.4.7. *The diameter of expanders is optimal in the sense that any family of graphs where the vertices have uniformly bounded degree must have at least logarithmic diameter. The proof of this is very similar to the proof of the previous proposition, noting that if k is an upper bound on degree then $|\partial A| \leq (k+1)|A|$ for all $A \subseteq V$.*

One might wonder if the converse to this corollary holds, that is, whether a family of graphs of logarithmic diameter is necessarily a family of expanders. The converse however is false, and it is not too difficult to find families of graphs with logarithmic diameter that are not expanders. Indeed, since examples is essentially the same problem as finding large regular graphs with logarithmic diameter.

A very natural way to construct such examples is to take symmetric trees of constant degree for internal vertices and join the leaves to fill them out to be regular.

As the simplest example, we can take balls of radius r inside the 3-regular tree, joining the leaf nodes in local cycles of length 4.

The number of vertices in the balls grows exponentially in r , whereas the diameters of the balls are $2r$, so that they have logarithmic diameter. On the other hand, if we take A in the combinatorial definition of expander, as one particular branch at the root, comprising roughly $1/3$ of the vertices but with boundary ∂A containing only the root, we see that these graphs will not form a family of expanders.

So, in summary, expansion is strictly stronger than logarithmic diameter.

This remark leads to the following result, which can be found in [CC].

Corollary 2.4.8. *Let G_i be an expander family of graphs. Then we have that*

$$\text{diam}(G_i) \ll \log(3|G_i|), \text{ for all } i,$$

where the implied constant depends only on the expansion parameters (h, u) of the family.

Note here that the examples of finite trees $T_{d,k}$ with $d \geq 3$ fixed, show that the converse to this statement is not true: $T_{d,k}$ for $k \geq 1$, form a sequence of graphs which have constant valency d , and diameter $2k \ll \log|T_{d,k}|$, but they are not expanders.

Proof. We remove from I those indices $i \in J$ such that $|G_i| \leq \frac{1}{3}e^3$, and then we just apply previous proposition.

So, denoting

$$u = \max_{i \in I} \max_{x \in G_i} \text{val}(x) < +\infty, h = \inf_{i \in I} h(G_i) > 0$$

and

$$\xi = \frac{1}{\log(1 + \frac{u}{h})} > 0.$$

And hence we get that,

$$\text{diam}(G_i) \geq 2\xi \log\left(\frac{1}{2}|G_i|\right) + 3 \leq 2\xi \log\left(\frac{1}{2}|G_i|\right) + \log(3|G_i|) \leq (2\xi + 1)\log(3|G_i|), \text{ for } i \notin J,$$

and we get an estimate valid for all i , e.g., by writing $\text{diam}(G_i) \leq C \log(3|G_i|)$

with $C = \max(2\xi + 1, \max_{j \in J} \text{diam}(G_j)), \forall i \in I$. ■

Although there infinite sequences of graphs with diameter growing asymptotically than the algorithm of the number of vertices, as we have seen with complete graphs, this result is in fact the best possible under the sparseness condition, as far as the order of magnitude is concerned :

Lemma 2.4.9. *Let G be any graph with maximal valency $\leq k$, where $k \geq 0$ is an integer. Then*

$$\text{diam } G \geq \frac{\log(|G|)}{\log k}.$$

Proof. The idea is simply that, from distance n to $n + 1$ of a fixed vertex x , we can at most multiply the number of vertices by the maximal valency, so the size of balls x can only grow exponentially fast. Precisely, let $x \in G$ be such a fixed vertex. We claim that

$$|B_x(n)| \leq k^n, \text{ for } n \geq 0$$

Indeed, from each $y \in B_x(n - 1)$, we reach at most k new vertices at distance n , so that

$$|B_x(n)| \leq k|B_x(n - 1)|,$$

which gives the bound above by induction. Now, if $d = \text{diam}(G)$, we have $B_x(d) = G$ by definition, and hence

$$|G| \leq k^d$$

as desired. ■

Thus, we see that, if they exist, expander families are essentially optimal graphs when it comes to combining sparsity and strong connectedness (or expansion) properties.

Another useful property of expanders arise from previous lemma, which is the following.

Proposition 2.4.10 (Growth of metric balls). *Let (G_i) be an expander family. Then the metric balls in (G_i) are uniformly exponentially expanding, in the sense that there exists $\gamma > 1$, independent of i , such that for any graph G_i , we have*

$$|B_x(n)| \geq \min\left(\frac{|G|}{2}, G^n\right), \text{ for all } x \in G_i \text{ and } n \geq 0.$$

In fact, one can take $\gamma = 1 + \frac{h}{u}$, where (h, u) are expansion parameters of the family.

Lemma 2.4.11. *Let $G = (V, E, ep)$ be a non-empty finite graph with maximal valency u and let $W \subset V$ be any subset. We have that*

$$\frac{1}{u}|\epsilon(W)| \leq |\partial W| \leq |\epsilon(W)|.$$

In particular, a family of (G_i) is an expander family if and only if it has bounded valency and there exist $\tilde{h} > 0$ such that

$$\tilde{h}(G_i) \geq \tilde{h}, \text{ for all } i \in I.$$

Proof. Consider the map $\epsilon(W) \rightarrow \partial W$, that is $\alpha \mapsto \text{ep}(\alpha) \cap (V - W)$, which sends an edge in $\epsilon(W)$ to the one among its extremities which is not in W . By definition, this map is surjective, which gives the second inequality of the lemma, and there are at most u edges which map to any given $x \in \partial W$, which means that $|\epsilon(W)| \leq u|\partial W|$. ■

The expander property can be thought of as relatively quantitative, and in particular, it is fairly robust to certain changes of the structure (especially of the edges) of the graphs. Here is a fairly convenient lemma in this direction without its proof :

Lemma 2.4.12 (Comparison of expansion ratios, [CC]). *Let $\Gamma_1 = (V_1, E_1, ep)$ and $\Gamma_2 = (V_2, E_2, ep)$ be non-empty finite graphs with distances d_1 and d_2 respectively, and maximal valencies bounded by u_1 and u_2 respectively, and let $f : V_1 \rightarrow V_2$ be a surjective map such that :*

1. $\forall y \in V_2$, the set $f^{-1}(y)$ has the same cardinality $d \geq 1$, in particular $|V_1| = d|V_2|$.
2. There exists $C > 0$ for which

$$d_2(f(x), f(y)) \leq C d_1(x, y), \forall x, y \in V_1.$$

Then, we have that

$$h(\Gamma_2) \geq \frac{h(\Gamma_1)}{w},$$

where $w > 0$ depends only on (C, u_1, u_2) , namely $w = u_1 \sum_{j=1}^{|C|} u_2^{j-1}$. We emphasize that f is not assumed to be a graph map, so that the condition $d_2(f(x), f(y)) \leq d_1(x, y)$ is not automatic (as it is for graph maps).

Chapter 3

Existence and Construction of Expanders

3.1 Applications to Expanders

Proposition 3.1.1. *Let (G_i) be a family of expander graphs, $G_i = (V_i, E_i, ep)$ with maximal valency bounded by u .*

1. *If G'_i is any graph with the same vertices as G_i and with ‘more edges’, i.e. , $E_{G'_i} \supset E_{G_i}$, then (G'_i) is also an expander graph provided the maximal valency of G'_i remains bounded by u .*
2. *The family of simple graphs (G_i^s) is a family of expanders with bounded valency.*
3. *More generally, assume that for any $i \in I$, we are given graphs $G'_i = (V'_i, E'_i, ep)$ with maximal valency bounded by w , and bijections $V_i \xrightarrow{f_i} V'_i$, such that*

$$d_{G'_i}(f_i(x), f_i(y)) \leq C d_{G_i}(x, y) \quad (\star)$$

for some fixed constant $C \geq 0$.

Then the family $(G'_i)_{i \in I}$ is also an expanding family. More precisely, it satisfies

$$\inf_{i \in I} h(G'_i) \leq \delta \inf_{i \in I} h(G_i)$$

where $\delta = \frac{1}{u \sum_{j=1}^{|C|} w^{j-1}}$, i.e., if (h, u) are expansion parameters for (G_i) , then $(\delta h, w)$ are expansion parameters for (G'_i) .

The example to keep in mind for the third part of the previous proposition, is when $V = V'$ and f_i is simply the identity. This means that G'_i is a graph with the same vertices, but with edges “rearranged” in some way, and the condition (\star) says that the distance between vertices in the new graphs (using the modified edges) is distorted at most by a constant factor from the distance in the original ones. This will be very useful for Cayley graphs.

Proof. In each case, we only need to apply lemma 2.4.13 to compare each Γ_i with a graph Γ'_i , which has in each case the same set of vertices, i.e., the graphs (Γ_1, Γ_2) according to the lemma 2.4.13 are (Γ_i, Γ'_i) , and with f being the identity (so that 1 of the lemma is automatic). In (1), because we added edges, we have that

$$d_{\Gamma'_i}(x, y) \leq d_{\Gamma_i}(x, y),$$

so we can take $C = 1$ in condition (2) of the lemma.

In (2), with $\Gamma'_i = \Gamma_i^s$, although we may have removed some edges, we have not changed the distance, so condition (2) holds again with $C = 1$.

Finally, in (3), condition (2) is precisely given by (\star) . ■

Remark 3.1.2. *Some of the previous results can be interpreted in two ways :*

- *First, as a rough qualitative expression of properties of expanders (logarithmic growth of diameter, exponential growth of balls, stability under “localized” changes of edge sets),*
- *but also as quantitative expressions of these properties, since in each case one can write down precise inequalities in terms of the expansion parameters of the family.*

As is often the case, the actual value of the constants appearing in such inequalities should not be considered as particularly significant for a first understanding of the intuitive meaning. Nevertheless, it is very important for certain applications that it is indeed possible to control these constants explicitly.

At this point, the most significant question is, *do expanders really exist?*

In all the easy examples of graphs (with bounded valency) for which we computed the expansion constant, it tends to 0 as the number of vertices goes to infinity, even in the case of finite trees where the diameter, at least, has the right order of magnitude. This is a bad sign. However, we might observe that, in the case of the best candidates so far (the trees $T_{d,k}$ with $d \geq 3$ fixed and $k \rightarrow \infty$), there are many subsets which do have large expansion ratio $|\epsilon(W)|/|W|$.

Roughly speaking, as long as W is a set of vertices that only contains a few elements at the maximal distance k from the root of the tree, there will be many edges “escaping” further away from the root, in fact typically as many as the size of W . In other words, one might imagine that adding some edges to the far vertices, reconnecting them to the middle of the tree, might have a chance of producing graphs with good expansion constant. We will not proceed this way, but indeed, expanders do exist, and in fact exist in cheerful abundance.

3.2 Existence of Expanders

We will briefly establish in this section, using probabilistic arguments, the existence of expander families. Pinsker first showed the existence of expander graphs in [E]. In the style of Erdos who used probabilistic methods to establish the existence of various combinatorial objects- Pinsker considered a model of random regular graphs, and showed that the probability that such a random graph is an expander is non-zero for sufficiently large n . In fact, the probability tends to 1. In this section, we follow the example of Pinsker and develop a model of random regular graphs, and use it to prove the existence of expanders. This line of thinking raises a natural but difficult question :

How should one model a random regular graph?

Perhaps a nice way to do this would be to consider each isomorphism class of k -regular graphs on n vertices to be equiprobable. However, there is not even a one known formula for the number of such isomorphism classes in general. So, we model a k -regular bipartite graphs on $2n$ vertices as follows.

The vertices are labelled $I = \{v_1, \dots, v_n\}$ and $O = \{w_1, \dots, w_n\}$. We then take k permutations $\pi_1, \dots, \pi_k \in S_n$, drawn uniformly (each permutation is chosen with probability $\frac{1}{n!}$) and independently. Then for each $1 \leq i \leq n$ and $1 \leq j \leq k$ we create an edge joining v_i and $w_{\pi_j(i)}$. Note that the random graphs generated are very likely to be multigraphs : even for $n = 2$ we are just asking about the probability that a random permutation is a derangement, which is approximately $\frac{1}{e}$.

The following theorem is adapted from [[F], Proposition 1.2.1]

Theorem 3.2.1 (Existence of expander families). *Let $k \geq 6$ be a positive integer and $c = \frac{1}{2}$. Then the probability that a random k -regular multigraph on n vertices, drawn from the model described above, is a c -expander tends to 1 as $n \rightarrow \infty$. In particular, families of c -expanders exist.*

The complete proof can be found in [[C], pp.478-481]. We will only sketch the proof here.

Proof. Consider sets $A \subseteq I$ with $|A| = t \leq \frac{n}{2}$ and $B \subseteq O$ with $|B| = m = \lfloor \frac{3}{2}t \rfloor$.

Let $P(t)$ be the probability that for a random k -regular bipartite expander, $\partial A \subseteq B$.

We compute that

$$P(t) = \left(\frac{m!(n-t)!}{(m-t)!n!} \right)^k.$$

Let $Q(t)$ be the number of choices of such A and B . Then

$$Q(t) = \binom{n}{t} \binom{n}{m}.$$

Put $R(t) = Q(t)P(t)$.

Then a very crude upper bound on the probability that a random k -regular bipartite graph on n vertices is *not* a $c = \frac{3}{2}$ expander is

$$P_n = \sum_{1 \leq t \leq \frac{n}{2}} R(t).$$

It now remains to show that $P_n \rightarrow 0$ as $n \rightarrow \infty$. For small values of t , the probability $P(t)$ is large whereas $Q(t)$ is small. The opposite is true for values more of the order of $\frac{n}{3} \leq t \leq \frac{n}{2}$. One can verify that $R(t)$ is roughly decreasing for small values, so the maximum occurs at $R(1)$.

For the larges values, we compare $R(t)$ to $R(\frac{n}{2})$ which we can approximate to within a constant factor by Stirling's formula. Then for all t we have that

$$R(t) \leq R(1) + R\left(\frac{n}{3}\right) + R\left(\frac{n}{2}\right) = o\left(\frac{1}{n}\right)$$

so that $P_n \rightarrow 0$. ■

These results lead us to the next section, where we use them to analyse explicitness.

3.3 Two levels of Explicitness

Towards discussing various notions of explicit constructions of graphs, we need to fix a representation of such graphs. Specifically, throughout this section, when referring to an infinite family of graphs $(G_N)_{N \in \mathbb{S}}$, we shall assume that the vertex set of G_N equals $[N]$.

Indeed, at times, we shall consider vertex sets having a different structure (for example, $[n] \times [n]$ for some $n \in \mathbb{N}$), but in all these cases there exists a simple isomorphism of these sets to the canonical representation (i.e., there exists an efficiently computable and invertible mapping of the vertex set of G_N to $[N]$).

Recall that a mild notion of explicit constructiveness refers to the *complexity of constructing the entire object* (i.e., the graph).

Applying this notion to our setting, we say that an infinite family of graphs $(G_N)_{N \in \mathbb{S}}$ is explicitly constructible if there exists a polynomial-time algorithm that, on input 1^N , where $N \in \mathbb{S}$ and $1^N = 11 \cdots 1$ (N times), outputs the list of the edges in the N -vertex graph G_N . That is, the entire graph is constructed in time that is polynomial in its size (i.e., in poly(N)-time).

The foregoing (mild) level of explicitness suffices when the application requires holding the entire graph and/or when the running-time of the application is lower-bounded by the size of the graph. In contrast, other applications refer to a huge virtual graph (which is much bigger than their running time), and only require the computation of the neighborhood relation in such a graph. In this case, the following stronger level of explicitness is relevant.

A strongly explicit construction of an infinite family of (d -regular) graphs $(G_N)_{N \in \mathbb{S}}$ is a polynomial-time algorithm that on input $N \in \mathbb{S}$ (in binary), a vertex v in the N -vertex graph G_N (i.e., $v \in [N]$), and an index $i \in [d]$, returns the i^{th} neighbor of v . That is, the neighbor query is answered in time that is polylogarithmic in the size of the graph. Needless to say, this strong level of explicitness implies the basic (mild) level.

An additional requirement, which is often forgotten but is very important, refers to the tractability of the set \mathbb{S} . Specifically, we require the existence of an efficient algorithm that given any $n \in \mathbb{N}$ finds an $s \in \mathbb{S}$ such that $n \leq s < 2n$.

Corresponding to the two foregoing levels of explicitness, efficient may mean either running in time poly(n) or running in time poly($\log n$). The requirement that $n \leq s < 2n$ suffices in most applications, but in some cases a smaller interval (e.g., $n \leq s < n + \sqrt{n}$) is required, whereas in other cases a larger interval (e.g., $n \leq s < \text{poly}(n)$) suffices.

Greater flexibility. In continuation to the foregoing paragraph, we comment that expanders can be combined in order to obtain expanders for a wider range of graph sizes.

For example, given two d -regular c -expanding graphs, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ where $|V_1| \leq |V_2|$ and $c \leq 1$, we can obtain a $(d+1)$ -regular $c/2$ -expanding graph on $|V_1| + |V_2|$ vertices by connecting the two graphs using a perfect matching of V_1 and $|V_1|$ of the vertices of V_2 (and adding self-loops to the remaining vertices of V_2). More generally, combining the d -regular c -expanding graphs $G_1 = (V_1, E_1)$ through $G_t = (V_t, E_t)$, where $N' = \sum_{i=1}^t |V_i| \leq |V_t|$,

yields a $(d+1)$ -regular $c/2$ -expanding graph on $\sum_{i=1}^{t-1} |V_i|$ vertices (by using a perfect matching of $\cup_{i=1}^{t-1} V_i$ and N' of the vertices of V_t).

3.4 Construction of Expander Families

Expander graphs solve the most basic problem in the design of networks, designing a robust network to connect a large number of disjoint sets of users. Expander graphs strike the ideal balance between the number of connections between nodes and the reliability of the network as

the number of nodes grows.

Now, in Pure Mathematics various explicit constructions of (d, λ) - expanders are known; The material is found in [CC], [EA], [AA], [Q].

3.4.1 The Margulis - Gabber - Galil Expander

For every natural number m , consider the graph with vertex set $\mathbb{Z}_m \times \mathbb{Z}_m$ and the edge set in which every $\langle x, y \rangle \in \mathbb{Z}_m \times \mathbb{Z}_m$ is connected to the vertices $\langle x \pm y, y \rangle, \langle x \pm (y+1), y \rangle, \langle x, y \pm x \rangle$ and $\langle x, y \pm (x+1) \rangle$, where the arithmetic is modulo m . This yields an extremely simple 8-regular graph with an eigenvalue bound that is a constant $\lambda < 8$ (which is independent of m).

Margulis's Construction

The following graph is a beautiful algebraic construction of Marguli's which was the earliest explicit expander known.

Let $V = \mathbb{Z}_n \times \mathbb{Z}_n$ and define an 8-regular graph on V as follows.

Let $T_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, T_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$.

Also, let $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Each vertex $v \in \mathbb{Z}_n \times \mathbb{Z}_n$ is adjacent to $T_1V, T_2V, T_1v + e_1, T_2v + e_2$ and four other vertices obtained by the inverse transformations: $T_1^{-1}v, T_2^{-1}v, T_1^{-1}(v - e_1), T_2^{-1}(v - e_2)$.

This is actually a multigraph with possible multiedges and loops. We will deepen our information of Marguli's construction in the last chapter.

By the above, we get the following theorem. We will not mention the proof. It can be found in [DA].

Theorem 3.4.1 (Gaber - Galil). *There exists a strongly explicit construction of a family of $(8, 7.9999)$ -expanders for a graph sizes $\{m^2 : m \in \mathbb{N}\}$. Furthermore, the neighbors of a vertex in these expanders can be computed in logarithmic-space.*

In fact, for m that is a power of two, the neighbors can be computed by an algorithm that uses a constant amount of space. An appealing property of this theorem is that, for every $n \in \mathbb{N}$, it directly yields expanders with vertex set $\{0, 1\}^n$. This is obvious in case n is even, but can be easily achieved also for odd n . For example, use two copies of the graph for $n - 1$ and connect the two copies by the obvious perfect matching.

Definition 3.4.2. *Let p be a prime and let $V = \mathbb{Z}_p$. We define a 3-regular graph $G = (V, E)$ where the edges are of two types: $(x, x+1)$ and (x, x^{-1}) , for all $x \in \mathbb{Z}_p$. Assume that $0^{-1} = 0$ for this purpose.*

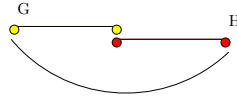
It is known that this is a $(3, \epsilon)$ -expander for some $\epsilon > 0$ fixed and p prime. The proof of this relies on deep results in number theory.

Remark 3.4.3. *These graphs are not Ramanujan graphs. For instance, their second eigenvalue is not on the order of \sqrt{d} . However, even such graphs can be constructed explicitly.*

The proof of Gaber - Galil theorem uses Harmonic analysis on $\mathbb{Z}_n \times \mathbb{Z}_n$.

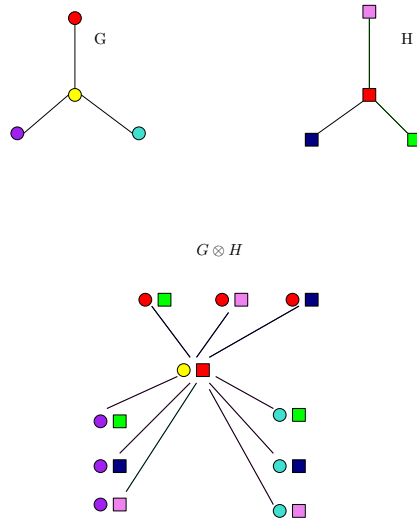
3.4.2 Construction of Constant Degree Products

The product of graphs lead to new graphs with different expansion properties. Let G and H be graphs on n vertices, whose vertices are identified.



This corresponds to multiplication of the adjacency matrices and we have that $\lambda_{G^2} = \lambda_G^2$. However, the number of vertices is the same, and the degree is squared.

That is :



Tensor Product

This corresponds to the typical Kronecker product of matrices. So, we have that $\lambda_{G \otimes H} = \max(\lambda_G, \lambda_H)$. The degree is the multiplication of degrees of G and H , and the number of vertices is the product of the number of vertices of G and H .

Graph Squaring

The square $G^2 = (V, E')$ is a graph that has the same set of vertices and $(u, w) \in E'$ if and only if there is a path of length 2 in G from u to w .

This leads to the Zig-Zag product.

The Zig-Zag Construction

The idea :

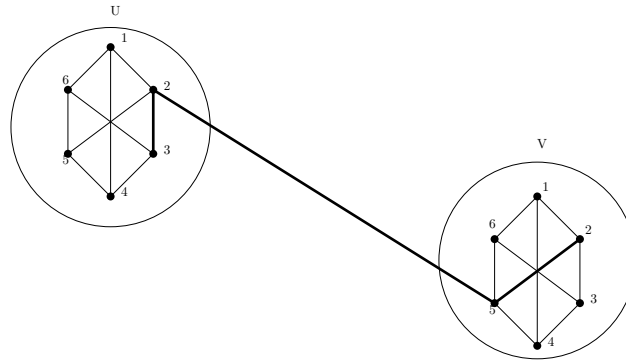
Let G be a d -regular graph on n vertices. Let H be a r -regular graph on d vertices.

Then the Zig-Zag product of G and H , denoted as $G \times_Z H$ is an r^2 -regular graph on nd vertices.

The edges around a vertex of G are identified with the vertices of H .

Now, let us describe this construction formally in details :

The starting point of the following construction is a very good expander G of constant size, which may be found by an exhaustive search. The construction of a large expander graph proceeds in iterations, where in the i^{th} iteration the current graph G_i and the fixed graph G are combined, resulting in a larger graph G_{i+1} . The combination step guarantees that the expansion property of G_{i+1} is at least as good as the expansion of G_i , while G_{i+1} maintains the degree of G_i and is a constant times larger than G_i . The process is initiated with $G_1 = G^2$ and terminates when we obtain a graph G_t of approximately the desired size (which requires a logarithmic number of iterations).



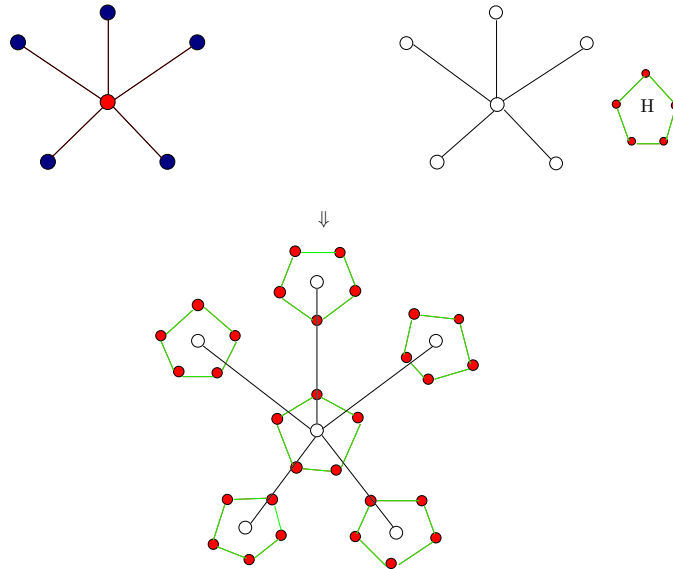
In this example, H is a 6-regular and G is a 3-regular having 6 vertices. In the graph H (not shown), the second edge of vertex u is incident at v , as its fifth edge. The wide 3-segment line shows one of the corresponding edges of $H \times_Z G$, which connects the vertices $\langle u, 3 \rangle$ and $\langle v, 2 \rangle$.

The Zig-Zag product. The heart of the combination step is a new type of graph product called Zig-Zag product. This operation is applicable to any pair of graphs $G = ([D], E)$ and $H = ([N], E')$, provided that H (which is typically larger than G) is D -regular.

For simplicity, we assume that G is d -regular (where typically $d \ll D$). The Zig-Zag product of H and G , denoted $H \times_Z G$, is defined as a graph with vertex set $[N] \times [D]$ and an edge set that includes an edge between $\langle u, i \rangle \in [N] \times [D]$ and $\langle v, j \rangle$ if and only if $\{i, k\}, \{l, j\} \in E$ and the k^{th} edge incident at u equals the l^{th} edge incident at v .

That is, $\langle u, i \rangle$ and $\langle v, j \rangle$ are connected in $H \times_Z G$ if there exists a three step sequence consisting of a G -step from $\langle u, i \rangle$ to $\langle u, k \rangle$ (according to the edge $\{i, k\}$ of G), followed by a H -step from $\langle u, k \rangle$ to $\langle v, l \rangle$ (according to the k^{th} edge of u in H (which is the l^{th} edge of v)), and a final G -step from $\langle v, l \rangle$ to $\langle v, j \rangle$ (according to the edge $\{l, j\}$ of G). We will not present further formalization but we will try to describe this construction through the following figures.

We begin with a neighborhood of a vertex G .



A very important proposition follows.

Proposition 3.4.4.

$$\lambda_{G \times_Z H} \leq \lambda_G + \lambda_H + \lambda_H^2.$$

Proof. [FA] ■

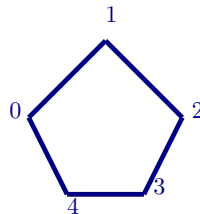
From this result, we can construct constant degree expanders.

3.5 Cayley Graphs

Cayley graphs give a means to construct graphs from groups. The construction of expanders due to Margulis is a family of Cayley graphs. There are many reasons why it is natural to use Cayley graphs to construct expanders. As well as being regular graphs by definition, they enable us to construct large graphs in an effective and concise manner; it can be much easier to describe the group than the graph. Before describing some previously-encountered graphs as Cayley graphs, we give the definition and some notes.

Definition 3.5.1. *The Cayley graph of a group G with respect to a generating set $S \subseteq G$ is the directed graph whose vertex set is G and whose edges are given by (g, gs) for each $g \in G$ and $s \in S$. It is denoted by $\text{Cay}(G, S)$*

For example, the following graph is a $\text{Cay}(\mathbb{Z}_5, \{-1, 1\})$:



Remark 3.5.2. *The Cayley graph will be simple, that is, loop free if and only if $1 \notin S$.*

Since for any distinct $s, s' \in S$ we have $gs \neq gs'$, it follows that the edges of $\text{Cay}(G, S)$ are distinct, so it is not a multigraph. Because S generates G , every $g \in G$ can be written as a word $s_1 s_2 \cdots s_l$ in the generators

$$1_G, s_1, s_1 s_2, s_1 s_2 s_3, \cdots, s_1 s_2 \cdots s_l = g.$$

Thus, $\text{Cay}(G, S)$ is connected.

Finally, we say that $S \subseteq G$ is *symmetric* if

$$S = S^{-1} := \{s^{-1} | s \in S\}.$$

If S is symmetric, then we can identify $\text{Cay}(G, S)$ with an undirected graph. This is because to each directed edge (g, gs) there corresponds a reversed edge

$$(gs, (gs)s^{-1}) = (gs, g).$$

This is still true if $s^{-1} = s$.

This undirected graph is $|S|$ -regular. Usually, we will have S a symmetric generating set which does not include the identity, so that the graph $\text{Cay}(G, S)$ is a simple, connected, undirected graph.

Remark 3.5.3. *As expanders are finite graphs, we are only interested in the Cayley graphs of finite groups. Note that the number of vertices in $\text{Cay}(G, S)$ is $|G|$. These finite groups might, however, be obtained as subgroups or quotients of infinite groups, so it is certainly not the case that we will consider only finite groups.*

At this point, we need to mention that it is more common to consider left group actions. The reason we define the Cayley graph by right-multiplication of the vertices $g \in G$ by the generators $s \in S$ is that the action of G on itself by left-multiplication induces a graph isomorphism, that is, the element $h \in G$ maps any edge (g, gs) to the edge (hg, hgs) .

Example 3.5.4. *The families of graphs from the example of graphs in the first chapter, can be constructed as Cayley graphs as follows.*

1. *The Cayley graph for the group $\mathbb{Z}/n\mathbb{Z}$ with respect to the generating set $S = \{1, -1\}$ is the cycle graph C_n .*
2. *The group $\mathbb{Z}/n\mathbb{Z}$ taken with the generating set $S = \{1, 2, \cdots, n-1\}$ consisting of all non-zero elements has the Cayley graph K_n .*
3. *The Cayley graph for the directed product of n groups of order 2,*

$$G = \bigoplus_{i=1}^n \mathbb{Z}/2\mathbb{Z},$$

with the n generators

$$S = \{(1, 0, \cdots, 0), \cdots, (0, \cdots, 1)\}$$

is the n^{th} hypercube graph Q_n .

As we have mentioned, none of these is an expander.

A very important theorem about the spectrum of the Cayley graphs is the following, which is found in [GA].

Theorem 3.5.5 (Diaconis-Shahshahani). *The eigenvalues of $\text{Cay}(G, S)$ are*

$$\lambda_{\mathcal{X}} = \frac{1}{|S|\mathcal{X}(1)} \sum_{s \in S} \mathcal{X}(s)$$

of multiplicity $\mathcal{X}(1)^2$, where \mathcal{X} runs over all the irreducible characters of G .

We will now turn to some negative results for expansion of Cayley graphs.

3.6 Some Negative Results for Cayley Expansion

As expanders resemble random graphs, we might expect that groups which are very uncomplicated would not give expanders. In this section, we will show that there exist results which establish the veracity of this intuition to some extent, first in the case that our notion of ‘uncomplicated’ is ‘abelian’.

Theorem 3.6.1 (Abelian groups do not give expanders). *Let (G_i) be a family of abelian groups with respective generating sets S_i of constant cardinality k . Then the graphs $\text{Cay}(G_i, S_i)$ are not a family of expanders.*

Proof. Without loss of generality, let S_i be symmetric, because we can take $S_i \cup S_i^{-1}$ otherwise. We could possibly take some generators more than once so as to get a $2k$ -regular Cayley graphs.

Let, $S = \{s_1, \dots, s_k\}$. Then the balls in the graph can be written as

$$B_m(0) = \left\{ \alpha_1 s_1 + \dots + \alpha_k s_k \mid \alpha_i \geq 0, \sum_{i=1}^k \alpha_i \leq m \right\}.$$

So, the size of $B_m(0)$ is bounded by the number of ways of writing an integer $l \leq m$ as the sum of non-negative integers α_1, α_k , summed over all possible l . For each fixed l , this is simply making an unordered selection of l objects of k types, with repetition allowed. The fact that this selection is unordered is very important, and is where the fact that the group is abelian enters. It is well-known that this is

$$\binom{l+k-1}{k-1}.$$

We can derive this result by noting that such selections of l objects of k types are in bijection with sequences of $l+k-1$ symbols, $k-1$ of which are $-$'s and which delimit the blocks comprising some of the l 's, the blocks encoding how many of that particular item appears in the selection.

For example, the selection $(1, 2, 0, 1)$ is represented by the sequence $\circ - \circ \circ - - \circ$. We have that

$$\binom{l+k-1}{k-1} \leq \frac{(l+k-1)^{k-1}}{(k-1)!} \leq \frac{2^k}{(k-1)!} l^{k-1}$$

for $l \geq k-1$. Hence, we then have that

$$|B_m(0)| \leq \sum_{l=0}^m \binom{l+k-1}{k-1} \leq \sum_{l=1}^{k-1} \binom{l+k-1}{k-1} + \sum_{l=k}^m \frac{2^k}{(k-1)!} l^{k-1} \leq \frac{2^k}{(k-1)!} m^k + C$$

for a constant C .

Thus, $|B_m(0)| = O(m^k)$, where the implied constant depends only on k , and not on the particular abelian groups whose Cayley graph we are considering.

However, the balls in an expander must grow exponentially until they comprise at least half vertices, which we will only be an problem for finitely many of the Cayley graphs, as we have seen in previous proof of proposition. Thus $\text{Cay}(G_i, S_i)$ is not a family of expanders. ■

We can generalize the previous theorem to solvable groups, which are one class of groups which can be understood as being ‘approximately abelian’.

Definition 3.6.2. *A group G is solvable is its derived series $G^{(n)}$, defined recursively by $G^{(1)} = G$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$, terminates at the trivial group after finitely many steps. That is, there exists a minimal positive integer l called the derived length of G such that $G^{(l)} = \{1\}$.*

Theorem 3.6.3. *Let (G_i) be a sequence of finite groups with respective generating sets S_i of constant cardinality k . Let l be a positive integer. Suppose that for all n , we have that G_i is solvable with derived length at most l . Then the graphs $\text{Cay}(G_i, S_i)$ are not a family of expanders.*

We will not prove this theorem, but details can be found in Krebs and Shaheen [[D], Theorem 4.47] and [[G], Corollary 3.3].

Example 3.6.4. *Finite dihedral groups do not give expanders, as they have derived series of length 2.*

Remark 3.6.5. *The last theorem is essentially the only result known about when it is impossible to choose generating sets to construct a family of expanders as the Cayley graphs of a given family of groups $[H]$. We will see that in contrast the Margulis construction does work for any fixed set of generators of the group with property (T) that is used.*

The preceding remark on generators makes the question whether expansion is a group property. Lubotzky and Weiss presented the problem as follows. The material can be found in [[G], p.536].

Problem 3.6.6. *Let (G_i) be a family of finite groups, with $\langle S_i \rangle = \langle S'_i \rangle = G_i$ and $|S_i|, |S'_i| \leq k$ for all i . Does the fact that $(\text{Cay}(G_i, S_i))$ is an expander family imply the same for $(\text{Cay}(G_i, S'_i))$?*

This question was answered in the negative by Alon, Lubotzky and Wigderson in [J]. Their counterexample is beyond the purpose of this thesis. As noted in [[C], p.536], the more recent paper [I] provides a simpler counterexample. In this paper Kassabov answered what had been a big open problem for decades, by demonstrating that for certain generating sets, the alternating groups A_n and symmetric groups Sym_n give expanders. However, one can show that for instance, the generating sets $S_n = \{(12), (12 \cdots n) \pm 1\}$ do not make $\text{Cay}(\text{Sym}_n, S_n)$ expanders.

Remark 3.6.7. *Expander families are defined by some texts to have uniformly bounded, rather than constant, degree. For instance, the original presentation of previous problem by Lubotzky and Weiss was for uniformly bounded degree. However, this does not change the problem essentially.*

In one direction, graphs with constant degree trivially have uniformly bounded degree. In the other direction, if a family of expanders has uniformly bounded degree, we can add edges to obtain constant degree graphs (or, perhaps necessarily, constant degree multigraphs), which only possibly improves their expansion properties.

As a technicality, if the bound on degree is k , we can generally just add an edge between vertices of degree less than k until each graph is k -regular, possibly introducing loops. However, when nk is odd we will have to go to a $(k+1)$ -regular graph (the handshaking lemma requires that the sum of degrees is even, being twice the number of edges). Since we are mostly concerned with Cayley graphs, we will restrict attention to regular graphs.

Chapter 4

Ramanujan Graphs

In general, a Ramanujan graph is a regular graph whose spectral gap is almost as large as possible. Such graphs are excellent expanders. Formally, let G be a connected d -regular graph with n vertices, and let $\lambda_1 \geq \dots \geq \lambda_{n-1}$ be the eigenvalues of the adjacency matrix of G . Recall that because G is connected and d -regular, its eigenvalues satisfy that $d = \lambda_1 > \lambda_2 \geq \dots \geq \lambda_{n-1} \geq -d$. Whenever there exists λ_i with $|\lambda_i| < d$, we define $\lambda(G) = \max_{|\lambda_i| < d} |\lambda_i|$. A d -regular graph is then a Ramanujan graph if

$$\lambda(G) \leq 2\sqrt{d-1}.$$

In other words :

Definition 4.0.8. *A graph is said to be a Ramanujan graph if $\hat{\mu}(G) \leq 2\sqrt{d-1}$.*

Recall that $\hat{\mu}(G) = \max\{|\mu_1|, |\mu_{n-1}|\}$, unless we want to allow bipartite Ramanujan graphs, in which case we have $\hat{\mu}(G) = \max_{|\mu_i| \neq d} |\mu_i|$.

This definition means that Ramanujan graphs are good expanders. We will see that this is the strictest definition we can have, while we still allowing for infinite families of such graphs to exist.

4.1 Motivation of the Definition

We will state the following important theorem which was the motivation of the definition of Ramanujan graphs. However, we will not represent the proof in this thesis as it is large enough and does not be considered as the main result of this section. Although it can be found in many texts such as [AE].

Theorem 4.1.1. *For any infinite family of d -regular graphs on n nodes, denoted as $\{G_n\}$, we have that $\hat{\mu}(G_n) \geq 2\sqrt{d-1} - o(1)$, where the last term is some number that goes to zero as n gets large.*

Given the importance of the second eigenvalue λ_2 , we should know how close it can be to d . Nilli in [AE] shows that it cannot be much closer than $2\sqrt{d-1}$. This leads to the following result.

Theorem 4.1.2. *Let G be a d -regular graph containing two edges (u_0, u_1) and (v_0, v_1) whose vertices are at distance at least $2k+2$ from each other. Then*

$$\lambda_2 \leq d - 2\sqrt{d-1} + \frac{2\sqrt{d-1} - 1}{k+1}.$$

4.2 Existence of Bipartite Ramanujan Graphs

In this section, we give a briefly overview of the arguments, where it is proven that there exist infinite families of bipartite Ramanujan graphs of all degrees at least 3.

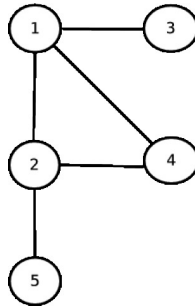
4.2.1 2-Lifts

Definition 4.2.1. A 2-lift is a process that acts on a graph, turning it into a graph with twice as many vertices. The process is as follows.

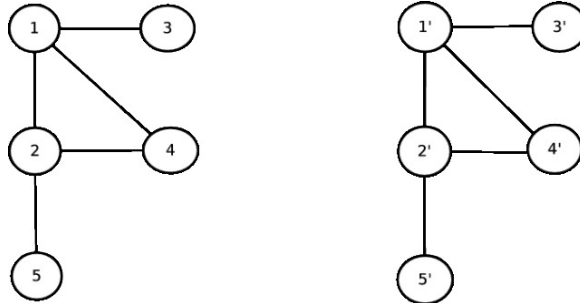
Take a graph, and draw a duplicate graph next to it. Then, every original edge $\{u, v\}$ now has two analogs : $\{u_0, v_0\}$ and $\{u_1, v_1\}$. Then, select some subset of the edge pairs $\{u_0, v_0\}$ and $\{u_1, v_1\}$ and transform the edges to : $\{u_0, v_1\}$ and $\{u_1, v_0\}$.

The process can be seen in the following figures.

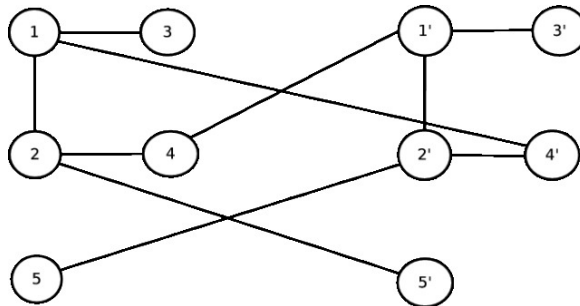
Step 1 - Original graph



Step 2 - after nodeduplication



Step 3 - cross some of the edges



Let us now call the original graph G , and the resulting graph after completing the 2-lift G' . We also say that G has an $n \times n$ matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

However, rather than considering A' , the $2n \times 2n$ adjacency matrix A .

Recall that in constructing the 2-lift, some of the original edges in G ended up being crossed with the analogous edges among new vertices.

Definition 4.2.2 (signed adjacency matrix). *We create the A_s matrix, which is the signed adjacency matrix for G' , by first copying the matrix A . Then if the edge $\{u, v\}$ was one of the crossed edges in the 2-lift, we replace the 1 at $(A_s)_{uv}$ and $(A_s)_{vu}$ with a -1 .*

In this small example, we have

$$A_s = \begin{bmatrix} 0 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}.$$

Notice that the following results in this section is for regular graphs, although this example is not regular.

Certainly, we can not immediately understand why it is useful to create such a matrix. However it is very important because its eigenvalues are closely related to the eigenvalues of G' , which are the A' eigenvalues.

The following theorem is very important in this section. However, it will be stated without its proof. It can be found in [R].

Theorem 4.2.3. *The eigenvalues of G' are the union of the eigenvalues of A and the eigenvalues of A_s (counting multiplicities)*

Chapter 5

Random Walks and Spectral Expansion

5.1 Random Walks

In this section we will show that expanders can be characterized by the property that a random walk on their vertices converges to the limit distribution quickly. This allows efficient pseudorandom sampling, which gives a means to reduce the probability of error for a randomized algorithm just as rapidly as repeated random sampling, but with very little additional use of the random resource.

We begin by defining the random graphs and analyze random walks on graphs, which is a very useful mathematical concept, as on a graph, a random walk moves between adjacent vertices at random.

5.1.1 Random Graphs

Random graphs is the general term to refer to probability distributions over graphs. Random graphs may be described simply by a probability distribution, or by a random process which generates them.

Random graphs are good expanders. It is easier to analyze bipartite expanders which we recall below.

Definition 5.1.1. *A bipartite graph G on $n+n$ vertices $L \sqcup R$ is called a (d, β) -expander, if the degrees in L are d and any set of vertices $S \subset L$ of size $|S| \leq \frac{n}{d}$ has at least $\beta|S|$ neighbors in R .*

Theorem 5.1.2. *Let $d \geq 4$ be a random bipartite graph obtained by choosing d random edges for each vertex in L . Then G is a $(d, d/4)$ -expander with constant positive probability.*

Proof. For each $S \subseteq L$ and $T \subseteq R$, let $E_{S,T}$ denote the event that all neighbors of S are in T . The probability of this is

$$P_r[E_{S,T}] = \left(\frac{|T|}{n}\right)^{d|S|}.$$

Let $\beta = d/4 \geq 1$.

By the union bound, and the standard estimate $\binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$,

$$\begin{aligned}
P_r[\exists S, T | |S| \leq \frac{n}{d}, |T| < \beta |S|] \\
\leq \sum_{S=1}^{n/d} \binom{n}{s} \binom{n}{\beta s} (\beta s)^{ds} &\leq \sum_{s=1}^{n/d} \binom{n}{\beta s}^2 (\beta s)^{ds} \leq \sum_{s=1}^{n/d} \binom{ne}{\beta s}^{2\beta s} (\beta s)^{ds} \\
= \sum_{s=1}^{n/d} \binom{4ne}{ds}^{ds/2} \binom{ds}{4n}^{ds} &= \sum_{s=1}^{n/d} \binom{eds}{4n}^{ds/2} \leq \sum_{s=1}^{n/d} \binom{e}{4}^{ds/2}.
\end{aligned}$$

This is bounded by $\sum_{s=1}^{\infty} (e/4)^{ds/2} = (e/4)^{d/2} / (1 - (e/4)^{d/2}) < 1$ for $d \geq 4$. ■

5.1.2 Random Walks

As we have already mentioned, there are several measures of expansion. We have already discussed a couple of them in previous chapters, so additionally, two other very important measures are :

- Random walks : Random walks converge quickly to the uniform distribution, i.e., if the second eigenvalue λ_2 is small.
- ‘Well-mixed edges’ : For every two sets S and T (say of constant density) the fraction of edges between S and T is roughly the product of their densities.

All these measures are very closely related to each other, and are even equivalent for certain settings of parameters. In this section we will analyze random walks only. The material is found in [Z].

Definition 5.1.3 (Random walk, [C] , Definition 3.1). *A random walk on a finite graph $G = (V, E)$ is a discrete-time stochastic process (G_0, G_1, \dots) taking values in V . The vertex G_0 is sampled from some initial distribution on V , and G_{i+1} is chosen uniformly at random from the neighbours of G_i .*

Perhaps, the first question to ask about a random walk is what its long term behaviour is. In order to study this, we will need to know how the probability distribution of the walk evolves with time.

Remark 5.1.4. *Let $G = (V, E)$ be a k -regular graph on n vertices. Suppose that x is a probability distribution vector that describes a random vertex on the graph at some point in a random walk, that is, $x = (x_1, \dots, x_n)$ where each $x_i \geq 0$ and $\sum_{i=1}^n x_i = 1$, and x_i is the probability that vertex v_i is chosen. The probability that the walk will be at vertex v_i at the next step is*

$$\frac{1}{k} \sum_{\{v_i, v_j\} \in E} x_j.$$

We see that this is a linear operator on \mathbb{R}^n .

This leads to the definition of the adjacency matrix, that we have already stated in the first chapter. Recall that Since we have assumed that the graphs are undirected, the adjacency matrix will be symmetric. Also, note that we assume A to be a linear operator. This interpretation still works for non-regular graphs, and for multigraphs, we just need to normalize each column individually.

For convenience, we will sometimes prefer to consider the normalized adjacency matrix $\hat{A} = \frac{1}{k}A$. Then, \hat{A}_{ij} is the probability that a random walk at vertex j will step to the vertex i , so \hat{A} is precisely the linear operator described in previous remark. The probability that the random walk will be at vertex i one time step after the distribution x is

$$(\hat{A}x)_i = \sum_{j=1}^n \hat{A}_{ij}x_j.$$

It is not hard to see that if $x = u = \frac{1}{n}(1, 1, 1, \dots, 1)$, the uniform distribution, then $\hat{A}x = x$, that is, x is an eigenvector of \hat{A} with eigenvalue 1. The uniform distribution is stationary. We might imagine that a random walk will always tend towards this stable distribution. To understand whether this is the case, that is, whether the sequence $x, \hat{A}x, \hat{A}^2x, \dots$ will always tend towards u , we need to study the other eigenvalues of \hat{A} .

Throughout the rest of the chapter, let

$$u = \left(\frac{1}{n}, \dots, \frac{1}{n} \right)$$

be the uniform distribution probability vector.

Now that we analyzed random walks, and before we get to the next main section of this chapter, which is spectral expansion, it would be interesting to consider a really important result about expanders :

5.1.3 Expanders are Sparse Approximations of Complete Graphs

Here, we will show a lemma in [AB] and [HA] that demonstrates that the better an expander is, the more quickly a random walk over its vertices mixes. This emphasizes a similarity between expanders and a complete graph (where the first step of a random walk is already uniformly distributed). Of course, unlike complete graphs, expanders are more sparse, which is most meaningful in the context of expander families of fixed degree.

Lemma 5.1.5. *Let p be a probability distribution over the vertices. Then, for a d -regular, non-bipartite, and connected graph G ,*

$$\left\| \left(\frac{1}{d}A \right)^s p - \omega \right\|_2 \leq \left(\frac{\hat{\mu}}{d} \right)^s.$$

Note that there is no need to specify which definition of $\hat{\mu}(G)$ we are using, since they agree in the non-bipartite case. Also, note that A is the adjacency matrix, s is the number of steps taken in the walk, and ω is the vector representing the uniform distribution $(1/|V|, \dots, 1/|V|)$.

Observe that we start the random walk at a vertex distributed according to p , and then proceed to neighbors uniformly at random. Thus, the s distribution over the vertices after s steps is $(\frac{1}{d}A)^s p$. So, this lemma demonstrates that the distribution over the vertices approaches the uniform distribution more quickly the better the expansion constant is.

Proof. Recall that for the normalized Laplacian graph, we can assume that $\psi_0 = \frac{D^{1/2}\mathbf{1}}{\|D^{1/2}\mathbf{1}\|}$, an eigenvector with eigenvalue 0. Also, note that $A\mathbf{1} = d\mathbf{1}$, implying that we can similarly assume that ω is an eigenvector of A with eigenvalue d (recall that G is a d -regular graph).

It is easy to check that

$$\langle \omega, \left(\frac{1}{d}A \right) v \rangle = \langle \left(\frac{1}{d}A \right) \omega, v \rangle,$$

since $A = A^T$ and $\sum_i \sum_j A_{ij} v_i$. Earlier, we mentioned that $A\omega = d\omega$, which means that $(\frac{1}{d}A)\omega = \omega$. So, we have that

$$\langle \omega, \left(\frac{1}{d}A\right)v \rangle = \langle \left(\frac{1}{d}A\right)\omega, v \rangle = \langle \omega, v \rangle.$$

Hence, $v \perp \omega \implies \left(\frac{1}{d}A\right)v \perp \omega$.

Recall that the orthogonal basis of eigenvectors for A is $\phi_0, \phi_1, \dots, \phi_{n-1}$. We can assume that (despite having norm not equal to 1) that $\phi_0 = \omega$. We also take G to be connected and non-bipartite, which means that there is only one eigenvalue with absolute value d .

Now, let v be in the space of all vectors perpendicular to ω , as above. Then, v is in the span of eigenvectors with eigenvalues less than or equal to $\hat{\mu}$, in absolute value, since $\hat{\mu}$ is the second largest eigenvalue in absolute value. So, we have that $\|Av\|_2 \leq \hat{\mu}\|v\|_2$, or equivalently, $\|\left(\frac{1}{d}A\right)v\|_2 \leq \frac{\hat{\mu}}{d}\|v\|_2$. Note that the vector $(\frac{1}{d}A)v$ is still orthogonal to ω , which means that we can induce from this argument that :

$$\left\| \left(\frac{1}{d}A\right)^s v \right\|_2 \leq \left(\frac{\hat{\mu}}{d}\right)^s \|v\|_2,$$

for $s \in \mathbb{Z}^+$.

Now let p be our probability distribution. Then, decompose p into parts in the direction of, and orthogonal to ω : $p = \alpha\omega + p'$. Since the entries of ω are constant, and p' is orthogonal to ω , we know that the sum of the entries of p' is 0. Also, we know that the sum of the entries of p is 1, and the sum of entries of $\alpha\omega$ is α . This means that α must equal 1. We then have $p = \omega + p'$. We need a couple more facts:

$$\left(\frac{1}{d}A\right)^s p = \left(\frac{1}{d}A\right)^s (\omega + p') = \left(\frac{1}{d}A\right)^s \omega + \left(\frac{1}{d}A\right)^s p' = \omega + \left(\frac{1}{d}A\right)^s p'.$$

Also, $\omega \perp p'$ and $p = \omega + p' \implies \|p\|_2^2 = \|\omega\|_2^2 + \|p'\|_2^2 \implies \|p'\|_2^2 \leq \|p\|_2^2 \leq 1$, where the last inequality follows from the fact that p has entries that are positive and sum to 1. So, we have $\|p'\|_2^2 \leq 1$ and thus :

$$\left\| \left(\frac{1}{d}A\right)^s p - \omega \right\|_2 = \left\| \left(\frac{1}{d}A\right)^s p' \right\|_2 \leq \left(\frac{\hat{\mu}}{d}\right)^s \|p'\|_2 \leq \left(\frac{\hat{\mu}}{d}\right)^s.$$

The last two inequalities follow from the facts that $p' \perp \omega$ and $\|p'\|_2^2 \leq 1$. ■

Note that since $\frac{\hat{\mu}}{d} < 1$ for any connected d -regular graph, we have just proven that the distribution of final vertex location over a s -step random walk converges to the uniform distribution as s gets large for any such graph. However, the statement is not true for bipartite graphs (which bounce back and forth forever) or non-regular graphs (which converge to a different stationary distribution).

This results can be very useful to even prove a bound on the diameter of graphs. This can be seen from the following corollary. The proof is based on the previous lemma.

Corollary 5.1.6. *For a connected, n -vertex, d -regular graph G , we have that*

$$\text{diam}(G) \leq \frac{\log(n)}{\log(d) - \log(\hat{\mu})}.$$

For another bound on the diameter, see [W].

Example 5.1.7. *Note that this means for infinite families of d -regular graphs, all with $\hat{\mu}(G)$ bounded away from d , then $\text{diam}(G) = O(\log n)$.*

Example 5.1.8. For the family of complete graphs on n vertices (all of which have diameter of 1), this bound gives

$$\text{diam}(G) \leq \frac{\log(n)}{\log(n-1)}.$$

5.2 Spectral Expansion

We have already discussed and analyzed the graph spectrum in Chapter 1. In this section, we study the spectrum of the graph further, mention its most basic properties, and relate the spectral gap to the combinatorial expansion of the graph. The material is found in [C], [T].

Intuitively, another way of saying that a graph is well-connected is to require that random walks on the graph converge quickly to the stationary distribution. The mixing lemma rates random walks, in turn is captured, by the second largest eigenvalue of the transition matrix, and it turns out to be a very useful measure of expansion.

Remark 5.2.1. Recall that for a regular directed graph G with random-walk matrix M , we define :

$$\lambda(G) = \max_{\pi} \frac{\|\pi M - u\|}{\|\pi - u\|} = \max_{x \perp u} \frac{\|xM\|}{\|x\|},$$

where the first maximization is over all probability distributions $\pi \in [0, 1]^n$ and the second is over all vectors $x \in \mathbb{R}^n$ such that $x \perp u$. We write that $\gamma(G) = 1 - \lambda(G)$.

Before we continue further, we need also to recall the Real Spectral Theorem [[K] , Theorem 7.13].

Theorem 5.2.2. Suppose that V is a linear product space and T is a linear operator on V . Then V has an orthonormal basis consisting of eigenvectors of T if and only if T is self-adjoint.

In the language of matrices, this means that since the adjacency matrix $A(G)$ of an undirected graph G on n vertices is symmetric, it will have n real eigenvalues (counting multiplicities) and that the corresponding eigenvectors are orthogonal. We can study a graph via its corresponding eigenvalues, which that actually is, the spectrum of the graph.

Remark 5.2.3. Spectral graph theory is an area in Mathematical research in its own right. Many useful properties of a graph can be inferred from its spectrum, as we shall soon see. Also, similar matrices have the same spectrum : If v is a λ -eigenvector of A and P is invertible, then $P^{-1}v$ is a λ -eigenvector for $P^{-1}AP$ since

$$(P^{-1}AP)(P^{-1}V) = P^{-1}AvP^{-1}(\lambda v) = \lambda(P^{-1}v).$$

In particular, this means that the spectrum of a matrix is invariant under conjugation by a permutation matrix. Thus it makes sense to talk about the spectrum of a graph, since it is independent of the particular labelling v_1, \dots, v_n of the vertices used to write down a particular adjacency matrix (as changing from the matrix corresponding to one particular labelling to another is achieved by conjugation by a permutation matrix).

Recall that our graphs are undirected, the adjacency operator is clearly self-adjoint, and the adjacency matrix is symmetric. Now, by the spectral theorem, A thus has n real eigenvalues

$$\lambda_1 \geq \dots \geq \lambda_n.$$

Definition 5.2.4. For $\lambda \in [0, 1]$, we say that G is a λ spectral expander if $\lambda(G) \leq \lambda$.

Smaller values of λ correspond to better expansion. Sometimes, we will state results in terms of the spectral gap $\gamma = 1 - \lambda$. We will write λ_i as $\lambda_i(G)$ whenever we need to emphasize the dependence on the graph G .

The largest eigenvalue λ_1 is easily understood for k -regular graphs :

Lemma 5.2.5. *If G is a k -regular graph, then*

$$k = \lambda_1 \geq \lambda_n \geq -k$$

Proof. Clearly $A1 = k1$, where we write $1 \in l^2(V)$ for the constant function, and so k is an eigenvalue of A with eigenvector 1 .

On the other hand, for any $f, g \in l^2(V)$ with norm one, one has

$$\begin{aligned} | \langle Af, g \rangle_{l^2(V)} | &= \left| \sum_{v,w \in V: \{v,w\} \in E} f(w)g(v) \right| \\ &\leq \frac{1}{2} \sum_{v,w \in V: \{v,w\} \in E} (|f(w)|^2 + |g(v)|^2) \\ &\leq \frac{1}{2}k \sum_{w \in V} |f(w)|^2 + \frac{1}{2}k \sum_{v \in V} |g(v)|^2 = k. \end{aligned}$$

And so, A has operator norm k or equivalently, all eigenvalues of A lie between $-k$ and k . ■

Now we turn to the next eigenvalue after λ_1 .

Definition 5.2.6. *Let $\epsilon > 0$ and $k \geq 1$. A finite k -regular graph is said to be a one-sided ϵ -expander if one has*

$$\lambda_2 \leq (1 - \epsilon)k$$

and a two-sided ϵ -expander if one also has

$$\lambda_n \geq -(1 - \epsilon)k.$$

A sequence $G_i = (V_i, E_i)$ of finite k -regular graphs is said to be a one-sided (resp. two-sided) expander family, if there is an $\epsilon > 0$ such that G_i is a one-sided (resp. two-sided) ϵ -expander for all sufficiently large i .

Remark 5.2.7. *The operator $\Delta := 1 - \frac{1}{k}A$ is sometimes known as the graph-Laplacian. Note though that it is also common to use the normalisation $k - A$ instead of $1 - \frac{1}{k}A$, particularly if one wishes to generalise to graphs that are not perfectly regular. This is a positive semi-definite operator, with at least one zero eigenvalue, corresponding to the eigenvector 1 .*

A graph is an ϵ -expander if and only if there is a spectral gap of size ϵ in Δ , in the sense that the first eigenvalue of Δ exceeds the second by at least ϵ .

Remark 5.2.8. *Strictly speaking, we have not defined the notion of a one or two-sided expander graph in the above definition. We have defined an ϵ -expander graph for any given parameter $\epsilon > 0$, and have the notion of an expander family, which is a sequence of graphs rather than for an individual graph.*

One could propose defining an expander graph to be a graph that is a one or two-sided ϵ -expander for some $\epsilon > 0$, or equivalently, a graph G such that the constant sequence G, G, \dots is an expander family, but this definition collapses to existing concepts in graph theory :

Definition 5.2.9. A graph is a one-sided expander for some $\epsilon > 0$ if and only if it is connected, and a two-sided expander for some $\epsilon > 0$ if and only if it is connected and not bipartite.

Theorem 5.2.10 (Trace formula). Let G be a k -regular graph on n vertices for some $n > k \geq 1$. We have :

- $\sum_{i=1}^n \lambda_i = 0$
- $\sum_{i=1}^n \lambda_i^2 = nk$
- $\max(|\lambda_1|, |\lambda_n|) \geq \sqrt{k}$ as $O_n \rightarrow \infty, k$, where $O_n \rightarrow \infty, k$ denotes quantity that goes to zero as $n \rightarrow \infty$ for fixed k .

Remark 5.2.11. Trace formula, places an upper bound on how strong of a two-sided expansion one can obtain for a large k -regular graph. It is not quite sharp. It turns out that one can certain the improvement

$$\max(|\lambda_1|, |\lambda_n|) \geq 2\sqrt{k-1} - O_n \rightarrow \infty, k(1).$$

The

$$\max(|\lambda_1|, |\lambda_n|) \leq 2\sqrt{k-1}$$

are actually the Ramanujan graphs that we have discussed, and we have connections to number theory, but we will not discuss this here.

We will give a probabilistic construction of an expander family later, but first, let us give an example of a family of regular graphs, that is not an expander family.

Example 5.2.12. For each $n \geq 3$, let G_n be the 2-regular graphs whose vertex set is the cyclic group $\mathbb{Z}/n\mathbb{Z}$, and whose edge set is the set of pairs $\{x, x+1\}$ for $x \in \mathbb{Z}/n\mathbb{Z}$. This is a basic example of a Cayley graph.

- The eigenvalues of the adjacency operator A_n associated to G_n are $2\cos(2\pi j/n)$ for $j = 0, 1, \dots, n-1$
- G_n is not a one-sided expander family, thus not a two-sided expander family either. This is despite G_n always being connected (and non-bipartite for n odd).

Definition 5.2.13. Let $G = (V, E)$ a k -regular graph on n vertices. Let $G^c = (V, (\frac{V}{2}) \setminus E)$ be the complement graph, consisting of all the edges connecting two vertices in V that are not in E , thus, G^c is an $(n-k-1)$ -regular graph. The

$$\lambda_i(G^c) = -1 - \lambda_{n+2-i}(G), \text{ for all } 2 \leq i \leq n.$$

Proposition 5.2.14. Let $n \geq 2$ be an even number, and let $G = K_{n/2, n/2}$ be the complete bipartite graph between two sets of $n/2$ vertices each, thus G is an $n/2$ -regular graph.

$$\lambda_n = -\frac{n}{2} \text{ and } \lambda_2 = \dots = \lambda_{n-1} = 0.$$

Thus, G is a one-sided 1-expander, but is not a two-sided expander at all.

These results lead to the following formal definition, which is a reformulation of 5.2.4 definition.

Definition 5.2.15 (Eigenvalue expansion). We say that a d -regular graph is a λ eigenvalue expander if $\lambda_2 \leq \lambda$, for $\lambda \in [0, 1]$.

Definition 5.2.16. We say that a d -regular graph is a λ absolute eigenvalue expander if $|\lambda_2|, |\lambda_n| \leq \lambda$.

If we have a family of graphs with n tending to ∞ and with d constant, we informally call these graphs expander graphs if they are all λ eigenvalue expanders for some constant $\lambda < d$.

Having made this definition. We can now show that families of expander graphs (and absolute expander graphs) exist.

Proposition 5.2.17. *For every $d \geq 3$, there exists $\lambda < d$ such that for all sufficiently large n , there exists an n -vertex d -regular λ absolute eigenvalue expander graph.*

Now that we have discussed about λ_2 and even define expanders in terms of eigenvalue expansion, we continue by stating lemma 5.2.5, but with an extra condition.

Lemma 5.2.18. *Let G be a k -regular graph on n vertices, with adjacency matrix A . Let the spectrum of A be $\lambda_1 \geq \dots \geq \lambda_n$. Then*

1. $\lambda_1 = k$
2. $\lambda_2 = k$ if and only if G is not connected
3. $\lambda_n \geq -k$ with equality if and only if G has a bipartite graph as one of its connected components.

Proof. In this proof we will use the probability distribution from the previous section of random walks. Let $(\frac{1}{n}, \dots, \frac{1}{n})$. The entry $(Au)_i$ is the sum over all vertices v_j of $\frac{1}{n}$ times the number of edges between v_i and v_j . Since v_i has degree k , we see that $(Au)_i = \frac{k}{n}$, and $Au = ku$.

Since the sum of all entries in A is nk , every eigenvalue of A must have absolute value at most k . If G is not connected, then A will have invariant subspaces corresponding to different connected component, and a suitably chosen (to be orthogonal to u) linear combination of characteristic functions for the spaces will give a second k -eigenvector.

Conversely, supposing that we a second k -eigenvector v , as u and v are orthogonal and v is non-zero, v must have both positive and negative entries. But then the vertices corresponding to the maximal (positive) entry in v can only be joined to other such vertices, and this gives a disconnection of G .

Similarly, if G is bipartite with edges between A and B , the regularity of the graph implies $|A| = |B|$ and then the vector $v_i = 1_A 1_B$ will be a $(-k)$ -eigenvector. Conversely, with a (k) -eigenvector v orthogonal to u , we can conclude that all vertices corresponding to the maximal entry of v must be joined only to the vertices corresponding to the minimal (negative) entry of v , and vice versa, so the graph has a bipartite connected component. ■

Proposition 5.2.19. *Let G be a regular graph. If G is connected and non-bipartite, then any random walk on G tends to the uniform distribution.*

Proof. By above lemma, $u = v_1$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 1$ of \hat{A} , and all other eigenvalues satisfy $|\lambda| < 1$.

As the eigenvectors form a basis, any initial probability vector can be written as

$$p = u + \alpha_2 v_2 + \dots + \alpha_n v_n$$

where the coefficient of u must be 1 since $\langle p, u \rangle = \|u\|^2$. Then,

$$\hat{A}^s p = u + \lambda_2^s \alpha_2 v_2 + \dots + \lambda_n^s \alpha_n v_n \rightarrow u$$

as $s \rightarrow \infty$. ■

The rate of convergence in the above proposition depends on the 'spectral gap' of the adjacency matrix, $k - \max\{|\lambda_2|, |\lambda_n|\}$. It turns out that having a large spectral gap is equivalent to being a combinatorial expander (in the sense of the combinatorial definition). Keeping this in mind, we give a definition of spectral expanders which is actually equivalent to the eigenvalue definition we gave before.

Definition 5.2.20 (Spectral expansion). *Let G be a k -regular graph on n vertices. Let the spectrum of G be $\lambda_1 \geq \dots \geq \lambda_n$. Then G is a (n, k, α) -expander if $|\lambda_2|, |\lambda_n| \leq \alpha\lambda_1$.*

Remark 5.2.21. *From the lemma in the beginning of this section, we know a regular graph will be an (n, k, α) -expander for some $\alpha < 1$ if and only if the graph is connected and not bipartite. This is exactly like a result we have discussed, that classifies graphs that are c -expanders for $c > 0$ as connected graphs, and again we note that we require infinite families for which the spectral gap is uniformly bounded away from zero.*

Theorem 5.2.22. *Let $G = (V, E)$ be a finite, connected, k -regular graph and let λ be its second eigenvalue. Then*

$$\frac{k - \lambda}{2} \leq h(G) \leq \sqrt{2k(k - \lambda)}.$$

Proposition 5.2.23 (Spectral expansion implies combinatorial expansion). *Let G, k and λ be as in the above theorem. Then*

$$h(G) \geq \frac{k - \lambda}{2}.$$

Proof. Consider a partition $V = A \sqcup B$ with $|A| \leq |B|$ such that $h(G) = |E(A, B)|/|A|$. The vector (v_i) defined by

$$v_i = \begin{cases} b & \text{if } v_i \in A \\ -\alpha & \text{if } v_i \in B \end{cases}$$

will be orthogonal to u . The rest of the proof is just straightforward combutational of relating $E(A, B)$ to λ via considering $Av - kv$. ■

Proposition 5.2.24 (Combinatorial expansion implies spectral expansion). *Let G, k and λ be as in the above theorem. Then*

$$\sqrt{2k(k - \lambda)} \geq h(G).$$

Proof. See [[C] , PP.475-477] ■

Remark 5.2.25. *Although qualitatively graphs are combinatorial expanders if and only if they are spectral expanders, there is no direct quantitative relationship between the particular expansion constants, namely the Cheeger constant and the spectral gap. By this we mean that although there are the bounds of the above theorem, neither is a function of the other.*

Indeed, there are efficient algorithms to compute the eigenvalues of a matrix, but computing the Cheeger constant is hard (as first was proved at [L]). Another way in which combinatorial and spectral expansion differ quantitatively is in terms of extremal properties. The best expansion for combinatorial expanders is an open problem [[F] , Problem 10.1.1], but there is an upper bound on the spectral gap, which is attained by the Ramanujan graphs of Lubotzky-Phillips-Sarnak [M] and independently Margulis [N].

The bound, which measures in some way how good an eigenvalue expander can a d -regular graph be, is as follows. We refer to [[F], Proposition 4.2.6].

Proposition 5.2.26 (Alon- Boppona). *Let X_n be a family of k -regular graphs, where k is fixed and n is the number of vertices of the graphs, and $n \rightarrow \infty$. Then*

$$\limsup_{n \rightarrow \infty} \lambda_1(X_n) \leq k - 2\sqrt{k-1}.$$

We will use this result at the end of this chapter to give an explicit construction through a spectral method.

5.3 Spectral Expansion Properties

Suprisingly, the linear-algebraic measure of expansion turns out to be equivalent the combinatorial measure of vertex expansion for common parameters of interest. One direction is given by the following.

Theorem 5.3.1 (Spectral Expansion implies Vertex Expansion). *If G is a spectral expander for some $\lambda \in [0, 1]$, then for all $\alpha \in [0, 1]$, G is an $(\alpha N, \frac{1}{(1-\alpha)\lambda^2+\alpha})$ vertex expander.*

In particular, for $\alpha = \frac{1}{2}$ and $\lambda = 1 - \gamma$, we conclude that G is a $(\frac{N}{2}, 1 + \gamma)$ expander.

Before proving this theorem, we will introduce some properties of probability distributions.

Definition 5.3.2. *For a probability distribution π , the collision probability of π is defined to be the probability that two independent samples from π are equal, namely*

$$CP(\pi) = \sum_x \pi_x^2.$$

The support of π is

$$\text{supp}(\pi) = \{x : \pi_x > 0\}.$$

Lemma 5.3.3. *For every probability distribution $\pi \in [0, 1]^n$, we have that :*

1. $CP(\pi) = \|\pi\|^2 = \|\pi - u\|^2 + \frac{1}{N}$
2. $CP(\pi) \geq \frac{1}{|\text{supp}(\pi)|}$, with equality iff π is uniform on $\text{supp}(\pi)$.

Proof. 1. The fact that $CP(\pi) = \|\pi\|^2$ follows immediately from the definition. Then writing $\pi = u + (\pi - u)$, and noting that $(\pi - u) \perp u$, we have that

$$\|\pi\|^2 = \|u\|^2 + \|\pi - u\|^2 = \frac{1}{N} + \|\pi - u\|^2.$$

2. By Cauchy - Schwartz, we get :

$$1 = \sum_{x \in \text{supp}(\pi)} \pi_x \leq \sqrt{|\text{supp}(\pi)|} \sqrt{\sum_x \pi_x^2}.$$

■

We will now show the proof of the theorem.

Proof. By the definition of spectral expansion and (1) of previous lemma, we have that : $CP(\pi M) - \frac{1}{N} \leq \lambda^2 \cdot (CP(\pi) - \frac{1}{N})$ for every probability distribution π .

Letting now S be any subset of the vertices of size at most αN and π the uniform distribution on S , we have that $CP(\pi) = \frac{1}{|S|}$ and $CP(\pi M) \geq \frac{1}{|\text{supp}(\pi M)|} = \frac{1}{|N(S)|}$, implying that

$$\left(\frac{1}{|N(S)|} - \frac{1}{N}\right) \leq \lambda^2 \left(\frac{1}{|S|} - \frac{1}{N}\right)$$

solving for $|N(S)|$ and using $N \geq \frac{|S|}{\alpha}$, we obtain

$$|N(S)| \geq \frac{|S|}{(\lambda^2(1-\alpha) + \alpha)}.$$

■

The opposite direction, i.e., obtaining Spectral expansion from vertex expansion, is more difficult.

Theorem 5.3.4 (Vertex expansion implies spectral expansion [LA]). *For every $\delta > 0$ and $D > 0$, there exists $\gamma > 0$ such that if G is a D -regular $(\frac{N}{2}, 1 + \delta)$ vertex expander, then it is also $(1 - \gamma)$ spectral expander. Specifically, we can take $\gamma = \Omega(\delta^2/D)$.*

Note first the dependence on absolute size being $n/2$: This is necessary, because a graph can have vertex expansion $(\alpha N, \Omega(1))$ for $\alpha < \frac{1}{2}$ and be disconnected (e.g. the disjoint union of two good expanders), thereby having no spectral expansion. The other problem is that the bound on γ depends on D . This is also necessary, because adding edges to a good expander cannot hurt its vertex expansion, but can hurt its spectral expansion.

Still, roughly speaking, these two results showed that vertex expansion and spectral expansion are closely related, and equivalent for many interesting settings of parameters. Indeed, when people informally use the term “expander”, they often mean a family of D -regular graphs for constant degree $D = o(1)$ satisfying one of the following equivalent conditions :

1. Every graph in the family is a λ spectral expander for some constant $\lambda < 1$.
2. Every graph in the family is an $(N/2, 1 + \delta)$ expander for some constant $\delta > 0$.

However, these two measures are no longer equivalent if one wants to optimize the expansion constants.

- For vertex expansion, we have already seen that if we allow α to be a small constant (depending on D), then there exists $(\alpha N, A)$ vertex expanders with A very close to D . For example, let $A = D - 1, 01$, and clearly one cannot have A to be any larger than D .
- The optimal value for the spectral expansion is also well-understood. first note that, by taking $\alpha \rightarrow 0$ in theorem 5.3.1, a λ spectral expander has vertex expansion $A \approx \frac{1}{\lambda^2}$ for small sets.
So, a lower bound on λ is $\frac{1}{\sqrt{D-o(1)}}$.

In fact, this lower bound can be improved. We can see that from the following theorem, which is found in [LA].

Theorem 5.3.5. *Any infinite family of D -regular graphs with spectral expansion λ has*

$$\lambda \geq \frac{2\sqrt{D-1}}{D} - o(1),$$

where the additive term vanishes as $|V(G)| \rightarrow \infty$.

Surprisingly, there exist explicit constructions giving $\lambda < \frac{2\sqrt{D-1}}{D}$. Graphs meeting this bound are called Ramanujan graphs. Random graphs almost match this bound as well :

Theorem 5.3.6. *For any constant D and any constant $\epsilon > 0$, a random D -regular graph has spectral expansion at most $\frac{2\sqrt{D-1}}{D} + \epsilon$ with probability*

$$1 - \frac{1}{N^{\Omega(D)}}$$

The previous theorem is found in [AD].

Now, let's see what these results for spectral expansion imply in the world of vertex expansion: With Ramanujan graphs ($\lambda = \frac{2\sqrt{D-1}}{D}$), our bound gives a vertex expansion of $A \approx \frac{D}{4}$ (for small sets). This is not tight, and it is known that Ramanujan graphs actually have vertex expansion $\frac{D}{2}$, which is tight in the sense that there are families of graphs with $\lambda \rightarrow \frac{2\sqrt{D-1}}{D}$ with vertex expansion at most $\frac{D}{2}$. Still, this is not as good as we showed by the probabilistic method, where we achieved vertex expansion $D - o(1)$. This means that we cannot obtain optimal vertex expansion by going through spectral expansion. Similarly the opposite. Hence, a vertex and spectral expansion are loosely equivalent, but only if we are not interested in optimizing the constants in the trade-offs between various parameters (and for some applications this is crucial).

5.3.1 Eigenvalue Bounds on Expansion

In general, random walks are very good expanders, so the existence of expanders is not hard to establish. However, the difficult question is how to construct expanders explicitly.

For now, I will explore the connection between expansion and eigenvalues.

Theorem 5.3.7. *For any d -regular graph G with the second eigenvalues λ_2 ,*

$$h(G) \geq \frac{1}{2}(d - \lambda_2)$$

Proof. For any subset of vertices S of size s , let $x = (n - s)1_S - s1_{\bar{S}}$. We get that

$$x^T x = (n - s)^2 s + s^2(n - s) = s(n - s)n$$

and

$$x^T A x = 2 \sum_{(i,j) \in E} x_i x_j = 2(n - s)^2 e(S) - 2s(n - s)e(S, \bar{S}) + 2s^2 e(\bar{S}).$$

To eliminate $e(S)$ and $e(\bar{S})$, observe that every degree is equal to d , and ds can be viewed as counting each edge in $e(S)$ twice and each edge in $e(S, \bar{S})$.

So, $ds = 2e(S) + e(S, \bar{S})$.

Similarly, $d(n - s) = 2e(\bar{S}) + e(S, \bar{S})$. This yields,

$$x^T A x = (n - s)^2(ds - e(S, \bar{S})) - 2s(n - s)e(S, \bar{S}) + s^2(d(n - s) - e(S, \bar{S})) = dns(n - s) - n^2 e(S, \bar{S}).$$

Since now $x \cdot 1 = 0$, we can use the variational definition of λ_2 to claim that

$$\lambda_2 \geq \frac{x^T A x}{x^T x} = \frac{dns(n - s) - n^2 e(S, \bar{S})}{s(n - s)n} = d - \frac{ne(S, \bar{S})}{s(n - s)}.$$

For any set S of size $s \leq \frac{n}{2}$, we have that

$$\frac{e(S, \bar{S})}{|S|} \geq \frac{n - s}{n}(d - \lambda_2) \geq \frac{1}{2}(d - \lambda_2)$$

(\star) we used exactly the same vector to prove our bound on the independence number. ■

This theorem shows that if $d - \lambda_2$ is large, for example $\lambda_2 \leq \frac{d}{2}$, then the graph is a $(d, d/4)$ -expander, very close to the best possible. The quantity $d - \lambda_2$ is called the spectral gap.

Theorem 5.3.8 (Cheeger - Alon - Milman, [U]). *For any d -regular graph with second eigenvalue λ_2 , we have*

$$h(G) \leq \sqrt{d(d - \lambda_2)}$$

How large can the spectral gap be?

We have seen that graphs where the maximum eigenvalue λ_1 dominates all other eigenvalues have very interesting properties. Now, we can ask the question. How small can the remaining eigenvalues possibly be? We know that the complete graph has eigenvalues $n - 1$ and -1 and therefore

$$\lambda = \max_{i \neq 1} |\lambda_i|$$

is dominated by λ_1 by a factor of $n - 1$, the degree in K_n . For a constant degree d and large n , this cannot happen.

Theorem 5.3.9. *For any constant $d > 1$, any d -regular graph has an eigenvalue $\lambda_i \neq d$ of absolute value*

$$\lambda = \max_{i \neq 1} |\lambda_i| \geq (1 - o(1))\sqrt{d}$$

where $o(1) \rightarrow 0$ as $n \rightarrow \infty$.

Proof. We refer to [LA] and [KA].

Consider the square of the adjacency matrix A^2 , where A^2 has d on the diagonal, and so $\text{Tr}(A^2) = dn$. On the other hand, the eigenvalues of A^2 are λ_i^2 and so

$$\text{Tr}(A^2) = \sum_{i=1}^n \lambda_i^2 \leq d^2 + (n-1)\lambda^2.$$

Putting this together, we get that

$$\lambda^2 \geq \frac{dn - d^2}{n-1} \geq (1 - d/n)d = (1 - o(1))d.$$

■

So, the best possible spectral gap that we can have is roughly between \sqrt{d} and d . More precisely, it is common that the second eigenvalue is always at least $2\sqrt{d-1} - o(1)$. This leads to a different definition of Ramanujan graphs via this notation.

Definition 5.3.10. *A d -regular graph is Ramanujan, if all eigenvalues in absolute value are either equal to d or at most $2\sqrt{d-1}$.*

5.4 Other Measures of Expansion

5.4.1 Expansion and Poincare Inequality

If $G = (V, E)$ is a k -regular graph and $f : V \rightarrow \mathbb{C}$ is a function, define the gradient magnitude

$$|\nabla f(v)| = \left(\sum_{w \in V, \{v,w\} \in E} |f(w) - f(v)|^2 \right)^{\frac{1}{2}}$$

G is a one-sided ϵ -expander if and only if it has the Poincare inequality

$$\|\nabla f\|_{l^2(v)}^2 \geq 2k\epsilon \|f\|_{l^2(v)}^2$$

whenever f has mean zero.

5.4.2 Connection with Edge Expansion

Given two subsets F_1 and F_2 of the vertex set V in a graph $G = (V, E)$, define $E(F_1, F_2) \subset F_1 \times F_2$ to be the set of all pairs $(v_1, v_2) \in F_1 \times F_2$ such that $\{v_1, v_2\} \in E$. We have to note at this point that the cardinality of this set can be expressed in terms of the adjacency operator as

$$|E(F_1, F_2)| = \langle A 1_{F_1}, 1_{F_2} \rangle_{L^2(V)}$$

Define the boundary ∂F of a subset F of V , to be the set

$$\partial F := E(F, V \setminus F)$$

and thus ∂F is essentially the set of all edges that connect an element of F to an element outside of F .

At this point, recall that the edge expansion ratio $h(G)$ of the graph G is

$$h(G) := \min_{F \subset V: |F| \leq \frac{|V|}{2}} \frac{|\partial F|}{|F|},$$

where F ranges over all subsets of V of cardinality at most

$$|F| \leq \frac{|V|}{2}.$$

The quantity $h(G)$ can be interpreted as a type of isoperimetric constant for G , analogous to the cheeger constant of a compact Riemannian manifold, and so $h(G)$ is known as cheeger constant of G . Note here that $h(G)$ is non-zero precisely when G is connected. If G is disconnected, at least one of the components F will have cardinality less than $|V|/2$.

5.4.3 Weak Discrete Cheeger Inequality

Let $k \geq 1$, and let G_n be a family of finite k -regular graphs.

Then the following are equivalent :

1. G_n form a one-sided expander family.
2. There exist $c > 0$ such that $h(G_n) \geq c$ for all sufficiently large n .

Remark 5.4.1. *There is a more precise relationship between the edge expansion ratio $h(G)$ and the best constant ϵ that makes G a one-sided ϵ -expander, namely the discrete cheeger inequality :*

$$\frac{\epsilon}{2}k \leq h(G) \leq \sqrt{2\epsilon k}.$$

5.4.4 Expander Mixing Lemma

We mention a very useful measure of expansion involving edges crossing cuts in the graph.

Recall for two sets $S, T \subset V(G)$, let $e(S, T) = \{(u, v) \in S \times T : \{u, v\} \in E\}$, where (u, v) is interpreted as an ordered pair.

Here S, T may not be disjoint, in which case some edges may be counted twice, corresponding to both orientations.

Definition 5.4.2. *A d -regular graph G is a (k, ϵ) -edge expander if for all sets S of at most k vertices, the cut $e(S, \bar{S})$ is of size at least $\epsilon \cdot |S| \cdot d$.*

That is, at least an ϵ fraction of the edges from S lead outside S . We need to note here that sometimes edge expansion is defined without the normalization factor of S , only requiring $|e(S, \bar{S})| \geq \epsilon|S|$. When viewed in terms of stationary distribution of the random walk on G , the ratio $|e(S, \bar{S})|/(|S| \cdot D)$ is the probability that, conditioned on being in S , the random walk leaves S in one step. It turns out that, if we let ϵ be the minimum of this quantity over all sets S of density at most $1/2$, i.e., ϵ is the largest value such that G is an $(N/2, \epsilon)$ edge expander, then this turns out to be even more closely related to spectral expansion than vertex expansion. Indeed, it is known that the spectral gap γ satisfies $2\epsilon \geq \gamma \geq \frac{\epsilon^2}{2}$. The intuition is that a large edge expansion ϵ implies that the random walk on the graph has no "bottlenecks" and so should mix rapidly.

There is yet another measure of expansion in terms of edges not just from a set S to its complement but between any two sets S and T . If we think of an expander as being like a random graph, we would expect the fraction of graph edges that are in $e(S, T)$ to be proportional to the fraction of nodes that are in S and T .

The following lemma, which is found in [C], shows that this intuition is correct.

Lemma 5.4.3 (Expander mixing lemma). *Let G be a D -regular λ spectral expander on N vertices. Then, for all sets of vertices S, T , we have :*

$$\left| \frac{e(S, T)}{N \cdot D} - \mu(S)\mu(T) \right| \leq \lambda \sqrt{\mu(S)\mu(T)}$$

where $\mu(R) = \frac{|R|}{N}$ for any set R .

Observe that the denominator $N \cdot D$ counts all edges of the graph (as ordered pairs). The lemma states that the difference between the fraction of edges in $e(S, T)$ and the expected value if G, S and T would be chosen randomly is "small", roughly the square root of this fraction. The error term should not be surprising, since the theorem holds for all sets S and T , so we would expect a standard deviation. Finally, observe that the error term is at most λ , which is an interesting result if λ is small.

Proof. Let X_S be the characteristic vector of S and X_T the characteristic of T . Let A be the adjacency matrix of G , and $M = A/D$ be the random - walk matrix for G . Note that, $e(S, T) = X_T^t A X_S = X_S^t (DM) X_T$. Let $\alpha = \mu(S)$ and $\beta = \mu(T)$. As usual, we can express X_S as the sum of two components, one parallel to the uniform distribution u , and the other a vector X_S^\perp , where $X_S^\perp \perp u$. The coefficient of u should be $\frac{X_S \cdot u}{\|u\|^2} = \sum_i (X_S)_i = |S| = \alpha N$. Then,

$$X_S = (\alpha N)u + X_S^\perp$$

and similarly,

$$X_T = (\beta N)u + X_T^\perp$$

Intuitively, X_S^\perp and X_T^\perp give the error term, while the component parallel to the uniform distribution "spreads" the weight of S and T uniformly over the entire graph.

Formally, we have that

$$\begin{aligned} \frac{e(S, T)}{N \cdot D} &= \frac{1}{N} ((\alpha N)u + X_S^\perp)^t M ((\beta N)u + X_T^\perp) \\ &= \frac{1}{N} (\alpha \beta N^2) u^t M u + \frac{1}{N} (\alpha N) u^t M X_T^\perp + \frac{1}{N} (\beta N) (X_S^\perp)^t M u + (X_S^\perp)^t M X_T^\perp. \end{aligned}$$

Since the component of X_T^\perp along u is zero, and u is an eigenvector of A , $M X_T^\perp$ also has a component of 0 along u , and so $u^t M X_T^\perp = 0$. In addition, $(X_S^\perp)^t M u = X_S^\perp \cdot u = 0$.

Our expression thus simplifies to :

$$(\alpha \beta N) u \cdot u + \frac{(X_S^\perp)^t M X_T^\perp}{N} = \alpha \beta + \frac{X_S^\perp \cdot (M X_T^\perp)}{N}.$$

Thus,

$$\begin{aligned} \left| \frac{e(S, T)}{N \cdot D} - \mu(S)\mu(T) \right| &= \frac{X_S^\perp \cdot (MX_T^\perp)}{N} \leq \frac{1}{N} \|X_S^\perp\| \|MX_T^\perp\| \\ &\leq \frac{1}{N} \|X_S^\perp\| \lambda \|X_T^\perp\| \leq \frac{\lambda}{N} \|X_S\| \cdot \|X_T\| = \frac{\lambda \sqrt{\alpha N} \sqrt{\beta N}}{N} = \lambda \sqrt{\alpha \beta}. \end{aligned}$$

■

A natural question is whether such “well-mixed” edges characterizes a good expander. We can see from the following theorem that this is indeed true. We refer to [NA].

Theorem 5.4.4 (Converse to Expander Mixing Lemma). *Let G be a D -regular graph. If for all $S, T \subset V(G)$,*

$$\left| \frac{e(S, T)}{N \cdot D} - \mu(S)\mu(T) \right| \leq \theta \sqrt{\mu(S)\mu(T)}, \text{ for fixed } \theta,$$

then G is a λ spectral expander for $\lambda = O(\theta \log \frac{D}{\theta})$.

Putting the two theorems together, we see that λ and θ differ by at most an $o(\log D)$ factor, and in particular $\lambda = \frac{1}{D^{\Omega(1)}}$ iff $\theta = \frac{1}{D^{\Omega(1)}}$ for constant. So, unlike the other connections we have seen, this connection is good for highly expanding graphs (as opposed to the case λ slightly less than 1). Unlike the case of vertex expansion vs. spectral expansion, here it is not known if the dependence of the relationship on the degree D is necessary.

We will sketch the idea of the proof :

It is known that $\lambda_2(A) = \max_{x, y \perp u} \frac{x^t A y}{\|x\| \|y\|}$. The proof relates this to the maximum taken when $x, y \in \{\pm 1\}^N$, and bounds this based on the relation from the hypothesis, for appropriate S and T .

5.4.5 Expansion as Pseudorandomness

Clearly, better expanders give a better bound on the difference between the number of edges between any two subsets of the vertices and the quantity $\frac{d|S||T|}{n}$. This quantity is significant because it represents the expected number of edges between S and T in a random graph. This is because the number of edges leaving a vertex set of size $|S|$ in a d -regular graph is $d|S|$, and if the edges lead to a vertex chosen uniformly at random, then each edge would lead to a node in T with probability $\frac{|T|}{n}$. Hence, the expected number of edges leading between S and T is $d|S|\frac{|T|}{n}$. We need to note here that this protocol for creating a random graph could lead to multiple edges between the same nodes.

Therefore, very good expanders will ensure that the number of edges between any two vertex subsets is close to this quantity.

The material of this section is found in [AH] and [AI].

5.5 Further Results

Proposition 5.5.1. *Let $k \geq 1$ and let G_n be a family of finite k -regular graphs. The following are equivalent :*

1. G_n form a two-sided expander family.

2. There exist $c > 0$ such that whenever n is sufficiently large and F_1, F_2 are subsets of V_n of cardinality at most $|V_n|/2$, then

$$|E(F_1, F_2)| \leq (k - c)\sqrt{|F_1||F_2|}$$

The below connect expansion to some other graph-theoretic properties. On a connected graph G , one can define the graph metric

$$d : G \times G \rightarrow \mathbb{R}^+$$

by defining the $d(v, w)$ to be the length of the shortest path from v to w using only edges of G . This is a metric on G .

- Expanders have high chromatic number

Let G be a k -regular graph on n vertices that is a two-sided ϵ -expander for some $n > k \geq 1$ and $\epsilon > 0$.

- Any independent set in G has cardinality at most $(1 - \epsilon)n$
- The chromatic number of G is at least $\frac{1}{1 - \epsilon}$.

- Expansion and concentration of Measure

Let G be a k -regular graph on n vertices that is a one-sided ϵ -expander for some $n > k \geq 1$ and $\epsilon > 0$. Let $f : G \rightarrow \mathbb{R}$ a function which is Lipschitz with some Lipschitz constant K , thus

$$|f(v) - f(w)| \leq Kd(v, w), \text{ for all } v, w \in V.$$

Let M be a median value of f (thus $f(v) \geq M$ for at least half of the vertices v , and $f(v) \leq M$ for at least half the vertices v),

$$|\{v \in V : |f(v) - M| \geq \lambda k\}| \leq Cn \exp(-c\lambda), \text{ for all } \lambda > 0$$

and some constants C , where $C > 0$ depending only on k, ϵ .

At this point, it would be remarkable to give an explicit construction through a spectral method, which is very important for applications

5.5.1 Spectral Construction

For applications one wants explicit constructions. A useful means to achieve this is the following spectral method :

Given an $X = X_{n,k}$, let A be the $n \times n$ symmetric matrix whose rows and columns are indexed by the vertices $v \in V$ and for which the v, w entry is 1 if v is joined to w , and 0 otherwise. The vector $(1, 1, \dots, 1)$ is an eigenvector of A with eigenvalue k . The eigenvalues of A are real and lie in the interval $[k, k]$. Let $\lambda_1(X)$ denote the next-to-largest eigenvalue of A after k .

The following inequalities, which are the discrete analogues of inequalities of Cheeger and Buser in differential geometry, relate $h(X)$ to the gap between $\lambda_1(X)$ and k and are due to Tanner, Alon, and Milman that we studied in previous section:

$$\frac{k - \lambda_1(X)}{2} \leq h(X) \leq \sqrt{2k(k - \lambda_1(X))}.$$

Thus $X_{n,k}$ is an expanding family if and only if it has a uniform lower bound on the spectral gap, and the larger the gap the better the expansion.

Using this and quotients of explicit infinite discrete groups which enjoy Kazhdans property

(T)(which is the property that the trivial representation is isolated in the space of all unitary representations), Margulis [Q] gave the first examples of explicit expanders. There is a limit as to how big the spectral gap can possibly be (Alon-Boppana): For k fixed we have

$$\liminf_{n \rightarrow \infty} \lambda_1(X_{n,k}) \geq 2\sqrt{k-1}.$$

The liminf is taken over all $X_{n,k}$ s. An $X_{n,k}$ (or rather a sequence of such $X_{n,k}$ s with $n \rightarrow \infty, k$ fixed) is, as we have already said, called a Ramanujan graph if $\lambda_1(X_{n,k}) \leq 2\sqrt{k-1}$.

In view of the above, such a graph is optimal, at least as far as the spectral gap measure of expansion is concerned. It is a pleasant fact that if $k = q + 1$ with q a prime power, then Ramanujan graphs exist. The constructions are in terms of very explicit descriptions as Cayley graphs of $PGL(2, \mathbb{F}_q)$ ([AA], [Q]) (Cayley graphs are those whose vertex sets are the elements of a group G and whose edges correspond to $g \rightarrow gg_1$ with g_1 in a set of generators).

For k not of this form it is an interesting question as to whether Ramanujan graphs exist. J. Friedman has recently shown that for k fixed and $\epsilon > 0$ the probability that $\lambda_1(X_{n,k}) \leq 2\sqrt{k-1} + \epsilon$ tends to 1 as $n \rightarrow \infty$.

So the random graph is asymptotically Ramanujan. Some very interesting numerical experimentation (T. Novikoff, <http://www.math.nyu.edu/Courses/V63.0393/>, honors math lab, projects) indicates that the probability that the random graph is Ramanujan is slightly bigger than 0.5, corresponding to the skewness in the Tracy-Widom distribution in random matrix theory (specifically the Gaussian Orthogonal Ensemble).

In [Co] it is shown how one may use their novel zig-zag graph product (a notion related to the semidirect product of groups when the graphs are Cayley graphs) to give new explicit constructions of expanders. Their construction of the family is inductive, with the zig-zag product being used at each step. It has the extra flexibility that allows them to construct explicit expanders that expand small sets almost optimally (they call these lossless expanders). This extra twist turns out to be very useful in a number of applications, and this feature cannot be deduced from a spectral gap analysis.

Chapter 6

Kazhdan's Property (T)

Property (T) was introduced by Kazhdan in his paper [O] . It was used to demonstrate that a large class of lattices in semisimple Lie groups are finitely generated.

Property (T) is defined in terms of unitary representations, so we will first give the necessary background on Hilbert spaces, group representations and topological groups.

6.1 Unitary Representation of Locally Compact Groups

We will recall the background of what is needed to define a unitary representation of a locally compact group.

Definition 6.1.1 (Inner product). *Let H be a vector space over \mathbb{C} . An inner product on H is a function $\langle \cdot, \cdot \rangle: H \times H \rightarrow \mathbb{C}$ such that for all $\alpha, \beta \in \mathbb{C}$ and $x, y, z \in H$, the following are satisfied :*

1. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$
2. $\langle x, \alpha y + \beta z \rangle = \bar{\alpha} \langle x, y \rangle + \bar{\beta} \langle x, z \rangle$
3. $\langle x, x \rangle \geq 0$ with equality if and only if $x = 0$
4. $\langle x, y \rangle = \overline{\langle y, x \rangle}$

A vector space H over \mathbb{C} together with an inner product on H will be called an inner product space. Usually the particular inner product will be implicit, and we refer to H by itself as an inner product space.

Example 6.1.2. *The space \mathbb{C}^n has the inner product*

$$\langle (z_1, \dots, z_n), (w_1, \dots, w_n) \rangle = z_1 \bar{w}_1 + \dots + z_n \bar{w}_n.$$

An inner product endows a vector space with a norm. We will present this result without proof . The material can be found in [[Co], Corollary 1.5].

Corollary 6.1.3 (norm). *Let $\langle \cdot, \cdot \rangle$ be an inner product on a vector space H and define $\|x\| = \sqrt{\langle x, x \rangle}$ for all x in H . Then $\|\cdot\|$ is a norm for H , that is :*

1. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in H$
2. $\|\alpha x\| = |\alpha| \|x\|$ for all α in \mathbb{C} and $x \in H$

3. $\|x\| \geq 0$ with equality if and only if $x = 0$.

A normed vector space V has a natural metric structure, given by defining $d(x, y) = \|x - y\|$ for all $x, y \in V$. The metric space structure then gives a topology on V .

We will now define what a Hilbert space is which is the space we will work from now on.

Definition 6.1.4. Let H be an inner product space with norm $\|\cdot\|$ given by its inner product. Then H is called a Hilbert space if it is complete with respect to the norm topology on H .

Remark 6.1.5. One can study inner product spaces over \mathbb{R} , but for purposes of developing theory around the property (T) will consider only Hilbert spaces over \mathbb{C} . So, from now on, we assume H is a Hilbert space over \mathbb{C} .

Example 6.1.6 (Finite-dimensional Hilbert spaces). The space \mathbb{C}^n with the inner product defined in previous example is a finite-dimensional Hilbert space. Moreover, any n -dimensional Hilbert space can be identified with \mathbb{C}^n , that is, they are isometrically isomorphic, which is the natural equivalence for Hilbert spaces.

Example 6.1.7 (L^2 spaces). Let (X, A, μ) be a measure space, where the measure μ is defined on the σ -algebra A of subsets of X . Let

$$L^2 = \{f : X \rightarrow \mathbb{C} \mid \int_X |f(x)|^2 d\mu(x) < \infty\}$$

be the vector space of square-integrable complex functions on X . If we quotient out by the equivalence relation

$$f \sim g \Leftrightarrow \int_X |f(x) - g(x)|^2 d\mu(x) = 0$$

then we get the Hilbert space $L^2(X)$. The inner product is

$$\langle f, g \rangle = \int_X f(x) \overline{g(x)} d\mu(x).$$

Remark 6.1.8. When dealing with $L^2(X)$, we will consistently abuse notation and refer to its elements as functions f , rather than equivalence classes of functions $[f]$.

Example 6.1.9 (Direct sum of Hilbert spaces). For a family $\{H_i : i \in I\}$ of Hilbert spaces, we can define the direct sum

$$\bigoplus_{i \in I} H_i$$

to be the complex vector space of $(x_i)_{i \in I}$ such that

$$\sum_{i \in I} \|x_i\|^2 < \infty.$$

When given the inner product

$$\langle (x_i), (y_i) \rangle = \sum_{i \in I} \langle x_i, y_i \rangle$$

this is a Hilbert space.

We will now recall some properties that follow immediately from the definition of an inner product space (we will not state their proofs).

Lemma 6.1.10 (Parallelogram rule). For $x, y \in H$ we have

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Lemma 6.1.11 (Apollonius). Let $a, b, p \in H$ and let $m = \frac{1}{2}(a + b)$. Then

$$\|p - m\|^2 + \frac{1}{4}\|a - b\|^2 = \frac{1}{2}\|p - a\|^2 + \frac{1}{2}\|p - b\|^2.$$

Proposition 6.1.12 (Cauchy-Schwarz inequality). Let $\langle \cdot, \cdot \rangle$ be an inner product on a vector space H . Then

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$$

for all x and y in H . Moreover, equality occurs if and only if there are elements $\alpha, \beta \in \mathbb{C}$, not both zero, such that $\alpha x = \beta y$.

Definition 6.1.13 (Topological group). A topological group G is a group together with a Hausdorff topology on G such that the maps

$$G \times G \rightarrow G : (g, h) \mapsto gh \text{ and } G \rightarrow G : g \mapsto g^{-1}$$

are continuous.

A locally compact topological space is a topological group such that the topology is locally compact, that is, each point has a compact neighbourhood.

Remark 6.1.14. The continuity conditions in previous definition can be thought of as the requirement that the algebraic and topological structures on G be compatible. Nonetheless, we note that local compactness is defined as a purely topological property that has no direct relation to the group structure (although the continuity requirement enforces an indirect relation, so for instance it suffices that any element of the group has a compact neighbourhood since G acts on itself transitively by homeomorphisms).

Remark 6.1.15. Not all authors require a topological group G to be Hausdorff. However, since the axioms imply that translation and inversion are homeomorphisms of G , it is a simple exercise to show that if G satisfies the T_0 separation axiom then it is actually T_2 (Hausdorff):

If U is an open neighbourhood of x but not y , then $yU^{-1}x$ is an open neighbourhood of y but not x . (In fact, the topology will be $T_{3\frac{1}{2}}$, that is, completely regular Hausdorff.) Stated differently, if a topological group is not Hausdorff, then the elements of G are not topologically distinguishable in general, so it is a coarse topology. So the convenient requirement that the topology be Hausdorff is rather mild.

Example 6.1.16. The following are several familiar topological groups. They are all locally compact.

1. Any group is a topological group when endowed with the discrete topology. (Any finite topological group is generally assumed to have the discrete topology.)
2. The additive groups $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are topological groups under the usual topology given by the Euclidean metric.
3. The general linear group

$$GL(n, \mathbb{R}) = \{T : \mathbb{R}^n \rightarrow \mathbb{R}^n : T \text{ is invertible}\}$$

is a topological group. The topology is defined by identifying $GL(n, \mathbb{R})$ with a subset of \mathbb{R}^{n^2} , as each operator T can be identified with an $n \times n$ matrix. (The euclidean metric topology on \mathbb{R}^{n^2} is then used).

Definition 6.1.17. A group representation of a group G on a vector space V is a group homomorphism

$$\pi : G \rightarrow GL(V)$$

of G into the general linear group on V , that is, the group of invertible linear operators on V . A representation π is irreducible if V has no non-trivial subspace W that is invariant under the action of G , that is, no subspace $\{0\} \neq W \subsetneq V$ such that

$$\pi(g)w \in W$$

for all $g \in G$ and $w \in W$.

We will mostly be interested in unitary representations. These are defined on Hilbert spaces and are required to preserve the geometric structure, that is, each $\pi(g)$ must be a surjective isometry. Furthermore, we demand a continuity condition.

Definition 6.1.18 (Isometry). A linear map $T : H \rightarrow H'$ between inner product spaces is an isometry if it preserves the inner product, that is,

$$\langle Tu, Tv \rangle = \langle u, v \rangle$$

for all $u, v \in H$.

Definition 6.1.19 (Unitary group). Let H be a Hilbert space over \mathbb{C} . The unitary group on H , denoted $U(H)$, is the set of all linear surjective isometries $T : H \rightarrow H$.

Lemma 6.1.20. The unitary group $U(H)$ under composition of maps is indeed a group.

Proof. It is clear that the identity of H is a surjective isometry. If $S, T \in U(H)$, then ST is surjective and

$$\langle STu, STv \rangle = \langle S(Tu), S(Tv) \rangle = \langle Tu, Tv \rangle = \langle u, v \rangle$$

for all $u, v \in H$, so that $ST \in U(H)$. Finally, each $T \in U(H)$ is an isometry, so $\|Tv\| = \|v\|$ for all $v \in H$ and thus T is injective. So, $T : H \rightarrow H$ is a linear bijection, and has an inverse T^{-1} . Its inverse is an isometry since

$$\langle T^{-1}u, T^{-1}v \rangle = \langle TT^{-1}u, TT^{-1}v \rangle = \langle u, v \rangle$$

for all $u, v \in H$. ■

We are now able to define unitary representations. We refer to [[P], Definition A.1.1].

Definition 6.1.21 (Unitary representation). A unitary representation of a topological group G on a Hilbert space H is a group homomorphism $\pi : G \rightarrow U(H)$ which is strongly continuous in the sense that the mapping

$$G \rightarrow H : g \mapsto \pi(g)\xi$$

is continuous for every vector $\xi \in H$.

Remark 6.1.22. Since any unitary operator is a bounded linear function, the map

$$H \rightarrow H : \xi \mapsto \pi(g)\xi$$

is continuous for each $g \in G$. We can actually say more : the evaluation map $G \times H \rightarrow H$ is jointly continuous.

Lemma 6.1.23. Let $\pi : G \rightarrow U(H)$ be a strongly continuous unitary representation. Then the map

$$G \times H \rightarrow H : (g, \xi) \mapsto \pi(g)\xi$$

is continuous.

We will not mention the proof here, however from now on we will assume that all unitary representations are strongly continuous.

6.2 Property (T)

We can now define the property (T).

Since the definition is quite involved and technical, we need to give some definitions regarding invariants first.

Definition 6.2.1. Let $\pi : G \rightarrow U(H)$ be a strongly continuous unitary representation of a locally compact group G . For a given $\epsilon > 0$ and $K \subseteq G$ compact, we say that a unit vector $\xi \in H$ is (ϵ, K) -invariant if

$$\sup\{\|\pi(g)\xi - \xi\| : g \in K\} < \epsilon. \quad (\star)$$

We say that π has almost invariant vectors if, for all (ϵ, K) , there exists an (ϵ, K) -invariant unit vector. Finally, we say that π has non-zero invariant vectors if there exists $\eta \in H$ with $\eta \neq 0$ such that $\pi(g)\eta = \eta$ for all $g \in G$.

Remark 6.2.2. We restrict to unit vectors in the definition of invariant vectors because otherwise by scaling down any given ξ to have sufficiently small norm we could always satisfy Equation (\star) . Alternatively, we could require that

$$\sup\{\|\pi(g)\xi - \xi\| : g \in K\} < \epsilon\|\xi\|$$

with no restriction on $\xi \in H$. We could then normalize such a ξ if desired. Indeed the restriction inequality means that $\xi \neq 0$.

We illustrate the definition of invariant vectors with the following lemmas which we will need later.

Lemma 6.2.3. If $\xi \in H$ is (ϵ, K) -invariant, then it is $(\epsilon, K \cup K^{-1})$ -invariant.

Proof. For each $g \in K$, since $\pi(g^{-1})$ is an isometry we have that

$$\|\pi(g)\xi - \xi\| = \|\pi(g^{-1})(\pi(g)\xi - \xi)\| = \|\pi(g^{-1}\xi - \xi\|.$$

■

Lemma 6.2.4. Suppose that $\xi \in H$ is (ϵ, K) -invariant. Let $n \in \mathbb{N}$ and

$$K^n = \{k_1 \cdots k_n \mid k_1, \dots, k_n \in K\}.$$

Then, ξ is $(n\epsilon, K^n)$ -invariant.

Proof. The proof of this lemma is pretty straightforward by letting $\delta = \sup\{\|\pi(g)\xi - \xi\| : g \in K\} < \epsilon$ and using triangle inequality.

■

Definition 6.2.5 (Property (T)). A locally compact group G has property (T), or is a Kazhdan group, if any (strongly continuous) unitary representation of G which has almost invariant vectors has a non-zero invariant vector.

Remark 6.2.6. The T in Property (T) stands for trivial. One can equip the unitary dual of G , consisting of equivalence classes of irreducible unitary representations of G and denoted \hat{G} , with a natural topology called the Fell topology.

Property (T) then amounts to saying by G that the trivial representation is an isolated point in the Fell topology on \hat{G} .

However, we will not consider \hat{G} and the Fell topology further, preferring instead the more hands on will not consider G invariant vectors definition.

Definition 6.2.7. *Given a finitely generated group Γ and a family of finite index normal subgroups $L = \{N_i\}$, let $R = \{\phi \in \Gamma \mid \ker \phi \supseteq N_i \text{ for some } i\}$. We say that Γ has property (T) if Γ has property (T : R) with respect to the family of all finite index normal subgroups.*

Theorem 6.2.8. *$SL_3(\mathbb{R})$ has property (T).*

Proof. Let p be a unitary representation of $SL_3(\mathbb{R})$ that weakly contains the trivial representation p_0 . We must find a nonzero invariant vector. Let

$$H = \left\{ \begin{pmatrix} \alpha & b & r \\ c & d & s \\ 0 & 0 & 1 \end{pmatrix} \in G \right\} \simeq J \times SL_2(\mathbb{R}) \simeq \mathbb{R}^2 \times SL_2(\mathbb{R}).$$

By assumption p restricted to H weakly contains the trivial representation. Also $p|_H$ contains a vector invariant under $J \simeq \mathbb{R}^2$ and this vector is invariant under $SL_2(\mathbb{R})$. Hence, $SL_3(\mathbb{R})$ has property (T). ■

With this condition we are finally ready to prove that we can construct expanders from the Cayley graphs of quotient subgroups satisfying property (T).

6.3 Kazhdan Expanders

In this section we present constructions of expanders using property (T).

Note that, property (T) can be understood as a pushing property : if a unitary representation of a Kazhdan group does not have non-zero invariant vectors then it does not have almost invariant vectors, that is, there is some ϵ such that all unit vectors are moved by at least ϵ by some element of a Kazhdan set. It is an essential result that there exists a universal ϵ that suffices for any unitary representation of the group as we studied at the previous section.

The construction of expanders using property (T) leads to the following result.

Proposition 6.3.1 ([F], Proposition 3.3.1). *Let Γ be a discrete Kazhdan group. Suppose that (N_i) be a family of finite index normal subgroups of Γ such that the indices $|\Gamma/N_i| \rightarrow \infty$ as $i \rightarrow \infty$.*

Suppose furthermore that $S \subset \Gamma$ is a finite symmetric, that is, $S^{-1} = S$, generating set for Γ , and let S_i be the image of S in Γ/N_i (under the quotient map). Then the family $\text{Cay}(\Gamma/N_i, S_i)$ is a family of (n_i, k, c) -expanders for some $c > 0$, $k = |S|$ and $n_i = |\Gamma/N_i|$.

Proof. By known corollary, there exists $\epsilon > 0$ such that any unitary representation of Γ with an (ϵ, S) -invariant vector has a non-zero invariant vector. When expressed in the contrapositive, this means that if a unitary representation $\pi : \Gamma \rightarrow U(H)$ does not have a non-zero invariant vector, then no unit vector is (ϵ, S) -invariant, and thus each $\xi \in H$ satisfies

$$\sup\{\|\pi(s)\xi - \xi\| : s \in S\} \geq \epsilon\|\xi\|.$$

As S is finite, for all $\xi \in H$ there exists $s \in S$ such that

$$\|\pi(s)\xi - \xi\| \geq \epsilon\|\xi\|.$$

Fix a particular N_I and S_i , and let $V_i = \Gamma/N_i$. Consider the representation of Γ on $H = L^2(V_i)$ defined by

$$(g \cdot f)(x) = f(xg)$$

for all $f \in L^2(V_i)$, $x \in V_i$. This is the pull back onto Γ of the right-regular representation of V_i . Since V_i is discrete, if a function $f \in H$ is invariant, then

$$(g \cdot f)(N_i) = f(N_i g)$$

for all $g \in \Gamma$, where $e \in V_i$ is the identity.

The action of Γ on V_i by right-multiplication is transitive, so we can make the argument $N_i g$ any element of V_i . Thus f is constant. So, we consider the subspace

$$H_0 = \left\{ f : V_i \rightarrow \mathbb{C} \mid \sum_{x \in V_i} f(x) = 0 \right\}.$$

The only constant function $f \in H_0$ is zero. Thus the unitary representation $\pi : \Gamma \rightarrow U(H)$ given by the right action of Γ on V_i does not have non-zero invariant vectors. Now, let $V_i = A \sqcup B$ where $|A| \leq |B|$, and write $\alpha = |A|$ and $b = |B|$. The characteristic function for A in H_0 is

$$f_A(x) = \begin{cases} b & \text{if } x \in A \\ -\alpha & \text{if } x \in B \end{cases}.$$

By the discussion above, since (ϵ, S) is a Kazhdan pair for Γ , there is some $s \in S$ such that

$$\|s \cdot f_A - f_A\| \geq \epsilon \|f_A\|. \quad (1)$$

We can easily evaluate both sides of this inequality. Since

$$(s \cdot f_A)(x) = \begin{cases} a + b & \text{if } x \in B \text{ and } xs \in A \\ -a - b & \text{if } x \in A \text{ and } xs \in B \\ 0 & \text{otherwise} \end{cases}.$$

Let $E_s(A, B)$ denote the set of edges between A and B that are due to the generator s . Then $|E_s(A, B)| = |\{x \in B \mid xs \in A\} \cup \{x \in A \mid xs \in B\}|$, or half that in the case that $s^2 = 1$. In either case, we have that

$$|E_s(A, B)| \geq \frac{1}{2} \|s \cdot f_A - f_A\|^2 / (\alpha + b)^2. \quad (2)$$

On the other hand,

$$\|f_A\|^2 = |A|b^2 + |B|\alpha^2 = \alpha b(\alpha + b). \quad (3)$$

Putting together equations (1), (2) and (3), we have that

$$|E(A, B)| \geq |E_s(A, B)| \geq \frac{1}{2} \epsilon^2 \alpha b(\alpha + b) / (\alpha + b)^2 = \frac{\epsilon^2}{2} \alpha b / (\alpha + b).$$

Since we assumed $|A| \leq |B|$, we have that $b / (\alpha + b) \geq \frac{1}{2}$, so this gives

$$\frac{|E(A, B)|}{\min\{|A|, |B|\}} \geq \frac{\epsilon^2}{4}.$$

As the partition $V = A \sqcup B$ was arbitrary, we can conclude that

$$h(\text{Cay}(V_i, S_i)) \geq \frac{\epsilon^2}{4}.$$

As ϵ is independent of N_i, S_i (we choose $\epsilon > 0$ so that (ϵ, S) is a Kazhdan pair for $SL(3, \mathbb{Z})$), it follows that the Cayley graphs form a family of expanders as required. ■

Remark 6.3.2. *There are several parallels between the above proof and the proof that spectral expanders are combinatorial expanders. In the same way that u is always a 1-eigenvector for \hat{A} , we saw that a constant f is always invariant. Considering the eigenvectors orthogonal to u is essentially the same as considering the invariant subspace H_0 .*

Remark 6.3.3. *At first it might seem unusual that we achieve the expansion of the above construction by considering only the edges $E_s(A, B)$ owing to one particular generator s for each choice of the partition $V = A \sqcup B$. However, since there are only k generators, some generator will always contribute at least $\frac{1}{k}$ of the edges $E(A, B)$ of a cut, so for a Cayley graph expander one can always just show that $|E_s(A, B)|$ is bounded from below by some constant multiple of $\min\{|A|, |B|\}$.*

Remark 6.3.4. *This construction does not use the full power of property (T), as highlighted by the fact that we are only considering finite-dimensional representations. In fact, all that is required is that any representation of Γ which factors through a finite quotient of Γ and has almost invariant vectors has a non-zero invariant vector.*

A group Γ has property (τ) relative to L , a particular family of finite index normal subgroups, if any unitary representation of Γ with almost invariant vectors that factors through some quotient Γ/N with $N \in L$ has a non-zero invariant vector.

Remark 6.3.5. *The group has Property (τ) if it has Property (τ) with respect to the family of all finite index normal subgroups. The family of Cayley graphs $\text{Cay}(\Gamma/N, S)$ for $N \in L$ will be a family of expanders if and only if Γ has Property (τ) with respect to L . Property (τ) is an interesting topic of study because although it is weaker than Property (T), consequently more groups have Property (τ) .*

This implies that $SL(2, \mathbb{Z})$ has Property (τ) with respect to the family of principle congruence subgroups, that is, the kernels of the natural maps $SL(2, \mathbb{Z}) \rightarrow SL(2, \mathbb{Z}/m\mathbb{Z})$. More details of this topic can be seen in the book by Lubotzky and Zuk.

Another comment that we can make on the above proof is that we did not actually use the fact that the N_i are normal in Γ . In order to call the graphs Cayley graphs, this is required. However, we can generally consider Schreier graph which is like a Cayley graph, except that the vertices are cosets of Γ/H .

Lemma 6.3.6. *Define*

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

Then $S = \{A, A^{-1}, B, B^{-1}\}$ generate $SL(3, \mathbb{Z})$.

Construction

Let $S = \{A^{\pm 1}, B^{\pm 1}\}$, where A and B are as in the previous lemma. We then define a family of graphs by taking the Cayley graphs $\text{Cay}(SL(3, \mathbb{Z}/p\mathbb{Z}), S)$ with those generators.

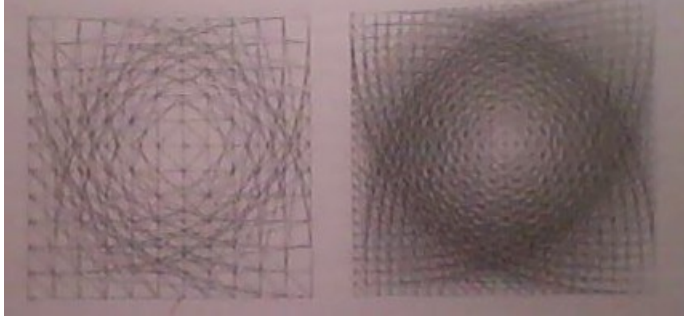
Proposition 6.3.7. *The above construction is a family of expanders.*

Proof. For any p prime, let $\phi : SL(3, \mathbb{Z}) \rightarrow SL(3, \mathbb{Z}/p\mathbb{Z})$ be the natural homomorphism defined by mapping each entry α_{ij} of $\alpha \in SL(3, \mathbb{Z})$ to $\alpha_{ij} \pmod{p}$. Because the map $\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$ is a ring homomorphism, and the determinant is defined only in terms of addition and multiplication of matrix entries, the image of any element of $SL(3, \mathbb{Z})$ will be indeed in $SL(3, \mathbb{Z}/p\mathbb{Z})$.

The group $SL(3, \mathbb{Z}/p\mathbb{Z})$ is finite, so the kernels of those homomorphisms define finite index normal subgroups N_i . This actually completes the proof by applying this to proposition(6.3.1) of Kazhdan groups. ■

6.3.1 Margulis's Construction of Expanders

We have already mentioned Margulis construction but we are now ready to give more details. We also give the original construction of Margulis, which illustrated for $m = 12$ and $m = 24$ in the below figure.



Construction[Margulis, [Q]]

Let m be a positive integer and $V_m = (\mathbb{Z}/m\mathbb{Z})^2$.

Define a graph on the set V_m by connecting every $(\alpha, b) \in V_m$ to $\sigma_1(\alpha, b) = (\alpha+1, b)$, $\sigma_2(\alpha, b) = (\alpha, b+1)$, $\sigma_3(\alpha, b) = (\alpha, \alpha + b)$ and $\sigma_4(\alpha, b) = (-b, b)$.

Proposition 6.3.8. *The graphs (V_m) in the above construction are a family of expanders.*

Proof. We cannot apply Kazhdan's group proposition directly since $\Gamma = \mathbb{Z}^2 \rtimes SL(\mathbb{Z}, 2)$ does not have the property (T), but only the relative property (T) with respect to the subgroup \mathbb{Z}^2 . The group Γ acts naturally on V_m by affine transformations :

$\left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} p & q \\ r & s \end{pmatrix} \right) \cdot \begin{pmatrix} \alpha \\ b \end{pmatrix} = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \begin{pmatrix} \alpha \\ b \end{pmatrix} + \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p\alpha + qb + x \\ r\alpha + sb + y \end{pmatrix}$. To see that this is indeed a group action, note that the identity $(0, 1)$ of $\mathbb{Z}^2 \rtimes SL(\mathbb{Z}, 2)$ does indeed act as the identity, and for arbitrary

$$(t_1, r_1), (t_2, r_2) \in \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$$

and

$$x \in (\mathbb{Z}/m\mathbb{Z})^2$$

we have that

$$\begin{aligned} (t_1, r_1) \cdot (t_2, r_2) \cdot x &= (t_1, r_1) \cdot (r_2x + t_2) \\ &= r_1(r_2x + t_2) + t_1 = r_1r_2x + (r_1t_2 + t_1) \\ &= (r_1r_2, r_1t_2 + t_1) \cdot x = ((r_1, t_1)(r_2, t_2)) \cdot x \end{aligned}$$

by the definition of the semidirect product (where we took the standard action of $SL(2, \mathbb{Z})$ on \mathbb{Z}^2).

We can express the σ_i as the actions of particular elements of the two groups used to construct the semidirect product $\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$:

$$\sigma_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{Z}^2, \sigma_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{Z}^2, \sigma_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{Z}), \sigma_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in$$

$SL(2, \mathbb{Z})$. It is obvious that σ_1 and σ_2 generate \mathbb{Z}^2 , and not too hard to see that σ_3 and σ_4 generate $SL(2, \mathbb{Z})$, so that together they generate the entire semidirect product $\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$.

The action of Γ on V_m gives rise to a unitary representation of Γ on $L^2(V_m)$, by

$$(g \cdot f)(x) = f(g \cdot x)$$

for all $x \in V_m$, $f \in L^2(V_m)$.

Now, similarly to the proof of Kazhdan group, the only \mathbb{Z}^2 -invariant vectors are the constant functions, since the action of \mathbb{Z}^2 on V_m is transitive. So if we quotient out by the constant functions, considering the subspace of functions that sum over V_m to zero, there is some ϵ such that each unit vector is moved at least ϵ by one of the four generators. Considering characteristic functions, it follows that these graphs form a family of expanders. ■

An interesting point of comparison between the last two constructions is that in the latter we do not require the generation of large primes. Moreover, given any particular vertex in these expanders, there is a polynomial time algorithm, in fact a linear time algorithm, to compute its neighbours. It is thus a 'very explicit construction. This conclusion can be seen in [[C], p.453].

Bibliography

- [A] Reinhard Diestel. Graduate texts in Mathematics. *Graph theory*, pages 2-28.
- [B] Varsos, Deriziotis, Emmanouil, Maliakas, Melas, Talelli. *Introduction to Linear Algebra* : National and Kapodistrian University of Athens. Sofia press, ISBN 978-960-6706-19-6, pages 34-61, 99-115, 200, 240-248.
- [C] S. Hoory, N. Linial, and A. Wigderson. *Expander graphs and their applications*. Bulletin of the American Mathematical Society, 43(4):439-562, 2006.
- [D] M. Krebs and A. Shaheen. *Expander families and Cayley Graphs* : A Beginner's Guide. pages: 95-102. Oxford University Press, USA, 2011.
- [E] M. Pinsker. On the complexity of a concentrator. In *7th Annual Teletraffic Conference*, pages 318/1-318/4, Stockholm, 1973.
- [F] A. Lubotzky. *Discrete Groups, Expanding Graphs and Invariant Measures*. Birkhauser, 1994.
- [G] A. Lubotzky and B. Weiss. *Groups and expanders*. pages:95-109. DIMACS series Vol. 10, 1993.
- [H] M. Kassabov. *Constructions of the dual space of a group with the structure of its close subgroups*. Functional analysis and its applications, 1(1):63-65, 1967.
- [I] M. Kassabov. *Symmetric groups and expander graphs*. Inventiones Mathematicae / Discrete Math lecture , Institute for Advanced study, 2009.
- [J] N. Alon, A. Lubotzky, and A. Wigderson. Semi-direct product in groups and zig-zag product in graphs : connections and applications. In *Foundations of computer Science*, 2001. Proceedings. 42nd IEEE Symposium on, pages 630-637. IEEE, 2001.
- [K] S. Axler. *Linear Algebra done right*. Springer, 1997.
- [L] M. Blum, R.M. Karp, O. Vorneberger, D.H. Papadimitriou, and M. Yannakakis. *The complexity of testing whether a graph is a superconcentrator*. Inform. Process. Lett., 13(4):164-167, 1981.
- [M] A. Lubotzky, R. Phillips, and P. Sarnak. *Ramanujan graphs*. Combinatorica, 8(3):261-277, 1988.
- [N] G. Margulis. *Explicit group-theoretic constructions of combinatorial schemes and their applications in the construction of expanders and concentrators*. Probl. of Inform. Transm., 24(1):51-60, 1988.
- [O] D.A. Kazhdan. *Connection of the dual space of a group with the structure of its close subgroups*. Functional analysis and its applications, 1(1) 63-65, 1967.

- [P] B. Bekka, P. de la Harpe, and A. Valette. *Kazhdan's Property (T)*. New mathematical monographs: 11. Cambridge University Press, 2008.
- [Q] G. Margulis. *Explicit constructions of concentrators*. Problems of Information Transmission, 1973, 9:4, 325-332.
- [R] Bilu, Yonatan, and Nathan Linial. *Lifts, discrepancy and nearly optimal spectral gap**. Combinatorica 26.5 (2006): 495-519
- [S] Branden, Petter. *Lecture Notes For Interlacing Families*. Royal Institute of Technology, Stockholm, Sweden, 2013. <https://people.kth.se/~pbranden/notes.pdf>
- [T] Brouwer, Andries E., and Willem H. Haemers. *Spectra of graphs*. Springer, pages : 1-12, 33-58, 67-72, 91. 2011. <http://www.win.tue.nl/~aeb/2WF02/spectra.pdf>
- [U] Chung, Fan RK. *Laplacians of graphs and Cheegers inequalities*. Proc. Int. Conf. *Combinatorics, Paul Erdos is Eighty*, Keszthely (Hungary). Vol. 2. 1993. <http://www.math.ucsd.edu/~fan/wp/cheeger.pdf>
- [V] Csikvari, Peter. *Graph Polynomials and Graph Transformations in Algebraic Graph Theory*. Diss. PhD thesis. Eotvos Lorand University 2011.:
- [W] Gharan, Shayan Oveis, and Luca Trevisan. *A Universal upper bound on Graph Diameter based on Laplacian Eigenvalues*. arXiv preprint arXiv:1212.2701 (2012). <http://arxiv.org/pdf/1212.2701v1.pdf>
- [X] Haemers, Willem H. *Interlacing eigenvalues and graphs*. Linear Algebra and its Applications 226 (1995): 593-616. <http://members.upc.nl/w.haemers/interlacing.pdf>
- [Y] Hoory, Shlomo, Nathan Linial, and Avi Wigderson. 2006. *Expander Graphs and Their Applications*. Bulletin of the American Mathematical Society, 43(4): 439-561. S 0273-0979(06)01126-8
- [Z] Laszlo Lovasz, *Lecture*. Eotvos Lorand University, Budapest. <http://www.cs.elte.hu/~lovasz/eigenvals-x.pdf>
- [AA] Lubotzky, Alexander, Ralph Phillips, and Peter Sarnak. *Ramanujan graphs*. Combinatorica 8.3 (1988): 261-277.
- [AB] Lubotzky, Alexander. *Expander graphs in pure and applied mathematics*. Bulletin of the American Mathematical Society 49.1 (2012): 113-162. <http://arxiv.org/pdf/1105.2389v1.pdf>
- [AC] Morgenstern, Moshe. *Existence and Explicit Constructions of $q + 1$ Regular Ramanujan Graphs for Every Prime Power q* . Journal of Combinatorial Theory, Series B 62.1 (1994): 44-62.
- [AD] Murty, M. Ram. *Ramanujan graphs*. Journal-Ramanujan Mathematical Society 18.1 (2003): 33-52. <http://www.mast.queensu.ca/~murty/ramanujan.pdf>
- [AE] Nilli, Alon. *On the second eigenvalue of a graph*. Discrete Mathematics 91.2 (1991): 207-210. <http://www.sciencedirect.com/science/article/pii/0012365X9190112F>
- [AF] Spielman, Daniel. *Spectral graph theory*. Lecture Notes, Yale University (2009): 740-0776.
- [AG] Trevisan, Luca. *Lecture*. CS366. Stanford University, Stanford, CA. 4 March 2013. <http://cs.stanford.edu/people/trevisan/cs366/lecture13.pdf>

- [AH] Trevisan, Luca. *Extractors and pseudorandom generators*. J. ACM, 48 (4) (2001), pp. 860-879 <http://theory.stanford.edu/~trevisan/pubs/extractor-full.pdf>
- [AI] Vadhan, Salil. *Randomness Extractors*. Pseudorandomness. : monograph in preparation for FnTCS, 2013. Print. <http://people.seas.harvard.edu/~salil/pseudorandomness/extractors.pdf>
- [BB] N. Alon. *Eigenvalues and expanders*. Combinatorica, Vol. 6, pages 80-96, 1986.
- [BC] N. Alon and J. H. Milman. λ_1 , *Isoperimetric Inequalities for Graphs and Superconcentrators*. J. Combinatorial Theory, Ser. B, Vol. 38, pages 73-88, 1985.
- [CC] E. Kowalski. *Expander graphs*. <https://people.math.ethz.ch/~kowalski/expander-graphs.pdf>
- [DA] O. Gabber and Z. Galil. *Explicit constructions of linear size superconcentrators*. Journal of Computer and System science, Vol. 22, pages 407-420, 1981.
- [EA] O. Reingold, S. Vadhan, and A. Wigderson. *Entropy Waves, the Zig-Zag Graph Product, and New Constant-Degree Expanders and Extractors*. Annals of Mathematics, Vol. 155 (1), pages 157-187, 2001. Preliminary version in 41st FOCS, pages 3-13, 2000.
- [FA] Andrew Brown. Thesis: *Codes, Graphs and Graph Based Codes*, EPFL.
- [GA] P. Diaconis and M. Shahshahani. *Generating a random permutation with random transpositions*. Z Wahrsch. Verw. Gebiete 57(1981) 159-179.
- [HA] Arora, Sanjeev, and Boaz Barak. *Computational complexity : a modern approach*. Cambridge University Press, 2009.
- [LA] Salil. Pseudorandomness. Chapter 4. <http://people.seas.harvard.edu/~salil/pseudorandomness/expanders.pdf>
- [KA] Robert Krauthgamer. Advanced Algorithms 2012A. Lecture 7: *Cheeger's inequality*. <http://www.wisdom.weizmann.ac.il/~robi/teaching/2012a-AdvancedAlgorithms/lecture7>
- [NA] Thomas Sauerwald and He Sun. *Spectral Graph Theory and Applications*. Lecture 3: Expander Mixing Lemma <https://resources.mpi-inf.mpg.de/departments/d1/teaching/ws11/SGT/Lecture3.pdf>
- [R - V - W] O. Reingold, S. Vadhan, and A. Wigderson, Ann. of Math. 155 (2002), 157-187.
- [Co] J.B. Conway. *A course in functional analysis*, volume 96. Springer, 1990.