Information and Default Risk in Financial Valuation

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Abstract

This thesis consists of an introduction and five articles in the field of financial mathematics. The main topics of the papers comprise credit risk modelling, optimal stopping theory, and Dynkin games. An underlying theme in all of the articles is valuation of various financial instruments. Namely, Paper I deals with valuation of a game version of a perpetual American option where the parties disagree about the distributional properties of the underlying process, Papers II and III investigate pricing of default-sensitive contingent claims, Paper IV treats CVA (credit value adjustment) modelling for a portfolio consisting of American options, and Paper V studies a problem motivated by model calibration in pricing of corporate bonds.

In each of the articles, we deal with an underlying stochastic process that is continuous in time and defined on some probability space. Namely, Papers I-IV treat stochastic processes with continuous paths, whereas Paper V assumes that the underlying process is a jump-diffusion with finite jump intensity.

The information level in Paper I is the filtration generated by the stock value. In articles III and IV, we consider investors whose information flow is designed as a progressive enlargement with default time of the filtration generated by the stock price, whereas in Paper II the information flow is an initial enlargement. Paper V assumes that the default is a hitting time of the firm's value and thus the underlying filtration is the one generated by the process modelling this value.

Moreover, in all of the papers the risk-free bonds are assumed for simplicity to have deterministic prices so that the focus is on the uncertainty coming from the stock price and default risk.

*Keywords:* pricing, valuation, American options, Dynkin games, optimal stopping problem, optimal stopping games, credit risk, default risk, information, filtration, enlargement of filtrations

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List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.


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1. Introduction

This thesis consists of an introduction and five papers, which deal with problems in financial mathematics. The main areas of the articles comprise credit risk modelling, optimal stopping theory, and Dynkin games. In the introduction, we present an overview of the key results in the aforementioned fields that can be considered as building blocks of the whole thesis.

The main theme in all of the papers is the valuation problem. Namely, Paper I deals with valuation of a game version of a perpetual American option where the parties do not agree on the distributional properties of the underlying process, Papers II and III study pricing of default-sensitive contingent claims, Paper IV treats CVA (credit value adjustment) modelling for a portfolio consisting of American options, and Paper V investigates an inverse problem coming from model calibration for pricing of corporate bonds.

In each of the articles, we have an underlying stochastic process in continuous time defined on some probability space. Specifically, Papers I-IV treat continuous stochastic processes, whereas Paper V assumes that the underlying process is a jump-diffusion with finite jump intensity. The information level in Paper I is the filtration generated by the stock value. In articles III and IV, we consider investors whose information flow is designed as a progressive enlargement with default time of the filtration generated by the stock price, whereas in Paper II the information flow is an initial enlargement. Paper V assumes that the default is a hitting time of the firm’s value and thus the underlying filtration is the one generated by the process modelling this value. Moreover, in all of the papers the risk-free bonds are assumed for simplicity to have deterministic prices so that the focus is on the uncertainty coming from the stock price and default risk.

The introduction is organised as follows. In Section 1.1, we present the martingale theory of option pricing, where the market consists of a risky asset, which is modelled by a strictly positive semi-martingale, and a deterministic risk-free bond. Specifically, we demonstrate the results about the connection between the completeness of the market and the existence of a unique equivalent martingale measure. Then, we consider the pricing problem of a European claim and show that its price is unique and given by the mathematical expectation of the discounted pay-off under the equivalent martingale measure. As an example, we assume the classical Black-Scholes model and give the price of a European call option.

Section 1.2 deals with credit risk. We discuss two approaches to default modelling: the structural approach and the reduced-form approach as well as study a number of associated examples.
In Section 1.3 we investigate the valuation of perpetual American options which reduces to an optimal stopping problem. Finally, Section 1.4 introduces zero-sum Dynkin games (optimal stopping games) and gives the definitions of a value of the game and a Nash equilibrium.

1.1 Martingale Theory of Derivative Pricing

Let $T > 0$ be a time horizon. Fix a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$, where $\mathbb{F}$ is a filtration satisfying the usual hypothesis generated by a continuous, positive semi-martingale $X = (X_t)_{t \in [0,T]}$. Consider a financial market consisting of a stock $X$ and a risk-free bond $B = (B_t)_{t \in [0,T]}$, where

$$dB_t = rB_t dt, \quad B_0 = 1,$$

and $r > 0$ is a constant risk-free interest rate.

A trading strategy is a pair $(\gamma, \theta) = (\gamma_t, \theta_t)_{t \in [0,T]}$ of $\mathbb{F}$-predictable stochastic processes, where $\gamma_t$ and $\theta_t$ represent the number of units at time $t$ of the risk-free bond $B_t$ and the stock $X_t$, respectively.

The value $V(\gamma, \theta) = (V_t(\gamma, \theta))_{t \in [0,T]}$ of the trading strategy $(\gamma, \theta)$ is given by

$$V_t(\gamma, \theta) := \gamma_t B_t + \theta_t X_t, \quad \forall t \in [0, T].$$

We say that a trading strategy $(\gamma, \theta)$ is self-financing, if

$$\int_0^t \gamma_s dB_s + \int_0^t \theta_s dX_s$$

is well-defined for any $t \in [0, T]$ and

$$V_t(\gamma, \theta) = V_0(\gamma, \theta) + \int_0^t \gamma_s dB_s + \int_0^t \theta_s dX_s, \quad \forall t \in [0, T].$$

Moreover, a self-financing strategy is admissible if it is lower bounded.

A trading strategy $(\gamma, \theta) \in \mathcal{A}$ is an arbitrage if its initial value equals zero, it gives an increase between time $t = 0$ and $t = T$ in value almost surely and a strict increase with a strictly positive probability, i.e. $V_0(\gamma, \theta) = 0$, $\mathbb{P}(V_T(\gamma, \theta) > 0) = 1$, and $\mathbb{P}(V_T(\gamma, \theta) > 0) > 0$.

A sufficient condition for the market to be arbitrage-free is the existence a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$, called an equivalent martingale measure, under which the discounted stock price process $(e^{-rt}X_t)_{t \in [0,T]}$ is a martingale (see for example Lemma 12.1.6. in [8]). In fact, it can be shown that the existence of an equivalent martingale measure implies a stronger condition No Free Lunch with Vanishing Risk (NFLVR). Conversely, the existence of an equivalent martingale measure follows from NFLVR. This result is known as The First Fundamental Theorem of Asset Pricing.
Another main result in financial mathematics uses the notion of completeness. Namely, we say that a market is complete if any $\mathcal{F}_T$-measurable, square-integrable contingent claim $\Phi_T$ is attainable, i.e. there exists $(\gamma, \theta) \in \mathcal{A}$ such that
\[ \Phi_T = V_0^{(\gamma, \theta)} + \int_0^T \gamma_s dB_s + \int_0^T \theta_s dX_s \quad \text{a.s.} \]
The Second Fundamental Theorem of Asset Pricing states that a market is complete if and only if there exists a unique equivalent martingale measure.

A classical problem in financial mathematics is the pricing of financial derivatives, i.e. contracts with pay-offs that depend on the traded underlying asset $X$. In this section we restrict ourselves to the so-called European $T$-claims, i.e. contracts that promise to pay the amount $\Phi_T$ at the maturity time $T > 0$. American claims are studied in Section 1.3.

We distinguish two agents: a seller and a buyer of a given European $T$-claim. The buyer purchases at time $t = 0$ the contract from the seller who guarantees the delivery of the promised pay-off $\Phi_T$ at time $t = T > 0$. The question we address is the following: what is the highest price $c(\Phi_T)$ that the buyer is willing to pay for this derivative and what is the lowest price $C(\Phi_T)$ which the seller accepts? Moreover, we are interested in a second problem: under which conditions are the prices $c(\Phi_T)$ and $C(\Phi_T)$ equal?

The values $c(\Phi_T)$ and $C(\Phi_T)$ are defined in terms of investors’ investment strategies. Assume that the buyer can find an admissible strategy $(\gamma, \theta)$ such that
\[ -x + \int_0^T \gamma_t dB_t + \int_0^T \theta_t dX_t + \Phi_T \geq 0 \quad \text{a.s.,} \]
where $V_0^{(\gamma, \theta)} = -x$. Then, the strategy of paying $x$ for the $T$-claim at time 0 and investing in $(\gamma, \theta)$ implies no risk for the buyer. We define the buyer’s price $c(\Phi_T)$ as the largest such $x$, i.e.
\[ c(\Phi_T) := \sup \{ x \in \mathbb{R}_+ : \exists (\gamma, \theta) \in \mathcal{A} \text{ s.t.} -x + \int_0^T \gamma_t dB_t + \int_0^T \theta_t dX_t + \Phi_T \geq 0 \quad \text{a.s.} \} \]
Similarly, the seller would be willing to sell at any price $z$ such that there exists an admissible strategy $(\gamma, \theta)$ with
\[ z + \int_0^T \gamma_t dB_t + \int_0^T \theta_t dX_t \geq \Phi_T, \quad \text{a.s.} \]
The seller’s price $C(\Phi_T)$ is defined as the smallest such value, i.e.
\[ C(\Phi_T) := \inf \{ z \in \mathbb{R}_+ : \exists (\gamma, \theta) \in \mathcal{A} \text{ s.t.} z + \int_0^T \gamma_t dB_t + \int_0^T \theta_t dX_t \geq \Phi_T \quad \text{a.s.} \} \]
Under the assumption that the market is arbitrage-free, we have that $c(\Phi_T) \leq C(\Phi_T)$ and if $c(\Phi_T) = C(\Phi_T)$, then the contract’s price $P(\Phi_T)$ is defined as
\[ P(\Phi_T) := c(\Phi_T) = C(\Phi_T). \]
It can be shown (see for example Theorem 12.3.2 in [8]) that the existence of an equivalent martingale measure $Q$ implies
\[ c(\Phi_T) \leq e^{-rT} \mathbb{E}_Q[\Phi_T] \leq C(\Phi_T), \]  
(1.1)
where $\mathbb{E}_Q[\cdot]$ denotes the mathematical expectation under measure $Q$. Let us assume that the underlying market is complete. Then, by Theorem 12.3.2 in [8] we have that
\[ c(\Phi_T) = e^{-rT} \mathbb{E}_Q[\Phi_T] = C(\Phi_T) \]
and hence the price $P(\Phi_T)$ of the contract with pay-off $\Phi_T$ is at time $t = 0$ given by
\[ P(\Phi_T) = e^{-rT} \mathbb{E}_Q[\Phi_T], \]
where $Q$ is the unique equivalent martingale measure.

**Example** Let us assume that $X = (X_t)_{t \in [0,T]}$ is a geometric Brownian motion, i.e.
\[ X_t := xe^{(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t}, \quad \forall t \in [0,T], \]
where $W = (W_t)_{t \in [0,T]}$ is a Brownian motion and $\mu, x > 0, \sigma > 0$ are constants. It is well-known that such a market is arbitrage-free and complete. In fact, the unique equivalent martingale measure $Q$ is given by
\[ \frac{dQ}{dP}|_{\mathcal{F}_T} = \exp\left\{-\frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2 T - \frac{\mu - r}{\sigma} WT \right\}. \]

The famous Black-Scholes formula, see [2], states that the price at time $t \in [0, T]$ of a European call option with pay-off function $(X_T - K)^+$, where $K$ is the strike price, is given by
\[ C(t,x) = e^{-r(T-t)} \mathbb{E}^Q_t \left[ (X_T - K)^+ \right] = x N'(d_1) - K e^{-r(T-t)} N'(d_2), \]  
(1.2)
where $\mathbb{E}^Q_t[\cdot]$ denotes the mathematical expectation under the unique martingale measure $Q$ conditional on $\{X_t = x\}$, $N(\cdot)$ denotes the probability distribution function of a standard normal random variable and
\[ d_1 := \frac{\ln \left( \frac{x}{K} \right) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}} \quad \text{and} \quad d_2 := \frac{\ln \left( \frac{x}{K} \right) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}}. \]

Furthermore, by the Feynman-Kac theorem we have that $C(t,x)$ satisfies
\[ \begin{cases} \frac{\partial C(t,x)}{\partial t} + rx \frac{\partial C(t,x)}{\partial x} + \frac{\sigma^2x^2}{2} \frac{\partial^2 C(t,x)}{\partial x^2} - rC(t,x) = 0, & (t,x) \in [0,T) \times (0, \infty) \\ C(T,x) = (x-K)^+, & x \in (0, \infty), \end{cases} \]
which is known as the Black-Scholes equation.

In Papers II and III we focus on the study of the sets of equivalent martingale measures in different filtrations.
1.2 Credit Risk

Credit risk, or default risk, deals with the case where the promised pay-off is not delivered if the default occurs before the maturity time $T > 0$. Two main approaches to default modelling have been studied in financial literature: the structural approach and the reduced-form approach.

The structural approach can be traced back to Merton (see [7]), who assumed that the default of a firm happens if the value of the firm $V = (V_t)_{t \in [0,T]}$ is insufficient to pay the debt at maturity time $T > 0$. Hence, the default can happen only at maturity. Moreover, it was presumed that the company is financed by zero-coupon defaultable corporate bonds with the same maturity time $T > 0$ and with face values that sum up to $L$. Then, at time $T > 0$, the bondholders receive $\min(V_T, L)$, which can be rewritten as

$$\min(V_T, L) = L - (L - V_T)^+,$$

where $V$ is modelled under the martingale measure $Q$ as a geometric Brownian motion, i.e.

$$dV_t = rV_t dt + \sigma V_t dW_t, \quad V_0 = v > L$$ (1.3)

for constants $r > 0$ and $L > 0$ and the default time is assumed to be a strictly positive random variable $\tau$ satisfying

$$\tau := T \mathbb{1}_{V_T \leq L} + \infty \mathbb{1}_{V_T > L}.$$

The value of all defaultable corporate bonds issued by the firm at time $t \in [0, T]$ is

$$\mathbb{E}_Q \left[ e^{-r(T-t)} \min(V_T, L) \right] = \mathbb{E}_Q \left[ e^{-r(T-t)} \min(V_T, L) | V_t = y \right]_{|y=V_t}
= Le^{-r(T-t)} - \mathbb{E}_Q \left[ e^{-r(T-t)} (L - V_T)^+ | V_t = y \right]_{|y=V_t}
= Le^{-r(T-t)} \mathcal{N}(c_1) + V_t \mathcal{N}(c_2),$$

where

$$c_1 := \frac{\ln(V_T)}{T} + \frac{(r - \frac{\sigma^2}{2})(T - t)}{\sigma \sqrt{T - t}} \quad \text{and} \quad c_2 := \frac{\ln(L)}{T} - \frac{(r + \frac{\sigma^2}{2})(T - t)}{\sigma \sqrt{T - t}}.$$ (1.4)

Merton’s model was extended by Black and Cox in [1] who assumed that default happens the first time when the firm’s value crosses a lower constant boundary $L$, i.e.

$$\tau := \inf \{ t \geq 0 : V_t \leq L \}, \quad V_0 \geq L,$$

where $V$ is a geometric Brownian motion with dynamics as in (1.3). Let $D(t, T)$ denote the price of a defaultable zero-coupon bond (DZC) with face value
value 1. Then,

\[
D(t, T) := e^{-r(T-t)} \mathbb{E}_Q[^\tau>T|\mathcal{F}_t] = e^{-r(T-t)} Q(\tau > T | \mathcal{F}_t)
\]

\[
= \mathbb{I}_{\tau>T} e^{-r(T-t)} Q(\tau > T | \mathcal{F}_t) = \mathbb{I}_{\tau>T} e^{-r(T-t)} \left( \mathcal{N}(c_1) - \frac{L}{V_t} \frac{2(r - \sigma^2)}{\sigma^2} \mathcal{N}(c_3) \right),
\]

where \( c_1 \) is as in (1.4) and

\[
c_3 := \frac{\ln \left( \frac{L}{V_t} \right) + (r - \frac{\sigma^2}{2})(T - t)}{\sigma \sqrt{T - t}}.
\]

An extension of the Black and Cox model was provided by Zhou in [11], who assumed that the firm’s value is a strictly positive jump-diffusion process with finite intensity of jumps \( \lambda > 0 \), i.e.

\[
\frac{dV_t}{V_t} = (\mu - \lambda \zeta) dt + \sigma dW_t + (\Pi - 1) dY_t,
\]

where \( \zeta > 0 \), \( \sigma > 0 \), and \( \mu \) are constants, \( Y \) is a Poisson process with intensity \( \lambda \), \( \Pi \) is the jump size with expected value equal to \( \zeta + 1 \) and all sources of randomness \( W, Y, \) and \( \Pi \) are mutually independent. Zhou investigated Merton’s problem in this setting and suggested approximations for the distribution of \( \tau \), where

\[
\tau := \inf\{t \geq 0 : V_t \leq L\}, \quad V_0 \geq L.
\]

and \( V \) satisfies (1.5).

We see that all the structural models assume that the default time \( \tau \) is a stopping time with respect to the filtration generated by the firm’s value.

The reduced-form approach assumes that default time \( \tau \) is an exogenous strictly positive random variable. Specifically, let \((\Omega, \mathcal{G}, \mathcal{F}, \mathbb{Q})\) be a filtered probability space, where \( \mathbb{F} = (\mathcal{F}_t)_{t \in [0, \infty]} \) is a right-continuous and complete filtration satisfying \( \mathcal{F}_t \subset \mathcal{G} \) for any \( t \in [0, \infty] \). Let \( \lambda = (\lambda_t)_{t \geq 0} \) be a non-negative \( \mathbb{F} \)-adapted process defined on the probability space and let \( \Theta \) be an exponentially distributed random variable with parameter 1 defined on \((\Omega, \mathcal{G}, \mathbb{Q})\) that is independent of \( \mathcal{F}_\infty \). We model the default time \( \tau \) as the first time the process \( \Lambda = (\Lambda_t)_{t \geq 0} \), where \( \Lambda_t = \int_0^t \lambda_s ds \), is above \( \Theta \), i.e.

\[
\tau := \inf\{t \geq 0 : \Lambda_t \geq \Theta\}.
\]

We also assume that \( \Lambda_t < \infty \) for all \( t < \infty \) and \( \Lambda_\infty = \infty \).

Since \( \tau \perp \mathcal{F}_\infty \), it is easy to see that for any \( s \leq t \)

\[
\mathbb{Q}(\tau > s | \mathcal{F}_t) = e^{-\Lambda_s}.
\]
This justifies that $\lambda$ can be referred to as the intensity of default. Let $\mathcal{G} = (\mathcal{G}_t)_{t \geq 0}$ denote the smallest right-continuous filtration containing $\mathcal{F}$ and making $\tau$ a $\mathcal{G}$-stopping time. Then we can write in short

$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t, \quad \forall t \geq 0,$$

where $\mathcal{H}_t := \sigma(\mathbb{1}_{\tau \leq s} : s \leq t)$.

Let us consider a defaultable contingent claim of the form

$$\phi = Z \mathbb{1}_{\tau > T},$$

where $Z$ is $\mathcal{F}_T$-measurable. Then, the value of $\phi$ at time $t \in [0, T]$ is

$$e^{-r(T-t)} \mathbb{E}_Q[Z \mathbb{1}_{\tau > T} | \mathcal{G}_t] = e^{-r(T-t)} \mathbb{E}_Q[Ze^{-\Lambda_t} | \mathcal{F}_t].$$

Papers II-V deal with problems connected to default modelling. The set-up of Papers II and III can be considered as a mixture of the structural and reduced-form approaches since the default time is modelled as a stopping time with respect to firm’s value which is not directly observable on the market. Paper IV assumes that the default times are some strictly positive random variables independent of the observable stock price. The model used in Paper V is the aforementioned Zhou’s extension of Black and Cox model and thus the framework is the structural approach.

1.3 Perpetual American Options and Optimal Stopping

A European option, considered in Section 1.1, is a financial contract that can be exercised only at the maturity time $T > 0$. In contrast, an American option gives the holder the right to exercise at any time before the maturity time. Since Papers I, IV and V deal with perpetual versions of these contracts, we limit our study to American options with infinite horizon.

Let the dynamics of $X$ under $\mathbb{Q}$ be

$$dX_t = rX_t dt + \sigma X_t dW_t, \quad X_0 = x > 0,$$

where $\sigma > 0$ and $r > 0$ is the risk-free interest rate. We note that the discounted stock price process is a martingale under measure $\mathbb{Q}$ and hence the market is arbitrage-free.

Consider a perpetual American option with pay-off $G(x)$. At any time $t \geq 0$, the holder of the option needs to decide whether to exercise the option immediately or to continue holding it based on observations of the stock price. Hence, the model for his/her strategy is a stopping time with respect to the filtration generated by the stock price up to time $t$. Then, the arbitrage-free price is given by

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_Q^x [e^{-rt} G(X_\tau)],$$

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where $\mathcal{T}$ is the set of stopping time with respect to the filtration generated by the stock price process $X$ (see [6]).

Solving the optimal stopping problem amounts to determining $V(x)$ (the value of the option) as well as an optimal stopping time $\tau^*$ such that

$$V(x) = \mathbb{E}_Q^x[e^{-r\tau^*}G(X_{\tau^*})].$$

From the possibility of exercising at time $t = 0$, we have that $V(x) \geq G(x)$ for any $x > 0$. Let us define the following two sets

$$C := \{x > 0 : V(x) > G(x)\}$$

and

$$D := \{x > 0 : V(x) = G(x)\}.$$ 

We call $C$ and $D$ the continuation set and the stopping set, respectively. Then, under the usual integrability condition of $G(x)$ (see for example [9]), we have that a stopping time $\tau^*$ defined as

$$\tau^* := \inf\{t > 0 : V(X_t) = G(X_t)\} = \inf\{t > 0 : X_t \in D\}$$

is optimal in (1.8).

Let us consider a perpetual American put, i.e. the pay-off function is $G(x) = (K - x)^+$. Then the arbitrage-free price is given by

$$V(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}_Q^x[e^{-r\tau}(K - X_{\tau})^+]$$

and since the holder has the possibility to stop at time $t = 0$, we have $V(x) \geq (K - x)^+$ for any $x > 0$. Moreover, the candidate for the optimal stopping strategy is

$$\tau := \inf\{t > 0 : V(X_t) = (K - X_t)^+\}.$$ 

To find the price, we consider two steps. Step 1 is to guess the solution and Step 2 is to prove that the predicted solution is the optimal one. Specifically, in Step 1 we infer that there exists a constant $b \in (0, K)$, such that

$$\tau_b := \inf\{t \geq 0 : X_t \leq b\}$$

is optimal, i.e.

$$V(x) = \sup_{\tau} \mathbb{E}_Q^x[e^{-r\tau}(K - X_{\tau})^+] = \mathbb{E}_Q^x[e^{-r\tau_b}(K - X_{\tau_b})^+],$$

and then by the Feynman-Kac formula we can write that

$$\begin{cases} \mathcal{L}V(x) - rV(x) = 0, & x > b \\ V(x) = (K - x)^+, & x \leq b, \\ V'(b) = -1, \end{cases}$$

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where \( \mathcal{L} \) is the infinitesimal generator of \( X \) and \( V'(b) = -1 \) is the so-called smooth-fit condition. Hence, the pricing problem reduces to the so-called free-boundary problem since we do not know the boundary \( b \) nor the function \( V(x) \).

Since

\[
\lim_{x \to \infty} V(x) = 0,
\]

we get that

\[
V(x) = \begin{cases} 
    d \left( \frac{K}{1+d} \right)^{1+d} x^{-d}, & x \geq b \\
    K-x, & x < b,
\end{cases}
\]

where

\[
d = \frac{\sigma^2}{2r}, \quad b = \frac{K}{1+d}, \quad \text{and} \quad C_2 = d \left( \frac{K}{1+d} \right)^{1+d}.
\]  

(1.10)

Then, Step 2 is to formulate and prove a verification theorem, which in our case has the following form. The arbitrage-free price \( V(x) \) of an American put option is given explicitly by (1.10). The stopping time \( \tau_b \) defined in (1.9), where \( b \) is given by (1.11) is optimal. The proof can be found for example in [9].

However, there are perpetual American-type contracts where no optimal stopping time exists. As an example, we investigate a perpetual American call option. On one hand, the process \( (e^{-rt}X_t)_{t \geq 0} \) is a martingale and hence for any \( n \in \mathbb{N} \), by the optional sampling theorem, we have that

\[
\mathbb{E}^x_Q \left[ e^{-r(\tau \wedge n)} X_{\tau \wedge n} \right] = x,
\]

which by Fatou’s lemma gives

\[ V(x) \leq x. \]

On the other hand, we have

\[
V(x) = \sup_{\tau \in \mathcal{F}} \mathbb{E}^x_Q \left[ e^{-r\tau} (X_{\tau} - K)^+ \right] \geq \mathbb{E}^x_Q \left[ e^{-rn} (X_n - K)^+ \right]
\]

\[ \geq x - Ke^{-rn} \to x \]

when \( n \to \infty \) and hence \( V(x) = x \). Furthermore, a closer look at the arguments above reveals that

\[
\mathbb{E}^x_Q \left[ e^{-r\tau} (X_{\tau} - K)^+ \right] < x = V(x)
\]

for any stopping time \( \tau \). Thus no optimal stopping time exists.

Paper IV deals with the problem of finding a value of a portfolio consisting of perpetual American options between two defaultable counterparties.
1.4 Zero-Sum Dynkin Games with Infinite Horizon

A zero-sum Dynkin game is a generalization of an optimal stopping problem. It considers two players among which one is a sup-player (Player 1) and the other one is an inf-player (Player 2).

An example of a zero-sum Dynkin game is a cancellable American option, where the issuer of the option can quit the contract at any time under the condition that he/she will pay an additional cancellation cost.

Let us fix a continuous Markov process \( X = (X_t)_{t \geq 0} \) starting at \( X_0 = x \in \mathbb{R} \) and let \( G_1(x) \) and \( G_2(x) \) be two real-valued functions satisfying \( G_1(x) \leq G_2(x) \) for any \( x \in \mathbb{R} \). Define the expected pay-off of the game

\[
R_x(\tau_1, \tau_2) := \mathbb{E}^x [G_1(X_{\tau_1}) \mathbb{1}_{\tau_1 \leq \tau_2} + G_2(X_{\tau_2}) \mathbb{1}_{\tau_2 < \tau_1}].
\]

If Player 2 chooses a stopping time \( \tau_2 \), then the aim of Player 1 is to choose a stopping time \( \tau_1^\ast \) such that

\[
R_x(\tau_1, \tau_2) = \sup_{\tau_1 \in \mathcal{T}} R_x(\tau_1, \tau_2).
\]

Analogously, if Player 1 chooses a stopping time \( \tau_1 \), then Player 2 is aiming to choose a stopping time \( \tau_2^\ast \) such that

\[
R_x(\tau_1, \tau_2) = \inf_{\tau_2 \in \mathcal{T}} R_x(\tau_1, \tau_2).
\]

A pair of stopping times \( (\tau_1^\ast, \tau_2^\ast) \) is called a Nash equilibrium if

\[
R_x(\tau_1, \tau_2) \leq R_x(\tau_1^\ast, \tau_2^\ast) \leq R_x(\tau_1^\ast, \tau_2)
\]

for any stopping times \( \tau_1 \) and \( \tau_2 \).

We define the upper and lower value of the game by

\[
\mathcal{V}(x) := \inf_{\tau_2 \in \mathcal{T}} \sup_{\tau_1 \in \mathcal{T}} R_x(\tau_1, \tau_2) \quad \& \quad \mathcal{V}(x) := \sup_{\tau_1 \in \mathcal{T}} \inf_{\tau_2 \in \mathcal{T}} R_x(\tau_1, \tau_2),
\]

where \( \mathcal{T} \) denotes the set of stopping times with respect to the filtration generated by the Markov process \( X \). Clearly, from the possibility of choosing \( \tau_i = 0 \) we have

\[
G_1(x) \leq \mathcal{V}(x) \leq \mathcal{V}(x) \leq G_2(x).
\]

If in addition

\[
\mathcal{V}(x) \leq \mathcal{V}(x),
\]

then

\[
\mathcal{V}(x) := \mathcal{V}(x) = \mathcal{V}(x)
\]

is called a value of the optimal stopping game (1.12). It is straight-forward to check that if a Nash equilibrium exists, then a value of the game exists.
Let us define the sets
\[ C := \{ x \in \mathbb{R} : G_1(x) < V(x) < G_2(x) \}, \]
and
\[ D_i := \{ x \in \mathbb{R} : V(x) = G_i(x) \}, \quad i = 1, 2. \]
Here \( C \) and \( D_i \) are the continuation region and the stopping region for the Player \( i \), respectively. Then, the candidate for a Nash equilibrium is a pair of stopping times \((\tau_1, \tau_2)\) such that
\[
\tau_i := \inf \{ t > 0 : V(X_t) = G_i(X_t) \} = \inf \{ t > 0 : X_t \in D_i \}, \quad i = 1, 2.
\]
A value of the game and a Nash equilibrium can be found by following two steps. The first step is to guess the solution and the second step is to formulate a verification theorem and prove it.

It was shown in [3] that if the pay-off functions \( G_i(x), i = 1, 2 \) are integrable, i.e.
\[
\mathbb{E}^x \left[ \sup_t |G_i(X_t)| \right] < \infty, \quad i = 1, 2,
\]
then there exists a value of the game.

Papers I and IV deal with zero-sum optimal stopping games, where the solutions are guessed with the use of variational inequalities and the optimality of the solutions is shown by associated verification theorems.
2. Summary of Papers

In this section we give short summaries of each of the papers included in this thesis.

2.1 Paper I

In this paper we study a zero-sum Dynkin game between two players that disagree about the drift of the underlying diffusion process $X$ and enter an optimal stopping game with pay-off

$$G_1(X_{\tau_1})I_{\{\tau_1 \leq \tau_2\}} + G_2(X_{\tau_2})I_{\{\tau_2 < \tau_1\}},$$

(2.1)

where $G_1(x) \leq G_2(x)$ and $\tau_i$ denotes the stopping time of Player $i$, for $i = 1, 2$.

At time $\tau := \tau_1 \wedge \tau_2$ the game is terminated and Player 1 receives the amount (2.1) from Player 2 and hence Player 1 wants to maximize the expected (discounted) future pay-off and Player 2 aims at minimizing it.

Let $r > 0$ denote a constant risk-free interest rate. Since the players do not agree about the distributional properties of the underlying process, their expected future pay-offs differ and are given by

$$R_i^x(\tau_1, \tau_2) := E^x_{P_i} \left[ e^{-r(\tau_1 \wedge \tau_2)} \left( G_1(X_{\tau_1})I_{\{\tau_1 \leq \tau_2\}} + G_2(X_{\tau_2})I_{\{\tau_2 < \tau_1\}} \right) \right],$$

where $E^x_{P_i}$ denotes the probability measure of Player $i$, for $i = 1, 2$.

We show that under some integrability conditions on the functions $G_i, i = 1, 2$, the zero-sum Dynkin game with heterogeneous beliefs can be reduced to a nonzero-sum game with homogeneous beliefs. Using recent developments for nonzero-sum games, see [5], this enables us to provide conditions under which a Nash equilibrium exists.

Moreover, we formulate a verification theorem which provides means to check whether a given pair of candidate value functions indeed is the value function. However, unlike the case of zero-sum games under homogeneous beliefs, different Nash equilibria may be associated with different value functions.

As an example we study a cancellable perpetual call option, where Player 1 is the buyer of the option and Player 2 is the seller of the option who has to pay an additional cancellation cost if he/she decides to cancel the contract.

We consider different regimes for the parameters of the underlying model and study Nash equilibria and associated value functions. In particular, we show that for some parameter values, multiple equilibria may exist, and the corresponding value functions may be different.
2.2 Paper II

We investigate the problem of pricing financial contracts with pay-offs that change their form when a default happens before the maturity time \( T > 0 \), i.e. the so-called default-sensitive contingent claims. We introduce an informed investor who observes the stock price fluctuations \( S \) as well as knows the default time from the very beginning. We model his/her information level \( \mathcal{G} \) by an initial enlargement of the filtration generated by the stock price with the default time \( \tau \), i.e. \( \mathcal{G} = (\mathcal{G}_t)_{t \in [0,T]} \):

\[
\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\tau),
\]

where \( \mathcal{F}_t \) denotes the information generated by \( S \) up to time \( t \), i.e.

\[
\mathcal{F}_t := \sigma(S_s : s \leq t).
\] (2.2)

The problem is studied in a general framework, since the stock price \( S \) is assumed to be any continuous and strictly positive process.

Let us consider a default-free market which consists of the risky asset \( S \), constant bank account and \( \mathcal{F}_T \)-measurable claims. We assume that this market is arbitrage-free and complete, and hence there exists a unique martingale measure. Moreover, we study a defaultable market comprising the risky asset \( S \), constant bank account, and default-sensitive contingent claims. We presume that this market is arbitrage-free and thus there exists at least one equivalent martingale measure. We show that under these conditions there exists a unique price of a default-sensitive contingent claim in the sense that it does not depend on the choice of equivalent martingale measure. Moreover, we prove that this price can be represented as prices of corresponding \( \mathcal{F}_T \)-measurable claims.

As an example, we study the Brownian filtration and show with the use of Yor’s method (see [10]) that the conditions on the sets of equivalent martingale measures are satisfied and hence the price of a default-sensitive contingent claim is unique.

2.3 Paper III

In this article we consider a regular investor who observes the stock price \( S \) and default time \( \tau \) when it happens. Hence, his/her information level \( \mathcal{G} = (\mathcal{G}_t)_{t \in [0,T]} \) is modelled by a progressive enlargement of the filtration generated by the stock price, i.e.

\[
\mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t,
\]

where \( \mathcal{F}_t \) is as in (2.2) and

\[
\mathcal{H}_t := \sigma(S_s : s \leq t) \cup \sigma(I_{\tau \leq s} : s \leq t).
\]
We define default-sensitive contingent claims, default-free market, and defaultable market analogously to Paper II and impose conditions under which there exists an equivalent martingale measure in $\mathcal{G}$. Moreover, we show that the set of equivalent martingale measures is infinite.

The choice of a pricing measure is executed by the so-called f-divergence methodology, which is equivalent to utility maximization. We choose the logarithmic utility function and find the equivalent martingale measure accordingly.

We provide the pricing formula at time $t \in [0, T]$ for a default-sensitive contingent in terms of $\mathcal{F}_t$-conditional expectations and apply it to two examples of modified European claims that pay a rebate in case of default before the maturity time $T$.

2.4 Paper IV

For portfolios of European options, CVA (credit valuation adjustment) is easily calculated as the difference of the defaultable value and the corresponding non-defaultable value. For portfolios of American options, however, the optimal exercise strategies will depend on the possibility of counterparty risk.

In this article, we consider two defaultable counterparties, Player 1 and Player 2, that enter the following contract. Player 1 issues an American claim with pay-off $G_1(x)$ and sells it to Player 2 and Player 2 issues an American contract with pay-off $G_2(x)$ and sells it to Player 2.

Furthermore, we assume that both players are subject to default, but with possibly different default rates.

We embed the credit risk directly in the expected pay-offs so that the optimal exercise strategies are automatically adjusted to the default possibilities. The problem of pricing such a contract between two defaultable parties reduces to a zero-sum Dynkin game. We impose some standard integrability conditions on the pay-off functions and formulate a verification theorem to study Nash equilibria and associated value functions.

Moreover, we define a CVA as the difference between a value of the game and the value of the two American claims if considered default-free.

As an example, we take $G_1(x) = (x - K)^+$ and $G_2(x) = (K - x)^+$ and assume that only one of the counterparties is defaultable. We characterise the value of the contract and provide the formula for CVA.

2.5 Paper V

In this paper we assume a structural model for the default time $\tau$. We namely assume that the default happens the first time a jump-diffusion process $X$ hits
a boundary $b(t)$, i.e.
\[ \tau := \inf\{t > 0 : X_t \leq b(t)\}, \]
where
\[ X_t = \sigma B_t + \mu t + \sum_{i=1}^{N_t} D_i, \quad X_0 = 0, \]
(2.3)

$B$, $N$, and $(D_i)_{i \geq 1}$ denote the Brownian motion, the Poisson process and a sequence of independent and identically distributed random variables, respectively, and are mutually independent.

Since the default probabilities can be estimated from the bond spreads, we study the so-called inverse first passage problem: for a given distribution function $g(t)$, find a boundary $b(t)$ such that
\[ \mathbb{P}(\tau > t) = g(t). \]

Moreover, we prove that if $g(t)$ has at $t = 0$ a strictly positive density $\Lambda$, satisfying $\Lambda < \lambda \mathbb{P}(D_1 \leq 0)$ where $\lambda$ is the jump intensity, then the solution of the inverse first passage problem is a barrier function which is bounded away from zero at $t = 0$.

Furthermore, we use the connection between the inverse first passage problem and an optimal stopping problem formulated in [4] and show numerically that jump-diffusion models lead to flatter boundaries than the diffusion models.
3. Summary in Swedish

Denna avhandling består av fem vetenskapliga artiklar som handlar om prissättning av finansiella instrument. Dessutom innehåller avhandlingen en introduktion till martingalteorin för finansiell värdering, kreditvärdighet, optimal stopptidsteori samt till Dynkinspel.


Artikel IV behandlar prissättning av en portfölj som består av två amerikanska optioner mellan två aktörer som båda är utsatta för defaultrisk. Vi bäddar in kreditvärdighet direkt i de förväntade utbetalningarna så att de optimala strategierna är justerade till att ta hänsyn till defaultrisken. Problemet kan skrivas som ett Dynkinspel av nollsummesort, och vi tillhandahåller villkor under vilka en Nashjämvikt i stoppstrategier existerar. Vi finner även explicita formler för priset samt för hur mycket defaultrisken påverkar portföljpriset.

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