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Synchronization in chaotic dynamical systems

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1   Abstract

Synchronization refers to a process where systems adjust so that certain properties of their motion is shared between them. This text seeks to clarify the notions of synchronization and chaos, present some tools for analysing dynamical systems, and then use those tools to investigate different classes of chaotic systems, with the goal of establishing conditions under which synchronization may occur in such systems. Though most of the results presented are well known, they are here clarified to be made available to non-experts of dynamical systems, and in certain cases extended on.

The main focus of study will be systems of the form

$$\dot{x} = f(x) + g_1(x, y)$$
$$\dot{y} = f(y) + g_2(x, y)$$

in which we will establish a sufficient criterion for complete synchronization between the x- and y-components.

2   Introduction

The first thing we will discuss is the notion of chaos, starting in section 3, after which we will make the notion of synchronization precise in section 4, after which we proceed with the intended analysis.

The reader is expected to be familiar with dynamical systems defined by ODEs, basic terminology regarding such, and methods of analysing stability of solutions to them. For the purposes of self containment however, the rest of section 2 consists of a very brief reminder of some fundamental concepts.

2.1 Basic terminology

Definition 2.1 (Continuous dynamical system). A *continuous dynamical system* is a tuple, (I, T, ϕ), where I is an open connected subset of real numbers, T a topological space, and $\phi : I \times T \to T$ a function, continuous in both arguments, that generates a semi-group as follows:

1. $\exists t_0 \in I : \phi(t_0, x) = x \quad \forall x \in T$

2. $\forall t, s \in I$ *such that* $(t + s) \in I$ *it holds that* $\phi(s, \phi(t, x)) = \phi(t + s, x) \quad \forall x \in T$

I can be thought of as a set of time over which the system evolves (commonly taken as $\mathbb{R}$ or $\mathbb{R}^+$), although no connection with time in the physical meaning is usually intended. In this text, the set I will always be assumed to have no largest element.

$\phi(t, x)$, or simply $\phi_t(x)$, is so thought of as the point in T to which x has evolved to at the time t. Applying ϕ_t to some point, is sometimes referred to as *applying the flow*. Every occurrence of t or t_j $(j \in \mathbb{N})$ in this text will refer to an element from the set corresponding to I.

T is referred to as the *phase space* of the system, representing the system. Within this space the dynamics we seek to study occurs.

The famous Picard's existance and uniquness theorem along with the Birkhoff Rota theorem allows us to define dynamical systems through linear ODEs [3].

We will study dynamical systems defined by autonomous ODEs. This means that the functions f above does not explicitly depend on the variable t $(\dot{x} = f(x(t)))$.

Of great interest when studying dynamical systems are equilibrium points, invariant sets and attractors (also called attracting sets). This is becuause they in different ways reflect the long term behaviour of the system.

Definition 2.2 (Equilibrium point). Let (I, T, ϕ) be a dynamical system. If for some $x^* \in T$, we have $\phi_t(x^*) = x^* \quad \forall t \in I$, then x^* is an *equilibrium point* of the system. For a system defined by ODEs, $\dot{x} = f(t, x)$, it corresponds to $f(t, x^*) = 0 \quad \forall t \in I$.

Definition 2.3 (Invariant set)**.** Let (I, T, ϕ) be a dynamical system. A set $A \subseteq T$ is said to be an *invariant set* (or simply *invariant*) if no point in A ever leaves A by the flow. I.e. $x_0 \in A \Rightarrow \phi_t(x_0) \in A \quad \forall t > t_0$.

Definition 2.4 (Attractor)**.** Let (I, T, ϕ) be a dynamical system. An *attractor* is a subset A of T, which satisfies the following properties:

1. A is a compact and invariant set.

2. There exists and invariant neighbourhood $B(A)$ of A, with the property that $\cap_{t>0} \phi_t(B(A)) = A$. ($B(A)$ is called the *basin of attraction of A*).

3. For any $x, y \in A$, and any neighbourhoods $U_x, U_y \subset B(A)$ about x, y respectively, it holds that $\exists t \in I$ such that $\phi_t(U_x) \cap U_y \neq \emptyset$.

An attractor whose *Hausdorff dimension* is a non-integer is said to be a *strange attractor*. This is to say that the attractor is fractal in nature. An exact definition of this concept is given in [19].

2.2 Stability analysis of dynamical systems defined by ODEs

Definition 2.5 (Lyapunov stable)**.** Let $x(t)$ be a solution to a system of ODEs. $x(t)$ is said to be *Lyapunov stable* if for any $\epsilon > 0$ and any $t_0 \geq 0$ there exists $\delta = \delta(t_0, \epsilon)$ such that

- all solutions $\tilde{x}(t)$ with $\|x_0 - \tilde{x}_0\| < \delta$ are defined for $t \geq t_0$.

- for such solutions, $\|x(t) - \tilde{x}(t)\| \leq \epsilon$ holds for all $t \geq t_0$.

We will refer to this simply as *stable.*

Definition 2.6 (Asymptotically Lyapunov stable)**.** If a solution $x(t)$ is Lyapunov stable and there is, for any $t_0 \geq 0$, a $\delta = \delta(t_0)$ such that for solutions $\tilde{x}(t)$ with $\|\tilde{x}_0 - x_0\| \leq \delta$ it holds that $\lim_{t \to \infty} \|x(t) - \tilde{x}(t)\| = 0$ then x^* is said to be *asymptotically stable.*

2.2.1 Stability analysis by linearisation

For $f : \mathbb{R}^n \to \mathbb{R}^n$, consider a system

$$\dot{x} = f(x) \tag{1}$$

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and x^* be an equilibrium point to the system. By the Hartman-Grobman theorem [7], we known that the linearisation

$$\dot{v} = D_x f(x^*) v \tag{2}$$

where $D_x f(x^*)$ is the Jacobian matrix evaluated at x^*, gives information about the behaviour of solutions to (1) near the equilibrium point, given that no eigenvalue of said matrix has real part zero. Furthermore the eigenvectors corresponding to eigenvalues with negative real part will locally describe the stable manifold of the equilibria, whereas those with positive real parts describe the unstable manifold. One may also see [3] for details on this topic.

2.2.2 Lyapunov functions

Another fundamental tool for understanding nonlinear dynamics are so called Lyapunov functions. Although no general scheme for finding such functions is known, their existance can be used to determine the stability of equilibrium points, even when the method of linearisation fails.

Definition 2.7 (Lyapunov function)**.** Let $\dot{x} = f(x)$ be a system of differential equations, where $f : \mathbb{R}^n \to \mathbb{R}^n$, and $\exists x^* \in U \subset \mathbb{R}^n$ such that $f(x^*) = 0$.
   Let $g : U \to \mathbb{R}$ such that the following holds:

1. g is a continuously differentiable in U.

2. $g(x^*) = 0$ and $g(x) > 0$ $\forall x \neq x^*$. (*g is positive definite.*)

3. $\dot{g}(x) \leq 0$ $\forall x \neq x^*$. (*g is non-increasing along trajectories.*)

   Then g is a *Lyapunov function* for x^*.

   Here $\dot{g}$ denotes the derivative of g w.r.t the system f, i.e. $\dot{g}(x) = \frac{d}{dt} g(x) = (\nabla g \cdot f)(x)$.

   If also the following holds:

4. $\dot{g}(x) < 0$ $\forall x \neq x^*$. (*g is decreasing along trajectories.*)

   Then g is a *strict Lyapunov function* for x^*.

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Depending on what the codomain is, different things can be said about the system, given the existance of a Lyapunov function. Two theorems are presented that utilize this kind of function (proofs of these theorems are included in the appendix):

Theorem 2.8 (Lyapunov's stability theorem). *Let $\dot{x} = f(x)$ where $f : \mathbb{R}^n \to \mathbb{R}^n$. Suppose now that there is a (strict) Lyapunov function g for an equilibrium point x^*, where $g : U \to \mathbb{R}$ for an open and bounded set U containing x^*.*

Then x^* is (asymptotically) stable.

Definition 2.9 (Global Lyapunov function). Consider a system $\dot{x} = f(x), f : \mathbb{R}^n \to \mathbb{R}^n$. If a function g is a strict Lyapunov function to the system f on the entire $\mathbb{R}^n$, as described in definition 2.7, but with the added criterion that g is also *radially unbounded*, meaning that $\|x\| \to \infty \Rightarrow \|g(x)\| \to \infty$, then g is said to be a global Lyapunov function for x^*.

Given the existence of a global Lyapunov function, one can draw conclusions about a system's global behaviour.

Theorem 2.10. *Suppose $\dot{x} = f(x)$, $f : \mathbb{R}^n \to \mathbb{R}^n$, with $f(x^*) = 0$ for some $x^* \in \mathbb{R}^n$. If there exists a global Lyapunov function for x^*, then x^* is globally asymptotically stable, meaning that every trajectory converges to x^* as $t \to \infty$.*

2.2.3 Lyapunov exponents

Another way of analysing the stability of solutions is through *Lyapunov exponents*. Although difficult to compute, they can sometimes provide insights not available through more elementary means.

Technically, there are for each solution to a system, as many Lyapunov exponents as the dimensionality of the system's phase-space. It is known however, that for autonomous systems, these exponents are a property of the system itself, rather than of its solutions. Also, for the sake of stability analysis it is sufficient to consider the largest Lyapunov-exponent λ. $\lambda > 0$ implies instability, whereas $\lambda < 0$ implies Lyapunov stability [28].

Definition 2.11 (Maximal Lyapunov-exponent). Consider a dynamical system with flow-function $\phi_t(x_0)$, and a compact attractor A. The maximal Lyapunov-exponent for trajectories starting in $B(A)$, is then

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$$\lambda = \sup_{x_0 \in B(A)} \lim_{t \to \infty} \frac{\ln \|D_{x_0} \phi_t(x_0)\|}{t}$$

It is known that for two solutions, $\tilde{x}(t), \hat{x}(t)$, to a system with maximal Lyapunov-exponent λ, and for any $\epsilon > 0$, there is a constant $C(\epsilon) < \infty$ such that $\|\tilde{x}(t) - \hat{x}(t)\| \le C(\epsilon) e^{(\lambda + \epsilon)t} \|\tilde{x}_0 - \hat{x}_0\|$. [4]

3   Chaotic dynamical systems

3.1   Intoduction

Linear autonomous ODEs, i.e. ones that can be written $\dot{x} = Ax$, where $x \in \mathbb{R}^n$ and A is an $n \times n$ matrix, are well known to have the very simple solution Ce^{tA} where C is a row-vector, simply determined by the initial condition. These equations have solutions that either tend to an equilibrium point, diverge to infinity, or settle at a periodic orbit. In short, their attractors are quite simple.

Also, solutions to any ODE of the form $\dot{x} = f(x)$, $f : \mathbb{R}^2 \to \mathbb{R}^2$ (a so called planar system) will by the famous Poincaré-Bendixon theorem have very simple long-term behaviour. For details and a proof, see [3].

As soon as one steps out of this comfort zone however, solutions may behave very erratic. In fact, as shown by the Rössler-system, one single non-linear term in a 3-dimensional system of ODEs may lead to chaotic behaviour on a strange attractor.

A few different definitions of chaos exists, but here we will use the one by Devaney, [3].

Definition 3.1 (Chaotic dynamics). Let $(\mathbb{R}, T, \phi)$ be a dynamical system. If for some invariant subset $\Omega \subset T$

- ϕ depends sensitively on its initial condition in Ω.

- the set of periodic orbits of ϕ is dense in Ω.

- ϕ is transitive on Ω.

then the system is *chaotic* on Ω.

Here, sensitive dependence of initial conditions means that points arbitrarily close to each other, develop onto significantly different trajectories. Formally, $x \in R \Rightarrow \lambda_{max}(x) > 0$.

Topological transitivity means that every pair of subsets $U, V \subset R$ will "mix" when the flow is applied to its points:

$$\exists T \in \mathbb{R}, T > t_0 : \phi_T(U) \cap V \neq \emptyset$$

3.2 Example: The Lorentz system, a brief analysis

The most classical example of a chaotic dynamical system is the *Lorentz system*, introduced by Edward Lorentz [1] as a simplified model for atmospheric convection. Lorentz discovered that slight perturbation in initial conditions would drastically alter the future states of a solution. The original equations are in 12 dimensions, but what is known today as the Lorentz system is his simplified version:

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - \beta z$$

Normally, $\sigma, r, \beta > 0$ is assumed, and the systems behaviour will vary depending on exactly what they are chosen to be. Here we will briefly analyse the system for the classical parameter values, $\sigma = 10$, $\beta = 8/3$, $r = 24$.

3.2.1 Type and location of equilibria

By solving $(\dot{x}, \dot{y}, \dot{z}) = (0, 0, 0)$ one finds that there is an equilibrium point, C_0, at the origin, and that there for $r > 1$ is a pair of equilibrium points, $C^{\pm} = (\pm\sqrt{\beta(r-1)}, \pm\sqrt{\beta(r-1)}, r-1)$.

The Jacobian matrix for the system is,

$$\begin{pmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -\beta \end{pmatrix}$$

which evaluated at the origin leads to the characteristic equation

$$0 = (\beta + \lambda)(\lambda^2 + (\sigma + 1)\lambda - \sigma(r - 1)) = 0 \tag{3}$$

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This corresponds to two negative eigenvalues and one positive, as $\sigma(r-1) > 0$. The origin so has a one-dimensional unstable manifold, and a two-dimensional stable manifold, which is a type of saddle.

For $C^\pm$ the calculations are quite lengthy, so here we will simply conclude (as in [5]) that the Jacobian matrix for these points has a pair of complex eigenvalues with positive real part, and one real negative eigenvalue. Thus, this linearised view shows $C^\pm$ as unstable spirals, with a one dimensional stable curve feeding into it.

3.2.2 Trapping region

We have seen that every equilibrium point is unstable, so one might expect solutions diverge to infinity. A common way to refute this is to consider a function $V(x, y, z) = rx^2 + \sigma y^2 + \sigma(z - 2r)^2$. Then

$$\dot{V} = 2rx\dot{x} + 2\sigma y\dot{y} + 2\sigma(z - 2r)\dot{z} = -2\sigma(rx^2 + y^2 + \beta(z - r)^2 - \beta r^2) \quad (4)$$

$\dot{V}$ is evidently negative for sufficiently large x, y, z. This show that there is some $k > 0$ such that every solution to the system must eventually enter the ellipsoid $V(x, y, z) = k$, and once it has, it can not escape. Note that (4) guarantees the existence of a trapping region at least for all $r, \beta, \sigma > 0$.

3.2.3 Dissapation of volume

We also not that the system is highly dissipative, as the divergence of the vector field is given by

$$\text{div } f = \nabla \cdot f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = -(\sigma + \beta + 1)$$

showing that any volume will shrink extremely fast under the flow. To see this, we consider an arbitrary closed bounded surface S (smooth enough to have a tangent plane in each point), with volume V and outward normal n. If one applies a flow to the points of the surface, we could regard V as a function of time. Consider an infinitesimal patch dA of S. If we apply the flow ϕ_{dt} to it, it will move $(f \cdot n)dt$ length units outwards. Integrating to get the total change of volume yields

$$\dot{V} = \int_S (f \cdot n)dA = \int_V \text{div } f dV$$

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The second equality is of course by the divergence theorem. For the Lorentz equations, the integral becomes simply

$$\dot{V} = -(\sigma + \beta + 1)V$$

which has the solution

$$V(t) = V_0 e^{-(\sigma+\beta+1)t}$$

3.2.4 Conclusions

These observations put a lot of restrictions on trajectories: Solutions must be attracted to some bounded set of zero volume, but never settle down at an equilibrium point.
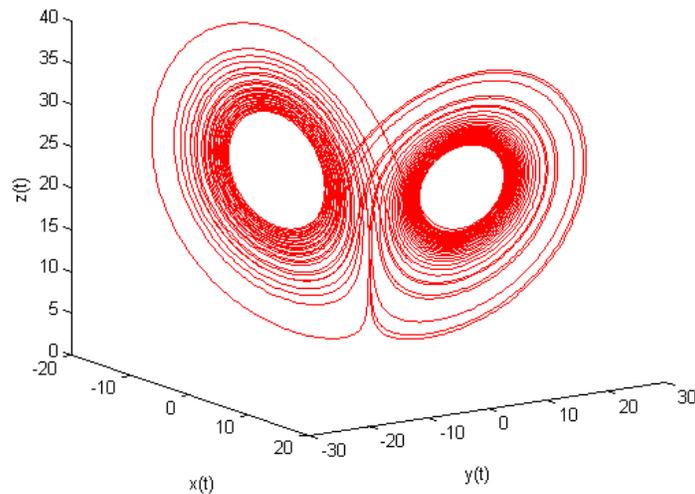

Figure 1: *A sample solution curve to the Lorentz equations.* $(x(0), y(0), z(0) = (0, 6, 22)$, $t \in [0\,,\,50]$, *with the classic parameters.*

The problem of showing that the attracting set for the Lorentz equation is not a stable limit cycle was chosen Stephen Smale to be on his list of 18 problems for the 21st century [6], and was solved by Warwick Tucker [5] in a computer assisted proof.

It is now well know that the attracting set is indeed a strange attractor with the shape reminiscent of the wings of a butterfly. Remarkably, every trajectory in phase space is attracted to this set, and they remain there for all time without ever intersecting themselves or other trajectories, yet not requiring any volume to do so.

4 Synchronization in chaotic systems

Synchronization means agreement or correlation in time of different processes. It could for instance be two pendulums that mirror each others movement, due to them hanging from the same beam (as was observed by the 17th century researcher C. Huygens [8]). This occurs in many other natural systems [9], so when dynamical systems where shown able to exhibit chaotic behaviour, a question naturally arising is whether such systems too would be able to synchronize, and if so, is the synchronization robust (able to withstand slight perturbations) and thus plausibly found in nature?

In the context of chaotic systems, due to their sensitivity to initial conditions, it is maybe not so obvious that this is at all possible. However, the aim of this chapter is, once the fundamental concepts are clarified, to show that such configurations are not only possible, but also in some cases very robust.

4.1 Definitions

There are many varieties of synchronization, and to attempt a generalized definition would be quite lengthy indeed. Instead we will simply define the most fundamental case, and roughly out-line how this can be generalized.

To this end, we will consider a system

$$\dot{x} = f(x) \tag{5}$$

which can be sub-divided into three parts $x = (u, v, w)$.

Definition 4.1. (Complete synchronization) If for a solution of such system, $(u(t), v(t), w(t))$ it holds that $\|v(t) - w(t)\| \to 0$ as $t \to \infty$, we say the the v- and w-components are in *complete synchronization*.

In other words, the concept of synchronization is closely related to stable manifolds: the v, w-components being synchronized, is in this case equivalent

to solutions to the system (5) approaching the subspace $\{(u, v, w) \in \mathbb{R}^n \mid v = w\}$, called the *synchronization manifold*.

In fact, many of the different synchronization properties studied are variations of this idea, but with different synchronization manifolds. An example could be the synchronization manifold $(u(t), v(t), w(t + \tau))$, for some $\tau \neq 0$. Then the v, w components are related as before, but with a shift in time τ. The sign of τ determines whether this is *lag synchronization* or *anticipating synchronization*. For a more comprehensive overview of the varieties of synchronization currently studied, see [10].

4.2   Dissipative coupling of identical systems

To achieve complete synchronisation, there are two main methods, that may or may not work depending on circumstances. The first is by adding a dissipative coupling, the second is the method of complete replacement, a concept we will return to later in the text.

First, we will study a kind of dissipative coupling which involves linear functions of the difference between the subspaces one would like to become synchronized. Specifically,

$$\dot{x} = f(x) + A(y - x)$$
$$\dot{y} = g(y) + B(x - y) \tag{6}$$

where A, B are positive semi-definite matrices of sizes that makes the equations well defined. If exactly one of the matrices A, B is the zero-matrix, the system is said to be *unidirectionally coupled*. If both are non-zero, the system is *bidirectionally coupled*.

One should note though, that adding bidirectional coupling to a system can change its behaviour other ways than to make it synchronized. Stability of equilibria might change, and so a system could seize to be chaotic under such coupling. We will see an example of this later. This effect is of course not present in the case of unidirectional coupling.

Commonly, $f \equiv g$, while A, B are just the identity matrix multiplied by some constant. We will restrict our focus to this simpler case, and attempt to answer the question whether there is always a way of coupling (6) so that x, y becomes synchronized, given a particular $f \equiv g$.

4.2.1 Establishing synchronization in dissipatively coupled systems, without distrupting the chaotic dynamics

Consider the system

$$\dot{x} = f(x) + \alpha_x(y - x)$$
$$\dot{y} = f(y) + \alpha_y(x - y) \tag{7}$$

where the real-analytic function f is such that the system $\dot{z} = f(z)$ (referred to as the underlying system) exhibits chaotic dynamics on a bounded attractor Ω, in accordance with definition (3.1). (Let x_0, y_0 be restricted to $\Omega \times \Omega$.)

We note right away that $M = \{(x, y) \in \mathbb{R}^{2n} \mid x = y\}$ is an invariant set, as for $(x, y) \in M$, the coupling terms vanishes, and the system so will reduce to two identical systems, which will so evolve identically $(x = y \Rightarrow \dot{x} = \dot{y})$.

As a consequence, one might suspect that as long as synchronization is stable, the dynamics of (7) is entirely inherited from the underlying system. We will however leave this for future investigation.

Defining instead $\Delta \equiv y - x$, we can transform the said system into the equivalent

$$\dot{x} = f(x) + \alpha_x \Delta$$
$$\dot{\Delta} = f(x + \Delta) - f(x) - (\alpha_x + \alpha_y)\Delta \tag{8}$$

In these coordinates, $M = \{(x, \Delta) \in \mathbb{R}^{2n} \mid \Delta = 0\}$.

Take now $\epsilon \ll 1$ and define the hyper-tube $S = \{(x, \Delta) \mid (x, \Delta) \in B(\Omega) \times \mathbb{R}^n, \|\Delta\| \leq \epsilon\}$. We will now show, that if α_x, α_y are chosen to be sufficiently large, M will attract nearby trajectories. This can be accomplished by showing that given such α_x, α_y; ∂S will act as a trapping region for the flow for any ϵ sufficiently small.

The outwards facing unit vector field on ∂S is $N(x, \Delta) = (0, \frac{\Delta}{\epsilon})$. S is a trapping region if the vector field of (8) always opposes $N(x, \Delta)$ on ∂S. In other words, if $N(x, \Delta) \cdot (\dot{x}, \dot{\Delta}) < 0$. Clearly,

$$N(x, \Delta) \cdot (\dot{x}, \dot{\Delta}) = 0 \cdot \dot{x} + \frac{\Delta}{\epsilon} \cdot \dot{\Delta} = \frac{1}{\epsilon} \dot{\Delta} \cdot \Delta =$$

$$= \frac{1}{\epsilon} (Df(x)\Delta - (\alpha_x + \alpha_y)\Delta + O(\|\Delta\|^2)) \cdot \Delta$$

where O represents the higher order terms of f, and $Df(x)$ is the linear term. S is bounded because $B(\Omega)$ is bounded, so we know that $\|Df(x)\| \leq \Lambda = \sup_{x \in B(\Omega)} \|Df(x)\|$. It follows that,

$$N(x, \Delta) \cdot (\dot{x}, \dot{\Delta}) \leq \frac{1}{\epsilon} ((\Lambda - (\alpha_x + \alpha_y))\|\Delta\|^2 + O(\|\Delta\|^3))$$

on ∂S. It is now clear that for any choice of α_x, α_y such that

$$(\alpha_x + \alpha_y) > \Lambda \tag{9}$$

there exists an $\epsilon > 0$ which will ensure $N(x, \Delta) \cdot (\dot{x}, \dot{\Delta}) < 0$, meaning that ∂S a trapping region given such α_x, α_y. In other words, for such ϵ,

$$\|\Delta(0)\| \leq \epsilon \Rightarrow \|\Delta(t)\| < \epsilon \tag{10}$$

for any $t > 0$.

Applying here the choice of $(\alpha_x + \alpha_y) > \Lambda + K'$ for some $K' > 0$ (This clearly satisfies (9)) to $\frac{d}{dt}\|\Delta(t)\|^2 = \frac{d}{dt}(\Delta \cdot \Delta) = 2\dot{\Delta} \cdot \Delta = 2\epsilon(N(x, \Delta) \cdot (\dot{x}, \dot{\Delta}))$, and utilizing (10), we arrive at

$$2\|\Delta\| \cdot \frac{d}{dt}\|\Delta\| \leq -2K'\|\Delta\|^2 + O(\|\Delta\|^3)$$

If ϵ is sufficiently small, there exists a $K > 0$ such that

$$2\|\Delta\| \cdot \frac{d}{dt}\|\Delta\| < -2K\|\Delta\|^2$$

By assuming the that $\|\Delta(t)\| > 0$ (the case where $\|\Delta(t)\| = 0$ is handled trivially by the invariance of the set M under the flow), it follows immediately that

$$\frac{d}{dt}\|\Delta(t)\| < -K\|\Delta(t)\|$$

which allows us to bound $\|\Delta(t)\|$ as follows:

$$\frac{d}{dt}\|\Delta(t)\| + K\|\Delta(t)\| < 0$$
$$\frac{d}{dt}(e^{Kt}\|\Delta(t)\|) < 0$$
$$\|\Delta(t)\| < \|\Delta_0\|e^{-Kt}$$

which shows that M attracts all nearby trajectories. We summarize these results in a theorem:

Theorem 4.2 (Sufficient coupling strength for identical systems). *For any system of the form (7), it is possible to choose values of α_x, α_y such that the x, y components will approach complete synchronization exponentially fast, given that $\|x_0 - y_0\|$ is sufficiently small.*

The value Λ used in the proof is quite wasteful, and synchronization is often observed in practice with significantly weaker coupling as we shall see later. Parts of this is because that the proof uses the entire $B(\Omega)$, whereas any forward invariant subset thereof would suffice, as long as in contains the initial condition. The actual number Λ is expected to be lower the smaller the set used is. This theorem however, is mainly intended to answer in the positive, the question whether synchronization is always possible in bidirectionally coupled identical systems. More thorough work on this can be found in [11] where among other things a more economic bound is established.

4.2.2 An overly optimistic, but practical approach to achieve complete synchronization

Consider again system (7), but suppose further that it has on $B(\Omega)$ the maximal Lyapunov exponent λ.

Introducing $\Delta \equiv y - x$, we note that

$$\dot{\Delta} = f(x + \Delta) - f(x) - (\alpha_x + \alpha_y)\Delta \tag{11}$$

will determine the time evolution of this difference. Now we linearize about a point (x, y) on M, while completely neglecting any higher-order terms. This gives a system which governs small scale perturbations from this manifold given that the effects of the non-linear terms is indeed negligible, *which they in general are not.* The resulting system is,

$$\dot{\Delta} = Df(x + \Delta)\Delta - (\alpha_x + \alpha_y)\Delta \tag{12}$$

If $\alpha_x = \alpha_y = 0$, equation (7) becomes

$$\begin{aligned} \dot{x} &= f(x) \\ \dot{y} &= f(y) \end{aligned} \tag{13}$$

while equation (12) is simply

$$\dot{\Delta} = Df(x + \Delta)\Delta \tag{14}$$

This is not very tangible, and not at all soluble, but we know from definition 2.11 that for any $\epsilon > 0$, there is a finite $C(\epsilon)$ such that

$$\|\Delta(t)\| \le C(\epsilon)\|\Delta(0)\|e^{(\lambda+\epsilon)t} \tag{15}$$

With this observation in mind, we seek to establish a relation between the solution in the uncoupled case (13), and solutions of the general in equation (12). We so make the ansatz $\Delta(t) = \widehat{\Delta}(t)e^{-(\alpha_x+\alpha_y)t}$, where $\widehat{\Delta}(t)$ is a solution to (14). Differentiation, then substituting in accordance with (14) yields

$$\begin{aligned} \dot{\Delta} &= \dot{\widehat{\Delta}}e^{-(\alpha_x+\alpha_y)t} - (\alpha_x + \alpha_y)\widehat{\Delta}e^{-(\alpha_x+\alpha_y)t} \\ &= Df(x + \Delta)\widehat{\Delta}e^{-(\alpha_x+\alpha_y)t} - (\alpha_x + \alpha_y)\widehat{\Delta}e^{-(\alpha_x+\alpha_y)t} \\ &= Df(x + \Delta)\Delta - (\alpha_x + \alpha_y)\Delta \end{aligned} \tag{16}$$

In other words, if we have a solution $\widehat{\Delta}(t)$ to (14), then $\Delta(t) = \widehat{\Delta}(t)e^{-(\alpha_x+\alpha_y)t}$ solves (12). Returning now to the key observation in (15), we see that

$$\begin{aligned} \|\widehat{\Delta}(t)\|e^{-(\alpha_x+\alpha_y)t} &\le C(\epsilon)\|\widehat{\Delta}(0)\|e^{(\lambda+\epsilon)t}e^{-(\alpha_x+\alpha_y)t} \Leftrightarrow \\ \|\Delta(t)\| &\le C(\epsilon)\|\Delta(0)\|e^{((\lambda+\epsilon)-\alpha_x-\alpha_y)t} \end{aligned} \tag{17}$$

where of course $\|\Delta(t)\|$ vanishes rapidly with increasing t, if $\alpha_x + \alpha_y > \lambda + \epsilon$. As ϵ could be chosen arbitrarily small and the inequality is strict, this is to say $\|\Delta(t)\| \to 0$ as $t \to \infty$ if $\alpha_x + \alpha_y > \lambda$.

Of utmost importance here, is to notice that this last Δ is *not* a solution to (7), but rather the overly simplified version of it, namely (12), where all non-linearities was simply ignored. However, it is not ridiculous to think that this observation may still in some limited sense be relevant to the question of how strong a coupling is needed for synchronization to occur.

This latter approach may not be pretty or rigorously shown to work, but it is in some sense practical. If the underlying system used is a known chaotic system, then information of its Lyapunov specrum is usually readily available, while maximizing $\|Df(x)\|$ where x ranges across some basin of attraction, or some other forward invariant set, might be a lot of work. Furthermore, this method should result in a weaker coupling term, and is so a more economical solution. The next example will illustrate that it may be good enough - at least in certain practical applications.

4.2.3   Example: The Rössler system

Another iconic chaotic system is the Rössler system [20], given by the equations

$$
\begin{aligned}
\dot{x} &= -y - z \\
\dot{y} &= x + ay \\
\dot{z} &= b + zx - cz
\end{aligned}
\tag{18}
$$

Here we will use, $a = 0.2$, $b = 0.2$, $c = 5.7$ is chosen, and the result is another chaotic attractor. We will not study the system in too much detail, but as to give an idea of how the system behaves, fig. 2 shows a sample trajectory.

When coupling two Rössler systems, as to make

$$
\begin{aligned}
\dot{x} &= -y - z + \alpha(u - x) \\
\dot{y} &= x + ay + \alpha(v - y) \\
\dot{z} &= b + zx - cz + \alpha(w - z)
\end{aligned}
$$

$$
\begin{aligned}
\dot{u} &= -v - w + \alpha(x - u) \\
\dot{v} &= u + av + \alpha(y - v) \\
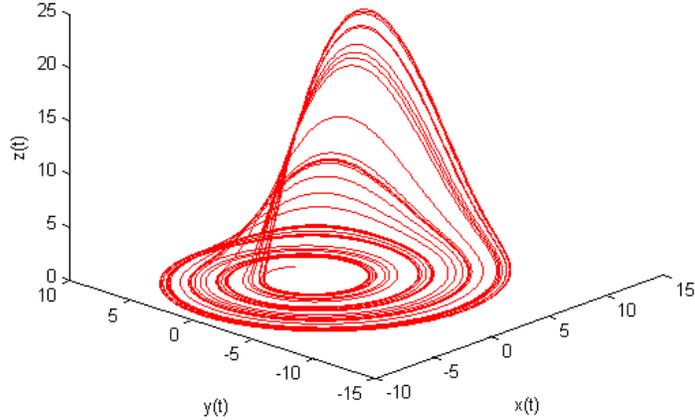\dot{w} &= b + wu - cw + \alpha(z - w)
\end{aligned}
$$

Figure 2: *A sample solution curve to the Rössler equations.* $(x(0), y(0), z(0)) = (1, 2, 0)$, $t \in [0, 200]$, *with* $a = 0.2$, $b = 0.2$, $c = 5.7$.

we would expect the (x, y, z) and (u, v, w) components to synchronize completely if α is chosen to be sufficiently large, given that the initial conditions are such that $\|(x, y, z) - (u, v, w)\|$ is sufficiently small. According to [21], the maximal Lyapunov exponent for the Rössler attractor for the parameters above is $\lambda = 0.0714$, and so the optimist might expect synchronization for $\alpha > 0.0714/2 = \alpha_c$. For comparison, $\|Df(5, 0, 20)\|_2 = 4.502 \ldots$ a number roughly two orders of magnitude greater than λ. $((x, y, z) = (5, 0, 20)$ is just an arbitrary point that is very close to the attracting set to illustrate the point.)

Fig. 3 shows the norm of the difference, $\|e(t)\|$ plotted against t for $\alpha = 1.2\alpha_c$ (left) and $\alpha = 0.8\alpha_c$ (right). Here $e(t) \equiv (x(t) - u(t), y(t) - v(t), z(t) - w(t))$. In both cases $(x(0), y(0), z(0), u(0), v(0), w(0) = (1, 2, 0, 1, 2, 0) + \frac{0.001}{\sqrt{3}}(0, 0, 0, 1, 1, 1)$. That is to say $\|e(0)\| = 0.001$.

4.2.4 Example: Lorentz equations

For the classical parameter values, the maximal Lyapunov exponent of the Lorentz attractor is $\lambda = 0.9056$ [21]. Numerical experiments suggests that
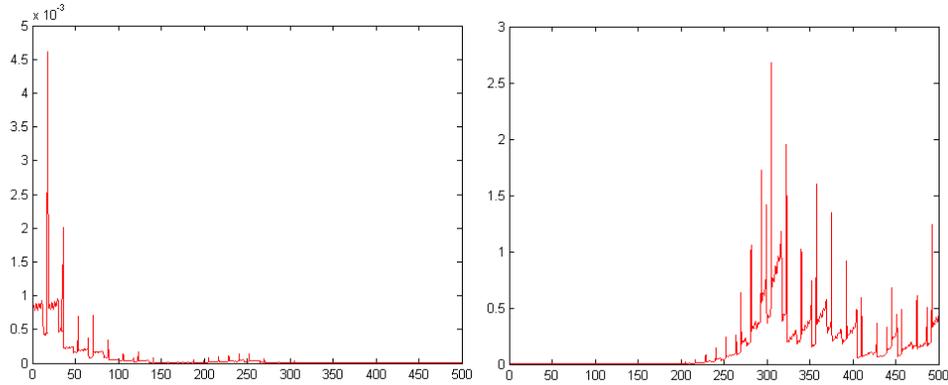
17

Figure 3: *The distance from the synchronization manifold vs. time when the coupling strength is sufficient(left), and insufficient (right). $t \in [0\,,\,500]\,.$ Note the difference in scales in the graphs.*

it may indeed be true that for Lorentz equations bidirectionally coupled in the same way as with the Rössler system above, synchronization occurs for $\alpha > \alpha_c = 0.9056/2$ (fig. 4).

However, a very interesting phenomenon occurs when decreasing the coupling strength to $\alpha = \frac{2}{3}\alpha_c$ (fig. 5). For a while, synchrony is observed, but eventually, the solutions come to a point on the attractor which is so sensitive to perturbations that the coupling term is overcome by the chaotic effects of the attractor, and so (x, y, z) diverge from (u, v, w) quite abruptly.

Both these figures use the classical parameters, $t \in [0\,,\,200]$, and $x(0) = 8.6507$, $y(0) = 20.6688$, $z(0) = -1.680$ (taken from a uniform distribution over $[-25\,,\,25]$) and $(u(0), v(0), w(0)) = (x(0), y(0), z(0)) + \frac{0.1}{\sqrt{3}}(1, 1, 1)$.

After diverging however, the solution settles down at the equilibrium point (C^+, C^-), and remains there. This is of course not at all chaotic behaviour, and might seem unexpected, since both C^+ and C^- are unstable if the coupling term is not present. What we observed appears to be the breaking down of a chaotic attractor due to the effects of adding the dissipative coupling.
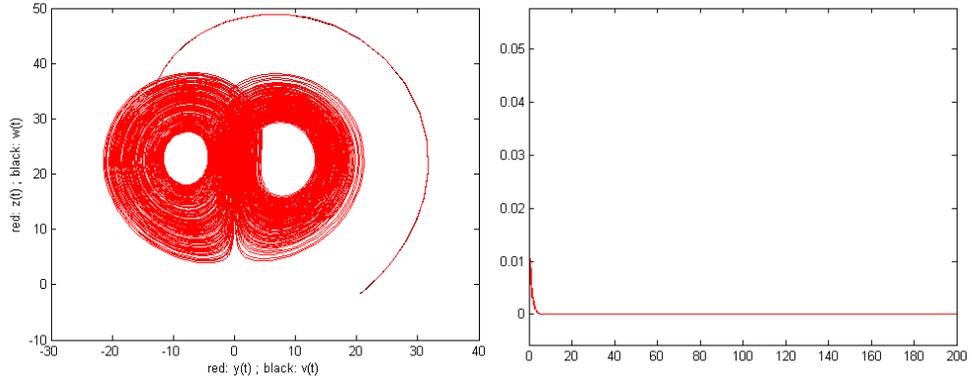
18

Figure 4: *Bidirectionally coupled Lorentz equations, with sufficient coupling strength. Black is representing (u, v, w), red (x, y, z). The red is plotted on top of the black plot, so as a result of the stable synchronization, the black plot is barely visible. The error, $e(t)$ remains close to zero.*
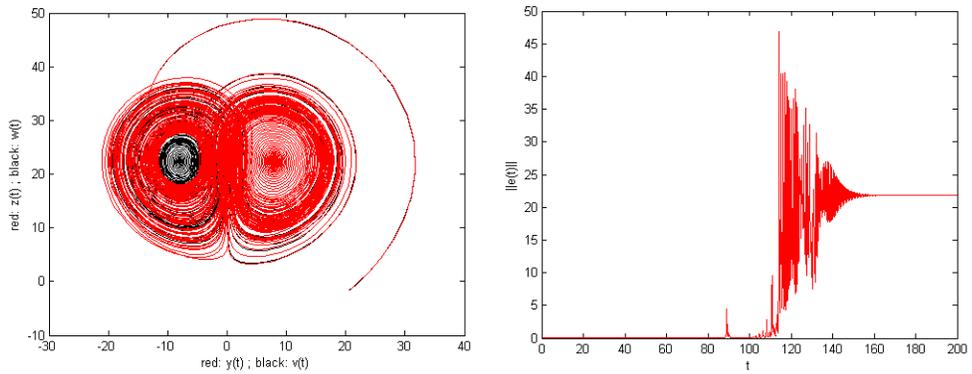

Figure 5: *When coupling strength is insufficient ($\alpha = \frac{2}{3}\alpha_c$), the synchronization is only temporary, and the chaotic dynamics seize as solutions tend to equilibria after diverging.*

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4.3 Method of complete replacement

As discovered in the pioneering work of L. M. Pecora and T. L. Carroll [22], another way of achieving complete synchronization between two systems is what is now called the method of complete replacement. Consider a system

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

One can sometime use one of the state vectors to drive another system,

$$\dot{u} = f(u, v)$$
$$\dot{v} = g(x, v)$$

as to make it synchronize with the first.

We see that the equations for $\dot{u}$ and $\dot{v}$ are almost identical with the ones for $\dot{x}$ and $\dot{y}$. If in the equation for $\dot{v}$ we had a u instead of x, we would just have two independent copies of the same system.

Although one could perhaps see how this could sometimes work, it is not at all obvious that it ever will. In fact, as opposed to the method of dissipative coupling that we proved works in a quite general setting, this method will in general not work, as we will see later.

A rather obvious criterion needed for this method to lead to complete synchronization between y and z is that the Lyapunov exponents of difference system $\dot{\Delta} \equiv g(x, y) - g(x, z)$ are all negative.

In addition to the difficulty of determining Lyapunov exponents for a given system, this criterion has the additional drawback that it is only necessary, and not sufficient [10]. Therefore, when attempting this kind of configuration, other methods of determining the synchronization property are usually preferred.

4.3.1 Applicability to the Lorentz equation

We demonstrate the use of the method of complete replacement using the Lorenz system, and show that a very robust complete synchronization is possible. First we use x to drive a copy of the Lorenz system as follows:

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - \beta z$$

$$\text{(19)}$$

$$\dot{u} = \sigma(v - u)$$
$$\dot{v} = rx - v - xw$$
$$\dot{w} = xv - \beta w$$

Here, x, y, z are of course the original state variables of the Lorentz system, and u, v, w are their copies.

The second case we will study is

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - \beta z$$

$$\text{(20)}$$

$$\dot{u} = \sigma(y - u)$$
$$\dot{v} = ru - v - uw$$
$$\dot{w} = uy - \beta w$$

where we say that y is the driving variable of the subsystem (u, v, w).

Proposition 4.3. *Regardless of initial conditions, the (x, y, z) and (u, v, w) components of equation (19) will synchronize exponentially fast. I.e. $\|(x, y, z) - (u, v, w)\| \leq \alpha e^{-at}$ for some $\alpha, a \in \mathbb{R}^+$ and all $t \geq t_0$.*

Proof. Similarly to the work in [23], we study the evolution of the difference between sender and receiver, $(\Delta_1, \Delta_2, \Delta_3) \equiv (x, y, z) - (u, v, w)$, given by

$$\dot{\Delta}_1 = \sigma(\Delta_2 - \Delta_1) \tag{21}$$
$$\dot{\Delta}_2 = -\Delta_2 - x(t)\Delta_3 \tag{22}$$
$$\dot{\Delta}_3 = x(t)\Delta_2 - \beta\Delta_3 \tag{23}$$

The equations for $\dot{\Delta}_2$ and $\dot{\Delta}_3$ are independent of Δ_1. Thus these components can be studied separately, and conclusions will still hold for the full system.

By introducing a positive definite function $V(\Delta_2, \Delta_3) = \frac{1}{2}\Delta_2{}^2 + \frac{1}{2}\Delta_3{}^2$ and differentiating w.r.t time, we arrive at

$$\begin{aligned}\dot{V} =& \Delta_2\dot{\Delta}_2 + \Delta_3\dot{\Delta}_3 = \Delta_2(-\Delta_2 - x\Delta_3) + \Delta_3(x\Delta_2 - \beta\Delta_3) \\ =& -\Delta_2{}^2 - \beta\Delta_3{}^2 \end{aligned} \tag{24}$$

As V is radially unbounded, it is a global Lyapunov function for (22) and (23).

$\beta > 0$ is assumed since the construction of the Lorenz system. If we further assume $\beta \geq 1$ we would get $\dot{V} = -V - (\beta - 1)\Delta_3{}^2 \leq -V$.

If instead $\beta \in (0, 1)$ we would get $\dot{V} = -\beta V - (1 - \beta)\Delta_2{}^2 \leq -\beta V$, as $1 - \beta > 0$. In either case we see that $\dot{V} \leq -kV$, with $k = \min(1, \beta)$.

$$\dot{V} \leq -kV \Leftrightarrow 0 \geq e^{kt}\dot{V} + e^{kt}kV = \frac{d}{dt}(e^{kt}V) \tag{25}$$

Integration yields

$$V(t) \leq Ce^{-kt}, C \in \mathbb{R}^+ \tag{26}$$

or equivalently

$$\frac{1}{2}(\Delta_2(t))^2 + \frac{1}{2}(\Delta_3(t))^2 \leq Ce^{-kt} \tag{27}$$

which shows that (Δ_2, Δ_3) is confined to the interior of a circle that shrinks exponentially fast. As a consequence $(\Delta_2, \Delta_3) \to (0, 0)$ exponentially fast as t increases.

This still says nothing about Δ_1, so to analyse this component we look at (21).

$$\dot{\Delta}_1 = \sigma(\Delta_2 - \Delta_1) \Leftrightarrow \frac{d\Delta_1}{\Delta_1 - \Delta_2} = -\sigma dt \tag{28}$$

After integration we see that

$$\ln|\Delta_1 - \Delta_2| = -\sigma t + \gamma \Leftrightarrow \Delta_1 = e^{\gamma - \sigma t} + \Delta_2 \leq e^{\gamma - \sigma t} + \sqrt{2C}e^{-\frac{k}{2}t}, \gamma \in \mathbb{R}^+ \tag{29}$$

where the last inequality is of course due to equation (27).

We conclude that $|\Delta_1(t)| \leq (e^{\gamma} + \sqrt{2C})e^{-\min(\sigma,\frac{k}{2})t}$. □

Proposition 4.4. *If $\sigma > \beta > 1$, the (x,y,z) and (u,v,w) components of equation (20) will synchronize exponentially fast, regardless of initial conditions. I.e. $\|(x,y,z) - (u,v,w)\| \leq Ae^{-t}$ for some $A \in \mathbb{R}^+$ and all $t \geq t_0$.*

Proof. Again, we define $\Delta = (\Delta_1, \Delta_2, \Delta_3) \equiv (x,y,z) - (u,v,w)$ and so,

$$\dot{\Delta}_1 = -\sigma\Delta_1$$
$$\dot{\Delta}_2 = r\Delta_1 - \Delta_2 - xz + uw$$
$$\dot{\Delta}_3 = -\beta\Delta_3 + y\Delta_1$$

Clearly, $\Delta_1(t) = \Delta_1(0)e^{-\sigma t}$.

Following the work in [16], we conclude that, $\dot{\Delta}_3 = -\beta\Delta_3 + y\Delta_1 \Leftrightarrow e^{\beta t}(\dot{\Delta}_3 + \beta\Delta_3) = e^{\beta t}y\Delta_1$. Equivalently,

$$\frac{d}{dt}\left(e^{\beta t}\Delta_3(t)\right) = e^{\beta t}y\Delta_1(0)e^{-\sigma t} \Leftrightarrow$$
$$e^{\beta t}\Delta_3(t) = \int_0^t y(s)\Delta_1(0)e^{-(\sigma-\beta)s}ds + \Delta_3(0)$$

We now try to find a bound on $\Delta_3(t)$ as follows:

$$|\Delta_3(t)| = e^{-\beta t}|\int_0^t y(s)\Delta_1(0)e^{-(\sigma-\beta)s}ds + \Delta_3(0)|$$
$$\leq e^{-\beta t}\left(|\int_0^t y(s)\Delta_1(0)e^{-(\sigma-\beta)s}ds| + |\Delta_3(0)|\right)$$
$$\leq e^{-\beta t}\left(\int_0^t |y(s)\Delta_1(0)|e^{-(\sigma-\beta)s}ds + |\Delta_3(0)|\right)$$

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$$\leq e^{-\beta t}\left(\int\limits_{0}^{\infty}|y(s)\Delta_1(0)|e^{-(\sigma-\beta)s}ds + |\Delta_3(0)|\right)$$

$$\leq e^{-\beta t}\left(|Y\Delta_1(0)|\int\limits_{0}^{\infty}e^{-(\sigma-\beta)s}ds + |\Delta_3(0)|\right)$$

where $Y \in \mathbb{R}^+$, is the maximum value that the function $|y(t)|$ attains. The existence of this Y is guaranteed by the trapping region of the Lorentz system, as this implies all solutions $(x(t), y(t), z(t))$ are boundend. As $\sigma - \beta > 0$ this last integral converges to $\frac{1}{\sigma-\beta}$, it is now clear that $|\Delta_3(t)| \leq Ke^{-\beta t}$ for some positive constant K.

Having now obtained this bound, the same method can be applied to bound $|\Delta_2(t)|$:

$$\dot{\Delta}_2 = r\Delta_1 - \Delta_2 - xz + uw = (r - w)\Delta_1 - \Delta_2 - x\Delta_3$$

By the definition of Δ. This leads to

$$\Delta_2(t) = e^{-t}\left(\Delta_2(0) + \int\limits_{0}^{t}e^{-(\sigma-1)s}(r-w)\Delta_1(0)ds - \int\limits_{0}^{t}e^s x\Delta_3(s)ds\right)$$

and so

$$|\Delta_2(t)| \leq e^{-t}\left(|\Delta_2(0)| + |\int\limits_{0}^{t}e^{-(\sigma-1)s}(r-w)\Delta_1(0)ds| + |\int\limits_{0}^{t}e^s x\Delta_3(s)ds|\right) \leq$$

$$e^{-t}\left(|\Delta_2(0)| + \int\limits_{0}^{\infty}e^{-(\sigma-1)s}|(r-w)||\Delta_1(0)|ds + \int\limits_{0}^{\infty}e^s|x||\Delta_3(s)|ds\right) \leq$$

$$e^{-t}\left(|\Delta_2(0)| + \int\limits_{0}^{\infty}e^{-(\sigma-1)s}(r+|z(s)|+|\Delta_3(s)|)|\Delta_1(0)|ds + \int\limits_{0}^{\infty}e^s|x(s)||\Delta_3(s)|ds\right)$$

In this last expression, the first integral converges to a constant as $\sigma-1 > 0$ and $r + |z(t)| + |\Delta_3| \leq r + Z + K$. Z is here the maximum value the function $|z(t)|$ attains, and K is as was previously defined. The second integral converges to a constant as $e^s|x||\Delta_3| \leq XKe^{-(\beta-1)s}$, where of course X is the maximum value that $|x(t)|$ attains.

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Thus each component of $(\Delta_1(t), \Delta_2(t), \Delta_3(t))$ is bounded from above by an exponentially decreasing function. It follows immediately that $\|\Delta(t)\| \leq Ae^{-t}$ for some $A \in \mathbb{R}^+$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

It is worth mention that the constraints on the parameters does not exclude the classic parameters, and so this is viable in at least parts of the chaotic part of parameter space.

Numerical simulations of the method of commplete replacement, using y as the drive signal sent from one set of Lorentz equations to another suggests that the exponent of this bound is optimal. Fig. 6 shows a simulation of equation (20), for initial conditions
$(x(0), y(0), z(0)) = (-20.1230, -11.0751, 7.3441)$ and $(u(0), v(0), w(0)) = (22.8753, -23.2444, -7.1193)$ (taken from a uniform distribution over $[-25\,,\,25]$) together with the logarithm of the error vector plotted against time. For reference, a linear function with slope -1 is shown.
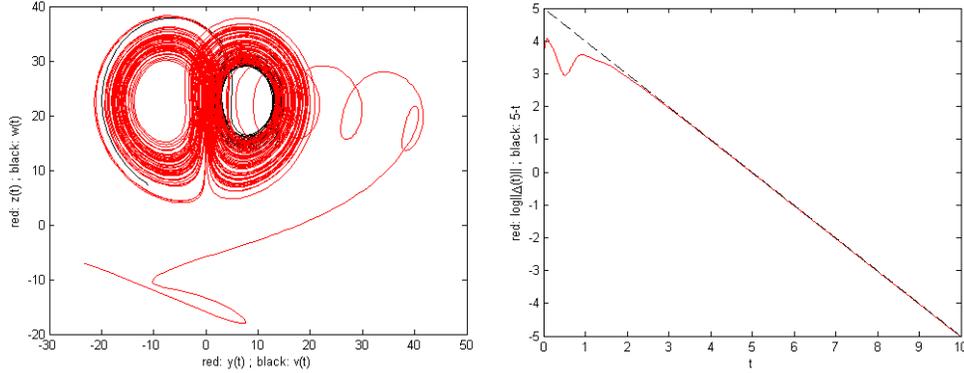

Figure 6: *Left: Phase portrait with (x, y, z) in red and (u, v, w) in black. Right: $log\|\Delta(t)\|$ and a linear decrease (slope -1) plotted against t.*

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4.3.2 Z-Drive

Attempting a similar configuration with z as the drive signal, as follows,

$$\dot{x} = \sigma(y - x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - \beta z$$

$$(30)$$

$$\dot{u} = \sigma(v - u)$$
$$\dot{v} = ru - v - uw$$
$$\dot{w} = xy - \beta w$$

will not result in synchronization.

As noted in [13], the maximal conditional Lyapunov exponent for this latter configuration is positive.

However, as Lyapunov exponents are seldom obtainable analytically, and approximations of them are hard to prove to be accurate, one might not want to rely on them for a proof. Two ways of proving that this configuration will not lead to synchronization is provided in [16], neither using Lyapunov exponents.

4.3.3 Synchronization and cryptology

One interesting potential application to synchronization of dynamical systems was devised by Kevin Cuomo and Alan Oppenheim [12]. The idea is to use a chaotic signal to mask a relatively low volume signal. The signal can then be sent to another identical system, which can use it to recover the message through synchronization.

Cuomo et al. was able to construct an electronic implementation of the Lorentz system, and were indeed able to send and decipher messages through it. The method used is similar to the method of complete replacement, but instead of transferring the value of x or y, one adds a message to it before transmitting. E.g. letting $s(t) \equiv x(t) + m(t)$, where m is the message and then setting up the receiving equations to be

$$\dot{u} = \sigma(v - u)$$
$$\dot{v} = rs(t) - y - xz$$
$$\dot{w} = s(t)y - \beta z$$

Of course if the message is entirely silent (i.e. $m(t) \equiv 0$), this is the same as (19). We have thus shown that without a message added, the synchronisation is very robust - for any initial condition, the sender and receiver will synchronize exponentially fast.

A thorough treatment of this problem was later done by Cuomo [13], and he showed using, among other things, stochastic calculus that the idea is theoretically sound at least when the signal is so-called white noise.

A   Lyapunov functions

A.1   Proof of theorem (2.8)

Proof: Take $r > 0$ small enough so that $B_r = \{x \in \mathbb{R}^n \mid \|x - x^*\| \le r\} \subset U$. ∂B_r is compact, so we can define $M = \{x \in B_r \mid g(x) < \min_{z \in \partial B_r}(g(z))\}$. Any trajectory $x(t)$ that ever enters M remains there for all time, otherwise $g(x(t))$ has *increased* along this trajectory, contradiction the definition of g.
   This is to say M is an invariant set of the system and x^* is indeed stable.

If we further suppose that g is a *strict* Lyapunov function, one can continue as follows to prove asymptotical stability:

Suppose, for a contradiction, that for some trajectory $x(t)$ with $x(t_0) \in M$ we have $\lim_{t \to \infty} x(t) \not\to x^*$. Then $\lim_{t \to \infty} g(x(t)) \not\to 0$. But g is strictly decreasing along trajectories, and also bounded from below by 0, so $\lim_{t \to \infty} g(x(t)) = \alpha > 0$ must hold.

We now define $\widehat{M_\alpha} = \{x \in B_r \mid \alpha \le g(x) \le \min_{z \in \partial B_r}(g(z))\}$, which is compact. Hence, $\dot{g}(x)$ attains its maximum value on $\widehat{M_\alpha}$.

Let $\max_{z \in \widehat{M_\alpha}} \dot{g}(z) = -\mu$. As a consequence of how $\widehat{M_\alpha}$ was constructed, and the negative definiteness of $\dot{g}$, we know that this $\mu > 0$.

It follows that $M_\alpha = \{x \in B_r \mid \alpha \leq g(x) < \min_{z \in \partial S_r}(g(z))\}$ is an invariant set. Hence, for any trajectory $x(t)$ with $x(t_0) \in M_\alpha$ it must hold that

$$g(x(t)) = g(x(t_0)) + \int_{t_0}^{t} \dot{g}(x(\tau))d\tau \leq g(x(t_0)) - \mu(t - t_0)$$

This is a contradiction since $g(x(t))$ is assumed positive definite, yet it is bounded from above by a function linearly decreasing w.r.t t, thus negative for sufficiently large t. $\qquad\square$

Note that the proof is entirely hinged on showing that if we have a Lyapunov function for an open bounded set U containing the equilibrium point x^*. Then embedded in U is an invariant open set M, containing the equilibria, on which boundary the function g is defined and constant. This was then sufficient to show (asymptotic) stability of x^*.

As a consequence, if we can show that some subset V of U has these desired properties, we would in addition to proving the stability of x^* also show that for every $x \in V$, $\phi_t(x) \to x^*$ as $t \to \infty$. This is to say V is a subset of the *basin of attraction of* x^*. With the aid of this observation, and one additional criterion, we are now ready to determine a sufficient condition to prove global stability of a point.

A.2   Proof of theorem (2.10)

Proof. Take any $x_0 \in \mathbb{R}^n$ and let $x(t_0) = x_0$. Define $\widehat{M} = \{x \in \mathbb{R}^n \mid g(x) \leq m\}$ for some $m > g(x_0)$. The condition that g is radially unbounded guarrantees first that such m exists, second that $\widehat{M}$ is bounded, and by construction, $\widehat{M}$ is closed. Also g is constant on $\partial\widehat{M}$. Thus $M = \{x \in \mathbb{R}^n \mid g(x) < m\}$ is an invariant set of the system. The claim of the theorem now follows by contradiction in the same way as in the previous proof, with M and $\widehat{M}$ taking the same roles. $\qquad\square$

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