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Legendrian Conormals of 2-Knots

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Abstract

Let K be a 2-knot. The conormal Lift Λ_K of K is a Legendrian submanifold in the unit cotangent bundle $U^*\mathbb{R}^4$. One can use Legendrian homology on Λ_K to define an invariant of K which is a differential graded algebra partly generated by the Reeb chords of Λ_K. We try to count and specify these Reeb chords by describing the conormal lifts, of 2-knots given as closed 2-dimensional braids, as 1-jet graphs of Λ_{S^2} and get a description for some of these chords.

Contents

1 Introduction

Knot theory is an exciting branch of mathematics, that is not only interesting for pure mathematicians, but has become helpful for physicists and biochemists over the years. Although knot theory has been studied for more than 100 years, people have not yet found an invariant that tells all non-equivalent knots apart.

A promising approach has been made quite recently by using tools of contact geometry. T. Ekholm, J. Etnyre, L. Ng and M. Sullivan used Legendrian homology to define an invariant for classical 1-dimensional links and computed the associated homology of the resulting differential graded algebra in their paper [EENS13].

Definition 1.0.1. A *differential graded algebra* (DGA) is a graded algebra $A = \bigoplus_i A^i$ with a map $d : A \to A$ of degree -1, i.e. $d(A^i) \subset A^{i-1}$, such that $d^2 = 0$ and

$$d(ab) = d(a)b + (-1)^{\deg(a)} ad(b), \quad \forall a, b \in A.$$

We will adapt this idea to 2-dimensional knots. After an introduction to symplectic and contact geometry we define the conormal lift Λ_K for every 2-knot K, which is a Legendrian submanifold in the unit cotangent bundle $U^*\mathbb{R}^4$ and defined to be the set of unit cotangent vectors which vanish on TK. We will see that this submanifold is suitable to define an invariant for 2-knots. This invariant is a DGA, whose algebra is the non-commutative unital algebra over the commutative ring $\mathbb{Z}[H_1(\Lambda)]$ generated by $R(\Lambda_K)$, the Reeb chords of Λ_K.
The Reeb chords $R(\Lambda_K)$ of Λ_K are flow line segments of the so called Reeb vector field, which start and end on Λ_K and have positive length.
We also show that Λ_K is topologically the product $S^2 \times S^1$ such that $H_1(\Lambda_K)$ consists of only one generator and is thus not of further interest.

Our aim is to give a count and description of these Reeb chords. In Chapter 4 we give a full description of Reeb chords for the unknotted S^2 and use this later on to relate some Reeb chords of almost arbitrary 2-knots to the Reeb chords of S^2. By almost arbitrary we mean 2-knots which can be represented as 2-dimensional braids as defined in Chapter 5. However, we will not give a description for all Reeb chords. With our technique we can only describe what happens outside a small disc where all the ramification happens. Chapter 5 is also a good reference for getting an insight of how one can represent 2-knots, or more generally surfaces, by using the motion picture method.

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2 Introduction to Symplectic and Contact Geometry

Before we start with our discussion about 2-dimensional knots we need some background knowledge about the spaces where our knots, or rather the conormal lifts of our knots, live in. We will start with a brief introduction to symplectic geometry and continue with the main objects of this section: contact manifolds. Readers who are already familiar with the basics of symplectic and contact geometry can certainly skip this chapter. The main references for this chapter are [MS98] and [Gei08].

2.1 Linear Symplectic Geometry

Since the tangent space of a symplectic manifold carries the structure of a symplectic vector space, we first recall some properties of these.

Definition 2.1.1. A *symplectic vector space* is a pair (V, ω) consisting of a finite dimensional vector space V and a non-degenerate, skew-symmetric bilinear form $\omega : V \times V \to \mathbb{R}$, i.e. from $\omega(v,w) = 0$ for all $w \in V$ follows $v = 0$, and $\omega(v,w) = -\omega(w,v)$ for all $v,w \in V$.

It is easy to see that a symplectic vector space has to have even dimension since skew-symmetric matrices have zero determinant in the odd-dimensional case and hence a kernel.

Example 2.1.2. Let $(x_1,y_1,\dots,x_n,y_n)$ be the standard basis of $\mathbb{R}^{2n}$. Then $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$ defines a symplectic structure on $\mathbb{R}^{2n}$.

We would like to show that this is, in some way, the only symplectic vector space.

Definition 2.1.3. The *symplectic complement* of a linear subspace $W \subset V$ of a symplectic vector space V is the space

$$W^\omega = \{v \in V \mid \omega(v,w) = 0 \text{ for all } w \in W\}.$$

Theorem 2.1.4. Let (V,ω) be a symplectic vector space of dimension $2n$. Then there exists a basis $u_1,\dots,u_n, v_1,\dots,v_n$ of V such that

$$\omega(u_j,u_k) = \omega(v_j,v_k) = 0, \quad \omega(u_j,v_k) = \delta_{jk},$$

and an isomorphism $\varphi : \mathbb{R}^{2n} \to V$ such that $\varphi^*\omega = \omega_0$, where $\varphi^*\omega(v,w) = \omega(\varphi(v),\varphi(w))$.

Proof. We will prove the theorem by induction on n. As ω is non-degenerate we find $u_1,v_1 \in V$ such that $\omega(u_1,v_1) = 1$, which fulfills the claim for $n = 1$. Let W be the symplectic complement of the subspace spanned by u_1 and v_1. Then (W,ω) is a symplectic vector space of dimension $2n-2$, and by induction hypothesis we find a desired basis $u_2,\ldots,u_n,v_2,\ldots,v_n$. But then $u_1,\ldots,u_n,v_1,\ldots,v_n$ constitute a desired basis of V. Now we define the isomorphism $\varphi : \mathbb{R}^{2n} \to V$ by $\varphi(x_1,\ldots,x_n,y_1,\ldots,y_n) = \sum_{i=1}^n x_i u_i + \sum_{i=1}^n y_i v_i$. By evaluating both sides on the standard basis of $\mathbb{R}^{2n}$ one gets $\varphi^*\omega = \omega_0$. □

We can immediately state:

Corollary 2.1.5. A skew-symmetric bilinear form ω on a $2n$-dimensional vector space V is a symplectic form iff $\omega^n = \omega \wedge \cdots \wedge \omega \neq 0$.

Proof. If ω is degenerate, we find $v_1 \neq 0$ such that $\omega(v_1,w) = 0$ for all $w \in V$. If we complete v_1 to a basis $v_1,\ldots,v_n$ of V we get $\omega^n(v_1,\ldots,v_n) = 0$. On the other hand if ω is non-degenerate we have $\omega^n \neq 0$ by the previous theorem since

$$\begin{aligned}\omega_0^n &= \sum_{i_1=1}^n \cdots \sum_{i_n=1}^n dx_{i_1} \wedge dy_{i_1} \wedge \cdots \wedge dx_{i_n} \wedge dy_{i_n} \\ &= \sum_{|\{i_1,\ldots,i_n\}|=n} dx_{i_1} \wedge dy_{i_1} \wedge \cdots \wedge dx_{i_n} \wedge dy_{i_n} \\ &= n!\; dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n.\end{aligned}$$

□

2.2 Symplectic Manifolds

Finally, a *symplectic manifold* is a differentiable manifold M together with a closed, non-degenerate 2-form $\omega \in \Omega^2(M)$, i.e. (T_qM,ω_q) is a symplectic vector space for all $q \in M$ and $d\omega = 0$.
It follows immediately that $\dim(M) = 2n$, and ω is non-degenerate iff $\omega^n \neq 0$.

Example 2.2.1. We can continue our previous example 2.1.2 and define a symplectic structure on $\mathbb{R}^{2n}$ by $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$.

Darboux's theorem tells us now that all symplectic manifolds are locally identical. But before we state the theorem we have to define the isomorphisms of symplectic manifolds:

Definition 2.2.2. A *symplectomorphism* is an isomorphism $\varphi : (M,\omega) \to (N,\omega')$ such that $\varphi^*\omega' = \omega$, where φ^* is the usual pullback, i.e.

$$(\varphi^*\omega')_x(X_1,X_2) = \omega'_{\varphi(x)}(d\varphi_x(X_1),d\varphi_x(X_2)).$$

Theorem 2.2.3 (Darboux)**.** Every symplectic manifold is locally symplectomorphic to $(\mathbb{R}^{2n},\omega_0)$, [MS98, Thm. 3.15].

2.3 Contact Geometry

Contact manifolds are somehow the odd-dimensional counterpart of symplectic manifolds. Before we will be able to define a contact structure on a manifold we have to observe a property of smooth tangent hyperplane fields.

Let M be a differentiable manifold and $\xi \subset TM$ a smooth subbundle of codimension 1.

Lemma 2.3.1. Locally ξ can be written as the kernel of a 1-form on M, and this equality can be written globally iff TM/ξ is trivial.

Proof. We choose a Riemannian metric g on M and thus have an isomorphism $TM/\xi \cong \xi^\perp$. Since $\xi^\perp$ is a vector bundle, we find a local trivialization $\xi^\perp\big|_U \cong U \times \mathbb{R}$ around every $p \in M$. So there is a nonvanishing section X on $\xi^\perp\big|_U$, and we can define a 1-form on U by $\alpha_U = g(x,-)$. $\xi\big|_U = \ker(\alpha_U)$ follows immediately.
If TM/ξ is trivial the above U can be chosen to be M. On the other hand, if $\xi = \ker(\alpha)$ for $\alpha \in \Omega^1(M)$, we find a section X of $\xi^\perp$ such that $\alpha(X) > 0$, i.e. a nonvanishing section that leads to a trivialization of $\xi^\perp \cong TM/\xi$. □

One can show that codimension 1 subbundles of TM with $\xi = \ker(\alpha)$ are integrable, i.e. it exists a submanifold $N \subset M$ of codimension 1 such that $T_pN = \xi_p$ for all $p \in N$, iff $\alpha \wedge d\alpha \equiv 0$. Contact manifolds are somewhat the opposite:

Definition 2.3.2. A *contact manifold* is a manifold (M,ξ) of dimension $2n+1$ together with a smooth codimension 1 subbundle $\xi \subset TM$ such that for every 1-form with $\xi = \ker(\alpha)$ (either locally or globally) holds $\alpha \wedge (d\alpha)^n \neq 0$. We call ξ a *contact structure* and α a *contact form.*

Notice that every contact form α' with $\xi = \ker(\alpha')$ has to be of the form $\alpha' = \lambda\alpha$ with $\lambda : M \to \mathbb{R} \setminus \{0\}$, and it follows

$$
\begin{aligned}
\alpha' \wedge (d\alpha')^n &= \lambda\alpha \wedge (d\lambda \wedge \alpha + \lambda d\alpha)^n \\
&= \lambda \sum_{k=0}^{n-1} \binom{n}{k} (d\lambda \wedge \alpha)^{n-k-1} \wedge d\lambda \wedge \alpha \wedge \alpha \wedge (\lambda d\alpha)^k \\
&\quad + \lambda\alpha \wedge (\lambda d\alpha)^n = 0 + \lambda^{n+1}\alpha \wedge (d\alpha)^n \neq 0.
\end{aligned}
$$

Moreover, if α is a global contact form, $\alpha \wedge (d\alpha)^n$ defines a volume form on M, and hence M is orientable.

Example 2.3.3. Let $\mathbb{R}^{2n+1}$ be given with coordinates $(x_1,y_1,\ldots,x_n,y_n,z)$. We define a 1-form via $\alpha = dz + \sum_{j=1}^n x_j\, dy_j$. This is indeed a contact form since for the vector

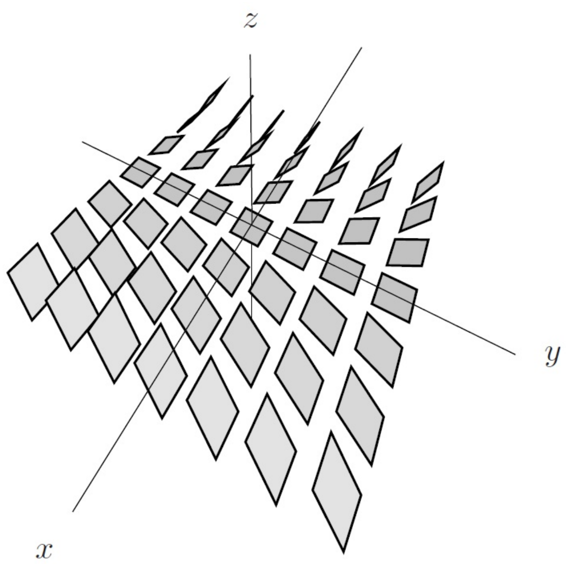


Figure 1: The contact structure $\ker(dz + xdy)$ of the contact form in Example 2.3.3 for the case $n = 1$, [Gei08].

field $X = \frac{\partial}{\partial z}$ we obtain $\iota_X(\alpha \wedge (d\alpha)^n) = (d\alpha)^n \neq 0$, where the last inequality is due to example 2.1.2.

For a contact manifold (M^{2n+1},ξ) we can not find a submanifold of dimension $2n$ that is everywhere tangent to ξ, but for submanifolds of dimension n there is more hope:

Definition 2.3.4. A submanifold $L \subset (M^{2n+1},\xi)$ of dimension n is called a *Legendrian submanifold* if $T_pL \subset \xi_p$ for all $p \in L$.

2.4 The Reeb Vector Field

To every contact form α we can associate a unique vector field R_α, the so called *Reeb vector field*. It is defined by $d\alpha(R_\alpha,-) \equiv 0$ and $\alpha(R_\alpha) \equiv 1$.

Proof. Let $\dim M = 2n+1$. For all $p \in M$, $d\alpha\big|_{T_pM}$ is a skew-symmetric bilinear form with maximal rank $2n$ since $\alpha \wedge (d\alpha)^n \neq 0$. This means that the kernel of this form must be 1-dimensional, and hence $d\alpha(R_\alpha,-) \equiv 0$ defines a nonzero vector field R_α uniquely up to scaling. Since α is smooth, R_α is smooth as well. By using again $\alpha \wedge (d\alpha)^n \neq 0$ we obtain $\alpha(R_\alpha) \neq 0$, i.e. R_α is unique. □

Definition 2.4.1. Let $L \subset (M,\alpha)$ be a Legendrian submanifold. The *Reeb chords* of L are flow line segments of the Reeb vector field of M which begin and end on L and have positive length.

Example 2.4.2. In the previous example 2.3.3 the Reeb vector field is ∂z, which can be checked easily. The flow lines of this vector field are of the form $\{p\} \times \mathbb{R}$ with $p \in \mathbb{R}^{2n}$. We also have a Legendrian submanifold $\mathbb{R}^n \subset \mathbb{R}^{2n+1}$ given by the coordinates $(x_1, \ldots, x_n)$, but all flow line segments which begin and end on $\mathbb{R}^n$ have length zero, and thus there are no Reeb chords. For a general Legendrian submanifold $L \subset \mathbb{R}^{2n+1}$ Reeb chords correspond to distinct points $q_1, q_2 \in L$ such that $q_1 - q_2 \in \{0\}^{2n} \times \mathbb{R}$.

3 The Conormal Construction

In this section we will define what we mean by a 2-knot and construct the conormal of a 2-knot which will be a Legendrian submanifold of some contact manifold.

3.1 The Liouville 1-Form

The Liouville 1-form will be the most important differential form for our discussion and also restricts to a contact form on the mentioned ambient manifold of the conormal lift which we are going to define in the next section. The references for this subsection are [Gei08], [MS98] and [EE05].

Let M be a manifold with local coordinates $q_1, \ldots, q_m$. For every vector field $X \in \mathfrak{X}(M)$ on M we define a map $P_X : T^*M \to \mathbb{R}$ by

$$P_X(q,p) = p(X_q), \qquad \forall p \in T_q^*M.$$

Then q_i and $p_j := P_{\partial/\partial q_j}$ define coordinates on T^*M.
We define a 1-form $\mathbf{p}d\mathbf{q}$ on T^*M locally by $\mathbf{p}d\mathbf{q} = \sum p_i dq_i'$, where $dq_i' := \pi^* dq_i$ for the projection $\pi : T^*M \to M$.
To see that $\mathbf{p}d\mathbf{q}$ does not depend on the choice of coordinates we give an equivalent coordinate free description of this form. Consider the commutative diagram

$$\begin{array}{ccc} TT^*M & \longrightarrow & T^*M \\ \big\downarrow{\scriptstyle d\pi} & & \big\downarrow{\scriptstyle \pi} \\ TM & \longrightarrow & M. \end{array}$$

We define another 1-form λ on T^*M by $\lambda_{(q,p)} = p \circ d\pi_{(q,p)}$.

Proposition 3.1.1. $\lambda = \mathbf{p}d\mathbf{q}$ and for every $\alpha \in \Omega^1(M)$ we have $\alpha^*\lambda = \alpha$. Moreover, λ is the only 1-form on T^*M with this property.

Proof. Since $\pi : T^*M \to M$ is the base projection, we have

$$d\pi_{(q,p)}\left(\frac{\partial}{\partial q_i}\right) = \frac{\partial}{\partial q_i} \quad \text{and} \quad d\pi_{(q,p)}\left(\frac{\partial}{\partial p_i}\right) = 0.$$

Thus we get

$$\lambda_{(q,p)}\left(\frac{\partial}{\partial q_i}\right) = p \circ d\pi_{(q,p)}\left(\frac{\partial}{\partial q_i}\right) = p\left(\frac{\partial}{\partial q_i}\right) = p_i(q,p) \quad \text{and}$$
$$\lambda_{(q,p)}\left(\frac{\partial}{\partial p_i}\right) = p \circ d\pi_{(q,p)}\left(\frac{\partial}{\partial p_i}\right) = p(0) = 0$$

which means nothing but $\lambda = \mathbf{p}d\mathbf{q}$.
For the second property notice that $\alpha \in \Omega^*(M)$ can be written locally as $\alpha = \sum \alpha_i \, dq_i$, and a direct calculation leads to

$$\begin{aligned}\alpha^*\lambda &= \sum \alpha^* p_i \, \alpha^* dq_i' = \sum p_i \circ \alpha \, d(q_i \circ \pi \circ \alpha) = \sum P_{\partial/\partial q_i}(\alpha) \, d(q_i \circ id) \\ &= \sum \alpha \left(\frac{\partial}{\partial q_i} \right) dq_i = \sum \alpha_i \, dq_i = \alpha.\end{aligned}$$

For the other way around take another 1-form μ on T^*M with the same property. This means that $\alpha^*(\mu - \lambda)$ vanishes for every $\alpha \in \Omega^1(M)$. Hence $\mu - \lambda$ has to vanish as well. □

Remark 3.1.2. We will sometimes abuse the notation and write dq_i instead of dq_i'. Obviously, $d\lambda$ defines a symplectic form on T^*M.

3.2 The Conormal Lift of a 2-Knot

Consider the unit cotangent bundle $U^*\mathbb{R}^4 = \{u \in T^*\mathbb{R}^4 | \, |u| = 1\}$ of $\mathbb{R}^4$ based on the standard metric on $\mathbb{R}^4$. Clearly there is an isomorphism $U^*\mathbb{R}^4 \cong \mathbb{R}^4 \times S^3$.

Proposition 3.2.1. The Liouville form λ on $T^*\mathbb{R}^4$ restricted to $U^*\mathbb{R}^4$ is a contact form.

Proof. We know that $d\lambda = \sum dp_i \wedge dq_i$ is a symplectic form on $T^*\mathbb{R}^4$ and hence $(d\lambda)^4$ a volume form. For the radial vector field $X = \sum p_i \frac{\partial}{\partial p_i}$ on $T^*\mathbb{R}^4$ we have

$$\frac{1}{4}\iota_X (d\lambda)^4 = (\iota_X d\lambda) \wedge (d\lambda)^3 = \lambda \wedge (d\lambda)^3,$$

where the first equality follows by induction and the second by $\iota_X d\lambda = \lambda$. Now, $\lambda \wedge (d\lambda)^3$ defines a volume form on every hypersurface in $T^*\mathbb{R}^4$ that is transverse to X and since $U^*\mathbb{R}^4$ is transverse to X, λ is a contact form on $U^*\mathbb{R}^4$. □

Definition 3.2.2. A *2-knot* is a smooth embedding of the sphere S^2 into the 4-dimensional euclidean space $\mathbb{R}^4$.
We call two 2-knots K_1 and K_2 *equivalent* if there exists a smooth ambient isotopy $H : \mathbb{R}^4 \times [0,1] \to \mathbb{R}^4$, i.e. the restriction of H to $t \in [0,1]$ is a diffeomorphism, $H(-,0)$ is the identity, and $H(K_1,1) = K_2$.

Let K be a 2-knot. Then the *conormal lift* of K is the set

$$\Lambda_K := \{u \in U^*\mathbb{R}^4 \,|\, \pi(u) = x \in K \text{ and } u(v) = 0, \quad \forall \, v \in T_xK\},$$

which is a circle bundle over S^2 and thus a 3-manifold.

Proposition 3.2.3. Λ_K is a Legendrian submanifold of $U^*\mathbb{R}^4$.

Proof. Pick $u \in \Lambda_K$ and $v \in T_u\Lambda_K$. Then u can be written locally as $u = \sum u^i \, dq_i$, and $p_i(u) = P_{\partial/\partial q_1}(u) = u\left(\frac{\partial}{\partial q_i}\right) = u^i$. Further we compute

$$\begin{aligned}\lambda_u(v) &= \sum p_i(u)\, \pi^* dq_{i_u}(v) = \sum u^i \, d(q_i \circ \pi)_u(v) = \sum u^i \, dq_{i_{\pi(u)}} \circ d\pi_u(v) \\ &= u(d\pi_u(v)) = 0,\end{aligned}$$

where the last equation holds since $d\pi_u(v) \in T_{\pi(u)}K$ and by definition of Λ_K. But that means nothing but $T\Lambda_K \subset \ker(\lambda)$, hence Λ_K is a Legendrian submanifold of $U^*\mathbb{R}^4$. □

By construction, for two equivalent knots K_1 and K_2 the related conormal lifts Λ_{K_1} and Λ_{K_2} will be Legendrian isotopic, i.e. all intermediate steps of the isotopy are Legendrian submanifolds.

We actually get an ambient isotopy of the conormal lifts. Before we construct this we give a slightly different description of the unit conormal bundle. This is $U^*\mathbb{R}^4 \cong W_{\mathbb{R}^4} := \{v \in T^*\mathbb{R}^4 | v \neq 0\}/\sim$, where we identify v and v' iff $v = cv'$ for a $c > 0$. The isomorphism $W_{\mathbb{R}^4} \to U^*\mathbb{R}^4$ is given by sending an equivalence class $[v]$ to $\frac{v}{|v|}$.

Now, if $H : \mathbb{R}^4 \times [0,1] \to \mathbb{R}^4$ is an ambient isotopy from K_1 to K_2, we get a lift $\tilde{H} : T^*\mathbb{R}^4 \times [0,1] \to T^*\mathbb{R}^4$ to the cotangent bundle by

$$\tilde{H}_t : T^*\mathbb{R}^4 \to T^*\mathbb{R}^4, \quad (q,p) \mapsto p \circ dH_t^{-1}{}_{H_t(q)}$$

which induces a well-defined map $W_{\mathbb{R}^4} \to W_{\mathbb{R}^4}$. For every $t \in [0,1]$ H_t is a diffeomorphism and we have a commutative diagram

$$\begin{array}{ccc} W_{\mathbb{R}^4} & \xrightarrow{\tilde{H}_t} & W_{\mathbb{R}^4} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathbb{R}^4 & \xrightarrow{H_t} & \mathbb{R}^4. \end{array}$$

We have to show that $\tilde{H}_t$ maps Λ_{K_1} to $\Lambda_{H_t(K_1)}$. But this follows from the fact that for $(q,p) \in \Lambda_{K_1}$ we have

$$\tilde{H}_t(q,p)(T_{H_t(q)}H_t(K_1)) = p(dH_t^{-1}{}_{H_t(q)}(T_{H_t(q)}H_t(K_1)) = p(T_qK_1) = 0,$$

and hence $\tilde{H}_t(\Lambda_{K_1}) \subset \Lambda_{H_t(K_1)}$. The other inclusion follows analogously. In particular we have $\tilde{H}_1(\Lambda_{K_1}) = \Lambda_{K_2}$. We obtain even more:

$$\begin{aligned}(\tilde{H}_t{}^*\lambda)_{(q,p)} &= \lambda_{\tilde{H}_t(q,p)} \circ d\tilde{H}_{t(q,p)} = p \circ dH_t^{-1}{}_{H_t(q)} \circ d\pi_{\tilde{H}_t(q,p)} \circ d\tilde{H}_{t(q,p)} \\ &= p \circ dH_t^{-1}{}_{H_t(q)} \circ d(\pi \circ \tilde{H}_t)_{(q,p)} = p \circ dH_t^{-1}{}_{H_t(q)} \circ dH_{tq} \circ d\pi_{(q,p)} \\ &= p \circ d\pi_{(q,p)} = \lambda_{(q,p)},\end{aligned}$$

which means that $\tilde{H}_t$ is even a contactomorphism for every $t \in [0,1]$.

Let us also show that the normal bundle to K in $\mathbb{R}^4$ is trivial and thus Λ_K will be topologically isomorphic to the trivial circle bundle over S^2, i.e. $\Lambda_K \cong S^2 \times S^1$. In order to prove this, at least partially, we need the notion of orientability for vector bundles. A good reference for this discussion is [Hat03].

Definition 3.2.4. A real vector bundle $p : E \to B$ has an *orientation* if every point in B has a neighborhood U and a local trivialization $h : p^{-1}(U) \to U \times \mathbb{R}^n$ carrying the orientations of the fibers in $p^{-1}(U)$ onto the standard orientation of $\mathbb{R}^n$ in the fibers of $U \times \mathbb{R}^n$.

Equivalently, one can say that the transition functions of local trivializations of the bundle take their values in $\mathrm{GL}_n^+(\mathbb{R})$ instead of just in $\mathrm{GL}_n(\mathbb{R})$.
Since S^2 can be given a CW-structure and $\pi_1(S^2) = 0$, it follows that the first Stiefel-Whitney class vanishes, and thus by [Hat03, Prop. 3.11] every vector bundle over S^2 is orientable.

Now let $S^2 = D_+ \cup D_-$, where $D_\pm$ is the upper- respectively lower hemisphere of S^2 such that $D_+ \cap D_- = S^1$. Also let $f : S^1 \to \mathrm{GL}_2^+(\mathbb{R})$ be a smooth map and $E_f := (D_+ \times \mathbb{R}^2 \sqcup D_- \times \mathbb{R}^2)/\sim$, where we identify $(x,v) \in \partial D_- \times \mathbb{R}^2$ with $(x,f(x)v) \in \partial D_+ \times \mathbb{R}^2$.
We obtain a natural projection $p : E_f \to S^2$, and this defines a 2-dimensional vector bundle (this might be easier to see if one extends D_+ and D_- slightly such that $D_+ \cap D_- = S^1 \times (-\epsilon,\epsilon)$).
We call f a *clutching function* for the vector bundle E_f. Now, let $f,g : S^1 \to \mathrm{GL}_2^+(\mathbb{R})$ be homotopic maps, i.e. there exists a homotopy $F : S^1 \times I \to \mathrm{GL}_2^+(\mathbb{R})$ from f to g. With a similar construction as above we obtain a vector bundle $E_F \to S^2 \times I$, and it restricts to vector bundles E_f over $S^2 \times \{0\}$ and E_g over $S^2 \times \{1\}$. Since S^2 is paracompact, E_f and E_g are isomorphic vector bundles, see [Hat03, Prop. 1.7] for instance. Thus we obtain a well-defined map

$$\varphi : \left[S^1,\mathrm{GL}_2^+(\mathbb{R})\right] \to \mathrm{Vect}^2(S^2), \quad f \mapsto E_f,$$

where $\mathrm{Vect}^2(S^2)$ is the set of isomorphism classes of 2-dimensional vector bundles over S^2 and $\left[S^1,\mathrm{GL}_2^+(\mathbb{R})\right]$ the set of homotopy classes of maps from S^1 to $\mathrm{GL}_2^+(\mathbb{R})$, i.e. $\pi_1(\mathrm{GL}_2^+(\mathbb{R}))$.

Proposition 3.2.5. φ is bijective.

Proof. Let $p : E \to S^2$ be an oriented vector bundle. We obtain trivial restrictions $p_\pm : E_\pm \to D_\pm$ since D_+ and D_- are contractible. We choose trivializations $h_\pm : E_\pm \to D_\pm \times \mathbb{R}^2$ and obtain a map $h_+ \circ h_-^{-1} : S^1 \to \mathrm{GL}_2^+(\mathbb{R})$ whose homotopy class $\psi(E) \in [S^1,\mathrm{GL}_2^+(\mathbb{R})]$ is our preimage.
We have to check that ψ is indeed a well-defined inverse to φ. Notice that every other choice of $h_\pm$ differs only by a map $D_\pm \to \mathrm{GL}_2^+(\mathbb{R})$. But since $D_\pm$ is contractible, every such map is homotopic to a constant. And as $\mathrm{GL}_2^+(\mathbb{R})$ is path connected, $h_\pm$ is unique up to homotopy and thus $h_+ \circ h_-^{-1}$ as well. □

So, every 2-dimensional vector bundle on S^2 is determined by $\pi_1(\mathrm{GL}_2^+(\mathbb{R}))$. Since SO(2) is a deformation retract of $\mathrm{GL}_2^+(\mathbb{R})$, we actually have $\pi_1(\mathrm{GL}_2^+(\mathbb{R})) \cong \pi_1(\mathrm{SO}(2)) \cong \mathbb{Z}$, and thus there is a bijection between $\mathrm{Vect}^2(S^2)$ and $\mathbb{Z}$.
Because of the deformation retract, every vector bundle is actually determined by a map $f : S^1 \to \mathrm{SO}(2)$. Take such a map, $u \in S^1$, and define the map $F : S^1 \to S^1$, $F(x) = f(s)u$. By a Lemma of G. Walschap [W$^+$02, Lem. 1.1] we have

Lemma 3.2.6. $\deg(F) = \pm e(E_F)$, where $e(E_F)$ is the Euler number of the vector bundle E_F.

The *Euler number* $e(E)$ of an oriented vector bundle $E \to S^2$ is defined to be the intersection number $\#(S^2,S^2;E)$. See [Hir94] for a precise definition of this number.
So, every 2-dimensional vector bundle S^2 with vanishing Euler number has to be isomorphic to the trivial vector bundle. The next proposition closes the discussion:

Proposition 3.2.7. For the normal bundle $\nu \to K$ from above we have $e(\nu) = 0$.

Proof. Since the normal bundle defines a tubular neighborhood of K, we can identify a neighborhood U of the zero section of ν with a neighborhood of K in $\mathbb{R}^4$. Then

$$e(\nu) = \#(K,K;U) = \#(K,K;\mathbb{R}^4) = 0,$$

where the last equation is due to the fact that K is compact, and thus we can homotope the embeddings $i_1 : S^2 \to \mathbb{R}^4$ and $i_2 : S^2 \to \mathbb{R}^4$ far away from each other such that there is no intersection. □

3.3 The 1-Jet Space of S^3

Since the Reeb vector field of $U^*\mathbb{R}^4$ is not so easy to compute and thus the Reeb chords of Λ_K are not easy to describe either, we give a slightly different description for the ambient manifold $U^*\mathbb{R}^4$.

Let $J^1(S^3) = T^*S^3 \times \mathbb{R}$ be the 1-jet space of S^3. The standard contact 1-form on $J^1(S^3)$ is given by $\alpha = dz - \tilde{\lambda}$, where $\tilde{\lambda}$ is the Liouville form on T^*S^3 and z the coordinate in the $\mathbb{R}$-direction. We define a map $\phi : U^*\mathbb{R}^4 \to J^1(S^3)$ by

$$\phi(q,p) = (p,q - \langle q,p\rangle p,\langle q,p\rangle).$$

Proposition 3.3.1. α is indeed a contact form and ϕ a contactomorphism.

Proof. By contracting the vector field $X = \frac{\partial}{\partial z}$ into $\alpha \wedge (d\lambda)^3$ we yield $(-d\lambda)^3$ which is a volume form, and thus α defines a contact form on $J^1(S^3)$.
The injectivity of ϕ can be shown easily. The surjectivity is a little bit more laborious: take $(a,b,c) \in J^1(S^3)$. Then we have to choose $p = a$. Let $q = b + cp$. Then $\langle q,p\rangle = \langle cp,p\rangle = c$ and $q - \langle q,p\rangle p = b + cp - cp = b$. So we have found a preimage and thus ϕ is a bijection. We also see quite easily that ϕ and ϕ^{-1} are smooth. The last thing is to check that $\phi^*\alpha = \lambda$ holds. We can write $\lambda_{(q,p)} = \langle p,dq\rangle$ and thus get

$$\begin{aligned}\phi^*\alpha = \phi^*dz - \phi^*\tilde{\lambda} &= d\langle q,p\rangle - \langle q - \langle q,p\rangle p,dp\rangle \\ &= \langle dq,p\rangle + \langle q,dp\rangle - \langle q,dp\rangle - \langle q,p\rangle\langle p,dp\rangle \\ &= \langle dq,p\rangle = \lambda,\end{aligned}$$

where we used that $\langle p,dp\rangle$ vanishes since $\langle p,p\rangle = 1$. □

We are interested in two projections: the *Lagrangian projection* $\Pi : J^1(S^3) \to T^*S^3$ and the *front projection* $\Pi_F : J^1(S^3) \to S^3 \times \mathbb{R}$.
Like in example 2.3.3 and 2.4.2 the Reeb vector field on $J^1(S^3)$ is $R_\alpha = \partial_z$ and flow lines of this vector field are $\{p\} \times \mathbb{R}$ with $p \in T^*S^3$. So for a Legendrian submanifold $L \subset J^1(S^3)$ the Reeb chords correspond to points $p_1,p_2 \in L$ such that $\Pi(p_1) = \Pi(p_2)$.

4 Reeb Chords of the Unknot

In this section we are going to investigate the Reeb chords of $\phi(\Lambda_{S^2}) \subset J^1(S^3)$, which correspond to double points of the Lagrangian projection. We slightly follow the discussion in [EENS13, 3.1]. To make things easier to see we can also investigate the double points of the base projection $\pi : J^1(S^3) \to S^3$, which have parallel tangent spaces while we still identify vectors and covectors by the flat metric on $\mathbb{R}^4$. However, it will be helpful to look at the front projection of $\phi(\Lambda_{S^2})$ first the image of which lies in $S^3 \times \mathbb{R}$, where we identify $S^3 \times \mathbb{R}$ in the pictures with $\mathbb{R}^4 \setminus \{0\}$ such that $S^3 \times \{0\}$ corresponds to S^3, $S^3 \times \{p\}$ is a concentric sphere inside S^3 for $p < 0$, and a concentric sphere outside of S^3 for $p > 0$.

Let $S^2 \subset \mathbb{R}^3$ be the standard embedding of S^2. By the canonical inclusion $\mathbb{R}^3 \hookrightarrow \mathbb{R}^4$ we get an embedding of S^2 in $\mathbb{R}^4$. Every point $q \in S^2$ can be written as

$$q = \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \\ 0 \end{pmatrix}$$

with $\theta,\varphi \in \mathbb{R}$. Now a point p is in the conormal lift $\Lambda_{S^2 q}$ iff $|p| = 1$ and $\langle p, T_q S^2\rangle = 0$. We can immediately find two solutions which are linearly independent and thus span the normal of S^2 at q: $p_a = q$ and $p_b = e_4$, the fourth canonical basis vector of $\mathbb{R}^4$. The restriction $|p| = 1$ gives us a full parametrization $p = p(t) = \sin t\ p_a + \cos t\ p_b$ of the conormal lift at q.
Now we apply ϕ to points q and p and obtain

$$(q,p) \mapsto (p, q - \langle q,p\rangle p, \langle q,p\rangle) = (p, q - p\sin t, \sin t).$$

Especially for $t = \frac{\pi}{2}$ we get $(q,p) \mapsto (q,0,1)$ and $(q,p) \mapsto -(q,0,1)$ for $t = \frac{3\pi}{2}$ and thus an entire S^2 of double points when we project onto T^*S^3. So we end up with an S^2 worth of Reeb chords which is quite degenerate. See Figure 2.
In order to get a more generic situation, we perturb the unknot a little bit such that it becomes an ellipsoid. We define the ellipsoid to be

$$E^2_\epsilon = \{(\sin\theta\cos\varphi, (1+\epsilon)\sin\theta\sin\varphi, (1-\epsilon)\cos\theta, 0)\}$$

for a fixed $1 >> \epsilon > 0$. In the same manner as in the S^2-case we get a basis for the

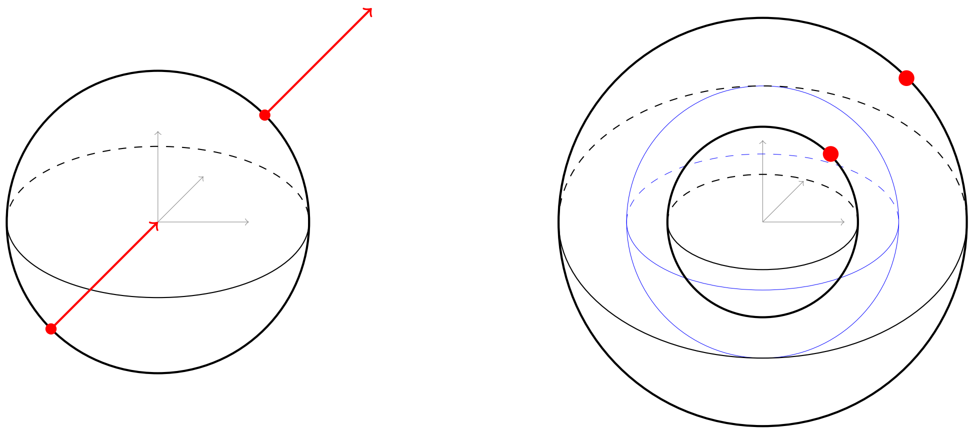


Figure 2: On the left you see two antipodal points and associated unit normal vectors. On the right their image is shown. The red dot in the inner S^2 corresponds to the inward pointing normal vector in the left image. The blue sphere in the middle indicates $S^2 \times \{0\}$.

normal space at q spanned by the vectors

$$p_a = \begin{pmatrix} (1-\epsilon^2)\sin\theta\cos\varphi \\ (1-\epsilon)\sin\theta\sin\varphi \\ (1+\epsilon)\cos\theta \\ 0 \end{pmatrix} \text{ and } p_b = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

However, for $p = \alpha p_a + \beta p_b \in \Lambda_{E^2 q}$ it is not so easy to give a nice expression for α and β anymore. Like before we first focus on the case $\beta = 0$. For $\theta = \frac{\pi}{2}$ the map $\Pi_F \circ \phi$ computes to

$$(q,p) \mapsto (\alpha p_a, \alpha\langle p_a,q\rangle) = \alpha(p_a, 1-\epsilon^2) = \pm|p_a|^{-1}(p_a, 1-\epsilon^2) \in S^1 \times \mathbb{R}.$$

If we vary φ over $[0,2\pi)$, we get two ellipses like in Figure 3.
Now, for a more general θ the only things that change are a fixed contribution to the x_3-direction in S^3 and a fixed difference in scaling. So the resulting picture is similar to the picture in Figure 3, except for scaling.
When we change the roles of θ and φ, i.e. we fix φ and vary θ, we have the same discussion as before and end up with pictures like in Figure 3.
Therefore, when the only restriction is $\beta = 0$ we get two ellipsoids inside each other as Figure 4 shows.
We get exactly 6 candidates a_i,b_i,c_i, $i = 1,2$ for Reeb chords as indicated in the figure. The 2-dimensional tangent planes in these pictures agree at these 6 points. In order to

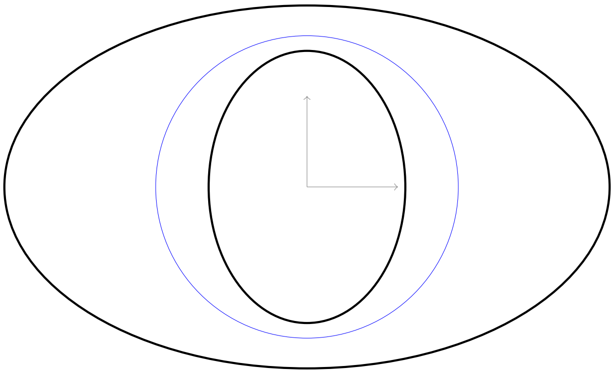


Figure 3: In blue the unit circle is indicated and we see the two ellipses that we obtain by fixing $\beta = 0$ and $\theta = \frac{\pi}{2}$.

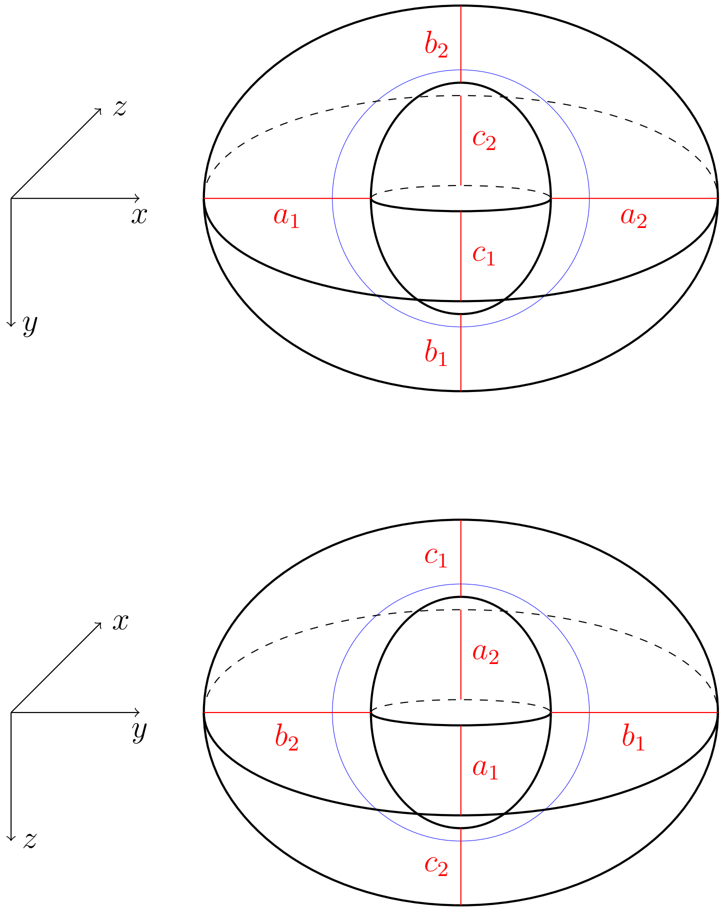


Figure 4: If we only fix β and let θ and φ run, we end up with these ellipsoids. Top-view, $\theta = \frac{\pi}{2}$ on the left. Side-view, $\varphi = \frac{\pi}{2}$ on the right. The 6 Reeb chords are shown in red. The unit sphere $S^2 \times \{0\}$ is indicated in blue.

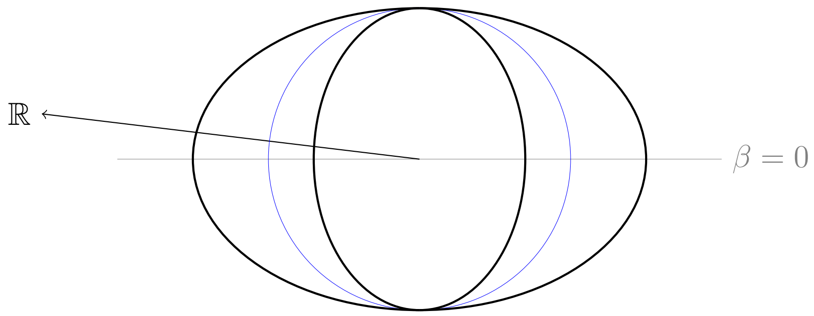


Figure 5: The picture is parametrized by α and β while θ and φ are fixed. One realizes that the inner- and outer arcs have parallel tangents only if $\alpha\beta = 0$.

see that the entire tangent spaces agree we have to look what happens when we alternate β. So we fix θ and φ and vary α and β. For negative α's and fitting β's we obtain two arcs inside S^1. For positive α's we obtain two arcs outside S^1 like in Figure 5. We see that only in the case $\beta = 0$ the one-dimensional tangent spaces are parallel, so there are Reeb chords if and only if $\beta = 0$, and we thus have found all of them.

5 Representations of Surfaces in $\mathbb{R}^4$

5.1 Visualization of the 4th-Dimension

Since we can not embed every 2-knot in $\mathbb{R}^3$ without self-intersections, we need to find a way of visualizing 2-knots in $\mathbb{R}^4$. There is a rather easy way to do this. We are going to consider so called "movies" as suggested by C. Adams in [Ada94]. However, there is a more formal approach to those "movies" developed in [KSS82] and [Kam94]. According to my advisor everything in this section can be done in the smooth category (using e.g. Morse theory). For simplicity, however, we will discuss this in the slightly weaker category of piecewise linear manifolds.

Definition 5.1.1. A submanifold N of dimension n embedded in a topological manifold M of strictly higher dimension m is said to be *locally flat* if every $x \in N$ has a neighborhood $U \subset M$ such that $(U, U \cap N)$ is homeomorphic to $(\mathbb{R}^m, \mathbb{R}^n)$.

Let us identify $\mathbb{R}^3 \setminus \{z{-}axis\}$ with the space $\mathbb{R}^2_+ \times S^1$, where $\mathbb{R}^2_+ = \{(x,0,z) \in \mathbb{R}^3 \mid x > 0\}$. As every link l in $\mathbb{R}^3$ can be represented as the closure of a braid, we can always assume that l lies in $\mathbb{R}^2_+ \times S^1$, that it is transverse to the fibers $\mathbb{R}^2_+ \times \{s\}$ for every $s \in S^1$, and the projection $l \to S^1$ induced from the natural projection $\mathbb{R}^2_+ \times S^1 \to S^1$ is a covering map. We call such a braid a *closed braid around the z-axis.*

Definition 5.1.2. An ambient isotopy H of $\mathbb{R}^3$ is called an $\mathfrak{R}$*-deformation* if each homeomorphism H_t fixes the z-axis and preserves $\mathbb{R}^2_+ \times \{s\}$ for all $s \in S^1$.

Let l be a link in $\mathbb{R}^3$ with fixed orientation and $\alpha, \alpha' \subset l$ arcs (or paths) on l.

Definition 5.1.3. A 2-disc (in this context also called an *oriented band*) B in $\mathbb{R}^3$ is said to *span the link l by attaching arcs* α, α' if $\alpha \cap \alpha' = \emptyset$, $\quad B \cap l = \partial B \cap l = \alpha \cup \alpha'$, and the orientation of $\overline{l \cup \partial B \setminus (\alpha \cup \alpha')}$ is compatible with the orientations of $l \setminus (\alpha \cup \alpha')$ and $\partial B \setminus (\alpha \cup \alpha')$ like illustrated in the first part of Figure 6.

More generally, let $B_1, \ldots, B_m$ be pairwise disjoint bands in $\mathbb{R}^3$ such that each B_i spans l by attaching arcs α_i, α'_i.

Definition 5.1.4. The oriented link

$$h(l; B_1, \ldots, B_m) = \overline{l \cup \partial B_1 \cup \cdots \cup \partial B_m \setminus (\alpha_1 \cup \alpha'_1 \cup \cdots \cup \alpha_m \cup \alpha'_m)}$$

is called the *link obtained from l by hyperbolic transformations along the bands* $B_1, \ldots, B_m$.

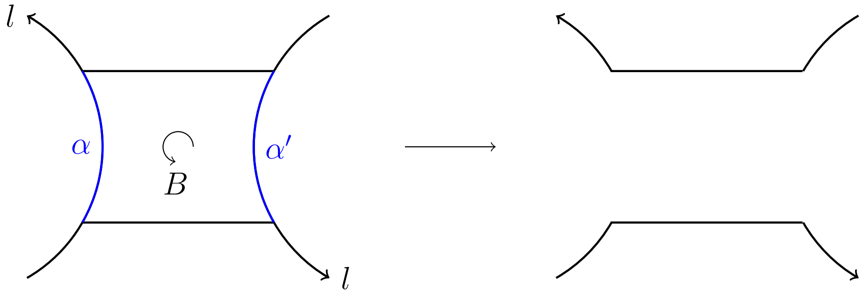


Figure 6: On the left the link l with attached oriented band B and the transition to $h(l;B)$ on the right-hand side.

Let $l_0 \to l_1 \to \cdots \to l_n$ be a sequence of oriented links such that $l_{i+1} = h(l_i; B_1^i, \ldots, B_{m_i}^i)$, $i = 0, \ldots, n-1$. For the closed interval $[a,b]$ let $a = t_0 < \cdots < t_n = b$ be an equidistant subdivision. We define a locally flat surface $F_{t_i}^{t_{i+1}} = F_{t_i}^{t_{i+1}}(l_i, l_{i+1}; B_1^i, \ldots, B_{m_i}^i)$ in $\mathbb{R}^3 \times [t_i, t_{i+1}] \subset \mathbb{R}^4$ by

$$F_{t_i}^{t_i+1} \cap \mathbb{R}^3 \times \{t\} = \begin{cases} l_i \times \{t\} & \text{if } t_i \ \leq t < \frac{t_i + t_{i+1}}{2} \\ (l_i \cup B_1^i \cup \cdots \cup B_{m_i}^i) \times \{t\} & \text{if } t = \frac{t_i + t_{i+1}}{2} \\ l_{i+1} \times \{t\} & \text{if } \frac{t_i + t_{i+1}}{2} < t \leq t_{i+1}. \end{cases}$$

We obtain an induced orientation on $F_{t_i}^{t_{i+1}}$ and $\mathbb{R}^3 \times [t_i, t_{i+1}]$ from orientations on $l_{i+1} \times \{t_{i+1}\} \subset \mathbb{R}^3 \times \{t_{i+1}\}$ and $\mathbb{R}^3 \times \{t_{i+1}\}$ by identifying $l_{i+1} \times \{t_{i+1}\} \subset \mathbb{R}^3 \times \{t_{i+1}\}$ with $l_{i+1} \subset \mathbb{R}^3$.
In the end we have $F_a^b := \bigcup_{i=0}^{n-1} F_{t_i}^{t_{i+1}}$, a locally flat oriented surface in $\mathbb{R}^3 \times [a,b]$.

Definition 5.1.5. We call F_a^b the *realizing surface* in $\mathbb{R}^3 \times [a,b]$ of the sequence $l_0 \to l_1 \to \cdots \to l_n$.

If l_0 and l_n are trivial links, i.e. they are isotopic to μ_0 and μ_n copies of S^1 respectively, then there exist mutually disjoint discs $D_1, \ldots, D_{\mu_0}$ and $D_1', \ldots, D_{\mu_n}'$ in $\mathbb{R}^3$ such that $\partial(D_1 \cup \cdots \cup D_{\mu_0}) = l_0$ and $\partial(D_1' \cup \cdots \cup D_{\mu_n}') = l_n$. We are now able to define a closed oriented surface $\bar{F}_a^b$ in $\mathbb{R}^3 \times [a,b]$ by

$$\bar{F}_a^b = F_a^b \cup (D_1 \cup \cdots \cup D_{\mu_0}) \times \{a\} \cup (D_1' \cup \cdots \cup D_{\mu_n}') \times \{b\}$$

such that the orientations of $\bar{F}_a^b$ and F_a^b agree. We call $\bar{F}_a^b$ the *closed realizing surface* in $\mathbb{R}^3 \times [a,b]$ of the sequence $l_0 \to l_1 \to \cdots \to l_n$ with trivial links l_0 and l_n.

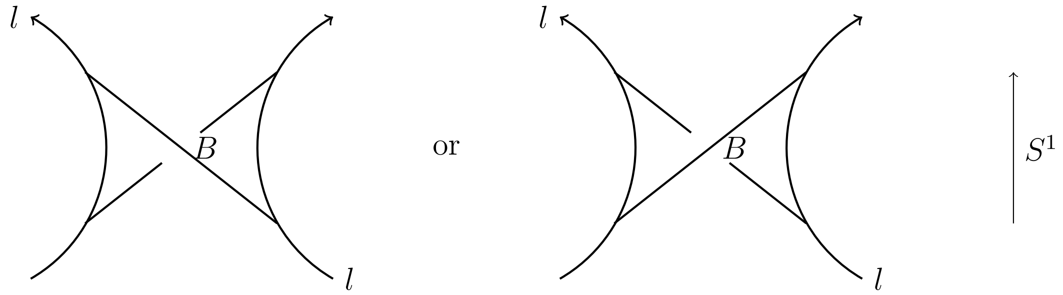
Remark 5.1.6. $\bar{F}_a^b$ depends on the choices of $D_1, \ldots, D_{\mu_0}$ and $D_1', \ldots, D_{\mu_n}'$. An example is given in [KSS82].

However, one can show the following Lemma. For a proof we refer to [KSS82] again.

Lemma 5.1.7. $\bar{F}_a^b$ is uniquely determined up to isotopies of $\mathbb{R}^4$ which fix $\mathbb{R}^3 \times [a+\epsilon, b-\epsilon]$ for $\epsilon > 0$ sufficiently small. □

Before we state the most important theorem of this section we have to give another definition:

Definition 5.1.8. For a link l around the z-axis we call a band B *simple* if there exists an $\mathfrak{R}$-deformation of l and B such that there is a neighborhood $D^2 \times [s,s'] \subset \mathbb{R}^2_+ \times S^1$ around B where the band looks like in the following picture:



Remark 5.1.9. A link l' obtained from a link l around the z-axis by attaching a simple band B is also a link around the z-axis.

Theorem 5.1.10. Every locally flat, connected, closed, and oriented surface in $\mathbb{R}^4$ is ambient isotopic to the closed realizing surface of the sequence $l_- \to l_+$ with only simple bands $B_1, \ldots, B_n$ such that

- $[s_i, s_i'] \cap [s_j, s_j'] = \emptyset$ for $i \neq j$, where B_i is contained in $D_i^2 \times [s_i, s_i']$ and
- l_- and l_+ are links around the z-axis, which can be deformed to trivial links by $\mathfrak{R}$-deformations.

A proof can be found in [Kam94] and [KSS82].

We call this procedure of representing a surface a *movie* since for every $t \in [a,b]$, thought as a point in time, we get a segment of the surface in $\mathbb{R}^3$, our 3-dimensional screen.

5.2 2-Dimensional Braids

There is a different way of representing surfaces introduced by O. Ya. Viro and further investigated by S. Kamada, [Kam94]. While the movies from above help us to visualize surfaces in the 4-dimensional euclidean space and hence our 2-knots, the description of surfaces as "2-dimensional braids" helps us to calculate the Reeb chords of a 2-knot as done in section 6. Here we will see how to go from a closed realizing surface of the type produced by Theorem 5.1.10 to a 2-dimensional braid.

Definition 5.2.1. Let D_1^2 and D_2^2 be two discs. A *2-dimensional m-braid* is a compact and oriented surface F embedded into $D_1^2 \times D_2^2$ such that

1. $pr_2 : F \to D_2^2$ is a branched covering of degree m and
2. $F \cap (D_1^2 \times \partial D_2^2) \cong X_m \times \partial D_2^2$,

where X_m is a set of m points in D_1^2.

We can extend F to a closed oriented surface $\bar{F}$ in a natural way such that $\bar{F}$ is embedded into $D_1^2 \times S^2 = D_1^2 \times (D_2^2 \cup D^2)$. Here, D^2 is a copy of D_2^2 with reversed orientation and we identify the boundaries of D_2^2 and D^2 in the canonical way. One might see this easier by identifying S^2 with $D_2^2/\partial D_2^2$ and notice $\partial F = X_m \times \partial D_2^2$.
If we identify $D_1^2 \times S^2$ with a tubular neighborhood of S^2 in $\mathbb{R}^4$, we call $\bar{F}$ a *closed 2-dimensional m-braid* in $\mathbb{R}^4$. Then $\bar{F} \to S^2$ is still an m-fold branched covering, and we can assume that $\bar{F} \cap (D_1^2 \times D^2) \cong X_m \times D^2$ with $D^2 \subset S^2$ sufficiently small.

We are going to sketch that every closed 2-dimensional braid can be represented as a closed realizing surface from section 5.1 and the other way around. At first we will see how one gets from a realizing surface to a 2-dimensional braid. For references of stated facts look at [Kam94].
Let $S^2 = D_- \cup A \cup D_+ \subset \mathbb{R}^4$ with $A = C \times [-1,1] \subset \mathbb{R}^4$ where C is the unit circle in the $xy-$plane and D_-, D_+ are the standard disc fillings of S^1 in $\mathbb{R}^3 \times \{-1\}$ and $\mathbb{R}^3 \times \{1\}$ respectively. Now let S be the closed realizing surface in $\mathbb{R}^3 \times [-1,1]$ obtained from a trivial m-link l_- by hyperbolic transformations to a trivial link l_+ along the simple and disjoint bands $B_1, \dots, B_n$. We impose that l_- and l_+ lie in $D_1^2 \times C$ and the realizing surface F of $l_- \to l_+$ lies in the tubular neighborhood $D_1^2 \times A = (D_1^2 \times S^2)\big|_A$ of A in $\mathbb{R}^3 \times [-1,1]$. We can now deform each of these simple bands in the following manner:
We end up with an m-fold branched covering $pr_2 : F \to A$. W.l.o.g. S is the closed realizing surface that we obtain from F by Theorem 5.1.10 with two families of m unknotted, parallel discs in $D_1^2 \times D_-$ and $D_1^2 \times D_+$ respectively. Then the induced

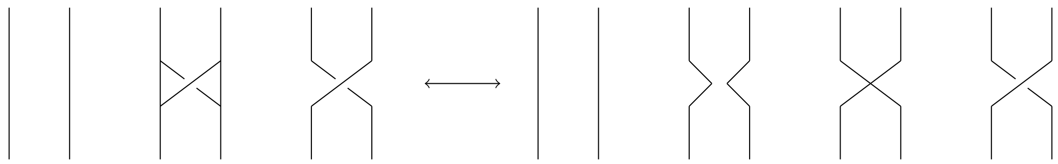


Figure 7:

ramified covering $S \to S^2$ from $D_1^2 \times S^2 \to S^2$ has degree m, and thus S is a closed 2-dimensional m-braid in $\mathbb{R}^4$. Moreover, the branched covering is simple.

For the other way around let S be a closed 2-dimensional m-braid in $D_1^2 \times S^2 \subset \mathbb{R}^4$ such that the associated branched covering is simple. We can deform $S \subset D_1^2 \times S^2$ such that all branch points lie in $C \times \{0\} \subset \mathbb{R}^3 \times \{0\}$ of A. For a branch point p let $\tilde{P}$ be the lift to S of multiplicity 2. Then S looks like the right side of Figure 7 around $\tilde{P}$.

Like before we can deform this ramification point to a simple band in $\mathbb{R}^3 \times \{0\}$. For $\epsilon > 0$ sufficiently small we define $l_- \times \{-\epsilon\} := S \cap (\mathbb{R}^3 \times \{-\epsilon\})$ and $l_+ \times \{\epsilon\} := S \cap (\mathbb{R}^3 \times \{\epsilon\})$ such that $S \cap (\mathbb{R}^3 \times [-\epsilon,\epsilon])$ is the realizing surface of hyperbolic transformations from l_- to l_+ along the simple bands we have defined in order to dissolve the ramification points. Since $S \to S^2$ restricts to $S \cap (\mathbb{R}^4 \setminus (\mathbb{R}^3 \times (-\epsilon,\epsilon))$ without any branch points, we see that l_- and l_+ are trivial closed m-braids, and $S \cap (\mathbb{R}^3 \times (-\infty, -\epsilon])$ as well as $S \cap (\mathbb{R}^3 \times [\epsilon,\infty))$ are families of parallel unknotted discs. Hence S is ambient isotopic to the closed realizing surface of $l_- \to l_+$.

Filling out the details in the discussion above together with Theorem 5.1.10 yields:

Theorem 5.2.2. Every locally flat, connected, closed, and oriented surface in $\mathbb{R}^4$ is ambient isotopic to a closed 2-dimensional braid whose induced branched covering is simple.

For our further discussion we assume that a 2-knot is given as a 2-dimensional braid. In Figure 8 a movie for a covering with two ramification points is shown.

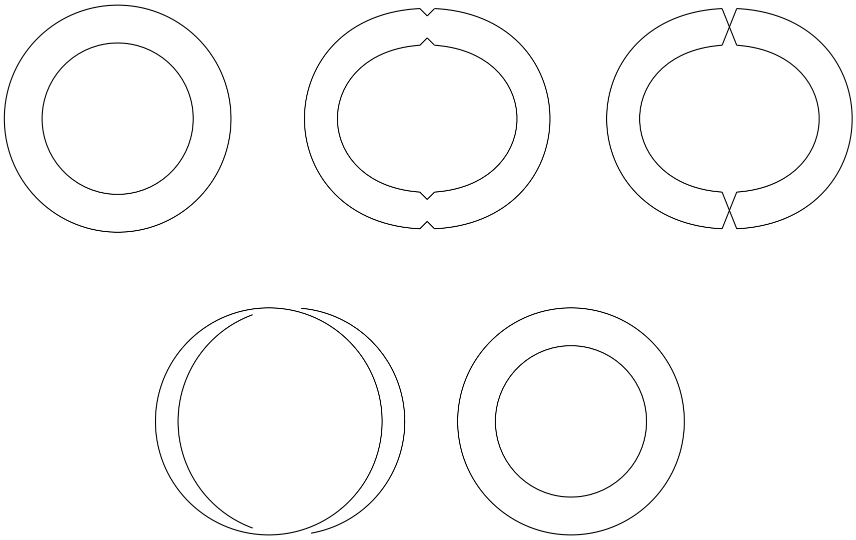


Figure 8: A movie representation for a covering of S^2 with two ramification points.

6 Reeb Chords of a 2-Knot

In this section we will discuss an idea of how to compute the Reeb chords for a 2-knot given by a closed 2-dimensional braid as discussed in the previous section.

6.1 The Unknotted Case

Let K be a 2-knot given as a closed 2-dimensional braid with associated branched covering $p : K \to S^2$ and degree $\deg(p) = n$. We choose K such that all ramification happens above a small open disc in S^2 and remove this disc in S^2 together with its preimages in K, and we end up with a proper covering $\tilde{p} : \tilde{K} \to D^2$ where D^2 is the standard compact unit disc in $\mathbb{R}^4$, which is ambient isotopic to S^2 with a small open disc removed. We thus use the notion of D^2 and S^2 with removed disc interchangeably. We will focus on the Reeb chords of $\tilde{K}$, but give an idea of how to compute the remaining chords in the end of this paper.
Following [EENS13] we construct n functions $f_i : D^2 \to D^2$ such that $f_i(s) \neq f_j(s)$, $i \neq j$ and such that $\tilde{K}$ is represented by the graph of the multi-function $f_B = \{f_1, \ldots, f_n\}$. The trivial multi-function consists of n functions mapping D^2 to 0. The distance of $\tilde{K}$ to S^2 can be controlled by the distance of f_B to the trivial multi-function. So if we fix a neighborhood N of Λ_{D^2} we can achieve $\Lambda_{\tilde{K}} \subset N$ provided f_B is sufficiently C^1-small.
Our idea is now to relate the Reeb chords of $\Lambda_{\tilde{K}}$ and Λ_{D^2} by identifying a neighborhood of the zero-section Λ_{D^2} in $J^1(\Lambda_{D^2})$ with a neighborhood of Λ_{D^2} in $U^*\mathbb{R}^4$. For that we take s to be a coordinate in D^2 and σ an associated cotangent coordinate. Also we think of T^*S^1 as $\{(\xi,\eta) \in \mathbb{R}^2 \times \mathbb{R}^2 | \; |\xi| = 1, \xi \cdot \eta = 0\}$. Our identifying map

$$\Phi : J^1(\Lambda_{D^2}) \to U^*\mathbb{R}^4 = \mathbb{R}^4 \times S^3$$

is defined to be

$$\begin{aligned}\Phi_1(s,\sigma,\xi,\eta,\zeta) &= s + \sqrt{1-|\sigma|^2}^{-1}(\eta_1 e_3 + \eta_2 e_4) \\ &\quad + \zeta\left(-\begin{pmatrix}\sigma\\0\end{pmatrix} + \sqrt{1-|\sigma|^2}(\xi_1 e_3 + \xi_2 e_4)\right) \\ \Phi_2(s,\sigma,\xi,\eta,\zeta) &= -\begin{pmatrix}\sigma\\0\end{pmatrix} + \sqrt{1-|\sigma|^2}(\xi_1 e_3 + \xi_2 e_4),\end{aligned}$$

where e_i is the $i-th$ standard vector of $\mathbb{R}^4$ as usual.
$\Phi\big|_{D^2\times 0\times S^1\times 0\times 0}$ is a parametrization of $D^2 \times S^1 \cong \Lambda_{D^2} \subset U^*\mathbb{R}^4$ then. Notice that Φ is only defined in a neighborhood of the zero section in $J^1(\Lambda_{D^2})$.

Proposition 6.1.1. The pullback of the Liouville-form λ on $U^*\mathbb{R}^4$ is the standard contact form $d\zeta - \sigma \cdot ds - \eta \cdot d\xi$ on $J^1(\Lambda_{D^2})$.

Proof. Since $\Phi^*\lambda = \Phi^*\langle p,dq\rangle = \langle \Phi_2, d\Phi_1\rangle$, we should compute $d\Phi_1$ first:

$$\begin{aligned}
d\Phi_1 &= e_1\, ds_1 + e_2\, ds_2 \\
&\quad + \sum \left(\partial_{\sigma_i}\left(\sqrt{1-|\sigma|^2}^{-1}\right)(\eta_1 e_3 + \eta_2 e_4) + \zeta(-e_i + \partial_{\sigma_i}(\sqrt{1-|\sigma|^2})(\xi_1 e_3 + \xi_2 e_4)\right) d\sigma_i \\
&\quad + \zeta\sqrt{1-|\sigma|^2}\, e_3\, d\xi_1 + \zeta\sqrt{1-|\sigma|^2}\, e_4\, d\xi_2 \\
&\quad + \sqrt{1-|\sigma|^2}^{-1} e_3\, d\eta_1 + \sqrt{1-|\sigma|^2}^{-1} e_4\, d\eta_2 + \Phi_2\, d\zeta.
\end{aligned}$$

Now,

$$\begin{aligned}
\langle \Phi_2, d\Phi_1\rangle &= -\sigma_1 ds_1 - \sigma_2 ds_2 + \sum \partial_{\sigma_i}\left(\sqrt{1-|\sigma|^2}^{-1}\right) \underbrace{\eta\cdot\xi}_{=0}\, d\sigma_i \\
&\quad + \zeta \sum \left(\sigma_i + \partial_{\sigma_i}\left(\sqrt{1-|\sigma|^2}\right)\sqrt{1-|\sigma|^2}\,|\xi|\right) d\sigma_i \\
&\quad + \zeta(1-|\sigma|^2)\xi\cdot d\xi + \xi\cdot d\eta + |\Phi_2| d\zeta \\
&= d\zeta - \sigma\cdot ds - \eta\cdot d\xi,
\end{aligned}$$

where we used that $\xi \cdot d\xi$ vanishes due to $|\xi| = 1$, and $\xi \cdot d\eta = -\eta \cdot d\xi$ due to the fact that $\xi \cdot \eta$ vanishes. $\square$

Since $\omega \wedge (d\omega)^3$ is a volume form for every contact form ω and the pullback commutes with the differential and the wedge product, Φ pulls back a volume form to another volume form. Hence, $d\Phi$ is injective and thus Φ is an immersion in a neighborhood of the zero section.

We have to check that there is also a neighborhood of the zero section where Φ is injective in order to get an embedding. So assume $\Phi(s,\sigma,\xi,\eta,\zeta) = \Phi(\tilde{s},\tilde{\sigma},\tilde{\xi},\tilde{\eta},\tilde{\zeta})$. Then it follows immediately that $s = \tilde{s}$, $\sigma = \tilde{\sigma}$, and we are left with the system

$$\begin{pmatrix} s - \zeta\sigma \\ \eta + \zeta\alpha^2\xi \end{pmatrix} = \begin{pmatrix} \tilde{s} - \tilde{\zeta}\sigma \\ \tilde{\eta} + \tilde{\zeta}\alpha^2\xi \end{pmatrix},$$

where $\alpha = \sqrt{1-|\sigma|^2}$. Now the injectivity follows from $\alpha^2 \neq 0$ and

$$0 = \eta\cdot\xi = \tilde{\eta}\cdot\xi + (\tilde{\zeta}-\zeta)\alpha^2\xi\cdot\xi = (\tilde{\zeta}-\zeta)\alpha^2.$$

So Φ is an embedding in a neighborhood of the Λ_{D^2}.

We will now use this map to show that $\Lambda_{\tilde{K}} \subset J^1(\Lambda_{D^2})$ is represented by the 1-jet extension of the multi-function $F_B = \{F_1, \ldots, F_n\}$ with $F_i : D^2 \times S^1 \to \mathbb{R}$ given by $(s,\xi) \mapsto f_i(s) \cdot \xi$.

Proof. Take a small neighborhood $N \subset U^*\mathbb{R}^4$ of Λ_{D^2} such that Φ^{-1} is defined on N and maps it to some neighborhood where Φ is an embedding. For f_B sufficiently C^1-small $\Lambda_{\tilde{K}}$ is in N. We want to show that $\Phi^{-1}(\Lambda_{\tilde{K}})$ is the 1-jet extension of F_B. Every point in the image of f_B is given by

$$(f_i(s))_x e_3 + (f_i(s))_y e_4$$

and using the identification of $U^*\mathbb{R}^4$ and $J^1(S^3)$ we see that the $\mathbb{R}$-component in $J^1(S^3)$ is given by

$$\langle (f_i(s))_x e_3 + (f_i(s))_y e_4, \xi_1 e_3 + \xi_2 e_4 \rangle = F_i(s,\xi).$$

The $\mathbb{R}$-component of $J^1(\Lambda_{D^2})$ maps to the $\mathbb{R}$-component of $J^1(S^3)$ by

$$\zeta \mapsto \pi_{\mathbb{R}} \langle \Phi_1, \Phi_2 \rangle = -s \cdot \sigma + \xi \cdot \eta + \zeta = \zeta.$$

Hence the $\mathbb{R}$-component in $J^1(\Lambda_{D^2})$ is given by $F_i(s,\xi)$ too.
Effectively we have computed the front projection of $\Lambda_{\tilde{K}}$ in $J^1(\Lambda_{D^2})$. But this is actually enough to determine $\Lambda_{\tilde{K}}$ completely and conclude that it is given by the 1-jet extension of F_B, which is the fact that a Legendrian in a jet bundle can be reconstruct from its front projection. □

Since Φ maps the Reeb flow of $J^1(\Lambda_{D^2})$ to the Reeb flow of $J^1(S^3)$, it maps Reeb chords of $\Lambda_{\tilde{K}} \subset J^1(\Lambda_{D^2})$ to Reeb chords of $\Lambda_{\tilde{K}} \subset J^1(S^3)$ and the other way around for Reeb chords in a neighborhood N like in the last proof. We thus get a notion of *short Reeb chords* which lie entirely in the neighborhood N and *long Reeb chords* which do not.
Now we can prove the main theorem of this section:

Theorem 6.1.2. We can choose $\tilde{K}$ such that it has exactly $6n^2$ long Reeb chords and $n(n-1)$ short chords.

Proof. For the short chords notice that we can perturb the multi-function f_B such that $|f_i - f_j|$ has exactly one critical point around $0 \in D^2$. This can, for instance, be achieved by changing f_i to $\tilde{f}_i(s) = f_i(s)(c||s||^2 + 1)$ for a suitable $c > 0$.
We are looking for the double points of $\Pi(\Lambda_{\tilde{K}}) \subset T^*\Lambda_{D^2}$. Let v be such a double point, i.e. there exist local defining functions $f, \tilde{f}$ such that $(p, df_p, f(p)), (p, d\tilde{f}_p, \tilde{f}(p)) \in \Lambda_{\tilde{K}}$ and $df_p = d\tilde{f}_p$. Then $p = \pi(\Pi^{-1}(v))$ is a critical point of two local defining functions for $\Lambda_{\tilde{K}}$ and the other way around. This means we are really looking for critical points of the functions

$$F_{ij}(s,t) = (f_i(s) - f_j(s)) \cdot (\cos(t), \sin(t)).$$

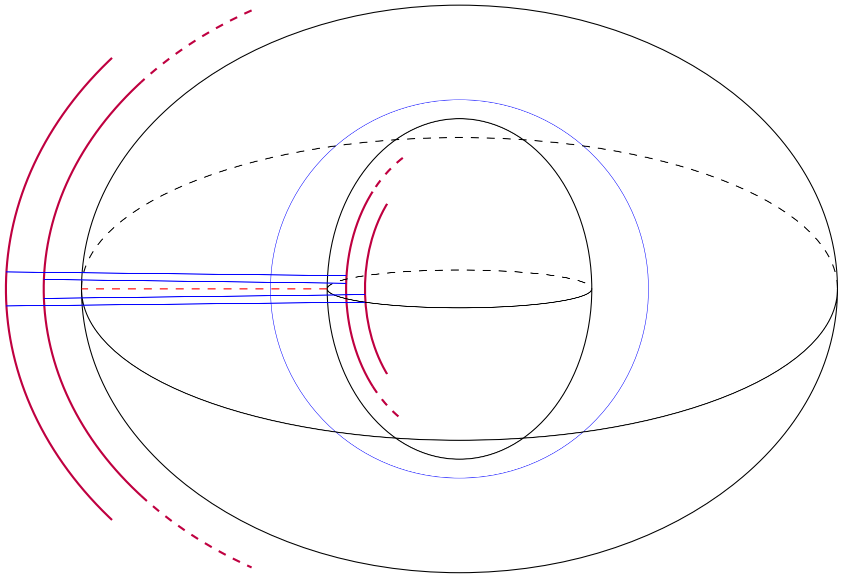


Figure 9: Here we see the relevant part of $\Lambda_{D^2} \subset J^1(S^3)$ in black with one Reeb chord in red. Purple and blue indicate parts of $\Lambda_{\tilde{K}}$ and it's long Reeb chords above the red Reed chord of Λ_{D^2}.

Now, $dF_{i,j}(s,t) = 0$ iff

$$\begin{aligned} (f_i'(s) - f_j'(s)) \cdot (\cos(t), \sin(t)) = 0 \quad &\text{and} \\ (f_i(s) - f_j(s)) \cdot (-\sin(t), \cos(t)) = 0. \end{aligned} \tag{1}$$

Since $f_i - f_j$ vanishes nowhere, s is critical for $|f_i - f_j|$ if and only if s is critical for $f_i - f_j$. So equation 1 is fulfilled if s is critical for $|f_i - f_j|$ and t takes one of the two values for which the second equation in 1 is true.
This means that we get a short chord for every pair (i,j) with $i \neq j$, hence $n(n-1)$. For the long chords we think of D^2 again as S^2 with a small disc removed. Since $\Lambda_{\tilde{K}}$ is the 1-jet extension of a multi-function with n sheets over Λ_{D^2} which has 6 Reeb chords, we get $6n^2$ long Reeb chords for $\Lambda_{\tilde{K}}$. □

6.2 Outlook

We have gained control about Reeb chords in parts of the knot where no ramification happens. But what about the small disc we have removed since all ramification, i.e. knottedness happens in that area? We could try to find a similar argument for this area as above. But then we have the problem that we can not make things C^1-close due to ramification.
A different approach would be to compute the Reeb vector field for $U^*\mathbb{R}^4$ to get a more

geometric description of the Reeb chords. Then one would have to investigate what happens in a vicinity of a ramification point.

References

[Ada94] Colin C Adams, *The knot book*, American Mathematical Soc., 1994.

[EE05] Tobias Ekholm and John B Etnyre, *Invariants of knots, embeddings and immersions via contact geometry*, Geometry and topology of manifolds **47** (2005), 77–96.

[EENS13] Tobias Ekholm, John B Etnyre, Lenhard Ng, and Michael G Sullivan, *Knot contact homology*, Geometry & Topology **17** (2013), no. 2, 975–1112.

[Gei08] Hansjörg Geiges, *An introduction to contact topology*, vol. 109, Cambridge University Press, 2008.

[Hat03] Allen Hatcher, *Vector bundles and k-theory*, http://www.math.cornell.edu/˜hatcher (2003).

[Hir94] Morris W Hirsch, *Differential topology, volume 33 of graduate texts in mathematics*, 1994.

[Kam94] Seiichi Kamada, *A characterization of groups of closed orientable surfaces in 4-space*, Topology **33** (1994), no. 1, 113–122.

[KSS82] Akio Kawauchi, Tetsuo Shibuya, and Shin'ichi Suzuki, *Descriptions on surfaces in four-space i normal forms*, Mathematics seminar notes **10** (1982), no. 1, 75–125.

[MS98] Dusa McDuff and Dietmar Salamon, *Introduction to symplectic topology*, Oxford University Press, 1998.

[RW98] Andrew Ranicki and Elmar Winkelnkemper, *High-dimensional knot theory: Algebraic surgery in codimension 2*, Springer Science & Business Media, 1998.

[W$^+$02] Gerard Walschap et al., *The euler class as a cohomology generator*, Illinois Journal of Mathematics **46** (2002), no. 1, 165–169.