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Analysis of stretched grids as buffer zones in simulations of wave propagation

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ABSTRACT

A zone of increasingly stretched grid is a robust and easy-to-use way to avoid unwanted reflections at artificial boundaries in wave propagating simulations. In such a buffer zone there are two main damping mechanisms, dissipation and under-resolution that turns a traveling wave into an evanescent wave. We present analysis in one and two space dimensions showing that evanescent decay through under-resolution is a very efficient way to damp waves. The analysis is supported by numerical computations.

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1. Introduction

Non-reflecting boundary conditions are of vital importance in numerical simulations of convective flows and the research on non-reflecting boundary conditions has long going traditions. In general exact non-reflecting boundary conditions are global, both in space and time. There are also many local non-reflecting boundary conditions. They can be divided into three main types namely, local approximations of exact non-reflecting boundary conditions, see e.g. [7,8,11], buffer zones, see e.g. [12,1] and Perfectly Matched Layers, see e.g. [3,2].

In [4] an overview of non-reflecting boundary conditions for compressible flows is given. In the paper the vast development of non-reflecting boundary conditions for linear problems is pointed out, whereas there is a need for enhanced development and knowledge of the performance of non-reflecting boundary conditions for non-linear flows such as turbulent shear flows.

Most buffer zones involve a stretching of the grid, but damping can also be enhanced by a forcing function or artificial viscosity. The popularity of buffer zones lays in their simplicity. They are easy to implement and in most cases, the change in the time step restriction due to stability is small. However, unwanted reflections from buffer zones can occur either at the entrance at the buffer zones, due to e.g. grid stretching or sudden increase of artificial viscosity and forcing functions, or reflections at the outflow boundary due to too low damping within the buffer zone.

In [1] reflections from buffer zones due to under-resolution of outgoing waves, and the influence of different orders of artificial viscosity terms on the damping of reflections are studied. It is shown by numerical experiments for linear hyperbolic systems that artificial viscosity terms based on high-order undivided difference substantially reduce the reflections from the buffer zone. For a dispersive scheme, under-resolution of a wave can be seen as equivalent to lowering the phase

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speed of the wave. Karni [14] suggested a slowing-down operator and a similar concept was investigated as the acoustic black hole layer boundary condition in [16].

A more simplistic approach is to use grid stretching in the buffer zone together with the same numerical scheme as in the interior computational domain. The main difficulty with this type of boundary zone is to determine how large grid stretching that should be applied and the number of grid points in the boundary zone. Often the choice is based on a rule of thumb from previous knowledge of the particular code used. Hence, a better understanding of the propagation and dissipation of waves on a stretched grid decreases the time spent on determining suitable buffer zones in a practical simulation.

In a series of papers Vichnevetsky et al., [19–21], analyzed the influence of grid stretching on wave propagation in one space dimension. The analysis is based on the discrete advection equation, where waves and wave packets are imposed as initial data and the discrete equation is Fourier transformed in space or in time. The conclusions from the papers can be summarized as follows, see [21]: a slowly and uniformly changing mesh creates no scattering as long as waves are well resolved, but a sharp change in a otherwise uniformly changing mesh creates scattering also for a well resolved wave. Hence, when a wave-like structure enters a region with a smoothly stretched grid, the reflection will be very small. Our experience agrees well with this conclusion. However, for wave packets the reflections due to a group velocity of opposite sign are of the same order as the amplitude of the wave packet, see [9,15].

In this paper, we focus on buffer zones based on pure grid stretching with a constant numerical viscosity coefficient. The focus of the paper is to analyze the damping mechanisms for waves in stretched grids. A main result is that damping from grid stretching is much more efficient than damping by dissipation when the grid stretching turns the propagating waves into evanescent waves. A similar result was briefly discussed in [13], but is studied in more detail in this paper. With this understanding, reflections of upstream waves originating from the outflow boundary can be avoided. The results of the analysis can in turn be used to formulate estimates on the number of cells needed in a buffer zone, for a given system and grid stretching ratio.

Two approaches can be taken when analyzing the effect of varying grid spacing on wave propagation. Either, the frequency in time is kept constant via imposing temporally periodic boundary data. This approach leads to analysis of a discrete boundary value problem. The other alternative, which so far has been the most common approach, is to impose a wave as periodic initial data on an infinite domain, see [6,10,20]. As will be seen, the two approaches are naturally equal for the continuous problem when the model includes only advection. For the semi-discrete equation, however, the two approaches lead to different insights.

To begin with we analyze semi-discrete scalar advection and advection–diffusion equations, respectively, where time periodicity is imposed as boundary data in a one dimensional setting. We find that physically propagating waves turn into evanescent waves when the grid resolution is low. We analyze the decay rate, and a main contribution of this work is the conclusion that the amplitude of waves is very efficiently reduced in the evanescent regime. This type of decay is compared with decay due to viscous damping, and we find that evanescent decay is a much more efficient way to reduce the amplitude of physical waves. In many cases there are also high frequency spurious waves present, and viscous damping can be important for such waves. We also extend the analysis to hyperbolic systems in two space dimensions, and apply the results to the linearized Euler equations in an aero-acoustic setting. Important contributions in the paper are the precise expressions for decay of waves, which can be used for determining the thickness of buffer zones both in one and two space dimension. Numerical computations demonstrate the validity of the analysis.

2. Background

In the main part of this paper, we analyze the discrete solution of linear hyperbolic equations in terms of boundary value problems. In [20,21] as well as in [6] the discrete solutions are studied in form of an initial value problem. In order to compare the insights from the two approaches, we describe the initial value approach in this section for completeness.

We consider

$$u_t + cu_x = 0, \quad -\infty < x < \infty, \quad (1)$$

with periodic initial data, that is

$$u(x, 0) = e^{-i\xi x}. \quad (2)$$

The solution of the continuous problem is

$$u(x, t) = e^{i(\omega t - \xi x)}, \quad (3)$$

where the temporal frequency ω and the spatial wave number ξ are related via the dispersion relation

$$\omega = c\xi. \quad (4)$$

We discretize in space using second order accurate central differences, yielding

$$(v_j)_t + cD_0 v_j = 0. \quad (5)$$

Here $x_j = jh$, $j = 0, \pm 1, \pm 2, \dots$ define the grid, and $v_j(t)$, $j = 0, \pm 1, \pm 2, \dots$ is a time dependent grid function, and D_0 is the usual central difference operator,

$$D_0 v_j = \frac{v_{j+1} - v_{j-1}}{2h}.$$

Corresponding to the continuous problem we will consider periodic initial data for the discrete equation,

$$v_j(0) = e^{i\xi x_j}. \quad (6)$$

The solution to (5) and (6) is

$$v_j = e^{i(\omega t - \xi x_j)}, \quad (7)$$

where

$$\omega = c \frac{\sin \xi h}{h}. \quad (8)$$

The relation (8) is the standard discrete dispersion relation, see e.g. [10]. In this setting the spatial wave number ξ is given by initial data and the resulting temporal frequency ω is always bounded by $|\omega| \leq c/h$. The highest wave number that can be represented on a grid corresponds to two points per wavelength and hence the frequency of the discrete initial data must satisfy $|\xi| \in [0, \pi/h]$. We note that there are two wave numbers, $|\xi_1| \in [0, \pi/(2h)]$ and $|\xi_2| \in [\pi/(2h), \pi/h]$ corresponding to the same temporal frequency.

With artificial dissipation added to the semi-discrete equation (5) we have

$$(v_j)_t + c D_0 v_j = \varepsilon h^2 D_+ D_- v_j \quad (9)$$

$$v_j(0) = e^{i\xi x_j}. \quad (10)$$

Here

$$D_+ D_- v_j = \frac{v_{j+1} - 2v_j + v_{j-1}}{h^2}.$$

The solution to (9) and (10) is

$$v_j = e^{i(\omega t - \xi x_j)}, \quad \text{where } i\omega = ci \frac{\sin \xi h}{h} - 4\varepsilon \sin^2 \frac{\xi h}{2}. \quad (11)$$

Clearly the viscous term only affects the amplitude and not the phase speed.

This analysis holds for constant grid spacing and is limited to cases with spatially periodic solutions. However, quantitative information can be obtained by a heuristic extension to settings when a wave propagates into a region with variable grid or variable viscous damping, and is no longer spatially periodic, see [5].

3. The boundary value problem

In the previous section the initial boundary value problem was studied. That approach is valid for solutions that are periodic in space. This is not the case in a buffer zone with stretched grids, where waves enter the buffer zone with a certain wave length and amplitude and are thereafter modified both in wave number and in amplitude as they propagate in the buffer zone. A more detailed analysis is obtained by considering a boundary value problem.

In this section we study solutions of two discrete model boundary value problems. From the first model problem an understanding of the effect of the grid resolution on wave propagation is obtained. The second model problem yields an insight of the influence of a viscous term on a propagating wave, for different resolutions in space.

3.1. Scalar advection

As a model problem we consider the advection equation in one space dimension

$$u_t + cu_x = 0, \quad (12)$$

with boundary data that is periodic in time

$$u(0, t) = e^{i\omega t}. \quad (13)$$

For simplicity we assume $c > 0$. The exact solution of (12) and (13) is unique and given by

$$u(x, t) = e^{i(\omega t - \xi x)}, \quad (14)$$

where the temporal frequency ω and the spatial wave number ξ are related via the dispersion relation

Fig. 1. Absolute value of the two characteristic roots a function of $\omega h/c$.

$$\omega = c\xi. \quad (15)$$

Naturally the solutions (14) and (3) are equal.

Consider a semi-discrete approximation of (12) using central differences in space

$$(v_j)_t + D_0 v_j = 0. \quad (16)$$

We prescribe the frequency in time, for instance by prescribing a periodic function as boundary data,

$$v_0(t) = e^{i\omega t} \hat{v}_0, \quad (17)$$

yielding the solution of the form

$$v_j(t) = e^{i\omega t} \hat{v}_j, \quad (18)$$

where $\hat{v}_j$ satisfies a difference equation with constant coefficients,

$$D_0 \hat{v}_j = -\frac{i\omega}{c} \hat{v}_j. \quad (19)$$

In the discrete setting another boundary condition is also needed. We will discuss this further below.

In order to solve Eq. (16) and (17) we make the ansatz

$$\hat{v}_j = \eta^j \quad (20)$$

where η satisfies the characteristic equation

$$\eta^2 + i2\frac{\omega h}{c}\eta - 1 = 0. \quad (21)$$

There are two roots of Eq. (21), namely

$$\eta_{\pm} = -i\frac{\omega h}{c} \pm \sqrt{1 - \left(\frac{\omega h}{c}\right)^2}. \quad (22)$$

In Fig. 1 we have plotted the absolute values of the two roots as functions of $\omega h/c$.

As long as $|\eta_+| = |\eta_-| = 1$ we can use the representation

$$\eta_+ = e^{-\xi_1 h}, \quad \eta_- = e^{-\xi_2 h}, \quad \xi_{1,2} \in \mathcal{R}. \quad (23)$$

The solution of the boundary value problem by the ansatz (18) is valid for all ω , but when $|\eta_{\pm}| \neq 1$ the discrete solution cannot be periodic in space. We have made the above precise in the following theorem.

Theorem 1. For different resolutions in space, the roots (22) satisfy

1. $0 < \omega h/c < 1$. The roots of (21) are distinct, and have absolute value unity, $|\eta_+| = |\eta_-| = 1$. We can express $\eta_{\pm}$ as (23), where we define $\xi_1 \in [0, \pi/2h]$ and $\xi_2 \in [\pi/2h, \pi/h]$ (spatial wave numbers) as the solutions to

$$\frac{\omega h}{c} = \sin \xi h. \quad (24)$$

2. $\omega h/c = 1$. There is a double root, $\eta_+ = \eta_- = -i$, and the corresponding solution to (24) is $\xi = \pi/(2h)$. In this case (18) yields a solution with a linearly growing component.
3. $\omega h/c > 1$. There are two roots, $|\eta_+| < 1$ and $|\eta_-| > 1$, corresponding to a decaying and a growing evanescent wave, respectively.

Proof. The proof follows straightforwardly from (22). We only include details for the relation between η and ξ in the first case. Remember that $\omega h/c = \sin(\xi_1 h)$, $h\xi_1 \in [0, \pi/2]$ and $h\xi_2 \in [\pi/2, \pi]$. Thus $\sin(\xi_1 h) = \sin(\xi_2 h) > 0$ and $0 < \cos(\xi_1 h) = -\cos(\xi_2 h)$, and we have

$$\begin{aligned}\eta_+ &= -i\frac{\omega h}{c} + \sqrt{1 - \left(\frac{\omega h}{c}\right)^2} = -i\sin(\xi_1 h) + \sqrt{1 - \sin^2(\xi_1 h)} = \\ &= -i\sin(\xi_1 h) + \cos(\xi_1 h) = i\sin(-\xi_1 h) + \cos(-\xi_1 h) = e^{-\xi_1 h}, \\ \eta_- &= -i\frac{\omega h}{c} - \sqrt{1 - \left(\frac{\omega h}{c}\right)^2} = -i\sin(\xi_1 h) - \sqrt{1 - \sin^2(\xi_1 h)} = \\ &= -i\sin(\xi_2 h) - \cos(\xi_2 h) = i\sin(-\xi_2 h) + \cos(-\xi_2 h) = e^{-\xi_2 h}.\end{aligned}$$

Note that in the propagating regime

$$(\eta_+)^j = e^{-i\xi_1 x_j}, \quad (\eta_-)^j = e^{-i\xi_2 x_j}. \quad (25)$$

The first root corresponds to a well resolved wave (more than 4 points per wavelength), and we call it the physical solution. The other one has a higher wave number, and will be called the spurious solution.

Corollary 1. The general solution to (16) in the time harmonic case with temporal frequency ω is

$$v_j(t) = \begin{cases} e^{i\omega t}(\alpha_+(\eta_+)^j + \alpha_-(\eta_-)^j), & \frac{\omega h}{c} \neq 1 \\ e^{i\omega t}(\alpha_+ + j\alpha_-)(-i)^j, & \frac{\omega h}{c} = 1. \end{cases} \quad (26)$$

Here $\eta_{\pm}$ given by (22). On a bounded interval $0 \leq i \leq N$, the coefficients $\alpha_{\pm}$ are uniquely determined by (17) and the corresponding condition

$$v_N(t) = e^{i\omega t} \hat{v}_{\infty}. \quad (27)$$

In the evanescent regime the infinite, half-line problem has a unique solution if the condition (27) is replaced by a boundedness condition, $|v_j| < \infty$, which leads to $\alpha_- = 0$. On a bounded interval both boundary conditions are needed also for the evanescent case, but they decouple, with each condition essentially determining the coefficient of one component.

3.1.1. Relation to initial value problem

There is a correspondence between the periodic boundary data problem with $0 < \omega h/c < 1$ (the well-resolved case), and the periodic initial data problem, expressed by (24). Note that there are two spatial wave numbers, ξ_1 and ξ_2 , corresponding to the same temporal frequency. Therefore it is not surprising that the boundary value problem doesn't have a unique solution in the well-resolved case. When considering a time-periodic problem in a realistic setting the problem is posed on a bounded spatial domain, and the downstream boundary treatment will determine the solution uniquely.

When $\omega h/c \geq 1$, which we will sometimes call the under-resolved case, the correspondence between the solutions of the periodic boundary data problem and periodic initial data problem, breaks down. The linearly growing component, or the evanescent waves cannot be represented as a spatially periodic function.

3.2. Scalar advection–diffusion

In this section we analyze the influence of an artificial viscosity term on the solution of the boundary value problem. First we study the roots of the relevant characteristic equation in a qualitative way. For some parameter values we prove quantitative results, which are formulated in Theorem 3.

With second order artificial viscosity added to the first-order wave equation we have the discrete convection–diffusion equation on an equidistant grid

$$(v_j)_t + cD_0 v_j = h^2 \varepsilon D_+ D_- v_j. \quad (28)$$

As before we prescribe periodic boundary data

$$v_0(t) = e^{i\omega t}, \quad \|v\|_h < \infty. \quad (29)$$

In a second order accurate computation ε is independent of h . With a typical temporal frequency ω the corresponding spatial scale is $\sim c/\omega$. The parameter ε is typically chosen small enough to ensure that $\varepsilon h^2 D_+ D_- v$ is small compared to the truncation error in the advection term discretization $c D_0 v$, which is $ch^2 u_{xxx}/6$. By introducing the size of spatial variations we see that this requirement leads to

$$\frac{\varepsilon h^2 \omega^2}{c^2} \ll \frac{h^2 \omega^3}{c^2}.$$

It follows that $\varepsilon \ll \omega$. In a well-resolved case $\omega h/c < 1$, and thus we expect $\varepsilon h/c \ll 1$ and $|\omega h/c - 1| \gg \varepsilon h/c$. When a stretching of the grid is introduced we expect that $|\omega h/c - 1| \gg \varepsilon h/c$, except in a region where $h \approx c/\omega$.

Eq. (28) has a solution of the form

$$v_j(t) = \eta^j e^{i\omega t}, \quad |\eta| < 1, \quad (30)$$

where η satisfies the difference equation

$$(1 - 2\frac{\varepsilon h}{c})\eta^2 + (i2\frac{\omega h}{c} + 4\frac{\varepsilon h}{c})\eta - 1 - 2\frac{\varepsilon h}{c} = 0. \quad (31)$$

For the roots we have

Theorem 2. With $\varepsilon > 0$ the roots of the difference equation (31) have the following properties:

1. If $2\varepsilon h/c \neq 1$ there are two distinct roots,

$$\eta_{\pm} = -\frac{i\frac{\omega h}{c} + 2\frac{\varepsilon h}{c}}{1 - 2\frac{\varepsilon h}{c}} \pm \frac{\sqrt{1 - \left(\frac{\omega h}{c}\right)^2 + 4\frac{i\omega\varepsilon h^2}{c^2}}}{1 - 2\frac{\varepsilon h}{c}}. \quad (32)$$

At least one of the roots has absolute value greater than unity.

2. If $2\varepsilon h/c = 1$ the only root is

$$\tilde{\eta} = \frac{1}{i\frac{\omega h}{c} + 1}, \quad |\tilde{\eta}| < 1. \quad (33)$$

The root in (33) corresponds to a very large viscosity, of the order $\mathcal{O}(h^{-1})$.

We note that the second result is similar to a result obtained in [5] in a different setting.

Proof.

1. The fact that at least one root has absolute value greater than unity follows directly from the zeroth order term in the characteristic equation for η :

$$|\eta_+||\eta_-| = \frac{1 + 2\frac{\varepsilon h}{c}}{1 - 2\frac{\varepsilon h}{c}} > 1. \quad (34)$$

2. The coefficient for the second order term in (31) vanishes.

A root with absolute value less than unity corresponds to a mode which decays to the right, while a root with absolute value greater than unity corresponds to a rightwards growing mode (or leftwards decaying mode). We have not been able to prove that for all values of ω , ε , c and h there is always one growing and one decaying mode. However, direct evaluation of the expressions (32) for two cases where $\omega \gg \varepsilon$ indicates that this is indeed the case, see Fig. 2. As in the inviscid case we call the wave corresponding to η_+ the physical wave, and the wave corresponding to η_- the spurious wave. As expected viscosity dampens the physical wave in the propagating regime and enhances the decay in the evanescent regime. However, as the stretching is increased beyond the transition point, $h = c/\omega$, it is clear that the effect of the viscous damping is small compared to the decay rate of the evanescent wave.

From (34) we immediately have

Corollary 2. For $h \neq c/2\varepsilon$ the two roots of (31) satisfy

$$|\eta_-^{-1}| < |\eta_+|. \quad (35)$$

Fig. 2. Absolute value of the two characteristic roots a function of h when $\omega = c = 1$ for $\varepsilon = 0.1$ and $\varepsilon = 0.01$, respectively. For comparison the corresponding inviscid roots are also plotted.

Fig. 3. Absolute value of η_+ and η_-^{-1} as a function of h when $\omega = c = 1$ for $\varepsilon = 0.1$ and $\varepsilon = 0.01$, respectively.

The decay rate to the right of the physical wave is given by $|\eta_+|$, and the decay rate to the left of the spurious wave is given by $|\eta_-^{-1}|$. Hence, the corollary implies that the decay rate to the right of the physical wave is slower than the decay rate to the left of the spurious wave. This statement is demonstrated in Fig. 3, which shows $|\eta_-^{-1}|$, and $|\eta_+|$ as functions of the grid spacing, for different values of ε . Note the significant effect on the spurious wave of the viscous damping in the propagating regime, $|\omega h/c| \leq 1$. It is not surprising that the damping effect of ε is stronger on the spurious mode than on the physical mode, since the spatial frequency of the spurious mode is higher.

It is possible to quantify the effect of the artificial viscosity term on the two waves. We summarize the result in the following theorem. The proof is based on Taylor expansions, and is rather technical. It is found in the appendix.

Theorem 3. With $1 \gg \frac{\varepsilon h}{2c} > 0$ the roots of the difference equation (31) have the following properties:

1. For $\frac{\omega h}{c} \leq C = \mathcal{O}(1)$ and $\delta_3 := \max\left(\frac{\frac{\varepsilon h}{c}}{|\frac{\omega h}{c} - 1|}, \frac{\varepsilon h}{c}\right) \ll 1$ the roots in (32) can be written as

$$\eta_+ = \eta_+^0 \left(1 - 2 \frac{\varepsilon h}{c} \frac{1 - \sqrt{1 - \left(\frac{\omega h}{c}\right)^2}}{\sqrt{1 - \left(\frac{\omega h}{c}\right)^2}} \right) + \mathcal{O}(\delta_3^2), \quad (36)$$

$$\eta_- = \eta_-^0 \left(1 + 2 \frac{\varepsilon h}{c} \frac{1 - \sqrt{1 - \left(\frac{\omega h}{c}\right)^2}}{\sqrt{1 - \left(\frac{\omega h}{c}\right)^2}} \right) + \mathcal{O}(\delta_3^2), \quad (37)$$

where $\eta_{\pm}^0$ are the roots in the inviscid case, given by (22). For sufficiently small δ_3 the roots have the property $|\eta_+| < 1$, $|\eta_-| > 1$.

2. For $\delta_4 := \frac{|\frac{\omega h}{c} - 1|}{\frac{\varepsilon h}{c}} \ll 1$ the roots in (32) can be written as

$$\eta_+ = \frac{\sqrt{2 \frac{\varepsilon h}{c}} - i}{1 + \sqrt{2 \frac{\varepsilon h}{c}}} + \mathcal{O}\left(\frac{\delta_4}{1 - 2 \frac{\varepsilon h}{h}}\right), \quad (38)$$

$$\eta_- = -\frac{\sqrt{2 \frac{\varepsilon h}{c}} + i}{1 - \sqrt{2 \frac{\varepsilon h}{c}}} + \mathcal{O}\left(\frac{\delta_4}{1 - 2 \frac{\varepsilon h}{h}}\right). \quad (39)$$

Also in this case $|\eta_+| < 1$, $|\eta_-| > 1$.

For well-resolved propagating waves ($\omega h/c < 1$) Case 1 in Theorem 3 is valid, and the modes are close to the inviscid propagating modes. When grid stretching is introduced to a problem with $\varepsilon \ll \omega$, Case 1 in Theorem 3 will be valid also in the evanescent regime, once the transition region corresponding to Case 2 is passed (where $\omega h/c \approx 1$). Also in this regime are the modes close to the inviscid evanescent modes.

3.3. Analysis for stretched grids

We now apply the results in the previous subsection to discrete equations on variable grids. We assume that the grid is stretched in one space dimension by factor s , that is $x_{j+1} = s x_j$. Typically the grid stretching is of the order 5 to 10%.

On a slowly varying grid we expect the solution to behave similarly to the constant grid case. Based on local, frozen coefficient analysis, we expect that for time-periodic boundary data the solution will have two components, which can be expressed as

$$v_j(t) = \left(\Pi_0^j \eta_{k+} \right) e^{i\omega t}, \quad (40)$$

$$v_j(t) = \left(\Pi_j^N \eta_{k-} \right) e^{i\omega t}, \quad (41)$$

where

$$\eta_{k+} = -\frac{i \frac{\omega h_k}{c} + 2 \frac{\varepsilon h_k}{c}}{1 - 2 \frac{\varepsilon h_k}{c}} + \frac{\sqrt{1 - \left(\frac{\omega h_k}{c}\right)^2 + 4 \frac{i\omega \varepsilon h_k^2}{c^2}}}{1 - 2 \frac{\varepsilon h_k}{c}},$$

$$\eta_{k-} = -\frac{i \frac{\omega h_k}{c} + 2 \frac{\varepsilon h_k}{c}}{1 - 2 \frac{\varepsilon h_k}{c}} - \frac{\sqrt{1 - \left(\frac{\omega h_k}{c}\right)^2 + 4 \frac{i\omega \varepsilon h_k^2}{c^2}}}{1 - 2 \frac{\varepsilon h_k}{c}}.$$

We denote (40) as the physical component, which decays to the right. In Sec. 5 we show that numerical solutions of the Euler equations show good agreement with the solution in (40) for physical waves. The corresponding spurious wave, given by (41) decays to the left. This spurious wave has a higher frequency in space and the viscous damping of this wave is of greater significance than the viscous damping of the physical wave.

4. Hyperbolic systems in two space dimensions

We consider a half space problem for hyperbolic systems in two space dimensions,

$$u_t + A u_x + B u_y = 0, \quad x \geq 0, \quad -\infty < y < \infty. \quad (42)$$

Here A is non-singular with eigenvalues a_k . To ensure a unique solution, initial and boundary data must also be provided. In the analysis of the 2 space dimensional system, we do not include viscous damping since we know from the previous section that the most efficient damping is related to under-resolution.

After Fourier transform in t and y we have

$$i\omega\hat{u} + A\hat{u}_x + i\beta B\hat{u} = 0, \quad x \geq 0, \quad (43)$$

where ω and β are the frequency in time and the wave number in the y -direction, respectively. For each set of ω, β this is a system of ordinary differential equations,

$$\hat{u}_x = -i\omega A^{-1} \left(I + \frac{\beta}{\omega} B \right) \hat{u}, \quad x \geq 0. \quad (44)$$

We can express the solution of this equation with the help of the eigen-solutions ξ_k, ϕ_k of the corresponding eigenvalue problem

$$A^{-1} \left(I + \frac{\beta}{\omega} B \right) \phi_k = \xi_k \phi_k. \quad (45)$$

The eigenvalues will be functions of β/ω , $\xi_k(\beta/\omega)$. By hyperbolicity ϕ_k are linearly independent, and ξ_k are purely real with sign determined by the sign of a_k , for all sufficiently small $|\beta/\omega|$. This regime is called the propagating regime. Note that for $\beta = 0$ we have, trivially, $\xi_k(1, 0) = a_k^{-1} \in \mathcal{R}$ and $\phi_k = \varphi_k$, where a_k, φ_k are the eigenvalues and eigenvectors of A . Corresponding solutions of (42) are

$$u_k(x, y, t) = e^{i\omega(t - \xi_k x - \frac{\beta}{\omega} y)} \phi_k. \quad (46)$$

As long as β and ω are in the propagating regime (46) is a propagating wave, convecting in the direction determined by the sign of a_k .

Next we consider the semi-discrete problem,

$$i\omega\hat{v}_j + AD_0\hat{v}_j + i\beta B\hat{v}_j = 0, \quad j = 1, 2, \dots \quad (47)$$

We consider the case where grid stretching is only introduced in the x -direction, i.e. normal to the boundaries at $x = \text{constant}$. We assume that the solution is well resolved in the y -direction, as well as in time. This motivates analyzing this semi-discrete problem. Rewrite (47) as

$$D_0\hat{v}_j = -i\omega A^{-1} \left(I + \frac{\beta}{\omega} B \right) \hat{v}_j. \quad (48)$$

Clearly this system can be diagonalized by the same eigenvectors ϕ_k as in (45). Each component of the diagonalized system satisfies

$$D_0\hat{z}_j^{(k)} = -i\omega\xi_k\hat{z}_j^{(k)}, \quad (49)$$

which is the same type of scalar difference equation as Eq. (19) in the previous section with $i\omega/c$ replaced by $i\omega\xi_k$. Here ξ_k is an eigenvalue defined in Eq. (45). As long as $|\omega h \xi_k| \neq 1$, for each component there are two roots of the corresponding characteristic equation,

$$\eta_{\pm}^{(k)} = -i\omega h \xi_k \pm \sqrt{1 - (\omega h \xi_k)^2}. \quad (50)$$

The corresponding solution to (47) is of the form

$$\hat{v}_j^{(k)} = (\sigma_1(\eta_+^{(k)})^j + \sigma_2(\eta_-^{(k)})^j) \phi_k.$$

As long as $\xi_k \in \mathcal{R}$ Theorem 1 is valid also for these roots.

When $|\omega h \xi_k| < 1$ there is a propagating physical wave and a propagating spurious wave. Here $|\eta_{\pm}^{(k)}| = 1$, and there is no damping. For components with $a_k > 0$ the physical wave is generated by $\eta_+^{(k)}$ and propagates to the right, while the corresponding spurious wave is of high spatial frequency and propagates to the left. When $a_k < 0$ the physical wave is generated by $\eta_-^{(k)}$, and propagates to the left, and the high frequency spurious wave propagates to the right.

At low resolution, i.e. $|\omega h \xi_k| > 1$, the corresponding component is in the evanescent regime, and the roots (50) correspond to either exponentially growing or exponentially decaying modes. For components with $a_k > 0$ the evanescent physical wave decays to the right, while for $a_k < 0$ the evanescent physical wave decays to the left. The corresponding spurious waves decay in the opposite directions.

As in the scalar case, efficient damping is achieved by stretching the grid in order to under-resolve outgoing waves. In analogy with (40) and (41) we have approximate expressions of the decaying and growing modes for each component in the diagonalized system,

$$v_n^{(k+)}(y, t) = \left(\prod_{j=0}^n \eta_{j+}^{(k)} \right) e^{i(\omega t + \beta y)} \phi_k, \quad (51)$$

$$\eta_{j+}^{(k)} = -i\omega h_j \xi_k + \sqrt{1 - (\omega h_j \xi_k)^2}, \quad (52)$$

and

$$v_n^{(k-)}(y, t) = \left(\prod_{j=n}^N \eta_{j-}^{(k)} \right) e^{i(\omega t + \beta y)} \phi_k, \quad (53)$$

$$\eta_{j-}^{(k)} = -i\omega h_j \xi_k - \sqrt{1 - (\omega h_j \xi_k)^2}. \quad (54)$$

In a realistic setting the domain is bounded and the domain is closed by some boundary conditions. These will typically cause reflections in the form of upstream physical waves or upstream spurious high frequency waves. If there is some viscous damping the spurious upstream waves will be damped. Important to note is that if a reflected physical wave is in the evanescent regime in at least part of the grid-stretched domain, it will be significantly damped before it reaches the computational domain.

4.1. Linear Euler equations

Consider the isentropic Euler equations in 2D linearized at a constant flow $U_0 > 0$ aligned with the x -axis. The system for particle velocity perturbations in the two coordinate directions and the pressure perturbation, is of the form (42) with

$$A = \begin{pmatrix} M_0 & 0 & \frac{1}{\gamma} \\ 0 & M_0 & 0 \\ \gamma & 0 & M_0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\gamma} \\ 0 & \gamma & 0 \end{pmatrix},$$

where the velocity and pressure perturbations are scaled with the speed of sound, C_0 , and the mean pressure, P_0 , respectively. Further, γ is the specific heat ratio, and $M_0 = U_0/C_0$ the free stream Mach number. We can explicitly express the eigenvalues of (45):

$$\xi_1 = \frac{1}{M_0}, \quad \xi_{2,3} = \frac{\left(M_0 \pm \sqrt{1 - \left(\frac{\beta}{\omega}\right)^2 (1 - M_0^2)} \right)}{1 - M_0^2}. \quad (55)$$

Here ξ_1 corresponds to the downstream propagating vorticity wave, while ξ_2 and ξ_3 correspond to the acoustic waves. Clearly all eigenvalues are purely real precisely if

$$\left(\frac{\beta}{\omega}\right)^2 \leq \frac{1}{1 - M_0^2}.$$

If this condition is not satisfied the acoustic waves do not propagate.

From the previous subsection we know that in order to efficiently damp a propagating acoustic wave at an outflow boundary, we need to stretch the grid until the grid size in the x -direction, h , fulfills $h > 1/|\omega \xi_2|$, where ξ_2 is given by (55). For such values of h , the outgoing acoustic wave is evanescent. We will see in Sec. 5.2 that the expression (51), which predicts damping of a propagating wave, agrees very well with solutions from numerical experiments.

5. Numerical experiments

In this section we present the results of numerical experiments conducted in order to evaluate the solutions of the model problems in the previous sections. We consider the linearized Euler equations in two space dimensions given in section 4.1. In order to include the influence of artificial viscosity, explicit viscous terms were added to Eq. (42) resulting in

$$\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} + B \frac{\partial U}{\partial y} = \Delta x^2 \epsilon_x U_{xx} + \Delta y^2 \epsilon_y U_{yy}. \quad (56)$$

Eq. (56) is solved using second order accurate central finite differences in space. An optimized 4-step third order time integration scheme as proposed in [17] was used. The time step was small enough such that the effect of the time integration could be considered as negligible. A one-dimensional version of the solver was used for the simulations in one space dimension.

The general setup of the computational domain is shown in Fig. 4. At the inflow boundary, characteristic boundary conditions were used, as in [18]. Likewise, at the outflow boundary, characteristic boundary conditions were applied. Periodic boundary conditions were applied in the y -direction.

5.1. Simulations in one space dimension

For the simulations in one space dimension, a base flow of $M_0 = 0.5$ was studied. In order to study the effect of grid stretching on an acoustic wave, the ingoing characteristic variable related to the eigenvalue $U_0 + C_0$ was perturbed by imposing an acoustic wave in the pressure perturbation as

$$p = \sin \omega t. \quad (57)$$

Fig. 4. Overview of the computational domain.

Fig. 5. Pressure as a function of x for different values of grid stretching in 1D. Solution of the linearized Euler equations (blue and red line), theoretical solution from (40) (green line). A constant cell size is used for $0 \leq x \leq 50$ and grid stretching with 5% (upper) 10% (middle) and 20% (lower) is applied for $x > 50$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

The frequency of the acoustic wave was chosen such that the acoustic wave was resolved with $N_x = 22$ points per wavelength on the unstretched grid. The grid consisted of constant cells sizes until $x = 50$. At $x = 50$, where the grid is stretched with a stretching factor s . Unless otherwise stated, all solutions of the linearized Euler equations presented in the figures are represented by the pressure as a function of x .

In Figs. 5 and 6 we have plotted numerical results for the linearized Euler equations in the presence of an artificial viscosity term, discretized using the second order central difference scheme, together with the solution predicted by (40), for different values of grid stretching ratio, s , and different artificial viscosity coefficients, ε_x , respectively. Note that the solution predicted by Eq. (40) yields a detailed description of the behavior of the acoustic wave on the unstretched and stretched grid, respectively. The numerical and predicted results are in excellent agreement for various stretching ratios and values of artificial viscosity. Hence, Eq. (40) can be used to estimate both the size of a buffer zone in a precise way, for an acoustic wave of a given frequency. In Theorem 3 it is shown that the artificial viscosity only has a smaller damping effect on the physical wave in the buffer zone for standard values of the artificial viscosity, i.e. the artificial viscosity term is tuned to stabilize the scheme in regions of pure wave propagation. This is also seen when comparing the results in Fig. 5, where the grid stretching factor is varied, with the results in Fig. 6, where the artificial viscosity coefficient is varied.

Fig. 6. Pressure as a function of x for different values of grid stretching in 1D. Solution of the linearized Euler equations (blue and red line), theoretical solution from (40) (green line). A constant cell size is used for $0 \leq x \leq 50$ and grid stretching with 5% is $x > 50$. Different values of numerical viscosity is used $\varepsilon_x = 0.00002$ (upper left) 0.005 (upper right), 0.005 (middle left), 0.01 (middle right) and 0.02 (lower center). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Fig. 7. Schematic picture of an oblique acoustic wave in two space dimensions.

5.2. Simulations in two space dimensions

We now continue to study the effect of a buffer zone in two space dimensions. Here, the mean flow is a steady $M = 0.5$ flow in the x -direction. At the inflow boundary, we set the ingoing characteristic variables such that an oblique acoustic wave is generated. See [6] for more details.

In Fig. 7 we show a schematic picture of an oblique acoustic wave in two space dimensions. Periodicity in the y -direction is imposed and at the outflow boundary the ingoing characteristic variable is set to zero. At $x = 0$ the wave takes values

Table 1

Parameters used in the numerical simulations of the solutions of Eq. (56). N_x and N_y are the number of grid points per wave length in the x - and y -directions, respectively.

Wave angle β	M_0	N_x	N_y	ω
0°	0.5	30	∞	0.31
20°	0.5	30	82	0.33
40°	0.5	30	36	0.38

$$p = \mathcal{R}(e^{i\omega(t - \frac{\beta}{\omega}y)}),$$

$$u = \left(\frac{1}{1/\xi - U_0}\right) \frac{C_0^2}{\gamma P_0} p,$$

$$v = \frac{\beta}{\omega\xi} u,$$

where $\mathcal{R}(z)$ is the real part of the complex number z , ξ is the wave number in the x direction, ω is the temporal frequency, and β is the wave number in the y -direction. For an oblique acoustic wave, the parameters ξ , ω and β are dependent as described by Eq. (55)

$$\xi = -\frac{\left(M_0 \pm \sqrt{1 - \left(\frac{\beta}{\omega}\right)^2(1 - M_0^2)}\right)}{1 - M_0^2}. \quad (58)$$

The wave angle, θ is related to β , ω and ξ , as

$$\tan \theta = \frac{\xi\omega}{\beta}. \quad (59)$$

The numerical solution of Eq. (56) and the theoretical results presented in Sec. 4 are now compared. As given in Table 1 three different values of ω were tested, resulting in three different positions in the stretched grid, where the acoustic wave should, according to the analysis, become evanescent. The grid resolution in the x -direction was 30 grid point per wave length for all test cases and the resolution in the y -direction per wave length before the grid stretching was applied is given in Table 1. As can be seen, the waves were well resolved both in the x - and the y -directions, respectively, before entering the buffer zone. The artificial viscosity was fixed and equal to 0.05 for all values of x .

In Fig. 8 the acoustic pressure is plotted as a function of x for a fixed y . The numerical results are compared to the theoretical solutions in Eq. (51). It is clear that the wave becomes evanescent for a certain value of grid spacing in the stretched grid and that the wave is thereafter dies out very fast down-stream of this point. In Fig. 8 the cut off grid cell size given by $h = 1/|\omega\xi_2|$, is indicated. By inspection, the solution predicted by Eq. (51) and the cut off criteria $h > 1/|\omega\xi_2|$, agree very well with the results in the numerical simulations.

Note that the effect of artificial viscosity is not included in the two-dimensional analysis, and hence the damping of the wave in the numerical solution is slightly faster than predicted by the analysis. However, for standard values of the artificial viscosity, the most efficient mechanism to damp upstream reflections, as for the one-dimensional case, is to achieve a cell size in the buffer zone for which the wave becomes evanescent. This conclusion is supported by the results on Fig. 8. Not shown here, we also performed simulations for acoustic waves with an inclination of 60° . The agreement between the numerical solution and the behavior predicted by analysis is degraded, which most likely was due to dispersion errors in the y -direction in the numerical solution, as the resolution in the y -direction became very low for the grid used. We conclude that the analysis in Sec. 4 can be used to predict the length of a buffer zone also for oblique waves.

As was discussed in Sec. 4 the cut-off criteria for the upstream traveling wave can also be taken into account for the design of a buffer zone at an outflow boundary. To illustrate this, we show computed reflections for two different buffer zones in Fig. 9.

In the first case the buffer zone is terminated very near the cut-off resolution for the up-stream wave, while in the second case the buffer zone is terminated very near the cut-off resolution for the down-stream wave. The same grid stretching is used in both cases, and we plotted the difference between the buffer-zone solutions and a reference solution on a much larger domain. We clearly see that a reflected up stream wave is only seen in the first case. This supports the conclusion from the analysis that the grid needs only to be stretched a bit down-stream of the cut-off resolution of the up-stream wave.

5.3. Discussion of results

The main message in the paper is that to get significant damping of a wave with frequency ω , the grid needs to be stretched until the wave becomes evanescent, that is until $h > \frac{c}{\omega}$. The number of grid points needed depends on the original resolution, say N points per wavelength, and the grid stretching, s . Then the original grid size is $h = 2\pi \frac{c}{(N\omega)}$. After m points

Fig. 8. Acoustic pressure in 2D, as a function of x for a fixed y . A constant cell size is used for $0 \leq x \leq 50$ and grid stretching with 5% is applied for $x > 50$. Wave angle 0° (upper), 20° (middle) and 40° (lower).

Fig. 9. Reflections from the outflow boundary for a plain wave (0° wave angle) in 2D as a function of x , for different x -positions of the outflow boundary. A constant cell size is used for $0 \leq x \leq 50$ and grid stretching with 5% is applied for $x > 50$. The position of the outflow boundary condition in the vicinity of the location of cut-off for the upstream traveling wave (left). The position of the outflow boundary condition in the vicinity of the location of cut-off for the downstream traveling wave (right).

we have a stretched grid size $h_m = s^m h$. To get efficient damping we need $h_m \frac{\omega}{c} > 1$, which corresponds to $s^m \frac{2\pi}{N} > 1$. As an example, we compute the values of m for the simulation in one space dimension presented in Fig. 5. The resulting values of m for a grid stretching of 5, 10 and 20%, respectively, with an original resolution of 22 points per wave length are 26, 14 and 8. It is clearly seen in Fig. 5, that for each grid stretching the waves decay rapidly downstream of these grid points.

For the Euler case in 2D, the analysis in Section 4.1 predicts that the downstream propagating acoustic wave will be efficiently damped when $h_m \omega \xi_2 > 1$. Here ξ_2 corresponds to the wave number in the stretching direction given by equation (55).

6. Conclusions

In this paper we have analyzed the effect of grid stretching in a buffer zone on propagating waves by considering a semi-discrete boundary value problem with time-harmonic boundary data. The setting includes numerical viscosity in the form of a second order undivided difference with a constant coefficient. In the semi-discrete setting physically propagating waves turn into evanescent waves if the grid resolution is low. From the analysis we conclude that a propagating wave that enters into a buffer zone decays significantly only when the wave is sufficiently under-resolved to be in the evanescent regime. The evanescent decay is a very efficient way to reduce the amplitude of physical waves, but the viscous damping is significant for high frequency spurious waves. We derive precise quantitative expressions for decay due to grid stretching, see Eq. (40) and Eq. (41), for a physical and on a spurious wave, respectively. These expressions can be used to design the buffer zone. We extend the analysis to hyperbolic systems in two space dimensions. Expressions for decay of physical and spurious waves for this case are given in section 4.1. The results are applied to the linearized Euler equations in an aero-acoustic setting. The numerical experiments in section 5 show very good agreement with the results predicted by analysis.

A simple analysis, not shown in the paper, yields that also for higher order accurate discretizations all the roots of the characteristic equation, corresponding to Eq. (21), will have absolute values strictly greater or strictly smaller than 1, for sufficiently large $\frac{\omega h}{c}$. This indicates that all waves will become evanescent by sufficient grid stretching.

In this paper we have only considered settings with waves propagating into a buffer zone. However, in many cases other structures, such as vortices etc. will also enter the buffer zone. An investigation of the behavior for wave packets is therefore also needed, but is not in the scope of this paper.

Appendix A. Proof of Theorem 3

1. Note that for $\frac{\omega h}{c} \leq C = \mathcal{O}(1)$ $\left| \frac{\frac{\varepsilon h}{c}}{\frac{\omega h}{c} - 1} \right| \ll 1$ implies that $\frac{\varepsilon h}{c}$ is also small compared to 1. Further

$$\frac{\frac{\varepsilon |\omega| h^2}{c^2}}{\left| 1 - \left(\frac{\omega h}{c} \right)^2 \right|} \leq \delta_3 \frac{\frac{\omega h}{c}}{1 + \frac{\omega h}{c}} \leq \delta_3.$$

By Taylor expansion

$$\begin{aligned} \sqrt{1 - \left(\frac{\omega h}{c} \right)^2 + 4 \frac{i \omega \varepsilon h^2}{c^2}} &= \sqrt{1 - \left(\frac{\omega h}{c} \right)^2} \sqrt{1 + 4 \frac{i \omega \varepsilon h^2}{c^2 (1 - \left(\frac{\omega h}{c} \right)^2)}} = \\ &= \sqrt{1 - \left(\frac{\omega h}{c} \right)^2} \left(1 + 2 \frac{i \omega \varepsilon h^2}{c^2 (1 - \left(\frac{\omega h}{c} \right)^2)} \right) + \mathcal{O}(\delta_3^2), \\ \frac{1}{1 - \frac{2 \varepsilon h}{c}} &= 1 + \frac{2 \varepsilon h}{c} + \mathcal{O}(\delta_3^2). \end{aligned}$$

It follows that

$$\begin{aligned} \eta_+ &= -\frac{i \frac{\omega h}{c} + 2 \frac{\varepsilon h}{c}}{1 - 2 \frac{\varepsilon h}{c}} + \frac{\sqrt{1 - \left(\frac{\omega h}{c} \right)^2 + 4 \frac{i \omega \varepsilon h^2}{c^2}}}{1 - 2 \frac{\varepsilon h}{c}} = \\ &= \eta_+^0 \left(1 - 2 \frac{\varepsilon h}{c} \frac{1 - \sqrt{1 - \left(\frac{\omega h}{c} \right)^2}}{\sqrt{1 - \left(\frac{\omega h}{c} \right)^2}} \right) + \mathcal{O}(\delta_3^2), \end{aligned}$$

$$\begin{aligned}\eta_- &= -\frac{i\frac{\omega h}{c} + 2\frac{\varepsilon h}{c}}{1 - 2\frac{\varepsilon h}{c}} - \frac{\sqrt{1 - \left(\frac{\omega h}{c}\right)^2 + 4\frac{i\omega\varepsilon h^2}{c^2}}}{1 - 2\frac{\varepsilon h}{c}} = \\ &= \eta_-^0 \left(1 + 2\frac{\varepsilon h}{c} \frac{1 + \sqrt{1 - \left(\frac{\omega h}{c}\right)^2}}{\sqrt{1 - \left(\frac{\omega h}{c}\right)^2}} \right) + \mathcal{O}(\delta_3^2).\end{aligned}$$

In the propagating regime $\omega h/c < 1$ and

$$0 < \frac{\varepsilon h}{c} \frac{1 - \sqrt{1 - \left(\frac{\omega h}{c}\right)^2}}{\sqrt{1 - \left(\frac{\omega h}{c}\right)^2}} = \frac{\sqrt{\frac{\varepsilon h}{c}}}{\sqrt{1 - \frac{\omega h}{c}}} \sqrt{\frac{\varepsilon h}{c}} \frac{1 - \sqrt{1 - \left(\frac{\omega h}{c}\right)^2}}{\sqrt{1 + \frac{\omega h}{c}}} < \delta_3 \ll 1. \quad (60)$$

Clearly $|\eta_+| < 1$, and by Theorem 2 $|\eta_-| > 1$.

In the evanescent regime $\omega h/c > 1$ a perturbation argument proves that $|\eta_+| < 1$ and $|\eta_-| > 1$ for sufficiently small δ_3 .

2. In this case we introduce

$$\delta_4 = \frac{\left|1 - \frac{\omega h}{c}\right|}{\frac{\varepsilon h}{c}}.$$

Note that by the assumption $2\varepsilon h/c < 1$ we have

$$\frac{\omega h}{c} = 1 + \delta, \quad |\delta| = \delta_4 \frac{\varepsilon h}{c} < \frac{1}{2}\delta_4,$$

and

$$\frac{\left|1 - \left(\frac{\omega h}{c}\right)^2\right|}{\frac{\varepsilon\omega h^2}{c^2}} = \frac{1 + \frac{|\omega|h}{c}}{\frac{|\omega|h}{c}} \delta_4 \leq \frac{2 + |\delta|}{1 - |\delta|} \delta_4 = \mathcal{O}(\delta_4).$$

Further

$$\begin{aligned}\sqrt{1 - \left(\frac{\omega h}{c}\right)^2 + 4\frac{i\omega\varepsilon h^2}{c^2}} &= \sqrt{\frac{4i\varepsilon\omega h^2}{c^2}} \sqrt{1 \pm \frac{1 + \left(\frac{\omega h}{c}\right)}{\frac{4i\omega h}{c}} \delta_4} = \\ &= \sqrt{\frac{4i\varepsilon h}{c}} (1 + \mathcal{O}(\delta_4)).\end{aligned} \quad (61)$$

Using (61), the formulas for the leading order terms are

$$\eta_+ \approx -\frac{i + 2\frac{\varepsilon h}{c}}{1 - 2\frac{\varepsilon h}{c}} + \frac{\sqrt{4\frac{i\varepsilon h}{c}}}{1 - 2\frac{\varepsilon h}{c}} = \frac{\sqrt{2\frac{\varepsilon h}{c}} - i}{1 + \sqrt{2\frac{\varepsilon h}{c}}}, \quad (62)$$

$$\eta_- \approx -\frac{i + 2\frac{\varepsilon h}{c}}{1 - 2\frac{\varepsilon h}{c}} - \frac{\sqrt{4\frac{i\varepsilon h}{c}}}{1 - 2\frac{\varepsilon h}{c}} = -\frac{\sqrt{2\frac{\varepsilon h}{c}} + i}{1 - \sqrt{2\frac{\varepsilon h}{c}}}, \quad (63)$$

and we can directly note that $|\eta_+| < 1$, $|\eta_-| > 1$.

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