A preconditioner for optimal control problems, constrained by Stokes equation with a time-harmonic control

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Abstract

In this article we construct an efficient preconditioner for solving the algebraic systems arising from discretized optimal control problems with time-periodic Stokes equations, based on a preconditioning technique for stationary Stokes-constrained optimal control problems, considered in an earlier paper by the authors. A simplified analysis of the derivation of the preconditioner and its properties is presented. The preconditioner is fully parameter-independent and the condition number of the corresponding preconditioned matrix is bounded by 2. The so-constructed preconditioner is favourably compared with another robust preconditioner for the same problem.

Keywords: optimal control, time-harmonic Stokes problem, preconditioning

1. Introduction

Optimal control problems, constrained by a state equation in the form of a partial differential equation (PDE) arise in many applications, where one wants to steer the modelled process in order to have a solution close to some given target (desired) function, see for example [1]. The process is steered by a control function, normally the source function to the differential equation.

Such optimal control problems lead to matrices in a special form,

$$\boldsymbol{A} = \begin{bmatrix} A & -b\,B_2 \\ a\,B_1 & A \end{bmatrix}, \tag{1}$$

with square blocks. Here a and b are scalars with the same sign. It has been shown ([2, 3, 4, 5]) that for $\boldsymbol{A}$ a particularly efficient preconditioner can be constructed, namely,

$$\boldsymbol{P}_F = \begin{bmatrix} A & -b\,B_2 \\ a\,B_1 & A + \sqrt{ab}(B_1 + B_2) \end{bmatrix}. \tag{2}$$

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In the earlier papers, mentioned above, the quality of $\boldsymbol{P}$ as well as how to apply $\boldsymbol{P}$ in a computationally easy way has been analysed via the explicit form of its inverse,

$$\boldsymbol{P}_F^{-1} = \begin{bmatrix} H_1^{-1} + H_2^{-1} - H_2^{-1} A H_1^{-1} & -\sqrt{\frac{b}{a}}(I - H_2^{-1}A)H_1^{-1} \\ \sqrt{\frac{a}{b}} H_2^{-1}(I - AH_1^{-1}) & H_2^{-1} A H_1^{-1} \end{bmatrix}. \tag{3}$$

Here the matrices $H_i = A + \sqrt{ab}B_i, i = 1, 2$ are assumed to be nonsingular.

Besides one matrix-vector multiplication and four vector updates, the action of $\boldsymbol{P}_F^{-1}$ requires only a single solution of a system with H_1 and H_2.

In Section 3 we show this in a simplified way, without use of the explicit form of the inverse of $\boldsymbol{P}_F^{-1}$.

In the earlier publication [5] we have considered the preconditioner, applied for Stokes-type control problems with a stationary state equation. In some applications a more general problem, with a time-harmonic desired state, arise. Such problems have been considered in e.g. [6, 7], see also the references therein. In the present paper we deal mainly with a Stokes time-harmonic problem as the state equation. We present an efficient and fully robust preconditioner that involves just two solutions of a regularized Stokes problem and that leads to a spectrum of the preconditioned matrix clustered around the unit value, with a condition number bounded by 2.

This holds uniformly with respect to all parameters involved, namely, the meshsize (h) in the discretization mesh, the frequency (ω) of the harmonic wave and the control parameter (β). This result generalizes previous results for optimal control problems, constrained by stationary PDEs, see e.g. [8, 9, 10, 11, 4, 5], to problems involving time-harmonic control functions and improves the results in [6].

The remainder of the paper is composed as follows. In Section 2 we present the type of optimal control problems and their discretized versions, dealt with in this paper. Thereby, as an introduction to time-harmonic Stokes control, we derive first the condition number of the preconditioner, applied to a parabolic problem. We first present the simplified derivation of the action of the preconditioner. We then apply it for the Stokes optimal control problem and derive the corresponding spectral condition number. This is followed by a simplified derivation of another robust preconditioner that has appeared earlier, as well as a short summary of some other preconditioners, used for time periodic optimal control problems. The general form of the preconditioner we aim at and its efficient implementation are recalled in Section 3 with a simplified derivation. Some important computational and implementation issues are presented in Section 4. We illustrate the numerical behaviour of the time-harmonic Stokes control preconditioner in Section 5 and give some concluding remarks.

2. Optimal control problems with a time-harmonic desired state

Following [6], consider first the optimal control problem of finding the state $u(x,t)$ and the control $f(x,t)$ that minimize the functional

$$\mathcal{J}(u,f) = \frac{1}{2}\int_0^T\int_\Omega |u(x,t) - u_d(x,t)|^2\,dx\,dt + \frac{1}{2}\beta\int_0^T\int_\Omega |f(x,t)|^2\,dx\,dt$$

subject to the time-dependent parabolic problem

$$\begin{aligned} \tfrac{\partial}{\partial t}u(x,t) - \Delta u(x,t) &= f(x,t) && \text{in}\,\Omega\times(0,T),\\ u(x,t) &= 0, && x\in\Gamma\times(0,T),\\ u(x,0) &= u(x,T) && \text{in}\,\Omega,\\ f(x,0) &= f(x,T) && \text{in}\,\Omega. \end{aligned}$$

Here Γ is the boundary of Ω and u_d is the desired state.

The target function is time-harmonic, $u_d(x,t) = u_d(x)e^{i\,\omega\,t}$ with period $\omega = 2\pi\,k/T$ for some positive integer $k\in\mathbb{Z}$. Then the solution and the control are also time-harmonic, $u(x,t) = u(x)e^{i\,\omega\,t}$ and $f(x,t) = f(x)e^{i\,\omega\,t}$. Hence, $u(x), f(x)$ are time-independent solutions of the following optimal control problem,

$$\text{minimize}\quad \frac{1}{2}\int_\Omega |u(x) - u_d(x)|^2\,dx + \frac{1}{2}\beta\int_\Omega |f(x)|^2\,dx$$

$$\text{subject to}\quad \begin{cases} i\omega u(x) - \Delta u(x) = f(x) & \text{in}\,\Omega,\\ u(x) = 0, & x\in\Gamma. \end{cases}$$

Here β is a positive regularization parameter. In the sequel we assume that $u(x)$ and $u_d(x)$ are real-valued but the control $f(x)$ must be complex-valued, $f(x) = f_0(x) + if_1(x)$.

Remark 1. *If also the function and its target have complex-valued coefficients then, as shown in [12], one can separate the real and imaginary parts of the equation, that leads to two systems of the form (4), one for the real and one for the complex part of the solution.*

Using an appropriate finite element subspace V_h for both u and f and introducing the corresponding, complex-valued, Lagrange multiplier λ, the Lagrangian functional (L) for the discretized constrained optimization problem is given by

$$L(u,f,\lambda) = \frac{1}{2}(u-u_d)^T M(u-u_d) + \frac{1}{2}\beta f^* Mf + \lambda^*(i\omega Mu + Ku - Mf).$$

Here M is the mass matrix, corresponding to the L_2-inner product in V_h, K is the negative discrete Laplacian and $*$ denotes conjugate transpose.

The first order necessary conditions, $\nabla L(u, f, \lambda) = 0$, which are also sufficient for the existence of the solution, lead to the system

$$\begin{bmatrix} M & 0 & K - i\omega M \\ 0 & \beta M & -M \\ K + i\omega M & -M & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{f} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} M\mathbf{u}_d \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}. \tag{4}$$

Using the relation $\boldsymbol{\lambda} = \beta\mathbf{f}$ we obtain the reduced system

$$\begin{bmatrix} M & \beta(K - i\omega M) \\ K + i\omega M & -M \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{f} \end{bmatrix} = \begin{bmatrix} M\mathbf{u}_d \\ \mathbf{0} \end{bmatrix}. \tag{5}$$

Here $\mathbf{u}$ is real-valued whereas $\mathbf{f}$ contains an imaginary part, i.e., $\mathbf{f} = \mathbf{f_0} + \mathbf{if_1}$. Thus, from the imaginary parts of (5) we have:

$$\begin{aligned} \beta K\mathbf{f_1} - \omega\beta M\mathbf{f_0} = \mathbf{0} \quad &\Rightarrow \quad K\mathbf{f_1} = \omega M\mathbf{f_0} \\ -M\mathbf{f_1} + \omega M\mathbf{u} = \mathbf{0} \quad &\Rightarrow \quad \mathbf{f_1} = \omega\mathbf{u}. \end{aligned} \tag{6}$$

By (6), $\beta\omega M\mathbf{f_1} - \beta\omega^2 M\mathbf{u} = \mathbf{0}$, that is, $\beta\omega M\mathbf{f_1} = \beta\omega^2 M\mathbf{u}$ and clearly $M\mathbf{f}_0 = K\mathbf{u}$. Finally, the real part of the reduced system (5) takes the form

$$\begin{bmatrix} (1 + \beta\omega^2)M & \beta K \\ K & -M \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{f}_0 \end{bmatrix} = \begin{bmatrix} M\mathbf{u}_d \\ \mathbf{0} \end{bmatrix}$$

or

$$\boldsymbol{A} \begin{bmatrix} \mathbf{u} \\ -\mathbf{f}_0 \end{bmatrix} \equiv \begin{bmatrix} M & -\widetilde{\beta}K \\ K & M \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ -\mathbf{f}_0 \end{bmatrix} = \frac{1}{1 + \beta\omega^2} \begin{bmatrix} M\mathbf{u}_d \\ \mathbf{0} \end{bmatrix}, \tag{7}$$

where $\widetilde{\beta} = \frac{\beta}{1+\beta\omega^2}$.

The matrix $\boldsymbol{A}$ in system (7) has the form, discussed, e.g., in [4] and in the introduction, and can therefore be efficiently preconditioned by

$$\boldsymbol{P}_F = \begin{bmatrix} M & -\widetilde{\beta}K \\ K & M + 2\sqrt{\widetilde{\beta}}K \end{bmatrix}. \tag{8}$$

The inverse of $\boldsymbol{P}_F$ exists and involves only two matrix inverses, $(M + \sqrt{\widetilde{\beta}}K)^{-1}$.

The theory, derived in [2, 3, 4, 5] shows that all eigenvalues of the preconditioned matrix $\boldsymbol{P}_F^{-1}\boldsymbol{A}$ are real and positive, and also belong to the interval $[0.5, 1]$. In particular, to see the influence of β on the spectrum, we briefly repeat the analysis. Consider the generalized eigenvalue problem

$$\mu \begin{bmatrix} M & -\widetilde{\beta}K \\ K & M + 2\sqrt{\widetilde{\beta}}K \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} M & -\widetilde{\beta}K \\ K & M \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix}, \quad \|\mathbf{v}\| + \|\mathbf{w}\| \neq 0.$$

Since the matrix $\boldsymbol{A}$ is nonsingular, here $\mu \neq 0$. It shows that $\mu = 1$ for the vector $[\mathbf{v}, \mathbf{w}]$, where $\mathbf{v} \neq \mathbf{0}$ and $\mathbf{w} = \mathbf{0}$ and

$$\left(\frac{1}{\mu} - 1\right) \begin{bmatrix} M & -\widetilde{\beta}K \\ K & M \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{w} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ 2\sqrt{\widetilde{\beta}}K\mathbf{w} \end{bmatrix}.$$

Further, for $\mu \neq 1$ it holds $M\mathbf{v} = \widetilde{\beta} K\mathbf{w}$, $\mathbf{w} \neq \mathbf{0}$ and after substituting this relation, we obtain

$$\left(\frac{1}{\mu} - 1\right) \mathbf{w}^T \left[\widetilde{\beta} K M^{-1} K + M\right] \mathbf{w} = 2\sqrt{\widetilde{\beta}} \mathbf{w}^T K \mathbf{w} = 2\sqrt{\widetilde{\beta}} \mathbf{w}^T K M^{-1/2} M^{1/2} \mathbf{w}. \quad (9)$$

That is, $\frac{1}{\mu} - 1 > 0$, or $\mu < 1$. Further, by the Cauchy-Schwarz inequality it follows that $\frac{1}{\mu} - 1 \leq 1$, i.e.,

$$\frac{1}{2} \leq \mu \leq 1.$$

It is seen from (9) and the relation between β and $\widetilde{\beta}$ that the eigenvalues cluster at $\mu = 1$ for small values of β and even further for large values of ω^2. Therefore, the convergence of the preconditioned iteration method can be expected to be rapid.

As systems with the matrices $M + \sqrt{\widetilde{\beta}} K$ are to be solved by inner iterations with variable accuracy, preferably, the preconditioner should be coupled with an iterative method that incorporates variable preconditioner, such as GCG ([13]) or FGMRES ([14]). Alternatively, one can use a fixed number of Uzawa steps in which case no flexible outer solver is needed. This requires, however, a good choice of the number of such steps, which must be found by trial and error. The numerical tests show that fewer iterations are needed for the flexible version of the outer solver.

Remark 2. *How to solve systems with a matrix of the form* $\begin{bmatrix} M & -\sigma K \\ K & M \end{bmatrix}$ *has been studied in earlier works, see, for instance [3] and the references therein. One method that proves to be very efficient for this type of matrices is the preconditioned modified Hermitian-skew-Hermitian splitting (PMHSS), applied also to solve a Poisson harmonic control problem in [15]. A comparison of the performance of PMHSS, although in a slightly different formulation, with the preconditioner* $\boldsymbol{P}_F$ *has been done in [3]. Since the present paper deals mainly with Stokes control and the preconditioner (8) is presented only as a motivation for the Stokes-type preconditioner presented in the sequel, we do not include comparisons with PMHSS here.*

Consider now the corresponding optimal control problem with a state equation of Stokes type,

$$\begin{array}{rcll} \mathcal{L}(u,p) \equiv \dfrac{\partial}{\partial t} u(x,t) - \Delta u(x,t) + \nabla p(x,t) & = & f(x,t) & \text{in } \Omega \times [0,T] \\ \nabla \cdot u(x,t) & = & 0 & \text{in } \Omega \times [0,T] \\ u(x,t) & = & 0 & \text{on } \Gamma \times [0,T] \\ u(x,0) & = & u(x,T) & \text{in } \Omega \\ p(x,0) & = & p(x,T) & \text{in } \Omega \\ f(x,0) & = & f(x,T) & \text{in } \Omega \end{array} \quad (10)$$

where f is the vectorial control function (for the velocity and the pressure component) and let $g(x,t)$ and $q(x,t)$ be Lagrangian multipliers.

The Lagrangian functional becomes

$$L(u,p,f,g,q) = \tfrac{1}{2}\int_0^T\int_\Omega |u(x,t)-u_d(x,t|^2\,dx\,dt + \int_0^T\int_\Omega g(x,t)(\mathcal{L}(u,p)-f)\,dx\,dt + \int_0^T\int_\Omega \nabla\cdot u(x,t)\,q(x,t)\,dx\,dt + \tfrac{1}{2}\beta\int_0^T\int_\Omega |f(x,t)|^2\,dx\,dt$$

As above, u_d and u are assumed to be time-harmonic, i.e., $u(x,t) = u(x)e^{i\omega t}$, $\omega = 2\pi\, k/T$, etc.

From $\nabla L(u,p,f,g,q) = 0$, after a suitable discretization, there arise five necessary conditions, written in the matrix form (11), see [6],

$$\begin{bmatrix} M & 0 & 0 & K-i\omega M & -D^T \\ 0 & 0 & 0 & D & 0 \\ 0 & 0 & \beta M & -M & 0 \\ K+i\omega M & -D^T & -M & 0 & 0 \\ -D & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \\ \mathbf{f} \\ \mathbf{g} \\ \mathbf{q} \end{bmatrix} = \begin{bmatrix} M\mathbf{u}_d \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad (11)$$

where M, K and D are the mass matrix, the discrete negative Laplacian and the discrete divergence matrix, correspondingly.

By use of the relation $M\mathbf{g} = \beta M\mathbf{f}$ we eliminate $\mathbf{g}$ and reduce to four equations which, together with some sign changes, take the form

$$\begin{bmatrix} M & 0 & \beta(K-i\omega M) & -\beta D^T \\ 0 & 0 & -\beta D & 0 \\ K+i\omega M & -D^T & -M & 0 \\ -D & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \\ \mathbf{f} \\ \widetilde{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} M\mathbf{u}_d \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad (12)$$

where $\widetilde{\mathbf{q}} = \mathbf{q}/\beta$.

Here $\mathbf{u}, \mathbf{p}$ and $\mathbf{q}$ are real-valued but $\mathbf{f}$ is complex-valued. With $\mathbf{f} = \mathbf{f}_0 + i\mathbf{f}_1$, by putting together the imaginary parts in the first and the third equations in (12), we see that $\beta K\mathbf{f}_1 = \beta\omega M\mathbf{f}_0$ and $\omega M\mathbf{u} = M\mathbf{f}_1$. Thus, $\mathbf{f}_1 = \omega\mathbf{u}$ and $K\mathbf{u} = M\mathbf{f}_0$. Hence, the matrix equation (12) takes the form

$$\begin{bmatrix} (1+\beta\omega^2)M & 0 & \beta K & -\beta D^T \\ 0 & 0 & -\beta D & 0 \\ K & -D^T & -M & 0 \\ -D & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \\ \mathbf{f}_0 \\ \widetilde{\mathbf{q}} \end{bmatrix} = \begin{bmatrix} M\mathbf{u}_d \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

or

$$\begin{bmatrix} M & 0 & -\widetilde{\beta}K & \widetilde{\beta}D^T \\ 0 & 0 & \widetilde{\beta}D & 0 \\ K & -D^T & M & 0 \\ -D & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \\ -\mathbf{f}_0 \\ -\widetilde{\mathbf{q}} \end{bmatrix} = \frac{1}{1+\beta\omega^2} \begin{bmatrix} M\mathbf{u}_d \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad (13)$$

with $\widetilde{\beta} = \beta/(1+\beta\omega^2)$.

In this way we obtain a block system of the type $\begin{bmatrix} A & -cB \\ B & A \end{bmatrix}$ and, following the idea of the method in the introduction, choose a preconditioner of the form

$$\mathcal{P}_F = \begin{bmatrix} M & 0 & -\widetilde{\beta}K & \widetilde{\beta}D^T \\ 0 & 0 & \widetilde{\beta}D & 0 \\ K & -D^T & M + 2\sqrt{\widetilde{\beta}}K & -2\sqrt{\widetilde{\beta}}D^T \\ -D & 0 & -2\sqrt{\widetilde{\beta}}D & 0 \end{bmatrix}. \tag{14}$$

Let denote the matrix in (13) by $\mathcal{A}$. An observation reveals that we can transform $\mathcal{A}$ into a spectrally equivalent matrix $\tilde{\mathcal{A}}$ with the help of a diagonal matrix $\mathcal{D}$, namely, we let $\widetilde{\mathcal{A}} = \mathcal{D}\mathcal{A}\mathcal{D}^{-1}$, where $\mathcal{D} = blockdiag(I, I, \sqrt{\widetilde{\beta}}, \sqrt{\widetilde{\beta}}\, I)$. After scaling, the transformed system reads

$$\begin{bmatrix} M & 0 & -\sqrt{\widetilde{\beta}}K & \sqrt{\widetilde{\beta}}D^T \\ 0 & 0 & \sqrt{\widetilde{\beta}}D & 0 \\ \sqrt{\widetilde{\beta}}K & -\sqrt{\widetilde{\beta}}D^T & M & 0 \\ -\sqrt{\widetilde{\beta}}D & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \\ -\sqrt{\widetilde{\beta}}\mathbf{f}_0 \\ -\sqrt{\widetilde{\beta}}\widehat{\mathbf{q}} \end{bmatrix} = \frac{1}{1+\beta\omega^2} \begin{bmatrix} M\mathbf{u}_d \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{15}$$

Numerical experiments show that solutions with the matrix $\widetilde{\mathcal{A}}$ require less iterations than that with the matrix $\mathcal{A}$.

3. Spectral properties of the preconditioned matrix for Stokes-type optimal control problems

We analyse now the preconditioner in Section 2 in the framework of Stokes control problems (cf. [5]) and amend it for the time-harmonic case. We compare it with a preconditioner as presented in [6], for which we also simplify the analysis.

As first we present a simplified derivation of the implementation of the preconditioner $\boldsymbol{P}$.

3.1. A simplified version of the preconditioner $\boldsymbol{P}_F$

An easy computation shows namely that $\boldsymbol{P}_F$ in (2) can be factorized as

$$\boldsymbol{P}_F = \begin{bmatrix} I & -\sqrt{\frac{b}{a}}I \\ 0 & I \end{bmatrix} \begin{bmatrix} A + \sqrt{ab}B_1 & 0 \\ aB_1 & A + \sqrt{ab}B_2 \end{bmatrix} \begin{bmatrix} I & \sqrt{\frac{b}{a}}I \\ 0 & I \end{bmatrix}. \tag{16}$$

This means that a system

$$\boldsymbol{P}_F \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{bmatrix}$$

can be solved as

$$\begin{bmatrix} A+\sqrt{ab}B_1 & 0 \\ aB_1 & A+\sqrt{ab}B_2 \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{g}_1 \\ \mathbf{f}_2 \end{bmatrix},$$

where $\mathbf{y}_1 = \mathbf{x}_1 + \sqrt{\frac{b}{a}}\mathbf{x}_2$ and $\mathbf{g}_1 = \mathbf{f}_1 + \sqrt{\frac{b}{a}}\mathbf{f}_2$.

The algorithm to compute $\mathbf{x}_1$ and $\mathbf{x}_2$ becomes as follows.

Algorithm 1 Solving the transformed $\boldsymbol{P}_F$

1: Compute $\mathbf{g}_1 = \mathbf{f}_1 + \sqrt{\frac{b}{a}}\mathbf{f}_2$.
2: Solve $H_1\mathbf{y}_1 = \mathbf{g}_1$.
3: Compute $\mathbf{h} = \mathbf{f}_2 - aB_1\mathbf{y}_1$.
4: Solve $H_2\mathbf{x_2} = \mathbf{h}$.
5: Compute $\mathbf{x}_1 = \mathbf{y}_1 - \sqrt{\frac{b}{a}}\mathbf{x}_2$.

For a comparison, we include below the algorithm, based on the inverse of $\boldsymbol{P}_F$ in (3), used in [4, 5] and in some earlier papers by the present authors.

Algorithm 2 Solving the factorized $\boldsymbol{P}_F$

1: Compute $\mathbf{g}_1 = \mathbf{f}_1 + \sqrt{\frac{b}{a}}\mathbf{f}_2$
2: Solve $H_1\mathbf{h} = \mathbf{g}_1$.
3: Compute $\mathbf{v} = \mathbf{f}_1 - A\mathbf{h}$.
4: Solve $H_2\mathbf{x}_2 = \mathbf{v}$.
5: Compute $\mathbf{x}_1 = \mathbf{h} + \mathbf{x}_2$ and $\mathbf{x}_2 = -\sqrt{\frac{a}{b}}\mathbf{x}_2$.

Note that the major difference between the algorithms is that in Algorithm 1 we perform a matrix-vector multiplication with B_1, while in Algorithm 2 we multiply by A. Also, with Algorithm 1 we need only one additional vector, compared with two for Algorithm 2.

3.2. The block matrix preconditioner applied to a Stokes optimal control problem

The arising matrix has the form (13). We analyse it below in a slightly more general form, replacing K by F, scaling and incorporating the coefficient β in the matrix blocks, as

$$\mathcal{A} = \begin{bmatrix} M & 0 & -F^T & -D^T \\ 0 & 0 & -D & 0 \\ F & D^T & M & 0 \\ D & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

indicating that the results hold also when $F \neq F^T$. It is preconditioned by a preconditioner, $\mathcal{P}_{FS}$, formed from the matrix in (17), to which we add

$$\begin{bmatrix} F & D^T \\ D & 0 \end{bmatrix} + \begin{bmatrix} F^T & D^T \\ D & 0 \end{bmatrix}$$

to the lower $(2,2)$ diagonal block, namely

$$\mathcal{P}_{FS} = \begin{bmatrix} M & 0 & -F^T & -D^T \\ 0 & 0 & -D & 0 \\ F & D^T & M+F+F^T & 2D^T \\ D & 0 & 2D & 0 \end{bmatrix}. \tag{18}$$

As we have shown in [5] and also follows from Section 3.1, an application of the preconditioner $\mathcal{P}_{FS}$ involves the solution of one system with the matrix $\begin{bmatrix} M+F & D^T \\ D & 0 \end{bmatrix}$ and one system with the matrix $\begin{bmatrix} M+F^T & D^T \\ D & 0 \end{bmatrix}$, both of Stokes type.

For completeness, we include a brief account for the analysis of the spectrum of the preconditioned matrix $\mathcal{P}_{FS}^{-1}\mathcal{A}$, where $\mathcal{A}$ is as in (17). The following result holds true.

Theorem 1. *Consider the generalized eigenvalue problem $\lambda\mathcal{P}_{FS}\mathbf{v} = \mathcal{A}\mathbf{v}$, where $\mathcal{A}$ and $\mathcal{P}_{FS}$ are defined in (17) and (18), correspondingly, Assume that M and $F+F^T$ are symmetric and positive definite and D has full rank. Then all eigenvalues λ are real and positive and*

$$1/3 \le \lambda \le 1.$$

If F is symmetric, then

$$1/2 \le \lambda \le 1.$$

A detailed proof can be found in [5].

3.3. An alternative preconditioner for the time-harmonic optimal control problem for Stokes equation

In [6], as a part of a more general analysis of saddle point matrices and related preconditioning techniques, a block-diagonal preconditioner is presented for the complex-valued matrix arising from the time-harmonic parabolic problem. As we compare our preconditioner to the preconditioner, derived in [6], for completeness we briefly describe their approach. Consider the system, arising from the first order necessary conditions in a compressed form and, compared to the system in (12), scaled and permuted as

$$\begin{bmatrix} M & \sqrt{\beta}(K-i\omega M) & 0 & -\sqrt{\beta}D^T \\ \sqrt{\beta}(K-i\omega M) & -M & -\sqrt{\beta}D & 0 \\ 0 & -\sqrt{\beta}D^T & 0 & 0 \\ -\sqrt{\beta}D & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{u} \\ \mathbf{p} \\ \frac{1}{\sqrt{\beta}}\mathbf{f} \\ \frac{1}{\sqrt{\beta}}\mathbf{q} \end{bmatrix} = \begin{bmatrix} M\mathbf{u}_d \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}$$

and denote the matrix by $\mathcal{M}$, where $\mathcal{M} = \begin{bmatrix} \mathbf{A} & \mathbf{B}^* \\ \mathbf{B} & -\mathbf{C} \end{bmatrix}$, $\mathbf{B} = -\sqrt{\beta}\begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix}$, $\mathbf{A} = \begin{bmatrix} M & \sqrt{\beta}(K-i\omega M) \\ \sqrt{\beta}(K-i\omega M) & -M \end{bmatrix}$, and $\mathbf{C} = \mathbf{0}$. Here, $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are

generic names for the matrix blocks of $\mathcal{M}$ and $\mathbf{A}$ and $\mathbf{B}$ are complex. In [6], the following block-diagonal preconditioner for $\mathcal{M}$ is analysed, based on non-standard norm techniques,

$$\mathcal{P}_{nsn} = \begin{bmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{S} \end{bmatrix}, \tag{19}$$

where $\mathbf{P} = \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix}$, $\mathbf{S} = \beta \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}$, $P = M + \sqrt{\beta}(F + \omega M)$ and $S = D(M + \sqrt{\beta}(F + \omega M))^{-1} D^T$. Clearly, $\mathbf{P}$ and $\mathbf{S}$ are real and symmetric positive definite. The analysis in [6] shows that the spectrum of $\mathcal{P}_{nsn}^{-1}\mathcal{M}$ is real and symmetric around zero, contained in the intervals (approximately) $[-1.618, -0.306] \cup [0.306, 1.618]$, correspondingly.

We present here a simplified analysis of the preconditioner $\mathbf{P}$, used to approximate the block $\mathbf{A}$ in $\mathcal{M}$. We recall that both F and M are symmetric positive definite. Consider the matrix

$$\mathbf{A_o} = \begin{bmatrix} M & F - i\omega M \\ F + i\omega M & -\frac{1}{\beta} M \end{bmatrix}.$$

We scale first $\mathbf{A_o}$ by multiplication from both sides by $\begin{bmatrix} I & 0 \\ 0 & \sqrt{\beta} I \end{bmatrix}$ to obtain

$$\mathbf{A} \equiv \begin{bmatrix} M & \sqrt{\beta}(F - i\omega M) \\ \sqrt{\beta}(F + i\omega M) & -M \end{bmatrix}.$$

For $\mathbf{A}$, a block-diagonal preconditioner is applied,

$$\mathbf{P} = \begin{bmatrix} (1 + \sqrt{\beta}\omega)M + \sqrt{\beta}F & 0 \\ 0 & (1 + \sqrt{\beta}\omega)M + \sqrt{\beta}F \end{bmatrix},$$

resulting in the preconditioned matrix

$$\mathbf{G} = \mathbf{P^{-1}A} = \begin{bmatrix} \frac{1}{1+\sqrt{\beta}\omega}\widetilde{M} & I - \widetilde{M} - i\frac{\sqrt{\beta}\omega}{1+\sqrt{\beta}\omega}\widetilde{M} \\ I - \widetilde{M} + i\frac{\sqrt{\beta}\omega}{1+\sqrt{\beta}\omega}\widetilde{M} & -\frac{1}{1+\sqrt{\beta}\omega}\widetilde{M} \end{bmatrix}, \tag{20}$$

where $\widetilde{M} = [(1+\sqrt{\beta}\omega)M + \sqrt{\beta}F]^{-1}(1+\sqrt{\beta}\omega)M$ and we have used that $\sqrt{\beta}F = \big((1 + \sqrt{\beta}\omega)M + \sqrt{\beta}F\big) - (1 + \sqrt{\beta}\omega)M$.

A calculation shows that $\mathbf{G^2} = \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix}$ where $G = \left(\frac{1}{1+\sqrt{\beta}\omega}\right)^2 \widetilde{M}^2 + (I - \widetilde{M})^2 + \frac{\beta\omega^2}{(1+\sqrt{\beta}\omega)^2}\widetilde{M}^2$.

Theorem 2. *Let* $\mathbf{G}$ *be defined by (20). Then the eigenvalues of* $\mathbf{G^2}$ *satisfy*

$$\frac{\alpha}{1+\alpha} \leq \lambda(\mathbf{G^2}) \leq \mathbf{1}, \tag{21}$$

where $\alpha = (1+\beta\omega^2)/(1+\beta\omega^2+2\sqrt{\beta}\omega)$, *i.e.,* $\frac{1}{2} \le \alpha \le 1$. *Hence, for the condition number of* $\mathbf{G^2}$ *there holds*

$$\varkappa(\mathbf{G^2}) \le \frac{\mathbf{1}+\alpha}{\alpha} \le \mathbf{3}.$$

Proof. Since $0 < \widetilde{M} \le I$, it follows that the eigenvalues of $\mathbf{G^2}$ satisfy

$$\lambda = \alpha\mu^2 + (1-\mu)^2,$$

where μ is an eigenvalue of $\widetilde{M}$. The minimal value of λ is taken for $\mu = \frac{1}{1+\alpha}$, namely,

$$\lambda = \frac{\alpha}{(1+\alpha)^2} + \left(1 - \frac{1}{1+\alpha}\right)^2 = \frac{\alpha}{1+\alpha}.$$

This shows that

$$\frac{\alpha}{1+\alpha} \le \lambda(\mathbf{G^2}) \le \mathbf{1}.$$

Since $\frac{1}{2} \le \alpha \le 1$ it follows that $\frac{1}{3} \le \lambda(\mathbf{G^2}) \le \mathbf{1}$ and the condition number of $\mathbf{G^2}$ is bounded by $\frac{1+\alpha}{\alpha} \le 3$. ∎

A minimal residual method to solve a system with $\mathbf{A}$, preconditioned by $\mathbf{P}$ will, hence, require twice the number of iterations for a CG-like method to solve a system with the matrix $\mathbf{G^2}$. The above is a simplification and clarification of the corresponding analysis in [6].

3.4. Some other preconditioners for time-periodic problems

For completeness, we include a short summary of some other already published methods, although those do not consider optimal control problems with a Stokes equation, but only a parabolic equation as state equation.

In [16], the solution method for the time-periodic heat equation uses backward Euler method that leads to a circular one-shot matrix. This can, for instance, be solved by a FFT algorithm. Various approximations of the arising Schur complement matrix, similar to the method in [10] are also discussed. A MINRES acceleration is used.

By eliminating the state and adjoint variables in the three-by-three block matrix for the optimality conditions, in [17] a Fredholm integral equation of the second kind arises. This can be seen as an infinite dimensional extension of the corresponding Schur complement matrix. The integral equation is discretized using a space-time finite element framework and then solved with a nested multigrid method.

The solution method in [18] for a parabolic partial differential constraint equation is based on the extension of a special decomposition of the solution operator and used both for the forward state equation and the backward in time adjoint equation. The problem is solved via a Newton-Picard fixed point iteration method, namely, via a truncated expansion as preconditioning operator. Assuming sufficient smoothing properties, it suffices then to capture the dominant, lower frequency part of the spectrum in order to obtain a fast rate

of convergence. Therefore, the Newton-Picard preconditioner can be implemented by use of a coarse grid approximation. Using a two-grid approach, the full discretized accuracy is obtained via a fine grid. Note, however, that for the Crank-Nicolson temporal discretization, no coarsening is used. The method seems to be quite competitive with the above two methods and simpler in terms of computational effort.

4. Computational complexity of the preconditioners $\mathcal{P}_F$ and $\mathcal{P}_{nsn}$

From the theoretical derivations in Section 3.2 it follows that the preconditioners $\mathcal{P}_{FS}$ and $\mathcal{P}_{nsn}$ are numerically very efficient. We present now some further details regarding their computational cost and implementation. First we point out that a scalar version of the preconditioner $\mathcal{P}_{nsn}$ is implemented in [6], involving a solution of the arising complex-valued system. In the extension [7], a decoupling into real-valued systems is done via a certain matrix transformation. In our method we use an elimination and substitution method to obtain a real-valued system.

4.1. The preconditioner $\mathcal{P}_{FS}$

As follows from Section 3, the solution of linear systems with $\mathcal{P}_{FS}$ involves the solution of two saddle point systems with the matrix

$$\mathcal{H} = \begin{bmatrix} M + \sqrt{\widetilde{\beta}}K & -\sqrt{\widetilde{\beta}}D^T \\ -\sqrt{\widetilde{\beta}}D & 0 \end{bmatrix}. \tag{22}$$

In order to efficiently solve systems with $\mathcal{H}$ we introduce

$$\mathcal{P}_{\mathcal{H}} = \begin{bmatrix} M + \sqrt{\widetilde{\beta}}K & 0 \\ -\sqrt{\widetilde{\beta}}D & -\widetilde{S} \end{bmatrix}, \tag{23}$$

where $\widetilde{S}$ is an approximation of the Schur complement S of H, $S = \widetilde{\beta}D(M + \sqrt{\widetilde{\beta}}K)^{-1}D^T$. Since the stiffness matrix K can be written in the form $D^T M_p^{-1} D$, where M_p is the pressure (i.e., scalar) mass matrix, we can use the relation

$$\begin{aligned} S^{-1} &= \left(D(M + \sqrt{\widetilde{\beta}}D^T M_p^{-1} D)^{-1} D^T \right)^{-1} = \sqrt{\widetilde{\beta}}M_p^{-1} + (DM^{-1}D^T)^{-1} \\ &\approx \sqrt{\widetilde{\beta}}M_p^{-1} + K_p^{-1} = \widetilde{S}^{-1}, \end{aligned}$$

where K_p is the scalar Laplace matrix. For details see [19]. Actually, the relation follows from

$$\left(\widetilde{D}(I + \widetilde{D}^T \widetilde{M}_p \widetilde{D}^T)^{-1} \widetilde{D} \right)^{-1} = \widetilde{M}_p^{-1} + (\widetilde{D}\widetilde{D}^T)^{-1},$$

where $\widetilde{D} = DM^{1/2}$, $\widetilde{M_p} = \frac{1}{\widetilde{\beta}} M_p$, which simply follows from

$$\widetilde{D}(I + \widetilde{D}^T \widetilde{M_p}^{-1} \widetilde{D})^{-1} \widetilde{D}^T (\widetilde{M_p}^{-1} + (\widetilde{D}\widetilde{D}^T)^{-1}) =$$
$$\widetilde{D}(I + \widetilde{D}^T \widetilde{M_p}^{-1} \widetilde{D})^{-1} (\widetilde{D}^T \widetilde{M_p}^{-1} \widetilde{D} + I) \widetilde{D}^T (\widetilde{D}\widetilde{D}^T)^{-1} = I.$$

Hence,

$$\widetilde{S}^{-1} = \sqrt{\widetilde{\beta}} M_p^{-1} + K_p^{-1}.$$

Moreover, the preconditioner $P_{\mathcal{H}}$ in (23) has been shown to possess a mesh-independent convergence, cf. [20, 21].

Solving systems with the two-by-two block matrix $\mathcal{H}$

Systems with $\mathcal{H}$ can be solved in various ways. We test here two such methods.

(i) Solve $\mathcal{H}$ by an (inner) FGMRES, preconditioned by $\mathcal{P}_{\mathcal{H}}$.

(ii) Solve $\mathcal{H}$ using the *inexact Uzawa* method, cf. i.e., [22, 23], in which case $\mathcal{P}_{\mathcal{H}}$ then acts as a splitting matrix for the iteration,

$$\boldsymbol{\rho_{n+1}} = \boldsymbol{\rho_n} + \mathcal{P}_{\mathcal{H}}^{-1} \mathcal{H} \mathbf{r_n} \tag{24}$$

where $\boldsymbol{\rho_n} = (\mathbf{x}, \mathbf{y})^{\mathbf{T}}$ and $\mathbf{r_n}$ are the solution and residual vectors at the n-th iteration. However, a parameter τ is introduced with $\widetilde{S}$ in $\mathcal{P}_{\mathcal{H}}$ to improve performance. This approach has been previously used in [8] in a similar context with $\tau = 3/5$.

We note that the choice of the Uzawa method is done to match the setting of the numerical tests in [8]. In recent works, more efficient Uzawa-like methods have been derived, that can be used as inner solvers.

4.2. The preconditioner $\mathcal{P}_{nsn}$

Following the approach described in [7], the complex-valued system is transformed into

$$\mathcal{A} = \mathbf{T}^* \mathcal{A}_T \mathbf{T} \tag{25}$$

where

$$\mathcal{A}_T = \begin{bmatrix} (1+\beta\omega^2)^{1/2} M & K & 0 & -D^T \\ K & \beta^{-1}(1+\beta\omega^2)^{1/2} M & -D & 0 \\ 0 & -D^T & 0 & 0 \\ -D & 0 & 0 & 0 \end{bmatrix} \tag{26}$$

and $\mathbf{T} = \begin{bmatrix} T \otimes I_n & 0 \\ 0 & T \otimes I_m \end{bmatrix}$ with $T = (1+\beta\omega^2)^{1/4} \begin{bmatrix} (1+\beta\omega^2)^{1/2} & -i \\ 0 & 1 \end{bmatrix}$. The operator $\otimes$ is the Kronecker product and $I_k \in R^k$ is the identity matrix. This transformation splits the original system into

$$\begin{aligned} \mathcal{A}_T \mathbf{y_1} &= \mathbf{c_1} \\ \mathcal{A}_T \mathbf{y_2} &= \mathbf{c_2}. \end{aligned} \tag{27}$$

where $\mathbf{c} = \mathbf{c}_1 + i\mathbf{c}_2 = (\mathbf{T}^*)^{-1}\mathbf{b}$ and $\mathbf{c} = \mathbf{c}_1 + i\mathbf{c}_2 = \mathbf{T}^{-1}\mathbf{x}$. Hence, one can solve two real systems, which may be done in parallel, instead of solving one single complex-valued linear system. As noted in [6], an effective preconditioner to solve (26) is

$$\mathcal{P}_{nsn} = \begin{bmatrix} \mathbf{P} & 0 \\ 0 & \mathbf{S} \end{bmatrix},$$

where $\mathbf{P} = \begin{bmatrix} P & 0 \\ 0 & \beta P \end{bmatrix}$, $\mathbf{S} = \beta \begin{bmatrix} S & 0 \\ 0 & \frac{1}{\beta} S \end{bmatrix}$, $P = M + \sqrt{\beta}(F + \omega M)$ and $S = D(M + \sqrt{\beta}(F + \omega M))^{-1} D^T$.

In [7], the action of the block S in $\mathbf{S}$ is replaced using two steps. In the first step each block S is replaced by

$$S = \widehat{S}_{CH} = (\sqrt{\beta} M_p^{-1} + (1 + \sqrt{\beta}\omega) K_p^{-1})^{-1} \tag{28}$$

using [19]. In the second step the action of S is replaced by the preconditioned Richardson iteration given by

$$\widehat{S} = S \left(I - \prod_{i=1}^{r} (I - \tau_i \widehat{S}_{CH} S)^i \right)^{-1} \tag{29}$$

for $i = 1, 2, 3, ...$ Here I is an identity matrix of proper order. Moreover, it is noted that the condition

$$1 - \prod_{i=1}^{r} (1 - \tau_i \lambda)^i > 0 \text{ for any } \lambda \in (0, 1] \tag{30}$$

is sufficient to guarantee that $\widehat{S}$ is positive definite. Furthermore, if $\tau_1 > 0$ and $\tau_i > 1$ then $\widehat{S}$ is symmetric positive definite and $\widehat{S} \sim S$. While replacing S with $\widehat{S}$ yields a good approximation, incorporating preconditioned Richardson iteration yields further improvements. This has been confirmed by the numerical experiments in [7].

4.3. Spectrally equivalent approximations of the involved matrix blocks

In the numerical tests in Section 5, the following approximations are used.

(i) Based on the analysis in [24], a system with a mass matrix, whether in the velocity space or the pressure space, i.e., $M_{\vec{y}}$ and M_p, is solved by the Chebyshev semi-iteration method, terminated after 20 iterations.

(ii) All blocks of the form K_p, $M_{\vec{y}} + \sqrt{\widetilde{\beta}} K_{\vec{y}}$, $M_{\vec{y}} + \sqrt{\beta}(K_{\vec{y}} + \omega M_{\vec{y}})$, $M_p + \sqrt{\widetilde{\beta}} K_p$ are replaced by one V-cycle of an Algebraic Multigrid (AMG) solver.

Table 1 summarizes the computational costs for each preconditioner per iteration.

Preconditioner	Operations
$\mathcal{P}_{nsn}$	Four AMG solves and two solves with Chebyshev semi-iterations.
$\mathcal{P}_{FS}(Uzawa)$	Four Uzawa iterations, each requiring one solution with $P_{\mathcal{H}}$ (eight AMG solves and four solves with Chebyshev semi-iterations).
$\mathcal{P}_{FS}(FGMRES)$	Four FGMRES iterations, preconditioned by $\mathcal{P}_{\mathcal{H}}$ (eight AMG solves and four solves with Chebyshev semi-iterations).

Table 1: Computational complexity of the preconditioners, summary

5. Numerical tests

We illustrate the performance of the preconditioners using the test problem from [6], see also [25].

Problem 1. *Consider the Stokes optimal control problem on the unit square. The target velocity is* $\vec{y}_d(x_1, x_2) = (y_{d,1}(x_1, x_2), y_{d,2}(x_1, x_2))$ *is chosen as*

$$y_{d,1}(x_1, x_2) = 10\frac{\partial}{\partial x_2}(\varphi(x_1)\varphi(x_2)) \quad \text{and} \quad y_{d,2}(x_1, x_2) = -10\frac{\partial}{\partial x_1}(\varphi(x_1)\varphi(x_2)),$$

where $\varphi(z) = (1 - cos(0.8\pi z))(1 - z)^2$.

As in [4, 5] we discretize the problem using the stable Taylor-Hood Q2-Q1 pair of finite element spaces, namely, the state $\mathbf{u}$, the control $\mathbf{f}$ and the adjoint $\mathbf{g}$ are discretized using piece-wise quadratic (Q2) basis functions, while the pressure $\mathbf{p}$ and its corresponding adjoint $\mathbf{q}$ are discretized using piece-wise linear (Q1) basis functions. For the meshsize we choose $h = 2^{-r}$, $r = 4, 5, 6, 7$.

The preconditioners are implemented in C++ and executed using the open source finite element library `deal.ii` ([26]) and the Algebraic Multigrid (AMG) solver from the Trilinos library ([27]). All experiments are performed on an Intel(R) Core(TM) i5 CPU 750 @ 2.67GHz-2.80GHz computer with 4GB RAM.

We compare the performance of $\mathcal{P}_{FS}$ with that of $\mathcal{P}_{nsn}$, to match the numerical experiments from [7]. We solve the system, preconditioned by $\mathcal{P}_{FS}$ with FGMRES and the system, preconditioned by $\mathcal{P}_{nsn}$ with MINRES.
The results are presented in Tables 2 and 3 for $\mathcal{P}_{FS}$, and in Table 4 for $\mathcal{P}_{nsn}$.

We also compare the performance of a block lower-triangular preconditioner, $\mathcal{P}_{BLT}$ with that of the block-diagonal $\mathcal{P}_{nsn}$, defined in (19). The preconditioner $\mathcal{P}_{BLT}$ is of the form

$$\mathcal{P}_{BLT} = \begin{bmatrix} \mathbf{P} & 0 \\ \mathbf{D} & -\mathbf{S} \end{bmatrix}, \tag{31}$$

where $\mathbf{P} = \begin{bmatrix} P & 0 \\ 0 & \beta P \end{bmatrix}$, $\mathbf{S} = \beta \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 0 & -D \\ -D & 0 \end{bmatrix}$, and P and S are as in $\mathcal{P}_{nsn}$.

	β								
Size	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}
$\omega = \mathbf{10^{-4}}$									
4934	**8**	**10**	**12**	**12**	**11**	**11**	**9**	**7**	**7**
	0.098	0.118	0.140	0.140	0.129	0.128	0.107	0.086	0.086
19078	**8**	**10**	**12**	**12**	**12**	**11**	**10**	**8**	**6**
	0.370	0.454	0.537	0.537	0.538	0.498	0.456	0.372	0.291
75014	**8**	**10**	**12**	**12**	**12**	**11**	**10**	**8**	**6**
	1.549	1.901	2.241	2.243	2.244	2.072	1.897	1.553	1.211
297478	**8**	**10**	**12**	**12**	**12**	**11**	**10**	**8**	**6**
	6.870	8.381	9.899	9.898	9.896	9.145	8.380	6.849	5.335
$\omega = \mathbf{1}$									
4934	**8**	**10**	**12**	**12**	**11**	**11**	**9**	**7**	**7**
	0.098	0.118	0.14	0.14	0.129	0.129	0.107	0.087	0.087
19078	**8**	**10**	**12**	**12**	**12**	**11**	**10**	**8**	**6**
	0.371	0.455	0.539	0.538	0.539	0.497	0.455	0.373	0.291
75014	**8**	**10**	**12**	**12**	**12**	**11**	**10**	**8**	**6**
	1.553	1.897	2.24	2.242	2.243	2.074	1.895	1.552	1.209
297478	**8**	**10**	**12**	**12**	**12**	**11**	**10**	**8**	**6**
	6.846	8.363	9.875	9.897	9.888	9.129	8.363	6.846	5.332
$\omega = \mathbf{10^4}$									
4934	**9**	**9**	**9**	**9**	**9**	**9**	9	**7**	**7**
	0.108	0.107	0.107	0.108	0.108	0.107	0.108	0.104	0.087
19078	**10**	**10**	**10**	**10**	**10**	**10**	**9**	**8**	**6**
	0.454	0.454	0.455	0.454	0.455	0.456	0.414	0.374	0.292
75014	**10**	**10**	**10**	**10**	**10**	**10**	**9**	**8**	**6**
	1.893	1.897	1.912	1.897	1.901	1.898	1.725	1.558	1.214
297478	**10**	**10**	**10**	**10**	**10**	**10**	**9**	**8**	**6**
	8.361	8.346	8.362	8.368	8.374	8.368	7.599	6.850	5.334

Table 2: Preconditioner $\mathcal{P}_{FS}$, outer solver FGMRES; each $\mathcal{H}$ block is approximated by 4 Uzawa iterations

	β								
Size	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}
$\omega = \mathbf{10^{-4}}$									
4934	**8**	**9**	**9**	**8**	**7**	**6**	**5**	**4**	2
	0.141	0.180	0.156	0.140	0.125	0.110	0.094	0.079	0.061
19078	**8**	**9**	**9**	**8**	**7**	**6**	**5**	**4**	**3**
	0.512	0.571	0.570	0.515	0.459	0.401	0.345	0.288	0.236
75014	**8**	**9**	**9**	**8**	**7**	**6**	**5**	**4**	**3**
	2.057	2.285	2.288	2.060	1.839	1.607	1.380	1.153	0.924
297478	**8**	**9**	**9**	**8**	**7**	**6**	**5**	**4**	**3**
	8.909	9.880	9.891	8.908	7.926	6.920	5.933	4.971	3.984
$\omega = \mathbf{1}$									
4934	**8**	**9**	**9**	**8**	**7**	**6**	**5**	**4**	2
	0.141	0.161	0.155	0.141	0.125	0.109	0.094	0.092	0.057
19078	**8**	**9**	**9**	**8**	**7**	**6**	**5**	**4**	**3**
	0.513	0.569	0.570	0.516	0.455	0.399	0.343	0.288	0.230
75014	**8**	**9**	**9**	**8**	**7**	**6**	**5**	**4**	**3**
	2.058	2.291	2.291	2.058	1.831	1.603	1.376	1.148	0.922
297478	**8**	**9**	**9**	**8**	**7**	**6**	**5**	**4**	**3**
	8.906	9.894	9.889	8.896	7.913	6.921	5.936	4.952	3.969
$\omega = \mathbf{10^{4}}$									
4934	**5**	**5**	**5**	**5**	**5**	**5**	**5**	**4**	2
	0.140	0.134	0.094	0.094	0.093	0.094	0.095	0.079	0.048
19078	**5**	**5**	**5**	**5**	**5**	**5**	**5**	**4**	**3**
	0.342	0.342	0.343	0.344	0.343	0.343	0.342	0.285	0.251
75014	**5**	**5**	**5**	**5**	**5**	**5**	**5**	**4**	**3**
	1.373	1.374	1.378	1.405	1.378	1.377	1.377	1.149	0.924
297478	**5**	**5**	**5**	**5**	**5**	**5**	**5**	**4**	**3**
	5.941	5.945	5.965	5.970	5.950	5.952	5.937	4.961	3.969

Table 3: Preconditioner $\mathcal{P}_{FS}$, outer solver FGMRES; each $\mathcal{H}$ block is approximated by four FGMRES iterations preconditioned by the block-lower triangular preconditioner $\mathcal{P}_{\mathcal{H}}$

	β								
Size	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}	10^{-10}
$\omega = \mathbf{10^{-4}}$									
4934	**36**	**37**	**37**	**35**	**33**	**29**	**27**	**25**	**23**
	0.188	0.206	0.191	0.182	0.186	0.151	0.141	0.131	0.121
19078	**38**	**40**	**41**	**39**	**37**	**33**	**29**	**27**	**25**
	0.75	0.788	0.828	0.767	0.731	0.652	0.573	0.533	0.498
75014	**40**	**44**	**45**	**43**	**39**	**37**	**33**	**29**	**25**
	3.327	3.65	3.73	3.573	3.248	3.073	2.747	2.413	2.083
297478	**44**	**45**	**47**	**45**	**41**	**39**	**35**	**31**	**27**)
	16.289	16.603	17.37	16.625	15.143	14.425	12.949	11.472	10.002
$\omega = \mathbf{1}$									
4934	**36**	**37**	**37**	**35**	**32**	**29**	**27**	**25**	**23**
	0.188	0.207	0.191	0.182	0.166	0.151	0.141	0.131	0.12
19078	**38**	**40**	**41**	**39**	**37**	**33**	**29**	**27**	**25**
	0.75	0.786	0.81	0.774	0.733	0.658	0.576	0.533	0.496
75014	**40**	**44**	**45**	**43**	**39**	**37**	**33**	**29**	**25**
	3.327	3.656	3.742	3.577	3.238	3.069	2.745	2.418	2.083
297478	**42**	**45**	**47**	**45**	**41**	**39**	**35**	**31**	**27**
	15.548	16.633	17.388	16.641	15.173	14.416	12.932	11.476	10.012
$\omega = \mathbf{10^4}$									
4934	**19**	**21**	**23**	**23**	**23**	**25**	**27**	**23**	**21**
	0.175	0.127	0.158	0.145	0.161	0.131	0.179	0.159	0.125
19078	**21**	**23**	**25**	**25**	**25**	**27**	**29**	**25**	**23**
	0.415	0.457	0.495	0.497	0.493	0.535	0.575	0.495	0.455
75014	**25**	**25**	**27**	**27**	**27**	**31**	**33**	**27**	**23**)
	2.083	2.08	2.248	2.247	2.236	2.586	2.748	2.249	1.919
297478	**27**	**28**	**29**	**31**	**31**	**33**	**37**	**29**	**25**
	9.995	10.373	10.731	11.47	11.473	12.221	13.695	10.738	9.265

Table 4: Preconditioner $\mathcal{P}_{nsn}$, outer solver MINRES; the Schur complement approximation S is approximated by two Richardson's iterations

Remark 3. *Denote* $\mathcal{P}_{BLT+} = \begin{bmatrix} \mathbf{P} & 0 \\ \mathbf{D} & \mathbf{S} \end{bmatrix}$. *As can be readily shown, see also [11], the eigenvalues of* $\mathcal{P}_{BLT}^{-1} \begin{bmatrix} \mathbf{P} & \mathbf{D}^T \\ \mathbf{D} & \mathbf{0} \end{bmatrix}$ *and of* $\mathcal{P}_{BLT+}^{-1} \begin{bmatrix} \mathbf{P} & \mathbf{D}^T \\ \mathbf{D} & \mathbf{0} \end{bmatrix}$ *equal* ± 1 *or* 1, *respectively. Theoretically, only three and two iterations, respectively, are needed for a GMRES method to converge, provided that* S *is the exact negative Schur complement* $\mathbf{DP^{-1}D^T}$ *and that we solve exactly with the diagonal blocks. However,* $\mathbf{P}$ *and* $\mathbf{S}$ *are approximations of the original matrix blocks, the matrices are, in general, not normal and when the arising systems with* $\mathbf{P}$ *and* $\mathbf{S}$ *are solved by inner iterations to some final tolerance, the eigenvalues are perturbed and more iterations are required. Numerical tests, not included here, indicate that the iterative solver, preconditioned with* $\mathcal{P}_{BLT+}$ *converges in about twice more iterations than when preconditioned with* $\mathcal{P}_{BLT}$.

For the nonsymmetric preconditioner $\mathcal{P}_{BLT}$ one has to use a GMRES-type of method instead of MINRES. Normally, for accurate pivot block preconditioner, a block-triangular version converges faster that the corresponding block-diagonal one. In the present case, however, as is seen in Table 5, it converges slower. This could be associated with the presumption that the block-diagonal matrices $\mathbf{P}$ and $\mathbf{S}$ are not accurate enough approximations of the true pivot matrix block and the exact Schur complement. The reported iteration counts in Table 5 are in line with the results in [11].

6. Conclusions

The block matrix preconditioner, previously applied by the authors to Stokes optimal control problems has been shown to have the same robust behaviour when applied for time-harmonic state equations. It outperforms previously published methods, such as the method proposed in [6].

The results show that our preconditioner converges with a number of iterations less than twice times the difference in number of iterations for a matrix with condition number 3 instead of condition number 2. In addition, the method in [6] is more complicated. It involves a block-diagonal preconditioner for the block 4×4 block matrix in (11). The preconditioning does not show a fully parameter-independent behaviour and needs many iterations, see [6, 7]. The method is not competitive with our method that involves solving two Stokes problems per outer iteration. Since for our method the spectrum of the outer preconditioned matrix is clustered and corresponds to a condition number, bounded by 2, it needs a few outer iterations.

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	β								
Size	10^{-2}	10^{-3}	10^{-4}	10^{-5}	10^{-6}	10^{-7}	10^{-8}	10^{-9}	
$\omega = \mathbf{10^{-4}}$									
4934	**31**	**37**	**43**	**45**	**47**	**47**	**44**	**40**	**33**
	0.184	0.237	0.286	0.3	0.304	0.303	0.279	0.248	0.198
19078	**32**	**38**	**45**	**48**	**48**	**47**	**47**	**44**	**40**
	0.688	0.822	0.99	1.064	1.064	1.043	1.042	0.968	0.872
75014	**33**	**38**	**46**	**48**	**48**	**48**	**47**	**46**	**44**
	2.92	3.373	4.118	4.314	4.317	4.315	4.225	4.129	3.95
297478	**35**	**38**	**46**	**48**	**49**	**49**	**48**	**47**	**46**
	13.709	14.902	18.09	18.914	19.33	19.27	18.915	18.495	18.085
$\omega = \mathbf{1}$									
4934	**31**	**37**	**43**	**45**	**47**	**47**	**44**	**40**	**33**
	0.187	0.235	0.3	0.298	0.304	0.303	0.279	0.248	0.199
19078	**33**	**38**	**45**	**48**	**48**	**47**	**47**	**44**	**40**
	0.721	0.841	1.011	1.066	1.067	1.04	1.038	0.969	0.872
75014	**35**	**38**	**46**	**48**	**48**	**48**	**47**	**46**	**44**
	3.177	3.459	4.229	4.322	4.317	4.312	4.228	4.137	3.948
297478	**35**	**38**	**46**	**48**	**49**	**49**	**48**	**47**	**46**
	14.067	15.216	18.062	18.894	19.283	19.281	18.88	18.471	18.065
$\omega = \mathbf{10^4}$									
4934	**25**	**30**	**31**	**34**	**37**	**38**	**40**	**38**	**32**
	0.151	0.206	0.198	0.204	0.226	0.246	0.249	0.234	0.192
19078	**28**	**30**	**33**	**36**	**39**	**41**	**42**	**42**	**40**
	0.598	0.642	0.709	0.775	0.862	0.898	0.919	0.922	0.876
75014	**29**	**31**	**35**	**37**	**40**	**45**	**47**	**44**	**43**
	2.568	2.749	3.104	3.287	3.592	4.055	4.263	3.967	3.865
297478	**30**	**31**	**34**	**39**	**42**	**46**	**51**	**48**	**45**
	11.712	12.085	13.282	15.267	16.439	18.101	20.14	18.893	17.684

Table 5: Preconditioner $\mathcal{P}_{BLT}$, outer solver FGMRES; the Schur complement approximation S is approximated by two Richardson's iterations

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