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Lebesgue Theory

A Brief Overview

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Abstract

We present different concepts in measure theory. We construct the Lebesgue measure on $\mathbb{R}$. We define the Lebesgue integral and prove some famous convergence theorems. We compare the Riemann integral with the Lebesgue integral and prove some famous results. We give some examples of how to use the Lebesgue integral.

By doing the above, we hope to give a brief overview of the Lebesgue theory.

Contents

1 Introduction

In 1904 Henri Lebesgue invented a new way of integrating functions. His theory of integration was a generalization of that of Riemann's—a larger set of functions could be integrated and the problem of limits interacting badly with integrals was solved.

In the center of the Lebesgue integral stands the following idea: *the limit of an integral should equal the integral of the limit.* In other words,

$$\lim_{n\to\infty}\int f_n = \int \lim_{n\to\infty} f_n. \tag{1}$$

(This equality holds true in the Riemann sense too, but under less mild conditions.) The aim of this paper is to establish (1). The following is some background that will provide a starting point.

Suppose we have (1), and consider any nonnegative function f. Let $0 \le f_1 \le f_2 \le \dots$ be any sequence of simple functions[1] approximating f from below. (Such a sequence is guaranteed to exist.) Since integrals should be sensitive to size,

$$\lim_{n\to\infty}\int f_n = \sup\{\int f_n\},\ n = 1, 2, 3, \dots.$$

But

$$\int \lim_{n\to\infty} f_n = \int f,$$

so for any nonnegative function f we are prone to define

$$\int f = \sup\{\int f_n\},$$

where (f_n) is any sequence of simple functions approximating f from below.

But this is to say that

$$\int f = \sup\{\int \varphi\}, \tag{2}$$

[1] A simple function is a function with finite range

where the supremum is taken over all simple function φ such that $0 \le \varphi \le f$. (For a negative function we simply consider $-f$.)

It remains to define $\int \varphi$ where φ is simple. As we know, Riemann defined his integral by partitioning the domain. Let us try something different. *Instead of partitioning the domain, let us partition the range.* This was Lebesgue's breakthrough idea. It is the Lebesgue integral in a nutshell.

See Figure 1.

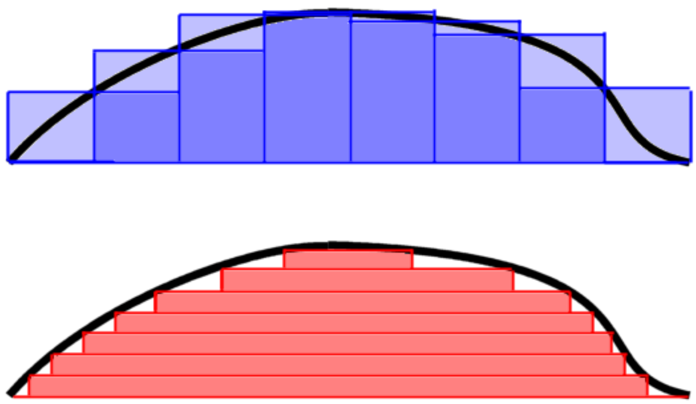

Figure 1: Riemann integration (blue) and Lebesgue integration (red). The Riemann integral partitions the domain of f, while the Lebesgue integral partitions the range of f

In a letter to Paul Montel, Lebesgue summarized his approach:

> I have to pay a certain sum, which I have collected in my pocket. I take the bills and coins out of my pocket and give them to the creditor in the order I find them until I have reached the total sum. This is the Riemann integral. But I can proceed differently. After I have taken all the money out of my pocket I order the bills and coins according to identical values and then I pay the several heaps one after the other to the creditor. This is my integral. —*Henri Lebesgue*

Inspired by Lebesgue, let us partition the range of φ. Then

$$\int \varphi = \sum a_i m(D_i)$$

where

- a_i is the different values φ assumes,
- D_i are the points where φ assumes a_i, and

- $m(D_i)$ is the size or *measure* of D_i.

Note that D_i can be written as

$$D_i = \{x : \varphi(x) = a_i\} = \varphi^{-1}(a_i).$$

We call D_i the *preimage* of a_i under φ.

If we want to integrate a large set of functions, measuring the preimage can be quite intricate. This is why Lebesgue invented measure theory.

2 Outline

Section 3, Measure on a σ-algebra
: We define a σ-algebra. We define sequences of sets and their limit. We define generated σ-algebras. We define the Borel σ-algebra. We define the concept of a measure on a σ-algebra. We define measure sequences and their limits, and some important results regarding them. We define the concept of measurable space and measure space.

Section 4, Outer measure
: We define the concept of an outer measure μ^*. We classify sets which are outer measurable, and show that the collection of all such sets is a σ-algebra. We show that μ^* is a measure when restricted to the collection of all outer measurable sets. We demonstrate a concrete way of constructing an outer measure.

Section 5, Lebesgue measure on $\mathbb{R}$
: We construct the Lebesgue measure on $\mathbb{R}$—using the Lebesgue outer measure. We show that all open sets in $\mathbb{R}$ are Lebesgue measurable. We prove some properties of the Lebesgue measurable space. We prove the existence of non Lebesgue measurable sets in $\mathbb{R}$.

Section 6, Measurable functions
: We define measurable functions. We define measurability of sequences of functions. We define the positive and negative part of f, the restriction and extension of f, and their measurability. We also define the concepts of real-valued and convergence almost everywhere.

Section 7, Almost everywhere
: In this section we introduce the concept of almost everywhere.

Section 8, The Lebesgue Integral
: We define the Lebesgue integral. We investigate some of its properties. We prove some famous convergence theorems, including Lebesgue's monotone convergence theorem, Fatou's theorem and Lebesgue's dominated convergence theorem.

Section 9, Comparison with the Riemann integral

We compare the Riemann integral with the Lebesgue integral. We show that if a bounded function is Riemann integrable over an interval $[a, b]$, it is Lebesgue integrable. We show that f is Riemann integrable if and only if the set of discontinuity points has measure 0. We prove the counterpart to the Lebesgue dominated convergence theorem in the Riemann sense. We demonstrate in a few examples how to use the Lebesgue integral.

3 Measure on a σ-algebra

Notations. $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$, $\mathbb{R}$ is the real line. $\mathcal{P}(X)$ is the collection of all subsets of a set X.

Let X be any set. Suppose we want a function that measures subsets of X. Let us call such a function a *measure*, and denote it by

$$m : \mathcal{P}(X) \to [0, \infty]. \tag{3}$$

m should meet certain criteria that we normally associate with measuring. (Note that one such criterion is already made implicit in (3); since no measure can be negative, the range of m is positive or zero.)

m should be *countably additive*, meaning that for any disjoint sets A and B in $\mathcal{P}(X)$,

$$m(A \cup B) = m(A) + m(B).$$

Furthermore, m should be *invariant under translation.* This means that when we translate a set in space, its measure should stay the same.

Ideally the domain of m would be $\mathcal{P}(X)$, so that every subset of X could be measured. Unfortunately this is not possible. In 1914 Felix Hausdorff showed that is impossible to define a measure on $\mathcal{P}(\mathbb{R})$ that is countably additive and at the same time invariant under translation. This is why we introduce *σ-algebras.* As we will see, a σ-algebra is a collection of subsets of X that is measurable with respect to some measure m.

3.1 σ-algebra of sets

Definition 1 (Algebra of subsets)**.** *Let X be a set. A collection $\mathcal{A}$ of subsets of X is called an algebra of subsets of X if it satisfies the following conditions:*

1° $X \in \mathcal{A}$,

2° $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$,

3° $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$.

Lemma 1. *If $\mathcal{A}$ is an algebra of subsets of X, then*

(1) $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$,

(2) $A, B \in \mathcal{A} \Rightarrow A \setminus B \in \mathcal{A}$.

Proof. $A \cap B = (A^c \cup B^c)^c \in \mathcal{A}$, $A \setminus B = A \cap B^c \in \mathcal{A}$. □

Definition 2 (σ-algebra of subsets)**.** *An algebra $\mathcal{A}$ of subsets of a set X is called a σ-algebra if is satisfies the following additional condition:*

4° $(A_n : n \in \mathbb{N}) \subset \mathcal{A} \Rightarrow \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$.

Example 1. $\mathcal{P}(X)$ is a σ-algebra of subsets of X. It is the greatest σ-algebra of subsets of X in the sense that if $\mathcal{A}$ is a σ-algebra of subsets of X and if $\mathcal{P}(X) \subset \mathcal{A}$, then $\mathcal{A} = \mathcal{P}(X)$.

Lemma 2. *If $\mathcal{A}$ is a σ-algebra of subsets of set X, then $(A_n : n \in \mathbb{N}) \subset \mathcal{A}$ $\Rightarrow \bigcap_{n\in\mathbb{N}} A_n \in \mathcal{A}$.*

Proof. $\bigcap_{n\in\mathbb{N}} A_n = \left(\bigcup_{n\in\mathbb{N}} A_n\right)^c \in \mathcal{A}$. □

3.2 Limits of sequences of sets

Definition 3 (Increasing and decreasing sequences of sets)**.** *Let $(A_n : n \in \mathbb{N})$ be a sequence of subsets of X. $(A_n : n \in \mathbb{N})$ is said to be an increasing sequence and we write $A_n \uparrow$ if $A_n \subset A_{n+1}$ for $n \in \mathbb{N}$. We say that $(A_n : n \in \mathbb{N})$ is a decreasing sequence and write $A_n \downarrow$ if $A_{n+1} \subset A_n$ for $n \in \mathbb{N}$. A sequence $(A_n : n \in \mathbb{N})$ is called monotone if it is either increasing or decreasing. For an increasing sequence, we define*

$$\lim_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} A_n = \{x \in X : x \in A_n \textit{ for some } n \in \mathbb{N}\}.$$

For a decreasing sequence $(A_n : n \in \mathbb{N})$, we define

$$\lim_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} A_n = \{x \in X : x \in A_n \textit{ for every } n \in \mathbb{N}\}.$$

In order to define the limit of sequences of sets, we must define the limit inferior and the limit superior of sequences of sets.

Definition 4 (Limit inferior, limit superior)**.** *The limit inferior and the limit superior of a sequence $(A_n : n \in \mathbb{N})$ of subsets of a set X is defined by*

$$\liminf_{n\to\infty} A_n = \bigcup_{n\in\mathbb{N}} \bigcap_{k\geq n} A_k,$$
$$\limsup_{n\to\infty} A_n = \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} A_k.$$

Definition 5 (Convergent sequence)**.** *Let* $(A_n : n \in \mathbb{N})$ *be an arbitrary sequence of subsets of a set* X*. We say that the sequence converges if* $\liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n$*, and we set*

$$\lim_{n\to\infty} A_n = \liminf_{n\to\infty} A_n = \limsup_{n\to\infty} A_n.$$

If this does not hold, we say that the sequence diverges.

Remark 1. It can be shown that for any sequence $(A_n : n \in \mathbb{N})$, both $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$ lies in $\mathcal{A}$. In particular if $(A_n : n \in \mathbb{N})$ converges, then $\lim_{n\to\infty} A_n \in \mathcal{A}$.

3.3 Generated σ-algebras

It is useful to define the idea of a *smallest* σ-algebra.

Definition 6 (Collection of indexed sets)**.** *Let* A *be any set. We call* $\{E_\alpha : \alpha \in A\}$ *a collection of sets indexed by* A*.*

Remark 2. Let $\{\mathcal{A}_\alpha : \alpha \in A\}$ be a collection of σ-algebras of subsets of X where A is an arbitrary indexing set. It is routine to verify that $\cap_{\alpha\in A} \mathcal{A}_\alpha$ is a σ-algebra of subsets of X.

Theorem 1. *Let* $\mathcal{C}$ *be any collection of subsets of* X*. There exists a smallest* σ*-algebra of* $\mathcal{A}_0$ *of subsets of* X *containing* $\mathcal{C}$*, smallest in the sense that if* $\mathcal{A}$ *is a* σ*-algebra of subsets of* X *containing* $\mathcal{C}$ *then* $\mathcal{A}_0 \subset \mathcal{A}$*. Similarly there exists a smallest algebra containing* $\mathcal{C}$*.*

Proof. $\mathcal{P}(X)$ is a σ-algebra of X containing $\mathcal{C}$. Let $\{\mathcal{A}_\alpha : \alpha \in A\}$ be the collection of all σ-algebras of subsets of X containing $\mathcal{C}$. By Remark 2, $\bigcap_{\alpha\in A} \mathcal{A}_\alpha$ is a σ-algebra that contains $\mathcal{C}$. It is the smallest such σ-algebra, since for any σ-algebra $\mathcal{A}$ containing $\mathcal{C}$, we have $\mathcal{A} \supset \bigcap_{\alpha\in A} \mathcal{A}_\alpha$. □

Definition 7 (σ-generated set)**.** *For any collection* $\mathcal{C}$ *of subsets of a set* X*, we write* $\sigma(\mathcal{C})$ *for the smallest* σ*-algebra of subsets of* X *containing* $\mathcal{C}$ *and call it the* σ*-algebra generated by* $\mathcal{C}$*.*

3.4 Borel σ-algebras

Let us review some topology.

Definition 8 (Topology, topological space)**.** *Let* X *be a set. A collection of subsets of* X *is called a topology on* X *if it satisfies:*

1° $\emptyset \in \mathcal{D}$,

2° $X \in \mathcal{D}$,

3° $\{E_\alpha : \alpha \in A\} \subset \mathcal{D} \Rightarrow \bigcup_{\alpha\in A} E_\alpha \in \mathcal{D}$,

4° $E_1, E_2 \in \mathcal{D} \Rightarrow E_1 \cap E_2 \in \mathcal{D}$.

The pair $(X, \mathcal{D})$ *is called a topological space, and the sets in* $\mathcal{D}$ *are called the open sets of the topological space.*

Definition 9 (Borel set)**.** *We call the* σ*-algebra of* $\mathcal{D}$ *the Borel* σ*-algebra of subsets of the topological space* $(X, \mathcal{D})$ *and we write* $\mathcal{B}_X$ *or* $\mathcal{B}(X)$ *for it. We call its members the Borel sets of the topological space.*

Definition 10. *Let* $(X, \mathcal{D})$ *be a topological space. A set* $E \subset X$ *is called a* G_δ*-set if it is the intersection of countably many open sets. A subset* E *of* X *is called an* F_σ*-set if it the union of countably many closed sets.*

If introducing the concept of topology seems mysterious, let us justify it here. We will use the Borel sets on $\mathbb{R}$ when we talk about the Lebesgue measure on $\mathbb{R}$. Since $\mathcal{B}(\mathbb{R})$ is a σ-algebra, it necessarily contains all open sets, all closed sets, all unions of open sets, all unions of closed sets, all intersections of closed sets, and all intersections of open sets. So by starting with open sets, we can—by the virtue of $\mathcal{B}(\mathbb{R})$ being a σ-algebra—generate a very large subset of $\mathcal{P}(\mathbb{R})$. The Borel sets on $\mathbb{R}$ can be thought of as all *conceivable* subsets of $\mathbb{R}$.

3.5 Measure on a σ-algebra

Notation 1. Let $\overline{\mathbb{R}} = \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$ be the extended real number system.

Definition 11. *Let* $\mathcal{C}$ *be a collection of subsets of a set* X*. Let* γ *be a nonnegative extended real-valued set function on* $\mathcal{C}$*. We say that*

(a) γ *is monotone on* $\mathcal{C}$ *if* $\gamma(E_1) \le \gamma(E_2)$ *for* $E_1, E_2 \in \mathcal{C}$ *such that* $E_1 \subset E_2$.

(b) γ *is additive on* $\mathcal{C}$ *if* $\gamma(E_1 \cup E_2) = \gamma(E_1) + \gamma(E_2)$ *for* $E_1, E_2 \in \mathcal{C}$ *such that* $E_1 \cap E_2 = \emptyset$.

(c) γ *is finitely additive on* $\mathcal{C}$ *if* $\gamma(\cup_{k=1}^n E_k) = \sum_{k=1}^n \gamma(E_k)$ *for every disjoint finite sequence* $(E_k : k = 1, \ldots, n)$ *in* $\mathcal{C}$ *such that* $\cup_{k=1}^n E_k \in \mathcal{C}$.

(d) γ *is countably additive on* $\mathcal{C}$ *if* $\gamma(\cup_{k=1}^n E_k) = \sum_{k=1}^n \gamma(E_k)$ *for every disjoint sequence* $(E_n : n \in \mathbb{N})$ *in* $\mathcal{C}$ *such that* $\cup_{n\in\mathbb{N}}^n E_n \in \mathcal{C}$.

(e) γ *is subadditive on* $\mathcal{C}$ *if* $\gamma(E_1 \cup E_2) \le \gamma(E_1) + \gamma(E_2)$ *for* $E_1, E_2 \in \mathcal{C}$ *such that* $E_1 \cup E_2 \in \mathcal{C}$.

(f) γ *is finitely subadditive on* $\mathcal{C}$ *if* $\gamma(\cup_{k=1}^n E_k) \le \sum_{k=1}^n \gamma(E_k)$ *for every finite sequence* $(E_k : k = 1, \ldots, n)$ *in* $\mathcal{C}$ *such that* $\cup_{k=1}^n E_k \in \mathcal{C}$.

(g) *γ is countable subadditive on $\mathcal{C}$ if $\gamma(\cup_{n\in\mathbb{N}}^{n} E_n) \leq \sum_{n\in\mathbb{N}} \gamma(E_n)$ for every sequence $(E_n : n \in \mathbb{N})$ in $\mathcal{C}$ such that $\cup_{n\in\mathbb{N}} E_n \in \mathcal{C}$.*

Lemma 3. *For any sequence $(E_n : n \in \mathbb{N})$ in an algebra $\mathcal{A}$ of subsets of X, there exists a disjoint sequence $(F_n : n \in \mathbb{N})$ in $\mathcal{A}$ such that*

$$(1) \ \bigcup_{n=1}^{N} E_n = \bigcup_{n=1}^{N} F_n \quad \text{and} \quad (2) \ \bigcup_{n\in\mathbb{N}} E_n = \bigcup_{n\in\mathbb{N}} F_n.$$

In particular, if $\mathcal{A}$ is a σ-algebra, then $\bigcup_{n\in\mathbb{N}} F_n \in \mathcal{A}$.

Proof. Let $F_1 = E_1$ and $F_n = E_n \setminus (\cup_{k=1}^{n-1} E_k) \in \mathcal{A}$, and use induction. □

Lemma 4. *Let γ be a nonnegative extened real-valued set function on an algebra $\mathcal{A}$ of subsets of a set X.*

(a) If γ is additive on $\mathcal{A}$, it is (1) finitely additive, (2) monotone and (3) finitely subadditive on $\mathcal{A}$.

(b) If γ is countably additive on $\mathcal{A}$, then it is countably subadditive on $\mathcal{A}$.

Proof. **(a).** The proof of (1) and (2) is routine and independent of (3).

Let us prove (3). Let $(E_k : k = 1, \ldots, n)$ be a finite sequence in $\mathcal{A}$, then using Lemma 3, (1) and (2),

$$\gamma(\bigcup_{k=1}^{n} E_k) = \gamma(\bigcup_{k=1}^{n} F_k) = \sum_{k=1}^{n} \gamma(F_k) \leq \sum_{k=1}^{n} \gamma(E_k).$$

This proves (a). The proof of (b) is done similarly. □

Proposition 1. *Let γ be a nonnegative extended real-valued set function on an algebra $\mathcal{A}$of subsets of a set X. If γ is additive and countably subadditive on $\mathcal{A}$ then γ is countably additive on $\mathcal{A}$.*

Definition 12 (Measure)**.** *Let $\mathcal{A}$ be a σ-algebra of subsets of a set X. A set function μ defined on $\mathcal{A}$ is called a measure if it satisfies the following conditions:*

1° $\mu(E) \in [0, \infty]$ *for every* $E \in \mathcal{A}$,

2° $\mu(\emptyset) = 0$,

3° μ *is countably additive.*

Remark 3. Note that

(1) μ is finitely additive,

(2) μ is monotone,

(3) $E_1, E_2 \in \mathcal{A}$, $E_1 \subset E_2$, $\mu(E_1) < \infty \Rightarrow \mu(E_2 \setminus E_1) = \mu(E_2) - \mu(E_1)$,

(4) μ is countably subadditive.

3.6 Measures of a sequence of sets

Theorem 2 (Monotone convergence theorem for sequences of measurable sets)**.** *Let μ be a measure on a σ-algebra $\mathcal{A}$ of subsets of a set X. Let $(E_n : n \in \mathbb{N})$ be a monotone sequence in $\mathcal{A}$.*

(a) If $E_n \uparrow$, then $\lim_{n\to\infty} \mu(E_n) = \mu(\lim_{n\to\infty} E_n)$.

(b) If $E_n \downarrow$, then $\lim_{n\to\infty} \mu(E_n) = \mu(\lim_{n\to\infty} E_n)$, provided that there exists a set $A \in \mathcal{A}$ with $\mu(A) < \infty$ such that $E_1 \subset A$.

Proof. **(a)** Suppose $E_n \uparrow$. Then $\mu(E_n) \uparrow$. If $\mu(E_{n_0}) = \infty$ for some $n_0 \in \mathbb{N}$, $\lim_{n\to\infty} \mu(E_n) = \infty$. Since E_n is increasing, $E_{n_0} \subset \bigcup_{n\in\mathbb{N}} E_n = \lim_{n\to\infty} E_n$, and so $\mu(\lim_{n\to\infty} E_n) \geq \mu(E_{n_0}) = \infty$.

If $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$, consider the disjoint sequence $(F_n : n \in \mathbb{N})$ in $\mathcal{A}$ defined by $F_n = E_n \setminus E_{n-1}$ for $n \in \mathbb{N}$ where $E_0 = \emptyset$. Then since $\bigcup_{n\in\mathbb{N}} E_n = \bigcup_{n\in\mathbb{N}} F_n$,

$$\begin{aligned}
\mu(\lim_{n\to\infty} E_n) &= \mu(\bigcup_{n\in\mathbb{N}} F_n) = \sum_{n\in\mathbb{N}} \mu(F_n) = \sum_{n\in\mathbb{N}} [\mu(E_n) - \mu(E_{n-1})] \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} [\mu(E_k) - \mu(E_{k-1})] = \lim_{n\to\infty} [\mu(E_n) - \mu(E_0)] \\
&= \lim_{n\to\infty} \mu(E_n).
\end{aligned}$$

(b) Let $(F_n : n \in \mathbb{N})$ be disjoint sequence in $\mathcal{A}$ defined by $F_n = E_n \setminus E_{n+1}$ for $n \in \mathbb{N}$. Then

$$E_1 \setminus \bigcap_{n\in\mathbb{N}} E_n = \bigcup_{n\in\mathbb{N}} F_n. \tag{4}$$

To see this, let $x \in E_1 \setminus \bigcap_{n\in\mathbb{N}} E_n$. Since $E_n \downarrow$, there exists a smallest $n_0 \in \mathbb{N}$ such that $x \notin E_{n_0+1}$. Then $x \in E_{n_0} \setminus E_{n_0+1} = F_{n_0} \subset \bigcup_{n\in\mathbb{N}} F_n$.

Conversely, if $x \in \bigcup_{n\in\mathbb{N}} F_n$, then $x \in F_{n_0} = E_{n_0} \setminus E_{n_0+1}$ for some $n_0 \in \mathbb{N}$. Thus $x \in E_{n_0} \subset E_1$, and since $x \notin E_{n_0+1}$, we have $x \notin \bigcap_{n\in\mathbb{N}} E_n$. Thus $x \in E_1 \setminus \bigcap_{n\in\mathbb{N}} E_n$. This proves (1).

By (1),

$$\mu\left(E_1 \setminus \bigcap_{n\in\mathbb{N}} E_n\right) = \mu\left(\bigcup_{n\in\mathbb{N}} F_n\right), \tag{5}$$

where, since $E_n \downarrow$,

$$\mu\left(E_1 \setminus \bigcap_{n\in\mathbb{N}} E_n\right) = \mu(E_1) - \mu\left(\bigcap_{n\in\mathbb{N}} E_n\right) = \mu(E_1) - \mu(\lim_{n\to\infty} E_n). \tag{6}$$

Now

$$\mu\left(\bigcup_{n\in\mathbb{N}} F_n\right) = \sum_{n\in\mathbb{N}} \mu(F_n) = \sum_{n\in\mathbb{N}} \mu(E_n \setminus E_{n+1}) \tag{7}$$

$$= \sum_{n\in\mathbb{N}} [\mu(E_n) - \mu(E_{n+1})] = \lim_{n\to\infty} \sum_{k=1}^{n} [\mu(E_n) - \mu(E_{n+1})] \tag{8}$$

$$= \lim_{n\to\infty} [\mu(E_1) - \mu(E_{n+1})] = \mu(E_1) - \lim_{n\to\infty} \mu(E_{n+1}). \tag{9}$$

Substituting (3) and (4) into (2), we arrive at the desired result. □

Theorem 3. *Let μ be a measure on a σ-algebra of subsets of a set X.*

(a) For an arbitrary sequence $(E_n : n \in \mathbb{N})$ in $\mathcal{A}$, we have

$$\mu(\liminf_{n\to\infty} E_n) \leq \liminf_{n\to\infty} \mu(E_n).$$

(b) If there exists $A \in \mathcal{A}$ with $\mu(A) < \infty$ such that $E_n \subset A$ for $n \in \mathbb{N}$, then

$$\mu(\limsup_{n\to\infty} E_n) \geq \limsup_{n\to\infty} \mu(E_n).$$

(c) If both $\lim_{n\to\infty} E_n$ and $\lim_{n\to\infty} \mu(E_n)$ exist, then

$$\mu(\lim_{n\to\infty} E_n) \leq \lim_{n\to\infty} \mu(E_n).$$

(d) If $\lim_{n\to\infty} E_n$ exists and if there exists $A \in \mathcal{A}$ with $\mu(A) < \infty$ such that $E_n \subset A$ for $n \in \mathbb{N}$ then $\lim_{n\to\infty} \mu(E_n)$ and

$$\mu(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} \mu(E_n).$$

Proof. **1.** $\liminf_{n\to\infty} E_n = \cup_{n\in\mathbb{N}} \cap_{k\geq n} E_k = \lim_{n\to\infty} \cap_{k\geq n} E_k$. Using (a) in Theorem 2, we have

$$\mu(\liminf_{n\to\infty} \bigcap_{k\geq n} E_k) = \liminf_{n\to\infty} \mu(\bigcap_{k\geq n} E_k) = \limsup_{n\to\infty} \mu(\bigcap_{k\geq n} E_k) \leq \liminf_{n\to\infty} \mu(E_n),$$

2. Assume what is given. Similarly as above,

$$\mu(\limsup_{n\to\infty} \bigcup_{k\geq n} E_k) = \limsup_{n\to\infty} \mu(\bigcup_{k\geq n} E_k) = \limsup_{n\to\infty} \mu(\bigcup_{k\geq n} E_k) \leq \limsup_{n\to\infty} \mu(E_n),$$

since $\cup_{k\geq n} E_k \subset E_n \subset A$.

3. By (a),

$$\mu(\lim_{n\to\infty} E_n) = \mu(\liminf_{n\to\infty} E_n) \leq \liminf_{n\to\infty} \mu(E_n) = \lim_{n\to\infty} \mu(E_n).$$

4. By (b) and (a),

$$\limsup_{n\to\infty} \mu(E_n) \le \mu(\limsup_{n\to\infty} E_n) = \mu(\liminf_{n\to\infty} E_n) \le \liminf_{n\to\infty} \mu(E_n).$$

But $\liminf_{n\to\infty} \mu(E_n) \le \limsup_{n\to\infty} \mu(E_n)$ for any sequence $(E_n : n \in \mathbb{N})$, and the result follows. □

3.7 Measurable space and measure space

Definition 13 (Measurable space, $\mathcal{A}$-measurable set)**.** *Let $\mathcal{A}$ be a σ-algebra of subsets of a set X. The pair $(X, \mathcal{A})$ is called a measurable space. A subset E of X is said to be $\mathcal{A}$-measurable if $E \in \mathcal{A}$.*

Definition 14 (Measure space)**.** *If μ is a measure on a σ-algebra of subsets of a set X, we call the triple $(X, \mathcal{A}, \mu)$ a measure space.*

Remark 4. Note that while $(X, \mathcal{P}(X))$ is a *measurable* space, it is not a *measure* space.

Definition 15 (Null set)**.** *Given a measure μ on a σ-algebra of $\mathcal{A}$ of subsets of a set X, a subset E of X is called a null set with respect to the measure μ if $E \in \mathcal{A}$ and $\mu(E) = 0$. In this case we also say that E is a null set in the measure space $(X, \mathcal{A}, \mu)$.*

Definition 16 (Complete σ-algebra)**.** *Given a measure μ on a σ-algebra of $\mathcal{A}$ of subsets of a set X, we say that the σ-algebra of $\mathcal{A}$ is complete with respect to the measure μ if an arbitrary subset E_0 of a null set with respect to μ is a member of $\mathcal{A}$.*

4 Outer measures

In this section we will introduce the *outer measure*. By its help, we will connect the two concepts measure and σ-algebra.

Definition 17 (Outer measure)**.** *Let X be a set. A set function μ^* defined on the σ-algebra $\mathcal{P}(X)$ of all subsets of X is called an outer measure on X if it satisfies the following conditions:*
(1) it is nonnegative extended real-valued,
(2) $\mu^*(0) = 0$,
(3) it is monotone,
(4) it is countably subadditive.

Let us define measurability with respect to a measure.

Definition 18 (μ^*-measurable set)**.** *Let μ^* be an outer measure on a set X. We say that $E \in \mathcal{P}(X)$ is measurable with respect to μ^* (or simply μ^*-measurable) if it satisfies the so-called Caratheodory condition:*

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c) \text{ for every } A \in \mathcal{P}(X).$$

The set A is called a testing set in the Caratheodory condition. We write $\mathcal{M}(\mu^*)$ for the collection of all μ^*-measurable sets $E \in \mathcal{P}(X)$.

In some sense a measurable set works as a "ruler" of other sets.

Remark 5. By subadditivity, to verify the Caratheodory condition it suffices to show that

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) \leq \mu^*(A) \text{ for every } A \in \mathcal{P}(X).$$

Lemma 5. *Let μ^* be an outer measure on a set X.*

(a) If $E_1, E_2 \in \mathcal{M}(\mu^*)$, then $E_1 \cup E_2 \in \mathcal{M}(\mu^*)$

(b) $\mu^*(E_1 \cup E_2) = \mu^*(E_1) + \mu^*(E_2)$ for every disjoint E_1, $E_2 \in \mathcal{M}(\mu^*)$.

Proof. Let us prove (b). Let $E_1 \cup E_2$ be the testing set in the Caratheodory condition. Since E_1 and E_2 are disjoint, $\mu^*(E_1 \cup E_2) = \mu^*((E_1 \cup E_2) \cap E_1) + \mu^*((E_1 \cup E_2) \cap E_1^c) = \mu^*(E_1) + \mu^*(E_2)$. □

Theorem 4. *Let μ^* be an outer measure on a set X. Then μ^* is additive on $\mathcal{P}(X)$ if and only if every member of $\mathcal{P}(X)$ is μ^*-measurable.*

Proof. ($\Rightarrow$) Suppose μ^* is additive and let $E \in \mathcal{P}(X)$. Then $A \cap E$ and $A \cap E^c$ are disjoint with union A, so the Caratheodroy condition is satisfied. Thus $\mathcal{P}(X) \subset \mathcal{M}(\mu^*)$. ($\Leftarrow$) Follows from (b) of Lemma 5. □

Remark 6. Let μ^* be an outer measure on a set X. It can be seen that $\mathcal{M}(\mu^*)$ is a σ-algebra of subsets of X.

Theorem 5. *Let μ^* be an outer measure on a set X. If we let μ be the restriction of μ^* to the σ-algebra $\mathcal{M}(\mu^*)$, then μ is a measure on $\mathcal{M}(\mu^*)$ and furthermore $(X, \mathcal{M}(\mu^*), \mu)$ is a complete measure space.*

Proof. μ^* is countably subadditive on $\mathcal{P}(X)$ and thus μ is countably subadditive on $\mathcal{M}(\mu^*)$. By Lemma 5, μ^* is additive on $\mathcal{M}(\mu^*)$, and thus countably additive on $\mathcal{M}(\mu^*)$ by Proposition 1. Let $E \in \mathcal{M}(\mu^*)$ and $\mu(E) = 0$. Let $E_0 \subset E$. By monotonicity, $\mu(E_0) = 0$, and

$$\mu^*(A \cap E_0) + \mu^*(A \cap E_0^c) \leq \mu^*(A),$$

so $E_0 \in \mathcal{M}(\mu^*)$. Thus the space is complete. □

This is take-away: *an outer measure induces a σ-algebra*. And since the measure is just the restriction of the outer measure, one may say that a measure induces a σ-algebra. This way, the concept of a σ-algebra becomes more clear.

Definition 19 (Borel outer measure)**.** *An outer measure μ^* on a topological space X is called a Borel outer measure if $\mathcal{B}_X \subset \mathcal{M}(\mu^*)$.*

4.1 Construction of outer measures

Definition 20 (Covering class)**.** *A collection $\mathcal{B}$ of subsets of a set X is called a covering class if it satisfies the following conditions:*

1° there exists $(V_n : n \in \mathbb{N}) \subset \mathcal{B}$ such that $\bigcup_{n\in\mathbb{N}} V_n = X$,

2° $\emptyset \in \mathcal{B}$.

For every $E \in \mathcal{P}(X)$, if $(V_n : n \in \mathbb{N}) \in \mathcal{B}$ such that $E \subset \bigcup_{n\in\mathbb{N}} V_n$ then $(V_n : n \in \mathbb{N})$ is a covering sequence for E.

Theorem 6. *Let $\mathcal{B}$ be a covering class of subsets of a set X. Let γ be an arbitrary set function on $\mathcal{B}$ such that*

1° γ is nonnegative extended real-valued,

2° $\gamma(\emptyset) = 0$.

Let us define a set function μ^* on $\mathcal{P}(X)$ by setting for every $E \in \mathcal{P}(X)$,

$$\mu^*(E) = \inf\{\sum \gamma(V_n : n \in \mathbb{N}) : (V_n : n \in \mathbb{N}) \subset \mathcal{B},\ E \subset \bigcup_{n\in\mathbb{N}} V_n\}.$$

Then μ^* is an outer measure on X, called the outer measure based on γ.

Proof. Let us verify that μ^* satisfies the conditions in Definition 17.

(1) Clearly $\mu^*(E) \in [0, \infty]$.

(2) $\emptyset \subset (\emptyset) \Rightarrow \mu^*(\emptyset) = 0$.

(3) For any $E_1, E_2 \in \mathcal{P}(X)$ such that $E_1 \subset E_2$, we indeed have $\mu^*(E_1) \le \mu^*(E_2)$ since any covering sequence of E_2 is a covering sequence of E_1.

(4) Let E_n be a sequence in $\mathcal{P}(X)$. Let $\epsilon > 0$ be given. Then for each $n \in \mathbb{N}$, there exists a sequence $V_{n,k}$ such that $E_n \subset \cup_{k\in\mathbb{N}} V_{n,k}$ and $\sum_{k\in\mathbb{N}} \gamma(V_{n,k}) \le \mu^*(E_n) + \frac{\epsilon}{2^n}$. Then

$$\mu^*\left(\bigcup_{n\in\mathbb{N}} E_n\right) \le \sum_{n\in\mathbb{N}}\sum_{k\in\mathbb{N}} \gamma(V_{n,k}) \le \sum_{n\in\mathbb{N}} \left(\mu^*(E_n) + \frac{\epsilon}{2^n}\right) = \sum_{n\in\mathbb{N}} \mu^*(E_n) + \epsilon.$$

The result now follows from the arbitrariness of ϵ. □

5 Lebesgue measure on $\mathbb{R}$

In this section we construct the Lebesgue measure on $\mathbb{R}$ (using the Lebesgue outer measure) and demonstrate some of its properties.

Definition 21. *Let $\mathcal{J}_0$ be the collection of $\emptyset$ and all open intervals in $\mathbb{R}$. Let $\mathcal{J}$ be the collection of all intervals in $\mathbb{R}$. For an interval I in $\mathbb{R}$ with endpoints $a, b \in \mathbb{R}$, $a < b$, we define $l(I) = b - a$. For an infinite interval I in $\mathbb{R}$ we define $l(I) = \infty$. We set $l(\emptyset) = 0$. For a countable disjoint collection $\{I_n : n \in \mathbb{N}\}$ we define $l(\cup_{n\in\mathbb{N}} I_n) = \sum_{n\in\mathbb{N}} l(I_n)$. As in the previous section, we set*

$$\mu^*(E) = \inf\{\sum \gamma(V_n) : (V_n : n \in \mathbb{N}) \subset \mathcal{J}_0,\ E \subset \bigcup_{n\in\mathbb{N}} V_n\},$$

and call it the Lebesgue outer measure on $\mathbb{R}$. We write $\mathcal{M}_L$ for the σ-algebra $\mathcal{M}(\mu_L^*)$ of μ_L^*-measurable sets $E \in \mathcal{B}(\mathbb{R})$ and call it the Lebesgue σ-algebra of subsets of $\mathbb{R}$. Members of the σ-algebra $\mathcal{M}_L$ are called $\mathcal{M}_L$-measurable or Lebesgue measurable sets. We call $(\mathbb{R}, \mathcal{M}_L)$ the Lebesgue measurable space. We write μ_L for the restriction of μ_L^* to $\mathcal{M}_L$ and call it the Lebesgue measure on $\mathbb{R}$. We call $(\mathbb{R}, \mathcal{M}_L, \mu_L)$ the Lebesgue measure space on $\mathbb{R}$.

Theorem 7. *$\mu_L^*(I) = l(I)$ for every interval I in $\mathbb{R}$.*

Proof. **1.** If I is finite and closed, $I = [a, b]$ for some $a, b \in \mathbb{R}$. Consider the covering sequence $((a-\epsilon, b+\epsilon), \emptyset, \emptyset, \ldots)$ in $\mathcal{J}_0$ for I. It follows that $\mu_L^*(I) \leq l(I)$.

Next we show that for any covering sequence $(I_n : n \in \mathbb{N})$ in $\mathcal{J}_0$ for I, we have

$$\sum_{n\in\mathbb{N}} l(I_n) \geq l(I). \tag{10}$$

If any of the intervals is infinite, (10) holds. Thus consider the case where every member is finite. Let us drop those members in the covering sequence that is disjoint from I and contained in any other member of the sequence. The resulting sequence J_n is a covering sequence of I, and since I is compact, J_n has a finite subcover. Renumber the members of J_n so that $J_k = (a_k, b_k)$ for $k = 1, \ldots, N$ and $a_1 \leq a_2 \leq \ldots \leq a_N$. In fact since none of the members are contained in another, we have $a_1 < a_2 < \ldots < a_N$. Let us show that $a_2 < b_1$. Assume not, then since J_1 and J_2 lies in I, there exist $x_1 \in (a_1, b_1) \cap I$ and $x_2 \in (a_2, b_2) \cap I$ such that $a_1 < x_1 < b_1 \leq a_2 < x_2 < b_2$. Note that $[x_1, x_2] \subset I$. Since $b_1 \leq a_2$, there exists at least on point in $[x_1, x_2]$

that is not covered by $(J_1, \ldots, J_n)$. Thus we have $a_2 < b_1$. Similarly,

$$\begin{aligned}
&a_1 < a_2 < b_1 \\
&a_2 < a_3 < b_2 \\
&\ldots \\
&a_{N-1} < a_N < b_{N-1} \\
&a_N < b_N.
\end{aligned}$$

We get

$$\begin{aligned}
\sum_{k=1}^{N} l(J_k) &= (b_1 - a_1) + (b_2 - a_2) + \ldots + (b_N - a_N) \\
&> (a_2 - a_1) + (a_3 - a_2) + \ldots + (b_N - a_n) \\
&= b_N - a_1 \geq b - a = l(I).
\end{aligned}$$

Thus $\sum_{n\in\mathbb{N}} l(I_n) \geq \sum_{k=1}^{N} l(J_k) \geq l(I)$, establishing (10). By the definition of infimum, (10) implies that $\mu_L^*(I) \geq l(I)$. Thus for any closed and finite interval I, we have $\mu_L^*(I) = l(I)$.

2. For any open interval $I = (a, b)$, we have

$$\mu_L^*((a,b)) \leq \mu_L^*([a,b]) \leq \mu_L^*(\{a\}) + \mu_L^*((a,b)) + \mu_L^*(\{b\}) = \mu_L^*((a,b)),$$

since the Lebesgue measure of a singleton is 0. Thus $\mu_L^*((a,b)) = \mu_L^*([a,b]) = l([a,b]) = l((a,b))$.

3. If I is finite and $I = (a,b]$, $\mu_L^*((a,b]) \leq \mu_L^*((a,b))$ and $\mu_L^*((a,b]) \geq \mu_L^*((a,b))$ by monotonicity. Similarly if $I = [a,b)$.

4. If I is infinite of the type $I = (a, \infty)$, $a \in \mathbb{R}$, then $(a,\infty) \subset (a,n)$ for every $n \in \mathbb{N}$ and thus $\mu_L^*((a,\infty)) \geq \mu_L^*((a,n)) = n - a$. By the arbitrariness of $n \in \mathbb{N}$, $\mu_L^*((a,\infty)) = \infty = l(a,\infty)$. Similarly for other types. □

Remark 7. From Theorem 7, one can show that every interval in $\mathbb{R}$ is Lebesgue measurable. This begs the question: are all sets in $\mathbb{R}$ Lebesgue measurable? As we will see, the answer is no. In practice, however, any subset of $\mathbb{R}$ *that we can think of* will be Lebesgue measurable. This is why we introduce the Borel σ-algebra of $\mathbb{R}$. Its role is so important that it deserves a separate section.

5.1 The Borel σ-algebra of $\mathbb{R}$

Definition 22 (Borel σ-algebra of $\mathbb{R}$)**.** *The Borel σ-algebra of $\mathbb{R}$, written $\mathcal{B}(\mathbb{R})$, is the σ-algebra generated by the open sets in $\mathbb{R}$. That is, if $\mathcal{D}$ is the collection of all open sets in $\mathbb{R}$, then $\mathcal{B}(\mathbb{R}) = \sigma(\mathcal{D})$.*

Theorem 8. *Every Borel set in $\mathbb{R}$ is a Lebesgue measurable set.*

Proof. Let $\mathcal{D}$ be the collection of all open sets in $\mathbb{R}$. Since any open set in $\mathbb{R}$ can be written as a union of countable many open intervals, we have $\mathcal{D} \subset \mathcal{M}_L$ and thus $\sigma(\mathcal{D}) = \mathcal{B}_{\mathbb{R}} \subset \mathcal{M}_L$. □

It will be useful to know that $\mathcal{B}(\mathbb{R})$ can be generated by intervals on $\mathbb{R}$:

Proposition 2. *The Borel σ-algebra on $\mathbb{R}$ can be generated by any of the following collections of intervals:*

$$\{(-\infty, b) : b \in \mathbb{R}\},\ \{(-\infty, b] : b \in \mathbb{R}\},\ \{(a, \infty) : a \in \mathbb{R}\},\ \{[a, \infty) : b \in \mathbb{R}\}.$$

By *generated*, we mean that for example $\mathcal{B}(\mathbb{R}) = \sigma(\{(-\infty, b) : b \in \mathbb{R}\})$.

Remark 8. We will not prove it here, but there are Lebesgue measurable sets on $\mathbb{R}$ that are not members of the Borel σ-algebra on $\mathbb{R}$, so the inclusion in Theorem 8 is a *true* inclusion.

5.2 The invariance of the Lebesgue measure space

The Lebesgue measure on $\mathbb{R}$ is invariant under different transformations.

Definition 23. *Let X be a linear space over the field of scalars $\mathbb{R}$.*

(a) For $E \subset X$ and $x_0 \in X$, we write

$$E + x_0 = \{x + x_0 : x \in E\}.$$

(b) For $a \in \mathbb{R}$, we write

$$\alpha E = \{\alpha x : x \in E\}.$$

(c) For a collection $\mathcal{C}$ of subsets of X, $x \in X$, and $\alpha \in \mathbb{R}$, we write $\mathcal{C} + x = \{E + x : E \in \mathcal{C}\}$ and $\alpha\mathcal{C} = \{\alpha E : E \in \mathcal{C}\}$.

Theorem 9 (Translation invariance of the Lebesgue measure space)**.** *The Lebesgue measure space $(\mathbb{R}, \mathcal{M}_L, \mu_L)$ is translation invariant, that is, for every $E \in \mathcal{M}_L$ and $x \in \mathbb{R}$ we have $E + x \in \mathcal{M}_L$ and $\mu_L(E + x) = \mu_L(E)$. Let $\mathcal{M}_L + x = \{E + x : E \in \mathcal{M}_L\}$. Then $\mathcal{M}_L + x = \mathcal{M}_L$ for every $x \in \mathbb{R}$.*

Proof. Let $E \in \mathcal{M}_L$ and $x \in \mathbb{R}$. Let us show that $E + x \in \mathcal{M}_L$ by verifying the Caratheodory condition for $E + x$. (In the proof we will use that $\mu_L^*(E + x) = \mu_L^*(E)$, for every $E \in \mathcal{B}(\mathbb{R})$, which is not difficult to prove.) We get

$$\begin{aligned}
&\mu_L^*(A \cap (E + x)) + \mu_L^*(A \cap (E + x)^c) \\
&= \mu_L^*(A \cap (E + x) - x) + \mu_L^*(A \cap (E + x)^c - x) \\
&= \mu_L^*((A - x) \cap E) + \mu_L^*((A - x) \cap E^c) \\
&= \mu_L^*(A - x) = \mu_L^*(A).
\end{aligned}$$

This shows that $E + x \in \mathcal{M}_L$ and therefore $\mu_L(E + x) = \mu_L^*(E + x) = \mu_L^*(E) = \mu_L(E)$.

Now since $E + x \in \mathcal{M}_L$ for every $E \in \mathcal{M}_L$, we have $\mathcal{M}_L + x \subset \mathcal{M}_L$. But $\mathcal{M}_L = \mathcal{M}_L + (-x) + x \subset \mathcal{M}_L + x$, and we are done. □

Theorem 10. *For $E \in \mathcal{B}(\mathbb{R})$ and $a \in \mathbb{R}$, let $\alpha E = \{y \in \mathbb{R} : y = ax \text{ for some } x \in E\}$. Then $\mu_L^*(\alpha E) = |\alpha|\mu_L^*(E)$.*

Proof. Let $S = (I_n : n \in \mathbb{N})$ be a sequence in $\mathcal{J}_0$, and let $\mathcal{S}$ be the collection of all such sequences. For $a \in \mathbb{R}$, $\alpha \neq 0$, define a function $M_\alpha : \mathcal{S} \to \mathcal{S}$ by setting $M_\alpha(S) = \alpha S$. Clearly M_α is one-to-one. For an arbitrary $E \in \mathcal{B}(\mathbb{R})$, let $\mathcal{S}_E$ be all sequences in $\mathcal{S}$ such that $E \subset \bigcup_{n\in\mathbb{N}} I_n$. Then $M_\alpha(\mathcal{S}_E)= \mathcal{S}_{\alpha E}$ is a one-to-one mapping of $\mathcal{S}_E$ onto $\mathcal{S}_{\alpha E}$, where $\mathcal{S}_{\alpha E}$ is all sequences in $\mathcal{S}$ such that $\alpha E \subset \bigcup_{n\in\mathbb{N}} \alpha I_n$.

Let λ be a nonnegative extended real-valued set function on $\mathcal{S}$ defined by $\lambda(S) = \sum_{n\in\mathbb{N}} l(I_n)$ for $S = (I_n : n \in \mathbb{N})$. Then

$$\lambda(M_\alpha(S)) = \sum_{n\in\mathbb{N}} l(\alpha I_n) = |\alpha| \sum_{n\in\mathbb{N}} l(I_n) = |\alpha|\lambda(S).$$

Now $\mu_L^*(E) = \inf_{S\in\mathcal{S}_E}\lambda(S)$ and $\mu_L^*(\alpha E) = \inf_{T\in\mathcal{S}_{\alpha E}}\lambda(T)$. But $M_\alpha(\mathcal{S}_E)$ is one-to-one, so $\inf_{T\in\mathcal{S}_{\alpha E}}\lambda(T) = \inf_{S\in\mathcal{S}_E}\lambda(M_\alpha(S)) = \inf_{S\in\mathcal{S}_E}|\alpha|\lambda(S) = |\alpha|\mu_L^*(E)$. Thus $\mu_L^*(\alpha E) = |\alpha|\mu_L^*(E)$ when $\alpha \neq 0$. For $\alpha = 0$ the equality is trivial. □

Theorem 11 (Positive homogeneity of the Lebesgue measure space)**.** *For every set $E \in \mathcal{M}_L$ and $\alpha \in \mathbb{R}$, we have $\alpha E \in \mathcal{M}_L$ and $\mu_L(\alpha E) = |\alpha|\mu_L(E)$. For every $\alpha \in \mathbb{R}$, let $\alpha\mathcal{M}_L = \{\alpha E : E \in \mathcal{M}_L\}$. Then $\alpha\mathcal{M}_L = \mathcal{M}_L$ for every $\alpha \in \mathbb{R}$ such that $\alpha \neq 0$.*

Proof. For $\alpha = 0$ the theorem is trivial. Assume therefore $\alpha \neq 0$. Let $E \in \mathcal{M}_L$ and $A \in \mathcal{B}(\mathbb{R})$, then $\frac{1}{\alpha}A \in \mathcal{B}(\mathbb{R})$ and

$$\mu_L^*(\frac{1}{\alpha}A) = \mu_L^*(\frac{1}{\alpha}A \cap E) + \mu_L^*(\frac{1}{\alpha}A \cap E^c). \tag{11}$$

By Theorem 10,

$$\begin{aligned}
&\mu_L^*(\frac{1}{\alpha}A) = \frac{1}{|\alpha|}\mu_L^*(A),\\
&\mu_L^*(\frac{1}{\alpha}A \cap E) = \frac{1}{|\alpha|}\mu_L^*(A \cap \frac{\alpha}{E}),\\
&\mu_L^*(\frac{1}{\alpha}A \cap E^c) = \frac{1}{|\alpha|}\mu_L^*(\frac{1}{\alpha}A \cap (\alpha E^c)).
\end{aligned}$$

Substituting these into (11), we get the Caratheodory condition for αE, so that $\alpha E \in \mathcal{M}_L$. Then by Theorem 10, $\mu_L(\alpha E) = |\alpha|\mu_L(E)$. Since $\alpha E \in \mathcal{M}_L$ for every $E \in \mathcal{M}_L$, we have $\alpha\mathcal{M}_L \subset \mathcal{M}_L$. But $\mathcal{M}_L = \alpha\frac{1}{\alpha}\mathcal{M}_\mathcal{L} \subset \alpha\,\mathcal{M}_L$, and we conclude. □

5.3 Existence of non Lebesgue measurable sets

Not all sets in $\mathbb{R}$ are Lebesgue measurable.

Let us define addition modulo 1 of $x, y \in [0,1)$ by

$$x \overset{\circ}{+} y = \begin{cases} x+y & \text{if } x+y<1, \\ x+y-1 & \text{if } x+y \geq 1. \end{cases}$$

It is easily seen that $\overset{\circ}{+}$ is commutative and associative. For $E \in [0,1)$ and $y \in [0,1)$, let $E \overset{\circ}{+} y = \{z \in [0,1) : z = x \overset{\circ}{+} y \text{ for some } x \in E\}$.

Remark 9. Let $E \subset [0,1)$ and $E \in \mathcal{M}_L$. It can be proven that $E \overset{\circ}{+} y \in \mathcal{M}_L$ and $\mu_L(E \overset{\circ}{+} y) = \mu_L(E)$ for every $y \in [0,1)$.

Theorem 12. $[0,1) \subset \mathbb{R}$ *contains a non Lebesgue measurable set.*

Proof. For $x, y \in [0,1)$, let us define a equivalence relation on $[0,1)$ by

$$x \sim y \text{ if and only if } x-y \text{ is a rational number.}$$

Let $\{E_\alpha : \alpha \in A\}$ be the collection of equivalence classes of $\sim$.

Let P be the subset of $[0,1)$ constructed by for each $\alpha \in A$, picking an element from E_α. Let $\{r_n : n \in \mathbb{Z}_+\}$ be an enumeration of the rational numbers in $[0,1)$ with $r_0 = 0$. Let

$$P_n = P \overset{\circ}{+} r_n \text{ for } n \in \mathbb{Z}_+.$$

Let us show that $\{P_n : n \in \mathbb{Z}_+\}$ is a disjoint collection. Assume not. Then for $m \neq n$, there exists $x \in P_m \cap P_n$. Then $x \in P_m$ and $x \in P_n$ so that $x = p_m \overset{\circ}{+} r_m = p_n \overset{\circ}{+} r_n$ for some $p_m, p_n \in P$. This implies that $p_m - p_n$ is rational, so $p_m \sim p_n$. By the construction of P, we must then have $p_m = p_n$, implying that $r_m = r_n$, and then $m = n$; contradiction, proving that indeed $\{P_n : n \in \mathbb{Z}_+\}$ is disjoint. Next, let us show that

$$\bigcup_{n \in \mathbb{Z}_+} P_n = [0,1). \tag{12}$$

Let us note that since $P_n \subset [0,1)$ for each $n \in \mathbb{N}$, we trivially have $\bigcup_{n \in \mathbb{Z}_+} P_n \subset [0,1)$. To prove the reverse inclusion, let $x \in [0,1)$. Then $x \in E_\alpha$ for some α. Since P contains an element from each equivalent class, there exists $p \in P$ such that $p \in E_\alpha$. Thus $x \sim p$ so that x and p differ by some rational number r_n, $n \in \mathbb{Z}_+$. If $x \geq p$ then $x = p + r_n \in P_n$. If $x < p$, $x = p - r_n$. Let $r_m = 1 - r_n \in [0,1)$, then $x = p + r_m - 1 = p \overset{\circ}{+} r_m$ so that $x \in P_m$. This shows that (12) holds.

Finally, let us show that $P \notin \mathcal{M}_L$. Assume the contrary, then by Remark 9, $P_n = P \overset{\circ}{+} r_n \in \mathcal{M}_L$ and $\mu_L(P_n) = \mu_L(P)$. Thus

$$1 = \mu_L([0,1)) = \mu_L\left(\bigcup_{n\in\mathbb{Z}_+} P_n\right) = \sum_{n\in\mathbb{Z}_+} \mu_L(P_n) = \sum_{n\in\mathbb{Z}_+} \mu_L(P). \qquad (13)$$

If $\mu_L(P) = 0$, then (13) says $1 = 0$. If $\mu_L(P) > 0$, then (13) says $1 = \infty$. Contradiction and we conclude. □

6 Measurable functions

It is natural to develop a concept of *measurable functions*, that is, functions whose preimage is measurable.

A measurable function pulls back measurable sets to measurable sets, much like a continuous function pulls back open sets to open sets.

6.1 Measurability of functions

Definition 24 (Measurable function)**.** *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces. A function $f : X \to Y$ is measurable if $f^{-1}(B) \in \mathcal{A}$ for every $B \in \mathcal{B}$.*

Definition 25 (Measurable function on $\mathbb{R}$, Borel set)**.** *If $(X, \mathcal{A})$ is a measurable space, then $f : X \to \mathbb{R}$ is measurable if $f^{-1}(B) \in \mathcal{A}$ for every Borel set $B \in \mathcal{B}(\mathbb{R})$.*

Since $\mathcal{B}(\mathbb{R}) = \sigma(\{[-\infty, b) : b \in \mathbb{R}\})$ by Proposition 2, we can make the following definition.

Definition 26 (Measurable function on $\mathbb{R}$)**.** *Let $(X, \mathcal{A})$ be an arbitrary measurable space and let $D \in \mathcal{A}$. An extended real-valued function f defined on D is said to be $\mathcal{A}$-measurable on D if $\{x \in D : f(x) \le \alpha\} \in \mathcal{A}$ for every $\alpha \in \mathbb{R}$.*

Every real-valued continuous function is measurable:

Theorem 13. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then f is measurable.*

Proof. Let O be an arbitrary open set in $\mathbb{R}$. Since f is continuous, $f^{-1}(O)$ is open in $\mathbb{R}$ and then $f^{-1}(O) \in \mathcal{B}_{\mathbb{R}}$, since any open set in $\mathbb{R}$ is a countable union of disjoint open intervals. By Theorem 8, $f^{-1}(O)$ is measurable. □

Theorem 14. *Each of the following conditions are equivalent.*
(a) $D_1 = \{x \in D : f(x) \le \alpha\} \in \mathcal{A}$ for every $\alpha \in \mathbb{R}$,
(b) $D_2 = \{x \in D : f(x) > \alpha\} \in \mathcal{A}$ for every $\alpha \in \mathbb{R}$,
(c) $D_3 = \{x \in D : f(x) \ge \alpha\} \in \mathcal{A}$ for every $\alpha \in \mathbb{R}$,
(d) $D_4 = \{x \in D : f(x) < \alpha\} \in \mathcal{A}$ for every $\alpha \in \mathbb{R}$.

Proof. **1.** (a) $\Leftrightarrow$ (b). D_1 and D_2 partition D. Thus if $D_1 \in \mathcal{A}$ then $D_2 = D \setminus D_1 \in \mathcal{A}$. Similarly, if $D_2 \in \mathcal{A}$ then $D_1 \in \mathcal{A}$.

2. (c) $\Leftrightarrow$ (d) as above.

3. (d) $\Rightarrow$ (a). Note that

$$D_1 = \bigcap_{n \in \mathbb{N}} \{D : f < \alpha + \frac{1}{n}\}. \tag{14}$$

Now if f satisfies (d), then every intersection in (14) lies in $\mathcal{A}$.

4. (b) $\Rightarrow$ (c). Note that

$$D_3 = \bigcap_{n \in \mathbb{N}} \{D : f > \alpha - \frac{1}{n}\}. \tag{15}$$

Now if f satisfies (b), every intersection in (15) lies in $\mathcal{A}$. □

Theorem 15. *If f is measurable, then $|f|$ is measurable.*

Proof. $\{x : |f(x)| < \alpha\} = \{x : f(x) < \alpha\} \cap \{x : f(x) > -\alpha\} \in \mathcal{A}$, by Theorem 14. □

6.2 Measurability of sequences of functions

Theorem 16. *Let $(f_n : n \in \mathbb{N})$ be a sequence of measurable functions. For $x \in X$, put*

$$\begin{aligned} g_1(x) &= \sup_{n \in \mathbb{N}} f_n(x), \\ g_2(x) &= \inf_{n \in \mathbb{N}} f_n(x), \\ g_3(x) &= \limsup_{n \to \infty} f_n(x), \\ g_4(x) &= \liminf_{n \to \infty} f_n(x). \end{aligned}$$

Then g_i $(i = 1, 2, 3, 4)$ are measurable.

Proof. **1.** Note that

$$\{x : g_1(x) > \alpha\} = \{x : \sup_{n \in \mathbb{N}} f_n(x) > \alpha\} = \bigcup_{n \in \mathbb{N}} \{x : f_n(x) > \alpha\}. \tag{16}$$

Since each set in (16) is measurable, it follows that g_1 is measurable.

2. Similarly as above,

$$\{x : g_2(x) < \alpha\} = \{x : \inf_{n \in \mathbb{N}} f_n(x) < \alpha\} = \bigcup_{n \in \mathbb{N}} \{x : f_n(x) < \alpha\}. \tag{17}$$

3. By the definition of lim sup,

$$g_3(x) = \lim_{n\to\infty} \left(\sup_{m\geq n} f_m(x) \right).$$

4. Similarly as above,

$$g_4(x) = \lim_{n\to\infty} \left(\inf_{m\geq n} f_m(x) \right).$$ □

Theorem 17. *Let* $(f_n : n \in \mathbb{N})$ *be a sequence of measurable functions. The functions*

$$\min_{n=1,\ldots,N} f_n \quad \textit{and} \quad \max_{n=1,\ldots,N} f_n$$

are measurable.

Proof. $\{\min_{n=1,\ldots,N} f_n > \alpha\} = \bigcap_{n=1}^{n}\{f_i > \alpha\}$, $\{\max_{n=1,\ldots,N} f_n < \alpha\} = \bigcap_{n=1}^{n}\{f_i < \alpha\}$. □

6.3 Measurability of the positive and negative part of f

Definition 27 (Positive and negative part of f)**.** *Let f be measurable. The positive part f^+ and the negative part f^- of f are nonnegative functions defined by*

$$f^+(x) = (f \vee 0)(x) = \max\{f(x), 0\}, \tag{18}$$
$$f^-(x) = -(f \wedge 0)(x) = -\min\{f(x), 0\}. \tag{19}$$

Remark 10. Note that $f(x) = f^+(x) - f^-(x)$.

Proposition 3. *Let f be measurable.*
(a) f^+ and f^- are measurable.
(b) The limit of a convergent sequence of measurable functions is measurable.

Proof. **1.** Consider the sequence $f_n(x) = (f_1(x) = f(x), 0, 0, \ldots)$. By Theorem 17,

$$\max_{n=1,\ldots,N} f_n(x) = \max\{f(x), 0\} = f^+$$

is measurable. Likewise for f^-.
2. If $(f_n : n \in \mathbb{N})$ converges, lim and lim sup are equal. □

6.4 Measurability of the restriction and extension of f

Lemma 6. *Let $(X, \mathcal{A})$ be a measurable space.*

(a) If f is an extended real-valued measurable function on a set $D \in \mathcal{A}$, then for every $D_0 \subset D$ such that $D_0 \in \mathcal{A}$, the restriction of f to D_0 is a measurable function on D_0.

(b) Let $(D_n : n \in \mathbb{N})$ be a sequence in $\mathcal{A}$ and let $D = \cup_{n \in \mathbb{N}} D_n$. Let f be an extended real-valued function on D. If the restriction of f to D_n is $\mathcal{A}$-measurable on D_n for every $n \in \mathbb{N}$, then f is $\mathcal{A}$-measurable on D.

Proof. **1.** $\{D_0 : f \leq \alpha\} = \{D : f \leq \alpha\} \cap D_0 \in \mathcal{A}$.

2. $\{D : f \leq \alpha\} = \{\cup_{n \in \mathbb{N}} D_n : f \leq \alpha\} = \cup_{n \in \mathbb{N}} \{D_n : f \leq \alpha\} \in \mathcal{A}$. □

Proposition 4. *Let $(X, \mathcal{A}, \mu)$ be a complete measure space.*

(a) Every extended real-valued function f defined on a null set N is $\mathcal{A}$-measurable on N.

(b) Let f and g be two extended real-valued functions defined on a set $D \in \mathcal{A}$ such that $f = g$ a.e. on D. If f is $\mathcal{A}$-measurable on D then so is g.

Proof. Let us prove (b). Suppose $f = g$ a.e. on D. Then there exists a null set N such that $N \subset D$ and $f = g$ on $D \setminus N$. Since f is measurable on D, it is measurable on the subset $D \setminus N$ by (a) of Lemma 6. Since $f = g$ on $D \setminus N$, g is measurable on $D \setminus N$. But since N is a null set and the measure space is complete, g is measurable on N by (a). Thus g is measurable on $D \setminus N$ and on N and therefore measurable on $(D \setminus N) \cup N = D$ according to (b) of Lemma 6. □

6.5 Almost everywhere

If a property P holds for every $x \in D \setminus A$, where A is a null set, it is customary to say that P holds almost everywhere on D.

Definition 28 (Real-valued a.e.)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space. Let f be an extended real-valued $\mathcal{A}$-measurable function on a set $D \in \mathcal{A}$. We say that f is real-valued a.e. on D if there exists a null set $(X, \mathcal{A}, \mu)$ such that $N \subset D$ and $f(x) \in \mathbb{R}$ for every $x \in D \setminus N$.*

Definition 29 (Existence and convergence of limit a.e.)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $(f_n : n \in \mathbb{N})$ be a sequence of extended real-valued $\mathcal{A}$-measurable functions on a set $D \in \mathcal{A}$. We say that $\lim_{n \to \infty} f_n$ exists a.e. (brief for* almost everywhere*) on D if there exists a null set $(X, \mathcal{A}, \mu)$ such that $N \subset D$ and $\lim_{n \to \infty} f_n(x)$ exists for every $x \in D \setminus N$. We say that $(f_n : n \in \mathbb{N})$ converges a.e. $\lim_{n \to \infty} f_n$ exists and $\lim_{n \to \infty} f_n \in \mathbb{R}$ for every $x \in D \setminus N$.*

Definition 30 (Uniform convergence a.e.)**.** *Let $(f_n : n \in \mathbb{N})$ be a sequence of extended real-valued measurable functions on a set $D \in \mathcal{A}$ and let f be a* *real-valued measurable function on D. We say that $(f_n : n \in \mathbb{N})$ converges almost uniformly on D to f if for every $\delta > 0$ there exists a measurable subset E of D such that $\mu(E) < \delta$ and $(f_n : n \in \mathbb{N})$ converges uniformly on $D \setminus E$ to f.*

Theorem 18 (D.E. Egoroff)**.** *Let $D \in \mathcal{A}$ and $\mu(D) < \infty$. Let $(f_n : n \in \mathbb{N})$ be a sequence of extended real-valued measurable functions on D and let f be a real-valued measurable function on D. If $(f_n : n \in \mathbb{N})$ converges to f a.e. on D, then $(f_n : n \in \mathbb{N})$ converges to f almost uniformly on D.*

7 The Lebesgue Integral

7.1 Simple functions and approximations

Definition 31 (Simple function)**.** *Let φ be a real-valued function defined on X. If the range of φ is finite, we say that φ is a simple function.*

Definition 32 (Canonical representation of a simple function)**.** *Let φ be a simple function on a set $D \subset X$. Let $\{a_i : i = 1, \dots, n\}$ be the set of distinct values assumed by φ on D and let $D_i = \{x \in D : \varphi(x) = a_i\}$ for $i = 1, \dots, n$. $\{D_i : i = 1, \dots, n\}$ is a disjoint collection and $\cup_{i=1}^n D_i = D$. The expression*

$$\varphi(x) = \sum_{i=1}^{n} a_i \mathbf{1}_{D_i}(x) \qquad \text{for } x \in D,$$

is called the canonical representation of φ.

The following important theorem shows that any measurable function can be approximated from below by a sequence of simple functions. [2]

Theorem 19. *Let $f : X \to [0, \infty]$ be measurable. There exists a sequence of real-valued simple functions $s_1, s_2, \dots$ on X such that $0 \le s_1 \le s_2 \le \dots f$, $s_n(x) \to f(x)$ pointwise.*

Proof. Let us construct a sequence of simple functions φ_1, $\varphi_2, \dots$ that converges to the identity from below. Then $s_n = \varphi \circ f$ is a sequence of simple functions that converges to f from below.

Step 1. Let $\lfloor\ \rfloor$ denote the floor function. Define $\varphi : [0, \infty] \to [0, \infty)$ by

$$\varphi_n(t) = \begin{cases} \lfloor 2^n t \rfloor / 2^n, & \text{if } t \in [0, n), \\ n, & \text{if } t \in [n, \infty]. \end{cases}$$

Note that φ_n is simple and therefore measurable. Let us show $\varphi_n \le \varphi_{n+1}$. Let $n \in \mathbb{N}$ and make a case study.

[2] Michael J. Fairchild, www.mikef.org/files/SimpleApproximation.pdf, retrieved 2016-03-17.

(*i*) If $t \in [0, n)$, then

$$\varphi_n(t) = \lfloor 2^n t \rfloor / 2^n = 2\lfloor 2^n t \rfloor / 2^{n+1} \leq \lfloor 2^{n+1} t \rfloor / 2^{n+1} = \varphi_{n+1}(t),$$

where we have used that $n\lfloor t \rfloor \leq \lfloor nt \rfloor$. (This can be proven by induction and using the fact that if $a < b$ then $\lfloor a \rfloor < \lfloor b \rfloor$. Idea proof: $\lfloor a \rfloor < a < b$, and $\lfloor b \rfloor < b$. Thus $\lfloor a \rfloor$ and $\lfloor b \rfloor$ are two integers not exceeding b, and so by the definition of floor, $\lfloor a \rfloor < \lfloor b \rfloor$.)

(*ii*) If $t \in [n, n+1)$, then

$$\varphi_n(t) = n \leq \lfloor t \rfloor = 2^{n+1} \lfloor t \rfloor / 2^{n+1} \leq \lfloor 2^{n+1} t \rfloor / 2^{n+1} \leq \varphi_{n+1}(t).$$

(*iii*) If $t \in [n+1, \infty]$, then

$$\varphi_n(t) = n < n + 1 = \varphi_{n+1}(t).$$

This shows that for all $t \in [0, \infty]$ we have $0 \leq \varphi_1(t) \leq \varphi_2(t) \leq \ldots \leq t$.

We show next that $\phi_n(t) \to t$ for $t \in [0, \infty]$. If $t \in [0, \infty)$, then $2^n t - 1 < \lfloor 2^n t \rfloor$ and therefore $\frac{2^n t - 1}{2^n} < \frac{\lfloor 2^n t \rfloor}{2^n}$, implying $t - \frac{1}{2^n} < \frac{\lfloor 2^n t \rfloor}{2^n}$. If $n > t$, then $t - \frac{1}{2^n} < \varphi_n(t) \leq t$. This implies $\varphi_n(t) \to t$ for $t \in [0, \infty)$. Lastly, if $t = \infty$ then $\varphi_n(t) = \infty$. Thus $\varphi_n(t) \to t$ for $t \in [0, \infty]$.

Step 2. $s_n : X \to [0, \infty)$ defined by $s_n = \varphi \circ f$ is simple and measurable, and $0 \leq s_n(x) = \varphi_n(f(x)) \leq \varphi_{n+1}(f(x)) = s_{n+1}(x)$. Moreover $s_n(x) = \varphi_n(f(x)) \leq f(x)$, since $f(x) \in [0, \infty]$, proving the first part of the theorem. Lastly, $s_n(x) = \varphi_n(f(x)) \to f(x)$. □

7.2 Integration of simple functions

Definition 33 (Lebesgue integral of a simple function)**.** *Let $\varphi = \sum_{i=1}^n a_i \mathbf{1}_{D_i}$ be the canonical representation of a simple function on a set $D \in \mathcal{A}$. The Lebesgue integral of φ on D with respect to μ is defined by*

$$\int_D \varphi(x)\, \mu(dx) = \sum_{i=1}^n a_i \mu(D_i),$$

or, briefly,

$$\int_D \varphi\, d\mu = \sum_{i=1}^n a_i \mu(D_i).$$

7.3 Integration of nonnegative functions

Definition 34 (Lebesgue integral of a nonnegative function)**.** *Let f be measurable and nonnegative on a set D. We define the Lebesgue integral on D with respect to μ by*

$$\int_D f\, d\mu = \sup\{\int_D \varphi\, d\mu : \varphi \in \Phi\},$$

where Φ is the collection of all simple functions φ such that $0 \leq \varphi \leq f$.

7.4 Integration of measurable functions

Definition 35 (Lebesgue integral)**.** *Let f be measurable on a set D. If at least one of the integrals $\int_D f^+ \, d\mu$ and $\int_D f^- \, d\mu$ are finite, we define*

$$\int_D f \, d\mu = \int_D f^+ \, d\mu - \int_D f^- \, d\mu.$$

If both integrals are finite, we say that f is integrable on D in the Lebesgue sense, with respect to μ, and write $f \in \mathcal{L}(\mu)$. (Note that μ need not be the Lebesgue measure.)

7.5 Properties of the Lebesgue integral

Remark 11. Let f be a function.

(a) If f is measurable and bounded on D, and if $\mu(D) < \infty$, then $f \in \mathcal{L}(\mu)$ on D.

(b) If $a \leq f(x) \leq b$ for $x \in D$ and $\mu(D) < \infty$, then

$$a\mu(D) \leq \int_D f \, d\mu \leq b\mu(D).$$

(c) If f and $g \in \mathcal{L}(\mu)$ on D, and if $f(x) \leq g(x)$ for $x \in D$, then

$$\int_D f \, d\mu \leq \int_D g \, d\mu.$$

(d) If $f \in \mathcal{L}(\mu)$ on D, then $cf \in \mathcal{L}(\mu)$ on D, for every finite constant c, and

$$c \int_D f \, d\mu = \int_D cf \, d\mu.$$

(e) If $\mu(D) = 0$ and f is measurable, then

$$\int_D f \, d\mu = 0.$$

(f) If $f \in \mathcal{L}(\mu)$ on D, $A \in \mathcal{A}$, and $A \subset D$, then $f \in \mathcal{L}(\mu)$ on A.

Proof. Let us prove *(c)*. Firstly, let f_1 and f_2 be nonnegative and measurable functions on a set D with $f_1 \leq f_2$. We claim that

$$\int_D f_1 \, d\mu \leq \int_D f_2 \, d\mu. \tag{20}$$

Indeed, for $i = 1$ and 2, let Φ_i be the collection of all nonnegative simple functions φ on D such that $0 \le \varphi \le f_i$. Since $0 \le \varphi \le f_1 \le f_2$ on D, we have $\Phi_1 \subset \Phi_2$. Then

$$\{\int_D \varphi\, d\mu : \varphi \in \Phi_1\} \subset \{\int_D \varphi\, d\mu : \varphi \in \Phi_2\}.$$

Let

$$S_1 = \{\int_D \varphi\, d\mu : \varphi \in \Phi_1\},$$
$$S_2 = \{\int_D \varphi\, d\mu : \varphi \in \Phi_2\}.$$

Let U_2 be *any* upper bound to S_2. Since $S_1 \subset S_2$, U_2 is an upper bound to S_1 too. In particular, then, $\sup S_2$ is an upper bound to S_1.

But $\sup S_1$ is the *smallest* upper bound to S_1, so

$$\sup S_1 \le \sup S_2,$$

or,

$$\int_D f_1\, d\mu \le \int_D f_2\, d\mu.$$

This proves (20). Note that the result now follows since

$$\int_D f\, d\mu = \int_D f^+\, d\mu - \int_D f^-\, d\mu \le \int_D g^+\, d\mu - \int_D g^-\, d\mu = \int_D g\, d\mu. \quad \square$$

Theorem 20. *(a) Suppose f is measurable and nonnegative on X. For $A \in \mathcal{A}$, define*

$$v(A) = \int_A f\, d\mu.$$

Then v is countably additive on $\mathcal{A}$.

(b) The same conclusion holds if f is Lebesgue measurable on X.

Proof. **1.** Let $(A_n : n \in \mathbb{N})$ be a disjoint sequence in $\mathcal{A}$ with $A = \bigcup_{n\in\mathbb{N}} A_n$. To prove (a), we have to show that

$$v(A) = \sum_{n\in\mathbb{N}} v(A_n). \tag{21}$$

Suppose f is the characteristic function on a set $D \subset X$. Then

$$v(A) = \int_A f\, d\mu = 1 \cdot \mu(A \cap D) + 0 \cdot \mu(A \cap D^c) = \mu(A \cap D),$$

and since μ is countably additive, v is too.

Suppose f is simple, and let $f = \sum_{i=1}^m a_i \mathbf{1}_{D_i}$ be its canonical representation.

Since $\{D_i \cap A, : i = 1, \dots, m\}$ is a disjoint collection with $\bigcup_{i=1}^m (D_i \cap A) = A$, the restriction of φ to A is given by

$$\varphi = \sum_{i=1}^m a_i \mathbf{1}_{D_i \cap A}.$$

Therefore

$$\begin{aligned} v(A) = \int_A f \, d\mu = \sum_{i=1}^m a_i \mu\,(D_i \cap A) = \sum_{i=1}^m a_i \sum_{n \in \mathbb{N}} \mu(D_i \cap A_n) \\ = \sum_{n \in \mathbb{N}} [\sum_{i=1}^m a_i \mu(D_i \cap A_n)] = \sum_{n \in \mathbb{N}} \int_{A_n} f \, d\mu = \sum_{n \in \mathbb{N}} v(A_n), \end{aligned}$$

where the fourth equality can be proven by induction.

In the general case, thus, we have that for every measurable simple function φ such that $0 \le \varphi \le f$,

$$\int_A \varphi \, d\mu = \sum_{n \in \mathbb{N}} \int_{A_n} \varphi(s) \, d\mu \le \sum_{n \in \mathbb{N}} \int_{A_n} f \, d\mu = \sum_{n \in \mathbb{N}} v(A_n). \tag{22}$$

Then by definition of the Lebesgue integral as a supremum,

$$v(A) = \int_A f \, d\mu \le \sum_{n \in \mathbb{N}} v(A_n).$$

Therefore, in order to establish (21), it remains to be shown that

$$v(A) = \int_A f \, d\mu \ge \sum_{n \in \mathbb{N}} v(A_n).$$

If $v(A_n) = \infty$ for some $n \in \mathbb{N}$, then $v(A) \ge \sum_{n=1}^\infty v(A_n)$ because

$$v(A) = \int_A f \, d\mu \ge \int_{A_n} f \, d\mu = v(A_n),$$

and f is nonnegative. Note that $\sum_{n=1}^\infty v(A_n) = \infty$, also because f is nonnegative. Thus $v(A) \ge \sum_{n=1}^\infty v(A_n)$ reads $\infty \ge \infty$.

In the case were $v(A_n) = \infty$ for some $n \in \mathbb{N}$, (21) is thus established.

Suppose $v(A_n) < \infty$ for every $n \in \mathbb{N}$. Given $\epsilon > 0$, we can choose a measurable function φ with $0 \le \varphi \le f$ such that

$$\int_{A_1} \varphi \, d\mu \ge \int_{A_1} f \, d\mu - \epsilon, \quad \int_{A_2} \varphi \, d\mu \le \int_{A_2} f \, d\mu - \epsilon. \tag{23}$$

Hence

$$v(A_1 \cup A_2) = \int_{A_1 \cup A_2} f\,d\mu \geq \int_{A_1 \cup A_2} \varphi\,d\mu = \int_{A_1} \varphi\,d\mu + \int_{A_2} \varphi\,d\mu$$
$$\geq v(A_1) + v(A_2) - 2\epsilon,$$

where we in the second equality used (22). By the arbitrariness of ϵ, this shows that

$$v(A_1 \cup A_2) \geq v(A_1) + v(A_2).$$

From this it follows that for every $n \in \mathbb{N}$,

$$v(A_1 \cup \ldots \cup A_n) \geq v(A_1) + \ldots + v(A_n). \tag{24}$$

Since $A \supset A_1 \cup \ldots \cup A_n$, (24) implies

$$v(A) \geq \sum_{n \in \mathbb{N}} v(A_n).$$

2. Let us write

$$v(A) = \int_A f\,d\mu = \int_A f^+\,d\mu - \int_A f^-\,d\mu,$$

and see that (b) follows from (a).

Let $A = A_1 \cup A_2$. We will prove that $v(A_1 \cup A_2) = v(A_1) + v(A_2)$. (From this the countable additivity follows by induction.) Indeed, if f is measurable then f^+ and f^- are measurable and nonnegative, so that

$$\begin{aligned} v(A_1 \cup A_2) &= \int_{A_1 \cup A_2} f^+\,d\mu - \int_{A_1 \cup A_2} f^-\,d\mu \\ &= \int_{A_1} f^+\,d\mu + \int_{A_2} f^+\,d\mu - \left(\int_{A_1} f^-\,d\mu + \int_{A_2} f^-\,d\mu \right) \\ &= \int_{A_1} f^+\,d\mu - \int_{A_1} f^-\,d\mu + \left(\int_{A_2} f^+\,d\mu - \int_{A_2} f^-\,d\mu \right) \\ &= v(A_1) + v(A_2). \end{aligned}$$ □

Corollary 1. *$v(A)$ is a measure.*

Corollary 2. *If $A \in \mathcal{A}$, $B \subset A$, and $\mu(A \setminus B) = 0$, then*

$$\int_A f\,d\mu = \int_B f\,d\mu.$$

Proof. By Theorem 20 and (e) of Remark 11,

$$\int_A f\,d\mu = \int_{B \cup (A \setminus B)} f\,d\mu = \int_B f\,d\mu + \int_{A \setminus B} f\,d\mu = \int_B f\,d\mu.$$ □

Theorem 21. *If $f \in \mathcal{L}(u)$ on E, then $|f| \in \mathcal{L}(u)$ on E and*

$$\left|\int_E f\,d\mu\right| \le \int_E |f|\,d\mu.$$

Proof. Write $E = A \cup B$, where $f(x) \ge 0$ on A and $f(x) < 0$ on B. Then A and B partitions E so by (b) of Theorem 20,

$$\int_E |f|\,d\mu = \int_A |f|\,d\mu + \int_B |f|\,d\mu = \int_A f^+\,d\mu + \int_B f^-\,d\mu.$$

But $A \subset E$ and $A \in \mathcal{A}$, so $f \in \mathcal{L}(u)$ on A by (f) of Remark 11, implying that

$$\int_A f^+\,d\mu < \infty.$$

Similarly,

$$\int_B f^-\,d\mu < \infty.$$

This shows that $|f| \in \mathcal{L}(\mu)$.

For the last part, note that since $f \le |f|$ and $-f \le |f|$, for all $x \in E$,

$$\int_E f\,d\mu \le \int_E |f|\,d\mu, \quad -\int_E f\,d\mu = \int_E (-f)\,d\mu \le \int_E |f|\,d\mu,$$

where we have used (c) and (d) of Remark 11. □

Theorem 22. *Suppose f is measurable on E, $|f| \le g$, and $g \in \mathcal{L}(\mu)$ on E. Then $f \in \mathcal{L}(\mu)$ on E.*

Proof. Note that $|f| \le g$ on E implies $f^+ \le g$ and $f^- \le g$ on E. Then

$$\int_E f^+\,d\mu < \int_E g\,d\mu < \infty,$$

and

$$\int_E f^-\,d\mu < \int_E g\,d\mu < \infty,$$

implying that $f \in \mathcal{L}(\mu)$ on E. □

7.6 Convergence theorems

Theorem 23 (Lebesgue's monotone convergence theorem)**.** *Suppose $E \in \mathcal{A}$. Let $(f_n : n \in \mathbb{N})$ be a sequence of measurable functions such that*

$$0 \le f_1(x) \le f_2(x) \le \dots, \quad x \in E. \tag{25}$$

Let f be defined by

$$f(x) = \lim_{n\to\infty} f_n(x), \quad x \in E.$$

Then

$$\lim_{n\to\infty} \int_E f_n\,d\mu = \int_E f\,d\mu.$$

In particular, if $f_n \in \mathcal{L}(\mu)$ is nondecreasing and $f_n \to f$ pointwise, then $f \in \mathcal{L}(\mu)$. As we will se, this is not true in the Riemann sense.

Proof. Since $(f_n : n \in \mathbb{N})$ is an increasing sequence of nonnegative functions,

$$\lim_{n\to\infty} f_n(x)$$

exists in $[0, \infty)$ for every $x \in E$ and thus f is measurable by Theorem 16.

Since $f_n \leq f$ on E, we have

$$\int_E f_n \, d\mu \leq \int_E f \, d\mu$$

by (c) of Remark 11. Also $f_n \leq f_{n+1}$ implies

$$\int_E f_n \, d\mu \leq \int_E f_{n+1} \, d\mu,$$

and thus $(\int_E f_n \, d\mu : n \in \mathbb{N})$ is an increasing sequence bounded above by $\int_E f \, d\mu$. Consequently,

$$\lim_{n\to\infty} \int_E f_n \, d\mu \leq \int_E f \, d\mu. \tag{26}$$

Choose c such that $0 < c < 1$. Let φ be a simple measurable function such that $0 \leq \varphi \leq f$. Put

$$E_n = \{x \in E : f_n(x) \geq c\varphi(x)\}, \quad n \in \mathbb{N}.$$

Since $(f_n : n \in \mathbb{N})$ is an increasing sequence, $f_n(x) \geq cs(x)$ implies $f_{n+1}(x) \geq c\varphi(x)$, and therefore $(E_n : n \in \mathbb{N})$ is an increasing sequence.

Since $E_n \subset E$ for every $n \in \mathbb{N}$, we have

$$\bigcup_{n\in\mathbb{N}} E_n \subset E. \tag{27}$$

In fact, the reverse inclusion of (27) also holds, that is,

$$\bigcup_{n\in\mathbb{N}} E_n \supset E. \tag{28}$$

To prove this, let us show that if $x \in E$ then $x \in E_n$ for some $n \in \mathbb{N}$. Let therefore $x \in E$. If $f(x) = 0$, then since $\varphi \leq f$ we have $\varphi(x) = 0$ and then trivially $x \in E_n$ for some $n \in \mathbb{N}$ since $f_n(x) \geq c\varphi(x) = 0$ and $f_n(x) \geq 0$.

Suppose $f(x) \neq 0$. Since $0 \leq f_n \leq f$ and $0 < c < 1$, we have $f(x) > c\varphi(x)$. Since $f_n(x) \uparrow f(x)$, there exists $N \in \mathbb{N}$, depending on x, such that $f_N(x) \geq c\varphi(x)$. Thus $x \in E_N$, so that (28) is established. This implies that

$$\bigcup_{n\in\mathbb{N}} E_n = E. \tag{29}$$

Let us define a set function v on $\mathcal{A}$ by setting

$$v(A) = \int_A \varphi \, d\mu,$$

for every $A \in \mathcal{A}$. By Theorem 20, v is countably additive and thus a measure on X.

For every $n \in \mathbb{N}$, since $E \supset E_n$ and $f_n \geq c\varphi$ on E_n, we have

$$\int_E f_n \, d\mu \geq \int_{E_n} f_n \, d\mu \geq c \int_{E_n} \varphi \, d\mu = cv(E_n). \tag{30}$$

Since $(E_n : n \in \mathbb{N})$ is an increasing sequence, we have

$$\lim_{n\to\infty} E_n = \bigcup_{n\in\mathbb{N}} E_n = E,$$

where the last equality comes from (29). Then by Theorem 2, we have

$$\lim_{n\to\infty} v(E_n) = v(\lim_{n\to\infty} E_n) = v(E).$$

Letting $n \to \infty$ in (30), we then see that

$$\lim_{n\to\infty} \int_E f_n \, d\mu \geq c \lim_{n\to\infty} v(E_n) = cv(E).$$

Since this holds for every $0 < c < 1$, letting $c \to 1$, we get

$$\lim_{n\to\infty} \int_E f_n \, d\mu \geq v(E) = \int_E \varphi \, d\mu. \tag{31}$$

Since (31) holds for an arbitrary nonnegative simple function φ on E such that $0 \leq \varphi \leq f$, and since f is nonnegative, we have

$$\lim_{n\to\infty} \int_E f_n \, d\mu \geq \sup_{0\leq\varphi\leq f} \int_E \varphi \, d\mu = \int_E f \, d\mu.$$

Combining this inequality with that of (26), we are done. □

Theorem 24. *Suppose $f = f_1 + f_2$ where $f_i \in \mathcal{L}(\mu)$ on E $(i = 1, 2)$. Then $f \in \mathcal{L}(\mu)$ on E, and*

$$\int_E f \, d\mu = \int_E f_1 \, d\mu + \int_E f_2 \, d\mu.$$

Proof. Suppose first that $f_1 \geq 0$, $f_2 \geq 0$. If f_1 and f_2 are simple, the result follows immediately. Otherwise, choose monotonically increasing sequences $(s_n^{'})$, $(s_n^{''})$ of nonnegative measurable simple functions which converge to f_1,

f_2. By Theorem 19, this is possible. Put $s_n = s_n^{'} + s_n^{''}$. Then by Theorem 20

$$\int_E s_n \, d\mu = \int_E s_n^{'} \, d\mu + \int_E s_n^{''} \, d\mu,$$

and letting $n \to \infty$ and applying Theorem 23 the result follows.

Next, suppose $f_1 \geq 0$, $f_2 \leq 0$. Put

$$A = \{x : f(x) \geq 0\}, \quad B = \{x : f(x) < 0\}.$$

Then f_1 and $-f_2$ are nonnegative on A. Hence

$$\int_A f_1 \, d\mu = \int_A (f + (-f_2)) \, d\mu = \int_A f \, d\mu + \int_A (-f_2) \, d\mu = \int_A f \, d\mu - \int_A f_2 \, d\mu, \tag{32}$$

where we have used Theorem 20.

Similarly, $-f$, f_1 and $-f_2$ are nonnegative on B, so that

$$\int_B (-f_2) \, d\mu = \int_B f_1 \, d\mu + \int_B (-f) \, d\mu,$$

or,

$$\int_B f_1 \, d\mu = \int_B f \, d\mu - \int_B f_2 \, d\mu. \tag{33}$$

Adding (32) and (33), we get

$$\int_A f_1 \, d\mu + \int_B f_1 \, d\mu = \int_A f \, d\mu - \int_A f_2 \, d\mu + \int_B f \, d\mu - \int_B f_2 \, d\mu,$$

or, again using Theorem 20,

$$\int_E f_1 \, d\mu = \int_E f \, d\mu - \int_E f_2 \, d\mu.$$

In the general case, E can be decomposed into four pairwise disjoint sets E_i where $f_1(x)$ and $f_2(x)$ have constant sign, and we proceed as above. □

Corollary 3. *Suppose $E \in \mathcal{M}$. If $(f_n : n \in \mathbb{N})$ is a sequence of nonnegative measurable functions and*

$$f(x) = \sum_{n=1}^{\infty} f_n(x), \quad (x \in E),$$

then

$$\int_E f \, d\mu = \sum_{n=1}^{\infty} \int_E f_n \, d\mu.$$

Proof. By Theorem 24, for any finite $n \in \mathbb{N}$, we have

$$\int_E \sum_{j=1}^{n} f_j(x)\, d\mu = \sum_{j=1}^{n} \int_E f_j(x)\, d\mu.$$

Since $(\sum_{j=1}^{n} f_j : j \in \mathbb{N})$ is a monotonically increasing sequence, Theorem 23 yields

$$\lim_{n\to\infty} \sum_{j=1}^{n} \int_E f_j(x)\, d\mu = \int_E \lim_{n\to\infty} \sum_{j=1}^{n} f_j(x)\, d\mu. \qquad \square$$

Theorem 25 (Fatou's theorem)**.** *Suppose $E \in \mathcal{M}$. If $(f_n : n \in \mathbb{N})$ is a sequence of nonnegative measurable functions and*

$$f(x) = \inf_{n\to\infty} f_n(x), \quad (x \in E),$$

then

$$\int_E f\, d\mu \leq \inf_{n\to\infty} \int_E f_n\, d\mu.$$

Proof. By definition,

$$\inf_{n\to\infty} f_n = \lim_{n\to\infty} \left(\inf_{k\geq n} f_k \right).$$

Since $(\inf_{k\geq n} f_k : n \in \mathbb{N})$ is a increasing sequence of nonnegative functions, Theorem 23 may be applied and

$$\begin{aligned} \int_E \inf_{n\to\infty} f_n\, d\mu &= \int_E \lim_{n\to\infty} \left(\inf_{k\geq n} f_k \right) d\mu \\ &= \lim_{n\to\infty} \int_E \left(\inf_{k\geq n} f_k \right) d\mu \\ &\leq \int_E \inf_{n\to\infty} f_n\, d\mu. \end{aligned} \qquad \square$$

We have now arrived at the most important theorem in this paper.

Theorem 26 (Lebesgue's dominated convergence theorem)**.** *Suppose $E \in \mathcal{A}$. Let $(f_n : n \in \mathbb{N})$ be a sequence of measurable functions such that*

$$f_n(x) \to f(x), \quad (x \in E)$$

as $n \to \infty$. If there exists a function $g \in \mathcal{L}(\mu)$ on E such that

$$|f_n(x)| \leq g(x) \quad (n = 1, 2, 3, \ldots,\ x \in E),$$

then

$$\lim_{n\to\infty} \int_E f_n\, d\mu = \int_E f\, d\mu.$$

Proof. Since $f_n + g \geq 0$, Fatou's theorem shows that

$$\int_E (f+g)\,d\mu \leq \liminf_{n\to\infty} \int_E (f_n+g)\,d\mu,$$

or,

$$\int_E f\,d\mu \leq \liminf_{n\to\infty} \int_E f_n\,d\mu,$$

Similarly $g - f_n \geq 0$, so by Fatou's,

$$\int_E (g-f)\,d\mu \leq \liminf_{n\to\infty} \int_E (g-f_n)\,d\mu,$$

or,

$$-\int_E f\,d\mu \leq \liminf_{n\to\infty} \left(-\int_E f_n\,d\mu\right),$$

or

$$\int_E f\,d\mu \geq \limsup_{n\to\infty} \int_E f_n\,d\mu.$$

Thus

$$\int_E f\,d\mu \geq \limsup_{n\to\infty} \int_E f_n\,d\mu \geq \liminf_{n\to\infty} \int_E f_n\,d\mu \geq \int_E f\,d\mu. \qquad \square$$

Theorem 27 (Bounded convergence theorem)**.** *Let $(f_n : n \in \mathbb{N})$ be a uniformly bounded sequence of real-valued measurable functions a on a set $D \in \mathcal{A}$ with $\mu(D) < \infty$. Let f a bounded real-valued measurable function on D. If $(f_n : n \in \mathbb{N})$ converges to f a.e. on D, then*

$$\lim_{n\to\infty} \int_D |f_n - f|\,d\mu = 0, \tag{34}$$

and in particular

$$\lim_{n\to\infty} \int_D f_n\,d\mu = \int_D f\,d\mu. \tag{35}$$

Proof. Since $(f_n : n \in \mathbb{N})$ is uniformly bounded on D, there exists $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in D$. Since f also is bounded on D, we may choose M so that $|f(x)| \leq M$ for all $x \in D$. Since $(f_n : n \in \mathbb{N})$ converges to f a.e. on D and $\mu(D) < \infty$, we may apply Theorem 18. According to this theorem, for every $\delta > 0$ there exists a measurable subset E of D such that $\mu(E) < \delta$ and such that $(f_n : n \in \mathbb{N})$ converges to f uniformly on $D \setminus E$, that is, for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \epsilon$ for all $x \in D \setminus E$ whenever $n \geq N$. Then

$$\begin{aligned}\int_D |f_n - f|\,d\mu &= \int_{D\setminus E} |f_n - f|\,d\mu + \int_E |f_n - f|\,d\mu = \int_{D\setminus E} \epsilon\,d\mu + \int_E 2M\,d\mu \\ &\leq \epsilon\,\mu(D\setminus E) + 2M\mu(E) < \epsilon\,\mu(D) + 2M\delta.\end{aligned}$$

Since this holds for all $n \geq N$, we have

$$\sup_{n \geq N} \int_D |f_n - f| \, d\mu \leq \epsilon \, \mu(D) + 2M\delta$$

which implies

$$\lim_{n\to\infty} \sup_{m \geq n} \int_D |f_n - f| \, d\mu \leq \epsilon \, \mu(D) + 2M\delta,$$

or,

$$\limsup_{n\to\infty} \int_D |f_n - f| \, d\mu \leq \epsilon \, \mu(D) + 2M\delta.$$

By the arbitrariness of of $\epsilon > 0$ and $\delta > 0$, this implies

$$\limsup_{n\to\infty} \int_D |f_n - f| \, d\mu = 0.$$

Now since $\int_D |f_n - f| \, d\mu \geq 0$ for every $n \in \mathbb{N}$, we have

$$\liminf_{n\to\infty} \int_D |f_n - f| \, d\mu \geq 0.$$

Then

$$0 \leq \liminf_{n\to\infty} \int_D |f_n - f| \, d\mu \leq \limsup_{n\to\infty} \int_D |f_n - f| \, d\mu = 0,$$

showing (60).

Let us note that (61) can be derived from (60). Indeed,

$$| \int_D f_n \, d\mu - \int_D f \, d\mu| = | \int_D (f_n - f) \, d\mu| \leq \int_D |f_n - f| \, d\mu. \qquad (36)$$

The equalities in (36) may need justification. For the first one, note that since f_n and f are measurable and bounded, and $\mu(D) < \infty$, we have $f_n, f \in \mathcal{L}(\mu)$ by Remark 11 (a). Then $-f_n \in \mathcal{L}(\mu)$ by (d) of the same Remark. We then use Theorem 24. For the last equality, we use Theorem 21.

Applying (60) to the right most member of (36), we get

$$\lim_{n\to\infty} | \int_D f_n \, d\mu - \int_D f \, d\mu| = 0.$$

This shows (61). □

Lemma 7. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Let f and g be bounded real-valued measurable functions on a set $D \in \mathcal{A}$ with $\mu(D) < \infty$.*

(a) If $f \geq 0$ a.e. on D and $\int_D f \, d\mu = 0$, then $f = 0$ a.e. on D.

(b) If $f \leq g$ a.e. on D and $\int_D f \, d\mu = \int_D g \, d\mu$, then $f = g$ a.e. on D.

Proof. **1.** Suppose $f \geq 0$ on D. Let

$$D_0 = \{x \in D : f(x) = 0\} \text{ and } D_1 = \{x \in D : f(x) > 0\}.$$

Let us show that

$$f = 0 \text{ a.e. on } D \Leftrightarrow \mu(D_1) = 0. \tag{37}$$

Suppose $f = 0$ a.e. on D. Then there exists a null set E such that $E \subset D$ and $f = 0$ on $D \setminus E$. Then $D \setminus E \subset D_0 = D \setminus D_1$, so that $D_1 \subset E$ and $\mu(D_1) = 0$.

Conversely suppose $\mu(D_1) = 0$, then $D_1 \subset D$ is a null set and $f = 0$ on $D_0 = D \setminus D_1$, so that $f = 0$ a.e. on D. Thus (37) is established.

Before proceeding, let us note that if $\mu(D) = 0$, then $\int_D f\, d\mu = 0$ and $\mu(D_1) = 0$ so that $f = 0$ a.e. on D by (37). So in order to show (a) may assume $\mu(D) > 0$.

To show that $f = 0$ a.e. on D, assume the contrary. By (37) this implies $\mu(D_1) > 0$. Now

$$D_1 = \{D : f > 0\} = \bigcup_{k \in \mathbb{N}} \{D : f \geq \frac{1}{k}\},$$

so that

$$0 < \mu(D_1) \leq \sum_{k \in \mathbb{N}} \mu\{D : f \geq \frac{1}{k}\}.$$

This implies that there exists $k_0 \in \mathbb{N}$ such that

$$\mu\{D : f \geq \frac{1}{k_0}\} > 0.$$

Let us define a simple function φ on D by

$$\varphi(x) = \begin{cases} \frac{1}{k_0} & \text{for } x \in \{D : f \geq \frac{1}{k_0}\}, \\ 0 & \text{for } x \in D \setminus \{D : f \geq \frac{1}{k_0}\}. \end{cases}$$

Then clearly $\varphi \leq f$ on D and

$$\int_D f\, d\mu \geq \int_D \varphi\, d\mu = \frac{1}{k_0} \mu\{D : f \geq \frac{1}{k_0}\} > 0;$$

a contradiction to the assumption that $\int_D f\, d\mu = 0$. Therefore $f = 0$ a.e. on D.

Consider finally the general case where $f \geq 0$ a.e. on D and $\int_D f\, d\mu = 0$. Then there exists a null set E such that $E \subset D$ and $f \geq 0$ on $D \setminus E$, and

$$0 = \int_D f\, d\mu = \int_E f\, d\mu + \int_{D \setminus E} f\, d\mu = \int_{D \setminus E} f\, d\mu.$$

This shows that $f \geq 0$ on $D \setminus E$ and $\int_{D\setminus E} f\, d\mu = 0$. But then $f = 0$ a.e. on $D \setminus E$ by our result above. Thus there exists a null set F such that $F \subset D \setminus E$ and such that $f = 0$ on $(D \setminus E) \setminus F = D \setminus (E \cup F)$. Thus $f = 0$ a.e. on D.

2. If $f \leq g$ a.e. on D, then $g - f \geq 0$ a.e. on D. If also $\int_D f\, d\mu = \int_D g\, d\mu$, then $\int_D (g - f)\, d\mu = 0$. Then by (a), $g - f = 0$ a.e. on D. □

8 Comparison with the Riemann integral

8.1 Riemann integrability

Definition 36 (Darboux sums, Darboux integrals)**.** *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Suppose $|f(x)| \leq M$ for $x \in I = [a, b]$ for some $M \geq 0$. Let $\mathcal{B}$ be the collection of all partitions of I. With $\mathcal{P} \in \mathcal{B}$ given by $\mathcal{P} = \{x_0, \ldots, x_n\}$, let $I_k = [x_{k-1}, x_k]$, $m_k = \inf_{x\in I_k} f(x)$ and $M_k = \sup_{x\in I_k} f(x)$ for $k = 1, \ldots, n$. The lower and upper Darboux sums of f corresponding to the partition $\mathcal{P}$ are defined by*

$$\underline{S}(f, \mathcal{P}) = \sum_{k=1}^{n} m_k l(I_k) \quad \textit{and} \quad \overline{S}(f, \mathcal{P}) = \sum_{k=1}^{n} M_k l(I_k).$$

Let

$$\underline{S}(f) = \sup_{\mathcal{P}\in\mathcal{B}} \underline{S}(f, \mathcal{P}) \quad \textit{and} \quad \overline{S}(f) = \inf_{\mathcal{P}\in\mathcal{B}} \underline{S}(f, \mathcal{P}).$$

We call $\underline{S}(f)$ and $\overline{S}(f)$ the lower and upper Darboux integrals of f on I.

Definition 37. *A bounded real-valued function f on $I = [a, b] \subset \mathbb{R}$ is Riemann integrable on I if and only if*

$$\underline{S}(f) = \overline{S}(f).$$

In this case we set

$$\int_a^b f(x)\, dx := \underline{S}(f) = \overline{S}(f).$$

Theorem 28. *If f is a continuous real-valued function on $I = [a, b]$, then $\underline{S}(f) = \overline{S}(f)$ and consequently f is Riemann integrable on I.*

Proof. If f is continuous on the compact set I, it is uniformly continuous on I. Thus for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x, x'' \in [a, b]$,

$$|x - x''| \leq \delta \Rightarrow |f(x') - f(x'')| \leq \frac{\epsilon}{b - a}. \tag{38}$$

Let $\mathcal{P}_0 \in \mathcal{B}$ be such that $|P_0| < \delta$. Let $\mathcal{P}_0 = \{x_0, \ldots, x_n\}$ where $a = x_0 < \ldots < x_n = b$. In preceding, let us note that (38) implies

$$\sup_{[x_{k-1},x_k]} f - \inf_{[x_{k-1},x_k]} f \leq \frac{\epsilon}{b - a}. \tag{39}$$

Indeed, let $\alpha, \beta \in [x_{k-1}, x_k]$. By (38),

$$f(\alpha) - f(\beta) \leq \frac{\epsilon}{b-a},$$

or,

$$f(\alpha) \leq \frac{\epsilon}{b-a} + f(\beta).$$

Then

$$\overset{f}{[x_k, x_{k-1}]} \leq \frac{\epsilon}{b-a} + f(\beta). \tag{40}$$

Thus the right hand side of (40) is an upper bound to $[x_k, x_{k-1}] f$. But $\sup_{[x_k, x_{k-1}]} f$ is the smallest upper bound to $[x_k, x_{k-1}] f$, so

$$\sup_{[x_k, x_{k-1}]} f \leq \frac{\epsilon}{b-a} + f(\beta). \tag{41}$$

Rewriting (41), we have

$$\sup_{[x_k, x_{k-1}]} f - \frac{\epsilon}{b-a} \leq f(\beta). \tag{42}$$

Thus the left hand side of (42) is a lower bound to $[x_k, x_{k-1}] f$, and since $\inf[x_k, x_{k-1}] f$ is the largest such lower bound,

$$\sup_{[x_k, x_{k-1}]} f - \frac{\epsilon}{b-a} \leq \inf_{[x_k, x_{k-1}]} f,$$

establishing (39). Then

$$\overline{S}(f, \mathcal{P}_0) - \underline{S}(f, \mathcal{P}_0) = \sum_{k=1}^{n} \left(\sup_{[x_{k-1}, x_k]} f - \inf_{[x_{k-1}, x_k]} f \right) (x_k - x_{k-1}) \tag{43}$$

$$\leq \frac{\epsilon}{b-a}(b-a) = \epsilon. \tag{44}$$

By a similar argument as above, we see that (43) implies

$$\inf_{\mathcal{P} \in \mathcal{B}} \overline{S}(f, \mathcal{P}) - \sup_{\mathcal{P} \in \mathcal{B}} \underline{S}(f, \mathcal{P}) \leq \epsilon. \tag{45}$$

By the arbitrariness of $\epsilon > 0$, (45) implies

$$\inf_{\mathcal{P} \in \mathcal{B}} \overline{S}(f, \mathcal{P}) = \underline{S}(f) = \overline{S}(f) = \sup_{\mathcal{P} \in \mathcal{B}} \underline{S}(f, \mathcal{P}).$$

This shows that f is Riemann integrable on I by Definition 37. □

8.2 The upper and lower envelope integrals

Definition 38 (Limit inferior, limit superior of f)**.** *For $x_0 \in \mathbb{R}$ and $\delta > 0$, let*

$$U(x_0, \delta) = (x_0 - \delta, x_0 + \delta),$$

and

$$U_0(x_0, \delta) = (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}.$$

Let f be an extended real-valued function on a set $D \in \mathbb{R}$ and let $x_0 \in \overline{D}$. The limit inferior and limit superior of f as x approaches x_0 are defined by

$$\liminf_{x \to x_0} f(x) = \lim_{\delta \to 0} \inf_{U_0(x_0,\delta)\cap D} f \quad \text{and} \quad \limsup_{x \to x_0} f(x) = \lim_{\delta \to 0} \sup_{U_0(x_0,\delta)\cap D} f.$$

We have

$$f \text{ is continuous at } x_0 \Leftrightarrow \begin{cases} f(x_0) \in \mathbb{R}, \\ \liminf_{x \to x_0} f(x) = \limsup_{x \to x_0} f(x) = f(x_0). \end{cases}$$

Definition 39 (Lower and upper envelope)**.** *Let f be an extended real-valued function on a set $D \in \mathbb{R}$. Let $x_0 \in \overline{D}$. We define the lower envelope and upper envelope of f at x_0 by*

$$f_*(x_0) = \lim_{\delta \to 0} \inf_{U(x_0,\delta)\cap D} f \quad \text{and} \quad f^*(x_0) = \lim_{\delta \to 0} \sup_{U_0(x_0,\delta)\cap D} f.$$

Remark 12. For an extended real-valued function f on a set $D \subset \mathbb{R}$ and for $x_0 \in \overline{D}$,

$$f_*(x_0) \le \liminf_{x \to x_0} f(x) \le \limsup_{x \to x_0} f(x) \le f^*(x_0). \tag{46}$$

$$f_*(x_0) \le f(x_0) \le f^*(x_0). \tag{47}$$

$$f \text{ is continuous at } x_0 \Leftrightarrow \begin{cases} f(x_0) \in \mathbb{R}, \\ f_*(x_0) = f^*(x_0). \end{cases} \tag{48}$$

Proof. Let us prove (48), since (46) and (47) follows directly from the definition.

Suppose f is continuous at $x_0 \in D$. Then for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in U(x_0, \delta) \cap D$,

$$f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon.$$

Note that the first of these inequalities implies

$$f(x_0) - \epsilon \le \inf_{U(x_0,\delta)\cap D} f(x).$$

Now since $\inf_{U(x_0,\delta)\cap D} f(x) \uparrow$ as $\delta \downarrow 0$ (as the infimum is taken over a smaller set), we have

$$f(x_0) - \epsilon \leq \inf_{U(x_0,\delta)\cap D} f(x) \leq \lim_{\delta\to 0} \inf_{U(x_0,\delta)\cap D} f(x) \leq f_*(x_0). \tag{49}$$

By a similar argument we show

$$f^*(x_0) \leq f(x_0) + \epsilon. \tag{50}$$

If we then combine (49) and (50), we get $f^*(x_0) - f_*(x_0) \leq 2\epsilon$. Since $\epsilon > 0$ was chosen arbitrarily, we have $f_*(x_0) = f^*(x_0)$.

Conversely, if $f(x_0) \in \mathbb{R}$ and $f_*(x_0) = f^*(x_0)$, then

$$f_*(x_0) = f^*(x_0) = f(x_0)$$

by (47). Then by (46),

$$\liminf_{x\to x_0} f(x) = \limsup_{x\to x_0} f(x) = f(x_0),$$

proving the continuity of f at x_0. □

Lemma 8. *Let f be a bounded real-valued function on $I = [a,b]$. Then $\int_I f_*\, d\mu = \underline{S}(f)$ and $\int_I f^*\, d\mu = \overline{S}(f)$.*

Proof. Let us prove $\int_I f^*\, d\mu_L = \overline{S}(f)$. (The equality $\int_I f_*\, d\mu_L = \underline{S}(f)$ is proved similarly.) Consider the upper Darboux integral of f on I,

$$\overline{S}(f) = \inf_{P\in\mathcal{B}} \overline{S}(f, \mathcal{P}).$$

By how infimum is defined, for every $m \in \mathbb{N}$, there exists $\mathcal{P}_m \in \mathcal{B}$ such that

$$\overline{S}(f) \leq \overline{S}(f, \mathcal{P}_m) < \overline{S}(f) + \frac{1}{m}. \tag{51}$$

For the sequence of partitions $(\mathcal{P}_m : m \in \mathbb{N})$, we then have

$$\lim_{n\to\infty} \overline{S}(f) = \overline{S}(f, \mathcal{P}_m). \tag{52}$$

If necessary, let us add partition points to $\mathcal{P}_m$ so that $|\mathcal{P}_m| \leq \frac{1}{m}$. Note that this not affect the validity of (51) and (52), since refining a partition does not increase the upper Darboux sum.

Let $\mathcal{P}_m = \{x_{m_1}, x_{m_2}, \ldots, x_{m_n}\}$ and let $I_{mk} = [x_{m_{k-1}}, x_{m_k}]$ for $k = 1, \ldots, n$. For each $m \in \mathbb{N}$, define a function ψ_m on I by

$$\psi_m = \sum_{k=1}^{n} \sup_{I_{mk}} f \cdot \mathbf{1}_{I_{mk}}. \tag{53}$$

Let E be the countable set consisting of all partition points in $(\mathcal{P}_m : m \in \mathbb{N})$, that is,

$$(\mathcal{P}_m : m \in \mathbb{N}) = (x_{11}, x_{22}, \ldots).$$

Let us show that

$$\lim_{m\to\infty} \psi_m(x) = f^*(x), \quad \text{for } x \in I \setminus E. \tag{54}$$

Let $x_0 \in I \setminus E$. Then for each $m \in \mathbb{N}$ there exists a subinterval in $\mathcal{P}_m$ which contain x_0 in the open interval between two partitions points, that is, for each $m \in \mathbb{N}$ there exists $k \in \mathbb{N}$, $1 \leq k \leq m_k$ such that $x_0 \in I^\circ_{m_k}$.

Pick a $\delta' > 0$ so small that for all $m \geq M'$, where $M' \in \mathbb{N}$, we have $U(x_0, \delta') \subset I^\circ_{m_k}$. Then

$$f^*(x_0) = \lim_{\delta\to 0} \sup_{U(x_0,\delta)} f \leq \sup_{U(x_0,\delta')} f \leq \sup_{I^\circ_{m_k}} f \leq \sup_{I_{m_k}} f = \psi_m(x_0), \tag{55}$$

where each inequality comes from the fact that we take the sup over successively greater sets, and the last equality from how we defined ψ_m.

Since $\sup_{U(x_0,\delta)\cap I} f$ "shrinks" from above to $f^*(x_0)$ as $\delta \downarrow 0$, we have that for an arbitrary $\epsilon > 0$ there exists $\delta > 0$ such that

$$|\sup_{U(x_0,\delta)\cap I} f - f^*(x_0)| = \sup_{U(x_0,\delta)\cap I} f - f^*(x_0) < \epsilon.$$

By the same reason,

$$\sup_{U(x_0,\delta)\cap I} f \geq f^*(x_0).$$

Combining these two inequalities, we get

$$f^*(x_0) \leq \sup_{U(x_0,\delta)\cap I} f < f^*(x_0) + \epsilon. \tag{56}$$

Since $|\mathcal{P}_m| \to 0$ as $m \to \infty$ and since $x_0 \in I^\circ_{m_k}$ for some $k \in \mathbb{N}$ for every $m \in \mathbb{N}$, there exists $M > 0$ such that for $m \geq M$,

$$I_{m_k} \subset U(x_0, \delta) \cap I. \tag{57}$$

Let $N = \max(M, M')$, then by (55), (57), and (56), we have that for all $m \geq N$

$$f^*(x_0) \leq \psi_m(x_0) = \sup_{I_{m_k}} f \leq \sup_{U(x_0,\delta)\cap I} f < f^*(x_0) + \epsilon,$$

proving (54).

Consider the sequence of simple functions $(\psi_m : m \in \mathbb{N})$ defined by (53). Then

$$\int_I \psi_m \, d\mu = \sum_{k=1}^{n} \sup_{I_{m_k}} f \cdot l(I_{m_k}) = \overline{S}(f, \mathcal{P}_m);$$

the upper Darboux sum of f corresponding to $\mathcal{P}_m$. Then by (52),

$$\lim_{n\to\infty} \int_I \psi_m \, d\mu = \overline{S}(f). \tag{58}$$

Since f is bounded on I, we have $|f(x)| \le M$ on I for some $M \ge 0$. Then by (53), we have $|\psi_m(x)| \le M$. By (54), we have that ψ_m converges to f^* a.e. on I. Then by Theorem 32,

$$\lim_{n\to\infty} \int_I \psi_m \, d\mu = \int_I f^* \, d\mu. \tag{59}$$

By combining (58) and (59), we have

$$\int_I f^* \, d\mu = \overline{S}(f).$$

□

Lemma 9. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Let f_1 and f_2 be bounded real-valued measurable functions on a set $D \in \mathcal{A}$ with $\mu(D) < \infty$. If $f_1 = f_2$ a.e. on D, then*

$$\int_D f_1, d\mu = \int_D f_2 \, d\mu.$$

8.3 Riemann integrability and Lebesgue integrability

Theorem 29. *Let f be a bounded real-valued function on $I = [a, b]$. If f is Riemann integrable on I, then f is $\mathcal{M}_L$-measurable and Lebesgue integrable on I and moreover*

$$\int_a^b f(x) \, dx = \int_I f \, d\mu.$$

Proof. If f is Riemann integrable on I, then $\underline{S}(f) = \overline{S}(f)$ by Definition 37. By Lemma 8,

$$\int_I f_* \, d\mu = \int_I f^* \, d\mu.$$

Since f_* and f^* are $\mathcal{B}_{\mathbb{R}}$-measurable on I, they are $\mathcal{M}_L$-measurable on I by Theorem 8. Since $f_* \le f^*$ and their Lebesgue integrals are equal, $f_* = f^*$ a.e. on I by (b) of Lemma 7. Since $f_* \le f \le f^*$ by Remark 12, we have $f = f_* = f^*$ a.e. on I. Since the Lebesgue measure space is complete, f is Lebesgue measurable on I by (b) of Remark 4.

Since f is Lebesgue measurable and $f = f^*$ a.e. on I, we have $\int_I f \, d\mu = \int_I f^* \, d\mu$ by Lemma 9. But by Lemma 8 and Definition 37,

$$\int_I f^* \, d\mu = \overline{S}(f) = \int_a^b f(x) \, dx.$$

Thus $\int_a^b f(x) \, dx = \int_I f \, d\mu$. □

8.4 Lebesgue's integrability condition

Theorem 30. *Let f be bounded real-valued function on $I = [a, b]$ and let E be the set of all points of discontinuity of f in I. Then*

$$f \text{ is Riemann integrable on } I$$
$$\Leftrightarrow$$
$$f_* = f^*, \text{ a.e. on } I$$
$$\Leftrightarrow$$
$$\mu_L(E) = 0.$$

Proof. **1.** If f is Riemann integrable on I, $f_* = f^*$ a.e. as seen above.

If $f_* = f^*$ a.e. on I, then by Lemma 9,

$$\int_I f_* \, d\mu = \int_I f^* \, d\mu,$$

implying Riemann integrability by Definition 37 and Theorem 8.

2. By Remark 12 we have that f is continuous at $x_0 \in I$ if and only if $f_*(x_0) = f^*(x_0)$. Then $f^* = f_*$ on $I \setminus E'$ for some null set E' if and only if f is continuous on $I \setminus E'$ for some set null E' if and only if f is discontinuous at E' for some set null E' if and only if $E' = E$. □

8.5 Where Riemann falls short

In a Riemann setting, uniform convergence tells us that we are allowed to switch places between the integral and the limit.

Theorem 31 (Uniform Convergence and Integration in the Riemann sense)**.** *Let $(f_n(x) : n \in \mathbb{N})$ be a sequence of continuous functions defined on the interval $[a, b]$ and assume that f_n converges uniformly to a function f. Then f is Riemann-integrable and*

$$\lim_{n\to\infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx.$$

Proof. Since the f_n's are continuous and converge uniformly to f, the limit function must be continuous. By Theorem 28, f is Riemann-integrable and

$$\begin{aligned}\left| \int_a^b f_n(x) \, dx - \int_a^b f(x) \, dx \right| &\le \left| \int_a^b f_n(x) \, dx - \int_a^b f(x) \, dx \right| \\ &\le \int_a^b |f_n(x) - f(x)| \, dx \\ &\le (b - a) \sup |f_n(x) - f(x)| \to 0. \end{aligned}$$

□

Note that Theorem 31 *does not* say $\neg$(uniform) $\Rightarrow$ $\neg$(interchange). However in practice, to be sure that interchanging is justified it is uniform convergence that we check.

Even when the sequence is uniformly bounded, pointwise convergence is not enough. Recall the Bounded convergence theorem:

Theorem 32 (Bounded convergence theorem)**.** *Let $(f_n : n \in \mathbb{N})$ be a uniformly bounded sequence of real-valued measurable functions a on a set $D \in \mathcal{A}$ with $\mu(D) < \infty$. Let f a bounded real-valued measurable function on D. If $(f_n : n \in \mathbb{N})$ converges to f a.e. on D, then*

$$\lim_{n\to\infty} \int_D |f_n - f| \, d\mu = 0, \tag{60}$$

and in particular

$$\lim_{n\to\infty} \int_D f_n \, d\mu = \int_D f \, d\mu. \tag{61}$$

The bounded convergence theorem does not hold in a Riemann setting. We will prove this in Proposition 5, but first we need the following Lemma.

Lemma 10. *Let f be the Dirichlet function on $[0,1]$, that is,*

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational}, \\ 0, & \text{otherwise}, \end{cases}$$

for all $x \in [0,1]$.

f is not Riemann integrable on $[0,1]$.

Proof. Take an arbitrary partition $\mathcal{P} = \{x_0, x_1, \ldots, x_n\}$ of the interval $[0,1]$.

Between any two points x_j and x_{j+1} there exists an irrational number. Therefore the inf over $[x_j, x_{j+1}]$ is 0, so that

$$\underline{S}(f) = \sup_{\mathcal{P}\in\mathcal{B}} \underline{S}(f, \mathcal{P}) = 0.$$

Between any two points x_j and x_{j+1} there exists a rational number. Therefore the sup over $[x_j, x_{j+1}]$ must be 1. This means that

$$\begin{aligned} \overline{S}(f, \mathcal{P}) &= \sum_{k=1}^{n} 1 \cdot (x_k - x_{k-1}) \\ &= (x_1 - x_0) + (x_2 - x_1) + \ldots + (x_n - x_{n-1}) \\ &= x_n - x_0 = 1 - 0 = 1, \end{aligned}$$

so that

$$\overline{S}(f) = \inf_{\mathcal{P}\in\mathcal{B}} \overline{S}(f, \mathcal{P}) = 1.$$

Then by Definition 37, f is not Riemann integrable. □

Proposition 5. *There exists a uniformly bounded sequence* $(f_n : n \in \mathbb{N})$ *of real-valued functions defined on the set* $[0,1]$ *that converges to a bounded function* f*, but where*

$$\lim_{n\to\infty} \int_0^1 f_n\,dx \neq \int_0^1 \lim_{n\to\infty} f_n\,dx. \tag{62}$$

Proof. Let f be the Dirichlet function on $[0,1]$. Then f is clearly bounded.

Let $\{r_n\}_{n=1}^{\infty}$ be any enumeration of the rationals in $[0,1]$, and define

$$f_n(x) = \begin{cases} 1, & \text{if } x = r_k, \text{ for some } 1 \leq k \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Then (f_n) is uniformly bounded by 1, and it converges pointwise to f.

Moreover,

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1.$$

However f is not Riemann integrable by Lemma 10, so (62) holds. □

8.6 Some examples of how to use the Lebesgue integral

Example 2. Compute

$$\lim_{n\to\infty} \int_0^1 \frac{1}{nx+1}\,d\mu.$$

Solution. Let $f_n(x) = \frac{1}{nx+1}$, then $f_n(x) \to 0$ and $|f_n(x)| \leq g(x) = 1$ for all $x \in [0,1]$ and for all $n \in \mathbb{N}$. Since g is continuous and bounded on $[0,1]$, it is Lebesgue integrable on $[0,1]$ by Theorem 28 and 29.

Therefore the integral is equal to 0. ■

Remark 13. Note that in the Riemann sense, we could not so easily be sure of the equality

$$\lim_{n\to\infty} \int_0^1 \frac{1}{nx+1}\,d\mu = \int_0^1 \lim_{n\to\infty} \frac{1}{nx+1}\,d\mu.$$

One way of *being sure*, is to know that $(f_n(x))$ converges uniformly to 0 and then use [0,1] Theorem 31. The problem is, it doesn't:

$$\sup |\frac{1}{nx+1}| = 1 \neq 0,$$

(To see this, note that $|f_n(x)| \leq 1$, so 1 is an upper bound to $|f_n(x)|$. Now since $f_n(0) = 1$, it is indeed the *smallest* upper bound.)

For the next example we will need the following result.

Proposition 6. *Let* $f_n(x) = \left(1 + \frac{x}{n}\right)^n$, $x \in [0, \infty)$.
Then (f_n) *is non-decreasing.*

Proof. Fix $x \in [0, \infty)$ and take $x_1 = 1$, $x_2 = x_3 = \ldots = x_{n+1} = 1 + \frac{x}{n}$ in the AM-GM inequality

$$\sqrt[n+1]{x_1 \cdot \ldots \cdot x_{n+1}} \leq \frac{x_1 + \ldots + x_{n+1}}{n+1}.$$

Then

$$\left(1 + \frac{x}{n}\right)^{\frac{n}{n+1}} \leq 1 + \frac{x}{n+1},$$

proving $f_n(x) \leq f_{n+1}(x)$. □

Example 3. Compute

$$\lim_{n\to\infty} \int_0^\infty \left(1 + \frac{x}{n}\right)^n e^{-2x}\, d\mu.$$

Solution. Let $f_n(x) = \left(1 + \frac{x}{n}\right)^n e^{-2x}$, then each f_n is measurable by Theorem 13, and

$$0 \leq f_1(x) \leq f_2(x) \leq \ldots, \quad x \in [0, \infty),$$

since $\left(1 + \frac{x}{n}\right)^n \to e^x$, monotonically, from below, by Proposition 6.

By Lebesgue's monotone convergence, the integral is equal to 1. ■

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