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Auction Theory

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Abstract

We study auction theory where bidders have independent private values. We describe different auction types, and derive Nash equilibria for the symmetric case in which all bidders have values drawn from the same distribution. We also study a case with uncertainty about the number of bidders, and examples with asymmetric distributions.

Contents

1 Introduction

We consider auctions with independent private values (IPV). We let the number of bidders be n, and the set of possible bids is $[0, \infty)$. A generic bid by player i is denoted b_i, and the value of player i is denoted v_i. The distribution function of a player's value is denoted by F.

1.1 Some Common Auction Forms

1.1.1 English Auction

The *English* auction is an open ascending price auction. This auction has a low start value, and increases with small increments until there is only one interested bidder left. This will be the same as the *Second Price Auction*, since when the second highest bidder drops off the highest bidder wins and pays the second highest price.

Bidders payoff function[1][2]: $u_i(b) = u_i(b_1, \ldots, b_n) = \begin{cases} v_i - \max\limits_{j \neq i} b_j, & \text{if } b_i > \max\limits_{j \neq i} b_j. \\ 0, & \text{otherwise.} \end{cases}$

1.1.2 Dutch Auction

The *Dutch* auction is the open descending price counterpart of the *English* auction. Here the auction has a start value higher than every interested bidder, and gradually lowered until there is a bidder that is interested at that price. This will be the same as the *First Price Auction*, since the highest bid wins and pays his own bid.

Bidders payoff function[1][2]: $u_i(b) = u_i(b_1, \ldots, b_n) = \begin{cases} v_i - b_i, & \text{if } b_i = \max\limits_{i \in N} b_i. \\ 0, & \text{otherwise.} \end{cases}$

1.1.3 Sealed-Bid First Price Auction (FPA)

In this form the bidders submit their bids in sealed envelopes. The bidder that has the highest bid when the auction has ended wins the auction and pays the price that he bids.

Bidders payoff function[1][2]: $u_i(b) = u_i(b_1, \ldots, b_n) = \begin{cases} v_i - b_i, & \text{if } b_i = \max\limits_{i \in N} b_i. \\ 0, & \text{otherwise.} \end{cases}$

1.1.4 Sealed-Bid Second Price Auction (SPA)

This auction is also known as a *Vickrey Auction*. Here also the bidders submit their bids in sealed envelopes, but in this case if you have the highest bid, you will pay the second highest bid.

Bidders payoff function[1][2]: $u_i(b) = u_i(b_1, \ldots, b_n) = \begin{cases} v_i - \max\limits_{j \neq i} b_j, & \text{if } b_i > \max\limits_{j \neq i} b_j. \\ 0, & \text{otherwise.} \end{cases}$

1.2 Nash Equilibrium

Nash Equilibrium is a stable state of a system involving the interaction of different participants, in which no participant can gain by a unilateral change of strategy if the strategies of the others remain unchanged.

Definition 1.2.1. ***Nash equilibrium of a strategic game***[4] $(N, (A_i)_{i \in N}, (u_i)_{i \in N})$ *is a profile* $a^* \in A$ *of actions with the property that for every player* $i \in N$*, we have*

$$u_i(a_i^*, a_{-i}^*) \geqslant u_i(a_i, a_{-i}^*) \; for \; all \; a_i \in A_i$$

where

- u_i is the payoff function.
- a_i^* is the strategy of player i, and a_{-i}^* if the strategies of all players except for player i.

2 Indepentent Private Values (IPV) Auctions with Symmetric Bidders

In the IPV case there are two important features that define an IPV auction, such as

- Bidder i's information is independent of j's information.
- Bidder i's value is independent of j's information.

2.1 First Price Auction Nash Equilibrium

Proposition 2.1.1. *In a First Price Auction the Nash Equilibrium is to bid*[1]

$$\beta(v) = v - \int_{\underline{v}}^{v} \left(\frac{F(x)}{F(v)} \right)^{n-1} dx$$

where

- *$\underline{v}$ is the lowest possible value*
- *$F(x)$ is the CDF of each player's value*

Proof.

- $N = \{1, \ldots, n\}$: The set of bidders.
- $\beta_i(v) = [0, \infty)$ for each $i \in N$: The set of possible bids by player i. A generic bid by player i is denoted $b_i(v) \in \beta_i$.
- Assume that $\beta_i(v) \equiv \beta(v)$ since this is a symmetric equilibrium, and all bidders will use the bid function $b = \beta(v)$.
- Bidders payoff function: $u_i(b) = u_i(b_1, \ldots, b_n) = \begin{cases} v_i - b_i, & \text{if } b_i = \max\limits_{i \in N} b_i. \\ 0, & \text{otherwise.} \end{cases}$
- A player will bid $\leqslant b$ if $v_i \leqslant \beta^{-1}(b)$.
- A player with valuation v_i and who bids b_i expects to earn $(v_i - b_i)\Big(Pr(b_{j \neq i} \leqslant b_i)\Big)^{n-1}$.
- $Pr(v_i \leqslant \beta^{-1}(b)) = F\big(\beta^{-1}(b)\big)$ and $\Big(Pr(v_{j \neq i} \leqslant \beta^{-1}(b))\Big)^{n-1} = \Big(F\big(\beta^{-1}(b)\big)\Big)^{n-1}$.
- A player with valuation v_i and who bids b then expects to earn $(v_i - b)\Big(F\big(\beta^{-1}(b)\big)\Big)^{n-1}$.

  To calculate the Nash Equilibrium we will take the derivative with respect to b.

$$\frac{\partial}{\partial b}\left((v-b)\Big(F\big(\beta^{-1}(b)\big)\Big)^{n-1}\right) = -\Big(F\big(\beta^{-1}(b)\big)\Big)^{n-1} + \frac{(v-b)(n-1)\Big(F\big(\beta^{-1}(b)\big)\Big)^{n-2}F'\big(\beta^{-1}(b)\big)}{\beta'\big(\beta^{-1}(b)\big)} = 0$$

$$\Leftrightarrow -\Big(F(v)\Big)^{n-1} + \frac{\big(v-\beta(v)\big)(n-1)\Big(F(v)\Big)^{n-2}F'(v)}{\beta'(v)} = 0$$

$$\Leftrightarrow \beta'(v)\Big(F(v)\Big)^{n-1} + \beta(v)(n-1)\Big(F(v)\Big)^{n-2}F'(v) =$$

$$= v(n-1)\Big(F(v)\Big)^{n-2}F'(v)$$

$$\Leftrightarrow \left[\beta(v)\Big(F(v)\Big)^{n-1}\right]' = \left(\int_{\underline{v}}^{v} x(n-1)\Big(F(x)\Big)^{n-2}F'(x)dx\right)'$$

$$\Leftrightarrow \beta(v)\Big(F(v)\Big)^{n-1} = \int_{\underline{v}}^{v} x\left[\Big(F(x)\Big)^{n-1}\right]'dx$$

Integraton by parts gives us:

$$\Leftrightarrow \beta(v)\Big(F(v)\Big)^{n-1} = v\Big(F(v)\Big)^{n-1} - \int_{\underline{v}}^{v}\Big(F(x)\Big)^{n-1}dx$$

This gives us the Nash Equilibrium:

$$\beta(v) = v - \int_{\underline{v}}^{v}\left(\frac{F(x)}{F(v)}\right)^{n-1}dx \tag{1}$$

□

Example 2.1.1. *If the value of each player is uniformly distributed over* $[0,1]$ *we have*

$$F(x) = x$$

We use Proposition 2.1.1. *that*

$$\beta(v) = v - \int_{\underline{v}}^{v} \left(\frac{F(x)}{F(v)}\right)^{n-1} dx = v - \int_{0}^{v} \left(\frac{x}{v}\right)^{n-1} dx$$

And by solving this integral we get that

$$\beta(v) = v - \frac{1}{n}v = \frac{(n-1)}{n}v$$

So if we use 2 players for this equation we get that

$$\beta(v) = \frac{1}{2}v$$

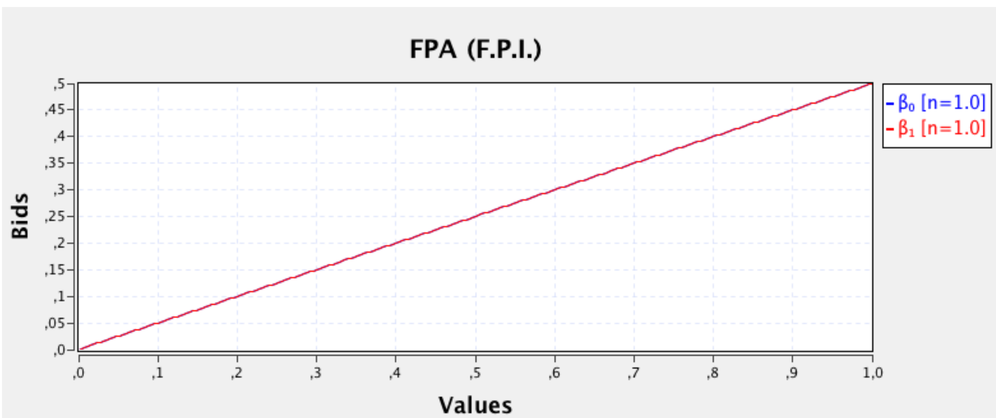

Figure 1: Nash Eqiulibrium for p_1 and p_2, (where both is drawn from the same distribution Uni[0,1]) for any given value at $[0,1]$

2.2 Second Price Auction Nash Equilibrium

Proposition 2.2.1. *In a Second Price Auction the Nash Equilibrium*[1] *is to bid his true value*

$$b_i = v_i$$

Proof. We want to show that the strategy $(b_1, \ldots, b_n) = (v_1, \ldots, v_n)$ i.e. a truthful bid.

- We valuate the bids as $(v_1 \geqslant, \ldots, \geqslant v_n) = (b_1 \geqslant, \ldots, \geqslant b_n)$.
- Then $b_1 = v_1$ wins the auction, he will pay the second highest bid $b_2 = v_2$.
- The payoff will then be $u_1 = v_1 - v_2 \geqslant 0$.
- Likewise, for all other bidders $v_i \neq v_1$ v_i will need to change her payoff from 0 and will bid higher than v_1, in which case the payoff will be $u_i = v_i - v_1 < 0$ and will result a negative payoff.
- Therefore no one will make a profit when deviate from the strategy.

□

2.3 The Envelope Theorem

Suppose that $b_i = b(v_i)$ is the symmetric equilibrium, then i's eqiulibrium payoff given value v_i is:

$$U(v_i) = (v_i - b(v_i))F^{n-1}(v_i)$$

since i is playing his best response we get

$$U(v_i) = \max_{b_i}(v_i - b_i)F^{n-1}(b^{-1}(b_i))$$

And if we take the derivative of this with respect of v, we get:

$$\frac{d}{dv}U(v)\bigg|_{v=vi} = F^{n-1}(b^{-1}(b(v_i))) = F^{n-1}(v_i)$$

and

$$U(v_i) = U(\underline{v}) + \int_{\underline{v}}^{v_i} F^{n-1}(\tilde{s})d\tilde{s}$$

and since a bidder with value $\underline{\text{v}}$ never will win the auction gives us that $U(\underline{v}) = 0$. And by combining the equations above we can solve for the equilibrium strategy[2][4] as

$$b(v) = v - \frac{\int_{\underline{v}}^{v_i} F^{n-1}(\tilde{v})d\tilde{v}}{F^{n-1}(v)}$$

2.4 Revenue Equivalence

Theorem 2.4.1. ***Revenue Equivalence Theorem (RET)***[2][4] *Suppose n bidders have values $v_1, \ldots, v_n$ identically and independently distributed with cumulative distribution function $F(\cdot)$. Then any equilibrium of any auction game in which*

1. *The bidder with the highest value wins the object.*
2. *The bidder with value $\underline{v}$ gets zero profits.*

With these settings the different auction types generate the same expected revenue, such that the seller will generate the same profit from any type of auction he choose.

Proof. We consider a general auction where bidders will place there bids $b_1, \ldots, b_n$. The auction rule specifies for all i, such that

$$\begin{aligned} x_i &: B_1 \times \ldots \times B_n \to \{0, 1\} \\ t_i &: B_1 \times \ldots \times B_n \to \mathbb{R}, \end{aligned} \tag{2}$$

where x_i is the probability that player i will win the object and t_i is players i's payment as a function of $(b_i, \ldots, b_n)$. Given this rule, bidder i's expected payoff as a function is:

$$E(U_i) = U_i(v_i, b_i) = v_i E_{b_{-i}}[x_i(b_i, b_{-i})] - E_{b_{-i}}[t_i(b_i, b_{-i})].$$

b_i, b_{-i} is the equilibrium of the auction, so bidder i's equilibrium payoff is therefore:

$$U_i(v_i, b_i) = v_i F^{n-1}(v_i) - E_{v_i}[t_i(b_i(v_i), b_{-i}(v_{-i})].$$

and here we use condition (1) from *Theorem* 2.4.1. where the highest value wins the object, so we can write $E_{v_i}[x_i(b_i(v_i), b_{-i}(v_{-i})] = F^{n-1}(v_i)$.

$$\frac{d}{dv}U_i(v)\bigg|_{v=v_i} = E_{b_{-i}}[x_i(b_i(v_i), b_{-i}(v_{-i})] = F^{n-1}(v_i),$$

and also that

$$U_i(v_i) = U_i(\underline{v}) + \int_{\underline{v}}^{v_i} F^{n-1}(\tilde{v})d\tilde{v} = \int_{\underline{v}}^{v_i} F^{n-1}(\tilde{v})d\tilde{v}$$

where we can use (2) from *Theorem* 2.4.1. to write $U_i(\underline{v}) = 0$ which uses the condition that the bidder with value $\underline{\text{v}}$ gets zero profits. If we combine our expressions for $U_i(v_i)$, we get bidder i's expected payment as:

$$E_{v_{-i}}[t_i(b_i, b_{-i})] = v_i F^{n-1}(v_i) - \int_{\underline{v}}^{v_i} F^{n-1}(\tilde{v})d\tilde{v} = \int_{\underline{v}}^{v_i} \tilde{v}dF^{n-1}(\tilde{v}),$$

and since x_i is not a part of this expression, bidder i's expected equilibrium payment given his value is the same under every auction that satisfies both (1) and (2) of *Theorem* 2.4.1., then the expected payment is:

$$E[V^{1:n-1} \mid V^{1:n-1} \leq v_i] = E[V^{2:n} \mid V^{1:n} = v_i].$$

So therefore the expected revenue is:

$$E[Revenue] = nE_{v_i}[i\text{'s expected payment} \mid v_i] = E[V^{2:n}].$$

□

Just to show how the expected revenue is for the different auctions we show this for the example where there are two bidders with values drawn from the same distribution $Uni[0, 1]$. Then the expected revenue is $\frac{1}{3}$ and the expected profit for bidder i with value v is $\frac{v^2}{2}$

2.4.1 First Price Auctions

- Bidder i has the value v_i
- $b_i = \frac{(n-1)}{n}v_i = \frac{v_i}{2}$
- The probability that b_i wins the auction is $P(v_i) = v_i$
- The expected payoff if b_i wins is then $\frac{v_i}{2}$
- So therefore the expected profit is

$$E[U_i(v_i)] = v_i\left(v_i - \frac{v_i}{2}\right) = \frac{v_i^2}{2}$$

2.4.2 Second Price Auctions

- Bidder i has the value v_i
- $b_i = v_i$
- The probability that b_i wins the auction is $P(v_i) = v_i$
- The expected payoff if b_i wins is then $\frac{v_i}{2}$
- So therefore the expected profit is

$$E[U_i(v_i)] = v_i\left(v_i - \frac{v_i}{2}\right) = \frac{v_i^2}{2}$$

2.4.3 Expected Revenue from FPA and SPA

Since the bidder with lowest value has the expected profit $E[U(v_{-i})] = 0$, therefore:

$$E[U(v_i)] = E[U(v_{-i})] + \int_0^v U'(x)dx = 0 + \int_0^v xdx = \frac{v^2}{2}$$

And the expected profit for each bidder is:

$$E[U_1(v)] = E[U_2(v)] = \int_0^1 U(v)dv = \int_0^1 \frac{v^2}{2}dv = \frac{1}{6}$$

This gives us the total expected profit for both bidders as:

$$E[\text{Total bidder profit}] = \frac{1}{3}$$

And hence expected revenue is given by:

$$E[\text{Revenue}] = E[\text{Surplus}] - E[\text{Total bidder profit}]$$

where

$$E[\text{Surplus}] = E[max\{v_i, v_j\}] = \frac{2}{3}$$

So this gives us that the expected revenue is

$$E[\text{Revenue}] = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

3 Uncertain number of bidders

3.1 First Price Auctions

Example 3.1.1. *If we let m be the number of possible bidders, and set the possibilities that player i will attend as $(1 - p_i)$ and p_i else.*

Let $m = 2$ and $p_1(1)$ be the probability that player 1 believes that there only will be 1 player (himself) in the auction. Since this is a symmetric auction with only two buyers we can write this as the following equation $p_1(1) = p \quad$ and $\quad p_1(2) = (1 - p)$

$$F(v) = p(v - b) + (1 - p)F(\beta^{-1}(b))(v - b)$$

Maximize by taking the derivative

$$\frac{dF(v)}{db} = -p - (1 - p)F(\beta^{-1}(b)) + (1 - p)(v - b)f(\beta^{-1}(b))\frac{1}{\beta'(\beta^{-1}(b))} = 0$$

If we use that $v = \beta^{-1}(b)$ we get

$$\begin{aligned} \frac{dF(v)}{db} &= -p - (1 - p)F(v) + \frac{(1 - p)(v - \beta(v)f(v))}{\beta'} = 0 \\ \frac{dF(v)}{db} &= (p + (1 - p)F)\beta' + (1 - p)f(v)\beta(v) = (1 - p)vf(v) \\ \frac{dF(v)}{db} &= \Big((p + (1 - p)F)\beta\Big)' = (1 - p)vf(v) \end{aligned} \tag{3}$$

So the equilibrium function is

$$\beta = \frac{(1 - p)\int_0^v uf(u)du}{p + (1 - p)F(v)}$$

So to see the price difference between full information about the number of players who will attend we can take the example with two players in both situations with $p_1(1) = p_1(2) = 0.5$ for the equation with uncertain number of bidders.

Uncertain number of bidders, FPA case gives us the Nash Equilibrium at

$$b_{i_{FPA}} = \frac{(1 - p)v^2}{2(p + (1 - p)v)}$$

and then will give us

$$b_{i_{FPA}} = \frac{(0.5)v^2}{2(0.5 + (0.5)v)} = \frac{v^2}{2(1 + v)}$$

Example 3.1.2. *If we here instead of Example 3.1.1. take three players instead of two where* $p_1(1) = p$, $p_1(2) = q$ *and* $p_1(3) = (1 - p - q)$ *we will get*

$$F(v) = p(v - b) + qF(\beta^{-1}(b))(v - b) + (1 - p - q)F(\beta^{-1}(x))^2(v - b)$$

Maximize by taking the derivative

$$\frac{dF(v)}{db} = -p - qF(\beta^{-1}(b)) + q(v - b)f(\beta^{-1}(b))\frac{1}{\beta'(\beta^{-1}(b))} - (1 - p - q)F(\beta^{-1}(b))^2 +$$
$$+ 2((1 - p - q)(v - b)F(\beta^{-1}(b))f(\beta(v)^{-1}\frac{1}{\beta'(\beta^{-1}(v))}) = 0$$

If we use that $v = \beta^{-1}(b)$ *we get*

$$\frac{dF(v)}{db} = -p - qF(v) + q(v - \beta(v)f(v))\frac{1}{\beta'(v)} - (1 - p - q)F(v)^2 +$$
$$+ 2(1 - p - q)(v - b)F(v)f(v)\frac{1}{\beta'(v)} = 0$$
$$\frac{dF(v)}{db} = -p - qF(v) - (1 - p - q)F(v)^2 + \frac{q(v - b)f(v) + 2(1 - p - q)(v - b)(F(v)f(v))}{\beta'(v)}$$
$$\frac{dF(v)}{db} = -(p + qF(v) + (1 - p - q)F(v)^2)\beta' - (qf(v)\beta(v) + 2(1 - p - q)F(v)f(v)\beta(v)) =$$
$$= -(qv + 2(1 - p - q)vF(v)f(v))$$
$$\frac{dF(v)}{db} = ((p + qF(v) + (1 - p - q)F(v)^2)\beta)' = qv + 2(1 - p - q)vF(v)f(v)$$

So the equilibrium function is

$$\beta = \frac{q\int_0^v udu + 2(1 - p - q)\int_0^v uF(u)f(u)du}{p + qF(v) + (1 - p - q)F(v)^2} \tag{4}$$

And here we use as above that $p_1(1) = p_1(2) = p_1(3) = 1/3$ *for the equation with uncertain number of bidders.*

Uncertain number of bidders, FPA case gives us the Nash Equilibrium at

$$b_{i_{FPA}} = \frac{q\frac{v^2}{2} + 2(1 - p - q)\frac{v^3}{3}}{p + q + (1 - p - q)v}$$

3.2 Second Price Auctions

For the Second Price Auction with uncertain number of bidders we still have the same strategy and Nash equilibrium since it will not change with the uncertainty of players attendance. So the weakly dominant strategy is to bid his true value $b_i = v_i$.

4 Asymmetric Auctions

Every auction we have studied so far has been symmetric in the way that all players have the same distribution and the same interval of values, so that all players draw every values equal likely. The scenario where it is an symmetric auction is not that common in practice, that is why I also want to study the asymmetric auctions where players are randomly drawn to have different distributions and values. We want to see how the Nash Equilibrium differ if we have this conditions in different type of auctions.

4.1 Strong and weak bidders[3]

When we talk about asymmetric auctions it is common to refer to strong and weak bidders, such that a strong bidder dominated the weak bidder with the values, for example

- The strong bidder has a shifted distribution such that $F_S \sim Uni[2,3]$ and $F_W \sim Uni[0,1]$ where F_S is the strong bidders distribution and F_W is the weak bidders distribution.
- The strong bidder has a stretched distribution such that $F_S \sim Uni[0,2]$ and $F_W \sim Uni[0,1]$.

4.1.1 Equilibrium for strong and weak bidders

If we let b_S and b_W be the equilibrium bid functions, then we have the problem

- For the **strong bidder** as
$$\max_b F_W(b_W^{-1}(b))(v_S - b)$$
which have the first-order condition
$$\frac{f_W(b_W^{-1}(b))}{F_W(b_W^{-1}(b))}(b_W^{-1})'(b) - \frac{1}{v_S - b}$$
and if we set this equal to 0 at $v_S = b_S^{-1}(b)$ we get
$$\frac{f_W(b_W^{-1}(b))}{F_W(b_W^{-1}(b))}(b_W^{-1})'(b) - \frac{1}{b_S^{-1}(b) - b}$$
- For the **weak bidder** as
$$\max_b F_S(b_S^{-1}(b))(v_W - b)$$
which gives the first-order condition
$$\frac{f_S(b_S^{-1}(b))}{F_S(b_S^{-1}(b))}(b_S^{-1})'(b) - \frac{1}{v_W - b}$$

and if we set this equal to 0 at $v_W = b_W^{-1}(b)$ we get

$$\frac{f_S(b_S^{-1}(b))}{F_S(b_S^{-1}(b))}(b_S^{-1})'(b) - \frac{1}{b_W^{-1}(b) - b}$$

Theorem 4.1.1. *Suppose F_S conditionally first-order stochastically dominates F_W, such that $F_S(x) \leq F_W(x)$[2]. Then comparing a first price auction and a second price auction, both who are uniform distributed,*

- *Every type of strong bidder prefers the second price auction since the expected payoff is higher in the second price auction for the strong bidder.*
- *Every type of weak bidder prefers the first price auction since the expected payoff is higher in the first price auction for the weak bidder.*

Proof. For this proof $b_S(v)$ and $b_W(v)$ have the same range, so we define a matching function as $m(v) \equiv b_W^{-1}(b_S(v))$ as a weak bidder who bids equal as the strong bidder in the **first price auction**. Since $b_S(v) < b_W(v)$ we know that $m(v) < v$. We calculate the strong bidders expected payoff as

$$E(U(v_i)) = Pr(b_W(v_W) < b)(v - b)$$

and by taking the derivative with respect to v this give us

$$E_v(U(v_i)) = Pr(b_W(v_W) < b)$$

and by replacing $b = b_S(v)$ this give us

$$E_v(U(v_i)) = Pr(b_W(v_W) < b_S(v)) = Pr(v_W < m(v)) = F_W(m(v))$$

since $Pr(v < a) = F(a)$ when uniform distributed.
By the envelope theorem we get

$$V_S^{FP}(v) = \int_{\underline{v}}^{\overline{v}} F_W(m(v))dv$$

For the second price auction, both bidders bid their true value, so

$$E_v(U(v_i)) = Pr(v_W < v) = F_W(v)$$

and so

$$V_S^{SP}(v) = \int_{\underline{v}}^{\overline{v}} F_W(v)dv$$

Since $m(v) < v$ and F_W is strictly increasing, the strong bidder prefers the second price auction. By the exact same logic, the weak bidders expected payoff for the first price auction is

$$V_W^{FP}(v) = \int_{\underline{v}}^{\overline{v}} F_S(m^{-1}(v))ds$$

and for the **second price auction**

$$V_W^{SP}(v) = \int_{\underline{v}}^{\overline{v}} F_S(v)dv$$

Since $m^{-1}(v) > v$ the expected payoff is higher with the first price auction for the weak bidder. □

4.1.2 Revenue Equivalence in Asymmetric Auctions

Proposition 4.1.1. *With asymmetric bidders, the expected revenue in a first price auction may exceed that in an English auction*[1][2].

Example 4.1.1. *Suppose that the weak buyers valuation is distributed as $b_W \sim Uni[0, \frac{1}{1+z}]$ and the strong buyers valuation is distributed as $b_S \sim Uni[0, \frac{1}{1-z}]$. So the strong buyer has a wider interval than the weak buyer.*
If $z = 0$, both $b_W, b_S \sim Uni[0, 1]$ and

$$b_i^{-1}(b) = 2b$$

is buyer $i's$ equilibrium inverse bid function in the first price auction. A buyer with valuation $2b$ has a probability to win

$$Pr(win \mid v_i = 2b) = 2b$$

and the expected payment is therefore

$$Pr(win \mid v_i = 2b)(v_i - b_i) = 2b(2b - b) = 2b^2$$

When z becomes positive, in the English auction the weak buyer with valuation $2b$ wins with probability $2b(1 - z)$ and the expected payment is $2b^2(1 - z)$ In a high-bid auction buyers do not use $b_i^{-1}(b) = 2b$, if they did the strong buyer would outbid the weaker buyer by $\frac{1}{2(1-z)}$ to $\frac{1}{2(1+z)}$, and so can reduce his bid and still win with probability 1. So for equilibrium the strong buyer must reduce his bid as a function of his valuation. A reduction would make the weak buyer to bid more aggressively than with $b_i^{-1}(b) = 2b$, since the strong buyers bids are distributed more densely than before. In equilibrium the weak and strong inverse bid functions are therefore

$$b_W^{-1}(b) = \frac{2b}{1 + z(2b)^2} \quad \text{and} \quad b_S^{-1}(b) = \frac{2b}{1 - z(2b)^2}$$

The cumulative distribution function for the winning bid of the first price auction is

$$F_{FPA}(b) = F_S(b_S^{-1}(b))F_W(b_W^{-1}(b)) = (1 - z)b_S^{-1}(b)(1 + z)b_W^{-1}(b) = \frac{(1 - z^2)(2b)^2}{1 - z^2(2b)^4}$$

For the English auction the second valuation is less than b iff it is not the case that both valuations are higher. So

$$F_{EA}(b) = 1-(1-F_S(b))(1-F_W(b)) = F_S+F_W-F_SF_W = (1-z)b+(1+z)b-(1-z^2)b^2 = 2b-(1-z)b^2$$

The cumulative distribution function in the open auction is increasing in z. If $z = 0$ the two distributions yield the same expected revenue. When $z > 0$ the expected revenue is strictly greater for the first price auction than for the English auction.

4.2 Different length of the interval of values, with uniform distributions

In this case we compare uniform distributions with two players where $p_1 \sim Uni(0,1)$ and $p_2 \sim Uni(0,2)$

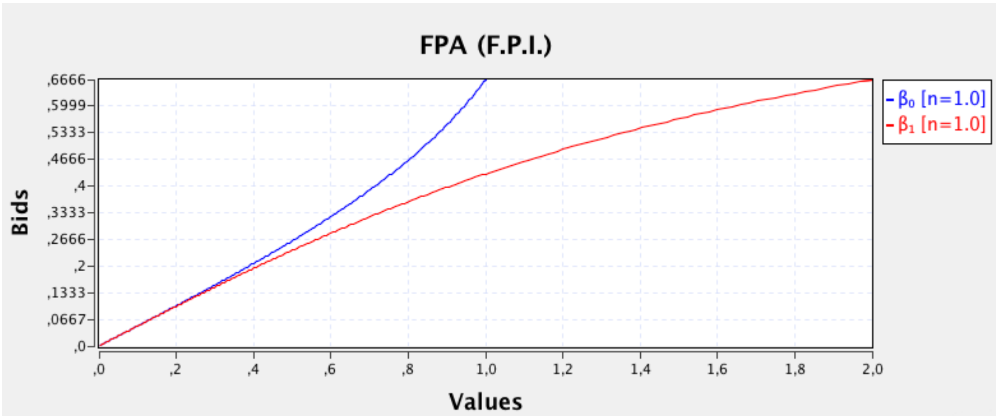

Figure 2: Nash Equilibrium for p_1 and p_2 for any given value at $[0,2]$

So here the Nash Equilibrium would be given by $b_i = \frac{2}{3}$ for both p_1 and p_2 if p_1 value is 1, and p_2 value is 2.

4.3 Same length of interval of values, but different distributions

When using normal and exponential distribution I have scaled these distributions such that the area under the interval is equal to 1.

The normal distribution is scaled to the interval $[0,1]$, so $\frac{C}{\sqrt{2\pi}}e^{-\frac{(x-0.5)^2}{2}}, \quad C > 1$
The exponential distribution is scaled to the interval $[0,1]$, so $Ce^{-x}, \quad C > 1$

4.3.1 Uniform and Normal, both with interval [0,1]

In this first case we compare uniform distribution with normal distribution, both with the interval $[0,1]$.

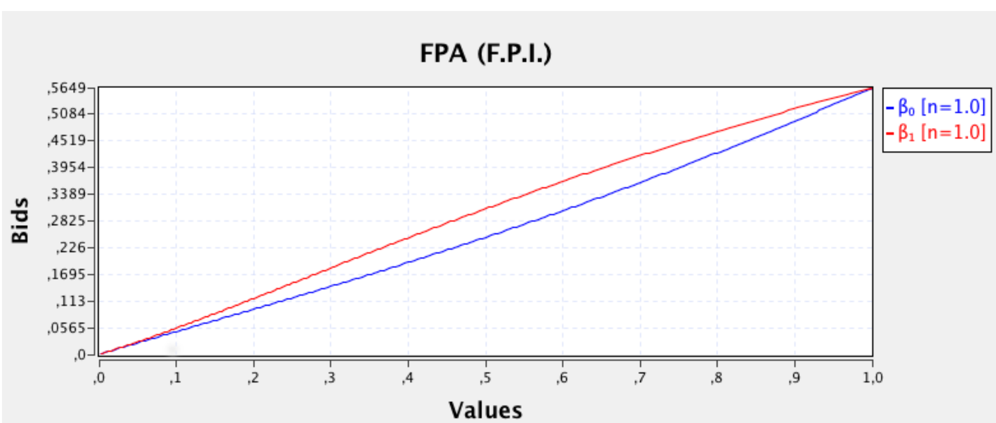

Figure 3: Nash Equilibrium for $p_1 \sim Normal(0.5,1)$ and $p_2 \sim Uni(0,1)$ for any given value at $[0,1]$

4.3.2 Uniform and Exponential, both with interval [0,1]

In this second case we compare uniform distribution with exponential distribution, both with the interval $[0, 1]$.

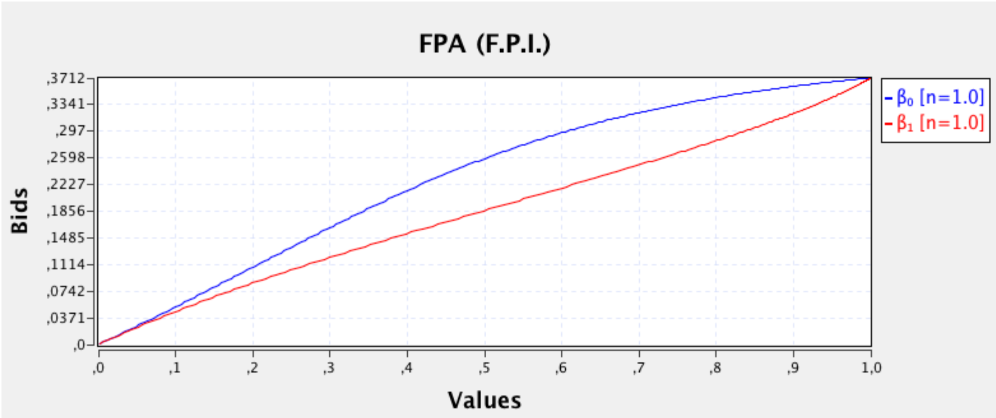

Figure 4: Nash Equilibrium for $p_1 \sim Exp(1)$ and $p_2 \sim Uni(0, 1)$ for any given value at $[0, 1]$

4.4 Different settings with three players

We now just have looked at asymmetric auctions with two players, but how will the equilibrium functions look when we instead uses three players? Here I also have used the scaled distributions for normal and exponential, such that

the normal distribution is scaled to the interval $[0, 1]$, so $\frac{C}{\sqrt{2\pi}} e^{-\frac{(x-0.5)^2}{2}}, \quad C > 1$
the exponential distribution is scaled to the interval $[0, 1]$, so $Ce^{-x}, \quad C > 1$

4.4.1 Uniform, Normal and Exponential, everyone with interval [0,1]

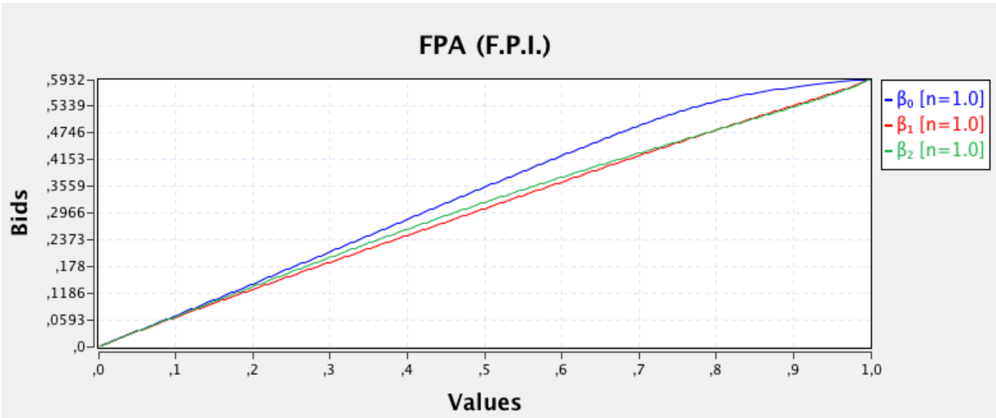

Figure 5: Nash Equilibrium for $p_1 \sim Exp(1)$, $p_2 \sim Normal(0.5, 1)$ and $p_3 \sim Uni(0, 1)$ for any given value at $[0, 1]$

4.4.2 Two players uniform distributed over [0,1] and one player uniform distributed over [0,2]

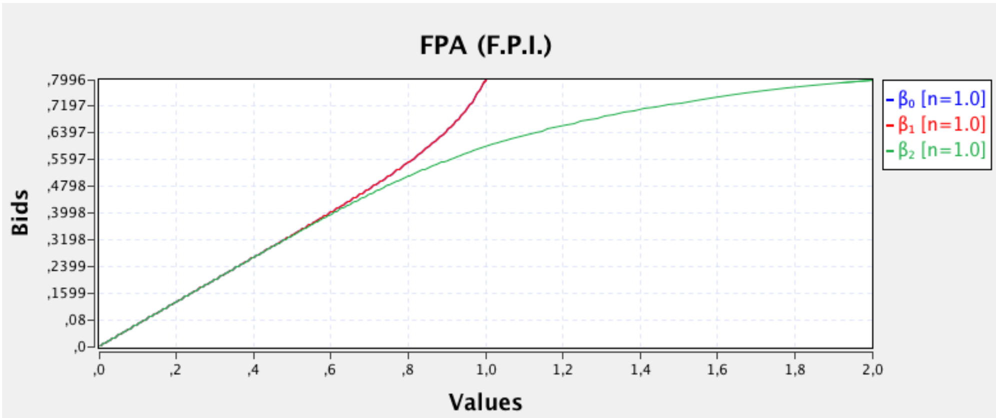

Figure 6: Nash Equilibrium for $p_1 = p_2 \sim Uni(0, 1)$ and $p_3 \sim Uni(0, 1)$ for any given value at $[0, 2]$

4.4.3 Uniform over [0,1], Uniform over [0,2] and Normal over [0,1]

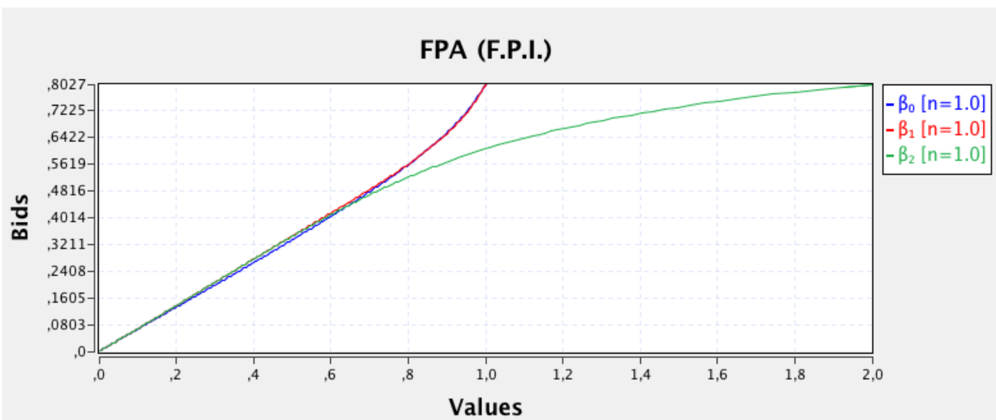

Figure 7: Nash Equilibrium for $p_1 \sim Normal(0.5, 1)$, $p_2 \sim Uni(0, 1)$ and $p_3 \sim Uni(0, 2)$ for any given value at $[0, 2]$

5 References

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