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On Completion of Modules and the Abelian Group Formed from Extension of Modules

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Abstract

Starting with an algebraic structure, such as a group, ring, module or even an algebra it is sometimes desirable to in some manner enlarge the object in question. The algebraic structure in question is therefore embedded in another that contains more elements. With our module embedded in another module an important question arises whether or not any homomorphism can be extended as well such that its domain may be the entire module that the original module is embedded inside. In many instances there exists multiple ways to extend a module but not all of them allow an extension of the domain of a homomorphism. The set of possible extensions can be made into an abelian group and has an intimate relation to the functor Ext *which is gone through in this thesis.*

Acknowledgements

I wish to give my explicit thanks to my family whom has helped making this possible through their support. I also wish to thank my supervisor Veronica for accepting becoming my supervisor and leading me into a subject that showed to be more interesting than I originally had anticipated. Despite a few bumps in the road she has been there and helped.

Thank you all!

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Preface

> The natural numbers are given by god, all else are made by man.
> (Kronecker, paraphrased)

Amongst animals there exists a numerical sense as they are capable to do basic arithmetic but only for small quantities. Man however counts beyond small quantities, a feature seen in all the early civilizations. It started with the natural numbers $\mathbb{N}$ upon which addition and multiplication was used, giving humanity a semiring to work with. This initial construction was expanded upon to form the semi-ring of positive rational numbers, $\mathbb{Q}$, which gave new elements such that multiplication had inverses. When mathematics matured in India this was further expanded such that negative numbers came about and we got proper rings and all of integers, $\mathbb{Z}$, eventually leading to the real numbers $\mathbb{R}$ and in turn $\mathbb{C}$.

Throughout these expansions in history three different methods have been used, localization to go from $\mathbb{Z}$ to $\mathbb{Q}$, completion from $\mathbb{Q}$ to $\mathbb{R}$ and extension from $\mathbb{R}$ to $\mathbb{C}$. All three methods provide means to construct elements previously not present within the given structure but in this thesis we will only focus on the latter two and in particular the extension case. This is because it can be generalized and in the generalization the set of extensions form an abelian group.

Section 1

Preliminaries

This section aims to provide the reader with the basic fundamental results that underpin this thesis. If the lemmas do not contain a proof it is considered too basic to be included but a reference to source material will be provided. The source of notation, definitions and lemmas, unless otherwise stated, is Rotman's book [1].

1.1 Prerequisite knowledge

The reader is assumed to be intimately familiar with groups, rings and module theory. Some knowledge about homology and category theory will be helpful but is not a requirement to understand the thesis. Many of topics brought up can be understood in categorical sense but here we limit it to purely algebraic and even more so remain within the category of all left R-modules, also written as R-**Mod**.

1.2 Diagram notation

Within this thesis there will be a heavy usage of commutative diagrams, diagrams of homomorphisms between modules where the chosen path does not matter for the end result, in order to more easily illustrate and argue for what is to proven, as such here is a list for what the arrows entails. In commutative diagrams we have the following meanings for respective arrow

- $\rightarrow$ represents a general homomorphism
- $\hookrightarrow$ represents a monomorphism
- $\twoheadrightarrow$ represents an epimorphism
- $\hookrightarrow\!\!\!\!\!\rightarrow$ represents an isomorphism
- $\dashrightarrow$ represents a supposed existence of a homomorphism.

1.3 Complexes

A key concept is that of a sequence and as such is what we start with.

1.3.1 Definition. *(Chain complex)*
A chain complex of modules, also called a complex of modules, is a sequence of modules $(\mathbf{M}_i)_{i\in\mathbb{N}}$ and homomorphisms $(\varphi_i)_{i\in\mathbb{N}}$ such that $\varphi_i : \mathbf{M}_{i+1} \to \mathbf{M}_i$ and $\varphi_{i+1} \circ \varphi_i = 0$. This is written as $(\mathbf{M}_\bullet, \varphi_\bullet)$ and is showed in diagram form as

$$\ldots \xrightarrow{\varphi_2} \mathbf{M}_2 \xrightarrow{\varphi_1} \mathbf{M}_1$$

When there is no risk for misunderstanding we may write simply $\mathbf{M}_\bullet$. The sequence is *exact* at $\mathbf{M}_i$ if $\ker \varphi_{i-1} = \operatorname{im} \varphi_i$ and if a sequence is exact at all $\mathbf{M}_i$ then we say that the sequence is an *exact sequence.*

The complex may be of any length, even infinitely long. However a commonly used special case is what is called a *short exact sequence* where $\mathbf{M}_i = 0$ for $i = 1$ and $i > 4$ with those in between being non-zero and is exact at all modules. In these cases we rarely include indexes and usually go for different letters.

1.3.2 Lemma. *For a short exact sequence*

$$0 \to \mathbf{A} \xrightarrow{\alpha} \mathbf{B} \xrightarrow{\gamma} \mathbf{C} \to 0$$

we have that α is a monomorphism and γ an epimorphism

which is why we often write

$$\mathbf{A} \hookrightarrow \mathbf{B} \twoheadrightarrow \mathbf{C}$$

instead of

$$0 \to \mathbf{A} \to \mathbf{B} \to \mathbf{C} \to 0$$

for short exact sequences, or any exact complex that begins and ends with the 0 module. By the nature of exactness we have this lemma.

1.3.3 Lemma. *For a short exact sequence*

$$\mathbf{A} \overset{\imath}{\hookrightarrow} \mathbf{B} \overset{\pi}{\twoheadrightarrow} \mathbf{C}$$

we have that

$$\mathbf{C} \cong \mathbf{B}/\imath(\mathbf{A}).$$

By abuse of notation we do write instead

$$\mathbf{C} \cong \mathbf{B}/\mathbf{A}$$

rather than using proper notation. That is because the image of $\imath$ and $\mathbf{A}$ are isomorphic. As with many structures we wish to have morphisms between them and for chain complexes we do the same.

1.3.4 Definition. *(Chain map)*
For two sequences $(\mathbf{A}_\bullet, \varphi_\bullet)$ and $(\mathbf{B}_\bullet, \epsilon_\bullet)$ we have that a morphism σ from $(\mathbf{A}_\bullet, \varphi_\bullet)$ to $(\mathbf{B}_\bullet, \epsilon_\bullet)$ is a collection of homomorphisms σ_i such that the following diagram commutes for all i.

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \mathbf{A}_{i+1} & \xrightarrow{\varphi_i} & \mathbf{A}_i & \xrightarrow{\varphi_{i-1}} & \mathbf{A}_{i-1} & \longrightarrow & \dots \\ & & \downarrow{\scriptstyle \sigma_{i+1}} & & \downarrow{\scriptstyle \sigma_i} & & \downarrow{\scriptstyle \sigma_{i-1}} & & \\ \dots & \longrightarrow & \mathbf{B}_{i+1} & \xrightarrow{\epsilon_i} & \mathbf{B}_i & \xrightarrow{\epsilon_{i-1}} & \mathbf{B}_{i-1} & \longrightarrow & \dots \end{array}$$

This may often be called a chain map.

By the definition of a chain complex we have that $\varphi_i \circ \varphi_{i+1} = 0$. This entails that $\operatorname{im} \varphi_{i+1} \subseteq \ker \varphi_i$, and as we deal with modules that means we can take their quotient, that is $\ker \varphi_i / \operatorname{im} \varphi_{i+1}$. This justifies the following definition.

1.3.5 Definition. *(Co/homology)*
For a chain complex

$$\dots \to \mathbf{C}_{i+1} \xrightarrow{\varphi_i} \mathbf{C}_i \xrightarrow{\varphi_{i-1}} \mathbf{C}_{i-1} \to \dots$$

we define the nth homology group of it as

$$H_n(\mathbf{C}_\bullet) = \ker \varphi_{n-1} / \operatorname{im} \varphi_n.$$

A dual concept is cohomology. In the homology case the sequence was descending over the index, in cohomology it is ascending instead. Otherwise the concept is identical. That is for a chain

$$\dots \to \mathbf{C}_{i-1} \xrightarrow{\phi_i} \mathbf{C}_i \xrightarrow{\phi_{i+1}} \mathbf{C}_{i+1} \to \dots$$

the cohomology is

$$H^n(\mathbf{C}_\bullet) = \ker \phi_{n+1} / \operatorname{im} \phi_n.$$

Both concepts are brought up for completion, in this thesis the only one that will be used is the cohomology in conjunction with the Hom functor, which will be described later. It should also be noticed that for homologies, along with cohomologies, we have that chain maps induce a homomorphism between the homologies. That is we have if $f : \mathbf{A}_\bullet \to \mathbf{B}_\bullet$ is given, then we have an induced $f^* : H_n(\mathbf{A}_\bullet) \to H_n(\mathbf{B}_\bullet)$. This leads us to the following definition.

1.3.6 Definition. *(Homotopic maps, [1], p.337)*
Let $\mathbf{A}_\bullet$ and $\mathbf{B}_\bullet$ be given complexes along with $f : \mathbf{A}_\bullet \to \mathbf{B}_\bullet$ and $g : \mathbf{A}_\bullet \to \mathbf{B}_\bullet$ being given chain maps. Then f and g are homotopic if there exists a chain map $s = (s_n) : \mathbf{A}_\bullet \to \mathbf{B}_\bullet$ such that we have $s_n : \mathbf{A}_n \to \mathbf{B}_{n+1}$ along with $f_n - g_n = q_n \circ s_n + s_{n-1} \circ p_{n-1}$ with this diagram commuting.

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathbf{A}_{n+1} & \xrightarrow{p_n} & \mathbf{A}_n & \xrightarrow{p_{n-1}} & \mathbf{A}_{n-1} & \longrightarrow & \cdots \\
 & \swarrow^{s_{n+1}} & \downarrow{\scriptstyle f_{n+1}} & \swarrow^{s_n} & \downarrow{\scriptstyle f_n} & \swarrow^{s_{n-1}} & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & \mathbf{B}_{n+1} & \xrightarrow{q_n} & \mathbf{B}_n & \xrightarrow{q_{n-1}} & \mathbf{B}_{n-1} & \longrightarrow & \cdots
\end{array}
$$

In algebraic topology homotopic chain maps, in the topological sense, induce the same map between homological groups. In the algebraic sense here we have the same property for homotopic maps.

1.3.7 Theorem. *Let $f, g : \mathbf{A}_\bullet \to \mathbf{B}_\bullet$ be two homotopic chain maps, then $f^* = g^*$.*

Proof. We only need to prove that $f_n^* = g_n^* : H_n(\mathbf{A}_\bullet) \to H_n(\mathbf{B}_\bullet)$ for any n. If $z \in \ker p_n \subseteq \mathbf{A}_{n+1}$, then clearly $p_n(z) = 0$ and

$$(f_n - g_n)(z) = q_n \circ s_n(z) + s_{n-1} \circ p_{n-1}(z) = q_n \circ s_n(z) \in \mathbf{B}_n$$

which gives us that $f_n(z) - g_n(z) \in \operatorname{im} q_n$ and hence we have $f_n(z) \cong g_n(z)$ in $H_n(\mathbf{B}_\bullet)$. Which means the induced homomorphisms are equivalent. Q.E.D.

The next lemma is another fundamental one that will be used frequently.

1.3.8 Lemma. (Five lemma, [1], p.90)
Let the following diagram be given with exact rows

$$
\begin{array}{ccccccccc}
\mathbf{A} & \longrightarrow & \mathbf{B} & \longrightarrow & \mathbf{C} & \longrightarrow & \mathbf{D} & \longrightarrow & \mathbf{E} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow & & \downarrow \\
\mathbf{A}' & \longrightarrow & \mathbf{B}' & \longrightarrow & \mathbf{C}' & \longrightarrow & \mathbf{D}' & \longrightarrow & \mathbf{E}'
\end{array}
$$

If it is equivalent to

$$
\begin{array}{ccccccccc}
\mathbf{A} & \longrightarrow & \mathbf{B} & \longrightarrow & \mathbf{C} & \longrightarrow & \mathbf{D} & \longrightarrow & \mathbf{E} \\
\twoheaddownarrow & & \hookdownarrow & & \downarrow{\scriptstyle \varphi} & & \hookdownarrow & & \downarrow \\
\mathbf{A}' & \longrightarrow & \mathbf{B}' & \longrightarrow & \mathbf{C}' & \longrightarrow & \mathbf{D}' & \longrightarrow & \mathbf{E}'
\end{array}
$$

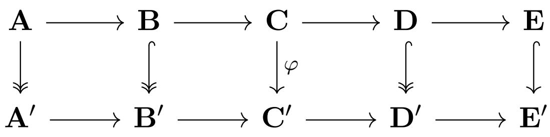
then we have that φ is a monomorphism.

If it is equivalent to

$$
\begin{array}{ccccccccc}
\mathbf{A} & \longrightarrow & \mathbf{B} & \longrightarrow & \mathbf{C} & \longrightarrow & \mathbf{D} & \longrightarrow & \mathbf{E} \\
\downarrow & & \twoheaddownarrow & & \downarrow{\scriptstyle \varphi} & & \twoheaddownarrow & & \hookdownarrow \\
\mathbf{A}' & \longrightarrow & \mathbf{B}' & \longrightarrow & \mathbf{C}' & \longrightarrow & \mathbf{D}' & \longrightarrow & \mathbf{E}'
\end{array}
$$

then φ is an epimorphism.

Consequently if it is equivalent to



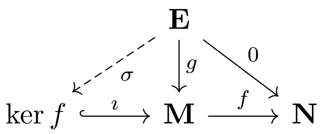
then φ is an isomorphism.

1.4 Kernels

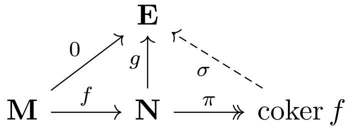
The kernel is a familiar concept, the cokernel is less familiar. The former can be seen as a measure of how injective a homomorphism is while the latter can be seen as a measure of how surjective the homomorphism is. The cokernel of a homomorphism $\varphi : \mathbf{M} \to \mathbf{N}$ is $\operatorname{coker} \varphi = \mathbf{N}/\operatorname{im} \varphi$. A thing to pay attention to is that this is not a sensible definition in all categories. One example is the category of groups at which the image may not be a normal subgroup and hence it requires to normalize the image of the homomorphism. However for these theorems and lemmas we will only remain in $R - \mathbf{Mod}$ and as such all images are submodules. From this it can then be seen that when the cokernel is trivial we have that the function is surjective. However both of them have each an important universal property.

1.4.1 Lemma. *Let $f : \mathbf{M} \to \mathbf{N}$ be given,*

1. *for the kernel of f we have that if for a module $\mathbf{E}$ with a homomorphism $g : \mathbf{E} \to \mathbf{M}$ such that $f \circ g = 0$ then there exists a homomorphism σ such that the following diagram commutes.*



2. *for the cokernel of f we have that if for a module $\mathbf{E}$ with a homomorphism $g : \mathbf{M} \to \mathbf{E}$ such that $g \circ f = 0$ then there exists a homomorphism σ such that the following diagram commutes.*

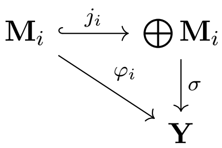


Proof. For the first, we observe that if $f \circ g = 0$ then $\operatorname{im} g \subseteq \ker f$, with $\operatorname{im} g$ being a submodule we have that $\sigma = \imath^{-1} \circ g$.

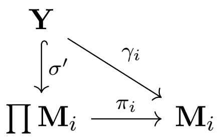
For second, we have that $\operatorname{im} f \subseteq \ker g$ and as such we can define $\sigma = g \circ \pi^{-1}$ and to see it is well defined, let $x = y \in \operatorname{coker} f$. Then we have that $x - y \in \operatorname{im} f \subseteq \ker g$ which gives us that $\sigma(x) = \sigma(y)$ Q.E.D.

Similar to how the kernel and cokernel can be put into commutative diagrams to define their universality, we can do with the direct sum and direct product.

1.4.2 Lemma. *Let* $(\mathbf{M}_i)$ *be a collection of modules and* (φ_i),(γ_i) *two collections of homomorphisms such that* $\varphi_i : \mathbf{M}_i \to \mathbf{Y}$ *and* $\gamma_i : \mathbf{Y} \to \mathbf{M}_i$ *then there exists a* σ *such that for the direct sum we have this diagram to commute*



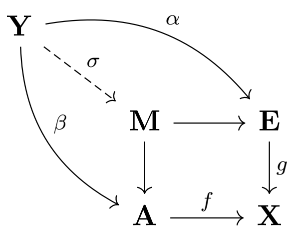
For the direct product we have instead a σ' *such that the following diagram commutes.*



The following two definitions, and their lemmas, will play a major role in the extension section of this thesis. They exist in the category theoretical sense but here we however focus on the module version.

1.4.3 Definition. *(Pullback)*
Let $f : \mathbf{A} \to \mathbf{X}$ and $g : \mathbf{E} \to \mathbf{X}$ be given homomorphisms, then we define the pullback of f and g as $\mathbf{M} = \{a \oplus e \in \mathbf{A} \oplus \mathbf{E} : f(a) = g(e)\}$.

1.4.4 Lemma. *For the pullback of two morphisms there exists a morphism* σ *such that it satisfies the following commutative diagram*



Proof. Clearly the morphisms from **M** are the natural projections,denote them π_E and π_A. Our σ is $\sigma = \beta \oplus \alpha$ and we naturally see that

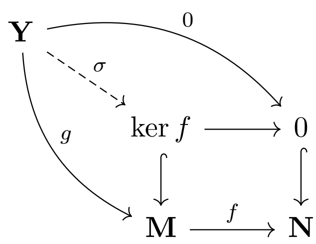
$$g \circ \pi_E \circ \sigma = g \circ \pi_E \circ (\beta \oplus \alpha) = g \circ \alpha$$

and

$$f \circ \pi_A \circ \sigma = f \circ \pi_A \circ (\beta \oplus \alpha) = f \circ \beta$$

which gives us that it is commutative. Q.E.D.

1.4.5 Corollary. *For a given homomorphisms $f : \mathbf{M} \to \mathbf{N}$ we have the kernel being a pullback of f and $s : 0 \to \mathbf{N}$ such that we get the diagram*



1.4.6 Lemma. *Let the diagram below be given*

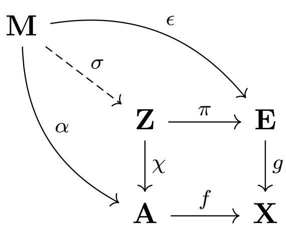
$$\begin{array}{ccc} \mathbf{M} & \xrightarrow{\epsilon} & \mathbf{E} \\ \downarrow \alpha & & \downarrow g \\ \mathbf{A} & \xrightarrow{f} & \mathbf{X} \end{array}$$

Then **M** *is a pullback if and only if the sequence*

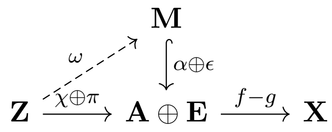
$$\mathbf{M} \overset{\alpha\oplus\epsilon}{\hookrightarrow} \mathbf{A} \oplus \mathbf{E} \overset{f-g}{\twoheadrightarrow} \mathbf{X}$$

is exact.

Proof. We need to show that the universal properties of kernel and pullback for $g - f$ overlap. By definition for any pullback **Z** of f and g, it will induce this diagram



This gives us that that $(f - g) \circ (\chi \oplus \pi) = 0$ and by the universal property of the kernel we have that



such that

$$(\alpha \oplus \epsilon) \circ \omega = \alpha \circ \omega \oplus \epsilon \circ \omega = \chi \oplus \pi.$$

But we also have that $\alpha \oplus \epsilon = (\chi \oplus \pi) \circ \sigma$ from the pullback diagram at which we get

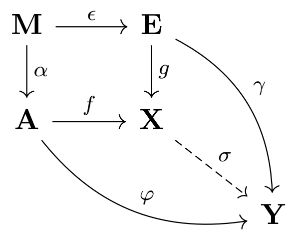
$$\chi \oplus \pi = (\alpha \oplus \epsilon) \circ \omega = (\chi \oplus \pi) \circ \sigma \circ \omega$$

and we have that $\sigma \circ \omega = \mathrm{id}$ and similarly do we get that $\omega \circ \sigma = \mathrm{id}$. Hence they are isomorphisms and with pullbacks and kernels being unique our statement is true. Q.E.D.

Next do we introduce the dual concept of a pullback.

1.4.7 Definition. *(Pushout)*
Let $\alpha : \mathbf{M} \to \mathbf{A}$ and $\epsilon : \mathbf{M} \to \mathbf{E}$ be given homomorphisms, then we define the pushout of α and β as the quotient of $\mathbf{A} \oplus \mathbf{E}$ and $\mathbf{N} = \{\alpha(n) \oplus -\epsilon(n) : n \in \mathbf{M}\}$. That is $\mathbf{X} = (\mathbf{A} \oplus \mathbf{E})/\mathbf{N}$ is our pushout.

1.4.8 Lemma. *For the pushout of two morphisms there exists a morphism σ such that it satisfies the following commutative diagram*

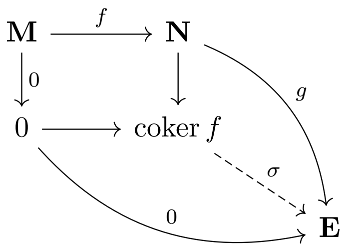


Proof. Clearly the morphisms to $\mathbf{X}$ are the natural injections combined with the surjections of quotients. We define $\sigma(a \oplus e + \mathbf{N}) = \varphi(a) + \gamma(e)$. It is well-defined as if $x \oplus y = x' \oplus y'$ we have $(x - x') \oplus (y - y') \in \mathbf{N}$ and as such we have $(x - x') \oplus (y - y') = \alpha(n) \oplus -\epsilon(n)$ some $n \in \mathbf{M}$. We get then

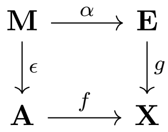
$$\begin{aligned} \sigma(x \oplus y) - \sigma(x' \oplus y') &= \sigma((x - x') \oplus (y - y')) \\ &= \sigma(\alpha(n) \oplus -\epsilon(n)) \\ &= \varphi \circ \alpha(n) - \gamma \circ \epsilon(n) \\ &= 0 \end{aligned}$$

So our σ is well-defined and satisfies our condition. Q.E.D.

1.4.9 Corollary. *The cokernel of $f : \mathbf{M} \to \mathbf{N}$ is a pushout of f and $\imath : \mathbf{M} \to 0$ and gives this diagram*



1.4.10 Lemma. *Let the diagram below be given*

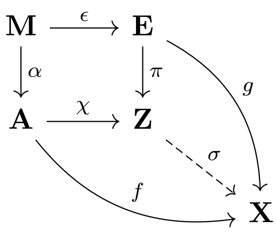


Then $\mathbf{X}$ *is a pushout if and only if the sequence*

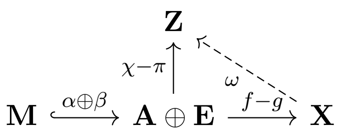
$$\mathbf{M} \xhookrightarrow{\alpha \oplus \epsilon} \mathbf{A} \oplus \mathbf{E} \xrightarrow{f-g} \mathbf{X}$$

is exact.

Proof. We must demonstrate that the universal property of cokernel of $\alpha \oplus \epsilon$ overlaps with the one of pushout. By definition for any pushout $\mathbf{Z}$ of α and ϵ, it will induce this diagram



This gives us that $(\alpha \oplus \beta) \circ (\chi - \pi) = 0$ and by the universal property of the cokernel we have that



such that

$$\chi - \pi = \omega \circ (f - g).$$

But we also have that $f - g = \sigma \circ (\chi - \pi)$ from the pushout diagram at which we get

$$\chi - \pi = \omega \circ \sigma \circ (\chi - \pi)$$

and we have that $\omega \circ \sigma = \text{id}$ and similarly do we get that $\sigma \circ \omega = \text{id}$. Hence they are isomorphisms and with pushout and cokernels being unique our statement is true. Q.E.D.

1.5 Functors

An ordinary function takes elements of a set to another set, while a homomorphism is a function that operates on algebraic structures and preserves the structure. We can define a "function" that is higher in abstraction and rather than operating on elements it operates on structures, that is takes a module to a module.

1.5.1 Definition. *(Functor)*
A functor F is a "function" from $R-\mathbf{Mod}$ to itself where it accepts entire modules and module homomorphisms such that for a module $\mathbf{M}$ and homomorphisms $\varphi : \mathbf{M} \to \mathbf{N}$ and $\sigma : \mathbf{N} \to \mathbf{A}$ we have that $F(\mathbf{M}) \in R-\mathbf{Mod}$ and $F(\varphi) : F(M) \to F(\mathbf{N})$ along with $F(\sigma \circ \varphi) = F(\sigma) \circ F(\varphi)$. This is often called a *covariant functor* and if the second condition on composition is replaced by $F(\sigma \circ \varphi) = F(\varphi) \circ F(\sigma)$ then it is a *contravariant.*

Normally a functor is defined in category theory much broader to be from one category to another, however we stay within just a single category of left R-modules. As mentioned before the concept of exact sequences will play a major role in this thesis and as such it is of importance how a functor will treat these.

1.5.2 Definition. Let the functor F be given and the following short exact sequence

$$0 \to \mathbf{A} \to \mathbf{B} \to \mathbf{C} \to 0$$

The functor F is left exact if when applied to our sequence we get

$$0 \to F(\mathbf{A}) \to F(\mathbf{B}) \to F(\mathbf{C})$$

is exact, and F called right exact if

$$F(\mathbf{A}) \to F(\mathbf{B}) \to F(\mathbf{C}) \to 0$$

is exact and finally, F is exact if

$$0 \to F(\mathbf{A}) \to F(\mathbf{B}) \to F(\mathbf{C}) \to 0$$

is exact.

This naturally entails that a left exact functor preserves monomorphisms and right exact functors preserves epimorphisms. This is seen by for a monomorphism we have the following short exact sequence

$$0 \to \mathbf{A} \overset{f}{\hookrightarrow} \mathbf{B} \twoheadrightarrow \mathbf{B}/\operatorname{im} f \to 0$$

and equivalently for an epimorphism

$$0 \to \ker g \hookrightarrow \mathbf{A} \xrightarrow{g} \mathbf{B} \to 0$$

and the rest follows. A functor is also said to preserve direct sum, or direct product, if

$$F(\bigoplus \mathbf{M}_i) = \bigoplus F(\mathbf{M}_i)$$

and

$$F(\prod \mathbf{M}_i) = \prod F(\mathbf{M}_i)$$

respectively. The main functor, besides Ext which we will get to later, we will work is the Hom functor and denotes the functor that takes a module to the module of all homomorphisms between it and another module. That is, $\mathrm{Hom}(\mathbf{A}, \mathbf{B})$ is the module of all homomorphisms with image in $\mathbf{B}$ and the domain being $\mathbf{A}$. For it we can define two functors, $\mathcal{F}(\mathbf{X}) = \mathrm{Hom}(\mathbf{X}, \mathbf{B})$ and $\mathcal{F}'(\mathbf{X}) = \mathrm{Hom}(\mathbf{B}, \mathbf{X})$ for a given $\mathbf{B}$. The former one is a covariant while the latter is contravariant as a functor. Let $f : \mathbf{A} \to \mathbf{X}$ and $g : \mathbf{X} \to \mathbf{A}$ be given, then $f^* = \mathcal{F}(f) = - \circ f$ and $g^* = \mathcal{F}'(g) = g \circ -$. These two are functions such that $f^* : \mathrm{Hom}(\mathbf{X}, \mathbf{B}) \to \mathrm{Hom}(\mathbf{A}, \mathbf{B})$ and $g^* : \mathrm{Hom}(\mathbf{B}, \mathbf{X}) \to \mathrm{Hom}(\mathbf{B}, \mathbf{A})$. We can therefore view f^* as a function that changes the domain of homomorphisms from $\mathbf{X}$ to $\mathbf{A}$ while g^* changes the image from within $\mathbf{X}$ to within $\mathbf{A}$. Hom is also what is called an additive functor, which means we have for $f, g : \mathbf{Y} \to \mathbf{X}$ that $\mathcal{F}(f + g) = \mathcal{F}(f) + \mathcal{F}(g)$. This can be seen as $f + g \in \mathrm{Hom}(\mathbf{Y}, \mathbf{X})$ and for any homomorphism $h : \mathbf{X} \to \mathbf{B}$ we have

$$\mathcal{F}(f + g) = h \circ (f + g) = h \circ f + h \circ g = \mathcal{F}(f) + \mathcal{F}(g)$$

and similarly for the contravariant version. This is again a phenomenon that holds only true for us as we are dealing with **R-Mod**, in the category of groups this is no longer true.

Section 2

Completion

As discussed before one of the methods to which expanding a ring, or in our case a module, is through the method of completion. This has typically been done in the analysis part of mathematics through the means of a metric. This process can be analogously done in algebra with a method that can give rise to a metric, and in turn a topology. Within this section we will deviate slightly and focus primarily on groups in general, not necessarily abelian unless explicitly stated. The reason for it is that the general case group underpins most algebraic structures and is hence the most generic. The definitions, lemmas and theorems here can easily be extended to include the richer structures.

2.1 Cauchy sequences

Consider the rational numbers $\mathbb{Q}$ with the ordinary norm $\|\cdot\|$. The completion of this field with respect to the norm utilizes Cauchy sequences in order to construct new elements. In abstract algebra we do not need to rely on a norm function $\|\cdot\| : \mathbf{G} \to \mathbb{R}$. Just like the metric underpins the Cauchy sequence for metric spaces we have a construct that underpins Cauchy sequences in algebraic structures, called filtrations.

2.1.1 Definition. *(Filtration, [3])*
Let $\mathbf{A}$ be a group and $\mathbf{A}_i$ be normal subgroups such that $\mathbf{A}_0 = \mathbf{A}$ and $\mathbf{A}_i \subseteq \mathbf{A}_j$ whenever $i \geq j$. This is then a *filtration* of $\mathbf{A}$ and gives us the chain

$$\mathbf{A}_0 \supseteq \mathbf{A}_1 \supseteq \mathbf{A}_2 \supseteq \dots$$

We may denote the filtration of $\mathbf{A}$ as $\mathbf{A}_\bullet$.

A filtration can therefore be viewed as it "filters" elements within our original group $\mathbf{A}$ such that for any element, not e, there comes a point where it is not included any further into the descending chain, similar to how real filters captures particles as they pass through the filter. An example we are familiar with is this one.

2.1.2 Example. Let $\mathbf{G} = \mathbb{Z}$ and $\mathbf{G}_i = p^i\mathbb{Z}$. We have then that $\mathbf{G}_i \supseteq \mathbf{G}_{i+1}$ as $p^{i+1}\mathbb{Z} = p(p^i\mathbb{Z}) \subseteq p^i\mathbb{Z}$. As mentioned before, for any $n \in \mathbb{Z}$ there exists an i such that $n \notin \mathbf{G}_i$ and consequently all the following normal subgroups. This filtration makes up the base for what is called p-adic numbers, which we will return to at a later part.

From this example we can see that a group can have many different filtrations, just like a metric space can have many metrics on it. Filtration is crucial to give a good definition of an algebraic Cauchy sequence.

2.1.3 Definition. *(Cauchy sequence, [3] p.52)*
Let $\mathbf{A}$ be a group and $\mathbf{A}_\bullet$ a filtration of $\mathbf{A}$. A sequence $(x_i)_{i\in\mathbb{N}}$, with $x_i \in \mathbf{A}$ is a Cauchy sequence if and only if for a given $m \in \mathbb{N}$ there exists an N such that for all $i, j > N$ we have that $x_i \cdot x_j^{-1} \in \mathbf{A}_m$. We denote the set of all Cauchy sequences as $\mathcal{C}(\mathbf{A}, \mathbf{A}_\bullet)$. If the filtration is understood from context we may simply write $\mathcal{C}_\mathbf{A}$ or just $\mathcal{C}$ if context allows us to.

A sequence is a null-sequence if for any given m we have an N such that for $i > N$ that $x_i \in \mathbf{A}_m$. The set of all null-sequences is denoted $\mathcal{C}_0(\mathbf{A}, \mathbf{A}_\bullet)$ and again, if context allows us we will write $\mathcal{C}_0$.

The parallels between the algebraic Cauchy sequence and the analytical one is fairly evident. While the analytical one depends on a given metric, the algebraic depends on a filtration. The dependence on distance between elements in the sequence being less than a given ϵ after a given point is replaced with the difference of elements are within the normal subgroup after a given point. This is in particular easy to see when a norm is used instead of a more generic metric as we then have

$$|x_i - x_j| < \epsilon \iff x_i - x_j \in \mathbf{A}_m$$

Naturally we wish to secure that this does indeed form a group and that the null-sequences form a normal subgroup.

2.1.4 Proposition. *Let* $\mathbf{A}$ *be a group and* $\mathbf{A}_\bullet$ *a filtration of it. Then* $\mathcal{C}_\mathbf{A}$ *is a group with* $(x_i) \cdot (y_i) = (x_i \cdot y_i)$ *and* $\mathcal{C}_0$ *is a normal subgroup.*

Proof. We want to check that $(x_i \cdot y_i)$ is a Cauchy sequence. For a given m we have that $i, j > N_x$ implies that $x_i x_j^{-1} \in \mathbf{A}_m$ and similarly for N_y. Then let $N = \max(N_x, N_y)$ and $i, j > N$

$$\begin{aligned} x_i y_i (x_j y_j)^{-1} &= x_i \overbrace{y_i y_j^{-1}}^{\in \mathbf{A}_m} x_j^{-1} \\ &= x_i y_i y_j^{-1} x_i^{-1} x_i x_j^{-1} \\ &= \overbrace{x_i y_i y_j^{-1} x_i^{-1}}^{\in \mathbf{A}_m} \underbrace{x_i x_j^{-1}}_{\in \mathbf{A}_m} \\ &\in \mathbf{A}_m \end{aligned}$$

Where in the first line we use the shoe-sock theorem and note from definition that $y_i y_j^{-1} \in \mathbf{M}$. In the third line we notice that $x_i y_i y_j^{-1} x_i^{-1} \in \mathbf{A}_m$, because $\mathbf{A}_m$ is normal and we already have that $y_i y_j^{-1} \in \mathbf{A}_m$. The inverse of (x_i) is (x_i^{-1}) as we have

$$(x_i) \cdot (x_i^{-1}) = (x_i x_i^{-1}) = (e)$$

and by nature of subgroups we have $e \in \mathbf{A}_i$. So it is closed with identity being (e) and hence is a proper group. For $\mathcal{C}_0$ we know it is non-empty as $(e) \in \mathcal{C}_0$ and let $(x_i), (y_i) \in \mathcal{C}_0$. Like before we have $N = \max(N_x, N_y)$ and $j > N$. Then we have $x_j \in \mathbf{A}_m$ and $y_j \in \mathbf{A}_m$, but as $\mathbf{A}_m$ is a subgroup we have $y_j^{-1} \in \mathbf{A}_m$ also and we have that $x_j y_j^{-1} \in \mathbf{A}_m$ by that $\mathbf{A}_m$ is closed, so it is a subgroup. For normality we let $(x_i) \in \mathcal{C}_\mathbf{A}$ and $(n_i) \in \mathcal{C}_0$. We then have, as before with $N = \max(N_n, N_x)$ and $j > N$, that $x_j n_j x_j^{-1} \in \mathbf{A}_m$ because we have that $n_j \in \mathbf{A}_m$ already and $\mathbf{A}_m$ is normal. Q.E.D.

While we are working with groups in general here we will use the addition notation hence forth. While it is normally reserved for abelian groups we will abuse it here as the thesis still primarily focuses on modules for the later parts. Now we finally define the completion the same way as it is done in analysis.

2.1.5 Definition. Let $\mathbf{A}$ be a group and $\mathbf{A}_\bullet$ a filtration, then we define the *completion of* $\mathbf{A}$ *with respect to* $\mathbf{A}_\bullet$ as the quotient structure

$$\widehat{\mathbf{A}} = \mathcal{C}_\mathbf{A} / \mathcal{C}_0$$

When the filtration is evident from context we will just call it the completion of $\mathbf{A}$. If there exists an isomorphism $\mathbf{A} \hookrightarrow\!\!\!\!\rightarrow \widehat{\mathbf{A}}$ then $\mathbf{A}$ is said to be complete.

This is identical to how it is in analysis and as such it shares many of the same properties of ordinary completion. However, to prove them we will require to acquire another tool in order to prove them easily which will come in the next section. There is a common example for the completion of a group, namely the p-adic integers.

2.1.6 Example. Let $\mathbf{G} = \mathbb{Z}$ be given. Then we have the normal subgroups $\mathbf{G}_i = p^i\mathbb{Z}$ which gives

$$\mathbb{Z} \supseteq p\mathbb{Z} \supseteq p^2\mathbb{Z} \supseteq p^3\mathbb{Z} \supseteq \dots$$

The completion according to this filtration is called the p-adic completion of integers where instead of a sequence being written as $(a_1, a_2, \dots)$ it is written as $\dots b_3b_2b_1$ where $b_i = a_i + p\mathbb{Z} \in \mathbb{Z}/p\mathbb{Z}$. This can be generalized further and depend on ideals amongst others.

A thing to notice is that $\mathbf{A}$ and $\mathbf{B}$ are groups along with a homomorphism $f : \mathbf{A} \to \mathbf{B}$. Then if $\mathbf{B}_\bullet$ is a filtration of $\mathbf{B}$, this will induce a filtration on $\mathbf{A}$ by having $\mathbf{A}_\bullet = f^{-1}(\mathbf{B}_\bullet)$ with $\mathbf{A}_m = f^{-1}(\mathbf{B}_m)$. This is because the inverse image of a normal subgroup is a a normal subgroup. This also means it induces a completion on $\mathbf{A}$ which also induces a homomorphism $f^* : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$ with $f((a_i)) = (f(a_i))$. As discussed previously we have the algebraic concept of Cauchy sequences being derived from the analytical one. However we will see in the next proposition that an algebraic completion also induces an analytical one with a metric.

2.1.7 Proposition. *Let $\mathbf{A}$ be a group and $\mathbf{A}_\bullet$ its filtration. We define then for $x, y \in \mathbf{A}$ the number $k = \sup\{m : x - y \in \mathbf{A}_m\}$ and $d(x, y) = p^{-k}$. Our d is then a metric.*

Proof. We clearly have that $d(x, y) \geq 0$ and $d(x, y) = 0$ implies $x = y$. For symmetry we observe that

$$y - x = -(x - y) \in \mathbf{A}_k$$

for which ever k it fits into by them being closed substructures. For subadditivity we have $\sup\{m : x - z \in \mathbf{A}_m\}$

$$x - z = x - y + y - z = (x - y) + (y - z)$$

Which gives that $x - y \in \mathbf{A}_k$ and $y - z \in \mathbf{A}_k$ and the rest follows. Q.E.D.

A concern in analysis when it came to Cauchy sequences, or a converging sequence in general, was whether or not the sequence would converge to an element within the space from which the elements making up the sequence originates. As seen with rational numbers this was not always the case. Spaces that had the property of sequences being unable to converge to elements outside of the given space would come to be considered as complete as there was no method of "going outside" using completion or localization when it came to real numbers and rational numbers. The process of creating spaces of Cauchy sequences would generate spaces which in turn always turned out to be complete and hence became known as completion. In the next section it will be shown that the algebraic usage of

Cauchy sequences does the same. We finish this section with a proof that the sequences matches up with any of module, ring or algebra as well as group.

2.1.8 Proposition. *Let* $\mathbf{A}$ *be an algebraic structure and* $\mathbf{A}_\bullet$ *a filtration of it. Then*

1. *If* $\mathbf{A}$ *is a ring (***not** *necessarily commutative) then* $\mathcal{C}_{\mathbf{A}}$ *is a ring and* $\mathcal{C}_0$ *is an ideal*

2. *If* $\mathbf{A}$ *is an* R*-module then* $\mathcal{C}_{\mathbf{A}}$ *is a* R*-module and* $\mathcal{C}_0$ *is a submodule.*

Proof. For 1 we want to check that $(x_i \cdot y_i)$ is a Cauchy sequence. For a given m we have that $i, j > N_x$ implies that $x_i x_j^{-1} \in \mathbf{A}_m$, and similarly for N_y. Then let $N = \max(N_x, N_y)$ and $i, j > N$

$$\begin{aligned} x_i y_i (x_j y_j)^{-1} &= x_i \overbrace{y_i y_j^{-1}}^{\in \mathbf{A}_m} x_j^{-1} \\ &= x_i y_i y_j^{-1} x_i^{-1} x_i x_j^{-1} \\ &= \overbrace{x_i y_i y_j^{-1} x_i^{-1}}^{\in \mathbf{A}_m} x_i x_j^{-1} \\ &\in \mathbf{A}_m \end{aligned}$$

For ring we have already shown for addition, as it is an abelian group and all subgroups of abelian groups are normal. We need to show that multiplication componentwise is closed and that $\mathcal{C}_0$ is an ideal. This is fairly simple as we again set $N = \max(N_x, N_y)$ for our (x_i), (y_i) and m. Then we have that for $j > N$ that $x_j, y_j \in \mathbf{A}_m$ and therefore, as $\mathbf{A}_i \lhd \mathbf{A}$, we have that $x_j y_j \in \mathbf{A}_m$. For $\mathcal{C}_0$ we have that for $(n_i) \in \mathcal{C}_0$, $(x_i) \in \mathcal{C}$ and with $j > N = \max(N_n, N_x)$ $n_j \in \mathbf{A}_m$. As $\mathbf{A}_m$ is an ideal we have that $a n_j \in \mathbf{A}_m$ for any $a \in \mathbf{A}$ and as such we get $x_i n_i \in \mathbf{A}_m$, same for right side.

The addition for modules is done as with addition so all is left is closed under our component wise R-action. Let $(x_i) \in \mathcal{C}$ and $r \in \mathbf{R}$. We have for any m and N such that for $i, j > N$ that $x_i - x_j \in \mathbf{A}_m$, that means as $\mathbf{A}_m$ is a submodule that $r x_i - r x_j = r(x_i - x_j) \in \mathbf{A}_m$ and as such we have $r(x_i) = (r x_i) \in \mathcal{C}$. Q.E.D.

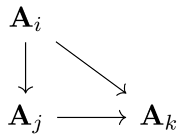
2.2 Inverse limit

Normally we can easily, through direct product, construct another group that is composed of the previous ones, in that case even infinitely many. However, as we discussed before we often work on sequences $(\mathbf{M}_\bullet, \varphi_\bullet)$ and as such this has an inherent structure which is depended on the groups internal relation to each other with respect to the homomorphisms. We wish therefore to have a method to combine these groups in a manner that respects the structure imposed by the homomorphisms, while like with the direct product that if there are homomorphisms from $\mathbf{A}$ to all $\mathbf{M}_i$ then there will be one into our constructed groups. This leads us to the inverse limit which first requires the inverse system.

2.2.1 Definition. *(Inverse system, [1] Section 5.2)*
Let

$$\ldots \mathbf{A}_3 \overset{\rightarrow}{\rightarrow} [\varphi_2]\mathbf{A}_2 \xrightarrow{\varphi_1} \mathbf{A}_1 \xrightarrow{\varphi_0} \mathbf{A}_0$$

be a sequence of groups and morphisms with the property that



commutes whenever $k < j < i$. This is then an *inverse system* and φ_i are called *transition morphisms*. We will denote it as $(\mathbf{A}_\bullet, \varphi_\bullet)$.

Some mathematicians have the morphisms being $\varphi_{ij} : \mathbf{A}_j \to \mathbf{A}_i$ with $i < j$ and the same diagram as above commuting, but not necessarily a chain, although from the commutative diagram iteratively applied we will see it forms a chain inherently. We will however have that for $i < j$ that

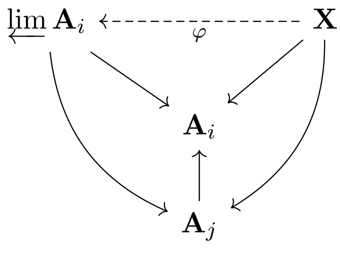
$$\varphi_{ij} = \varphi_i \circ \varphi_{i+1} \circ \ldots \circ \varphi_j$$

Such that our notation coincide and for ease. Now we can do the inverse limit.

2.2.2 Definition. *(Inverse limit, [1] Section 5.2)*
Let $(\mathbf{A}_\bullet, \varphi_\bullet)$ be an inverse system. We define then the *inverse (or projective) limit*, denoted $\varprojlim_n \mathbf{A}_n$, as a subset of $\prod_n \mathbf{A}_n$. We have that a sequence $(a_i) \in \varprojlim \mathbf{A}_n$ if $\varphi_i(a_i) = a_{i-1}$ for all i.

$$\varprojlim \mathbf{A}_n = \left\{ (a_i) \in \prod \mathbf{A}_n : \varphi_i(a_i) = a_{i-1} \right\}$$

This means that for the inverse limit there exists a morphism $\varphi : \mathbf{X} \to \varprojlim \mathbf{A}_i$ such that this diagram commutes for $j > i$



This is evident as if we have $f_i : \mathbf{X} \to \mathbf{A}_i$, which we know exists for all i as the trivial homomorphisms works for any group, then we can define

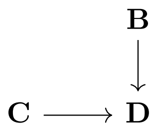
$$\varphi(x) = (f_i(x))_{i\in\mathbb{N}}$$

We have here that the inverse limit is dependent on the set of morphisms and as such one should always write

$$\varprojlim(\mathbf{A}_n, \varphi_n)$$

However, when the morphism is evident from context it will be omitted and written as before.

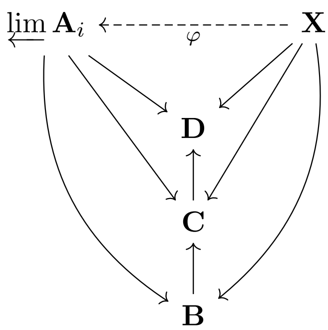
2.2.3 Example. An example of an inverse limit is for $\mathbf{A}_1 = \mathbf{D}$, $\mathbf{A}_2 = \mathbf{C}$ and $\mathbf{A}_3 = \mathbf{B}$ with morphisms $\mathbf{B} \to \mathbf{D}$ and $\mathbf{C} \to \mathbf{D}$ such that we have



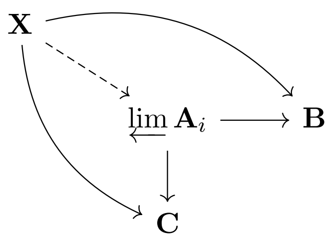
which can be given the sequence

$$\mathbf{B} \to \mathbf{C} \to \mathbf{D}.$$

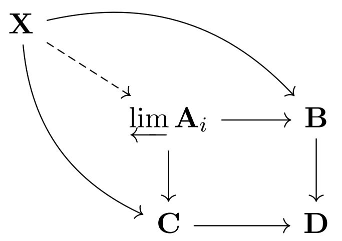
Using the map from before we get



and with a mere rearrangement we get



Adding in our original diagram we get



and we see that this is a pullback of our original morphisms.

From earlier we could see that pullback could be seen as a generalization of a kernel and here we can in turn see that inverse limits can generalize pullbacks to a much larger quantity of groups. From Def. 1.3.4 it can be seen that we have homomorphisms between inverse systems as well. If f is chain map between inverse systems then f induces a homomorphism between the inverse limits. We denote it f^* and have $f^*((a_i)) = (f_i(a_i))$.

2.2.4 Lemma. *Let $f : (\mathbf{A}_\bullet, \alpha_\bullet) \to (\mathbf{B}_\bullet, \beta_\bullet)$ be a homomorphism between inverse systems. The induced homomorphism $f^* : \varprojlim \mathbf{A}_k \to \varprojlim \mathbf{B}_k$ is a monomorphism if each f_n is a monomorphism.*

Proof. We have for the induced homomorphism f^* that $f^*((a_i)) = (f_i(a_i)) = 0 = (0)$ and as such we have that each $\ker f^* = (\ker f_i)$ and with f_i being a monomorphism we get the kernel to be trivial for each and for f^* as well and hence a monomorphism Q.E.D.

And as before short sequences are always of importance and for inverse limits there are relations to them as well.

2.2.5 Proposition. *Let*

$$(\mathbf{A}_\bullet, \alpha_\bullet) \to (\mathbf{B}_\bullet, \beta_\bullet) \to (\mathbf{C}_\bullet, \gamma_\bullet)$$

be a sequence of inverse systems. If for all $n \in \mathbb{N}$ we have

1. *$0 \to \mathbf{A}_n \to \mathbf{B}_n \to \mathbf{C}_n$ is exact, then $0 \to \varprojlim \mathbf{A}_k \to \varprojlim \mathbf{B}_k \to \varprojlim \mathbf{C}_k$ is exact.*

2. *$0 \to \mathbf{A}_n \to \mathbf{B}_n \to \mathbf{C}_n$ is exact and α_n is an epimorphism, then $0 \to \varprojlim \mathbf{A}_k \to \varprojlim \mathbf{B}_k \to \varprojlim \mathbf{C}_k \to 0$ is exact.*

Proof. We have this diagram given from our assumption

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \beta_2} & & \downarrow{\scriptstyle \gamma_2} & & \\
0 & \longrightarrow & \mathbf{A}_2 & \overset{\imath_2}{\hookrightarrow} & \mathbf{B}_2 & \overset{\pi_2}{\twoheadrightarrow} & \mathbf{C}_2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \gamma_1} & & \\
0 & \longrightarrow & \mathbf{A}_1 & \overset{\imath_1}{\hookrightarrow} & \mathbf{B}_1 & \overset{\pi_1}{\twoheadrightarrow} & \mathbf{C}_1 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle \gamma_0} & & \\
0 & \longrightarrow & \mathbf{A}_0 & \overset{\imath_0}{\hookrightarrow} & \mathbf{B}_0 & \overset{\pi_0}{\twoheadrightarrow} & \mathbf{C}_0 & \longrightarrow & 0
\end{array}
$$

We need to show that the induced $\imath^* : \varprojlim \mathbf{A}_k \to \varprojlim \mathbf{B}_k$ is a monomorphism and that it is exact at $\varprojlim \mathbf{B}_k$ for π^*. That $\imath^*$ is a monomorphism follows from the previous lemma as each $\imath_n$ is monomorphic. For exactness at $\varprojlim \mathbf{B}_k$ we notice that $\pi^* \circ \imath^* = (\pi_n \circ \imath_n) = (0) = 0$ and hence $\operatorname{im} \imath^* \subseteq \ker \pi^*$. For the reverse inclusion let $(x_n) \in \ker \pi^*$, this means that $x_n \in \ker \pi_n$ for each $n \in \mathbb{N}$. As each row is exact that means we have $x_n \in \operatorname{im} \imath_n$, as such there exists a_n such that $\imath_n(a_n) = x_n$ which is unique in $\mathbf{A}_n$. We already have that $(x_n) \in \varprojlim \mathbf{B}_n$ by default assumption, as such we have $\beta_{n-1}(x_n) = x_{n-1}$. Commutativity of the diagram gives us that $\beta_{n-1} \circ \imath_n = \imath_{n-1} \circ \alpha_{n-1}$ and we get

$$
\begin{aligned}
\beta_{n-1} \circ \imath_n(a_n) &= \beta_{n-1}(x_n) \\
&= x_{n-1} \\
&= \imath_{n-1}(a_{n-1}).
\end{aligned}
$$

As $\imath_n$ is a monomorphism we must have that $\alpha_{n-1}(a_n) = a_{n-1}$ and we have that $(a_n) \in \varprojlim \mathbf{A}_k$. Hence we get that $(x_n) \in \operatorname{im} \imath^*$ and it is exact at $\varprojlim \mathbf{B}_k$ and therefore is left exact.

For the last part all we need to do is confirm that π^* is an epimorphism. Let $(c_n) \in \varprojlim \mathbf{C}_k$, then for each c_n we have that $\pi_n^{-1}(c_n) = b_n + \mathbf{A}_n$ for some $b_n \in \mathbf{B}$ as we have $\mathbf{C}_n \cong \mathbf{B}_n/\mathbf{A}_n$. This implies that $c_n \cong b_n + \mathbf{A}_n$ and as such we have a bijection $f_n : \mathbf{A}_n \to \pi_n^{-1}(c_n)$, this in turn gives the following diagram

$$
\begin{array}{ccc}
\mathbf{A}_{n+1} & \overset{f_{n+1}}{\hookrightarrow\!\!\!\!\twoheadrightarrow} & \pi_{n+1}^{-1}(c_{n+1}) \\
\downarrow\!\!\downarrow{\scriptstyle \alpha_n} & & \downarrow\!\!\downarrow{\scriptstyle g_n} \\
\mathbf{A}_n & \overset{f_n}{\hookrightarrow\!\!\!\!\twoheadrightarrow} & \pi_n^{-1}(c_n)
\end{array}
$$

where we have $g_n = f_n \circ \alpha_n \circ f_{n+1}^{-1}$, for all $n \in \mathbb{N}$ which makes g_n naturally surjective. *Pay attention to that these are are set functions and not homomorphisms.* This means that that $(\pi_n^{-1}(c_n))$ form an inverse system where each g_n is surjective. Let

$$\mathbf{S}_n = \bigcap_{m \geq n} \beta_{mn}(\pi_m^{-1}(c_m))$$

We know this is non-empty as we have this commutative diagram

$$\begin{array}{ccc} \mathbf{B}_j & \xrightarrow{\pi_j} & \mathbf{C}_j \\ \downarrow{\scriptstyle \beta_{ij}} & & \downarrow{\scriptstyle \gamma_{ij}} \\ \mathbf{B}_i & \xrightarrow{\pi_i} & \mathbf{C}_i \end{array}$$

for all $j > i$. The collection $(\mathbf{S}_n)$ is a subset of $(\pi_n^{-1}(c_n))$ and as such forms an inverse system with the connecting morphisms being restrictions of β_{ij}. This implies that it is a subset of the inverse limit $\varprojlim \mathbf{B}_k$ and as such we have a $(b_n) \in \varprojlim \mathbf{B}_k$ such that $\pi^*((b_n)) = (c_n)$. Q.E.D.

The inverse limit has a close connection to the concept of completion within algebra. As a matter of fact it is isomorphic to the completion with respect to a filtration when one takes the inverse limit of the respective quotient groups.

2.2.6 Theorem. *For a group $\mathbf{A}$ and its filtration $\mathbf{A}_\bullet$, the completion through Cauchy sequences and through inverse limit of $(\mathbf{A}/\mathbf{A}_\bullet, \pi_\bullet)$, with $\pi_i : \mathbf{A}/\mathbf{A}_i \to \mathbf{A}/\mathbf{A}_{i-1}$, are equivalent.*

$$\widehat{\mathbf{A}} \cong \varprojlim \mathbf{A}/\mathbf{A}_m$$

Proof. We first observe that the Cauchy sequence (x_i) in our quotient $\mathbf{A}/\mathbf{A}_m$ will be constant. That is because by for some N we have that $x_i - x_j \in \mathbf{A}_m$ when $i, j > N$. This means that $x_i = x_j$ in $\mathbf{A}/\mathbf{A}_m$. We denote this value as

$$\lim_{\mathbf{A}_m} x_i$$

This is a function $\widehat{\mathbf{A}} \to \mathbf{A}/\mathbf{A}_m$. Next we define our function $f : \widehat{\mathbf{A}} \to \varprojlim \mathbf{A}/\mathbf{A}_m$ as

$$f((x_i)) = \left(\lim_{\mathbf{A}_i} x_j\right)$$

This is evidently a homomorphism. We demonstrate first that it is a monomorphism, by letting $f((x_i)) = 0 = (0, 0, 0, \ldots)$. This means that for each m we have,for some N and $N < j$, that $x_j \in \mathbf{A}_m$, which is the definition of our null sequence and, hence, means that $(x_i) = 0$ and $\ker f = 0$ which makes it a monomorphism.

For epimorphism we let $(y_i) \in \varprojlim \mathbf{A}/\mathbf{A}_m$ be given. For each n we pick an $x_n \in \mathbf{A}$ such that $x_n + \mathbf{A}_n = y_n \in \mathbf{A}/\mathbf{A}_n$. This sequence (x_i) is Cauchy because for $i, j > N = n$ and without loss of generality we have $i < j$ which gives

$$\begin{aligned}\varphi_{ni}(x_i) - \varphi_{nj}(x_j) &= \varphi_{ni}(y_i) - \varphi_{nj}(y_j) \\ &= \varphi_{ni}(y_i) - \varphi_{ni} \circ \varphi_{ij}(y_j) \\ &= \varphi_{ni}(y_i - \varphi_{ij}(y_j)) \\ &= \varphi_{ni}(y_i - y_i) \\ &= \varphi_{ni}(0) \\ &= 0\end{aligned}$$

and as such we have $x_i - x_j \in \mathbf{A}_n$ and hence is a Cauchy sequence, which gives us that it is an epimorphism and in turn an isomorphism. Q.E.D.

With that connection being so strong we can, when the filtration is understood and hence the quotient group $\mathbf{A}/\mathbf{A}_m$ is understood, write $\widehat{\mathbf{A}}$ to mean the inverse limit. Within the next theorem we observe an interesting phenomenon when it comes to the inverse limit and completion. Namely, while the completion through inverse limit enriches our group, the quotient groups of the completion remain isomorphic to the original quotient groups.

2.2.7 Proposition. *Let $\mathbf{A}$ be a group and $\mathbf{A}_\bullet$ its filtration. Let $\widehat{\mathbf{A}}_m$ be the completion of $\mathbf{A}_m \in \mathbf{A}_\bullet$ with respect to the filtration*

$$\mathbf{A}_m \supseteq \mathbf{A}_{m+1} \supseteq \mathbf{A}_{m+2} \supseteq \dots$$

Then we have that

$$\mathbf{A}/\mathbf{A}_m \cong \widehat{\mathbf{A}}/\widehat{\mathbf{A}}_m$$

Proof. Let $k > m$, we have the following short exact sequence

$$0 \to \mathbf{A}_m/\mathbf{A}_k \to \mathbf{A}/\mathbf{A}_k \to \mathbf{A}/\mathbf{A}_m \to 0$$

where we have $\mathbf{A}/\mathbf{A}_m \cong \frac{\mathbf{A}/\mathbf{A}_k}{\mathbf{A}_m/\mathbf{A}_k}$ by isomorphism theorems. The homomorphisms are the natural inclusion and projection. We have that

$$\dots \twoheadrightarrow \mathbf{A}_m/\mathbf{A}_{k+1} \twoheadrightarrow \mathbf{A}_m/\mathbf{A}_k$$

is a surjective system as each morphism is surjective. As the bottom set of elements is always $\mathbf{A}_m$ and hence from Proposition 2.2.5 we have that inverse limit is an exact functor which gives us that

$$0 \to \overbrace{\varprojlim_k \mathbf{A}_m/\mathbf{A}_k}^{\cong \widehat{\mathbf{A}}_m} \to \overbrace{\varprojlim_k \mathbf{A}/\mathbf{A}_k}^{\cong \widehat{\mathbf{A}}} \to \overbrace{\varprojlim_k \mathbf{A}/\mathbf{A}_m}^{= \mathbf{A}/\mathbf{A}_m} \to 0$$

Where the last one remains constant as the inverse limit is over the subsets $\mathbf{A}_k$ and not $\mathbf{A}_m$. This gives us the sequence

$$0 \to \widehat{\mathbf{A}}_m \to \widehat{\mathbf{A}} \to \mathbf{A}/\mathbf{A}_m \to 0$$

which in turn gives us that

$$\widehat{\mathbf{A}}/\widehat{\mathbf{A}}_m \cong \mathbf{A}/\mathbf{A}_m$$

Q.E.D.

With this we can now show that our attempt of a completion by algebraic means will retain the property of the analytical method, namely that completion of a completion will result in nothing new.

2.2.8 Theorem. *For a module* $\mathbf{A}$ *we have that* $\widehat{\mathbf{A}}$ *is complete with respect to the filtration* $\mathbf{A}_\bullet \cong \widehat{\mathbf{A}}_\bullet$*, that is*

$$\widehat{\widehat{\mathbf{A}}} \cong \widehat{\mathbf{A}}$$

Proof. We have the following set of isomorphisms

$$\begin{aligned} \widehat{\widehat{\mathbf{A}}} &\cong \varprojlim \widehat{\mathbf{A}}/\widehat{\mathbf{A}}_k \\ &\cong \varprojlim \mathbf{A}/\mathbf{A}_k \\ &\cong \widehat{\mathbf{A}} \end{aligned}$$

Q.E.D.

This is just like the analytical sense of completion where the completion of a completion of a space is isomorphic to the completion of the space. In the final part of this section we will demonstrate under which circumstances that inverse limit commutes with a functor.

2.2.9 Theorem. *Let* F *be a left exact functor that preserves direct product, then we have*

$$F(\varprojlim \mathbf{M}_i) = \varprojlim F(\mathbf{M}_i)$$

Proof. Let $(\mathbf{M}_\bullet, \varphi_\bullet)$ be our inverse system and $\pi_i : \varprojlim_k \mathbf{M}_k \to \mathbf{M}_i$ be the homomorphisms from the inverse limit to the constituents in. For any $i < j$ we have, by virtue of inverse limit, that $\varphi_{ij} \circ \pi_{j+1} = \pi_i$. Let $\mathbf{D}$ be a module such that there exist homomorphisms $\delta_i : \mathbf{D} \to F(\mathbf{M}_i)$ such that this diagram commutes for all $j > i$

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{\delta_{j+1}} & F(\mathbf{M}_{j+1}) \\ & \searrow^{\delta_i} & \downarrow{\scriptstyle F(\varphi_{ij})} \\ & & F(\mathbf{M}_i) \end{array}$$

We have F preserving direct product

$$F(\prod \mathbf{M}_i) \cong \prod F(\mathbf{M}_i),$$

which gives us that there exists a homomorphism σ such that this diagram commutes

$$\begin{array}{ccc} \mathbf{D} & & \\ \downarrow{\scriptstyle\sigma} & \searrow^{\delta_j} & \\ F(\prod \mathbf{M}_i) & \xrightarrow{F(p_j)} & F(\mathbf{M}_j) \end{array}$$

where p_j are the canonical projections. As we have $\varprojlim \mathbf{M}_i$ being a submodule of $\prod \mathbf{M}_i$ it follows we have a monomorphism $\alpha : \varprojlim \mathbf{M}_i \to \prod \mathbf{M}_i$, the canonical embedding. For this embedding we have that the following diagram commutes

$$\begin{array}{ccc} \varprojlim \mathbf{M}_k & \xrightarrow{\alpha} & \prod \mathbf{M}_k \\ & \searrow^{\pi_j} & \downarrow{\scriptstyle p_j} \\ & & \mathbf{M}_j \end{array}$$

This gives $\pi_j = p_j \circ \alpha$ and applying our functor F onto it we get $F(\pi_j) = F(p_j) \circ F(\alpha)$ for any j. However as F is left exact it preserves monomorphisms and in turn we have $F(\alpha)$ being a monomorphism also. By

$$\begin{array}{ccc} & & \mathbf{D} \\ & \swarrow^{g} & \downarrow{\scriptstyle\sigma} \\ F(\varprojlim \mathbf{M}_k) & \xrightarrow{F(\alpha)} & F(\prod \mathbf{M}_k) \end{array}$$

We can see that the following diagram is equivalent to the above

$$\begin{array}{ccc} & & \mathbf{D} \\ & \swarrow^{g} & \downarrow{\scriptstyle\sigma} \\ F(\varprojlim \mathbf{M}_k) & \xrightarrow{F(\alpha)} & \prod F(\mathbf{M}_k) \end{array}$$

As such we have that, by definition of inverse limit, that $\varprojlim F(\mathbf{M}_k) = \mathbf{D}$. By the definition of inverse limit our $\mathbf{D}$ is the largest structure satisfying the given criteria and as such we have that our g must exist and by the uniqueness of inverse limit our g is an isomorphism. Q.E.D.

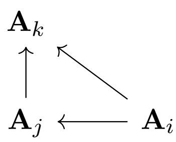
2.3 Direct limit

With the inverse limit we wanted an analogous construction of a sequence of modules to that of direct product while respecting our sequence structure. We can just as easily want the same for our direct sum and that introduces the direct limit. To make this construction though we need to establish a direct system first.

2.3.1 Definition. *(Direct system)*
Let

$$\mathbf{A}_0 \xrightarrow{\varphi_0} \mathbf{A}_1 \xrightarrow{\varphi_1} \mathbf{A}_2 \xrightarrow{\varphi_2} \mathbf{A}_3 \to \dots$$

be a sequence of algebraic structures and morphisms with the property that



commutes whenever $i < j < k$. This is then a *direct system* and φ_i are called *transition morphisms*. We will denote it as $(\mathbf{A}_\bullet, \overrightarrow{\varphi_\bullet})$.

As with the inverse limit we will use φ_{ij} to be the composition of all homomorphisms in the correct order. That is

$$\varphi_{ij} = \varphi_j \circ \varphi_{j-1} \circ \dots \circ \varphi_i.$$

Attention should be paid to the fact that the direction of the homomorphisms are the other way around here than for our inverse limit.

2.3.2 Definition. *(Direct limit)*
Let $(\mathbf{A}_\bullet, \overrightarrow{\varphi_\bullet})$ be an direct system. We define then the *direct (or inductive) limit*, denoted $\varinjlim_n \mathbf{A}_n$, as a quotient of $\bigoplus_n \mathbf{A}_n$ and the submodule $\mathbf{N}$ generated by all sequences

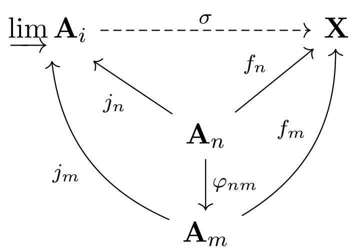
$$(\dots, 0, 0, x, 0, 0, \dots, 0, 0, -\varphi_{ij}(x), 0, 0, \dots)$$

with $i < j$ and x is in the i'th position, φ_{ij} in $j+1$'th for all $x \in \mathbf{A}_i$ and all φ_{ij}.

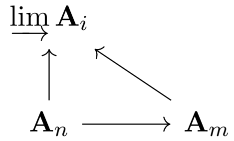
$$\varinjlim \mathbf{A}_i = \bigoplus_i \mathbf{A}_i / \mathbf{N}$$

We will henceforth write x_n for the x in n'th place and as such $(\dots, 0, 0, x_n, 0, 0, \dots) = x_n$, in this notation we have $x_n - \varphi_{nm}(x_n) \in \mathbf{N}$.

2.3.3 Remark. This means that for the direct limit there exists a morphism $\varphi : \varinjlim \mathbf{A}_i \to \mathbf{X}$ such that this diagram commutes for $j > i$



This can be seen as we have for each $\mathbf{A}_n$ the canonical injection into $\bigoplus \mathbf{A}_i$, $\imath_n : \mathbf{A}_n \to \bigoplus \mathbf{A}_i$, and the canonical surjection $\pi : \bigoplus \mathbf{A}_i \to \bigoplus \mathbf{A}_i/\mathbf{N}$, as such we have the morphism $\jmath_n = \pi \circ \imath_n : \mathbf{A}_n \to \varinjlim \mathbf{A}_i$. This clearly commutes the



part of the diagram. This is because $x \in \mathbf{A}_n$ is being sent to x_m through this chain $x \to \varphi_{nm}(x) \to \jmath_m \circ \varphi_{nm}(x) = x_m = \varphi_{nm}(x_n)$. From that we get $\jmath_n(x) = \jmath_m \circ \varphi_{nm}(x)$ and it commutes. We construct our σ as

$$\sigma((x_i)_{i\in\mathbb{N}}) = \sum_i f_i(x_i)$$

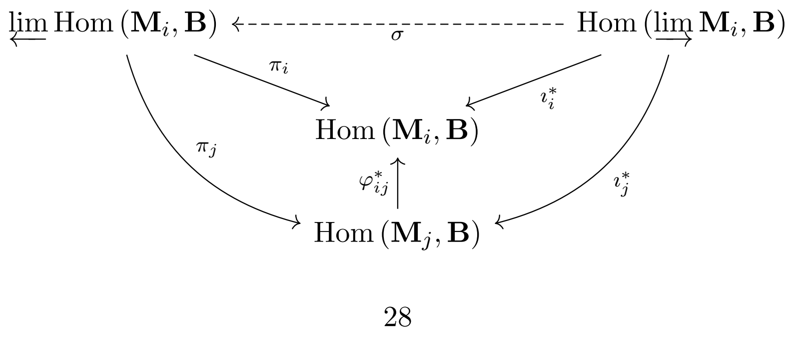
this is well-defined as $\bigoplus \mathbf{A}_i$ disallows infinitely many non-zero terms and we clearly have the commutativity.

From this it can be seen that the direct limit always exists in some form as we can have $f_n = f_m = \sigma = 0$ which will naturally satisfy our conditions. Similar to the inverse limit the direct limit is inherently dependent upon the homomorphisms in our sequence and as such one should write out them to make things as unambiguous as possible. However when there is no risk of confusion we will omit them. There is a very close relation between direct and inverse limit.

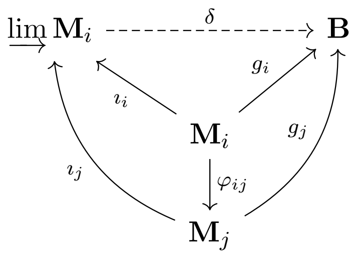
2.3.4 Theorem. *Let* $(\mathbf{M}_\bullet, \varphi_\bullet)$ *be a direct system of modules. Then for every module* $\mathbf{B}$ *we have*

$$\varprojlim \operatorname{Hom}(\mathbf{M}_i, \mathbf{B}) \cong \operatorname{Hom}(\varinjlim \mathbf{M}_i, \mathbf{B})$$

Proof. As $\operatorname{Hom}(-, \mathbf{B})$ is a contravariant functor we have that $(\operatorname{Hom}(\mathbf{M}_\bullet, \mathbf{B}), \varphi_\bullet^*)$ is an inverses system, where $\varphi_i^* = - \circ \varphi_i$ From that we get



with the inverse limit $\varprojlim \operatorname{Hom}(\mathbf{M}_i, \mathbf{B})$ and π_i being the projections. We define our σ as $\sigma(f) = (f \circ \imath_i)$. This is clearly a homomorphism and the diagram commutes as we have $\pi_j \circ \sigma(f) = \pi_j((f \circ \imath_i)) = f \circ \imath_j = \imath_j^*$. We will show that this is an isomorphism. Let $\sigma(f) = 0$, then this gives us that $f \circ \imath_j = 0$ for all j. This means that $\operatorname{im} f \subseteq \ker \imath_j$, but $\imath_j$ is a monomorphism so we have $\operatorname{im} f = 0$ and, hence, $f = 0$ so the kernel is trivial and it is a monomorphism. Next let $g = (g_i) \in \varprojlim \operatorname{Hom}(\mathbf{M}_i, \mathbf{B})$ be given. This gives us that we have $g_i \in \operatorname{Hom}(\mathbf{M}_i, \mathbf{B})$ and that the following diagram commutes therefore



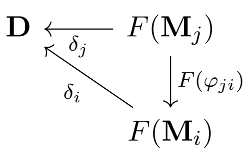
for some δ. Hence, we have $(g_i) = (\delta \circ \imath_i) = \sigma(\delta)$ and it is an epimorphism. Q.E.D.

This means that the two limits are related through the Hom-functor. It was earlier established that inverse limit commuted with functors satisfying certain properties. With direct limit being the dual notion of inverse limit, it seems reasonable to think that the dual criteria would make direct limit commute with a functor.

2.3.5 Theorem. *Let F be a right exact functor that preserves direct sum, then we have*

$$F(\varinjlim \mathbf{M}_i) = \varinjlim F(\mathbf{M}_i)$$

Proof. Let $(\mathbf{M}_\bullet, \varphi_\bullet)$ be our direct system and $\imath_i : \varinjlim_k \mathbf{M}_k \to \mathbf{M}_i$ be the homomorphisms from the the constituents to the direct limit. For any $i < j$ we have, by virtue of direct limit, that $\imath_j \circ \varphi_{ij} = \imath_i$. Let $\mathbf{D}$ be a module such that there exist homomorphisms $\delta_i : F(\mathbf{M}_i) \to \mathbf{D}$ such that this diagram commutes for all $j < i$



From F preserving direct sum we have that that

$$F(\bigoplus \mathbf{M}_i) \cong \bigoplus F(\mathbf{M}_i),$$

which gives us that there exists a homomorphism σ such that this diagram commutes

$$
\begin{array}{ccc}
\mathbf{D} & & \\
\sigma\uparrow & \nwarrow^{\delta_j} & \\
F(\bigoplus \mathbf{M}_i) & \xleftarrow[F(s_j)]{} & F(\mathbf{M}_j)
\end{array}
$$

where s_j are the canonical inclusions. As we have $\varinjlim \mathbf{M}_i$ being a quotient-module of $\bigoplus \mathbf{M}_i$ it follows we have an epimorphism $\alpha : \bigoplus \mathbf{M}_i \to \varinjlim \mathbf{M}_i$, the canonical quotient homomorphism. For this quotient do we have that the following diagram commutes

$$
\begin{array}{ccc}
\varinjlim \mathbf{M}_k & \xleftarrow[\alpha]{} & \bigoplus \mathbf{M}_k \\
 & \nwarrow_{\imath_j} & \uparrow s_j \\
 & & \mathbf{M}_j
\end{array}
$$

This gives $\imath_j = \alpha \circ s_j$ and applying our functor F onto it we get $F(\imath_j) = F(\alpha) \circ F(s_j)$ for any j. However, as F is right exact it preserves epimorphisms and in turn we have $F(\alpha)$ being a epimorphism also. By

$$
\begin{array}{ccc}
 & & \mathbf{D} \\
 & \nearrow^{g} & \sigma\uparrow \\
F(\varprojlim \mathbf{M}_k) & \xleftarrow[F(\alpha)]{} & F(\bigoplus \mathbf{M}_k)
\end{array}
$$

We can see that the following diagram is equivalent to the above

$$
\begin{array}{ccc}
 & & \mathbf{D} \\
 & \nearrow^{g} & \sigma\uparrow \\
F(\varprojlim) & \xleftarrow[F(\alpha)]{} & \bigoplus F(\mathbf{M}_k)
\end{array}
$$

As such we have by definition of direct limit, that $\varinjlim F(\mathbf{M}_k) = \mathbf{D}$. By the definition of direct limit our $\mathbf{D}$ is the smallest structure satisfying the given criteria and as such we have that our g must exist and by the uniqueness of inverse limit our g is an isomorphism. Q.E.D.

Section 3

Extensions

In this section we will take on the main subject of this thesis and a secondary method to expand algebraic structures. While the former section dealt with primarily groups, and could easily be extended to include most known main algebraic structures, in the remaining parts of the thesis we limit ourselves to primarily just modules. So all objects, when not stated, will be left **R**-modules. Extensions are made through an exact sequence of modules, where our original module is embedded through a monomorphism into another module along an exact sequence that ends in another non-trivial module (that has an epimorphism to it). Hence, the nature of the extension is determined by the homomorphisms along the sequence.

3.1 Extensions of length one

Sequences can be of any length and as our initial exploration we will focus on sequences possessing our original module, the module by which we extend it to which we have an epimorphism, and a single one in between that marks the actual extension.

3.1.1 Definition. For a given **B** we say that **E** is an extension of **B** by **A** if we have the following short exact sequence.

$$0 \to \mathbf{B} \xrightarrow{\imath} \mathbf{E} \xrightarrow{\pi} \mathbf{A} \to 0$$

Some authors consider the extension to be of **A** by **B** instead. In this thesis we pick the one where **B** is extended as it can be embedded within **E** through the monomorphism. Next we will observe a few examples of how these extensions are made.

3.1.2 Example. If we have $\mathbf{B} = \mathbb{Z}$, then $\mathbf{E} = \mathbb{Z} \oplus \mathbb{Z}$ then **E** is an extension of **B** by $\mathbf{A} = \mathbb{Z}$. Seen here,

$$\mathbb{Z} \xhookrightarrow{\imath} \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}$$

with $a \oplus b \in \mathbb{Z} \oplus \mathbb{Z}$, with $\imath(a) = a \oplus 0$ and $\pi(a \oplus b) = b$.

In the following example we observe an extension of a ring, but these too can be seen as an $\mathbf{R}$-module.

3.1.3 Example. In general, we let $\mathbf{R}$ be a ring and $\mathbf{E}$ a finitely generated extension of $\mathbf{R}$, that is $\mathbf{E} = \sum_{i=0}^{n} \mathbf{R}x_i$ for $x_i \in \mathbf{E}$ and $x_0 = 1$. Then $\mathbf{E}$ is an extension of $\mathbf{R}$ by the module $\mathbf{I} = \sum_{i=1}^{n} \mathbf{R}x_i$. From this we have the extension

$$\mathbf{R} \overset{\imath}{\hookrightarrow} \mathbf{E} \overset{\pi}{\twoheadrightarrow} \mathbf{I}$$

With $\imath(r) = r + 0 \cdot x_1 + 0 \cdot x_2 + \ldots + 0 \cdot x_n$ and $\pi(\sum_{i=0}^{n} r_i x_i) = \sum_{i=1}^{n}(r_i x_i)$.

3.1.4 Example. If we let $\mathbf{R}$ be a ring and $\mathbf{E}$ a finite type extension of $\mathbf{R}$, that is $\mathbf{E} = \mathbf{R}[x_1, \ldots, x_n]$ for $x_i \in \mathbf{E}$. Then $\mathbf{E}$ is an extension of $\mathbf{R}$ by the submodule $\mathbf{I} = \langle x_1, \ldots, x_n \rangle$.

A module can be extended in more than one way even through using the same epimorphism or final module. However, not all of them will naturally be distinct.

3.1.5 Definition. We define that two extensions of a module $\mathbf{B}$ by $\mathbf{A}$ are equivalent if and only if in the following commutative diagram the homomorphism φ exists

$$\begin{array}{ccccc} \mathbf{B} & \hookrightarrow & \mathbf{E} & \twoheadrightarrow & \mathbf{A} \\ \updownarrow & & \downarrow \varphi & & \updownarrow \\ \mathbf{B} & \hookrightarrow & \mathbf{F} & \twoheadrightarrow & \mathbf{A} \end{array}$$

From the five lemma we have that if φ exists then it is an isomorphism. It is important to pay attention to that the crucial part of the definition is the existence of φ, which is demonstrated in the following example.

3.1.6 Example. The group $\mathbb{Z}_2$ we can extend in two ways by $\mathbb{Z}_2$, either we have

$$\mathbb{Z}_2 \overset{\cdot 2}{\hookrightarrow} \mathbb{Z}_4 \overset{\pi}{\twoheadrightarrow} \mathbb{Z}_2$$

which is easily seen being exact. The other one is

$$\mathbb{Z}_2 \overset{\mathrm{id} \oplus 0}{\hookrightarrow} \mathbb{Z}_2 \oplus \mathbb{Z}_2 \overset{p}{\longrightarrow} \mathbb{Z}_2$$

with $p(a \oplus b) = b$. Clearly, we have

$$\mathbb{Z}_4 \ncong \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

In order to have the diagram commute with φ we must have

$$f(x) = \begin{cases} 0 \to 0 \oplus 0 \\ 1 \to 1 \oplus 1 \text{ or } 0 \oplus 1 \\ 2 \to 1 \oplus 0 \\ 3 \to 0 \oplus 1 \text{ or } 1 \oplus 1 \end{cases}$$

We have then that

$$1 \oplus 0 = f(2) = f(1+1) \neq f(1)+f(1) = 1 \oplus 1+1 \oplus 1 = (1+1) \oplus (1+1) = 0 \oplus 0 = f(0)$$

so φ is not a homomorphism. The existence of a homomorphism is not guaranteed even if a set function can be found.

However there are times where seemingly distinct modules are in fact the same. The following example is a classical one.

3.1.7 Example. Let our extensions of $\mathbb{Z}_q$ by $\mathbb{Z}_p$, with $\gcd(p, q) = 1$, be

$$\mathbb{Z}_q \hookrightarrow \mathbb{Z}_{pq} \twoheadrightarrow \mathbb{Z}_p$$

and

$$\mathbb{Z}_q \hookrightarrow \mathbb{Z}_p \oplus \mathbb{Z}_q \twoheadrightarrow \mathbb{Z}_p$$

From earlier theorems we have that a homomorphism $\varphi : \mathbb{Z}_p \oplus \mathbb{Z}_q \to \mathbb{Z}_{pq}$ does exist and is $\varphi(a \oplus b) = pa + qb$, which is also an isomorphism.

We have that an extension is considered trivial if $\mathbf{E} = \mathbf{A} \oplus \mathbf{B}$ as that extension always exists. However, just because we have $\mathbf{B} \subseteq \mathbf{E}$ it does not necessarily mean that $\mathbf{E}$ is a proper extension of $\mathbf{B}$.

3.1.8 Example. Consider this sequence

$$\mathbb{Z}_2 \xhookrightarrow{\cdot 3} \mathbb{Z}_6 \xrightarrow{p} \mathbb{Z}_2$$

We have that $\mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$ so we have $\mathbb{Z}_2 \subseteq \mathbb{Z}_6$. However it is not exact at $\mathbb{Z}_6$ as $\operatorname{im}(\cdot 3) = \{0, 3\}$ and $\ker p = \{0, 2, 4\}$.

The requirement of commutativity is important in homological algebra in order to the definition of equivalence to make sense. Especially in the light of that it results in an isomorphism where modules are virtually the same anyway. We previously discussed that extensions can be made into not only a group structure, but an abelian group structure and for that we require a binary operation of two extensions to result in a third one. We define the following operation on extensions.

3.1.9 Definition. *(Baer sum)*
Let

$$\mathbf{B} \overset{\imath_e}{\hookrightarrow} \mathbf{E} \overset{\pi_e}{\twoheadrightarrow} \mathbf{A}$$

and

$$\mathbf{B} \overset{\imath_f}{\hookrightarrow} \mathbf{F} \overset{\pi_f}{\twoheadrightarrow} \mathbf{A}$$

be two extensions of the module $\mathbf{B}$ by $\mathbf{A}$. Next define

$$\mathbf{N} = \{\imath_e(b) \oplus -\imath_f(b) : b \in \mathbf{B}\} \subseteq \mathbf{E} \oplus \mathbf{F}$$

and let $\mathbf{X}$ be the pullback of π_e and π_f in the following commutative diagram.

$$\begin{array}{ccc} \mathbf{X} & \longrightarrow & \mathbf{E} \\ \downarrow & & \downarrow{\scriptstyle \pi_e} \\ \mathbf{F} & \xrightarrow{\pi_f} & \mathbf{A} \end{array}$$

This gives us a short exact sequence

$$\mathbf{N} \hookrightarrow \mathbf{X} \twoheadrightarrow \mathbf{Y},$$

where evidently $\mathbf{Y} \cong \mathbf{X}/\mathbf{N}$, and is the Baer sum of extensions $\mathbf{E}$ and $\mathbf{F}$.

We may also express it by $\mathbf{X} = \{e \oplus f \in \mathbf{E} \oplus \mathbf{F} : \pi_e(e) = \pi_f(f)\}$ and $\mathbf{Y} = \mathbf{X}/\mathbf{N}$. As $\mathbf{N}$ is a kernel of $\mathbf{X}$ by virtue of having $\imath_e(\mathbf{B}) = \ker \pi_e$ and $\imath_f(\mathbf{B}) = \ker \pi_f$ we see that $\mathbf{N}$ is a kernel and hence the quotient is well-defined. This will be denoted as $\mathbf{Y} = \mathbf{E} +_B \mathbf{F}$.

For this definition to be meaningful in the sense of being able to be used as the binary operation of a group, then the resulting sequence must be exact.

3.1.10 Proposition. *Let*

$$\mathbf{B} \overset{\imath_e}{\hookrightarrow} \mathbf{E} \overset{\pi_e}{\twoheadrightarrow} \mathbf{A}$$

and

$$\mathbf{B} \overset{\imath_f}{\hookrightarrow} \mathbf{F} \overset{\pi_f}{\twoheadrightarrow} \mathbf{A}$$

be extensions of $\mathbf{B}$ *by* $\mathbf{A}$ *such that* $\mathbf{Y}$ *is the Baer sum of* $\mathbf{E}$ *and* $\mathbf{F}$*. Then we have that*

$$\mathbf{B} \hookrightarrow \mathbf{Y} \twoheadrightarrow \mathbf{A}$$

is exact

Proof. By virtue of the diagram in Def 3.1.9 we have induced homomorphism $\pi^* : \mathbf{Y} \to \mathbf{A}$. We can originally see it from $\mathbf{X} \to \mathbf{A}$, being $\pi^*(e \oplus f) = \pi_e(e) = \pi_f(f)$. However, we can see by the nature of the quotient that it applies on our $\mathbf{Y}$. If we have $e \oplus f = e' \oplus f' \in \mathbf{Y}$ then $(e - e') \oplus (f - f') = \imath_e(b) \oplus \imath_f(b)$ for some $b \in \mathbf{B}$, which gives

$$\pi^*(e \oplus f) - \pi^*(e' \oplus f') = \pi_e(e) - \pi_e(e') = \pi_e(e - e') = \pi_e(\imath_e(b)) = 0$$

Equivalently

$$\pi^*(e \oplus f) - \pi^*(e' \oplus f') = \pi_f(f) - \pi_f(f') = \pi_f(f - f') = \pi_f(\imath_f(b)) = 0$$

hence π^* is well-defined on $\mathbf{Y}$. We also get $\imath^* : \mathbf{B} \to \mathbf{Y}$ by $\imath^*(b) = 0 \oplus \imath_f(b) = \imath_e(b) \oplus 0$.

Now we will check that in

$$0 \to \mathbf{B} \xrightarrow{\imath^*} \mathbf{Y} \xrightarrow{\pi^*} \mathbf{A} \to 0$$

we have that π^* is an epimorphism and $\imath^*$ is a monomorphism as well as the exactness at $\mathbf{Y}$. For π^* we observe that $\operatorname{im} \pi_e = \operatorname{im} \pi_f = \mathbf{A}$ which gives $\operatorname{im} \pi^* = \mathbf{A}$. For $\imath^*$ we have $\ker \imath^* = \ker \imath_e = \ker \imath_f = 0$.

For exactness we have

$$\pi^* \circ \imath^* = \pi^* \circ (\imath_e \oplus 0) = \pi_e \circ \imath_e = 0$$

so $\operatorname{im} \imath^* \subseteq \ker \pi^*$. Next we have that

$$\begin{aligned} \ker \pi^* &\subseteq \ker \pi_e \oplus \ker \pi_f \\ &= \operatorname{im} \imath_e \oplus \operatorname{im} \imath_f \\ &= \operatorname{im} \imath_e \oplus 0 + 0 \oplus \operatorname{im} \imath_f \\ &= \operatorname{im} \imath^* + \operatorname{im} \imath^* \\ &= \operatorname{im} \imath^* \end{aligned}$$

So $\ker \pi^* \subseteq \operatorname{im} \imath^*$, and the equality follows. Q.E.D.

Further we must establish that the Baer sum of modules does not depended upon the choice of representation.

3.1.11 Proposition. *The Baer sum is well-defined.*

Proof. Let $\mathbf{E}' \cong \mathbf{E}$ and $\mathbf{F}' \cong \mathbf{F}$ be extensions of $\mathbf{B}$ by $\mathbf{A}$ with $\mathbf{E} +_B \mathbf{F} = \mathbf{Y}$ and $\mathbf{E}' +_B \mathbf{F}' = \mathbf{Y}'$. Remember from before that we have $\mathbf{N} = \{\imath_e(b) \oplus -\imath_f(b)\}$ and $\mathbf{N}' = \{\imath_{e'}(b) \oplus -\imath_{f'}(b)\}$. However we have the diagrams

$$\begin{array}{ccccc}
\mathbf{B} & \xhookrightarrow{\imath_e} & \mathbf{E} & \twoheadrightarrow & \mathbf{A} \\
\downarrow & & \downarrow \varphi & & \downarrow \\
\mathbf{B} & \xhookrightarrow{\imath_{e'}} & \mathbf{E'} & \twoheadrightarrow & \mathbf{A}
\end{array}$$

and

$$\begin{array}{ccccc}
\mathbf{B} & \xhookrightarrow{\imath_f} & \mathbf{F} & \twoheadrightarrow & \mathbf{A} \\
\downarrow & & \downarrow \phi & & \downarrow \\
\mathbf{B} & \xhookrightarrow{\imath_{f'}} & \mathbf{F'} & \twoheadrightarrow & \mathbf{A}
\end{array}$$

and as such we have $\varphi \circ \imath_e = \imath_{e'}$ and $phi \circ \imath_f = \imath_{f'}$ which gives us

$$\begin{aligned}
\mathbf{N'} &= \{\imath_{e'}(b) \oplus -\imath_{f'}(b)\} \\
&= \{\varphi \circ \imath_e(b) \oplus -\phi \circ \imath_f(b)\} \\
&\cong \{\imath_e(b) \oplus -\imath_f(b)\} \\
&= \mathbf{N}
\end{aligned}$$

For $\mathbf{X}$ we have

$$\begin{array}{ccc}
\mathbf{X} & \longrightarrow & \mathbf{E} \\
\downarrow & & \downarrow \pi_e \\
\mathbf{F} & \xrightarrow{\pi_f} & \mathbf{A}
\end{array}$$

and $\mathbf{X'}$ we have

$$\begin{array}{ccc}
\mathbf{X'} & \longrightarrow & \mathbf{E'} \\
\downarrow & & \downarrow \pi'_e \\
\mathbf{F'} & \xrightarrow{\pi'_f} & \mathbf{A}
\end{array}$$

but we have $\mathbf{E} \cong \mathbf{E'}$ and $\mathbf{F} \cong \mathbf{F'}$ and as we have pushbacks being unique up to isomorphism then $\mathbf{X} \cong \mathbf{X'}$, which gives us then that

$$\mathbf{Y'} = \mathbf{X'}/\mathbf{N'} \cong \mathbf{X}/\mathbf{N} = \mathbf{Y}$$

Q.E.D.

We will explore a few examples of the Bear sum and what it yields.

3.1.12 Example. For $\mathbf{A} = \mathbf{B} = \mathbb{Z}_4$ we have two evident extensions, namely

$$\mathbb{Z}_4 \xhookrightarrow{\cdot 4} \mathbb{Z}_{16} \xrightarrow{\pi} \mathbb{Z}_4$$

and

$$\mathbb{Z}_4 \xhookrightarrow{\mathrm{id} \oplus 0} \mathbb{Z}_4 \oplus \mathbb{Z}_4 \xrightarrow{\pi_2} \mathbb{Z}_4.$$

Using the Baer sum we can find another extension by adding the first one to itself, that is, $\mathbb{Z}_{16} +_B \mathbb{Z}_{16}$. Our pullback $\mathbf{X}$ is

$$\mathbf{X} = \bigcup_{i=0}^{3} (\mathbb{Z}_4 + i) \oplus (\mathbb{Z}_4 + i)$$

and our $\mathbf{N}$ is

$$\mathbf{N} = 4\mathbb{Z}_{16}(1 \oplus -1)$$

We have $|\mathbf{X}| = 64$ and $|\mathbf{N}| = 4$, as such we get $|\mathbf{X}/\mathbf{N}| = 16$ and we can show that our module is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ with the sequence

$$\mathbb{Z}_4 \xhookrightarrow{\mathrm{id} \oplus \cdot 2} \mathbb{Z}_2 \oplus \mathbb{Z}_8 \xrightarrow{a+2b} \mathbb{Z}_4.$$

However, one should pay attention to that while this might yield a central module clearly distinct from any of the original ones, this is not always the case. As we discussed earlier the homomorphisms are playing a key role in the structure of the extension, which the following example demonstrates.

3.1.13 Example. For $\mathbf{A} = \mathbf{B} = \mathbb{Z}_3$ we have two evident extensions, namely

$$\mathbb{Z}_3 \xhookrightarrow{\cdot 3} \mathbb{Z}_9 \xrightarrow{\pi} \mathbb{Z}_3$$

and

$$\mathbb{Z}_3 \xhookrightarrow{\mathrm{id} \oplus 0} \mathbb{Z}_3 \oplus \mathbb{Z}_3 \xrightarrow{\pi_2} \mathbb{Z}_3.$$

Using the Baer sum we can find another extension by adding the first twice, that is, $\mathbb{Z}_9 +_B \mathbb{Z}_9$. Our pullback $\mathbf{X}$ is

$$\mathbf{X} = \bigcup_{i=0}^{2} (\mathbb{Z}_3 + i) \oplus (\mathbb{Z}_3 + i)$$

and our $\mathbf{N}$ is

$$\mathbf{N} = 3\mathbb{Z}_9(1 \oplus -1)$$

We have $|\mathbf{X}| = 27$ and $|\mathbf{N}| = 3$ and hence $|\mathbf{X}/\mathbf{N}| = 9$, which is isomorphic to either $\mathbb{Z}_9$ or $\mathbb{Z}_3 \oplus \mathbb{Z}_3$. Since $1 \oplus 1$ generates our entire module, it is $\mathbb{Z}_9$. It is however not the same extension as we started with as it takes the form of

$$\mathbb{Z}_3 \xhookrightarrow{\cdot 3} \mathbb{Z}_9 \xrightarrow{\cdot 2} \mathbb{Z}_3,$$

while the one we started with is

$$\mathbb{Z}_3 \xhookrightarrow{\cdot 3} \mathbb{Z}_9 \xrightarrow{\cdot 1} \mathbb{Z}_3.$$

For the extension to be equivalent we must have a homomorphism φ in

$$\begin{array}{ccccc}
\mathbb{Z}_3 & \xhookrightarrow{\cdot 3} & \mathbb{Z}_9 & \xrightarrow{\cdot 1} & \mathbb{Z}_3 \\
\big\downarrow & & \Big\downarrow{\scriptstyle \varphi} & & \big\downarrow \\
\mathbb{Z}_3 & \xhookrightarrow{\cdot 3} & \mathbb{Z}_9 & \xrightarrow{\cdot 2} & \mathbb{Z}_3
\end{array}$$

such that the diagram commutes. Here, φ does not exist. Let $p_1 = \cdot 1$ and $p_2 = \cdot 2$ in our diagram. We have $p_1(1) = 1$ and $p_2(2) = 1$ so we must have $\varphi(1) = 2$ which means that $\varphi = \cdot 2$ as $\mathbb{Z}_9$ is cyclic. However, from our left morphisms we have $1 \mapsto 3$ but $\varphi(3) = 6$ and hence it does not commute.

We denote the Baer sum by $+_B$. Usually additional notation is used for commutative binary operations, and fortunately for us it is true here.

3.1.14 Proposition. *The Baer sum of two extensions* $\mathbf{F}$ *and* $\mathbf{E}$ *of* $\mathbf{B}$ *by* $\mathbf{A}$ *is commutative.*

Proof. Let $\varphi : \mathbf{F} \oplus \mathbf{E} \to \mathbf{E} \oplus \mathbf{F}$ be given by $\varphi(a \oplus b) = b \oplus a$. This is clearly an isomorphism. When φ is restricted to the submodules in question we see that

$$\begin{aligned}
\mathbf{Y} &= \mathbf{X}/\mathbf{N} \\
&\cong \mathbf{X}'/\mathbf{N} \\
&= \mathbf{Y}'
\end{aligned}$$

Q.E.D.

3.1.15 Proposition. *The Baer sum of extensions is associative.*

$$\mathbf{E} +_B (\mathbf{F} +_B \mathbf{G}) = (\mathbf{E} +_B \mathbf{F}) +_B \mathbf{G}$$

Proof. Let $\mathbf{E}$,$\mathbf{F}$ and $\mathbf{G}$ be extensions of $\mathbf{B}$ by $\mathbf{A}$

$$\mathbf{B} \xhookrightarrow{\imath_e} \mathbf{E} \xrightarrow{\pi_e} \mathbf{A},$$

$$\mathbf{B} \xhookrightarrow{\imath_f} \mathbf{F} \xrightarrow{\pi_f} \mathbf{A},$$

$$\mathbf{B} \xhookrightarrow{\imath_g} \mathbf{G} \xrightarrow{\pi_g} \mathbf{A}.$$

We set $\mathbf{X}_e$ as the pullback of $\mathbf{F}$ and $\mathbf{G}$, and $\mathbf{X}_g$ is the pullback of $\mathbf{F}$ and $\mathbf{E}$. That is

$$\begin{array}{ccc}
\mathbf{X}_e & \longrightarrow & \mathbf{G} \\
\big\downarrow & & \big\downarrow{\scriptstyle \pi_g} \\
\mathbf{F} & \xrightarrow{\pi_f} & \mathbf{A}
\end{array}$$

and

$$\begin{array}{ccc} \mathbf{X}_g & \longrightarrow & \mathbf{E} \\ \downarrow & & \downarrow{\scriptstyle \pi_e} \\ \mathbf{F} & \xrightarrow{\pi_f} & \mathbf{A} \end{array}$$

Note that we have $\mathbf{X}_e = \{(f, g) \in \mathbf{F} \oplus \mathbf{G} : \pi_f(f) = \pi_g(g)\}$ and similarly for the second pullback. Next we let $\mathbf{X}$ and $\mathbf{X'}$ be pullbacks of $\mathbf{E}$ and $\mathbf{X}_e$:

$$\begin{array}{ccc} \mathbf{X} & \longrightarrow & \mathbf{X}_e \\ \downarrow & & \downarrow{\scriptstyle \pi_e^*} \\ \mathbf{E} & \xrightarrow{\pi_e} & \mathbf{A} \end{array}$$

respectively $\mathbf{G}$ and $\mathbf{X}_g$:

$$\begin{array}{ccc} \mathbf{X'} & \longrightarrow & \mathbf{X}_g \\ \downarrow & & \downarrow{\scriptstyle \pi_g^*} \\ \mathbf{G} & \xrightarrow{\pi_g} & \mathbf{A} \end{array}$$

We define our $\mathbf{N}$ as following

$$\mathbf{N}_e = \{\imath_g(b) \oplus -\imath_f(b) : b \in \mathbf{B}\}$$

$$\mathbf{N}_g = \{\imath_e(b) \oplus -\imath_f(b) : b \in \mathbf{B}\}$$

$$\mathbf{N} = \{\imath_e(b) \oplus -\imath_e^*(b) : b \in \mathbf{B}\}$$

$$\mathbf{N'} = \{\imath_g^*(b) \oplus -\imath_g(b) : b \in \mathbf{B}\}$$

In such way we have these sequences

$$\mathbf{B} \xhookrightarrow{\imath_e^*} \mathbf{X}_e/\mathbf{N}_e \xrightarrow{\pi_e^*} \mathbf{A}$$

$$\mathbf{B} \xhookrightarrow{\imath_g^*} \mathbf{X}_g/\mathbf{N}_g \xrightarrow{\pi_g^*} \mathbf{A}$$

$$\mathbf{B} \xhookrightarrow{\imath^*} \mathbf{X}/\mathbf{N} \xrightarrow{\pi^*} \mathbf{A}$$

$$\mathbf{B} \xhookrightarrow{\imath_*} \mathbf{X'}/\mathbf{N'} \xrightarrow{\pi_*} \mathbf{A}$$

From this we then have that

$$\begin{aligned} \mathbf{X} &= \{e \oplus f \oplus g : \pi_e(e) = \pi_e^*(f \oplus g)\} \\ &= \{e \oplus f \oplus g : \pi_e(e) = \pi_f(f) = \pi_g(g)\} \\ &= \{e \oplus f \oplus g : \pi_g^*(e \oplus f) = \pi_g(g)\} \\ &= \mathbf{X'} \end{aligned}$$

and also

$$\mathbf{N} = \{\imath_e(b) \oplus -\imath_e^*(b)\} = \{\imath_e(b) \oplus 0 \oplus -\imath_g(b)\} = \{\imath_g^*(b) \oplus -\imath_g(b)\} = \mathbf{N'}$$

and, hence, we have

$$\mathbf{E} + (\mathbf{F} + \mathbf{G}) = \mathbf{X}/\mathbf{N} = \mathbf{X}'/\mathbf{N}' = (\mathbf{E} + \mathbf{F}) + \mathbf{G}$$

Q.E.D.

We had discussed before that the trivial extension is always present in the set of all extensions. It is also the neutral element in the Baer sum.

3.1.16 Proposition. *The extension* $\mathbf{A} \oplus \mathbf{B}$ *of* $\mathbf{B}$ *by* $\mathbf{A}$ *is neutral with respect to the Baer sum, that is*

$$\mathbf{A} \oplus \mathbf{B} +_B \mathbf{E} = \mathbf{E}.$$

Proof. Let $\mathbf{X}$ be the pullback in

$$\begin{array}{ccc} \mathbf{X} & \longrightarrow & \mathbf{A} \oplus \mathbf{B} \\ \downarrow & & \downarrow{\scriptstyle \pi_+} \\ \mathbf{E} & \xrightarrow{\pi_e} & \mathbf{A} \end{array}$$

and $\mathbf{N} = \{\imath_e(b) \oplus \imath_+(b)\}$ where $\imath_+$ and π_+ are the canonical injection and projection. We then have

$$\begin{aligned} \mathbf{X} &= \{e \oplus (a \oplus b) : \pi_e(e) = \pi_+(a \oplus b)\} \\ &= \{e \oplus a \oplus b : \pi_e(e) = a\} \\ &= \{(e \oplus a) \oplus b : \pi_e(e) = a\} \\ &\cong \{e \oplus b\} \\ &= \mathbf{E} \oplus \mathbf{B} \end{aligned}$$

and

$$\begin{aligned} \mathbf{N} &= \{\imath_e(b) \oplus \imath_+(b)\} \\ &= \{\imath_e(b) \oplus 0 \oplus b\} \\ &\cong \{\imath_e(b) \oplus 0 \oplus b\} \\ &\cong \{\imath_e(b) \oplus b\} \\ &\cong \{b\} \\ &\cong \mathbf{B} \\ &\cong 0 \oplus \mathbf{B} \end{aligned}$$

which gives us that

$$\begin{aligned}\mathbf{X}/\mathbf{N} &\cong \frac{\mathbf{E}\oplus\mathbf{B}}{0\oplus\mathbf{B}}\\ &\cong \frac{\mathbf{E}}{0}\oplus\frac{\mathbf{B}}{\mathbf{B}}\\ &\cong \frac{\mathbf{E}}{0}\oplus 0\\ &\cong \mathbf{E}\end{aligned}$$

. Q.E.D.

3.1.17 Proposition. *For any extension* $\mathbf{E}$ *of* $\mathbf{B}$ *by* $\mathbf{A}$ *there exists an extension* $\mathbf{F}$ *such that* $\mathbf{E} +_B \mathbf{F} = \mathbf{A}\oplus\mathbf{B}$.

Proof. Let $\mathbf{E}$ be the extension

$$\mathbf{B} \overset{\imath}{\hookrightarrow} \mathbf{E} \overset{\pi}{\twoheadrightarrow} \mathbf{A},$$

then we define our extension $\mathbf{F}$ as

$$\mathbf{B} \overset{\imath}{\hookrightarrow} \mathbf{F} \overset{-\pi}{\twoheadrightarrow} \mathbf{A}.$$

We set our pullback $\mathbf{X}$

$$\mathbf{X} = \{x \oplus x' : \pi(x) = -\pi(x')\}$$

which means that $\pi(x)+\pi(x') = \pi(x+x') = 0$. In turn we have $x+x' \in \ker\pi$ which means that there exists $b \in \mathbf{B}$ such that $\imath(b) = x + x'$.

From this we define $\varphi : \mathbf{X}/\mathbf{N} \to \mathbf{A}\oplus\mathbf{B}$ by $\varphi(x, x') = \pi(x) \oplus \imath^{-1}(x + x')$, where $\mathbf{N} = \{\imath(b) \oplus -\imath(b) : b \in \mathbf{B}\}$. This is evidently a homomorphism, it is well defined as if $x \oplus x' = y \oplus y' \in \mathbf{X}/\mathbf{N}$, then $(x \oplus x') - (y \oplus y') = (x - y) \oplus (x' - y') \in \mathbf{N}$. Inherently we get $\imath(b) = x - y = -(x' - y')$, and in turn $x \oplus x' - y \oplus y' = \imath(b) \oplus -\imath(b)$. This gives us then

$$\begin{aligned}\varphi(x \oplus x') - \varphi(y \oplus y') &= \varphi(x \oplus x' - y \oplus y')\\ &= \varphi(\imath(b) \oplus -\imath(b))\\ &= \pi \circ \imath(b) \oplus \imath^{-1}(\imath(b) - \imath(b))\\ &= 0(b) \oplus \imath^{-1}(0)\\ &= 0\end{aligned}$$

Hence, it is well defined and a proper homomorphism. By the five lemma φ is an isomorphism and the proof is concluded. Q.E.D.

With all these lemmas we see that all extensions of length one under the Baer sum form an abelian group. While it is evident that π and $-\pi$ has the same kernel and hence both give an extension, we may wonder if

$$\mathbf{B} \overset{\imath}{\hookrightarrow} \mathbf{E} \overset{\pi}{\twoheadrightarrow} \mathbf{A}$$

and

$$\mathbf{B} \overset{\imath}{\hookrightarrow} \mathbf{E} \overset{-\pi}{\twoheadrightarrow} \mathbf{A}$$

are the same. In most cases the extensions are not the same as we saw in Example 3.1.13.

3.2 Extensions of length n

Throughout this section of the thesis, MacLane [4] is the source of definitions, notation, and lemmas. We have now explained an extensions of length one, that is between $\mathbf{A}$ and $\mathbf{B}$ we have a single module, by which we are extending $\mathbf{A}$. However, we may want to extend our definition of extension so that an exact sequence can be of any length. That is if we extend $\mathbf{B}$ by $\mathbf{A}$ we define the extension to be the exact sequence

$$\mathbb{E} : \mathbf{B} \hookrightarrow \mathbf{E}_n \to \mathbf{E}_{n-1} \to \ldots \to \mathbf{E}_1 \twoheadrightarrow \mathbf{A}$$

This is an extension of length n. We will denote the extension as $\mathbb{E}$ where the letters coincide with the choice of modules between $\mathbf{A}$ and $\mathbf{B}$. An example of an extension of length two is:

3.2.1 Example. Let $\mathbf{F}$ be a field and $\mathbf{F}[x]$ the ring of polynomials, then we have $\mathbf{F}[x]/(x^2)$ be the quotient ring of it. We then get the following extension of $\mathbf{F}[x]$ modules.

$$\mathbf{F} \overset{\cdot x}{\hookrightarrow} \mathbf{F}[x]/(x^2) \overset{\cdot x}{\longrightarrow} \mathbf{F}[x]/(x^2) \overset{\pi}{\twoheadrightarrow} \mathbf{F}$$

where π is the quotient homomorphism.

3.2.1 Equivalence

To generalize equivalence from length one we will first focus on the case of length two. If we have two extensions where one begins with the module that the other ends with, we can splice them together using what is known as *Yoneda composition.*

3.2.2 Definition. *(Yoneda composition)*
Let

$$\mathbb{E} : \mathbf{N} \overset{n}{\hookrightarrow} \mathbf{E} \overset{f}{\twoheadrightarrow} \mathbf{B}$$

and

$$\mathbb{F} : \mathbf{B} \overset{g}{\hookrightarrow} \mathbf{F} \overset{a}{\twoheadrightarrow} \mathbf{A}$$

be two extensions. Noting that the former ends where the latter starts, then the *Yoneda composition*, or just composition, is

$$\mathbb{E} \circ \mathbb{F} : \mathbf{N} \overset{n}{\hookrightarrow} \mathbf{E} \overset{g \circ f}{\longrightarrow} \mathbf{F} \overset{a}{\twoheadrightarrow} \mathbf{A}$$

This composition of extensions may not be exact at the modules **E** or **F**. However, we have that they are exact at both instances making it an exact sequence.

3.2.3 Proposition. *For two extensions*

$$\mathbb{E} : \mathbf{B} \overset{\beta}{\hookrightarrow} \mathbf{E} \overset{f}{\twoheadrightarrow} \mathbf{N}$$

$$\mathbb{F} : \mathbf{N} \overset{g}{\hookrightarrow} \mathbf{F} \overset{\alpha}{\twoheadrightarrow} \mathbf{A}$$

we have that their composition

$$\mathbb{E} \circ \mathbb{F} : \mathbf{N} \overset{\beta}{\hookrightarrow} \mathbf{E} \overset{g \circ f}{\longrightarrow} \mathbf{F} \overset{\alpha}{\twoheadrightarrow} \mathbf{A}$$

is exact.

Proof. We note by the epimorphisms that we have $\operatorname{im} f = \mathbf{N}$ which gives us $\operatorname{im} g \circ f = \operatorname{im} g$. Equivalently from the monomorphisms $\ker g = 0$ implies that $\ker g \circ f = \ker f$. These together give us that $\operatorname{im} \beta = \ker f = \ker g \circ f$ so it is exact at **E**. At **F** we have $\operatorname{im} g \circ f = \operatorname{im} g = \ker \alpha$ and we are done. Q.E.D.

We should note that this composition, while demonstrated previously in sequences of length one, works for extensions of any length. This process can be done in the opposite manner, that is if we have

$$\mathbf{B} \hookrightarrow \mathbf{E} \overset{\varphi}{\longrightarrow} \mathbf{F} \twoheadrightarrow \mathbf{A}$$

then we can split it into

$$\mathbf{B} \hookrightarrow \mathbf{E} \twoheadrightarrow \operatorname{im} \varphi$$

and

$$\operatorname{im} \varphi \hookrightarrow \mathbf{F} \twoheadrightarrow \mathbf{A}$$

This decomposition can be done ad infinitum. While we could extend our equivalence by using the one length definition for all intermediary modules of an extension, this would be far too restrictive and not yield satisfactory properties and equivalence classes. We shall see next that there is a more natural way to extend the equivalence of extensions. We start this by exploring a few properties of short exact sequences in combination with pushouts and pullbacks discussed earlier.

3.2.4 Lemma. *Let the digram below be given.*

$$\begin{array}{ccc} \mathbf{P} & \xrightarrow{\varphi} & \mathbf{E} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ \mathbf{A} & \xrightarrow{\alpha} & \mathbf{X} \end{array}$$

Then

1. *If* $\mathbf{P}$ *is a pullback and* g *is an epimorphism, then* f *is an epimorphism*
2. *If* $\mathbf{P}$ *is a pullback, then we have* $\ker f \cong \ker g$
3. *If* $\mathbf{X}$ *is a pushout and* f *is a monomorphism, then* g *is an monomorphism*
4. *If* $\mathbf{X}$ *is a pushout, then we have* $\operatorname{coker} f \cong \operatorname{coker} g$

Proof. For the first, let $a \in \mathbf{A}$ be given and we have this diagram

$$\begin{array}{ccc} \mathbf{P} & \xrightarrow{\varphi} & \mathbf{E} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ \mathbf{A} & \xrightarrow{\alpha} & \mathbf{X} \end{array}$$

Next we consider this exact sequence

$$\mathbf{P} \xhookrightarrow{f \oplus \varphi} \mathbf{A} \oplus \mathbf{E} \xrightarrow{\alpha - g} \mathbf{X}$$

As g is an epimorphism we have that there exists an element e such that $g(e) = \alpha(a)$. This in turn gives $e \oplus a \in \ker(\alpha - g)$ and by exactness we have that there exists some $p \in \mathbf{P}$ such that $(f \oplus \varphi)(p) = a \oplus e$ and hence f is epimorphic.

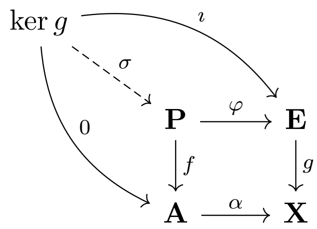
For the second we have the following diagram

$$\begin{array}{ccccc} \ker f & \xhookrightarrow{v} & \mathbf{P} & \xrightarrow{f} & \mathbf{A} \\ \downarrow & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \alpha} \\ \ker f & \xrightarrow{u} & \mathbf{E} & \xrightarrow{g} & \mathbf{X} \end{array}$$

where we have $u = \varphi \circ v$. By definition of a pullback we have that $\mathbf{P} \subseteq \mathbf{A} \oplus \mathbf{E}$ and as such we have the monomorphism $f \oplus \varphi : \mathbf{P} \to \mathbf{A} \oplus \mathbf{E}$. Clearly we have that v is a monomorphism as it is just the natural injection. This gives us that $(f \oplus \varphi) \circ v = (f \circ v) \oplus (\varphi \circ v) = 0 \oplus u$ is a monomorphism and, hence, u is a monomorphism. Next observe that

$$\begin{aligned} 0 &= \alpha \circ f \circ v \\ &= g \circ \varphi \circ v \\ &= u \circ g. \end{aligned}$$

Hence $\operatorname{im} u \subseteq \ker g$. For the reverse inclusion observe that for a pullback the following diagram



commutes for some σ. This in turn gives for $g \circ \imath = 0$ along with $\imath = \varphi \circ \sigma$ and $f \circ \sigma = 0$. That means that $\operatorname{im} \sigma \subseteq \ker f \cong \operatorname{im} u$ as u is a monomorphism. Ergo we have that $\ker f \cong \ker g$.

For the third we are given this commutative diagram.

$$\begin{array}{ccc} \mathbf{P} & \overset{f}{\hookrightarrow} & \mathbf{A} \\ \big\downarrow{\scriptstyle \varphi} & & \big\downarrow{\scriptstyle \alpha} \\ \mathbf{E} & \overset{g}{\longrightarrow} & \mathbf{X} \end{array}$$

Let $e \in \ker g$ be given. We wish to show that $e = 0$. We have the exact sequence

$$\mathbf{P} \overset{f \oplus \varphi}{\hookrightarrow} \mathbf{A} \oplus \mathbf{E} \overset{\alpha - g}{\twoheadrightarrow} \mathbf{X}$$

From our assumption we have $g(e) = 0$ and with $\alpha - g$ being an epimorphism, that there exists some a such that $\alpha(a) - g(e) = 0$. Which means that $a \oplus e \in \ker(\alpha - g) = \operatorname{im}(f \oplus \varphi)$. One of the elements in $\ker \alpha - g$ is $0 \oplus e$ as such it is in the image of $f \oplus \varphi$. This means that for some p we have that $(f \oplus \varphi)(p) = 0 \oplus e$ and with f being monomorphic, we have $p = 0$ and in turn $e = \varphi(p) = 0$. Hence the kernel is trivial and g is a monomorphism.

For the fourth this diagram is given

$$\begin{array}{ccccc} \mathbf{P} & \overset{f}{\longrightarrow} & \mathbf{A} & \overset{u}{\longrightarrow} & \operatorname{coker} g \\ \big\downarrow{\scriptstyle \varphi} & & \big\downarrow{\scriptstyle \alpha} & & \big\downarrow \\ \mathbf{E} & \overset{f}{\longrightarrow} & \mathbf{X} & \overset{\pi}{\twoheadrightarrow} & \operatorname{coker} g \end{array}$$

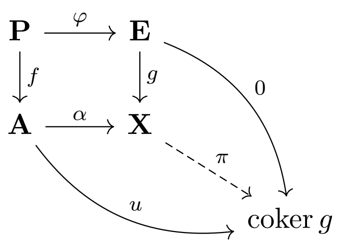
with $u = \pi \circ \alpha$. By definition of pushout we have the epimorphism $\alpha \oplus g : \mathbf{A} \oplus \mathbf{E} \to \mathbf{X}$. Also π is an epimorphism as it is the natural projection of quotients. With composition of epimorphisms being an epimorphism along with

$$\pi \circ (\alpha \oplus g) = \pi \circ \alpha \oplus \pi \circ g = \pi \circ \alpha \oplus 0 = u \oplus 0$$

we have that u is an epimorphism. Next observe that

$$u \circ f = \pi \circ g \circ \varphi$$

which means that $\operatorname{im} f \subseteq \ker u$. By definition of a pushout we have



Let $a \in \ker u$, then we have that $\alpha(a) \in \ker \pi$, by commutativity. We have $\ker \pi = \operatorname{im} g$ and as such we have some $e \in \mathbf{E}$ such that $g(e) = \alpha(a)$. This gives us that $a \oplus e \in \ker(\alpha - g)$. From the exact sequence

$$\mathbf{P} \overset{f\oplus\varphi}{\hookrightarrow} \mathbf{A} \oplus \mathbf{E} \overset{\alpha - g}{\twoheadrightarrow} \mathbf{X}$$

we get that $a \oplus e \in \operatorname{im}(f \oplus \varphi)$ and hence there exists some p such that $f(p) = a$. This implies that $\operatorname{im} f \subseteq \ker u$ and the rest follows. Q.E.D.

In the previous lemma we could see hints of the relation between pushouts, pullbacks and short exact sequences. In the next two lemmas we will see that short exact sequences and commutativity of homomorphisms makes it so that one of the modules in question must be either a pushout or a pullback.

3.2.5 Lemma. *Let the diagram below be given*

$$\begin{array}{ccccc} \mathbf{B} & \overset{f}{\hookrightarrow} & \mathbf{E} & \overset{g}{\twoheadrightarrow} & \mathbf{A} \\ \downarrow & & \downarrow \varphi & & \downarrow \alpha \\ \mathbf{B} & \overset{f'}{\hookrightarrow} & \mathbf{E}' & \overset{g'}{\twoheadrightarrow} & \mathbf{A}' \end{array}$$

Then $\mathbf{E}$ *is a pullback of* $\alpha : \mathbf{A} \to \mathbf{A}'$ *and* $g' : \mathbf{E}' \to \mathbf{A}'$

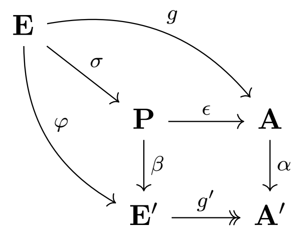
Proof. Let $\mathbf{P}$ be the pullback of $\alpha : \mathbf{A} \to \mathbf{A}'$ and $g' : \mathbf{E}' \to \mathbf{A}'$ such that we have the diagram

$$\begin{array}{ccc} \mathbf{P} & \overset{\epsilon}{\longrightarrow} & \mathbf{A} \\ \downarrow \beta & & \downarrow \alpha \\ \mathbf{E}' & \overset{g'}{\twoheadrightarrow} & \mathbf{A}' \end{array}$$

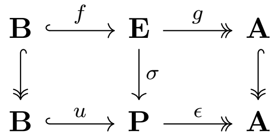
By Lemma 3.2.4 ϵ is an epimorphism and there exists isomorphism between $\ker \epsilon \cong \ker g'$. Hence we have

$$\begin{array}{ccccc} \mathbf{B} & \overset{u}{\hookrightarrow} & \mathbf{P} & \overset{\epsilon}{\twoheadrightarrow} & \mathbf{A} \\ \downarrow & & \downarrow \beta & & \downarrow \alpha \\ \mathbf{B} & \overset{f'}{\hookrightarrow} & \mathbf{E}' & \overset{g'}{\twoheadrightarrow} & \mathbf{A}' \end{array}$$

By the universal property of a pullback, as there exists $\mathbf{E} \xrightarrow{g} \mathbf{A}$ and $\mathbf{E} \xrightarrow{\varphi} \mathbf{E}'$, there must exist a $\sigma : \mathbf{E} \to \mathbf{P}$ such that this diagram commutes



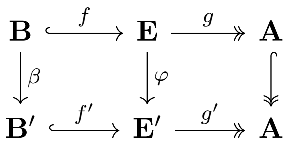
We get the following diagram



at which we get that σ is an isomorphism from the five lemma. Q.E.D.

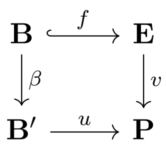
If we denote the upper row in the previous lemma for $\mathbb{E}$ and the bottom row for $\mathbb{E}'$ then we will denote this relation as $\mathbb{E} = \mathbb{E}'\alpha$

3.2.6 Lemma. *Let the digram below be given*

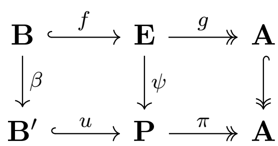


Then $\mathbf{E}'$ *is a pushout of* $\beta : \mathbf{B} \to \mathbf{B}'$ *and* $f : \mathbf{B} \to \mathbf{E}$.

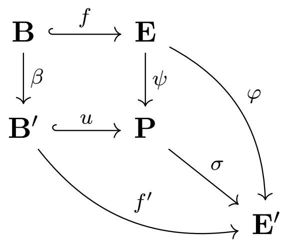
Proof. Let $\mathbf{P}$ be the pushout in this diagram



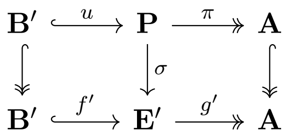
From Lemma 3.2.4 u is a monomorphism and $\operatorname{coker} f \cong \operatorname{coker} u$, which gives the following diagram, by $\operatorname{coker} f \cong \mathbf{A}$.



By the universal property of a pushout of φ and f' there exists a homomorphism $\sigma : \mathbf{P} \to \mathbf{E}'$ such that this diagram commutes.



This in turn gives this diagram



and we get that σ is an isomorphism from the five lemma. Q.E.D.

Let us denote the rows as before, then we will denote this relation as $\beta\mathbb{E} = \mathbb{E}'$

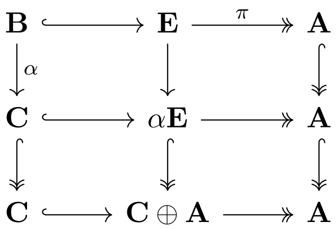
3.2.7 Corollary. *We have for an extension* $\mathbb{E}$ *of* $\mathbf{B}$ *by* $\mathbf{A}$ *and homomorphisms* $\sigma_1 : \mathbf{B} \to \mathbf{B}'$, $\sigma_2 : \mathbf{B}' \to \mathbf{B}^*$, $\eta_1 : \mathbf{A} \to \mathbf{A}'$ *and* $\eta_2 : \mathbf{A}' \to \mathbf{A}^*$ *that the following equations hold*

1. $\mathbb{E}\sigma_2 \circ \sigma_1 = (\mathbb{E}\sigma_2)\sigma_1$
2. $\eta_2 \circ \eta_1\mathbb{E}^* = \eta_1(\eta_2\mathbb{E}^*)$

This can easily be seen in the diagrams and the uniqueness of pushouts an pullbacks.

With these tools at our hands we can solve the question of when a homomorphism can have its domain extended as the module is extended.

3.2.8 Theorem. *Let* $\mathbf{B}$ *be a given module that is extended by* $\mathbf{A}$ *to give* $\mathbf{E}$. *A homomorphism* $\alpha : \mathbf{B} \to \mathbf{C}$ *is extendible if and only if the corresponding extension* $\alpha\mathbf{E}$ *splits. That is we have this diagram to commute*



Proof. First assume that $\alpha : \mathbf{B} \to \mathbf{C}$ is extendible so that $\widehat{\alpha} : \mathbf{E} \to \mathbf{C}$. Then define $\sigma : \mathbf{E} \to \mathbf{C} \oplus \mathbf{A}$ with $\sigma(e) = \widehat{\alpha}(e) \oplus \pi(e)$. This results in the following diagram

$$\begin{array}{ccccc}
\mathbf{B} & \hookrightarrow & \mathbf{E} & \xrightarrow{\pi} & \mathbf{A} \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \sigma} & & \| \\
\mathbf{C} & \hookrightarrow & \mathbf{C} \oplus \mathbf{A} & \twoheadrightarrow & \mathbf{A}
\end{array}$$

which by Lemma 3.2.6 shows that it is the pushout of $\alpha\mathbf{E}$.

Next assume that $\alpha\mathbf{E} \cong \mathbf{C}\oplus\mathbf{A}$ by $\theta : \alpha\mathbf{E} \hookrightarrow\!\!\!\!\rightarrow \mathbf{C}\oplus\mathbf{A}$. This means there exists a homomorphism $\sigma : \mathbf{C} \oplus \mathbf{A} \to \mathbf{C}$ and as such we can define $\widehat{\alpha} = \sigma \circ \theta$ Q.E.D.

We are going to need morphisms between extensions to move forward and as we continuously deal with commutativity of them the following definition becomes natural.

3.2.9 Definition. A homomorphism between extensions $\varphi : \mathbb{E} \to \mathbb{F}$ is a set of homomorphisms $(\alpha, \sigma_i, \omega)$ such that this diagram commutes

$$\begin{array}{ccccccccc}
\mathbf{B} & \xrightarrow{\epsilon_n} & \mathbf{E}_n & \xrightarrow{\epsilon_{n-1}} & \dots & \xrightarrow{\epsilon_2} & \mathbf{E}_1 & \xrightarrow{\epsilon_1} & \mathbf{A} \\
\downarrow{\scriptstyle \omega} & & \downarrow{\scriptstyle \sigma_n} & & & & \downarrow{\scriptstyle \sigma_1} & & \downarrow{\scriptstyle \alpha} \\
\mathbf{D} & \xrightarrow{\varphi_n} & \mathbf{F}_n & \xrightarrow{\varphi_{n-1}} & \dots & \xrightarrow{\varphi_2} & \mathbf{F}_1 & \xrightarrow{\varphi_1} & \mathbf{C}
\end{array}$$

This is just a special case of a general chain map where notation is slightly different. From this we can quickly see that for any such morphism between extensions we have that $\operatorname{im}\left(\sigma_i|_{\operatorname{im}\epsilon_i}\right) \subseteq \operatorname{im}\varphi_i$ by virtue of the diagram being commutative.

3.2.10 Lemma. *Let $\mathbb{E}$ and $\mathbb{F}$ be two extensions of length one and let (α, σ, γ) be a morphism from $\mathbb{E}$ to $\mathbb{F}$ then we have*

$$\alpha\mathbb{E} = \mathbb{F}\gamma$$

Proof. This can be see from the diagram

$$\begin{array}{ccccc}
\mathbf{B} & \hookrightarrow & \mathbf{E} & \twoheadrightarrow & \mathbf{A} \\
\downarrow{\scriptstyle \alpha} & & \downarrow & & \| \\
\mathbf{B}' & \hookrightarrow & \mathbf{C} & \twoheadrightarrow & \mathbf{A} \\
\| & & \downarrow & & \downarrow{\scriptstyle \gamma} \\
\mathbf{B}' & \hookrightarrow & \mathbf{F} & \twoheadrightarrow & \mathbf{A}'
\end{array}$$

where the upper row is $\mathbb{E}$ and the lower $\mathbb{F}$. From before it becomes evident that $\mathbf{C} = \alpha\mathbf{E} = \mathbf{F}\gamma$ and in turn $\alpha\mathbb{E} = \mathbb{F}\gamma$ Q.E.D.

Now we wish to extend our definition of equivalence and have the tools to do it. Let these extensions be given.

$$\mathbb{E} : \mathbf{B} \hookrightarrow \mathbf{E}_2 \xrightarrow{\epsilon} \mathbf{E}_1 \twoheadrightarrow \mathbf{A}$$

$$\mathbb{F} : \mathbf{B} \hookrightarrow \mathbf{F}_2 \xrightarrow{\varphi} \mathbf{F}_1 \twoheadrightarrow \mathbf{A}$$

We can decompose these so that we have following.

$$\mathbb{E} : \mathbf{B} \hookrightarrow \mathbf{E}_2 \twoheadrightarrow \operatorname{im} \epsilon \hookrightarrow \mathbf{E}_1 \twoheadrightarrow \mathbf{A}$$

$$\mathbb{F} : \mathbf{B} \hookrightarrow \mathbf{F}_2 \twoheadrightarrow \operatorname{im} \varphi \hookrightarrow \mathbf{F}_1 \twoheadrightarrow \mathbf{A}$$

From here we can see due to the previous lemmas that if we have a morphism, $\sigma : \operatorname{im} \epsilon \to \operatorname{im} \varphi$ such that it commutes, we have one of the elements in each chain being either a pushout or a pullback. This gives us a natural way to define the equivalence

3.2.11 Definition. Let the following extensions be given

$$\mathbb{E} : \mathbf{B} \hookrightarrow \mathbf{E}_1 \xrightarrow{\epsilon} \mathbf{E}_2 \twoheadrightarrow \mathbf{A}$$

$$\mathbb{F} : \mathbf{B} \hookrightarrow \mathbf{F}_1 \xrightarrow{\varphi} \mathbf{F}_2 \twoheadrightarrow \mathbf{A}$$

Let $\mathbf{C} = \operatorname{im} \epsilon$ and $\mathbf{D} = \operatorname{im} \varphi$ and define these extensions.

$$\mathbb{E}_1 : \mathbf{B} \hookrightarrow \mathbf{E}_1 \twoheadrightarrow \mathbf{C}$$

$$\mathbb{E}_2 : \mathbf{C} \hookrightarrow \mathbf{E}_2 \twoheadrightarrow \mathbf{A}$$

$$\mathbb{F}_1 : \mathbf{B} \hookrightarrow \mathbf{F}_1 \twoheadrightarrow \mathbf{D}$$

$$\mathbb{F}_2 : \mathbf{D} \hookrightarrow \mathbf{F}_2 \twoheadrightarrow \mathbf{A}$$

We then define that $\mathbb{E} \cong \mathbb{F}$ if there exists a $\sigma : \mathbf{C} \to \mathbf{D}$ such that this diagram commutes

$$\begin{array}{ccccccccc}
\mathbf{B} & \hookrightarrow & \mathbf{E}_1 & \twoheadrightarrow & \mathbf{C} & \hookrightarrow & \mathbf{E}_2 & \twoheadrightarrow & \mathbf{A} \\
\downarrow & & & & \downarrow{\scriptstyle \sigma} & & & & \downarrow \\
\mathbf{B} & \hookrightarrow & \mathbf{F}_1 & \twoheadrightarrow & \mathbf{D} & \hookrightarrow & \mathbf{F}_2 & \twoheadrightarrow & \mathbf{A}
\end{array}$$

and we have that $\mathbb{E}_1 = \mathbb{F}_1\sigma$ and $\mathbb{E}_2 = \sigma\mathbb{F}_2$.

3.2.12 Corollary. *The definition of equivalence of extensions of length two gives us the formula*

$$\mathbb{E}_1 \circ \sigma\mathbb{F}_2 = \mathbb{E}_1\sigma \circ \mathbb{F}_2$$

We may denote $\mathbb{E} \cong_\sigma \mathbb{F}$ to indicate with respect to which homomorphism they are congruent. The previous lemmas we went through shows us the reasoning behind this definition. The definition gives us a commutative diagram of the two extensions.

$$\begin{array}{ccccccccc}
\mathbf{B} & \hookrightarrow & \mathbf{E}_1 & \twoheadrightarrow & \operatorname{im} \epsilon & \hookrightarrow & \mathbf{E}_2 & \twoheadrightarrow & \mathbf{A} \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \sigma} & & \downarrow & & \downarrow \\
\mathbf{B} & \hookrightarrow & \mathbf{F}_1 & \twoheadrightarrow & \operatorname{im} \varphi & \hookrightarrow & \mathbf{F}_2 & \twoheadrightarrow & \mathbf{A}
\end{array}$$

That is we get the above diagram being commutative. This is however only for extensions of length two. We want it to be independent of the length for the most general case. We noted before any extension of length n

$$\mathbb{E} : \mathbf{B} \hookrightarrow \mathbf{E}_n \to \ldots \to \mathbf{E}_1 \twoheadrightarrow \mathbf{A}$$

can be decomposed into a Yoneda composition of extensions of length one so that we have

$$\mathbb{E} = \mathbb{E}_n \circ \ldots \circ \mathbb{E}_1$$

From this can we define an equivalence relation that is as generic as possible while possessing properties that is desired.

3.2.13 Definition. Let $\mathbb{E}$ and $\mathbb{F}$ be extensions of $\mathbf{B}$ by $\mathbf{A}$ of length n such that we have

$$\mathbb{E} = \mathbb{E}_n \circ \ldots \circ \mathbb{E}_1$$

and

$$\mathbb{F} = \mathbb{F}_n \circ \ldots \circ \mathbb{F}_1$$

Then we have that $\mathbb{F} = \mathbb{E}$, if through a finite number of steps of the following operations

1. Replace $\mathbb{F}_i$ with $\mathbb{E}_i$ when $\mathbb{E}_i \cong \mathbb{F}_i$
2. Replace $\mathbb{E}_i\sigma \circ \mathbb{F}_{i-1}$ with $\mathbb{E}_i \circ \sigma\mathbb{F}_{i-1}$ when $\mathbb{E}_i \circ \mathbb{E}_{i-1} \cong_\sigma \mathbb{F}_i \circ \mathbb{F}_{i-1}$ for some σ
3. Replace $\mathbb{F}_i \circ \sigma\mathbb{E}_{i-1}$ with $\mathbb{F}_i\sigma \circ \mathbb{E}_{i-1}$ when $\mathbb{E}_i \circ \mathbb{E}_{i-1} \cong_\sigma \mathbb{F}_i \circ \mathbb{F}_{i-1}$ for some σ

we can go from $\mathbb{F}$ to $\mathbb{E}$.

The reason for this choice of definition will be more apparent later on. One of the reasons reason is that we want, using our previous notation, to have that

$$\mathbb{E}\alpha \circ \mathbb{F} = \mathbb{E} \circ \alpha\mathbb{F}.$$

Which our definition gives. We will take a look at an easy example that illustrates it.

3.2.14 Example. We have the following two extensions being equivalent, for a given module $\mathbf{M}$.

$$\mathbf{A} \xhookrightarrow{\imath} \mathbf{E}_n \xrightarrow{\epsilon_{n-1}} \ldots \to \mathbf{E}_1 \xrightarrow{\pi} \mathbf{B}$$

and

$$\mathbf{A} \hookrightarrow \mathbf{E}_n \oplus \mathbf{M} \to \ldots \to \mathbf{E}_1 \oplus \mathbf{M} \twoheadrightarrow \mathbf{B}$$

through the following diagram

$$\begin{array}{ccccccccc}
\mathbf{A} & \xhookrightarrow{\imath\oplus 0} & \mathbf{E}_n \oplus \mathbf{M} & \xrightarrow{\epsilon_{n-1}\oplus 1} & \ldots & \xrightarrow{\epsilon_1\oplus 1} & \mathbf{E}_1 \oplus \mathbf{M} & \xrightarrow{\pi_e} & \mathbf{B} \\
\downarrow & & \downarrow{\scriptstyle \pi_n} & & & & \downarrow{\scriptstyle \pi_1} & & \downarrow \\
\mathbf{A} & \xhookrightarrow{\imath} & \mathbf{E}_n & \xrightarrow{\epsilon_{n-1}} & \ldots & \xrightarrow{\epsilon_1} & \mathbf{E}_1 & \xrightarrow{\pi_e} & \mathbf{B}
\end{array}$$

where π_i are the respective projections and π_e is the epimorphism of the extension.

Even though each one in the upper sequence is the direct sum with the module **M**, the upper extension is under this definition equivalent to the lower extension. Next we observe that the notation of $\alpha\mathbb{E}$ and $\mathbb{E}\alpha$ can be used and extended in these longer chains.

3.2.15 Corollary. *Let* $\mathbb{E}$ *be an extension of* $\mathbf{B}$ *by* $\mathbf{A}$ *of length* n *along with the homomorphisms* $\sigma : \mathbf{A} \to \mathbf{A}'$ *and* $\omega : \mathbf{B} \to \mathbf{B}'$. *Then the following two identities hold*

$$\omega\mathbb{E} = (\omega\mathbb{E}_n) \circ \mathbb{E}_{n-1} \circ \ldots \circ \mathbb{E}_1$$

$$\mathbb{E}\sigma = \mathbb{E}_n \circ \ldots \circ \mathbb{E}_2 \circ (\mathbb{E}_1\sigma)$$

so that we have these commutative diagrams

$$\begin{array}{ccccccccc}
\mathbf{B} & \hookrightarrow & \mathbf{E}_n & \longrightarrow & \mathbf{E}_{n-1} & \longrightarrow & \mathbf{E}_{n-2} & \longrightarrow & \ldots \\
\downarrow{\scriptstyle \omega} & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbf{B}' & \hookrightarrow & \mathbf{D} & \longrightarrow & \mathbf{E}_{n-1} & \longrightarrow & \mathbf{E}_{n-2} & \longrightarrow & \ldots
\end{array}$$

and

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & \mathbf{E}_3 & \longrightarrow & \mathbf{E}_2 & \longrightarrow & \mathbf{C} & \longrightarrow & \mathbf{A} \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \sigma} \\
\ldots & \longrightarrow & \mathbf{E}_3 & \longrightarrow & \mathbf{E}_2 & \longrightarrow & \mathbf{E}_1 & \longrightarrow & \mathbf{A}'
\end{array}$$

The module $\mathbf{C}$ *is a pullback and* $\mathbf{D}$ *is a pushout of respective pairs morphisms.*

We saw earlier that for an extension of length one, $\mathbb{E}$, and a chain map $(\alpha, \epsilon, \omega)$ we had that $\alpha\mathbb{E} = \mathbb{E}\omega$. This can be extended naturally to include extensions of any length.

3.2.16 Lemma. *Let* $\mathbb{E}$ *be an extension of* $\mathbf{B}$ *by* $\mathbf{A}$ *of length* n *and let* $\sigma = (\alpha, \sigma_i, \omega)$ *be a chain map from* $\mathbb{E}$ *to* $\mathbb{F}$. *Then we have that*

$$\alpha\mathbb{E} = \mathbb{F}\omega$$

Proof. Let

$$\mathbb{E} : \mathbf{B} \xrightarrow{\epsilon_n} \mathbf{E}_n \xrightarrow{\epsilon_{n-1}} \ldots \xrightarrow{\epsilon_2} \mathbf{E}_2 \xrightarrow{\epsilon_1} \mathbf{E}_1 \xrightarrow{\epsilon_0} \mathbf{A}$$

$$\mathbb{E}' : \mathbf{B}' \xrightarrow{\varphi_n} \mathbf{F}_n \xrightarrow{\varphi_{n-1}} \ldots \xrightarrow{\varphi_2} \mathbf{F}_2 \xrightarrow{\varphi_1} \mathbf{F}_1 \xrightarrow{\varphi_0} \mathbf{A}'$$

be our extensions and $\mathbf{C}_i = \operatorname{im} \epsilon_i$ along with $\mathbf{D}_i = \operatorname{im} \varphi_i$. In this we set $\mathbf{C}_n \cong \mathbf{B}$, $\mathbf{C}_0 = \mathbf{A}$, $\mathbf{D}_n \cong \mathbf{B}'$ and $\mathbf{D}_0 = \mathbf{A}'$. This means that the factors of $\mathbb{E}$, $\mathbb{E} = \mathbb{E}_n \circ \ldots \circ \mathbb{E}_1$, come in the form of

$$\mathbb{E}_i : \mathbf{C}_i \overset{\imath}{\hookrightarrow} \mathbf{E}_i \overset{\epsilon_{i-1}}{\twoheadrightarrow} \mathbf{C}_{i-1}$$

with $\imath$ being the natural injection. Similarly for $\mathbb{F}$, each of the factors are of the form

$$\mathbb{F}_i : \mathbf{D}_i \overset{\imath}{\hookrightarrow} \mathbf{F}_i \overset{\varphi_{i-1}}{\twoheadrightarrow} \mathbf{D}_{i-1}.$$

Next let $\sigma_i^* = \sigma_i|_{\mathbf{C}_i}$ along with $\sigma_n^* = \alpha$ and $\sigma_0^* = \omega$. By commutativity σ_i^* will be from $\mathbf{C}_i$ to $\mathbf{D}_i$. These form the following commutative diagram

$$\begin{array}{ccccc} \mathbf{C}_i & \hookrightarrow & \mathbf{E}_i & \twoheadrightarrow & \mathbf{C}_{i-1} \\ \downarrow{\scriptstyle \sigma_i^*} & & \downarrow & & \downarrow{\scriptstyle \sigma_{i-1}^*} \\ \mathbf{D}_i & \hookrightarrow & \mathbf{F}_i & \twoheadrightarrow & \mathbf{D}_{i-1} \end{array}$$

which by Lemma 3.2.10 gives $\sigma_i^* \mathbb{E}_i = \mathbb{F}_i \sigma_{i-1}^*$. This ultimately implies the following equalities.

$$\begin{aligned} \alpha\mathbb{E} &= \alpha(\mathbb{E}_n \circ \ldots \circ \mathbb{E}_1) \\ &= (\alpha\mathbb{E}_n) \circ \mathbb{E}_{n-1} \circ \ldots \circ \mathbb{E}_1 \\ &= \mathbb{F}_n \sigma_{n-1}^* \circ \mathbb{E}_{n-1} \circ \ldots \circ \mathbb{E}_1 \\ &= \mathbb{F}_n \circ \sigma_{n-1}^* \mathbb{E}_{n-1} \circ \ldots \circ \mathbb{E}_1 \\ &= \mathbb{F}_n \circ \mathbb{F}_{n-1} \sigma_{n-2}^* \circ \ldots \circ \mathbb{E}_1 \\ &\vdots \\ &= \mathbb{F}_n \circ \ldots \circ \mathbb{F}_2 \sigma_1^* \circ \mathbb{E}_1 \\ &= \mathbb{F}_n \circ \ldots \circ \mathbb{F}_2 \circ \sigma_1^* \mathbb{E}_1 \\ &= \mathbb{F}_n \circ \ldots \circ \mathbb{F}_2 \circ \mathbb{F}_1 \omega \\ &= \mathbb{F}\omega \end{aligned}$$

Q.E.D.

This illustrates partially the choice of equivalences, properties that holds true for extensions of length one can easily be extended to include extensions of any length. These notations and definitions gives us a set of extensions. From here we can now define an operation on it, using the tools that was given in this section.

3.2.2 Addition

To the set of extensions of length n will we define an operation that we will denote as $+_B$. This operation is called the Baer sum as before. Tradition holds that $+$ is reserved for commutative operations and it will be shown to be commutative later. First out we define the following homomorphism.

3.2.17 Definition. The diagonal homomorphism $\Delta_A : \mathbf{A} \to \mathbf{A} \oplus \mathbf{A}$ is defined as $\Delta_A(a) = a \oplus a$.

This homomorphism gives us the diagonal relation. A related homomorphism is the following one.

3.2.18 Definition. The codiagonal homomorphism $\nabla_A : \mathbf{A} \oplus \mathbf{A} \to \mathbf{A}$ is defined as $\nabla(a \oplus b) = a + b$.

These two homomorphisms are quite useful as they can reduce writing in many ways, an interesting application is on the addition of homomorphisms. If we let $f, g : \mathbf{A} \to \mathbf{B}$ be given homomorphisms, then we can write the sum of the f and g using our new homomorphisms.

3.2.19 Lemma. *For $f, g \in \mathrm{Hom}\,(\mathbf{A}, \mathbf{B})$ we have that*

$$f + g = \nabla_B \circ (f \oplus g) \circ \Delta_A$$

Proof. The proof is straightforward as we have $(f+g)(x) = f(x)+g(x)$ and

$$\begin{aligned}
\nabla_B \circ (f \oplus g) \circ \Delta_A(x) &= \nabla_B \circ (f \oplus g)(x \oplus x) \\
&= \nabla_B(f(x) \oplus g(x)) \\
&= f(x) + g(x)
\end{aligned}$$

Q.E.D.

We should also gently note the following relationship with composition of our diagonal and co-diagonal homomorphism.

3.2.20 Lemma. *Let $\mathbb{E}$ and $\mathbb{F}$ be short exact sequences that are extensions of $\mathbf{B}$ by $\mathbf{A}$ and $\alpha : \mathbf{A} \to \mathbf{A}'$ and $\beta : \mathbf{B} \to \mathbf{B}'$, then the following holds.*

1. $\beta \circ \nabla_B = \nabla_B \circ (\beta \oplus \beta)$
2. $\Delta_A \circ \alpha = (\alpha \oplus \alpha) \circ \Delta_A$

Proof. In both cases we want to show that they give the same result for any element. Let $a \oplus b \in \mathbf{A} \oplus \mathbf{A}$ be given, then we have

$$\begin{aligned}
\beta \circ \nabla_B(a \oplus b) &= \beta(a + b) \\
&= \beta(a) + \beta(b) \\
&= \nabla_B(\beta(a) \oplus \beta(b)) \\
&= \nabla_B \circ (\beta \oplus \beta)(a \oplus b).
\end{aligned}$$

For the other, let $a \in \mathbf{A}$ be given, then

$$\begin{aligned}\Delta_A \circ \alpha(a) &= \alpha(a) \oplus \alpha(a) \\ &= (\alpha \oplus \alpha)(a \oplus a) \\ &= (\alpha \oplus \alpha) \circ \Delta_A(a)\end{aligned}$$

Q.E.D.

We have defined direct sums of modules and along with it the direct sum of homomorphisms. By combining these do we define the direct sum of extensions.

3.2.21 Definition. Let $\mathbb{E}$ and $\mathbb{F}$ be two n-extensions such that

$$\mathbb{E} : \mathbf{B} \overset{i}{\hookrightarrow} \mathbf{E}_n \overset{\epsilon_{n-1}}{\longrightarrow} \dots \overset{\epsilon_1}{\longrightarrow} \mathbf{E}_1 \overset{\pi}{\twoheadrightarrow} \mathbf{A},$$

$$\mathbb{F} : \mathbf{B}' \overset{j}{\hookrightarrow} \mathbf{F}_n \overset{\varphi_{n-1}}{\longrightarrow} \dots \overset{\varphi_1}{\longrightarrow} \mathbf{F}_1 \overset{p}{\twoheadrightarrow} \mathbf{A}'.$$

Then the direct sum of $\mathbb{E}$ and $\mathbb{F}$ the following exact sequence.

$$\mathbb{E} \oplus \mathbb{F} : \mathbf{B} \oplus \mathbf{B}' \overset{i \oplus j}{\hookrightarrow} \mathbf{E}_n \oplus \mathbf{F}_n \overset{\epsilon_{n-1} \oplus \varphi_{n-1}}{\longrightarrow} \dots \overset{\epsilon_1 \oplus \varphi_1}{\longrightarrow} \mathbf{A} \oplus \mathbf{A}'$$

We have then that $\mathbb{E} \oplus \mathbb{F}$ is an extension of $\mathbf{B} \oplus \mathbf{B}'$ by $\mathbf{A} \oplus \mathbf{A}'$. The direct sum of extensions interacts in a convenient manner with the previously established pullback and pushout.

3.2.22 Lemma. *Let the following extensions be given*

$$\mathbb{E} : \mathbf{B} \hookrightarrow \mathbf{E} \twoheadrightarrow \mathbf{A}$$

$$\mathbb{E}' : \mathbf{B}' \hookrightarrow \mathbf{E}' \twoheadrightarrow \mathbf{A}'$$

along with these homomorphisms $\sigma : \mathbf{A} \to \mathbf{X}$, $\sigma' : \mathbf{A}' \to \mathbf{X}'$, $\eta : \mathbf{B} \to \mathbf{Y}$ *and* $\eta' : \mathbf{B}' \to \mathbf{Y}'$. *Then we have*

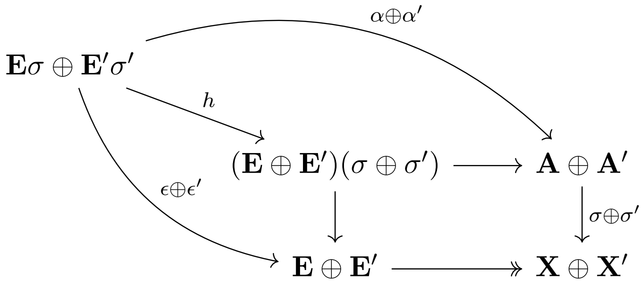
1. $\mathbb{E}\sigma \oplus \mathbb{E}'\sigma' = (\mathbb{E} \oplus \mathbb{E}')(\sigma \oplus \sigma')$
2. $\eta\mathbb{E} \oplus \eta'\mathbb{E}' = (\eta \oplus \eta')(\mathbb{E} \oplus \mathbb{E}')$

Proof. We have these diagrams

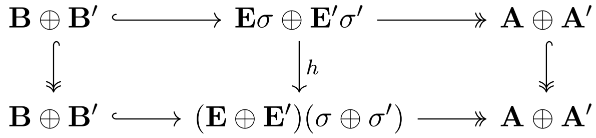
$$\begin{array}{ccc} \mathbf{E}\sigma & \overset{\alpha}{\longrightarrow} & \mathbf{A} \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \sigma} \\ \mathbf{E} & \overset{p}{\longrightarrow} & \mathbf{X} \end{array}$$

$$\begin{array}{ccc} \mathbf{E}'\sigma' & \overset{\alpha'}{\longrightarrow} & \mathbf{A}' \\ \downarrow{\scriptstyle \epsilon'} & & \downarrow{\scriptstyle \sigma'} \\ \mathbf{E}' & \overset{p}{\longrightarrow} & \mathbf{X}' \end{array}$$

by that they are pullbacks which in turn give the following diagram

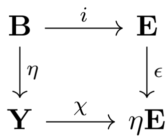


for some h. This in turn leads to this diagram

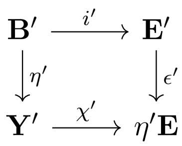


and then h is an isomorphism by the five lemma.

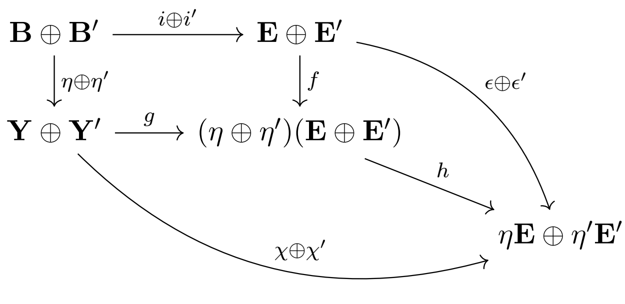
For the second part we have these diagrams



and



as both are pushouts, therefore we have get the following diagram



for some h. Similarly to before does it result in the following diagram

$$\begin{array}{ccccc}
\mathbf{X}\oplus\mathbf{X}' & \hookrightarrow & (\eta\oplus\eta')(\mathbf{E}\oplus\mathbf{E}') & \twoheadrightarrow & \mathbf{A}\oplus\mathbf{A}' \\
\downarrow & & \downarrow\sigma & & \downarrow \\
\mathbf{X}\oplus\mathbf{X}' & \hookrightarrow & \eta\mathbf{E}\oplus\eta'\mathbf{E}' & \twoheadrightarrow & \mathbf{A}\oplus\mathbf{A}'
\end{array}$$

where h is an isomorphism by the five lemma. Q.E.D.

This lemma is only valid on extensions of length one. However, our definition of the same operation on longer extensions gives that it holds true for extensions of any length. With the direct sum of extensions defined an issue arises. Are $(\mathbb{E}_1\oplus\mathbb{F}_1)\circ(\mathbb{E}_2\oplus\mathbb{F}_2)$ and $\mathbb{E}_1\circ\mathbb{E}_2\oplus\mathbb{F}_1\circ\mathbb{F}_2$ equivalent under our definition of equivalence?

3.2.23 Lemma. *Let $\mathbb{E}_1$ and $\mathbb{E}_2$ be extensions such that $\mathbb{E}_2\circ\mathbb{E}_1$ is defined, and equally so for $\mathbb{F}_1$ and $\mathbb{F}_2$ giving $\mathbb{F}_2\circ\mathbb{F}_1$. Then direct sum and Yoneda composition commute, that is*

$$(\mathbb{E}_2\oplus\mathbb{F}_2)\circ(\mathbb{E}_1\oplus\mathbb{F}_1) = (\mathbb{E}_2\circ\mathbb{E}_1)\oplus(\mathbb{F}_2\circ\mathbb{F}_1)$$

Proof. Let

$$\mathbb{E}_2\oplus\mathbb{F}_2 : \mathbf{B}\oplus\mathbf{D}\hookrightarrow\mathbf{E}_2\oplus\mathbf{F}_2\twoheadrightarrow\mathbf{X}\oplus\mathbf{Y}$$

and

$$\mathbb{E}_1\oplus\mathbb{F}_1 : \mathbf{X}\oplus\mathbf{Y}\hookrightarrow\mathbf{E}_1\oplus\mathbf{F}_1\twoheadrightarrow\mathbf{A}\oplus\mathbf{C}.$$

Then we get the composition $(\mathbb{E}_2\oplus\mathbb{F}_2)\circ(\mathbb{E}_1\oplus\mathbb{F}_1)$ as

$$\mathbf{B}\oplus\mathbf{D}\hookrightarrow\mathbf{E}_2\oplus\mathbf{F}_2\to\mathbf{E}_1\oplus\mathbf{F}_1\twoheadrightarrow\mathbf{A}\oplus\mathbf{C}$$

which is clearly $(\mathbb{E}_2\circ\mathbb{E}_1)\oplus(\mathbb{F}_2\circ\mathbb{F}_1)$. Q.E.D.

From this we can observe the following. For simplicity's sake we will focus on extensions of length one, but as before it works on any length. Let $\mathbb{E}$ and $\mathbb{F}$ be such that

$$\mathbb{E} : \mathbf{B}\xhookrightarrow{i}\mathbf{E}\xrightarrow{\pi}\mathbf{A}$$
$$\mathbb{F} : \mathbf{B}\xhookrightarrow{j}\mathbf{F}\xrightarrow{p}\mathbf{A}$$

which naturally gives us the direct sum

$$\mathbb{E}\oplus\mathbb{F} : \mathbf{B}\oplus\mathbf{B}\xhookrightarrow{i\oplus j}\mathbf{E}\oplus\mathbf{F}\xrightarrow{\pi\oplus p}\mathbf{B}\oplus\mathbf{B}$$

However, using our previous notation we can now make sense of $\nabla_B(\mathbb{E}\oplus\mathbb{F})$. Namely, it is the short exact sequence such that this diagram commutes

$$\begin{array}{ccccc}
\mathbf{B}\oplus\mathbf{B} & \xhookrightarrow{i\oplus j} & \mathbf{E}\oplus\mathbf{F} & \xrightarrow{\pi\oplus p} & \mathbf{A}\oplus\mathbf{A} \\
\downarrow\nabla_B & & \downarrow & & \downarrow \\
\mathbf{B} & \hookrightarrow & \mathbf{D} & \twoheadrightarrow & \mathbf{A}\oplus\mathbf{A}
\end{array}$$

where $\mathbf{D}$ is the pushout of $i \oplus j$ ∇_B. Similarly we can now make sense of $(\mathbb{E} \oplus \mathbb{F})\Delta_A$ as the extension satisfying the following commutative diagram

$$\begin{array}{ccccc} \mathbf{B} \oplus \mathbf{B} & \hookrightarrow & \mathbf{D} & \twoheadrightarrow & \mathbf{A} \\ \downarrow & & \downarrow & & \downarrow \Delta_A \\ \mathbf{B} \oplus \mathbf{B} & \xhookrightarrow{i \oplus j} & \mathbf{E} \oplus \mathbf{F} & \xrightarrow{\pi \oplus p} & \mathbf{A} \oplus \mathbf{A} \end{array}$$

where $\mathbf{D}$ is the pullback of $\pi \oplus p$ and Δ_A. These observations make it possible to see that our definition for addition of extensions of length one can be rewritten as

$$\mathbb{E} +_B \mathbb{F} = \nabla_B(\mathbb{E} \oplus \mathbb{F})\Delta_A$$

where the central module is pushout and pullback at the same time. As this notation is not length sensitive it may be useful to utilize it as our sum.

3.2.24 Definition. Let $\mathbb{E}$ and $\mathbb{F}$ be two extensions of $\mathbf{B}$ by $\mathbf{A}$ of length n. Then their Baer sum is

$$\mathbb{E} +_B \mathbb{F} = \nabla_B(\mathbb{E} \oplus \mathbb{F})\Delta_A =$$

$$\nabla_B(\mathbb{E}_n \oplus \mathbb{F}_n) \circ (\mathbb{E}_{n-1} \oplus \mathbb{F}_{n-1}) \circ \ldots \circ (\mathbb{E}_2 \oplus \mathbb{F}_2) \circ (\mathbb{E}_1 \oplus \mathbb{F}_1)\Delta_A$$

From the previous observation we see that the extension $\nabla_B(\mathbb{E} \oplus \mathbb{F})\Delta_A$ will look like

$$\mathbf{B} \hookrightarrow \mathbf{E}_n \amalg \mathbf{F}_n \to \mathbf{E}_{n-1} \oplus \mathbf{F}_{n-1} \to \ldots \to \mathbf{E}_2 \oplus \mathbf{F}_2 \to \mathbf{E}_1 \sqcap \mathbf{F}_1 \twoheadrightarrow \mathbf{A}$$

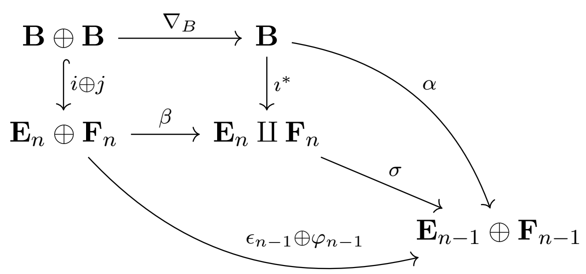
where $\mathbf{E}_n \amalg \mathbf{F}_n$ is the pushout of ∇_B and $\imath_E \oplus \imath_F$, and $\mathbf{E}_1 \sqcap \mathbf{F}_1$ is the pullback of $\pi_E \oplus \pi_F$ and Δ_A. This notation will be used from now on without mentioning. However we need to establish that this is indeed a proper extension.

3.2.25 Lemma. *For two extensions of* $\mathbf{B}$ *by* $\mathbf{A}$ *of length* n *we have that their Baer sum is an extension of* $\mathbf{B}$ *by* $\mathbf{A}$ *of length* n*.*

Proof. For our extensions $\mathbb{F}$ and $\mathbb{E}$ we have the supposed sum extension being

$$\mathbf{B} \hookrightarrow \mathbf{E}_n \amalg \mathbf{F}_n \to \mathbf{E}_{n-1} \oplus \mathbf{F}_{n-1} \to \ldots \to \mathbf{E}_2 \oplus \mathbf{F}_2 \to \mathbf{E}_1 \sqcap \mathbf{F}_1 \twoheadrightarrow \mathbf{A}$$

The sequence is known to be exact at all direct sum modules. It is the pushout and pullback that need to be confirmed. The pushout $\mathbf{E}_n \amalg \mathbf{F}_n$ has the following diagram.



As the diagram commutes it means that

$$0 = (\epsilon_{n-1} \oplus \varphi_{n-1}) \circ (i \oplus j) = \alpha \circ \nabla_B$$

However, we know that $\nabla_B \neq 0$, which means that we have $\operatorname{im} \nabla_B \subseteq \ker \alpha$. The image of ∇_B is $\mathbf{B}$ and we get $\operatorname{im} \imath^* \subseteq \ker \sigma$. For the reverse inclusion let $e \oplus f \oplus b \in \ker \sigma$. We have, by definition of pushout, that

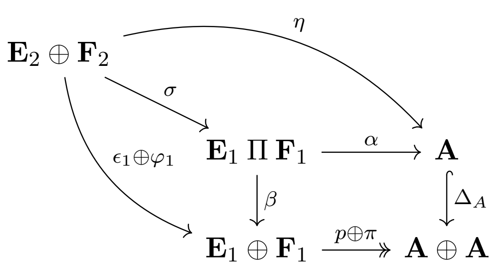
$$\mathbf{E}_n \amalg \mathbf{F}_n = \frac{\mathbf{E}_n \oplus \mathbf{F}_n \oplus \mathbf{B}}{\mathbf{N}}$$

with $\mathbf{N} = \{i(a) \oplus j(b) \oplus -\nabla_B(a \oplus b) : a \oplus b \in \mathbf{B} \oplus \mathbf{B}\}$. We have therefore

$$\begin{aligned}\sigma(e \oplus f \oplus b) &= (\epsilon_{n-1} \oplus \varphi_{n-1})(e \oplus f) + \sigma \circ \imath^*(b) \\ &= (\epsilon_{n-1} \oplus \varphi_{n-1})(e \oplus f) \\ &= 0\end{aligned}$$

which means that $e \oplus f \in \mathbf{E}_n \oplus \mathbf{F}_n$ along with $e \oplus f \in \ker(\epsilon_{n-1} \oplus \varphi_{n-1})$. This implies by exactness that $e \oplus f \in \operatorname{im}(i \oplus j)$ and in turn we have an element $\nabla_B(e \oplus f)\mathbf{B}$ such that $e \oplus f \oplus \imath^*(b) = e \oplus f \oplus \imath^*(b) \in \operatorname{im} \imath^*$ and, hence, $\ker \sigma \subseteq \operatorname{im} \imath^*$. To see it is a monomorphism we observe that $\imath^*(b) = 0 \oplus 0 \oplus b + \mathbf{N}$. For $\imath^*(b) = 0$ to be true we must have $0 \oplus 0 \oplus b \in \mathbf{N}$, but as both i and j are injective there is only the possibility that $b = 0$. Hence, the kernel is trivial.

The pullback $\mathbf{E}_1 \amalg \mathbf{F}_1$ gives the following diagram



which gives us $0 = (p \oplus \pi) \circ (\epsilon_1 \varphi_1) = \eta \circ \Delta_A$. However, we have that Δ_A is a monomorphism and hence we have $\eta(e \oplus f) = 0$ for $e \oplus f \in \mathbf{E}_2 \oplus \mathbf{F}_2$. From commutativity we get $\operatorname{im} \sigma \subseteq \ker \alpha$. For the reverse inclusion let $e \oplus f \oplus a \in \ker \alpha$ be given. By commutativity we then have that $(p \oplus \pi) \circ \beta(e \oplus f \oplus a) = 0$ as $\alpha(e \oplus f \oplus a) = 0$. From there $\beta(e \oplus f \oplus a) \in \ker p \oplus \pi$ and by exactness of the direct sum sequence we get that $\beta(e \oplus f \oplus a) \in \operatorname{im} \epsilon_1 \oplus \varphi_1$. Thus there exists $e' \oplus f' \in \mathbf{E}_2 \oplus \mathbf{F}_2$ such that $(\epsilon_1 \oplus \varphi_1)(e' \oplus f') = \beta(e \oplus f \oplus a)$ and by commutativity we get that $\sigma(e' \oplus f') = e \oplus f \oplus a$. Hence $\ker \alpha \subseteq \operatorname{im} \sigma$. To check that α is an epimorphism, let $a \in \mathbf{A}$ be given. Then we have $\Delta_A(a) = a \oplus a \in \mathbf{A} \oplus \mathbf{A}$, which by the fact of $p \oplus \pi$ being an epimorphism, gives us an element $e \oplus f \in \mathbf{E}_1 \oplus \mathbf{F}_1$ but that gives us $(p \oplus \pi)(e \oplus f) = \Delta_A(a)$ and hence we have that $e \oplus f \oplus a \in \mathbf{E}_1 \amalg \mathbf{F}_1$. Thus $\alpha(e \oplus f \oplus a) = a$ and we are done. Q.E.D.

This secures that $\mathbb{E} +_B \mathbb{F}$ is a proper n-extension of $\mathbf{B}$ by $\mathbf{A}$. We established earlier the relation

$$(\alpha \oplus \alpha')(\mathbb{E} \oplus \mathbb{E}') = \alpha\mathbb{E} \oplus \alpha'\mathbb{E}'$$

and its dual counterpart. We may wonder now if there is a relation as coincidental as this with our Baer sum. Here we had the direct sum in both instances. As it turns out there is a relation, in which cases we use the Baer sum and sum of homomorphisms.

3.2.26 Lemma. *Let $\mathbb{E}$ and $\mathbb{F}$ be given extensions of $\mathbf{B}$ by $\mathbf{A}$ of length n, then we have for homomorphisms $\alpha, \alpha' : \mathbf{A} \to \mathbf{A}'$ and $\beta, \beta' : \mathbf{B} \to \mathbf{B}'$ that the following hold*

1. $\beta(\mathbb{E} +_B \mathbb{F}) = \beta\mathbb{E} +_B \beta\mathbb{F}$
2. $(\beta + \beta')\mathbb{E} = \beta\mathbb{E} +_B \beta'\mathbb{E}$
3. $(\mathbb{E} +_B \mathbb{F})\alpha = \mathbb{E}\alpha +_B \mathbb{F}\alpha$
4. $\mathbb{E}(\alpha + \alpha') = \mathbb{E}\alpha +_B \mathbb{E}\alpha'$

Proof. 1:
We have initially that $\mathbb{E} +_B \mathbb{F}$ is the extension

$$\mathbf{B} \hookrightarrow \mathbf{E}_n \amalg \mathbf{F}_n \to \mathbf{E}_{n-1} \oplus \mathbf{F}_{n-1} \to \ldots \to \mathbf{E}_2 \oplus \mathbf{F}_2 \to \mathbf{E}_1 \amalg \mathbf{F}_1 \twoheadrightarrow \mathbf{A}$$

and therefore we have that $\beta(\mathbb{E} +_B \mathbb{F})$ is the pushout of

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{i} & \mathbf{E}_n \amalg \mathbf{F}_n \\ \downarrow{\scriptstyle\beta} & & \downarrow \\ \mathbf{B}' & \longrightarrow & \beta(\mathbf{E}_n \amalg \mathbf{F}_n) \end{array}$$

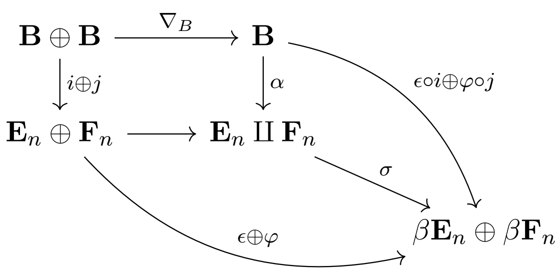
However for $\beta\mathbf{E}_n$ and $\beta\mathbf{F}_n$ we have the following diagrams

$$\begin{array}{ccc} \mathbf{B} & \overset{i}{\hookrightarrow} & \mathbf{E}_n \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\epsilon} \\ \mathbf{B}' & \xrightarrow{\epsilon'} & \beta\mathbf{E}_n \end{array}$$

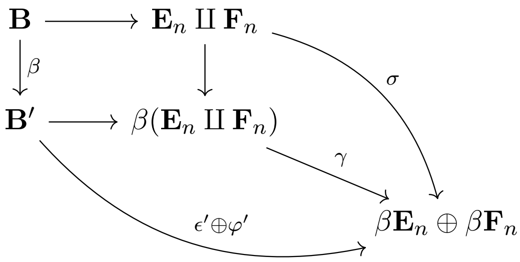
and

$$\begin{array}{ccc} \mathbf{B} & \overset{j}{\hookrightarrow} & \mathbf{F}_n \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\varphi} \\ \mathbf{B}' & \xrightarrow{\varphi'} & \beta\mathbf{F}_n \end{array}$$

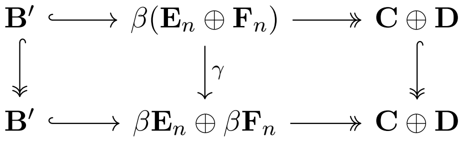
These give us the following commutative diagram for $\mathbf{E}_n \amalg \mathbf{F}_n$



for some σ. This in turn results in this diagram

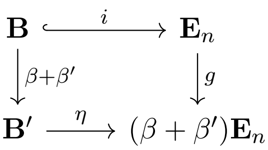


for some γ homomorphism. This gives us the following diagram, where $\mathbf{C}$ is the image from $\mathbf{E}_n$ and $\mathbf{D}$ is the image from $\mathbf{F}_n$.

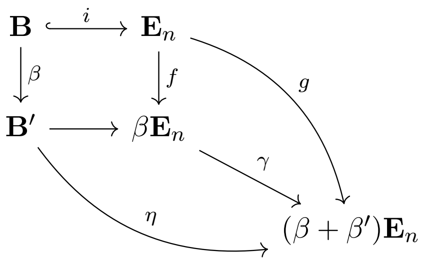


The five lemma gives that γ is an isomorphism.

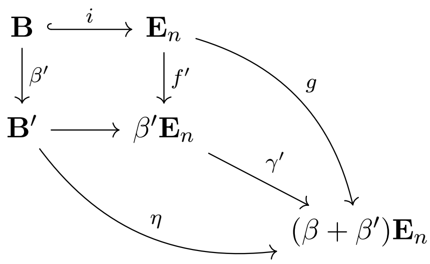
2:
We have that $(\beta + \beta')\mathbf{E}_n$ is the pushout in



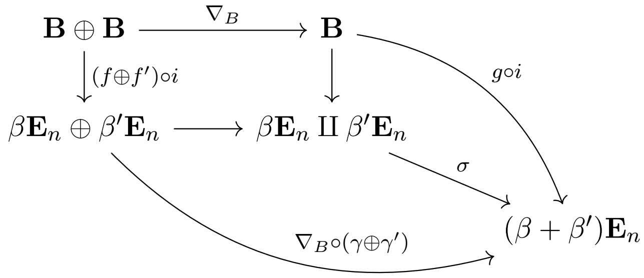
and we have for $\beta\mathbf{E}_n$ a $\beta'\mathbf{E}_n$ the following diagrams as they are pushouts



and



These combined give



for some σ. This in turn gives us the following short exact sequences, with $\mathbf{C}$ being the image of the homomorphism from $\mathbf{E}_n$ in our extension.

$$\begin{array}{ccccc}
\mathbf{B}' & \hookrightarrow & \beta\mathbf{E}_n \amalg \beta'\mathbf{E}_n & \twoheadrightarrow & \mathbf{C} \\
\big\downarrow & & \big\downarrow{\scriptstyle \sigma} & & \big\downarrow \\
\mathbf{B}' & \hookrightarrow & (\beta+\beta')\mathbf{E}_n & \twoheadrightarrow & \mathbf{C}
\end{array}$$

From the five lemma we get σ being an isomorphism.

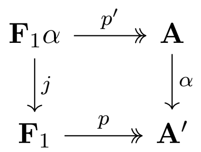
3:
For the pullback in $(\mathbb{E}+_B\mathbb{F})\alpha$ we have

$$\begin{array}{ccc}
(\mathbf{E}_1 \amalg \mathbf{F}_1)\alpha & \twoheadrightarrow & \mathbf{A} \\
\big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\
\mathbf{E}_1 \amalg \mathbf{F}_1 & \twoheadrightarrow & \mathbf{A}'
\end{array}$$

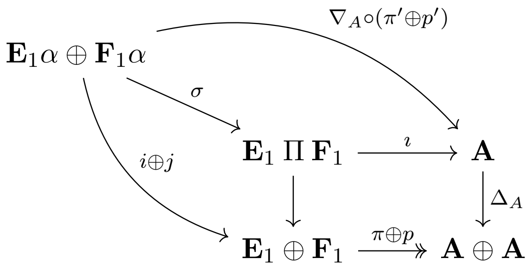
and for $\mathbf{E}_1\alpha$ and $\mathbf{F}_1\alpha$ we have

$$\begin{array}{ccc}
\mathbf{E}_1\alpha & \xrightarrow{\pi'} & \mathbf{A} \\
\big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle \alpha} \\
\mathbf{E}_1 & \xrightarrow{\pi} & \mathbf{A}'
\end{array}$$

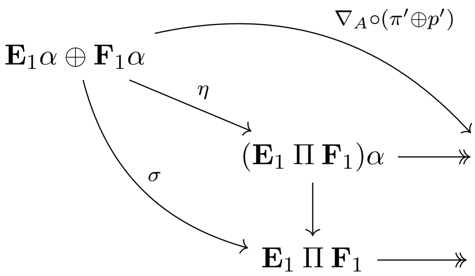
and



which combines into

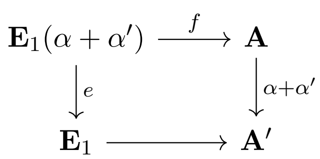


This combined with our first commutative diagram gives that η must exist in this diagram

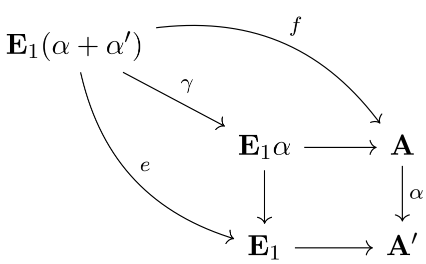


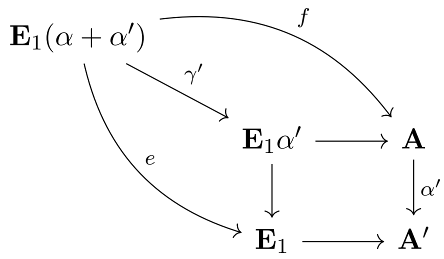
and, as before, the five lemma shows that η is an isomorphism.

4:
As before we have this commutative diagram for the pullback $\mathbf{E}_1(\alpha + \alpha')$

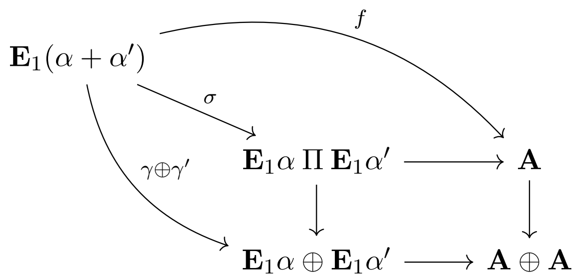


and for the pullbacks $\mathbf{E}_1\alpha$ and $\mathbf{E}_1\alpha'$ we have





which combine into



for some σ. Short exact sequence and five lemma give that σ is an isomorphism. Q.E.D.

We defined an equivalence of extensions. As with any function it is important to secure that the Baer sum is well-defined.

3.2.27 Lemma. *The extended Baer sum*

$$\mathbb{E} +_B \mathbb{F} =$$

$$\nabla_B(\mathbb{E}_n \oplus \mathbb{F}_n) \circ (\mathbb{E}_{n-1} \oplus \mathbb{F}_{n-1}) \circ \ldots \circ (\mathbb{E}_2 \oplus \mathbb{F}_2) \circ (\mathbb{E}_n \oplus \mathbb{F}_n)\Delta_A$$

is well-defined with respect to the equivalence relation of extensions.

Proof. Let $\mathbb{E} = \mathbb{E}'$ and $\mathbb{F} = \mathbb{F}'$ be four extensions of $\mathbf{B}$ by $\mathbf{A}$ of length n. Then we have the following sequence of equalities.:

$$\begin{aligned}
\mathbb{E} +_B \mathbb{F} &= \nabla_B(\mathbb{E} \oplus \mathbb{F})\Delta_A \\
&= \nabla_B(\mathbb{E}_n \oplus \mathbb{F}_n) \circ \ldots \circ (\mathbb{E}_1 \oplus \mathbb{F}_1)\Delta_A \\
&= \nabla_B(\mathbb{E}_n \circ \ldots \circ \mathbb{E}_1) \oplus (\mathbb{F}_n \circ \ldots \circ \mathbb{F}_1)\Delta_A \\
&= \nabla_B(\mathbb{E}'_n \circ \ldots \circ \mathbb{E}'_1) \oplus (\mathbb{F}'_n \circ \ldots \circ \mathbb{F}'_1)\Delta_A \\
&= \nabla_B(\mathbb{E}'_n \oplus \mathbb{F}'_n) \circ \ldots \circ (\mathbb{E}'_1 \oplus \mathbb{F}'_1)\Delta_A \\
&= \nabla_B(\mathbb{E}' \oplus \mathbb{F}')\Delta_A \\
&= \mathbb{E}' +_B \mathbb{F}'
\end{aligned}$$

Q.E.D.

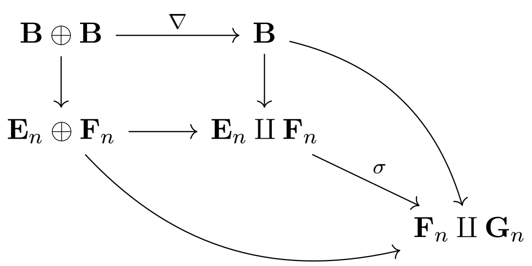
The following lemmas establish that the Baer sum has all the properties we want it to have.

3.2.28 Lemma. *The extended Baer sum*

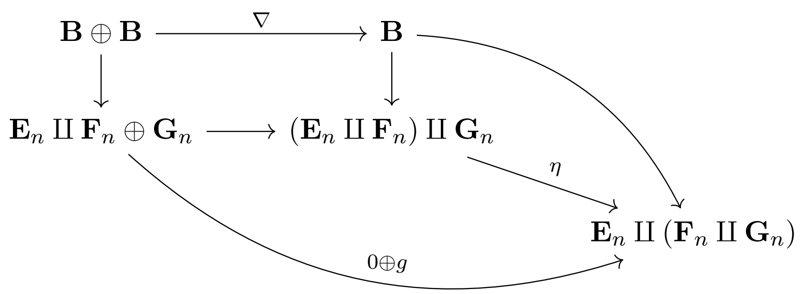
$$\mathbb{E} +_B \mathbb{F}$$

is associative.

Proof. Let $\mathbb{E}, \mathbb{F}$ and $\mathbb{G}$ be three extensions of $\mathbf{B}$ by $\mathbf{A}$ of length n. We wish to show that $(\mathbb{E} +_B \mathbb{F}) +_B \mathbb{G} = \mathbb{E} +_B (\mathbb{F} +_B \mathbb{G})$. As previously we only need to focus on the pushout and pullback at the ends of the sequences as we clearly have $(\mathbf{E}_i \oplus \mathbf{F}_i) \oplus \mathbf{G}_i \cong \mathbf{E}_i \oplus (\mathbf{F}_i \oplus \mathbf{G}_i)$. We start with the pushout $\nabla((\nabla(\mathbb{E}_n \oplus \mathbb{F}_n)) \oplus \mathbb{G}_n)$ and show it is equivalent to $\nabla(\mathbb{E}_n \oplus (\nabla(\mathbb{F}_n \oplus \mathbb{G}_n)))$. For these we have the following diagrams



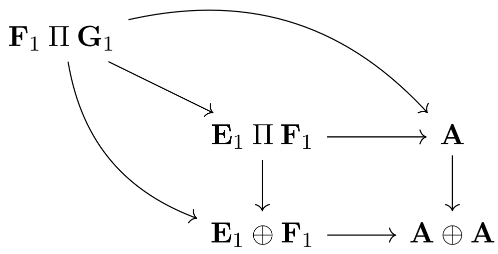
which gives us, for $g : \mathbf{E}_n \amalg \mathbf{F}_n \to \mathbf{F}_n \amalg \mathbf{G}_n$, the following diagram.

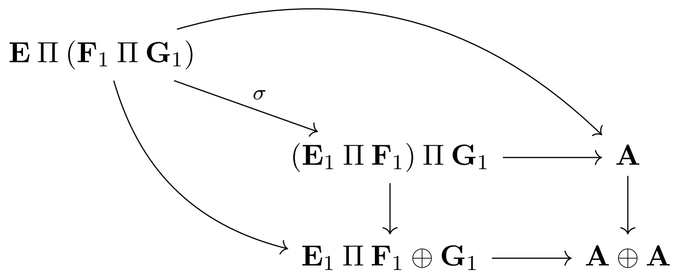


From this we get the diagram

$$\begin{array}{ccccc} \mathbf{B} & \hookrightarrow & (\mathbf{E}_n \amalg \mathbf{F}_n) \amalg \mathbf{G}_n & \twoheadrightarrow & \operatorname{im} \epsilon_{n-1} \oplus \operatorname{im} \varphi_{n-1} \oplus \operatorname{im} \gamma_{n-1} \\ \downarrow & & \downarrow \eta & & \downarrow \\ \mathbf{B} & \hookrightarrow & \mathbf{E}_n \amalg (\mathbf{F}_n \amalg \mathbf{G}_n) & \twoheadrightarrow & \operatorname{im} \epsilon_{n-1} \oplus \operatorname{im} \varphi_{n-1} \oplus \operatorname{im} \gamma_{n-1} \end{array}$$

which together with the five lemma yields that η is an isomorphism. For the pullback we have the following two diagrams





and with short exact sequences we get that σ is an isomorphism. Q.E.D.

3.2.29 Lemma. *The extended Baer sum is commutative.*

Proof. For each $\mathbf{E}_i \oplus \mathbf{F}_i$ we have the isomorphism $\varphi_i : \mathbf{E}_i \oplus \mathbf{F}_i \to \mathbf{F}_i \oplus \mathbf{E}_i$ with $\varphi_i(a \oplus b) = b \oplus a$, which gives us that $\mathbb{E} +_B \mathbb{F} \cong \mathbb{F} +_B \mathbb{E}$. Q.E.D.

With the Baer sum being well-defined and commutative, one might wonder if for any extension $\mathbb{E}$ there exists an extension $\mathbb{S}$ such that $\mathbb{E} +_B \mathbb{S} = \mathbb{E}$. We had such an extension in the case of simple extensions, where it was the split extension. We know that at least one such extension exists for each $\mathbb{E}$ as we have

$$\mathbb{E} +_B 0\mathbb{E} = (1+0)\mathbb{E} = 1\mathbb{E} = \mathbb{E}$$

with 1 being the identity homomorphism and 0 being the 0 homomorphism. Equally we have

$$\mathbb{E} +_B \mathbb{E}0 = \mathbb{E}(1+0) = \mathbb{E}1 = \mathbb{E}$$

If we let $\sigma = (0, 0, 0, ...)$ be an extension homomorphism from $\mathbb{E}$ to $\mathbb{F}$ and $\mathbb{F} = \mathbb{E}$, then we get from that $0\mathbb{E} = \mathbb{E}0$ and the previous 0's are actually the same element. When $\mathbb{F} \neq \mathbb{E}$ we get that

$$0\mathbb{E} = \mathbb{F}0$$

and as such we have that any $0\mathbb{E}$ must be neutral to any other extension with respect to the Baer sum. By calculating the pushout and pullback we can find out that our $\mathbb{S}$ will be something like

$$\mathbb{S} : \mathbf{B} \overset{1}{\hookrightarrow} \mathbf{B} \to \ldots \to \mathbf{A} \overset{1}{\twoheadrightarrow} \mathbf{A}$$

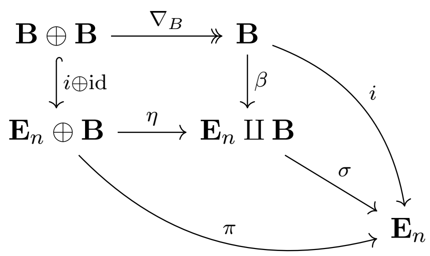
with modules in between. As it turns out those modules in between are all the trivial module.

3.2.30 Lemma. *For the extended Bear sum the extension*

$$0 : \mathbf{B} \overset{\text{id}}{\hookrightarrow} \mathbf{B} \to 0 \to 0 \to \ldots \to 0 \to 0 \to \mathbf{A} \overset{\text{id}}{\twoheadrightarrow} \mathbf{A}$$

is the neutral element.

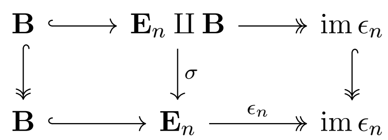
Proof. In $\mathbb{E} +_B 0$ we clearly have that $\mathbf{E}_i \oplus 0 \cong \mathbf{E}_i$ and as such it is again the pushout and the pullback that is of interest. We will start with the pushout which comes in the form of $\mathbf{E}_n \amalg \mathbf{B}$. By definition of the pushout we have that this commutative diagram exists for some σ



where $i : \mathbf{B} \to \mathbf{E}_n$ comes from the extension and π is the natural projection. Next we observe that we have the Bear sum extension being

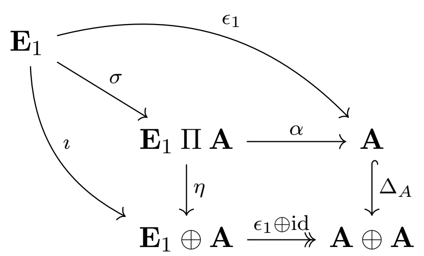
$$\mathbf{B} \to \mathbf{E}_n \amalg \mathbf{B} \xrightarrow{f} \mathbf{E}_{n-1} \xrightarrow{\epsilon_{n-1}} \ldots$$

which is exact and as such we have that $\operatorname{im} f = \ker \epsilon_{n-1} = \operatorname{im} \epsilon_n$. This gives us this diagram



which from the five lemma gives us that σ is an isomorphism.

Similarly for the pullback $\mathbf{E}_1 \Pi \mathbf{A}$ we have the diagram



with $\imath$ being the natural inclusion. and we get the short exact sequences

$$\begin{array}{ccccc} \ker \epsilon_1 & \hookrightarrow & \mathbf{E}_1 & \twoheadrightarrow & \mathbf{A} \\ \downarrow & & \downarrow \sigma & & \downarrow \\ \ker \epsilon_1 & \hookrightarrow & \mathbf{E}_1 \, \Pi \, \mathbf{A} & \twoheadrightarrow & \mathbf{A} \end{array}$$

and again by the five lemma we get that σ is an isomorphism. Ultimately we get $\mathbb{E} +_B 0 = \mathbb{E}$. Q.E.D.

If we denote this extension by 0, we can say that for any $\mathbb{E}$ we have $0 = 0\mathbb{E} = \mathbb{E}0$. This can be seen by using our previous extension homomorphism $(0, 0, \ldots, 0, 0)$ from $\mathbb{E}$ to 0. We then get that $0\mathbb{E} = 00 = 0$. Next we have that not only there is a neutral element for the Bear sum but that there are negative ones also such that $\mathbb{E} +_B \mathbb{S} = 0$.

3.2.31 Lemma. *For any given n-extension $\mathbb{E}$ of $\mathbf{B}$ by $\mathbf{A}$ there exists an extension $-\mathbb{E}$ such that $\mathbb{E} +_B (-\mathbb{E}) = 0$.*

Proof. Let -1 be the extension homomorphism from $\mathbf{B}$ to itself. Then $(-1)\mathbb{E}$ is our extension. We see this as

$$\mathbb{E} +_B (-1)\mathbb{E} = 1\mathbb{E} +_B (-1)\mathbb{E} = (1-1)\mathbb{E} = 0\mathbb{E} = 0$$

Q.E.D.

With all these lemmas we have proven that the extended Baer sum gives another extension; it is commutative, has a neutral element and even negative elements for each element. It therefore satisfies every criterion to be an abelian group. We will denote the abelian group of all extension of $\mathbf{B}$ by $\mathbf{A}$ of length n as $E^n(\mathbf{B}, \mathbf{A})$.

Section 4

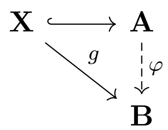
The Ext functor

The two most fundamental functors are Ext and Tor. In this thesis we will only study the first as it is intimately related to the extensions we have previously explored.

4.1 Foundational

In order to set up the Ext functor properly, we need a few tools. We start by defining a special type of a module.

4.1.1 Definition. *(Free module [2])*
A *free module* $\mathbf{A}$ over a set $\mathbf{X}$ is the smallest module such that for any set function (**not** necessarily homomorphism) $g : \mathbf{X} \to \mathbf{B}$, where $\mathbf{B}$ is a left module, there exists a homomorphism φ such that the following diagram commutes.



In some instances this is denoted as $\mathbf{F}(\mathbf{X}) = \mathbf{A}$.

Traditionally free modules are defined as being isomorphic to $\mathbf{R}^n$ for some n. The diagram definition gives us a diagram to work with, the ordinary definition falls within it with $\mathbf{X}$ being the basis of the module. Here we will see a few examples and counterexamples.

4.1.2 Example. $\mathbb{Z}$ is a free module over $\mathbf{X} = \{1\}$ or $\mathbf{X} = \{-1\}$. This is evident as for any homomorphism we must have that $f(n) = n \cdot f(1)$.

The definition only require the existence of a morphism. While for example we have many morphisms $f_n : \mathbb{Z} \to \mathbb{Z}_p$ such that $f_n(x) = nx$, all we need is the one, namely f_1 to exist.

4.1.3 Example. $\mathbb{Z} \oplus \mathbb{Z}$ is a free module over $\mathbf{X} = \{1 \oplus 0, 0 \oplus 1\}$. As before any homomorphism will be determined by how it sends $1 \oplus 0$ and $0 \oplus 1$m as we have

$$a \oplus b = a \oplus 0 + 0 \oplus b = a(1 \oplus 0) + b(0 \oplus 1)$$

4.1.4 Example. The module $\mathbb{Z}_p$ is not free. It may be generated by $\mathbf{X} = \{1\}$ but in

$$\begin{array}{ccc} 1 & \hookrightarrow & \mathbb{Z}_p \\ & \searrow^{g} & \downarrow{\scriptstyle \varphi} \\ & & \mathbb{Z} \end{array}$$

our supposed φ shall preserve 1 so that $\varphi(1) = 1$. However, the only viable homomorphism is $\varphi(x) = 0$ and as such it will not make the diagram commute. Hence, $\mathbb{Z}_p$ is not free.

From this we can see that there must exist an epimorphism from a free module to some other module.

4.1.5 Proposition. *For any module* $\mathbf{A}$ *there exists a free module* $\mathbf{F}(\mathbf{X})$*, with* $\mathbf{X}$ *being the underlying set of our module, such that we have*

$$\mathbf{F}(\mathbf{X}) \twoheadrightarrow \mathbf{A}$$

Proof. Let $\mathbf{X}$ be the underlying set of our module $\mathbf{A}$. Then we clearly have the diagram

$$\begin{array}{ccc} \mathbf{A} & \hookrightarrow & \mathbf{F}(\mathbf{A}) \\ & \searrow^{g} & \downarrow{\scriptstyle \varphi} \\ & & \mathbf{A} \end{array}$$

with g being a bijection. As such our φ must be an epimorphism as $\mathbf{F}(\mathbf{A})$ must at least contain all elements of $\mathbf{A}$. Q.E.D.

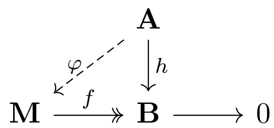
An interesting property that can be observed from this is that any module must be the quotient of a free module.

4.1.6 Corollary. *The first theorem yields the diagram*

$$\begin{array}{ccc} \mathbf{F}(\mathbf{A}) & & \\ \downarrow & \searrow^{\varphi} & \\ \mathbf{F}(\mathbf{A})/\ker\varphi & \hookrightarrow & \mathbf{A} \end{array}$$

Another important type of module is projective modules, which we will use extensively.

4.1.7 Definition. *(Projective module)*
Let $\mathbf{M}$ and $\mathbf{B}$ be modules with an epimorphism $f : \mathbf{M} \to \mathbf{B}$ be given. A module $\mathbf{A}$ is projective if for any homomorphism $h : \mathbf{A} \to \mathbf{B}$ there exists a homomorphism $\varphi : \mathbf{A} \to \mathbf{M}$ such that this diagram commutes.

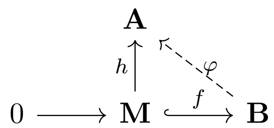


A proposition exists, see [1] Prop 3.2 , that states a module is projective if it can be through direct sum be combined with another module to form a free module. The following example demonstrates it.

4.1.8 Example. Let our ring for the module be $\mathbf{R} = \mathbb{Z}_6$. The module $\mathbb{Z}_2$ is then projective as we have $\mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong \mathbb{Z}_6 = \mathbf{R}$

Like often in category theory there always exists a dual, in our category of left R-modules there exist the dual notion which is injective modules.

4.1.9 Definition. *(Injective module)*
Let $\mathbf{M}$ and $\mathbf{B}$ be modules with a monomorphism $f : \mathbf{M} \to \mathbf{B}$ be given. A module $\mathbf{A}$ is injective if for any homomorphism $h : \mathbf{M} \to \mathbf{A}$ there exists a homomorphism $\varphi : \mathbf{B} \to \mathbf{A}$ such that this diagram commutes.



Having these special types of modules, along with our reliance on exact sequence, from previous sections, it is natural to ponder if they can be combined.

4.1.10 Definition. *(Resolution)*
A resolution of a module $\mathbf{A}$ is an exact sequence that ends in $\mathbf{A}$

$$\ldots \to \mathbf{B}_3 \to \mathbf{B}_2 \to \mathbf{B}_1 \to \mathbf{A}$$

This may be denoted as $\mathbf{B}_\bullet \to \mathbf{A}$ instead.

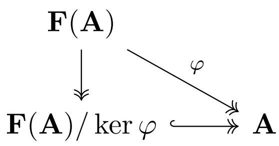
A resolution $\mathbf{B}_\bullet$ is called a *projective resolution* if and only if all $\mathbf{B}_i$ are projective. Respectively we have $\mathbf{B}_\bullet$ being an *injective resolution* if and only if all $\mathbf{B}_i$ are injective. While we primarily focus on projective resolutions here, the lemmas have their dual equivalence. A natural question is if these projective modules exist for a specific collection of modules, or all modules.

4.1.11 Proposition. *For any module* $\mathbf{A}$ *there exists a projective* $\mathbf{P}$ *such that the sequence*

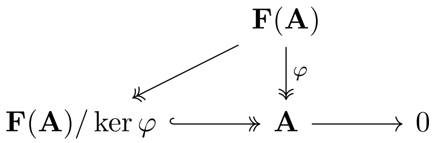
$$\mathbf{P} \twoheadrightarrow \mathbf{A} \to 0$$

is exact.

Proof. We have that for any module $\mathbf{A}$ there exists a free module $\mathbf{F}(\mathbf{A})$ such that this diagram commutes



for some morphism φ. As such we can extend the diagram and get



which gives us that $\mathbf{F}(\mathbf{A})$ is projective, and from this diagram we also get the exact sequence

$$\mathbf{F}(\mathbf{A}) \twoheadrightarrow \mathbf{A} \to 0$$

which gives us that any module has a projective. Q.E.D.

From this we can say that for any module we have a projective resolution available for it.

4.1.12 Proposition. *For any module* $\mathbf{A}$ *we have a projective resolution* $\mathbf{B}_\bullet$.

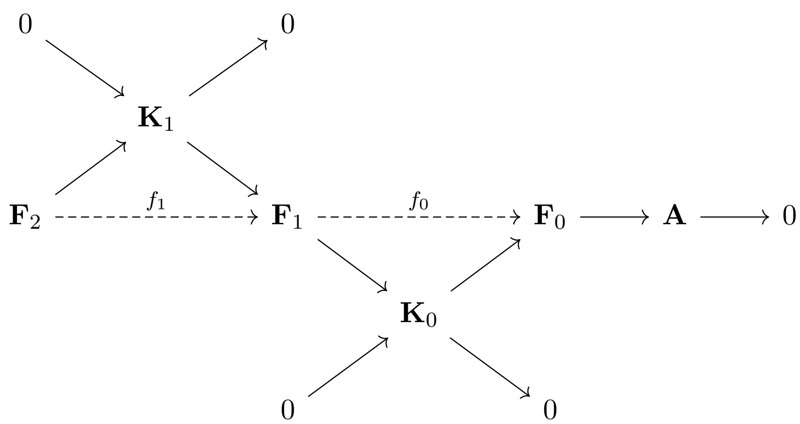
Proof. We have that for any module $\mathbf{A}$ there is a free module $\mathbf{F}_0$ such that $\mathbf{F}_0 \overset{\epsilon_0}{\twoheadrightarrow} \mathbf{A} \to 0$, if we let $\mathbf{K}_0 = \ker \epsilon_0$ we get the short sequence

$$0 \to \mathbf{K}_0 \overset{\imath}{\hookrightarrow} \mathbf{F}_0 \overset{\epsilon_0}{\twoheadrightarrow} \mathbf{A} \to 0$$

In a similar manner we can construct this sequence

$$0 \to \mathbf{K}_1 \overset{\imath}{\hookrightarrow} \mathbf{F}_1 \overset{\epsilon_1}{\twoheadrightarrow} \mathbf{K}_0 \to 0$$

reasoning in the same way and continue this ad infinitum. Combining these sequences we get the following commutative diagram



where f_0 and f_1 are the evident compositions. This diagram continues in a similar manner backward ad infinitum. We get that the induced sequence

$$\ldots \to \mathbf{F}_2 \stackrel{f_2}{\to} \mathbf{F}_1 \stackrel{f_1}{\to} \mathbf{F}_0$$

That $\operatorname{im} f_{i+1} \subseteq \ker f_i$ is evident as we have

$$\begin{aligned} f_i \circ f_{i+1} &= \imath_{i-1} \circ \epsilon_i \circ \imath_i \circ \epsilon_{i+1} \\ &= \imath_{i-1} \circ 0 \circ \epsilon_{i+1} \\ &= 0 \end{aligned}$$

The sequences

$$0 \to \mathbf{K}_i \xrightarrow{\imath_i} \mathbf{F}_i \xrightarrow{\epsilon_i} \mathbf{K}_{i-1} \to 0$$

are exact. For the reverse inclusion we notice that $\ker f_i = \ker \imath_{i-1} \circ \epsilon_i$ which means that $\epsilon_i(\ker f_i) \subseteq \ker \imath_{i-1} = 0$, as $\imath_i$ are all monomorphisms. This in turn gives us that $\ker f_i \subseteq \ker \epsilon_i$ which, again by exact sequence, gives us that $\ker f_i \subseteq \operatorname{im} \imath_i$. This combined with that

$$\operatorname{im} f_{i+1} = \operatorname{im} \imath_i \circ \epsilon_{i+1} = \operatorname{im} \imath_i,$$

as ϵ_i are epimorphisms, gives $\ker f_i \subseteq \operatorname{im} f_{i+1}$. Q.E.D.

In order to established the aforementioned relation between the Ext functor and the abelian group of extensions, the following lemma will play an important role and must therefore be established.

4.1.13 Theorem. (Comparison theorem)
Consider the following diagram

$$\begin{array}{ccccccccc} \ldots & \xrightarrow{p_2} & \mathbf{P}_2 & \xrightarrow{p_1} & \mathbf{P}_1 & \xrightarrow{\rho} & \mathbf{M} & \longrightarrow & 0 \\ & & & & & & \downarrow{\scriptstyle f} & & \\ \ldots & \xrightarrow{q_2} & \mathbf{Q}_2 & \xrightarrow{q_1} & \mathbf{Q}_1 & \xrightarrow{\epsilon} & \mathbf{N} & \longrightarrow & 0 \end{array}$$

If each $\mathbf{P}_i$ is projective and the diagram is exact at each $\mathbf{Q}_i$ then there exists a chain homomorphism $g : \mathbf{P}_\bullet \to \mathbf{Q}_\bullet$ such that this diagram commutes

$$\begin{array}{ccccccccc} \ldots & \xrightarrow{p_2} & \mathbf{P}_2 & \xrightarrow{p_1} & \mathbf{P}_1 & \xrightarrow{\rho} & \mathbf{M} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle f} & & \\ \ldots & \xrightarrow{q_2} & \mathbf{Q}_2 & \xrightarrow{q_1} & \mathbf{Q}_1 & \xrightarrow{\epsilon} & \mathbf{N} & \longrightarrow & 0 \end{array}$$

Moreover, if there exists another $h : \mathbf{P}_\bullet \to \mathbf{Q}_\bullet$, then h and g are homotopic.

Proof. We will demonstrate this through induction. The base is case $n = 1$. From the initial diagram we have the following diagram

$$
\begin{array}{ccccc}
 & & \mathbf{P}_1 & & \\
 & \swarrow^{g_1} & \downarrow{\scriptstyle f\circ\rho} & & \\
\mathbf{Q}_1 & \xrightarrow{\epsilon} & \mathbf{N} & \longrightarrow & 0
\end{array}
$$

where g_1 must exist as $\mathbf{P}_1$ is projective. For the induction step, consider the following diagram

$$
\begin{array}{ccccccc}
\mathbf{P}_{n+1} & \xrightarrow{p_n} & \mathbf{P}_n & \xrightarrow{p_{n-1}} & \mathbf{P}_{n-1} & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle g_n} & & \downarrow{\scriptstyle g_{n-1}} & & \\
\mathbf{Q}_{n+1} & \xrightarrow{q_n} & \mathbf{Q}_n & \xrightarrow{q_{n-1}} & \mathbf{Q}_{n-1} & \longrightarrow & \dots
\end{array}
$$

where g_n and g_{n-1} exist by the induction hypothesis. By commutativity we have that

$$q_{n-1} \circ g_n \circ p_n = g_{n-1} \circ p_{n-1} \circ p = g_{n-1} \circ 0 = 0$$

and as such we have $\operatorname{im} g_n \circ p_n \subseteq \ker q_{n-1} = \operatorname{im} q_n$ by exactness at $\mathbf{Q}_i$. From this we get the diagram

$$
\begin{array}{ccccc}
 & & \mathbf{P}_{n+1} & & \\
 & \swarrow^{g_{n+1}} & \downarrow{\scriptstyle g_n\circ p_n} & & \\
\mathbf{Q}_{n+1} & \xrightarrow{q_n} & \operatorname{im} q_n & \longrightarrow & 0
\end{array}
$$

where the existence of g_{n+1} assured as $\mathbf{P}_{n+1}$ is projective.

For the homotopic part, let h be another given chain map along with g from $\mathbf{P}_\bullet$ to $\mathbf{Q}_\bullet$. We will construct our homotopy chain map s through induction. For our base case $n = 0$ we have $s_0 = s_{-1} = 0$ as our $g_0 = h_0 = f$ where $\mathbf{P}_0 = \mathbf{M}$ and $\mathbf{Q}_0 = \mathbf{N}$, further we have then $g_0 - h_0 = f - f = 0$ and

$$\rho \circ s_0 - s_{-1} \circ 0 = 0$$

which fulfils the criterion of homotopy. For the induction step we consider

$$
\begin{array}{ccccc}
\mathbf{P}_{n+1} & \xrightarrow{p_n} & \mathbf{P}_n & \xrightarrow{p_{n-1}} & \mathbf{P}_{n-1} \\
{\scriptstyle g_{n+1}}\downarrow\downarrow{\scriptstyle h_{n+1}} & & {\scriptstyle g_n}\downarrow\downarrow{\scriptstyle h_n} & \swarrow_{s_{n-1}} & {\scriptstyle g_{n-1}}\downarrow\downarrow{\scriptstyle h_{n-1}} \\
\mathbf{Q}_{n+1} & \xrightarrow{q_n} & \mathbf{Q}_n & \xrightarrow{q_{n-1}} & \mathbf{Q}_{n-1}
\end{array}
$$

It suffices for us to show that $\operatorname{im} g_n - h_n - s_{n-1} \circ p_{n-1} \subseteq \operatorname{im} q_n$, because then we have

$$
\begin{array}{ccccc}
 & & \mathbf{P}_n & & \\
 & \swarrow^{s_n} & \downarrow{\scriptstyle g_n-h_n-s_{n-1}\circ p_{n-1}} & & \\
\mathbf{Q}_{n+1} & \xrightarrow{q_n} & \operatorname{im} q_n & \longrightarrow & 0
\end{array}
$$

as $\mathbf{P}_n$ is projective. As the lower row is exact we have that $\operatorname{im} q_n = \ker q_{n-1}$, so it suffices to show

$$q_{n-1} \circ (g_n - h_n - s_{n-1} \circ p_{n-1}) = 0.$$

We have at the same time

$$\begin{aligned} q_{n-1} \circ (g_n - h_n - s_{n-1} \circ p_{n-1}) &= q_{n-1} \circ (g_n - h_n) - q_{n-1} \circ s_{n-1} \circ p_{n-1} \\ &= q_{n-1} \circ (g_n - h_n) - (g_{n-1} - h_{n-1} - s_{n-2} \circ p_{n-2}) \circ p_{n-1} \\ &= q_{n-1} \circ (g_n - h_n) - (g_{n-1} - h_{n-1}) \circ p_{n-1} + s_{n-2} \circ p_{n-2} \circ p_{n-1} \\ &= q_{n-1} \circ (g_n - h_n) - (g_{n-1} - h_{n-1}) \circ p_{n-1} \\ &= 0 \end{aligned}$$

where the last terms disappear by commutativity as they are chain maps. Q.E.D.

A trivial thing to see from the comparison theorem is the following result.

4.1.14 Corollary. *If* $\mathbf{M}$ *and* $\mathbf{N}$ *have projective resolutions* $\mathbf{P}_\bullet$ *and* $\mathbf{Q}_\bullet$ *along with a homomorphism* $f : \mathbf{M} \to \mathbf{N}$ *then there exists a chain map* $g : \mathbf{P}_\bullet \to \mathbf{Q}_\bullet$.

With these things we have the fundamental tools that are required in order to work with the Ext functor itself.

4.2 Some properties

The final functor and section of this thesis will be the Ext-functor. The name of it derives from Extension. The connection was discovered from the study of extending abelian groups. While primarily done in extensions of length one the functors relation to extensions can be extended here too. However we will first go through what Ext is and its properties. Finally we will prove there is an isomorphism between Ext and the group of extensions.

4.2.1 Definition. Let $\mathbf{M}$ and $\mathbf{B}$ be modules and $\mathbf{P}_\bullet$ a projective resolution of $\mathbf{M}$. This gives us the sequence

$$0 \to \operatorname{Hom}(\mathbf{P}_1, \mathbf{M}) \to \operatorname{Hom}(\mathbf{P}_2, \mathbf{M}) \to \dots$$

and from this we define

$$\operatorname{Ext}^n_{\mathbf{R}}(\mathbf{M}, \mathbf{B}) = H^n(\operatorname{Hom}(\mathbf{P}_\bullet, \mathbf{B}))$$

Projective resolutions may differ from one another by substituting modules and homomorphisms. However, the next lemma shows that our Ext-functor is independent of the choice of projective resolution.

4.2.2 Lemma. *If* $\mathbf{P}_\bullet$ *and* $\mathbf{Q}_\bullet$ *are projective resolutions of* $\mathbf{M}$*, then* $\operatorname{Ext}^n(\mathbf{B}, \mathbf{M})$ *of both projections are isomorphic. That is*

$$H^n(\operatorname{Hom}(\mathbf{P}_\bullet, \mathbf{B})) \cong H^n(\operatorname{Hom}(\mathbf{Q}_\bullet, \mathbf{B}))$$

Proof. We note first that the functor $\mathcal{F} = \operatorname{Hom}(-, \mathbf{B})$ is additive, that is $\operatorname{Hom}(f+g, \mathbf{B}) = \operatorname{Hom}(f, \mathbf{B}) + \operatorname{Hom}(g, \mathbf{B})$ where f and g are homomorphisms. By the comparison theorem we have that there exists $s : \mathbf{P}_\bullet \to \mathbf{Q}_\bullet$ which extends from id_M and equally $t : \mathbf{Q}_\bullet \to \mathbf{P}_\bullet$. Thus we have $s \circ t : \mathbf{Q}_\bullet \to \mathbf{Q}_\bullet$. We also have $\mathrm{id} : \mathbf{Q}_\bullet \to \mathbf{Q}_\bullet$. As both extend id_M to a chain map from $\mathbf{Q}_\bullet$ to $\mathbf{Q}_\bullet$, by comparison theorem, we have that $s \circ t$ and id are homotopic by comparison theorem. Hence, so are $\mathcal{F}(s \circ t)$ and $\mathcal{F}(\mathrm{id})$ as we have

$$\begin{aligned}\mathcal{F}(s_n \circ t_n - \mathrm{id}) &= \mathcal{F}(s_n \circ t_n) - \mathcal{F}(\mathrm{id}) \\ &= \mathcal{F}(q_{n+1} \circ s_n + s_{n-1} \circ p_n) \\ &= \mathcal{F}(q_{n+1} \circ s_n) + \mathcal{F}(s_{n-1} \circ p_n)\end{aligned}$$

That means that the induced homomorphisms between homologies are equivalent, that is, $\mathcal{F}(s)^* \circ \mathcal{F}(t)^* = \mathcal{F}(s \circ t)^* = \mathcal{F}(\mathrm{id})^*$. Applying similar argument we get that $\mathcal{F}(t)^* \circ \mathcal{F}(s)^* = \mathcal{F}(t \circ s)^* = \mathcal{F}(\mathrm{id})^*$, and as such $\mathcal{F}(t)^*$ and $\mathcal{F}(s)^*$ are both isomorphisms. Q.E.D.

These lemmas and definitions are all we need to finalize this thesis and see where the Ext functor got its name from. Namely, from the fact that $\operatorname{Ext}^n(\mathbf{B}, \mathbf{A}) \cong E^n(\mathbf{B}, \mathbf{A})$, which is proven in the following theorem and finalize this thesis.

4.2.3 Theorem. *Let* $E^n(\mathbf{B}, \mathbf{A})$ *denote the abelian group of* n*-extensions of* $\mathbf{B}$ *by* $\mathbf{A}$*. Then we have* $E^n(\mathbf{B}, \mathbf{A}) \cong \operatorname{Ext}^n(\mathbf{B}, \mathbf{A})$*.*

Proof. We will first construct a function $\psi : E^n(\mathbf{B}, \mathbf{A}) \to \operatorname{Ext}^n(\mathbf{B}, \mathbf{A})$ and an inverse $\theta : \operatorname{Ext}^n(\mathbf{B}, \mathbf{A}) \to \psi : E^n(\mathbf{B}, \mathbf{A})$. Afterwards we show that both of them are homomorphisms. The proof is finalized by showing that the composition of them, in any order, yields the identity homomorphism.

ψ **is well-defined**:

Let $\mathbf{P}_\bullet$ by a projective resolution for $\mathbf{A}$ and $\mathbb{E}$ an n-extension, then we have the following commutative diagram by comparison theorem

$$\begin{array}{ccccccccccccc} \dots & \longrightarrow & \mathbf{P}_{n+1} & \xrightarrow{p_n} & \mathbf{P}_n & \xrightarrow{p_{n-1}} & \mathbf{P}_{n-1} & \longrightarrow & \dots & \xrightarrow{p_0} & \mathbf{P}_0 & \xrightarrow{\alpha} & \mathbf{A} \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n-1}} & & & & \downarrow{\scriptstyle \varphi_0} & & \downarrow{\scriptstyle 1} \\ \dots & \longrightarrow & 0 & \longrightarrow & \mathbf{B} & \xhookrightarrow{\imath} & \mathbf{E}_n & \longrightarrow & \dots & \longrightarrow & \mathbf{E}_1 & \xrightarrow{\pi} & \mathbf{A} \end{array}$$

From here we can see that $\varphi_n \circ p_n = 0$. For the cohomology of Ext^n we have

$$\ldots \to \operatorname{Hom}(P_{n-1}, \mathbf{B}) \xrightarrow{p_{n-1}^*} \operatorname{Hom}(P_n, \mathbf{B}) \xrightarrow{p_n^*} \operatorname{Hom}(P_{n+1}, \mathbf{B}) \to \ldots$$

where $p_n^* = - \circ p_n$. This means that $\varphi_n \circ p_n = p_n^*(\varphi) = 0$ and as such we have $\varphi_n \in \ker p_n^*$. We therefore define $\psi(\mathbb{E}) = \varphi_n + \operatorname{im} p_{n-1}^*$. We will now check that this is well-defined so that it does not depend on neither the choice of projective resolution nor the choice of our extension representation. For the choice of projection we have that the comparison theorem gives us that any two chain maps will be homotopic and hence be equivalent. Let $\mathbb{F}$ be another extension equivalent to $\mathbb{E}$, as such we have a chain map $\sigma = (1, \sigma_n, \ldots, \sigma_1, 1) : \mathbb{E} \to \mathbb{F}$ with this commutative diagram

$$\begin{array}{ccccccccc}
\mathbf{B} & \hookrightarrow & \mathbf{E}_n & \longrightarrow & \ldots & \longrightarrow & \mathbf{E}_1 & \twoheadrightarrow & \mathbf{A} \\
\downarrow & & \downarrow{\scriptstyle \sigma_n} & & & & \downarrow{\scriptstyle \sigma_1} & & \downarrow \\
\mathbf{B} & \hookrightarrow & \mathbf{F}_n & \longrightarrow & \ldots & \longrightarrow & \mathbf{F}_1 & \twoheadrightarrow & \mathbf{A}
\end{array}$$

This produces following diagram

$$\begin{array}{ccccccccccccc}
\ldots & \longrightarrow & \mathbf{P}_{n+1} & \xrightarrow{p_n} & \mathbf{P}_n & \xrightarrow{p_{n-1}} & \mathbf{P}_{n-1} & \longrightarrow & \ldots & \xrightarrow{p_0} & \mathbf{P}_0 & \xrightarrow{\alpha} & \mathbf{A} \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n-1}} & & & & \downarrow{\scriptstyle \varphi_0} & & \downarrow \\
\ldots & \longrightarrow & 0 & \longrightarrow & \mathbf{B} & \xhookrightarrow{\imath} & \mathbf{E}_n & \longrightarrow & \ldots & \longrightarrow & \mathbf{E}_1 & \xrightarrow{\pi} & \mathbf{A} \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \sigma_n} & & & & \downarrow{\scriptstyle \sigma_1} & & \downarrow \\
\ldots & \longrightarrow & 0 & \longrightarrow & \mathbf{B} & \hookrightarrow & \mathbf{F}_n & \longrightarrow & \ldots & \longrightarrow & \mathbf{F}_1 & \twoheadrightarrow & \mathbf{A}
\end{array}$$

which gives us that $\psi(\mathbb{E}) = \varphi_n + \operatorname{im} p_{n-1} = \psi(\mathbb{F})$ by commutativity. It is therefore well-defined.

ψ **is a homomorphism**:

To check if ψ is a homomorphism we need to confirm that $\psi(\nabla(\mathbb{E} \oplus \mathbb{F})\Delta) = \nabla \circ (\psi(\mathbb{E}) \oplus \psi(\mathbb{F})) \circ \Delta$. We will do this in three steps.

Step 1

In the first step we will show that $\psi(\nabla(\mathbb{E} \oplus \mathbb{F})\Delta) = \psi(\nabla(\mathbb{E} \oplus \mathbb{F})) \circ \Delta$. Let $\mathbf{P}_\bullet$ be a projective resolution of $\mathbf{A}$. Then $\mathbf{P}_\bullet \oplus \mathbf{P}_\bullet$ is a projective resolution of $\mathbf{A} \oplus \mathbf{A}$ and we get this diagram.

$$
\begin{array}{ccccccccccc}
\dots & \longrightarrow & \mathbf{P}_n & \longrightarrow & \mathbf{P}_{n-1} & \longrightarrow & \dots & \longrightarrow & \mathbf{P}_0 & \longrightarrow & \mathbf{A} \\
 & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} & & & & \downarrow{\scriptstyle \Delta} & & \downarrow{\scriptstyle \Delta} \\
\dots & \longrightarrow & \mathbf{P}_n \oplus \mathbf{P}_n & \longrightarrow & \mathbf{P}_{n-1} \oplus \mathbf{P}_{n-1} & \longrightarrow & \dots & \longrightarrow & \mathbf{P}_0 \oplus \mathbf{P}_0 & \longrightarrow & \mathbf{A} \oplus \mathbf{A} \\
 & & \downarrow{\scriptstyle \beta} & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle 1\oplus 1} \\
\dots & \longrightarrow & \mathbf{B} & \longrightarrow & \mathbf{E}_n \amalg \mathbf{F}_n & \longrightarrow & \dots & \longrightarrow & \mathbf{E}_1 \oplus \mathbf{F}_1 & \longrightarrow & \mathbf{A} \oplus \mathbf{A}
\end{array}
$$

We have a homomorphism $\sigma : \mathbf{P}_n \to \mathbf{B}$ given from comparison theorem. However, as we have two chain maps from the top row to the bottom row, one directly and the other through the middle row, we then have that $\sigma + \operatorname{im} p_n^* \cong \beta \circ \Delta + \operatorname{im} p_n^*$. With $\beta = \psi(\nabla(\mathbb{E} \oplus \mathbb{F}))$ we get the desired result.

Step 2
In this step we will show that $\psi(\nabla(\mathbb{E}\oplus\mathbb{F})) = \nabla \circ \psi(\mathbb{E}\oplus\mathbb{F})$. This one follows from the following diagram

$$
\begin{array}{ccccccccccc}
\dots & \longrightarrow & \mathbf{P}_n \oplus \mathbf{P}_n & \longrightarrow & \mathbf{P}_{n-1} \oplus \mathbf{P}_{n-1} & \longrightarrow & \dots & \longrightarrow & \mathbf{P}_0 \oplus \mathbf{P}_0 & \longrightarrow & \mathbf{A} \oplus \mathbf{A} \\
 & & \downarrow{\scriptstyle \gamma} & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle 1\oplus 1} \\
\dots & \longrightarrow & \mathbf{B} \oplus \mathbf{B} & \longrightarrow & \mathbf{E}_n \oplus \mathbf{F}_n & \longrightarrow & \dots & \longrightarrow & \mathbf{E}_1 \oplus \mathbf{F}_1 & \longrightarrow & \mathbf{A} \oplus \mathbf{A} \\
 & & \downarrow{\scriptstyle \nabla} & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle 1} \\
\dots & \longrightarrow & \mathbf{B} & \longrightarrow & \mathbf{E}_n \amalg \mathbf{F}_n & \longrightarrow & \dots & \longrightarrow & \mathbf{E}_1 \oplus \mathbf{F}_1 & \longrightarrow & \mathbf{A} \oplus \mathbf{A}
\end{array}
$$

with $\beta : \mathbf{P}_n \oplus \mathbf{P}_n \to \mathbf{B}$ is given. Therefore we have $\beta + \operatorname{im} p_n^* \oplus p_n^* \cong \nabla \circ \gamma + \operatorname{im} p_n^* \oplus p_n^*$. With $\gamma = \psi(\mathbb{E} \oplus \mathbb{F})$ the desired result comes out.

Step 3
For the final step we show that $\psi(\mathbb{E} \oplus \mathbb{F}) = \psi(\mathbb{E}) \oplus \psi(\mathbb{F})$. This can be seen in the following diagram

$$
\begin{array}{ccccccccccc}
\dots & \longrightarrow & \mathbf{P}_n \oplus \mathbf{P}_n & \longrightarrow & \mathbf{P}_{n-1} \oplus \mathbf{P}_{n-1} & \longrightarrow & \dots & \longrightarrow & \mathbf{P}_0 \oplus \mathbf{P}_0 & \longrightarrow & \mathbf{A} \oplus \mathbf{A} \\
 & & \downarrow{\scriptstyle \epsilon\oplus\phi} & & \downarrow & & & & \downarrow & & \downarrow{\scriptstyle 1\oplus 1} \\
\dots & \longrightarrow & \mathbf{B} \oplus \mathbf{B} & \longrightarrow & \mathbf{E}_n \oplus \mathbf{F}_n & \longrightarrow & \dots & \longrightarrow & \mathbf{E}_1 \oplus \mathbf{F}_1 & \longrightarrow & \mathbf{A} \oplus \mathbf{A}
\end{array}
$$

where $\epsilon = \psi(\mathbb{E})$ and $\phi = \psi(\mathbb{F})$. From all these we have

$$\begin{aligned}
\psi(\mathbb{E} +_B \mathbb{F}) &= \psi(\nabla(\mathbb{E} \oplus \mathbb{F})\Delta) \\
&= \psi(\nabla(\mathbb{E} \oplus \mathbb{F})) \circ \Delta \\
&= \nabla \circ \psi(\mathbb{E} \oplus \mathbb{F}) \circ \Delta \\
&= \nabla \circ (\psi(\mathbb{E}) \oplus \psi(\mathbb{F})) \circ \Delta \\
&= \psi(\mathbb{E}) + \psi(\mathbb{F})
\end{aligned}$$

So it is a homomorphism.

θ is well-defined

Next we construct the inverse of ψ which we label $\theta : \operatorname{Ext}^n(\mathbf{B}, \mathbf{A}) \to E^n(\mathbf{B}, \mathbf{A})$. Let $\mathbf{P}_\bullet$ be a projective resolution of $\mathbf{A}$ again. Next pick $\alpha \in \operatorname{Ext}^n(\mathbf{B}, \mathbf{A}) = \ker p_n^* / \operatorname{im} p_{n-1}^*$ arbitrarily. That means we have $0 = \alpha \circ p_n = p_n^*(\alpha)$ and α therefore induces another homomorphism $\alpha' : \mathbf{P}_n / \operatorname{im} p_n \to \mathbf{B}$ with $\alpha'(x + \operatorname{im} p_n) = \alpha(x)$. Next let $\mathbb{X}$ denote the following extension

$$\mathbb{X} : \mathbf{P}_n / \operatorname{im} p_n \to \mathbf{P}_{n-1} \to \ldots \to \mathbf{P}_1 \to \mathbf{A}$$

We therefore define $\theta(\alpha) = \alpha' \mathbb{X}$ which is an extension of $\mathbf{B}$ by $\mathbf{A}$. It is well-defined by homotopy from the comparison theorem.

θ is a homomorphism

To see that θ is a homomorphism we notice that $(\alpha + \beta)' = \alpha' + \beta'$ and as such we have

$$\begin{aligned}
\theta(\alpha + \beta) &= (\alpha + \beta)' \mathbb{X} \\
&= (\alpha' + \beta') \mathbb{X} \\
&= \alpha' \mathbb{X} +_B \beta' \mathbb{X} \\
&= \theta(\alpha) +_B \theta(\beta)
\end{aligned}$$

Composition is identity

Now we will show that $\theta \circ \psi = \operatorname{id}$ and $\psi \circ \theta = \operatorname{id}$. Let $\alpha \in \operatorname{Ext}^n(\mathbf{B}, \mathbf{A})$ be given, then we have the following commutative diagram

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & \mathbf{P}_n & \longrightarrow & \mathbf{P}_{n-1} & \longrightarrow & \ldots & \longrightarrow & \mathbf{P}_0 & \longrightarrow & \mathbf{A} \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow & & & & \downarrow & & \downarrow \\
\ldots & \longrightarrow & \mathbf{B} & \longrightarrow & \mathbf{E}_n & \longrightarrow & \ldots & \longrightarrow & \mathbf{E}_1 & \longrightarrow & \mathbf{A}
\end{array}$$

where the bottom row is $\theta(\alpha)$. From this diagram it can also be seen that $\psi \circ \theta(\alpha) = \alpha$. Hence, we have that $\psi \circ \theta = \operatorname{id}$. Next, let $\mathbb{E}$ be a given extension. We then have the following diagram

$$
\begin{array}{ccccccccc}
\mathbf{P}_n/\operatorname{im} p_n & \longrightarrow & \mathbf{P}_{n-1} & \longrightarrow & \dots & \longrightarrow & \mathbf{P}_0 & \longrightarrow & \mathbf{A} \\
\downarrow{\scriptstyle \alpha'} & & \downarrow & & & & \downarrow & & \downarrow \\
\mathbf{B} & \longrightarrow & \mathbf{E}_n & \longrightarrow & \dots & \longrightarrow & \mathbf{E}_1 & \longrightarrow & \mathbf{A}
\end{array}
$$

with $\psi(\mathbb{E}) = \alpha$, then we have that $\mathbb{E} = \alpha'\mathbb{X}$ and as such $\theta \circ \psi = \text{id}$ which gives us that ψ is an isomorphism, therefore $\text{Ext}^n(\mathbf{B}, \mathbf{A})$ and $E^n(\mathbf{B}, \mathbf{A})$ are isomorphic.

Q.E.D.

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