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Powers and Products of Monomial Ideals

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A large, faint watermark of the Uppsala University seal is visible in the bottom right corner of the page. The seal features a sun with rays, a cross, and the Latin motto "ALERE FLAMMAM VERITATIS" (to feed the flame of truth).

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Abstract

This thesis is about powers and products of monomial ideals in polynomial rings. We find necessary and sufficient conditions on powers and products of monomial ideals on the polynomial ring $K[x, y]$ for their graphs to take certain staircase-like shapes. In the case of powers, these shapes are repeated for each higher power, so that knowledge of the conditions simplifies the calculation of the powers. We also explore connections to areas of commutative algebra where powers of ideals are important, notably the Hilbert and Hilbert-Samuel functions in dimension theory.

Contents

1	Introduction and preliminary concepts	3
1.1	Rings and ideals	3
1.2	Modules	6
2	Monomial ideals and conditions on their powers and products	8
2.1	Polynomial rings and monomial ideals	8
2.2	Powers of monomial ideals	15
2.3	Products of monomial ideals	27
3	Connections with other areas: the Hilbert and Hilbert-Samuel functions	31
3.1	Graded rings and the Hilbert function	31
3.2	Local rings, localization, and Noetherian rings	32
3.3	The Hilbert-Samuel function	35

1 Introduction and preliminary concepts

The thesis consists of three sections. This first section contains definitions and explanations of concepts and some basic results required to understand the second section, and some of the concepts required for the third section. In the second section, we find and prove conditions on monomial ideals for their powers and products to have a certain form. In the third and last section, we define and explain the Hilbert and Hilbert-Samuel function, before showing how they may be calculated in a simple way on the polynomial ring $K[x, y]$ by applying the conditions found in section 2.

The first section consists of three subsections. The first subsection is concerned with basic properties and operations on rings and ideals. The second subsection introduces the concept of a module, which will be fundamental for the later discussion of the Hilbert and Hilbert-Samuel functions. In the third subsection, polynomial rings and monomial ideals are introduced, as well as their graphs and the notion of a staircase ideal. Most of the definitions in this section will follow David Eisenbud, *Commutative algebra*. [1]

1.1 Rings and ideals

A ring is an algebraic structure where the operations addition and multiplication are defined in such a way that they work similarly to the arithmetics on the integers.

Definition 1. *A ring is an abelian group $\langle R, + \rangle$ together with an operation called multiplication and an identity element 1, satisfying the following conditions for all $a, b, c \in R$:*

1. $a \cdot (b \cdot c) = a \cdot b \cdot (c)$ *(associativity)*
2. $a \cdot (b + c) = a \cdot b + a \cdot c$
3. $(b + c) \cdot a = b \cdot a + c \cdot a$ *(distributivity)*
4. $1 \cdot a = a \cdot 1 = a$ *(identity)*

If the multiplication operation is commutative, the ring is said to be a commutative ring. Usually, the multiplication sign is omitted, so that e.g. $a \cdot b$ is written ab .

In the following, all rings will be considered to be commutative.

A paradigmatic example of a commutative ring is the set of integers \mathbb{Z} together with the usual operations of addition and multiplication. Another example of a ring, prefiguring the discussion in section 1.3, is the ring $\mathbb{R}[X]$

of polynomial functions in one variable $p(x)$ on the real numbers \mathbb{R} , with addition and multiplication defined as usual in calculus: $(p+r)(x) = p(x) + r(x)$, and $(pr)(x) = p(x)r(x)$. The identity element is the constant function $p(x) = 1$.

It is easily seen that this satisfies the axioms of associativity, distributivity and commutativity.

Definition 2. *An ideal in a commutative ring R is a non-empty subset $I \subseteq R$ such that I is closed under addition and $rx \in I$ for all $x \in I$ and all $r \in R$.*

An example of an ideal in \mathbb{Z} is the set $6\mathbb{Z}$ consisting of all numbers divisible by 6.

An example of an ideal in $\mathbb{R}[X]$ is the set of polynomial functions with only powers greater or equal than l of the variable, together with the 0-element. All elements except for 0 may be written $p(x) = \sum_{i=l}^n a_i x^i$, with $a \in \mathbb{R}$. With $r(x) = \sum_{j=l}^n b_j x^j$ we have that $(p+r)(x) = p(x) + r(x) = \sum_{k=l}^n (a_k + b_k) x^k$, so the set is closed under addition. With $h(x) = \sum_{j=0}^m c_j x^j$ we have $(fh)(x) = f(x)h(x) = \sum_{k=l}^{m+n} (\sum_{i+j=k} a_i c_j) x^k$, so the set is closed under multiplication with elements of $\mathbb{R}[X]$.

In the following, ideals will frequently be identified by their generating sets.

Definition 3. *An ideal I on a ring R is said to be generated by a subset S if every element $a \in I$ can be written on the form $a = \sum_{i=1}^n s_i r_i$, with each $s_i \in S$ and each $r_i \in R$.*

An ideal is principal if it can be generated by one element.

The ideals which we are going to investigate will in general be finitely generated ideals, that is, ideals generated by finite subsets. It is then sometimes convenient to write $I = R\langle s_1, s_2, \dots, s_n \rangle$ for the ideal generated by the subset $S = \{s_1, s_2, \dots, s_n\}$ of the ring R . When there is no doubt as to which ring is intended, we will in the following use the simplified notation $\langle s_1, s_2, \dots, s_n \rangle$ for the ideal generated by the subset.

It is possible to define various operations on ideals, such that the resulting set is also an ideal.

Proposition 1. *Let I and J be ideals on a commutative ring R . Then the following operations on ideals give rise to an ideal:*

- (i) *The intersection $I \cap J$.*

- (ii) The sum $I + J = J + I = \{a + b \mid a \in I \wedge b \in J\}$.
- (iii) The product $IJ = JI = \{\sum_{i=1}^n a_i b_i \mid a_i \in I \wedge b_i \in J\}$.
- (iv) The quotient ideal $I : J = \{r \in R \mid rb \in I \text{ for all } b \in J\}$.
- (v) The radical $\sqrt{I} = \{x \in R \mid x^n \in I \text{ for some } n \leq 1\}$.

Proof. (i) In the intersection case, any elements a and b of $I \cap J$ will be elements of both I and J . Since these are ideals, they are closed under addition, so the sum $a + b$ will be an element of both I and J as well, and consequently of $I \cap J$. Similarly, the product ra of any element $r \in R$ with any element $a \in I \cap J$ will be in $I \cap J$, since it must be in I and J , because these are ideals.

- (ii) In the sum case, let $a, c \in I$ and $b, d \in J$, so that $a + b \in I + J$ and $c + d \in I + J$. Then $(a + b) + (c + d) = (a + c) + (b + d)$ because of associativity and commutativity, and since $a + c \in I$ and $b + d \in J$, $(a + c) + (b + d) \in I + J$. With $r \in R$, we have from distributivity that $r(a + b) = ra + rb \in I + J$, since $ra \in I$ and $rb \in J$.
- (iii) Let $r, s \in IJ$ with $r = \sum_{i=1}^n a_i b_i$ and $s = \sum_{i=n+1}^{n+m} a_i b_i$, such that all $a_i \in I$ and all $b_i \in J$. Then $r + s = \sum_{i=1}^{n+m} a_i b_i$, and since all $a_i \in I$ and all $b_i \in J$, $r + s \in IJ$. Furthermore, let $t \in R$. Then, $tr = t \sum_{i=1}^n a_i b_i = \sum_{i=1}^n (ta_i) b_i$. Since I is an ideal in R , all $ta_i \in I$, so $tr \in IJ$.
- (iv) Let $r, s \in I : J$. Then, from distributivity, $(r + s)b = rb + sb \in I$ for all $b \in J$, since $rb \in I$ for all $b \in J$ and $sb \in I$ for all $b \in J$, so $r + s \in I : J$. Let $t \in R$. Then $(tr)b = t(rb) \in I$ for all $b \in J$, since $rb \in I$ for all $b \in J$ and I is an ideal in R .
- (v) Let $r, s \in \sqrt{I}$, with $r^n \in I$ and $s^m \in I$. Then, the binomial theorem gives that $(r + s)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} r^k s^{m+n-k} \in \sqrt{I}$. This is because for all k , either $k \geq m$ or $m + n - k > m + n - m = n$, so each term in the sum will have one factor in I and the other factors in R . Let $t \in R$. Then, since R is commutative, $r^n \in I$, and $t^n \in R$, $(tr)^n = t^n r^n \in I$. Thus, $tr \in \sqrt{I}$. ■

This essay will be primarily concerned with products and the special case of powers of ideals. If I is an ideal in a commutative ring, its n -th power $I^n = \{a_1 a_2 \cdots a_n \mid a_1, a_2, \dots, a_n \in I\}$.

The concept of an ideal allows us to define an important kind of ring, the quotient ring.

Definition 4. For an ideal I on a commutative ring R , we may define the relation \sim : $a \sim b$ iff $a - b \in I$. The equivalence class with respect to the relation \sim of an element $a \in R$ is $[a] = \{a + r : r \in I\}$. The quotient ring R/I consists of the set of such equivalence classes of elements of R , with addition on R/I defined as $[a] + [b] = [a + b]$ and multiplication defined as $[a][b] = [ab]$.

For example, for the ring of integers \mathbb{Z} and the ideal $6\mathbb{Z}$, the quotient ring $\mathbb{Z}/6\mathbb{Z} = \{[0], [1], [2], [3], [4], [5]\}$.

For section 3, we will need the concepts of maximal and primary ideals.

Definition 5. Let R be a ring. An ideal $I \subsetneq R$ is maximal if there is no ideal J in R such that $I \subsetneq J \subsetneq R$.

Definition 6. An ideal $I \subseteq R$ is primary if $I \neq R$ and, for every $x, y \in R$, $xy \in I$ implies that $x \in I$ or $y^n \in I$ for some $n > 0$.

Furthermore, if $\sqrt{I} = m$, where $m \subseteq R$ is a maximal ideal, then I is said to be m -primary. [4][p. 275]

The ideal $9\mathbb{Z}$ (generated by a prime power) is a primary ideal. Assume that an element $a \in 9\mathbb{Z}$ is written $a = xy$. Then either $9|x$, in which case $x \in I$, $9|y$, in which case $y^1 \in I$, or $3|x$ and $3|y$, in which case $y^2 \in I$. $9\mathbb{Z}$ is also a $3\mathbb{Z}$ -primary ideal, since $\sqrt{9\mathbb{Z}} = 3\mathbb{Z}$, which is maximal.

The ideal $6\mathbb{Z}$ is not a primary ideal, since e. g. the element 6 may be written $6 = 2 \cdot 3$, with $2 \notin 6\mathbb{Z}$ and $3^n \notin 6\mathbb{Z}$ for any $n > 0$. Notice that $\sqrt{6\mathbb{Z}} = 6\mathbb{Z}$, which is not maximal, so $6\mathbb{Z}$ is also not m -primary for any m .

1.2 Modules

Modules will be important for the discussion in the third section of this thesis. The notion of a module is a generalization of the notion of a vector space. The main difference between modules and vector spaces is that the scalars of a module need only form a ring, whereas the scalars of a vector space must form a field.

Definition 7. Let R be a commutative ring. An R -module M consists of an abelian group M with an operation $\cdot : R \times M \rightarrow M$ satisfying the following conditions for all $x, y \in M$ and for all $r, s \in R$:

1. $r \cdot (x + y) = r \cdot x + r \cdot y$
2. $(r + s) \cdot x = r \cdot x + s \cdot x$
3. $(rs) \cdot x = r \cdot (s \cdot x)$
4. $1_R \cdot x = x$, where 1_R is the multiplicative identity of the ring.

The ring R itself is always an R -module. (This follows from the definition of a ring.)

Ideals and quotient rings of a ring R may be considered as modules over R . For instance, the ideal $6\mathbb{Z}$ of \mathbb{Z} and the quotient ring $\mathbb{Z}/6\mathbb{Z} = \{[0], [1], [2], [3], [4], [5]\}$ are modules over \mathbb{Z} .

Another example of a module is the vector space of Euclidean vectors (ordered triples) (a, b, c) in \mathbb{R}^3 over the field of real numbers \mathbb{R} .

Definition 8. Let R be a commutative ring, M be an R -module and N be a subgroup of M . Then N is a R -submodule of M if, for any $x \in N$ and $r \in R$, $r \cdot x \in N$.

For example, the subgroup $12\mathbb{Z}$ of the ideal $6\mathbb{Z}$ seen as a module over \mathbb{Z} is a submodule, since it is an additive subgroup which is also an ideal over \mathbb{Z} .

As in the case of rings, it is possible to construct quotient modules.

Definition 9. For a submodule N of a module M over a commutative ring R , we may define the relation $\sim: a \sim b$ iff $a - b \in N$. The equivalence class with respect to the relation \sim of an element $a \in M$ is $[a] = \{a + r : r \in N\}$. The quotient module M/N consists of the set of such equivalence classes of elements of M , with addition on M/N defined as $[a] + [b] = [a + b]$ and module multiplication with elements of R defined as $r[a] = [ra]$, for all $a, b \in M$ and $r \in R$.

An example is the quotient module $6\mathbb{Z}/12\mathbb{Z} = \{[0], [6]\}$ over \mathbb{Z} .

Definition 10. The length ℓ of a module is the length of its longest chain of proper submodules.

For example, the length of the quotient ring $\mathbb{Z}/6\mathbb{Z}$ seen as a module over \mathbb{Z} is 2, since the proper submodules (ideals) of $\mathbb{Z}/6\mathbb{Z}$ are $\{[0], [2]\}$ and $\{[0], [3]\}$, and neither of these submodules has any other proper submodule than the zero module $\{[0]\}$.

2 Monomial ideals and conditions on their powers and products

This section contains the main results of the thesis. In the first subsection, we explain what monomial ideals on polynomial rings are, how they can be graphically represented, and what it means for their powers and products to have forms which can be graphically represented as repeated staircases. Most of the definitions here will follow Moore, Rogers and Wagstaff in [2]. In the second subsection, we find conditions on ideals for their graphical representations of their powers to take such shapes, and in the third subsection, we find corresponding conditions on powers. Some of these results are special cases in the polynomial ring $K[x, y]$ of Veronica Crispin Quinoñez' results in [3].

2.1 Polynomial rings and monomial ideals

This essay is concerned with a special kind of ring, called a polynomial ring. These rings are formed from sets of polynomials with coefficients in a ring or a field. In the following, we will only consider polynomial rings with coefficients in fields. A polynomial ring in several variables $K[x_1, \dots, x_n]$ over a field K is an extension of K , called the coefficient field, with the elements x_1, \dots, x_n . These elements are external to K and commute with each of its elements.

Monomials (short for monomials) are polynomials with only one term. Hence, on a polynomial ring $K[x_1, \dots, x_n]$, a monomial is an element of the ring of the form $a_0 \prod_{k=1}^n x_k^{m_k}$, with $a_0 \in K$ and $m_k \in \mathbb{N}_0$ for each k . An example is the monomial $5.7x_1^4x_3$ on $\mathbb{R}[x_1, x_2, x_3]$.

In this thesis, we will focus primarily on the polynomial ring in two variables $K[x, y]$.

The question of whether a polynomial divides another will be important in the following.

Definition 11. *Let $\bar{u} = u_0 \prod_{k=1}^n x_k^{a_k}$ and $\bar{v} = v_0 \prod_{k=1}^n x_k^{b_k}$ be monomials. Then \bar{u} is said to divide \bar{v} if $a_k \leq b_k$ for all k . In that case, the quotient $\frac{\bar{v}}{\bar{u}} = \frac{v_0}{u_0} \prod_{k=1}^n x_k^{b_k - a_k}$.*

For example, the monomial $12y^4x^3$ on $K[x, y]$ divides the monomial $36y^4x^6$, and the quotient $\frac{36y^4x^6}{12y^4x^3} = 3x^3$.

Definition 12. An ideal in the polynomial ring $K[x_1, \dots, x_n]$ is called monomial if it is generated by monomials.

We notice that a monomial is redundant in the generating set of an ideal if some other monomial in the generating set divides it. For instance, in the generating set $y^3, x^2y^2, x^3y^3, x^4y, x^5$ the monomial x^3y^3 is redundant, since it is divisible by x^2y^2 . Since the divisibility relation is antisymmetric, there is never more than one way of removing all redundant monomials from a generating set which consists only of monomials. It follows that each monomial ideal has a unique minimal generating set of monomials.

The same ideal may, however, be generated by different minimal sets of polynomials. For instance, the monomial ideal $\langle y^3, x^2 \rangle$ may also be generated by the sequence $y^3, x^2 + y^3$ or the sequence $2x^2 + y^3, x^2$.

In the second section of this thesis, it will be important that the products of two generating sets $ST = \{st \mid s \in S, t \in T\}$ will be a generating set for the product of the ideals generated by each set.

Theorem 2. Let I and J be two ideals on a ring R , and S and T their respective generating sets. Then the set ST is a generating set of the product of ideals IJ , that is, $IJ = \langle ST \rangle$.

Proof. Let $a_\alpha = \sum_0^n s_i p_i$, $s_i \in S$, $p_i \in R$ be some element in I , and $b_\alpha = \sum_0^m t_j q_j$, $t_j \in S$, $q_j \in R$ be some element in J . Then we have the product $a_\alpha b_\alpha = \sum_{k=0}^{m+n} (\sum_{i+j=k} s_i p_i t_j q_j) = \sum_{k=0}^{m+n} (\sum_{i+j=k} s_i t_j p_i q_j)$. Since $\sum_{i+j=k} s_i t_j p_i q_j$ is a linear combination of elements on the form $s_i t_j p_i q_j$, with $s_i t_j \in ST$ and $p_i q_j \in R$, it is an element of $\langle ST \rangle$ for each k . But then $a_\alpha b_\alpha = \sum_{k=0}^{m+n} (\sum_{i+j=k} s_i t_j p_i q_j)$ is a finite sum of elements of $\langle ST \rangle$, so it is an element of $\langle ST \rangle$ for each α . Then we in turn have that $\sum_{\alpha=0}^n a_\alpha b_\alpha$ is a finite sum of elements of $\langle ST \rangle$, so it is also an element of $\langle ST \rangle$. But IJ is precisely the set of such sums, so $IJ \subseteq \langle ST \rangle$.

Conversely, let $rst, r \in R, st \in ST$ be some element of $\langle ST \rangle$. Since $s \in I$ and $t \in J$, $st \in IJ$, and since IJ is an ideal in R , $rst \in IJ$. Therefore, $\langle ST \rangle \subseteq IJ$. ■

This is very useful for monomial ideals, which have minimal generating sets of monomials, since it means that we can find and denote the product of two monomial ideals by multiplying pairwise the monomials of the generating sets.

Example 1. The product of $I = \langle y^3, x^3y^2, x^4 \rangle$ och $J = \langle y^6, x^2y^4, x^4y, x^5 \rangle$ in $K[x, y]$ is $IJ = \langle y^9, x^3y^8, x^4y^6 \rangle + \langle x^2y^7, x^5y^6, x^6y^4 \rangle + \langle x^4y^4, x^7y^3, x^8y \rangle + \langle x^5y^3, x^8y^2, x^9 \rangle = \langle y^9, x^2y^7, x^4y^4, x^5y^3, x^8y, x^9 \rangle$.

The square of $L = \langle y^3, x^2y^2, x^4y, x^5 \rangle$ is $L^2 = LL = \langle y^6, x^2y^5, x^4y^4, x^5y^3, x^6y^3, x^7y^2, x^8y^2, x^9y, x^{10} \rangle = \langle y^6, x^2y^5, x^4y^4, x^5y^3, x^7y^2, x^9y, x^{10} \rangle$.

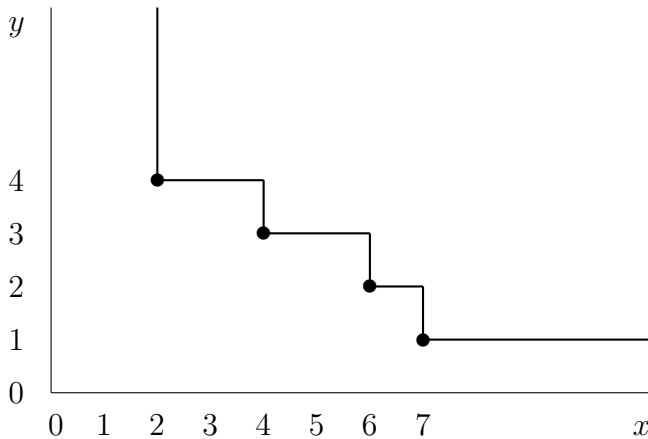
Monomial ideals in polynomial rings may be graphically represented by sets of ordered n -tuples of natural numbers.

Definition 13. The graph Γ of a monomial ideal I in the polynomial ring $K[x_1, \dots, x_n]$ is $\Gamma(I) = \{(a_1, \dots, a_n) \in \mathbb{N}^n \mid \prod_{k=1}^n x_k^{a_k} \in I\}$. [2, p. 5]

Since the monomial ideal contains all monomials with exponents of the respective variables larger than the corresponding exponents of the monomials in the generating set, the following notation is helpful: $[(a_1, \dots, a_n)] = \{(b_1, \dots, b_n) \in \mathbb{N}^n \mid b_k \geq a_k \text{ for all } k\}$.

The graph of a monomial ideal I on $K[x, y]$ may be easily visualized in a two-dimensional diagram.

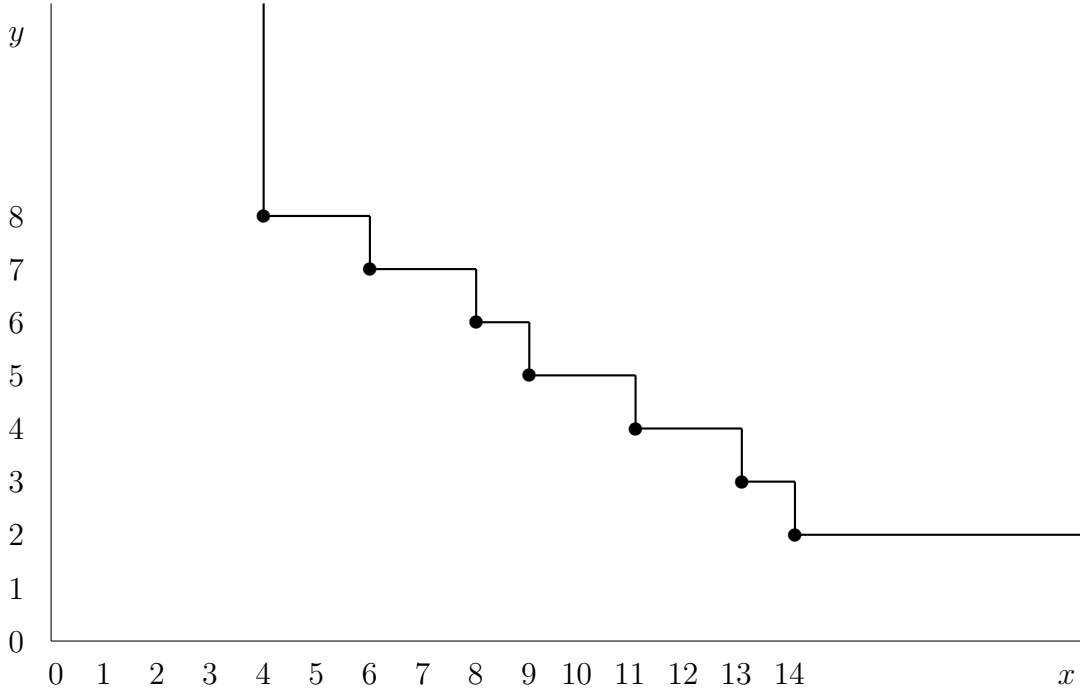
Example 2. The monomial ideal $I = \langle x^2y^4, x^4y^3, x^6y^2, x^7y \rangle$ in $K[x, y]$ has the graph $\Gamma(I) = [(2, 4)] \cup [(4, 3)] \cup [(6, 2)] \cup [(7, 1)]$. Notice that the monomials with the highest y -exponents are written first, which makes it easier to compare the graph $\Gamma(I)$ with the following visual representation:



We notice that the graph has the shape of a staircase.

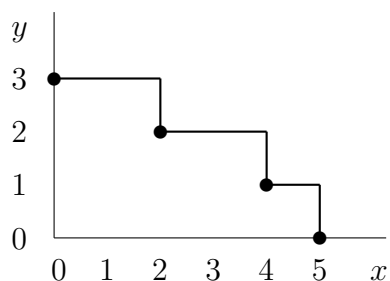
In certain cases, the graph of the square of the ideal will have the same staircase shape, repeated twice and translated to the double distance from

the y -axis and the x -axis, respectively. This happens to be the case with the ideal I above. We have that $I^2 = \langle x^4y^8, x^6y^7, x^8y^6, x^9y^5, x^{11}y^4, x^{13}y^3, x^{14}y^2 \rangle$. This square ideal is represented by the following diagram:

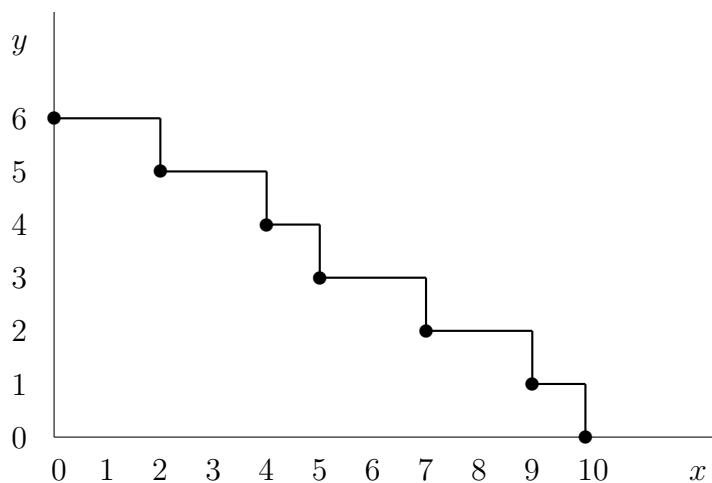


We notice that the ideal I may be written $I = x^2y\langle y^3, x^2y^2, x^4y, x^5 \rangle$, and, correspondingly, that $I^2 = x^4y^2\langle y^6, x^2y^5, x^4y^4, x^5y^3, x^7y^2, x^9y, x^{10} \rangle$. This means that squaring the ideal will, in effect, do two things: translate it by the vector corresponding to the exponents of the common factor, and create new points on (or outside) the new staircase graph by multiplying the monomials with each other. In the second respect, the original monomial ideal will behave identically to a monomial ideal generated by the set formed by dividing the minimal generating set of the original ideal by its greatest common factor.

Example 3. We may compare the graphs of I and I^2 to the graphs of the ideal $J = \langle y^3, x^2y^2, x^4y, x^5 \rangle$, with the common factors of the generating set of I quotiented out. $\Gamma(J) = [(0, 3)] \cup [(2, 2)] \cup [(4, 1)] \cup [(5, 0)]$. This is represented by the following diagram:



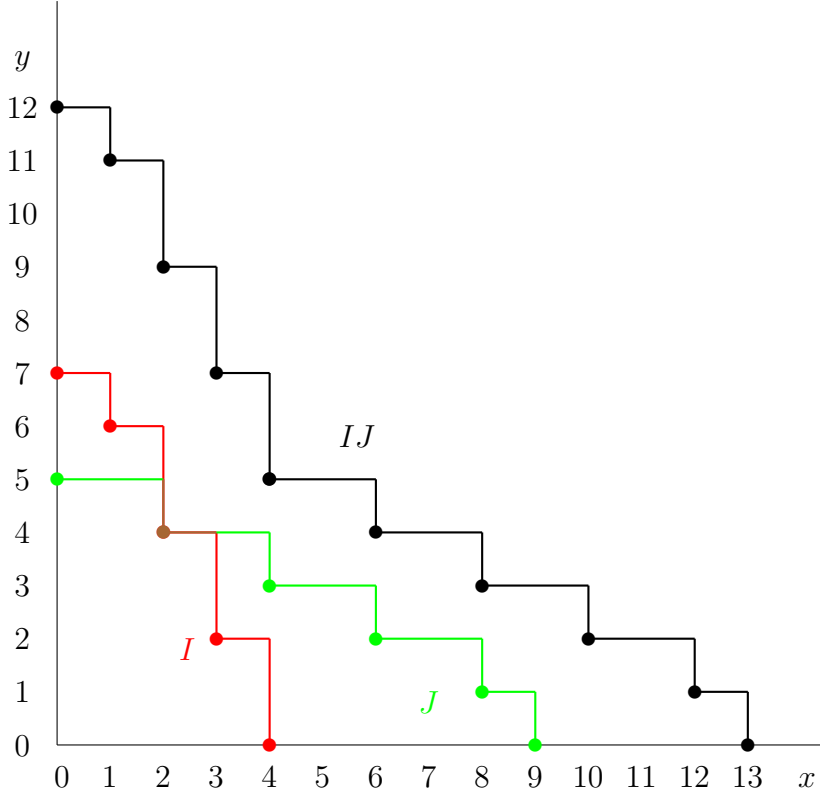
Squaring J , we get $J^2 = \langle y^6, x^2y^5, x^4y^4, x^5y^3, x^6y^3, x^7y^2, x^8y^2, x^9y, x^{10} \rangle = \langle y^6, x^2y^5, x^4y^4, x^5y^3, x^7y^2, x^9y, x^{10} \rangle$. This square ideal is represented by the following diagram:



We will in the following only study ideals with no common factors in the minimal generating set. Any such ideal can be written on the form $I = \langle y^{b_n}, x^{a_1}y^{b_{n-1}}, \dots, x^{a_{n-1}}y^{b_1}, x^{a_n} \rangle$.

Similarly to the case of powers, the graphs of some products of two different ideals have the shape of the graph of one of the ideals followed by the graph of the other ideal.

Example 4. Let $I = \langle y^7, xy^6, x^2y^4, x^3y^2, x^4 \rangle$ and $J = \langle y^5, x^2y^4, x^4y^3, x^6y^2, x^8y, x^9 \rangle$ be ideals on $K[x, y]$. Then their product is $IJ = \langle y^{12}, xy^{11}, x^2y^9, x^3y^7, x^4y^5, x^6y^4, x^8y^3, x^{10}y^2, x^{12}y, x^{13} \rangle$. This is represented in the following diagram:



The aim of this paper is to discover conditions for the n -th power of an ideal or the product of two ideals to be equal to the ideal represented by the corresponding repeated staircase graph.

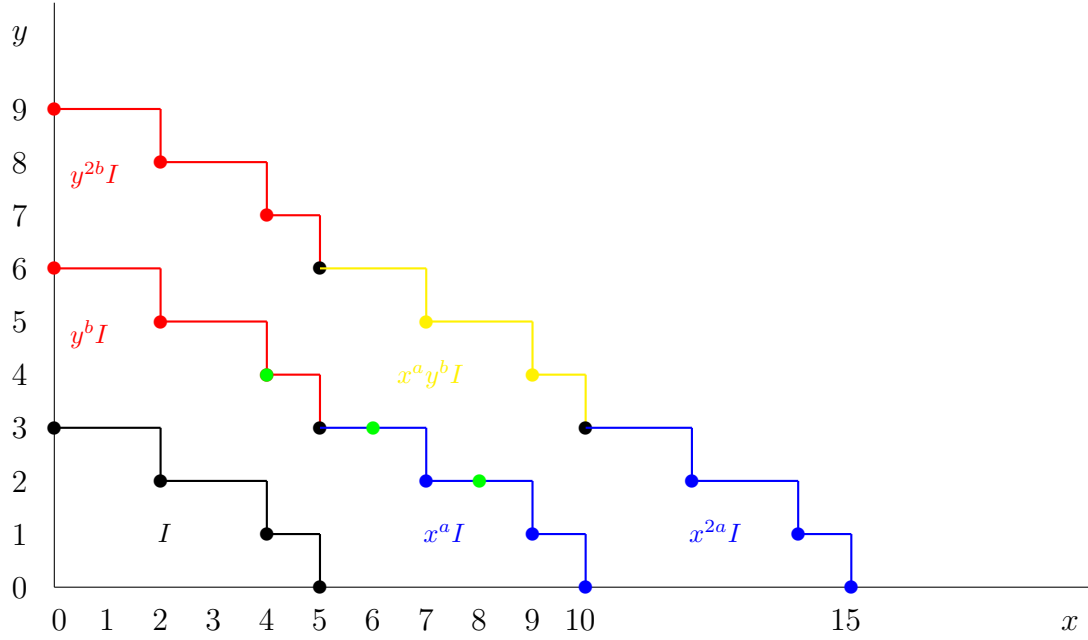
Definition 14. (a) Let I be a monomial ideal in $K[x, y]$, and assume that y^B and x^A belong to its minimal generating set of monomials, with y^B the monomial with the largest y -exponent and x^A the monomial with the largest x -exponent. Then we define the r -th staircase $T_r(I)$ of order r of an ideal I and its powers I^r as $T_r(I) = I\langle y^B, x^A \rangle^{r-1}$.

Alternatively, we may write:

$$T_1(I) = I, T_2(I) = \langle y^B T_1, x^A T_1(I) \rangle, T_3(I) = \langle y^B T_2(I), x^A T_2(I) \rangle, \dots \text{ or} \\ T_1(I) = I, T_2(I) = \langle y^B I, x^A I \rangle, T_3(I) = \langle y^{2B} I, x^A y^B I, x^{2A} I \rangle, \dots$$

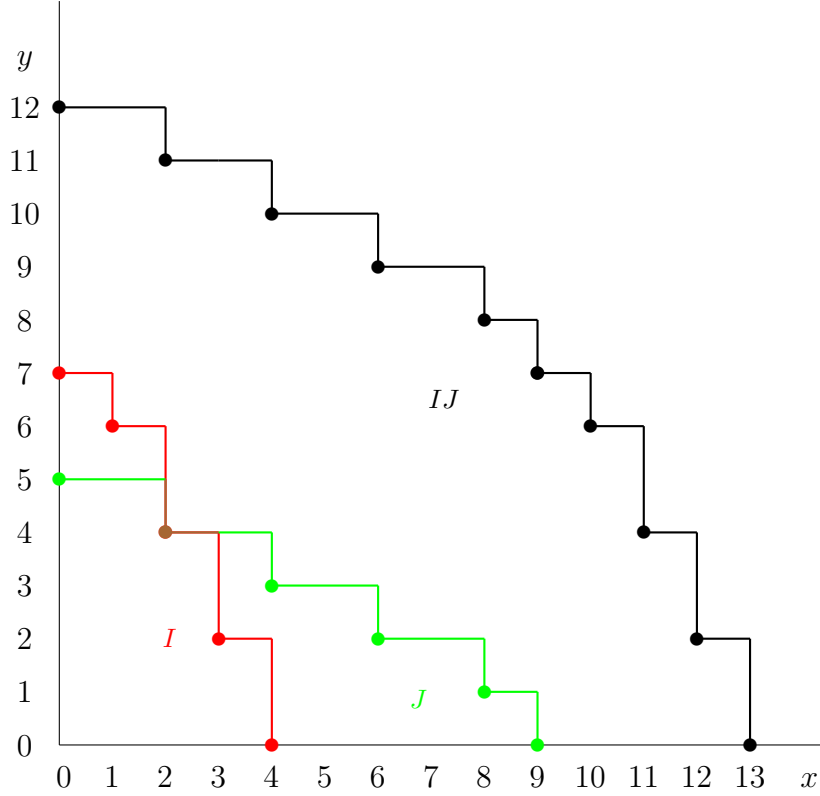
(b) Let I and J be monomial ideals on $K[x, y]$. Assume that y^B and x^A belong to the minimal generating set of I , and that y^D and x^C belong to the minimal generating set of J . Then we define the staircase $T(I, J) = \langle y^D I, x^A J \rangle$, and the staircase $T_{JI} = \langle y^B J, x^C I \rangle$.

Example 5. (a) Consider the ideal $I = \langle y^b, \dots, x^a \rangle = \langle y^3, x^2y^2, x^4y, x^5 \rangle$. We have that $T_1(I) = I$, $T_2(I) = \langle y^b, x^a \rangle I$, and $T_3(I) = \langle y^{2b}, x^a y^b, x^{2a} \rangle I$. These staircase ideals are shown in the figure below:



(b) With the ideals $I = \langle y^7, xy^6, x^2y^4, x^3y^2, x^4 \rangle$ and $J = \langle y^5, x^2y^4, x^4y^3, x^6y^2, x^8y, x^9 \rangle$ from Example 3, $IJ = JI = T(I, J) = \langle y^{12}, xy^{11}, x^2y^9, x^3y^7, x^4y^5, x^6y^4, x^8y^3, x^{10}y^2, x^{12}y, x^{13} \rangle$.

For comparison, in this case $T(J, I) = \langle y^{12}, x^2y^{11}, x^4y^{10}, x^6y^9, x^8y^8, x^9y^7, x^{10}y^6, x^{11}y^4, x^{12}y^2, x^{13} \rangle$, which is represented by the following diagram:



2.2 Powers of monomial ideals

In the first part of this section, we will study powers of a single ideal. We want to discover conditions, under which I satisfies the formula

$$I^r = T_r(I) \quad (*).$$

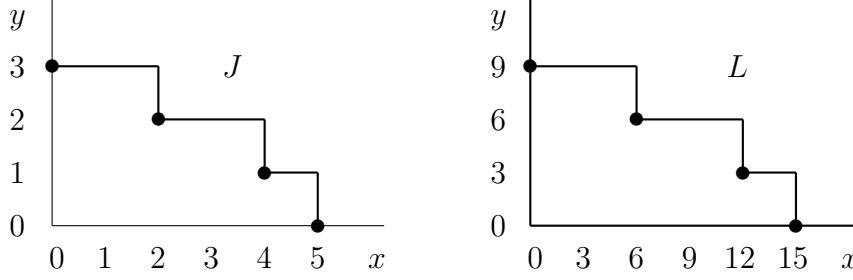
We start with the following result:

Theorem 3. *If $I^2 = T_2(I)$, then $I^r = T_r(I)$ for all r .*

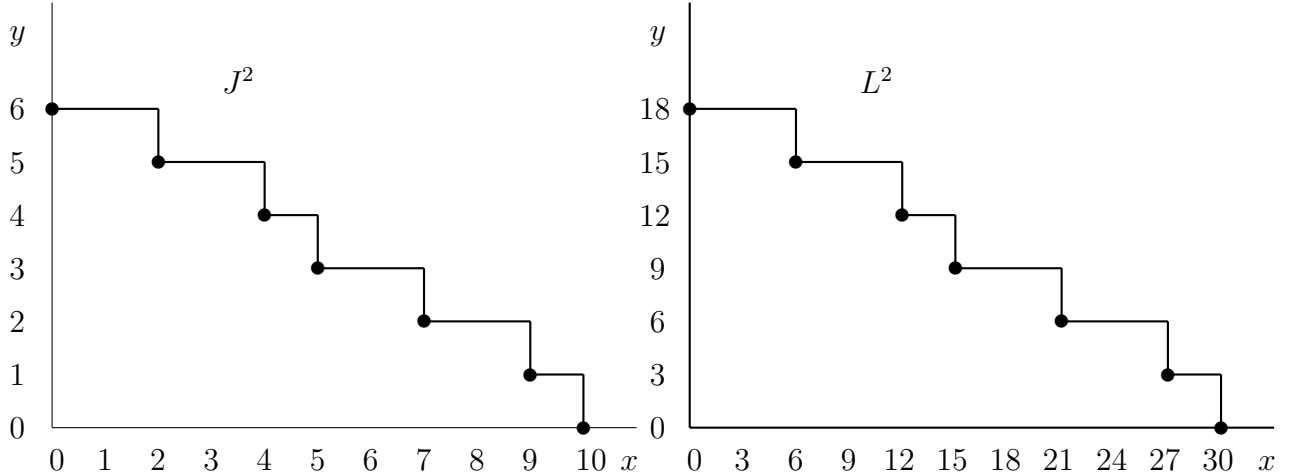
Proof. We use induction on r . The base case is: $I^3 = II^2 = IT_2(I) = I\langle y^B I, x^A I \rangle = \langle y^B I^2, x^A I^2 \rangle = \langle y^B T_2(I), x^A T_2(I) \rangle = T_3$. Now suppose that $I^{r-1} = T_{r-1}(I)$. Then $T_r(I) = \langle y^B T_{r-1}(I), x^A T_{r-1}(I) \rangle = \langle y^B I^{r-1}, x^A I^{r-1} \rangle = \langle y^B II^{r-2}, x^A aII^{r-2} \rangle = I^{r-2} \langle y^B I, x^A I \rangle = I^{r-2} T_2(I) = I^{r-2} I^2 = I^r$. ■

To find conditions on the shape of an ideal to satisfy (*), we turn to a basic consideration of symmetry. We expect ideals whose graphs have the same shape but different scales to behave identically.

Example 6. We compare the ideal $J = \langle y^3, x^2y^2, x^4y, x^5 \rangle$ of Example 3 with the ideal $L = \langle y^9, x^6y^6, x^{12}y^3, x^{15} \rangle$, where all the exponents of the monomials of J have been multiplied by three:



We have $J^2 = \langle y^6, x^2y^5, x^4y^4, x^5y^3, x^6y^3, x^7y^2, x^8y^2, x^9y, x^{10} \rangle$
 $= \langle y^6, x^2y^5, x^4y^4, x^5y^3, x^7y^2, x^9y, x^{10} \rangle$, and similarly $L^2 = \langle y^{18}, x^6y^{15}, x^{12}y^{12}, x^{15}y^9, x^{21}y^6, x^{27}y^3, x^{30} \rangle$:



In general, we may consider ideals I that are created from an ideal $J = \langle y^{b_n}, y^{b_{n-1}}x^{a_1}, \dots, y^{b_1}x^{a_{n-1}}, x^{a_n} \rangle$ having the required property, such that $I = \langle y^{kb_n}, y^{kb_{n-1}}x^{ka_1}, \dots, y^{kb_1}x^{ka_{n-1}}, x^{ka_n} \rangle$, where k is a positive integer.

Proposition 4. Let $I = \langle y^{kb_n}, y^{kb_{n-1}}x^{ka_1}, \dots, y^{kb_1}x^{ka_{n-1}}, x^{ka_n} \rangle$, where k is a positive integer, and $J = \langle y^{b_n}, y^{b_{n-1}}x^{a_1}, \dots, y^{b_1}x^{a_{n-1}}, x^{a_n} \rangle$ be ideals such that $J^i = T_i(J)$ for all i . Then $I^r = T_r(I)$ for all r .

Proof. We want to show that every point in the graph of I^2 is also a point in the graph of $T_2(I)$. Since the graph of $T_2(I)$ contains all points above and to the right of some point in the graph of some generating set of $T_2(I)$, it

suffices to show that each point in the graph of some generating set of I^2 lies above and to the right of some point in the graph of some generating set of $T_2(I)$. Formally, the condition is that for each monomial $y^{k(b_{n-i}+b_{n-j})}x^{k(a_i+a_j)}$ in the generating set of I^2 , there is some monomial $y^D x^C$ in the generating set of $T_2(I)$ such that $k(a_i + a_j) \geq c$ and $k(b_{n-i} + b_{n-j}) \geq d$.

If $i+j \leq n$, we consider the element $y^{k(b_n+b_{n-l})}x^{ka_l}$ such that $b_{n-i}+b_{n-j} \geq b_n + b_{n-l}$ and $a_i + a_j \geq a_l$. Such an element exists since $J^i = T_i(J)$ for all i . Then, $k(a_i + a_j) \geq k(b_n + b_{n-l})$ and $k(b_{n-i} + b_{n-j}) \geq ka_l$.

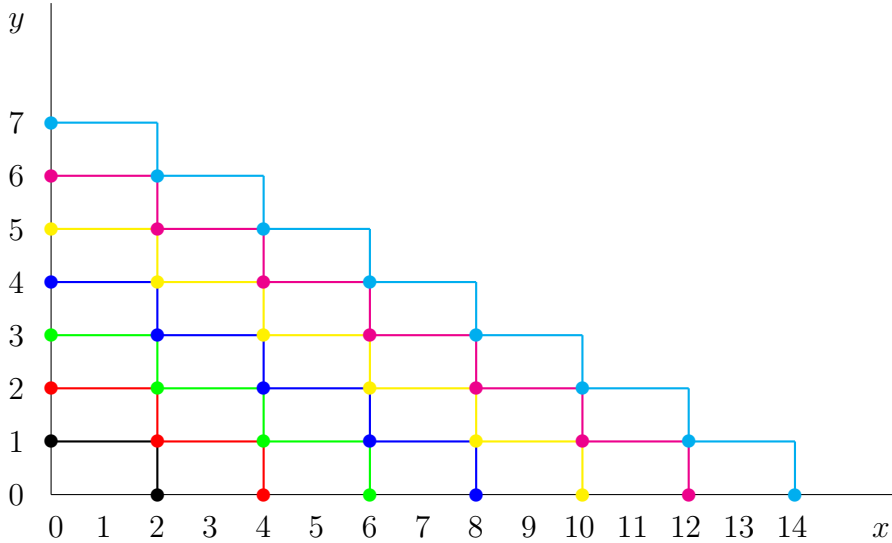
If $i+j > n$, we consider the element $y^{k(b_{n-l})}x^{k(a_n+a_l)}$ such that $b_{n-i}+b_{n-j} \geq b_{n-l}$ and $a_i + a_j \geq a_n + a_l$. Such an element exists since $J^i = T_i(J)$ for all i . Then, $k(b_{n-i} + b_{n-j}) \geq kb_{n-l}$ and $k(a_i + a_j) \geq k(a_n + a_l)$. ■

We continue by finding classes of ideals that satisfy $I^i = T_i(I)$. A simple case is given by ideals on the form $\langle y^b, x^a \rangle$.

Proposition 5. *Let $I = \langle y^b, x^a \rangle \subseteq K[x, y]$. Then $I^r = T_r(I)$ for all r .*

Proof. We have $I^i = \langle y^b, x^a \rangle^i = I \langle y^b, x^a \rangle^{i-1} = T_i(I)$. ■

An example is the ideal $I = \langle y, x^2 \rangle$. We may follow how the powers of I have staircase graphs with an increasing number of repetitions of the same kind of step (height 1, length 2) in this diagram:



Proposition 5 may be generalized by considering ideals of the form $I = \langle x^A, y^B \rangle^n$. We notice that ideals of this type correspond to staircase graphs where all the steps have the same size.

Corollary 6. *If $I = \langle y^b, x^a \rangle^n$, then $I^r = T_r(I)$ for all r .*

Proof. $I = \langle y^b, x^a \rangle^n = \langle y^{bn}, x^a y^{b(n-1)}, \dots, x^{a(n-1)} y^b, x^{an} \rangle. T_2(I) = \langle y^{bn} I, x^{an} I \rangle = \langle y^{2bn}, x^a y^{b(2n-1)}, \dots, x^{a(n-1)} y^{b(n+1)}, x^{an} y^{bn}, x^{a(n+1)} y^{b(n-1)}, \dots, x^{a(2n-1)} y^b, x^{2an} \rangle.$ An arbitrary element in the generating set of I may be written as $x^{ai} y^{b(n-i)}$, with $0 \leq i \leq n$. This means that every element in the generating set of I^2 may be written $x^{ai} y^{b(2n-i)} x^{aj} y^{bj(2n-j)} = x^{a(i+j)} y^{b(2n-i-j)}$, with $0 \leq i \leq n, 0 \leq j \leq n, i \leq j$, but if we write $i + j = k$ it is obvious that $x^{ak} y^{bk(2n-k)}$ is an element in the generating set of $T_2(I)$. By Theorem 3 above, this means that $I^r = T_r(I)$ for all r . ■

In Proposition 5, we saw that a certain class of ideals satisfies (*), and in Corollary 6, we saw that powers of ideals in this class also satisfy (*). It can actually be shown that if any ideal satisfies (*), then powers of that ideal also satisfy (*).

Theorem 7. *Let I be a monomial ideal such that $I^r = T_r(I)$ for all r . Then the ideal $J = I^k$ is such that $J^r = T_r(J)$ for all r .*

Proof. We have that $I^2 = I \langle y^b, x^a \rangle$, with $a = A$ and $b = B$ from Definition 14. Then $J^2 = (I^k)^2 = (I^2)^k$, and by the assumption, $(I^2)^k = I^k \langle y^b, x^a \rangle^k$. We want to show that this is equal to $T_2(J) = I^k \langle y^{bk}, x^{ak} \rangle$.

We have that $T_2(J) = I^k \langle y^{bk}, x^{ak} \rangle \stackrel{\text{assump.}}{=} I \langle y^b, x^a \rangle^{k-1} \langle y^{bk}, x^{ak} \rangle = I \langle y^b, x^a \rangle^{k-1} \langle y^{bk}, x^{ak} \rangle = I \langle y^{b(k-1)}, x^a y^{b(k-2)}, \dots, x^{a(k-2)} y^b, x^{a(k-1)} \rangle \langle y^{bk}, x^{ak} \rangle = \langle y^{b(2k-1)}, x^a y^{b(2k-2)}, \dots, x^{a(k-1)} y^{bk}, x^{ak} y^{b(k-1)}, \dots, x^{a(2k-2)} y^b, x^{a(2k-1)} \rangle = I \langle x^a y^b \rangle^{2k-1} \stackrel{\text{assump.}}{=} I^k \langle y^b, x^a \rangle^k = J^2$, by the above. Hence $T_2(J) = J^2$ and the result follows by Theorem 3. ■

There is also another way in which the ideals from Proposition 5 is a special case of a more general class of ideals that possess the required property. In fact, if the width Δx and the height Δy of each step of the staircase graph do not increase as x and y (the natural number exponents of the x -part and the y -part of the monomial, respectively) increase, the corresponding ideal satisfies $I^r = T_r(I)$ for all r . We may write such an ideal as $I = \langle y^{b_1+\dots+b_n}, y^{b_1+\dots+b_{n-1}} x^{a_1}, y^{b_1} x^{a_1+\dots+a_{n-1}}, x^{a_1+\dots+a_n} \rangle$, with $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n$ and $b_1 \geq b_2 \geq b_3 \geq \dots \geq b_n$.

Theorem 8. *Let $I = \langle y^{b_1+\dots+b_n}, x^{a_1} y^{b_1+\dots+b_{n-1}}, \dots, x^{a_1+\dots+a_{n-1}} y^{b_1}, x^{a_1+\dots+a_n} \rangle \subseteq K[x, y]$ be a monomial ideal, with $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n$ and $b_1 \geq b_2 \geq b_3 \geq \dots \geq b_n$. Then $I^r = T_r(I)$ for all r .*

Proof. The staircase ideal $T_2(I) = \langle y^{2(b_1+\dots+b_n)}, x^{a_1}y^{2(b_1+\dots+b_{n-1})+b_n}, \dots, x^{a_1+\dots+a_n}y^{b_1+\dots+b_n}, x^{2a_1+a_2+\dots+a_n}y^{b_1+\dots+b_{n-1}}, \dots, x^{2(a_1+\dots+a_n)} \rangle$. We want to show that any element of I^2 is included in $T_2(I)$. An element is included in $T_2(I)$ if there, for the corresponding monomial $x^a y^b$, is some monomial $x^c y^d$ in the generating set of $T_2(I)$ such that $a \geq c$ and $b \geq d$.

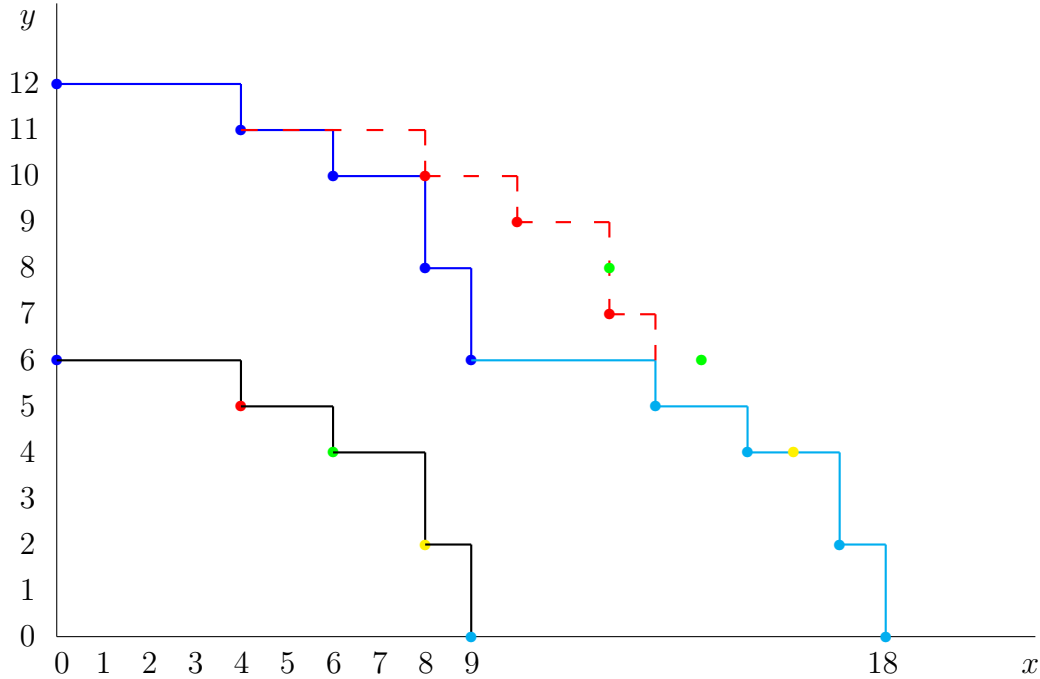
An arbitrary element in the generating set of I^2 may be written $y^{b_1+\dots+b_{n-i}}x^{a_1+\dots+a_i}y^{b_1+\dots+b_{n-j}}x^{a_1+\dots+a_j}$. We may without loss of generality assume that $j \geq i$. Then $y^{b_1+\dots+b_{n-i}}x^{a_1+\dots+a_i}y^{b_1+\dots+b_{n-j}}x^{a_1+\dots+a_j} = y^{2(b_1+\dots+b_{n-j})+b_{n-j+1}+\dots+b_{n-i}}x^{2(a_1+\dots+a_i)+a_{i+1}+\dots+a_j}$. We compare this to the element in the generating set of $T_2(I)$ with the same number $i+j$ of occurrences of a_s in the exponent of the x -term, and the same number $2(n-j) + n - i - (n-j) = 2n - i - j$ of b_t in the exponent of the y -term.

- (i) If $i+j < n$, the element in $T_2(I)$ with the same number of occurrences of a_s and b_t in the exponents as the arbitrary element in the generating set of I^2 is $x^{a_1+\dots+a_{i+j}}y^{2(b_1+\dots+b_{n-i-j})+b_{n-i-j+1}+\dots+b_n}$. We write $S_{1x} = 2(a_1 + \dots + a_i) + a_{i+1} + \dots + a_j$ for the exponent of the x -term of the I^2 -element, and $S_{2x} = a_1 + \dots + a_{i+j}$ for the exponent of the x -term of the $T_2(I)$ -element. If we disregard the common terms $a_1 + \dots + a_j$, we are left with the terms $a_1 + \dots + a_i$ in S_{1x} and the terms $a_{j+1} + \dots + a_{i+j}$ in S_{2x} . Since there is an equal number of terms in both sums, and $a_1 \geq \dots \geq a_i \geq \dots \geq a_{j+1} \geq \dots \geq a_{i+j}$, $S_{1x} \geq S_{2x}$. Furthermore, we write $S_{1y} = 2(b_1 + \dots + b_{n-j}) + b_{n-j+1} + \dots + b_{n-i}$ for the exponent of the y -term of the I^2 -element, and $S_{2y} = 2(b_1 + \dots + b_{n-i-j}) + b_{n-i-j+1} + \dots + b_n$ for the exponent of the y -term of the $T_2(I)$ -element. Disregarding the common terms $b_1 + \dots + b_{n-i}$, we are left with $S_{1y} = b_{n-i-j+1} + \dots + b_{n-j}$, and $S_{2y} = b_{n-j+1} + \dots + b_n$. Since there is an equal number of terms in both sums, and $b_{n-i-j+1} \geq \dots \geq b_{n-j} \geq b_{n-j+1} \geq \dots \geq b_n$, $S_{1y} \geq S_{2y}$.
- (ii) If $i+j \geq n$, the element in $T_2(I)$ with the same number of occurrences of a_s and b_t in the exponents as the arbitrary element in the generating set of I^2 is $x^{2(a_1+\dots+a_{i+j-n})+a_{i+j-n+1}+\dots+a_n}y^{b_1+\dots+b_{2n-i-j}}$. We write $S_{1x} = 2(a_1 + \dots + a_i) + a_{i+1} + \dots + a_j$ for the exponent of the x -term of the I^2 -element, and $S_{2x} = 2(a_1 + \dots + a_{i+j-n}) + a_{i+j-n+1} + \dots + a_n$ for the exponent of the x -term of the $T_2(I)$ -element. If we disregard the common terms $2(a_1 + \dots + a_{i+j-n})$ and $a_{i+1} + \dots + a_j$, we are left with the terms $a_{i+j-n+1} + \dots + a_i$ in S_{1x} and the terms $a_{j+1} + \dots + a_n$

in S_{2x} . Since there is an equal number of terms in both sums, and $a_{i+j-n+1} \geq \dots \geq a_i \geq \dots \geq a_{j+1} \geq \dots \geq a_n$, $S_{1x} \geq S_{2x}$. Furthermore, we write $S_{1y} = 2(b_1 + \dots + b_{n-j}) + b_{n-j+1} + \dots + b_{n-i}$ for the exponent of the y-term of the I^2 -element, and $S_{2y} = b_1 + \dots + b_{2n-i-j}$ for the exponent of the y-term of the $T_2(I)$ -element. Disregarding the common terms $b_1 + \dots + b_{n-i}$, we are left with $S_{1y} = b_1 + \dots + b_{n-j}$, and $S_{2y} = b_{n-i+1} + \dots + b_{2n-i-j}$. Since there is an equal number of terms in both sums, and $b_1 \geq \dots \geq b_{n-j} \geq \dots \geq b_{n-i+1} \geq \dots \geq b_{2n-i-j}$, $S_{1y} \geq S_{2y}$.

We thus have that, for an arbitrary monomial $x^a y^b$ of the generating set of I^2 , there is some monomial $x^c y^d$ in the generating set of $T_2(I)$ such that $a \geq c$ and $b \geq d$. This means that $I^2 \subseteq T_2(I)$. Obviously, $T_2(I) \subseteq I^2$, so $I^2 = T_2(I)$. By Theorem 3 above, we then have $I^r = T_r(I)$ for all r . ■

Example 7. Consider the ideal $I = \langle y^6, x^4 y^5, x^6 y^4, x^8 y^2, x^9 \rangle$, which has steps of non-increasing width from the left and of non-increasing height from below. The figure below shows, in red, green, and yellow, the points of the graph corresponding to monomials in the generating set of I^2 . All these points are also inside the area representing T_2 , because of the shape of I .



Another group of ideals that satisfy (*) may be called line ideals. An ideal is a line ideal if its graph is given by the points with integer coordinates on a straight line between points $(0, y)$ and $(x, 0)$ or to the right of and above the line.

Before we define the line ideals, we need the notion of x -tight and y -tight ideals. The following definition originates from Definition 2.1 in [3].

Definition 15. *Let I be a monomial ideal in $K[x, y]$. Let A be the largest x -exponent of any monomial in the minimal generating set of monomials of I , and B be the largest y -exponent. I is said to be x -tight if every integer between 0 and A is represented as the x -exponent of some monomial in the minimal generating set. Similarly, I is said to be y -tight if every integer between 0 and B is represented as the y -exponent of some monomial in the minimal generating set.*

We can now define the line ideals. The definition given here corresponds to that of simple integrally closed ideals in [3].

Definition 16. *Let I be a monomial ideal. Assume that $I = \langle y^{b_n}, x^{a_1}y^{b_{n-1}}, \dots, x^{a_{n-1}}y^{b_1}, x^{a_n} \rangle$, with $b_{n-i} \geq b_n - \frac{b_n}{a_n}a_i$ such that each b_{n-i} is a positive integer, I is y -tight if $b_n \geq a_n$, I is x -tight if $a_n \geq b_n$, $a_i - a_n + \frac{a_n}{b_n}b_{n-i} < 1$, and $b_{n-i} - b_n + \frac{b_n}{a_n}a_i < 1$. Then we call I a line ideal.*

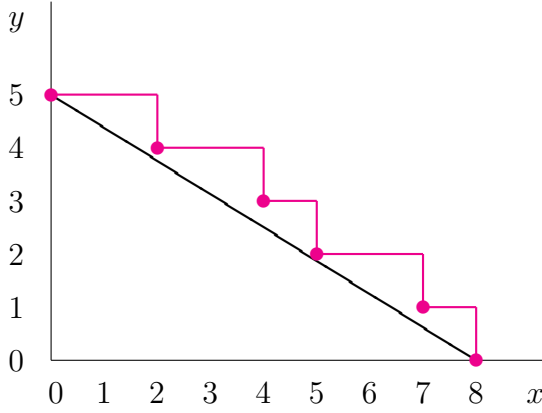
Theorem 9. *Let I be a line ideal. Then $I^r = T_r(I)$ for all r .*

Proof. We notice that the graph of $T_2(I)$ will be delimited to the left and below by a line with the same slope $-\frac{b_n}{a_n}$ as the line delimiting the ideal, running from $(0, 2b_n)$ to $(2a_n, 0)$. The equation of this line is $y = 2b_n - \frac{b_n}{a_n}x$. Since the graph of $T_2(I)$ contains two copies of the graph of I that have only been translated by an integer vector, the graph of $T_2(I)$ will contain all points with integer coordinates to the right and above its delimiting line. Formally, we have $T_2(I) = \langle y^{2b_n}, x^{a_1}y^{b_n+b_{n-1}}, \dots, x^{a_n}y^{b_n}, x^{a_n+a_1}y^{b_{n-1}}, \dots, x^{2a_n} \rangle$. We add b_n to both sides of the inequality $b_{n-i} \geq b_n - \frac{b_n}{a_n}a_i$ to get $b_n + b_{n-i} \geq 2b_n - \frac{b_n}{a_n}a_i$, so all points in the left half of the graph of $T_2(I)$ lie above and to the right of the delimiting line. If we instead add a_n to the inequality written as $a_i \geq a_n - \frac{a_n}{b_n}b_{n-i}$, we get $a_n + a_i \geq 2a_n - \frac{a_n}{b_n}b_{n-i}$, or $b_{n-i} \geq 2b_n - \frac{b_n}{a_n}(a_n + a_i)$, so all points in the right half of the graph of $T_2(I)$ lie above and to the right of the delimiting line. Since $a_i - a_n + \frac{a_n}{b_n}b_{n-i} \leq 1$, $a_n + a_i - 2a_n + \frac{a_n}{b_n}b_{n-i} \leq 1$. Similarly, $b_n + b_{n-i} - 2b_n + \frac{b_n}{a_n}a_i \leq 1$, so the points in the generating set of

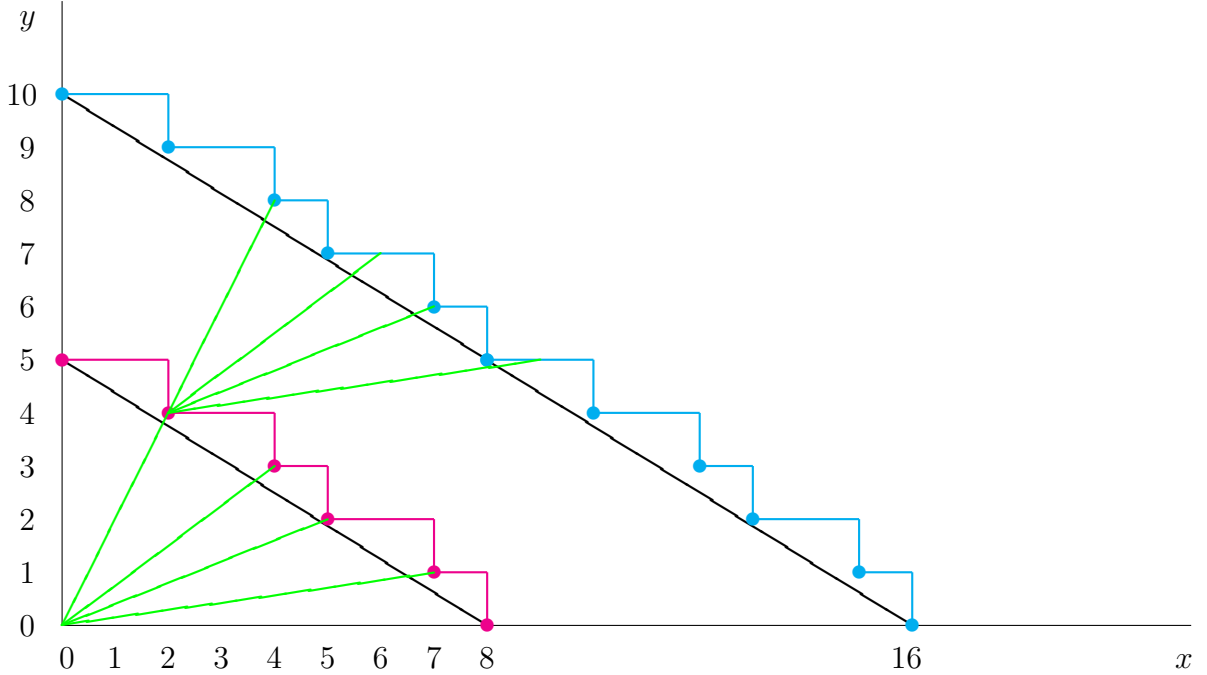
$T_2(I)$ are the closest points with integer coordinates to the right and above the delimiting line.

We thus only need to show that any point in I^2 , which by definition has integer coordinates, also lies to the right and above this delimiting line. Any element in the generating set of I^2 may be written $x^{a_i+a_j}y^{b_{n-i}+b_{n-j}}$. The corresponding point on the graph lies above and to the right of the delimiting line if and only if $b_{n-i} + b_{n-j} \geq 2b_n - \frac{b_n}{a_n}(a_i + a_j)$. Moreover, if we add the inequalities $b_{n-i} \geq b_n - \frac{b_n}{a_n}a_i$ and $b_{n-j} \geq b_n - \frac{b_n}{a_n}a_j$ from the definition of I , we get the desired inequality $b_{n-i} + b_{n-j} \geq 2b_n - \frac{b_n}{a_n}(a_i + a_j)$. ■

Example 8. The ideal $I = \langle y^5, x^2y^4, x^4y^3, x^5y^2, x^7y, x^8 \rangle$ is a line ideal, with the line going from $(0, 5)$ to $(8, 0)$. It is y -tight.



In the figure below, we see the graph of $T_2(I) = I^2$ in cyan. The monomial in I^2 obtained by multiplying the monomial x^2y^4 with other monomials in I is always contained in $T_2(I)$, since it corresponds to translating a point to the right of and above the first diagonal line (of $I = T_1(I)$) to a point to the right of and above the second diagonal line (of $T_2(I)$). This is shown by the green lines. The same is true for all other monomials in I .



It is important to note that I does not satisfy the conditions of Theorem 8, as the width of the third step from the left is smaller than the width of the fourth step from the left. In fact, there are many ideals which satisfy the conditions of both Theorem 8 and Theorem 9, as well as many ideals which only satisfy the conditions of one of the theorems.

Ideals with repeated steps comprise yet another class that satisfies (*), subject to certain limitations. As shown in Corollary 6, the simplest class of such ideals, those where there is only one kind of repeated step, satisfies (*).

Another way of addressing the problem is to find conditions for monomial ideals generated by a limited number of monomials. In Proposition 4, we saw that monomial ideals generated by only two monomials satisfy the condition $I^r = T_r(I)$. What about ideals generated by three monomials?

Lemma 10. *Let $I = \langle y^{b_1+b_2}, x^{a_1}y^{b_1}, x^{a_1+a_2} \rangle$. Then $I^r = T_r(I)$ if and only if $b_1 \geq b_2$ and $a_1 \geq a_2$.*

Proof. We have $I^2 = \langle y^{2(b_1+b_2)}, x^{a_1}y^{2b_1+b_2}, x^{a_1+a_2}y^{b_1+b_2}, x^{2a_1}y^{2b_1}, x^{a_1+2a_2}y^{b_1}, x^{2(a_1+a_2)} \rangle$, and $T_2(I) = \langle y^{2(b_1+b_2)}, x^{a_1}y^{2b_1+b_2}, x^{a_1+a_2}y^{b_1+b_2}, x^{a_1+2a_2}y^{b_1}, x^{2(a_1+a_2)} \rangle$. The only element of the generating set of I^2 not in the generating set of $T_2(I)$ is

$x^{2a_1}y^{2b_1}$. $I^2 = T_2(I)$ if and only if all elements of the generating set of I^2 have y - and x -exponents larger or equal than the y - and x -exponents of some element in $T_2(I)$. The only element of $T_2(I)$ which $x^{2a_1}y^{2b_1}$ may have larger or equal exponents than is $x^{a_1+a_2}y^{b_1+b_2}$, and it follows that $I^2 = T_2(I)$ if and only if $2b_1 \geq b_1 + b_2$ and $2a_1 \geq a_1 + a_2$, that is, if and only if $b_1 \geq b_2$ and $a_1 \geq a_2$. ■

In combination with Theorem 6 about powers of powers of ideals, this allows us to state the following theorem about ideals whose graphs consist of alternating pairs of repeated steps:

Corollary 11. *If $I = \langle y^{n(b_1+b_2)}, x^{a_1}y^{nb_1+(n-1)b_2}, x^{a_1+a_2}y^{(n-1)(b_1+b_2)}, \dots, x^{na_1+(n-1)a_2}y^{b_1}, x^{n(a_1+a_2)} \rangle$, with $a_1 \geq a_2$ and $b_1 \geq b_2$, then $I^r = T_r(I)$ for all r .*

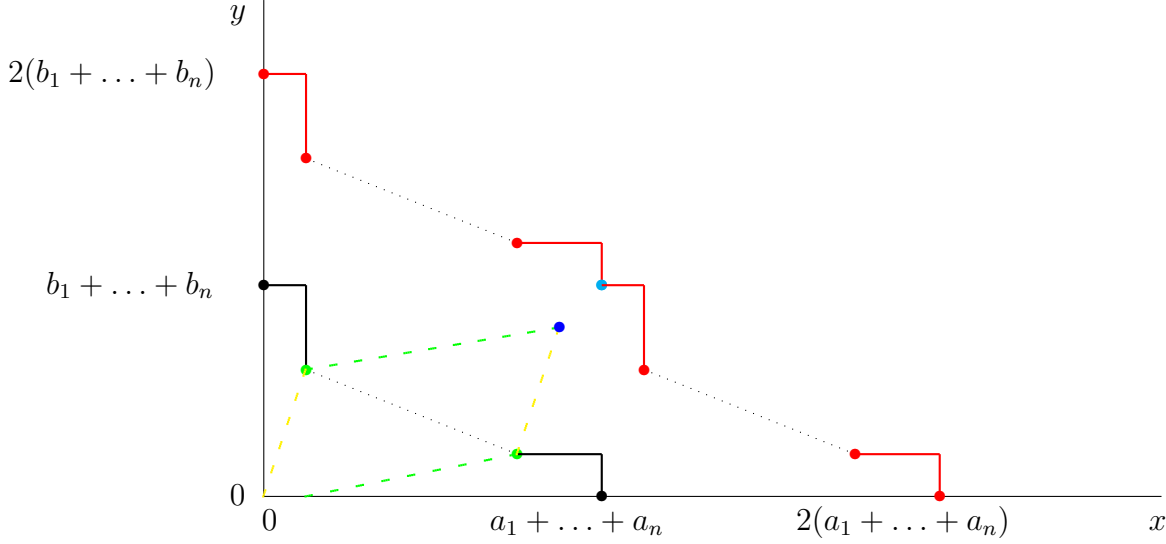
Proof. $I = J^n$, where $J = \langle y^{b_1+b_2}, x^{a_1}y^{b_1}, x^{a_1+a_2} \rangle$. By Theorem 9, $J^i = T_i(J)$ for all i . Then, by Theorem 6, $I^r = T_r(I)$ for all r . ■

The necessary and sufficient condition of Lemma 10 may be expanded to a necessary condition for more general monomial ideals.

Proposition 12. *Let $I = \langle y^{b_1+\dots+b_n}, x^{a_1}y^{b_1+\dots+b_{n-1}}, x^{a_1+\dots+a_{n-1}}y^{b_1}, x^{a_1+\dots+a_n} \rangle$ be a monomial ideal. Assume that $I^r = T_r(I)$ for all r . Then the width a_1 of the first step from the left of the staircase graph is greater or equal to the width a_n of the last step to the right, and the height b_n of the first step from the left is smaller or equal than the height b_1 of the last step to the right.*

Proof. We have $T_2(I) = \langle y^{2(b_1+\dots+b_n)}, x^{a_1}y^{2(b_1+\dots+b_{n-1})+b_n}, \dots, x^{a_1+\dots+a_n}y^{b_1+\dots+b_n}, x^{2a_1+a_2+\dots+a_n}y^{b_1+\dots+b_{n-1}}, \dots, x^{2(a_1+\dots+a_n)} \rangle$. Consider the element $x^{a_1}y^{b_1+\dots+b_{n-1}}x^{a_1+\dots+a_{n-1}}y^{b_1} = x^{2a_1+a_2+\dots+a_{n-1}}y^{2b_1+b_2+\dots+b_{n-1}}$ of I^2 . If $I^2 = T_2(I)$, then there is some element $x^c y^d$ in the generating set of $T_2(I)$ such that $2a_1 + a_2 + \dots + a_{n-1} \geq c$ and $2b_1 + b_2 + \dots + b_{n-1} \geq d$. Each element $x^c y^d$ in the generating set of $T_2(I)$ either has $2a_1 + a_2 + \dots + a_{n-1} \leq c$ or $2b_1 + b_2 + \dots + b_{n-1} \leq d$, except for the element $x^{a_1+\dots+a_n}y^{b_1+\dots+b_n}$. This means that we must have $2a_1 + a_2 + \dots + a_{n-1} \geq a_1 + \dots + a_n$, and $2b_1 + b_2 + \dots + b_{n-1} \geq b_1 + \dots + b_n$. It follows that $a_1 \geq a_n$ and $b_1 \geq b_n$. ■

We may visualize the theorem through the diagram below. The blue dot represents the monomial formed by multiplying the monomials represented by the green dots. This monomial will be either to the left of or below every monomial in $T_2(I)$, with the only possible exception of the monomial $x^{a_1+\dots+a_n}y^{b_1+\dots+b_n}$.



However, the approach of Proposition 12 quickly becomes impractical for sets of more than three monomials in the necessary and sufficient case, and for intermediate steps a_i and b_j in the more general, necessary case. If we study the element $y^{b_1+\dots+b_{n-2}}x^{a_1+a_2}y^{b_1+b_2}x^{a_1+\dots+a_{n-2}} = y^{2b_1+2b_2+b_3+\dots+b_{n-2}}x^{2a_1+2a_2+a_3+\dots+a_{n-2}}$ in I^2 , we notice that it may have greater x - or y -exponents than any of the elements $y^{2b_1+\dots+b_n}x^{a_1+\dots+a_{n-1}}$, $y^{b_1+\dots+b_n}x^{a_1+\dots+a_n}$ or $y^{b_1+\dots+b_{n-1}}x^{2a_1+\dots+a_n}$ in $T_2(I)$. We may thus have that $b_2 \geq b_{n-1} + b_n$ and $a_1 + a_2 \geq a_n$, or that $b_1 + b_2 \geq b_{n-1} + b_n$ and $a_1 + a_2 \geq a_{n-1} + a_n$, or that $a_2 \geq a_{n-1} + a_n$ and $b_1 + b_2 \geq b_n$. As we investigate the lengths of steps closer to the middle of the staircase, the conditions of this type become unmanageable.

The situation becomes less complicated if we consider only ideals whose graphs have a fixed step length in either the x -direction or the y -direction. If the step length is fixed in the y -direction, we may write such an ideal as $I = \langle y^{\kappa n}, x^{a_1}y^{\kappa(n-1)}, \dots, x^{a_{n-1}}y^{\kappa}, x^{a_n} \rangle$, with κ a constant. If the step length is fixed in the x -direction, we may write the ideal as $I = \langle y^{b_n}, x^{\kappa}y^{b_{n-1}}, \dots, x^{\kappa(n-1)}y^{b_1}, x^{\kappa n} \rangle$.

Theorem 13. (a) Let $I = \langle y^{\kappa n}, x^{a_1}y^{\kappa(n-1)}, \dots, x^{a_{n-1}}y^{\kappa}, x^{a_n} \rangle \subseteq K[x, y]$ be a monomial ideal. Then I satisfies $I^r = T_r(I)$ for all r if and only if

- (i) $a_j + a_k \geq a_{j+k}$ for all j and k such that $j + k \leq n$, and
- (ii) $a_j + a_k \geq a_n + a_{j+k-n}$ for all j and k such that $j + k > n$.

(b) Similarly, let $I = \langle y^{b_n}, x^{\kappa}y^{b_{n-1}}, \dots, x^{\kappa(n-1)}y^{b_1}, x^{\kappa n} \rangle$. Then I satisfies $I^r = T_r(I)$ for all r if and only if

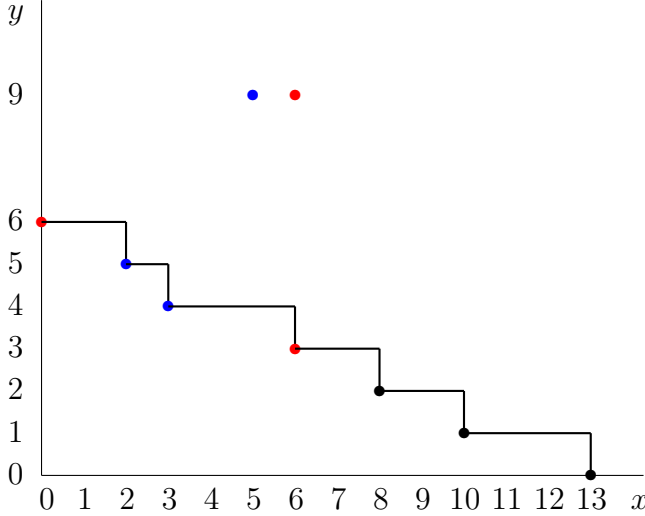
- (i) $b_j + b_k \geq b_{j+k}$ for all j and k such that $j + k \leq n$, and
(ii) $b_j + b_k \geq b_n + b_{j+k-n}$ for all j and k such that $j + k > n$.

Proof. (a) $I = \langle y^{\kappa n}, x^{a_1} y^{\kappa(n-1)}, \dots, x^{a_{n-1}} y^{\kappa}, x^{a_n} \rangle$. We may write an arbitrary element in the generating set of I^2 as $x^{a_j+a_k} y^{2n-j-k}$. We know that it is a necessary and sufficient condition for $I^2 = T_2(I)$ that each element of the generating set of I^2 has x - and y -exponents each larger than or equal to the x - and y -exponents of some element in the generating set of $T_2(I)$. We want to compare the arbitrary element $u = x^{a_j+a_k} y^{2n-j-k}$ of I^2 with some element v of $T_2(I)$. If the y -exponent of u is smaller than the y -exponent of v , v cannot be such an element as we are looking for.

If the y -exponent of u is greater than the y -exponent of v , the x -exponent of v can only be greater than the x -exponent of u if the same is true for the element w of $T_2(I)$ whose y -exponent is equal to the y -exponent of u , since x -exponent of v is smaller than the x -exponent of w . It is thus sufficient to compare each element of the generating set of I^2 to the element of $T_2(I)$ which has the same y -exponent. We consider two cases: when $j + k \leq n$, and when $j + k > n$.

- (i) Assume that $j + k \leq n$. Then, the element of $T_2(I)$ which has the same y -exponent may be written as $x^{a_{j+k}} y^{2n-j-k}$. The x -exponent of the element of I^2 is greater or equal to the x -exponent of this element of $T_2(I)$ if and only if $a_j + a_k \geq a_{j+k}$.
- (ii) Assume that $j + k > n$. Then, the element of $T_2(I)$ which has the same y -exponent may be written as $x^{a_n+a_{j+k-n}} y^{2n-j-k}$. The x -exponent of the element of I^2 is greater or equal to the x -exponent of this element of $T_2(I)$ if and only if $a_j + a_k \geq a_n + a_{j+k-n}$.
- (b) The proof for the case where the length of the steps in the x -direction is fixed is analogous to the proof in (a). ■

The content of Theorem 12 may be grasped intuitively through the following diagram:



The product of the monomials $x^{a_1}y^{\kappa(n-1)} = y^5x^2$ and $x^{a_2}y^{\kappa(n-2)} = x^3y^4$ (represented by blue dots in the diagram) will be the element $x^{a_1+a_2}y^{\kappa(n-3)} = x^5y^9$ (the uppermost blue dot) in the generating set of I^2 . Since I is y -tight, there will be one and only one monomial with the same y -coordinate $\kappa(n-3)$ in $T_2(I)$, corresponding to the monomial $x^{a_3}y^{\kappa(n-3)} = x^{a_3}y^{2\kappa-3} = x^6y^9$. Since $a_1 + a_2 < a_3 = a_{1+2}$, the monomial $x^{a_1+a_2}y^{\kappa(n-3)} = y^9x^5$ (blue dot) in I^2 will have a smaller x -exponent than the monomial $x^{a_3}y^{2\kappa-3} = x^6y^9$ (red dot) in $T_2(I)$, so $I^2 \neq T_2(I)$.

2.3 Products of monomial ideals

The conditions for $IJ = T_{IJ}$ are in many ways similar to the conditions for $I^2 = T_2(I)$.

We begin by looking at the simple case where $I = \langle x^a, y^b \rangle$ and $J = \langle x^c, y^d \rangle$, with a, b, c, d positive integers. It turns out that

Theorem 14. *If $I = \langle y^b, x^a \rangle$ and $J = \langle y^d, x^c \rangle$, then $IJ = T_{IJ}$ if and only if either $a \geq c$ and $d \geq b$, or $c \geq a$ and $b \geq d$.*

Proof. If $b \geq d$, then $T_{IJ} = \langle y^{b+d}, x^a y^d, x^{a+c} \rangle$. $IJ = \langle y^{b+d}, x^c y^b, x^a y^d, x^{a+c} \rangle$. $IJ = T_{IJ}$ if and only if each element in the generating set of IJ is contained in T_{IJ} , that is, there is some element $x^s y^t$ of T_{IJ} for each element $x^g y^h$ of IJ such that $g \geq s$ and $h \geq t$. Since the generating sets of IJ and T_{IJ} in our case contain the same elements with the exception of $x^c y^b$, we must have

$c \geq a$ and $b \geq d$.

If $d \geq b$, then $T_{IJ} = \langle y^{b+d}, x^c y^b, x^{a+c} \rangle$. Since the generating sets of IJ and T_{IJ} in this case contain the same elements with the exception of $x^a y^d$, we must have $a \geq c$ and $d \geq b$. \blacksquare

There are also analogues to the other theorems about the square of an ideal. We remember Theorem 8, which was valid for ideals with non-increasing widths and heights of steps. A similar theorem for products is the following:

Theorem 15. *Let $I = \langle y^{b_1+\dots+b_n}, x^{a_1} y^{b_1+\dots+b_{n-1}}, \dots, x^{a_1+\dots+a_{n-1}} y^{b_1}, x^{a_1+\dots+a_n} \rangle$, $J = \langle y^{d_1+\dots+d_m}, x^{c_1} y^{d_1+\dots+d_{m-1}}, \dots, x^{c_1+\dots+c_{m-1}} y^{d_1}, x^{c_1+\dots+c_m} \rangle$.*

(a) *Assume that $b_i \leq d_j$ for all i and j , and $c_k \leq a_l$ for all k and l . Then $IJ = T_{IJ}$.*

(b) *Assume that $d_i \leq b_j$ for all i and j , and $a_k \leq c_l$ for all k and l . Then $IJ = T_{JI}$.*

Proof. (a) We have that $b_1 + \dots + b_n \geq d_1 + \dots + d_m$ and $c_1 + \dots + c_m \geq a_1 + \dots + a_n$. We compare IJ to T_{IJ} . We have $T_{IJ} = \langle y^{b_1+\dots+b_n+d_1+\dots+d_m}, x^{a_1} y^{b_1+\dots+b_{n-1}+d_1+\dots+d_m}, \dots, x^{a_1+\dots+a_n} y^{d_1+\dots+d_m}, x^{c_1+a_1+\dots+a_n} y^{d_1+\dots+d_{m-1}}, \dots, x^{c_1+\dots+c_{m-1}+a_1+\dots+a_n} y^{d_1}, x^{c_1+\dots+c_m+a_1+\dots+a_n} \rangle$. An arbitrary element in IJ may be written $x^{a_1+\dots+a_i+c_1+\dots+c_j} y^{b_1+\dots+b_{n-i}+d_1+\dots+d_{m-j}}$. We will compare this element to an element in T_{IJ} . There are two cases: one where $n + m - i - j \geq m$, and one where $n + m - i - j < m$.

(i) Assume that $n + m - i - j \geq m$. We compare the arbitrary element of IJ to the element $x^{a_1+\dots+a_{i+j}} y^{b_1+\dots+b_{n-i-j}+d_1+\dots+d_m}$ of T_{IJ} . We write $S_{IJx} = a_1 + \dots + a_i + c_1 + \dots + c_j$ for the exponent of the x -term of the IJ -element, and $S_{Tx} = a_1 + \dots + a_{i+j}$ for the exponent of the x -term of the T_{IJ} -element. If we disregard the common terms $a_1 + \dots + a_i$, we are left with the terms $c_1 + \dots + c_j$ in S_{IJx} and the terms $a_{i+1} + \dots + a_{i+j}$ in S_{Tx} . Since there is an equal number of terms in both sums, and $c_k \leq a_l$ for all k and l , $S_{IJx} \geq S_{Tx}$.

Furthermore, we write $S_{IJy} = b_1 + \dots + b_{n-i} + d_1 + \dots + d_{m-j}$ for the exponent of the y -term of the IJ -element, and $S_{Ty} = b_1 + \dots + b_{n-i-j} + d_1 + \dots + d_m$ for the exponent of the y -term of the $T_2(I)$ -element. Disregarding the common terms $b_1 + \dots + b_{n-i-j}$

and $d_1 + \dots + d_{m-j}$, we are left with $S_{IJy} = b_{n-i-j+1} + \dots + b_{n-i}$, and $S_{Ty} = d_{m-j+1} + \dots + d_m$. Since there is an equal number of terms in both sums, and $b_i \leq d_j$ for all i and j , $S_{IJy} \geq S_{Ty}$.

- (ii) Assume that $n+m-i-j < m$. We compare the arbitrary element of IJ to the element $x^{a_1+\dots+a_n+c_1+\dots+c_{i+j-n}}y^{d_1+\dots+d_{n+m-i-j}}$ of T_{IJ} . We write $S_{IJx} = a_1 + \dots + a_i + c_1 + \dots + c_j$ for the exponent of the x -term of the IJ -element, and $S_{Tx} = a_1 + \dots + a_n + c_1 + \dots + c_{i+j-n}$ for the exponent of the x -term of the T_{IJ} -element. If we disregard the common terms $a_1 + \dots + a_i$ and $c_1 + \dots + c_{i+j-n}$, we are left with the terms $c_{i+j-n+1} + \dots + c_j$ in S_{IJx} and the terms $a_{i+1} + \dots + a_n$ in S_{Tx} . Since there is an equal number of terms in both sums, and $c_k \leq a_l$ for all k and l , $S_{IJx} \geq S_{Tx}$.

Furthermore, we write $S_{IJy} = b_1 + \dots + b_{n-i} + d_1 + \dots + d_{m-j}$ for the exponent of the y -term of the IJ -element, and $S_{Ty} = d_1 + \dots + d_{n+m-i-j}$ for the exponent of the y -term of the $T_2(I)$ -element. Disregarding the common terms $d_1 + \dots + d_{m-j}$, we are left with $S_{IJy} = b_1 + \dots + b_{n-i}$, and $S_{Ty} = d_{m-j+1} + \dots + d_{n+m-i-j}$. Since there is an equal number of terms in both sums, and $b_i \leq d_j$ for all i and j , $S_{IJy} \geq S_{Ty}$.

- (b) The proof is similar to that of (a). ■

We may also consider products of line ideals. (Remember that these were defined in Definition 16.)

Theorem 16. *Let $I = \langle y^{b_n}, x^{a_1}y^{b_{n-1}}, \dots, x^{a_{n-1}}y^{b_1}, x^{a_n} \rangle$ and $J = \langle y^{d_m}, x^{c_1}y^{d_{m-1}}, \dots, x^{c_{m-1}}y^{d_1}, x^{c_m} \rangle$ be line ideals. Then*

(a) $IJ = T_{IJ}$ if $\frac{b_n}{a_n} \geq \frac{d_m}{c_m}$, and

(b) $IJ = T_{JI}$ if $\frac{b_n}{a_n} \leq \frac{d_m}{c_m}$.

Proof. (a) Assume that $\frac{b_n}{a_n} \geq \frac{d_m}{c_m}$. As in the proof of Theorem 9, we first show that all points in the graph of T_{IJ} are above and to the right of the delimiting line with equation $y = b_n + d_m - \frac{b_n}{a_n}x$ from $x = 0$ to $x = a_n$, and equation $y = d_m - \frac{d_m}{c_m}x$ from $x = a_n$ to $x = a_n + c_m$. We have $T_{IJ} = \langle y^{b_n+d_m}, x^{a_1}y^{b_{n-1}+d_m}, \dots, x^{a_n}y^{b_n+d_m}, x^{a_n+c_1}y^{d_{m-1}}, \dots, x^{a_n+c_n} \rangle$. We add d_m to both sides of the inequality $b_{n-i} \geq b_n - \frac{b_n}{a_n}a_i$ to get $d_m + b_{n-i} \geq$

$b_n + d_m - \frac{b_n}{a_n}a_i$, so all points in the left half of the graph of T_{IJ} lie above and to the right of the delimiting line. If we instead add a_n to the inequality $d_{m-i} \geq d_m - \frac{d_m}{c_m}c_i$ written as $c_i \geq c_m - \frac{c_m}{d_m}d_{m-i}$, we get $a_n + c_i \geq a_n + c_m - \frac{c_m}{d_m}d_{m-i}$, or $d_{m-i} \geq \frac{d_m}{c_m}a_n + d_m - \frac{d_m}{c_m}(a_n + c_i) \geq d_m - \frac{d_m}{c_m}(a_n + c_i)$, so all points in the right half of the graph of T_{IJ} lie above and to the right of the delimiting line.

Since $d_m + b_{n-i} - (d_m + b_n - \frac{b_n}{a_n}a_i) = b_{n-i} - b_n + \frac{b_n}{a_n}a_i < 1$, and $a_n + c_i - (a_n + c_m - \frac{c_m}{d_m}d_{m-i}) = c_i - c_m + \frac{c_m}{d_m}d_{m-i} < 1$, the points in the generating set of T_{IJ} are the closest points with integer coordinates to the right and above the delimiting line. Since I and J are each either y -tight or x -tight, there are no points with integer coordinates corresponding to the exponents of monomials not in IJ to the right of and above the delimiting line.

We now only need to show that all points in the graph of IJ lie to the right and above the delimiting line. An arbitrary element of the generating set of IJ may be written $y^{b_{n-i}+d_{m-j}}x^{a_i+c_j}$. The corresponding point on the graph lies above and to the right of the left part of the delimiting line if $b_{n-i} + d_{m-j} \geq b_n + d_m - \frac{b_n}{a_n}(a_i + c_j)$. From the definition of a line ideal, we have that $b_{n-i} \geq b_n - \frac{b_n}{a_n}a_i$, and $d_{m-j} \geq d_m - \frac{d_m}{c_m}c_j$. Adding these inequalities, we get $b_{n-i} + d_{m-j} \geq b_n + d_m - \frac{b_n}{a_n}a_i - \frac{d_m}{c_m}c_j \geq b_n + d_m - \frac{b_n}{a_n}(a_i + c_j)$, since $\frac{b_n}{a_n} \geq \frac{d_m}{c_m}$.

Similarly, the point on the graph corresponding to the arbitrary element in IJ lies above and to the right of the right part of the delimiting line if $a_i + c_j \geq a_n + c_m - \frac{c_m}{d_m}(b_{n-i} + d_{m-j})$. From the definition of a line ideal, we have the inequalities $a_i \geq a_n - \frac{a_n}{b_n}b_{n-i}$ and $c_j \geq c_m - \frac{c_m}{d_m}d_{m-j}$. Adding these together, we get $a_i + c_j \geq a_n + c_m - \frac{a_n}{b_n}b_{n-i} - \frac{c_m}{d_m}d_{m-j} \geq a_n + c_m - \frac{c_m}{d_m}(b_{n-i} + d_{m-j})$, since $\frac{c_m}{d_m} \geq \frac{a_n}{b_n}$.

- (b) The proof of the case $IJ = T_{JI}$ is similar to that of (a). ■

3 Connections with other areas: the Hilbert and Hilbert-Samuel functions

The Hilbert function measures the growth of the dimension of the homogeneous components of a graded ring or module. It is an important numerical invariant of projective algebraic sets, which are basic objects of study in algebraic geometry.[1, pp. 39-42] On local rings, the counterpart of the Hilbert function is the Hilbert-Samuel function, which is defined in terms of powers of ideals. In this section we will explain how the calculation of the Hilbert-Samuel function of monomial ideals on localizations of the polynomial ring $K[x, y]$ is made easier by application of the conditions described in the previous sections. Before doing this, we will introduce the relevant concepts. As in section 1, most definitions will follow David Eisenbud, *Commutative algebra*.

3.1 Graded rings and the Hilbert function

The direct sum $A \oplus B$ of two abelian groups A and B is an abelian group consisting of ordered pairs of the form (a, b) , with $a \in A$ and $b \in B$. For two elements (a, b) and $(c, d) \in A \oplus B$, their sum is defined as $(a + c, b + d)$, where $a + c$ is the sum of the elements a and c of the group A , and $b + d$ is the sum of the elements b and d of the group B .

Definition 17. *Let R be a commutative ring. Assume that there exists a family of subgroups $\{R_n\}$, $n \in \mathbb{N}$ (the non-negative integers), such that $R = R_0 \oplus R_1 \oplus \cdots \oplus R_n$ as abelian groups, and $R_i R_j \subseteq R_{i+j}$ for all i and j . Then we say that R is a graded ring, with the grading given by the direct sum decomposition $R = R_0 \oplus R_1 \oplus \cdots \oplus R_n$. [1, p. 29]*

An element of some R_i of the decomposition is called a homogeneous element of order i of R . [1, p. 30]

Example 9. *The polynomial ring $K[x, y]$ together with the decomposition $K[x, y] = K \oplus (Kx + Ky) \oplus (Kx^2 + Kxy + Ky^2) \oplus \cdots$ is a graded ring. Elements of K are homogeneous elements of degree 0, elements of $Kx + Ky$ are homogeneous elements of degree 1, elements of $Kx^2 + Kxy + Ky^2$ are homogeneous elements of degree 2, and so on. For example, $x^3 + 2x^2y - 4y^3$ is a homogeneous element of order 3, and $x^5 + 2xy + 7xy^{13}$ is not a homogeneous element with respect to this grading.*

A homogeneous ideal is an ideal of R generated by homogeneous elements. [1, p. 30]

Definition 18. Let $R = R_0 \oplus R_1 \oplus \cdots$ be a graded ring, and M be a module on R . Assume that $M = \bigoplus_0^\infty M_i$, where each M_i is an abelian group, such that $M_i R_j \subseteq M_{i+j}$ for all i and j . Then we say that M is a graded module, with the grading given by the direct sum decomposition $M = \bigoplus_0^\infty M_i$. [1, p. 42]

Definition 19. Let M be a finitely generated graded module over $K[x_1, \dots, x_n]$. Then the Hilbert function of M is the map $H_M : \mathbb{N} \rightarrow \mathbb{N}$ such that $H_M(s) = \dim_K M_s$, where $\dim_K M_s$ is the dimension of M_s seen as a vector space over K . [1, p. 42]

Example 10. Consider the polynomial ring $K[x, y]$ with the grading described above. The ring seen as a module over itself will have the Hilbert function $H_{K[x, y]}(s) = \dim_K K[x, y]_s$. On the standard grading, we have $\dim_K K[x, y]_0 = \dim_K K = 1$, since K has dimension 1 seen as a vector space over itself. Similarly, $\dim_K K[x, y]_1 = \dim_K(Kx + Ky) = 2$, $\dim_K K[x, y]_2 = \dim_K(Kx^2 + Kxy + Ky^2) = 3$, and, in general, $\dim_K K[x, y]_s = s + 1$.

3.2 Local rings, localization, and Noetherian rings

The Hilbert-Samuel function is a generalization of the Hilbert function for local rings. Before defining it, we will introduce the concepts of a local ring and of localization.

Definition 20. Let R be a commutative ring. Assume that there is only one maximal ideal m on R . Then R is said to be a local ring.

The local ring R with maximal ideal m is often denoted by the expression (R, m) .

It is possible to find local counterparts of non-local rings by a process known as localization. The aim is to introduce inverses to all maximal ideals except one, so that this maximal ideal becomes unique. In general, this may be done by choosing a subset S of the non-local ring R such that S consists of the elements of R that one wants to become invertible, and then finding a ring homomorphism ϕ from R to some other ring R^* such that the image of S under ϕ is the group of units of R^* . Then R^* will be a local ring, known

as the localization of R . It is customary to write $R^* = S^{-1}R$, where S^{-1} is the set of inverses to S .

For example, the ring \mathbb{Z} is non-local, because every ideal $p\mathbb{Z}$ generated by a prime number p is maximal. If we want the ideal generated by a particular prime number p to become the unique maximal ideal, we may construct a localization according to the above, with $S = \mathbb{Z} - p\mathbb{Z}$ and $S^{-1} = \{1/s : s \in S\}$.

We may also choose the zero ideal as the new unique maximal ideal. Then the subset S will contain all the elements of R except the zero element. The resulting localization R^* will be a field, known as the field of fractions of R . For example, we may construct a localization of \mathbb{Z} such that $S = \mathbb{Z} - \{0\}$ and $S^{-1} = \{1/s : s \in S\}$. Then the localization $R^* = S^{-1}\mathbb{Z} = \mathbb{Q}$, the field of rational numbers.

In the case of monomial ideals on a polynomial ring $K[x_1, \dots, x_n]$, there are several natural choices of localization. To find a suitable localization, we first need to find the maximal ideals of the polynomial ring. By a corollary to the famous theorem known as Hilbert's Nullstellensatz, if K is algebraically closed (such as the field of complex numbers \mathbb{C}) these are the ideals of the form $\langle x_1 - a_1, \dots, x_n - a_n \rangle$.

The Nullstellensatz shows that there is a connection between a polynomial in $K[x_1, \dots, x_n]$, an algebraic object, and the set of its roots in K^n , a geometric object. Here, we will limit ourselves to stating the Nullstellensatz without proof and showing that the corollary follows from it. To state the Nullstellensatz, we first need the notion of an algebraic set, and a certain inverse construction of an ideal from an algebraic set.

Definition 21. *Let $K[x_1, \dots, x_n]$ be a polynomial ring and $S \subseteq K[x_1, \dots, x_n]$ be a subset. Then the corresponding algebraic subset $Z(S) = \{(a_1, \dots, a_n) \in K^n \mid f(a_1, \dots, a_n) = 0 \text{ for all } f \in S\}$. The algebraic subset is often called an algebraic set, and denoted by $X = Z(S)$. [1, p. 32]*

Proposition 17. *Let $X \subseteq K^n$ be any set. We construct the set $I(X) = \{f \in K[x_1, \dots, x_n] \mid f(a_1, \dots, a_n) = 0 \text{ for all } (a_1, \dots, a_n) \in X\}$. $I(X)$ is an ideal, since $(f + g)(a_1, \dots, a_n) = f(a_1, \dots, a_n) + g(a_1, \dots, a_n) = 0 + 0 = 0$ with $f, g \in I(X)$, and $(rf)(a_1, \dots, a_n) = (r)f(a_1, \dots, a_n) = r \cdot 0 = 0$, with $f \in I(X)$ and $r \in K[x_1, \dots, x_n]$. [1, p. 33]*

Theorem 18 (Hilbert's Nullstellensatz). *Let K be an algebraically closed field, and $a \subseteq K[x_1, \dots, x_n]$ be an ideal. Then $I(Z(a)) = \sqrt{a}$. [1, p. 33]*

Corollary 19. *Let K be an algebraically closed field. Then, every maximal ideal of $K[x_1, \dots, x_n]$ has the form $m_p = \langle x_1 - a_1, \dots, x_n - a_n \rangle$, corresponding to some $p = (a_1, \dots, a_n) \in K^n$.*

Proof. Let $m \subseteq K[x_1, \dots, x_n]$ be a maximal ideal. Since m is a maximal ideal, it is also a prime ideal. Prime ideals are their own radical ideals, so the Nullstellensatz gives that $m = I(Z(m)) = \{f \in K[x_1, \dots, x_n] \mid f(a_1, \dots, a_n) = 0 \text{ for all } (a_1, \dots, a_n) \in Z(m)\}$. Since $Z(m) \subseteq K^n$, there will always be some $p \in K^n$ such that $p \in Z(m)$. Now, m is the set of polynomials whose value is 0 when evaluated at any point in $Z(m)$, and m_p is the set of polynomials whose value is 0 when evaluated at the specific point $p \in Z(m)$. So, for any polynomial $f \in K[x_1, \dots, x_n]$, we have that $f \in m_p$ if $f \in m$, and thus $m \subseteq m_p$. But m was assumed to be maximal, so we must have that $m = m_p$. [1, p. 35] ■

Importantly, the maximal ideal $\langle x_1, \dots, x_n \rangle$ contains all monomial ideals generated by monomials of degree greater than zero on the polynomial ring, since it contains all monomials of degree greater than zero in the polynomial ring. If we want the properties of the ideals on the localization to be similar to the properties of the monomial ideals on the polynomial ring, it is natural to choose $\langle x_1, \dots, x_n \rangle$ as the unique maximal ideal of the localization. Then we have $S = K[x_1, \dots, x_n] - \langle x_1, \dots, x_n \rangle$, and we may define $S^{-1} = \{\frac{1}{f(x_1, \dots, x_n)} : f(x_1, \dots, x_n) \in S\}$.

A final concept which is necessary for understanding the Hilbert-Samuel function is that of a Noetherian ring.

Definition 22. *A ring R is Noetherian if every ideal in R can be generated by a set with finitely many elements. [1][p. 27]*

The condition that every ideal is finitely generated is equivalent to the condition that each ascending chain of ideals terminates. This ascending chain condition can be expressed in the following way:

For any chain of ideals $I_1 \subseteq \dots \subseteq I_k \subseteq I_{k+1} \subseteq \dots$, there is an n such that $I_n = I_{n+1} = \dots$

All fields are Noetherian rings, since they only have two ideals: the ring itself and the zero ideal. Also, the ring of integers \mathbb{Z} is a Noetherian ring, since every ideal can be generated by a single element (the ring is a principal ideal domain).

Importantly, if R is a polynomial ring in finitely many variables over a field, as in our case, R is Noetherian. This result is a special case of the Hilbert Basis Theorem:

Theorem 20. *If a ring R is Noetherian, then the polynomial ring $R[x]$ is Noetherian.*

See proof of Theorem 1.2 in [1]. It is easy to see that it follows from the theorem that a polynomial ring in any finite number of variables over a Noetherian ring is Noetherian: if R is Noetherian, then $R[x_1]$ is Noetherian, but then $R[x_1, x_2] = R[x_1][x_2]$ is Noetherian, and so on.

3.3 The Hilbert-Samuel function

On a Noetherian local ring, we may define the following version of the Hilbert function:

Definition 23. *On a Noetherian local ring R with maximal ideal m , the Hilbert function of a R -module M is the map $H_M : \mathbb{N} \rightarrow \mathbb{N}$ such that $H_M(n) = \dim_{R/m} m^n M / m^{n+1} M$. [1, p. 271]*

While working on intersection theory, Pierre Samuel developed a generalization of this function for ideals primary to the maximal ideal. This function is known as the Hilbert-Samuel function. The following definition comes from Eisenbud, but has been simplified by defining the function for primary ideals.

Definition 24. *Let R be a commutative Noetherian local ring with the maximal ideal m , M be a finitely generated R -module, and I be a m -primary ideal in R . Then the Hilbert-Samuel function of M with respect to I is $\chi_M^I : \mathbb{N} \rightarrow \mathbb{N}$ such that $\chi_M^I(n) = \ell(I^n M / I^{n+1} M)$, where ℓ denotes the length of the resulting quotient module seen as a module over R . [1, p. 272]*

Example 11. *Consider the localization of the ring of integers \mathbb{Z} with respect to the prime ideal $2\mathbb{Z}$. Choosing $S = \mathbb{Z} - 2\mathbb{Z}$ and $S^{-1} = \{1/s : s \in S\}$, we get the localization $R^* = S^{-1}\mathbb{Z}$. An arbitrary element $r^* \in R^*$ may be written $r^* = \frac{a}{b}$, with a an integer and b an odd integer. Ideals on R^* are generated by the powers of 2. Since inverses exist for all other prime numbers, ideals generated by numbers with a power of 2 and another prime number as factors will be identical to an ideal generated by a power of 2.*

We may now, for instance, calculate the Hilbert-Samuel function of the ring R^* seen as a module over itself with respect to the ideal $2R^*$. The quotient module $I^n M / I^{n+1} M = (2R^*)^n / (2R^*)^{n+1} = 2^n R^* / 2^{n+1} R^*$. The elements of the quotient module will be the equivalence classes $[0]$ and $[2^n]$. It is clear that the quotient module will contain precisely one submodule, the zero module $\{[0]\}$, so the length of the longest chain of proper submodules is 1 for any n , and the Hilbert-Samuel function $\chi_{R^*}^{2R^*}(n) = 1$ for all n .

This essay is particularly concerned with the Hilbert-Samuel function of ideals on the localization $K[x, y]_m$ described above on the polynomial ring $K[x, y]$, and with the Hilbert-Samuel function of ideals on the ring of formal power series $K[[x, y]]$.

On the localization $K[x, y]_m$, where m is the maximal ideal $\langle y, x \rangle$, the ideals are precisely the monomial ideals. To calculate the Hilbert-Samuel function $\chi_M^I(n) = \ell(I^n M / I^{n+1} M)$ of some module M with respect to some monomial ideal I , we must find the powers of the monomial ideal.

Example 12. An example is the simple case where $M = K[x, y]_m$, and $I = m$. For $n = 0$, the quotient module $m^0 K[x, y]_m / m^{0+1} K[x, y]_m = K$, since any polynomial in $K[x, y]_m$ can be written as the sum of a polynomial in $mK[x, y]_m$ and an element in K . K is a field, so it has precisely one proper ideal (and one proper submodule if it is seen as a module over $K[x, y]_m$), namely the zero ideal $\{0\}$. Consequently, $\chi_{K[x, y]_m}^m(0) = 1$.

For $n = 1$, we have $m^1 K[x, y]_m / m^2 K[x, y]_m = Kx + Ky$, since any polynomial in $mK[x, y]_m$ can be written as the sum of a polynomial in $m^2 K[x, y]_m$, one element in Kx , and one element in Ky . (Remember that $m^2 = \langle y^2, xy, x^2 \rangle$.) A proper submodule of $Kx + Ky$ will consist of K times one variable, for instance Kx or $K(x + y)$. Similarly to the $n = 0$ case, the submodule in one variable will only have one proper submodule, the zero module. Consequently, $\chi_{K[x, y]_m}^m(1) = 2$. In general, we have $\chi_{K[x, y]_m}^m(n) = n + 1$, since the difference between the number of monomials not included in m^{n+1} and m^n is $n + 1$. We notice that the length of the quotient modules seen as modules over $K[x, y]_m$ is the same as the dimension of the quotient modules seen as vector spaces over K .

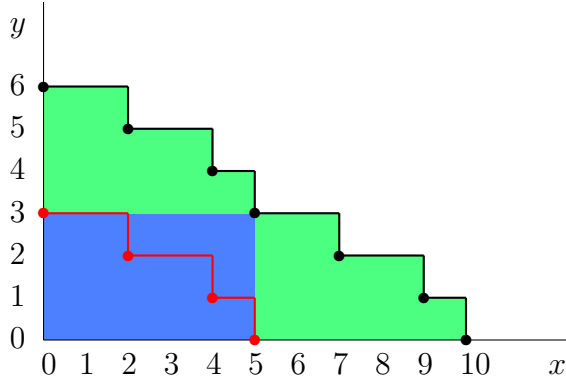
We may find a more general rule by considering another example. Let $M = K[x, y]_m$, and $I = \langle y^3, x^2 y^2, x^4 y, x^5 \rangle$. For $n = 0$, the quotient module $K[x, y]_m / \langle y^3, x^2 y^2, x^4 y, x^5 \rangle^1 K[x, y]_m = K + Ky + Ky^2 + Kx + Kyx + Ky^2 x + Kx^2 + Kyx^2 + Kx^3 + Kyx^3 + Kx^4$. The proper additive subgroups of the

quotient module are formed by proper subsets of the set of Ky^Bx^A . In turn, the proper subgroups of a subgroup are formed by proper subsets of the set of Kx^Ay^B that forms the subgroup. The longest chain of proper submodules will therefore have a length equal to the size of the set $K + Ky + Ky^2 + Kx + Kyx + Ky^2x + Kx^2 + Kyx^2 + Kx^3 + Kyx^3 + Kx^4$, namely 11. Consequently, $\chi_{K[x,y]_m}^I(0) = 11$.

For $n = 1$, the quotient module $IK[x,y]_m/I^2K[x,y]_m = Ky^3 + Ky^4 + Ky^5 + Ky^3x + Ky^4x + Ky^5x + Ky^2x^2 + Ky^3x^2 + Ky^4x^2 + Ky^2x^3 + Ky^3x^3 + Ky^4x^3 + Kyx^4 + Ky^2x^4 + Ky^3x^4 + Kx^5 + Kyx^5 + Ky^2x^5 + Kx^6 + Kyx^6 + Ky^2x^6 + Kx^7 + Kyx^7 + Kx^8 + Kyx^8 + Kx^9$, since $I^2 = \langle y^6, x^2y^5, x^4y^4, x^5y^3, x^7y^2, x^9y, x^{10} \rangle$ (see the example in section 1.2). Additive subgroups are formed in the same way as above, so the longest chain of submodules has a size equal to the set of Kx^Ay^B , namely 26, and $\chi_{K[x,y]_m}^I(1) = 26$.

If we had to find the powers of ideals of order n and $n + 1$ for each n , the calculation of the Hilbert-Samuel function would be very complicated. Fortunately, the ideal $I = \langle y^3, x^2y^2, x^4y, x^5 \rangle$ is such that $I^r = T_r(I)$ for all r , where $T_i(I)$ is the staircase ideal formed from I of order i . Since the value of the Hilbert-Samuel function $\chi_{K[x,y]_m}^I(n)$ is equal to the difference between the number of sets Kx^Ay^B not included in the monomial ideal I^{n+1} and the number of such sets not included in the ideal I^n , we need to find a formula for this difference.

We may find a pattern for the difference in the following diagram. The staircase ideal of order 1 will contain one copy of the area not in the graph of ideal I . The staircase ideal of order 2 will contain two copies of the area not in the graph of the ideal I , above and to the right of a rectangle with area ab . The staircase ideal of order 3 will contain three copies of the area not in the graph of the ideal I , above and to the right of three rectangles with area ab , and so on.



Generally, we have the following formula: if $I = T_1(I)$ does not include r sets of monomials, then $T_i(I)$ will not include $ri + ab\frac{i(i-1)}{2}$ monomials, where a is the largest x -exponent in the minimal generating set of I , and b is the largest y -exponent in the minimal generating set of I .

Consequently, $\chi_{K[x,y]_m}^I(n) = 11(n+1) + 15\frac{(n+1)n}{2} - (11n + 15\frac{n(n-1)}{2}) = 11 + 15n$ for $I = \langle y^3, x^2y^2, x^4y, x^5 \rangle$, and $\chi_{K[x,y]_m}^I(n) = r(n+1) + ab\frac{(n+1)n}{2} - (rn + ab\frac{n(n-1)}{2}) = r + abn$ for general I .

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