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Introduction to Representation Theory of Quivers

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Abstract

By thoroughly solving matrix problems of equivalence relation requirements and connecting these solutions to methods of exhibiting isomorphisms between representations of quivers, the basics of representation theory of quivers is displayed. The techniques used are taking both matrices and representations respectively and decomping their normal forms into indecomposables and comparing these forms to establish isomorphism/equivalence or the lack of it.

Keywords: Matrix, Quiver, Representation theory, Direct sum, Indecompsable

Introduction

This work focuses on illuminating the preliminaries that are useful when getting familiar with the first few concepts of the theory of quivers and representations of quivers. The aim is that this paper should be both readable and understandable for students that are in their first or second year of studies. Throughout the text the same approach of explaining the basics of representations theory as in [2] is used, however efforts are made to concretize the material further and being very explicit about which matrices are used for what purposes. For graduate students I hope that they can follow the text, and work out examples of their own as they go along.

The second chapter, following the preliminaries, exhibits two matrix problems. These are to determine when matrices are equivalent and when they are not, respectively when pairs of matrices are or are not equivalent. These problems are solved by multiplication with invertible matrices (the same as row and column operations) and identifying normal forms. The answer is found to be that matrices (or pairs) are equivalent if and only if they have the same normal forms. A notation of direct sums of matrices and indecomposability is then introduced to simplify notation and connect these problems to latter topics.

The notation of quivers and representations is established and another problem is introduced; whether two representations are isomorphic or not. As it turns out this problem can be solved by chosing bases for all vector spaces in the representations and solving the corresponding matrix problems for the matrices that correspond to each linear map in the representations. A simliar notation for direct sums of representations and indecomposability is used to prove that each problem (for certain types of quivers at least) of isomorphism of representations is in bijection with the matrix problems. One could consider indecomposable representations as building blocks to make up arbitrary reprensentations.

The last two chapters summarizes what is currently known about the types of quivers studied is this paper. It also goes slightly deeper into the quivers of type $\mathbb{A}_n$ and its properties. In these chapters one comes across Gabriel's Theorem regarding what type of quivers have a finite list of indecomposables, as well as the Krull-Schmidt's Theorem which states that two different decompositions into indecomposables differs only by permutation. However, for students that only wish to get a glance of represenation theory of quivers it should be suffiecent to exclude the theory in chapter 6.

1 Preliminaries

The reader is assumed to be familiar with linear algebra and algebraic structures through out this document. Standard notation; Unless otherwise stated, K is an arbitrary field. $K^{m\times n}$ is the set of all matrices of size $m \times n$ (having m rows and n columns) over K. I is the identity matrix. The set $\underline{n}$ is the set $\{1, 2, \dots, n\}$.

The following definitions are taken from the courses in linear algebra.

Definition 1.1. *Let $A \in K^{m\times n}, A' \in K^{m\times n}$ then A, A' lie in the same equivalence class by the relation $\sim$ if A can be transformed to A' by elementary row and column operations. Notation: $A \sim A'$*

Definition 1.2. *Let $A \in K^{m\times n}$ then the **rank** of A is given by the number of non-zero rows in the reduced row echelon form of A.*

The following notation will be used for totally reduced form;

$$I_{r,m,n} := \begin{bmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in K^{m\times n}$$

where $\mathbf{1}_r = I \in K^{r\times r}$ and $\mathbf{0}$ is the zero matrix of suitable size. $I_{r,m,n}$ is denoted briefly as I_r whenever the context is clear. Also $\mathbf{1}_1 \in K^{1\times 1}$ will be denoted $\mathbf{1}$, not to be confused with the number $1 \in K$.

Proposition 1.3. *Given $A \in K^{m\times n}$*

$$rank(A) = r \Leftrightarrow A \sim I_{r,m,n}$$

Proof:
Applying Gauss-Jordan elimination on A followed by suitable column operations yields that $A \sim I_{r,m,n}$ for some r. By Definition 1.2 it follows that $\text{rank}(A) = \text{rank}(I_{r,m,n}) = r$. □

Since $\text{rank}(I_{r,m,n}) = r$ it follows that $A \sim I_{r,m,n}$ for every matrix $A \in K^{m\times n}$ with $\text{rank}(A) = r$. Thus $I_{r,m,n}$ is called the **normal form** for the equivalence classes of matrices in $K^{m\times n}$. In other words; two matrices of the same size $m \times n$ and rank r are equivalent, and belong to the same equivalence class as $I_{r,m,n}$. Thus the question of equivalence between A and A' now translates to if A and A' have the same normal form.

1.1 Elementary matrices

The goal of this chapter is to prove that the relation $A \sim A'$ can be expressed by the equivalent statement $A = M_1 A' M_2$ for some invertible matrices M_1 and M_2.

Definition 1.4. *Let $\varepsilon_{i,j}$ be the square zero-matrix except for a 1 at index i, j, with it's size granted from the context, much like the identity matrix. Let $k \in K$, then*

$$E_{i,j}(k) := (I + k\varepsilon_{i,j}) \qquad i \neq j$$

$$E_i(k) := (I - \varepsilon_{i,i} + k\varepsilon_{i,i}) \qquad k \neq 0$$

$$E_{i\leftrightarrow j} := (I - \varepsilon_{i,i} - \varepsilon_{j,j} + \varepsilon_{i,j} + \varepsilon_{j,i})$$

*are called **elementary matrices**.*

These elementary matrices each correspond to a certain row or column operation e.g. when multiplying from the left with a certain elementary matrix;

(*i*) $E_{i,j}(k)A$ corresponds to "add k multiples of row j to row i"

(*ii*) $E_i(k)A$ corresponds to "multiply row i by k" $\quad (k \neq 0)$

(*iii*) $E_{i\leftrightarrow j}A$ corresponds to "swapping row i with row j"

and when multiplying from the right;

(*iv*) $AE_{i,j}(k)$ corresponds to "add k multiples of column i to column j",

(*v*) $AE_i(k)$ corresponds to "multiply column i by k" $\quad (k \neq 0)$,

(*vi*) $AE_{i\leftrightarrow j}$ corresponds to "swapping column i with column j",

Note the difference in (*i*) and (*iv*).

To establish a more convenient notation define:

Definition 1.5. $\mathbb{E}$ *is the set of all elementary matrices;*

$$\mathbb{E} := \{E_{i,j}(k), E_i(k), E_{i\leftrightarrow j} : i, j \in \mathbb{N}^+, k \in K\}$$

Definition 1.6. $\mathbb{M}$ *is the set of arbitrary products (which can be thought of as compositions of operations) of matrices in* $\mathbb{E}$*;*

$$\mathbb{M} := \{M : M = \prod_{i=1}^{n} E_i, E_i \in \mathbb{E}, n \in \mathbb{N}^+\}$$

also, let $S_1 \subset \mathbb{N}^+$ *and* $S_2 \subset \mathbb{N}^+$ *, then*

- *if* M *affects* ***at most*** *the rows/columns in* S_1*, and fetching information from* ***at most*** *the rows/columns in* S_2 *the notation* $\underset{S_2}{\overset{S_1}{M}} \in \mathbb{M}$ *is used,*
- *if* S_1 *and* S_2 *respectively make up all rows/columns of a matrix* A *the notation* $\underset{A}{\overset{A}{M}}$ *will instead be used for convenience.*

Example: Let $M = E_{3\leftrightarrow 6}E_{4,2}(10)E_4(7)$, then MA is obtained by performing the following operations on A;

(*i*) multiply row 4 by 7 and then;

(*ii*) add 10 times row 2 to row 4 and then;

(*iii*) swap rows 3 and 6,

which implies that $M = \underset{\{4,3,6\}}{\overset{\{2,3,4,6\}}{M}} \in \mathbb{M}$.

Example: Let $M = E_{3,1}(\frac{1}{2})$, then MA is obtained by adding $\frac{1}{2}$ times row 1 to row 3 which implies that $M = \underset{\{1\}}{\overset{\{3\}}{M}} \in \mathbb{M}$

It is advised that the reader should make sure that these examples are completely understood before moving on.

Proposition 1.7. *Let $M \in \mathbb{M}$ and $S = \{1, \dots, k\}$. If $M = \underset{S}{\overset{S}{M}}$ then M is on the form;*

$$M = \begin{bmatrix} N & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \qquad , \qquad N \in \mathbb{M} \cap K^{k \times k}$$

and moreover;

$$M^{-1} = \begin{bmatrix} N^{-1} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \qquad , \qquad M^T = \begin{bmatrix} N^T & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix}$$

Proof:
Let $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ such that $A_1 \in K^{k \times k}$. The matrix M can be split into $\begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}$. Now the blockwise multiplication MA yields;

$$MA = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} M_1A_1 + M_2A_2 \\ M_3A_1 + M_4A_2 \end{bmatrix}$$

and simultaneously since M does not fetch information from A_2 it follows that $M_2 = \mathbf{0}$. Similarly since M does not affect A_2 it follows that $M_3 = \mathbf{0}$ and $M_4 = I$.

The proofs of transpose and inverse are trivial, and therefore left out. □

Corollary 1.8. *Let $M \in \mathbb{M}$, then if;*

$$MI_r = \underset{\mathbf{1}_r}{\overset{\mathbf{1}_r}{M}} \begin{bmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

it follows that $MI_rM^{-1} = I_r$.

Proof:
Blockwise multiplication yields;

$$\begin{bmatrix} N & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} \begin{bmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} N^{-1} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} = \begin{bmatrix} N & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} N^{-1} & \mathbf{0} \\ \mathbf{0} & I \end{bmatrix} = \begin{bmatrix} NN^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$$

□

The above statement is a way of expressing that any row-operations performed on a totally reduced matrix have canceling column-operations. These column-operations can, when viewed as a matrix, be seen as the inverse matrix of the row-operations. It turns out that there is a lot more to be said about the elementary matrices of $\mathbb{E}$.

Proposition 1.9. $\forall E \in \mathbb{E}$, $det(E) \neq 0$ *thus E is invertible.*

Proof.
(These are all easy excercises using co-factor expansion)

$$\det(E_{i,j}(k)) = \det \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} = 1 \qquad (i \neq j)$$

$$\det(E_i(k)) = \det \begin{bmatrix} k \end{bmatrix} = k \neq 0$$

$$\det(E_{i \leftrightarrow j}) = \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$$

□

Corollary 1.10. $\forall M \in \mathbb{M}$, $det(M) \neq 0$ *thus M is invertible.*

Proof.
This follows from Proposition 1.9 using $\det(E_1E_2) = \det(E_1)\det(E_2)$ and that K is a field so that $K \setminus \{0\}$ is multiplicatively closed. $\square$

Proposition 1.11. *Let $E \in \mathbb{E}$. Then $E^T, E^{-1} \in \mathbb{E}$.*

Proof.
Taking inverses of matrices as in linear algebra yields;

$$E_{i,j}(k)^{-1} = E_{i,j}(-k)$$
$$E_i(k)^{-1} = E_i(k^{-1})$$
$$E_{i\leftrightarrow j}^{-1} = E_{i\leftrightarrow j}$$

Since K is a field $\{-k : k \in K\} = K$ and $\{k^{-1} : k \in K \setminus \{0\}\} = K \setminus \{0\}$ hence;

$$\{E_{i,j}(-k) : k \in K\} = \{E_{i,j}(k) : k \in K\}$$
$$\{E_i(k^{-1}) : k \in K \setminus \{0\}\} = \{E_i(k) : k \in K \setminus \{0\}\}$$

Thus $E \in \mathbb{E}$ implies $E^{-1} \in \mathbb{E}$.

Transposing yields;

$$E_{i,j}(k)^T = (I + k\varepsilon_{i,j})^T = (I^T + k\varepsilon_{i,j}^T) = (I + k\varepsilon_{j,i}) = E_{j,i}(k)$$

By establishing a bijection between $\{i, j\}$ and $\{j, i\}$ by transposition, whenever i and j are taken arbitrairly from the same set, it follows that $\{E_{i,j}(k) : i, j \in N \subset \mathbb{N}\} = \{E_{j,i}(k) : i, j \in N \subset \mathbb{N}\}$. The two remaining cases are transposes of symmetric matrices;

$$E_i(k)^T = E_i(k)$$
$$E_{i\leftrightarrow j}^T = E_{i\leftrightarrow j}$$

Thus $E \in \mathbb{E}$ implies $E^T \in \mathbb{E}$. $\square$

Proposition 1.12. *Let $M \in \mathbb{M}$. Then $M^T, M^{-1} \in \mathbb{M}$.*

Proof.
Let $M = E_1E_2 \cdots E_n$ where $E_i \in \mathbb{E}$ then;

$$M^{-1} = {E_n}^{-1} \cdots {E_2}^{-1}{E_1}^{-1}$$

$$M^T = {E_n}^T \cdots {E_2}^T{E_1}^T$$

the rest follows from Proposition 1.11. $\square$

Corollary 1.13. *Let $S_j = \{j_1, j_2, \ldots, j_n\} \subset \mathbb{N}^+$ and $S_i = \{i_1, i_2, \ldots, i_n\} \subset \mathbb{N}^+$ so that $S_i \cap S_j = \emptyset$. Then $\forall M \in \mathbb{M}$ satisfying*

$$M = \prod_{p=1}^{n} E_{i_p,j_p}(k_p)$$

$\exists N \in \mathbb{M}$ so that

$$N = (M^T)^{-1}$$

and in particular $M = \underset{S_i}{\overset{S_j}{M}} \Rightarrow N = \underset{S_j}{\overset{S_i}{N}}$.

Proof.

$$(M^T)^{-1} = ((\prod_{p=1}^{n} E_{i_p,j_p}(k_p))^T)^{-1} = (\prod_{p=1}^{n}(E_{i_{n-p},j_{n-p}}(k_{n-p}))^T)^{-1} =$$

$$= (\prod_{p=1}^{n} E_{j_{n-p},i_{n-p}}(k_{n-p}))^{-1} = \prod_{p=1}^{n}(E_{j_p,i_p}(k_p))^{-1} = \prod_{p=1}^{n} E_{j_p,i_p}(-k_p)$$

□

At this point it is possible to connect normal forms to equivalence relations as follows.

Proposition 1.14. *The following statements are equivalent;*

(*i*) $A \sim A'$

(*ii*) $\exists M_1, M_2 \in \mathbb{M} : A = M_1 A' M_2$

(*iii*) A, A' *have the same normal form.*

Proof:
$(i) \Rightarrow (ii)$:
By Definition 1.1 some sequences of matrices $E_1E_2\cdots E_p$ and $E'_1E'_2\cdots E'_{p'}$ (where $E_i, E'_j \in \mathbb{E}$) will correspond to the row and column operations that takes A' to A thus; $E_1E_2\cdots E_pA'E'_1E'_2\cdots E'_{p'} = M_1A'M_2 = A$ for some $M_1, M_2 \in \mathbb{M}$. This implies (ii).
$(ii) \Rightarrow (iii)$:
$A \sim I_{r,m,n}$ implies that $M_3AM_4 = I_{r,m,n}$ and so $M_3M_1A'M_2M_4 = N_1A'N_2 = I_{r,m,n}$ for some $N_1, N_2 \in \mathbb{M}$. Thus $A' \sim I_{r,m,n}$. This implies (iii).
$(iii) \Rightarrow (i)$:
$A \sim I_{r,m,n}$ and $A' \sim I_{r',m',n'}$ implies that $r = r'$, $m = m'$ and $n = n'$ thus $A \sim I_{r,m,n} = I_{r',m',n'} \sim A'$ which implies (i). □

Proposition 1.15. $\mathbb{M}$ *is the set of all invertible matrices.*

Proof:
Let A be invertible, then $A \sim I_{r,m,n}$ where $r = m = n$. By Proposition 1.14 $\exists M_1, M_2 \in \mathbb{M} : M_1AM_2 = I$ and thus $A = {M_1}^{-1}I{M_2}^{-1} = (M_2M_1)^{-1} \in \mathbb{M}$. The opposite inclusion is stated in Corollary 1.10. □

2 Matrix problems

As has been shown in Proposition 1.14 it is possible to reduce the problem of finding proper conditions for $A \sim A'$ to determination of normal forms. From that state the problem is again reducible since the normal form of a matrix is uniquely determined by its size and rank. This problem will be refered to as the *first matrix problem*.

2.1 Second problem

The next problem is slighly more complex and the preliminaries gone through will now prove useful for finding a solution. The idea is rather simple though; extend the equivalence relation to a pair of matrices and connect them as follows.

Definition 2.1. *Given two pairs of matrices (A, B) and (A', B'), such that $A, A' \in K^{m\times n}$ and $B, B' \in K^{l\times m}$, define an equivalence relation $\sim$ by*

$$(A, B) \sim (A', B') \Leftrightarrow \begin{cases} A' = M_2^{-1} A M_1 \\ B' = M_3^{-1} B M_2 \end{cases}$$

Note that $(A, B) \sim (A', B')$ implies $A \sim A'$ and $B \sim B'$ but not vice versa. In other words $A \sim A'$ and $B \sim B'$ are necessary but not sufficient conditions for $(A, B) \sim (A', B')$ to hold. Additionally in order for the matrix multiplication $M_2^{-1} A M_1$ and $M_3^{-1} B M_2$ to be defined the number of rows in A and the number of columns in B should both equal the size of M_2. Hence the problem can be formulated as follows;

Given matrices $A \in K^{m\times n}$ and $B \in K^{l\times m}$ along with matrices A', B' such that $A \sim A'$ and $B \sim B'$, find proper conditions for $(A, B) \sim (A', B')$.

solution:
It is now necessary to start performing simultaneous operations, since the equivalence relation have connected the row operations on A with the column operations on B. In order to make both notation and understanding less of an issue consider working on $B^T \in K^{m\times l}$ instead of B;

$$B' = M_3^{-1} B M_2 \Leftrightarrow B'^T = M_2^T B^T (M_3^{-1})^T$$

by Proposition 1.12 the following substitutions are valid

$$\begin{aligned} M_1 &= N_1 \in \mathbb{M} \\ M_2^T &= N_2 \in \mathbb{M} \\ (M_3^{-1})^T &= N_3 \in \mathbb{M} \end{aligned}$$

and thus this system of equations have been obtained;

$$(A, B) \sim (A', B') \Leftrightarrow \begin{cases} A' = ({N_2}^T)^{-1} A N_1 \\ B'^T = N_2 B^T N_3 \end{cases}$$

The rule for solving this problem is now that for every row operation performed on B^T the inverse of the transpose of that operation must be performed (at the same time) to A. (Also vice versa with swapped transpose and inverse) The notation $\mathcal{M}$ will be used to indicate that there exists a matrix $M \in \mathbb{M}$ that will satisfy the stated equality. The notation $\mathcal{F}$ will be used to indicate that a forced response by the rule has been applied.

Let $A \in K^{m\times n}$ and $B^T \in K^{m\times l}$ so that $\big([A],\,[B^T]\big) \in K^{m\times(n+l)}$ then by Proposition 1.3 A can be reduced to normal form;

$$\big([A],\,[B^T]\big) \sim \left(\overset{A}{\underset{A}{\mathcal{M}}}\,[A]\,\overset{A}{\underset{A}{\mathcal{M}}},\,\overset{B^T}{\underset{B^T}{\mathcal{F}}}\,[B^T]\right) = \left(\begin{bmatrix}\mathbf{1}_s & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix},\,\begin{bmatrix}B_1\\ B_2\end{bmatrix}\right)$$

Next by Corollary 1.8 B_1 can be reduced to normal form;

$$\left(\begin{bmatrix}\mathbf{1}_s & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix},\,\begin{bmatrix}B_1\\ B_2\end{bmatrix}\right) \sim \left(\overset{\mathbf{1}_s}{\underset{\mathbf{1}_s}{\mathcal{F}}}\begin{bmatrix}\mathbf{1}_s & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}(\overset{\mathbf{1}_s}{\underset{\mathbf{1}_s}{\mathcal{F}}})^{-1},\,\overset{B_1}{\underset{B_1}{\mathcal{M}}}\begin{bmatrix}B_1\\ B_2\end{bmatrix}\overset{B_1}{\underset{B_1}{\mathcal{M}}}\right) = \left(\begin{bmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix},\,\begin{bmatrix}\mathbf{1}_p & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\\ B_{2,1} & B_{2,2}\end{bmatrix}\right)$$

by Corollary 1.13 $B_{2,1}$ can be reduced to $\mathbf{0}$

$$\left(\begin{bmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix},\,\begin{bmatrix}\mathbf{1}_p & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\\ B_{2,1} & B_{2,2}\end{bmatrix}\right) \sim \left(\overset{\mathbf{1}_p}{\underset{\mathbf{0}}{\mathcal{F}}}\begin{bmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix},\,\overset{B_{2,1}}{\underset{\mathbf{1}_p}{\mathcal{M}}}\begin{bmatrix}\mathbf{1}_p & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\\ B_{2,1} & B_{2,2}\end{bmatrix}\right) =$$
$$= \left(\begin{bmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix},\,\begin{bmatrix}\mathbf{1}_p & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\\ \mathbf{0} & B_{2,2}\end{bmatrix}\right)$$

$B_{2,2}$ can be reduced to normal form since $\overset{\mathbf{0}}{\underset{\mathbf{0}}{\mathcal{F}}}$ is an operation on the zero-matrix:

$$\left(\begin{bmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix},\,\begin{bmatrix}\mathbf{1}_p & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\\ \mathbf{0} & B_{2,2}\end{bmatrix}\right) \sim \left(\overset{\mathbf{0}}{\underset{\mathbf{0}}{\mathcal{F}}}\begin{bmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix},\,\overset{B_{2,2}}{\underset{B_{2,2}}{\mathcal{M}}}\begin{bmatrix}\mathbf{1}_p & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\\ \mathbf{0} & B_{2,2}\end{bmatrix}\overset{B_{2,2}}{\underset{B_{2,2}}{\mathcal{M}}}\right) =$$
$$= \left(\begin{bmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix},\,\begin{bmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}_r & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}\right)$$

Note now that $\operatorname{rank}(A) = p + q, \operatorname{rank}(B) = p + r$. The above algoritm exhibits $N_1, N_2, N_3 \in \mathbb{M}$ such that

$$\begin{bmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix} = ({N_2}^T)^{-1}AN_1 \qquad \text{and} \qquad \begin{bmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}_r & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix} = N_2B^TN_3$$

holds. Furthermore since

$$(N_2B^TN_3)^T = \begin{bmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}_r & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix} = {N_3}^TB{N_2}^T$$

and this together with Proposition 1.12 yields;

$$(A,B) \sim \left(\begin{bmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix},\,\begin{bmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}_r & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}\right) \Leftrightarrow \begin{cases} ({N_2}^T)^{-1}AN_1 = \begin{bmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix} \\ {N_3}^TB{N_2}^T = \begin{bmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}_r & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix} \end{cases}$$

Once uniqueness is proven this normal form will be used to determine if pairs are equivalent or not.

Proposition 2.2. *Given a pair of matrices (A, B), there exists a **unique** normal form such that*

$$\left(\begin{bmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}, \begin{bmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}_r & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}\right) \sim (A, B)$$

Proof:

$$A = {M_2}^{-1}\begin{bmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix} M_1 \quad , \quad B = {M_3}^{-1}\begin{bmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}_r & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix} M_2$$

$$BA = {M_3}^{-1}\begin{bmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}_r & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix} M_2 {M_2}^{-1}\begin{bmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix} M_1 =$$

$$= {M_3}^{-1}\begin{bmatrix}\mathbf{1}_p & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix} M_1$$

So $\text{rank}(BA) = \text{rank}(\begin{bmatrix}\mathbf{1}_p & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}) = p$. Since $\text{rank}(A) = p + q$ and $\text{rank}(B) = p + r$ it is possible to recover (p, q, r) from $\text{rank}(A), \text{rank}(B)$ and $\text{rank}(BA)$. These numbers are unique for any pair of matrices (A, B), hence the normal form for (A, B) is unique. □

Next the realtion requires trasitivity in order for the unique normal forms to connect different pairs;

Proposition 2.3. *The following statements are equivalent;*

(i) $(A, B) \sim (A', B')$

(ii) (A, B) *and* (A', B') *have the same normal form.*

Proof:
$(i) \Rightarrow (ii)$:
Definition 2.1 and multiplying matrices B' and A' yields

$$(i) \Leftrightarrow \begin{cases} {M_2}^{-1}AM_1 = A' \\ {M_3}^{-1}BM_2 = B' \end{cases} \Leftrightarrow \begin{cases} {M_2}^{-1}AM_1 = A' \\ {M_3}^{-1}BM_2 = B' \\ {M_3}^{-1}BAM_1 = B'A' \end{cases}$$

next observe that $\text{rank}(A) = \text{rank}(A')$, $\text{rank}(B) = \text{rank}(B')$ and $\text{rank}(BA) = \text{rank}(B'A')$ and hence the normal forms obtained by these numbers are the same.

$(ii) \Rightarrow (i)$
Same normal form yields

$$\left.\begin{aligned} {M_2}^{-1}AM_1 &= \begin{bmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{1}_q & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix} = {M_2'}^{-1}A'M_1' \\ {M_3}^{-1}BM_2 &= \begin{bmatrix}\mathbf{1}_p & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{1}_r & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix} = {M_3'}^{-1}B'M_2' \end{aligned}\right\} \Rightarrow \begin{cases} {M_2}^{-1}AM_1 = {M_2'}^{-1}A'M_1' \\ {M_3}^{-1}BM_2 = {M_3'}^{-1}B'M_2' \end{cases}$$

next manipulate the equations to get only A and B on the right hand sides and note the slight difference of the expressions before and after the implication

$$\left.\begin{aligned} M_2 {M'_2}^{-1} A' M'_1 {M_1}^{-1} = A \\ M_3 {M'_3}^{-1} B' M'_2 {M_2}^{-1} = B \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} (M'_2 {M_2}^{-1})^{-1} A' M'_1 {M_1}^{-1} = A \\ (M'_3 {M_3}^{-1})^{-1} B' M'_2 {M_2}^{-1} = B \end{aligned}\right.$$

Next replace every instance of $M'_i {M_i}^{-1}$ with N_i as such

$$\left.\begin{aligned} N_2^{-1} A' N_1 = A \\ N_3^{-1} B' N_2 = B \end{aligned}\right\} \Leftrightarrow (i)$$

$\square$

As may have been noticed by the reader the normal form for the second matrix problem is rather clumsy. However the fact that the unique normal form can determine if two different pairs of matrices are equivalent together with the fact that the normal form is determined by the six integers $p + q = \text{rank}(A), p + r = \text{rank}(B), p = \text{rank}(BA)$ and m, n, l where $A \in K^{m\times n}, B \in K^{l\times m}$ introduces a possibility to reduce the normal form notation. This will be the topic of the next chapter.

3 Decomposition into indecomposables

It is now clear that obtaining the normal forms of the first and second matrix problems via the solving algorithms in Chapter 2 is equivalent to identifying the three respectively six numbers that are ranks and sizes of given matrices. Decomposition of these normal forms will give clear notation and association between normal forms and these numbers.

3.1 Direct sum

The idea is to write some matrix A as a direct sum of other matrices say B and C in order to clarify what information is contained within A. This is very similar to the concept of prime factorization, so it is therefore fitting to introduce the "smallest" matrices at hand.

Definition 3.1. *Any matrix $A \in K^{0\times 1} \cup K^{1\times 0} \cup K^{0\times 0}$ has no entries, but have size. It is therefore unique up to it's size. These three matrices are denoted by;*

$$\begin{cases} A := \boldsymbol{nul} & \text{if} \quad A \in K^{0\times 0} \\ A := \boldsymbol{row} & \text{if} \quad A \in K^{1\times 0} \\ A := \boldsymbol{col} & \text{if} \quad A \in K^{0\times 1} \end{cases}$$

Definition 3.2. *The direct sum of two matrices A and B is;*

$$A \oplus B := \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{bmatrix}$$

along with the exponential notation;

$$A^{\oplus 0} := \boldsymbol{nul}$$

$$A^{\oplus n} := A \oplus A^{\oplus n-1} \qquad n \geq 1$$

One immediate consequence of this definition is;

Proposition 3.3. *Given $A \sim I_{a,m,n}$ and $A' \sim I_{a',m',n'}$ it follows that;*

$$(A \oplus A') \sim (I_{a,m,n} \oplus I_{a',m',n'}) \sim I_{a+a',m+m',n+n'}$$

The proof of this is an easy excercise using elementary matrices.

Proposition 3.4. *Basic properties of direct sums;*
For any matrix $A \in K^{m\times n}$

$$\begin{cases} \boldsymbol{nul} \oplus A = A \oplus \boldsymbol{nul} = A \\ \boldsymbol{col} \oplus A = \begin{bmatrix} 0 & A \end{bmatrix} \in K^{m\times(n+1)} \\ A \oplus \boldsymbol{col} = \begin{bmatrix} A & 0 \end{bmatrix} \in K^{m\times(n+1)} \\ \boldsymbol{row} \oplus A = \begin{bmatrix} 0 \\ A \end{bmatrix} \in K^{(m+1)\times n} \\ A \oplus \boldsymbol{row} = \begin{bmatrix} A \\ 0 \end{bmatrix} \in K^{(m+1)\times n} \\ \boldsymbol{col} \oplus \boldsymbol{row} = \boldsymbol{row} \oplus \boldsymbol{col} = \begin{bmatrix} 0 \end{bmatrix} \end{cases}$$

3.2 Indecomposables and decomposition of the first matrix problem

Much like primes do not have a factorization consisting of smaller primes, there are certain matrices that cannot be decomposed in a non-trivial way. Formally;

Definition 3.5. *A is called indecomposable if* $A \neq \boldsymbol{nul}$ *and*

$$A \sim B \oplus C \text{ implies that } B = \boldsymbol{nul} \text{ or } C = \boldsymbol{nul}$$

Consider the normal form of the first matrix problem determined by r, m, n;

$$\begin{bmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in K^{m \times n}$$

It can be decomposed into indecompolsables as follows:

$$\begin{bmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \mathbf{1}^{\oplus r} \oplus \boldsymbol{row}^{\oplus m-r} \oplus \boldsymbol{col}^{\oplus n-r}$$

At this point is is natural to make the substitutions $r = a, m - r = b, n - r = c$ since the only task at the moment is to keep track of each normal form decomposition.

$$\begin{bmatrix} \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \mathbf{1}^{\oplus a} \oplus \boldsymbol{row}^{\oplus b} \oplus \boldsymbol{col}^{\oplus c}$$

Proposition 3.6. *The complete list of indecomposables associated to the first matrix problem are* $\mathbf{1}, \boldsymbol{row}$ *and* $\boldsymbol{col}$.

Proof:
Assume $\mathbf{1} \sim A \oplus B$ then since $A = \mathbf{1}^{\oplus a} \oplus \boldsymbol{row}^{\oplus b} \oplus \boldsymbol{col}^{\oplus c}$ and $B = \mathbf{1}^{\oplus a'} \oplus \boldsymbol{row}^{\oplus b'} \oplus \boldsymbol{col}^{\oplus c'}$ it follows that $a + a' = 1, b + b' = 0, c + c' = 0$ which implies that $A = \boldsymbol{nul}$ or $B = \boldsymbol{nul}$. Thus $\mathbf{1}$ is indecomposable. The same argument is used to show that $\boldsymbol{row}$ and $\boldsymbol{col}$ are imdecomposable.
Next assume that A is indecomposable. Since $A \sim \begin{bmatrix} \mathbf{1}_a & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \mathbf{1}^{\oplus a} \oplus \boldsymbol{row}^{\oplus b} \oplus \boldsymbol{col}^{\oplus c}$ since $\boldsymbol{row}$ and $\boldsymbol{col}$ commutes, a list of three possible ways (that are of interest) of writing A would be;

$$\begin{aligned} A &= (\mathbf{1}^{\oplus a} \oplus \boldsymbol{row}^{\oplus b}) \oplus \boldsymbol{col}^{\oplus c} \\ A &= \mathbf{1}^{\oplus a} \oplus (\boldsymbol{row}^{\oplus b} \oplus \boldsymbol{col}^{\oplus c}) \\ A &= (\mathbf{1}^{\oplus a} \oplus \boldsymbol{col}^{\oplus c}) \oplus \boldsymbol{row}^{\oplus b} \end{aligned}$$

This implies by Definition 3.5 that one of the following statements are true;

$$\begin{aligned} A &= \mathbf{1}^{\oplus a} \oplus \boldsymbol{row}^{\oplus b} \\ A &= \boldsymbol{col}^{\oplus c} \\ A &= \mathbf{1}^{\oplus a} \oplus \boldsymbol{col}^{\oplus c} \\ A &= \boldsymbol{row}^{\oplus b} \\ A &= \boldsymbol{row}^{\oplus b} \oplus \boldsymbol{col}^{\oplus c} \\ A &= \mathbf{1}^{\oplus a} \end{aligned}$$

By applying Definition 3.5 again on each of these, the list is reduced to;

$$A = \mathbf{1}^{\oplus a}$$
$$A = \boldsymbol{row}^{\oplus b}$$
$$A = \boldsymbol{col}^{\oplus c}$$

At this point it should be clear that regardless of which case, the sum $a + b + c = 1$ (e.g. if $b \geq 2$ then $A = \boldsymbol{row}^{\oplus 1} \oplus \boldsymbol{row}^{\oplus(b-1)}$, and $\boldsymbol{row} \neq \boldsymbol{nul}$ hence a contradiction). Thus $A = \mathbf{1}$, $A = \boldsymbol{row}$ or $A = \boldsymbol{col}$. □

Example: Let the normal form of some matrix $A \in K^{5\times 4}$ be $\begin{bmatrix} \mathbf{1}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, then the normal form can be decomposed into indecomposables as follows:

$$\begin{bmatrix} \mathbf{1}_3 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ 0 \end{bmatrix} =$$

$$= \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \boldsymbol{row} \oplus \begin{bmatrix} 0 \end{bmatrix} = \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \boldsymbol{row} \oplus \boldsymbol{row} \oplus \boldsymbol{col} =$$

$$= \mathbf{1}^{\oplus 3} \oplus \boldsymbol{row}^{\oplus 2} \oplus \boldsymbol{col}^{\oplus 1}$$

Note how given $m = 5, n = 4$ and $r = 3$ the above decomposition matches $\mathbf{1}^{\oplus m} \oplus \boldsymbol{row}^{\oplus n-m} \oplus \boldsymbol{col}^{\oplus r-m}$. Now it is also possible to assign the triple $(3, 2, 1)$ to the matrix A as its decomposition index.

3.3 Indecomposables and decomposition of the second matrix problem

In order to work with decompositions of the second matrix problem, first define a direct sum of paired matrices;

Definition 3.7. *The direct sum of two paired matrices (A, C) and (B, D) is;*

$$(A, C) \oplus (B, D) := (A \oplus B, C \oplus D)$$

Consider the normal form of the second matrix problem determined by p, q, r, m, n and l such that

$$\left(\begin{bmatrix} \mathbf{1}_p & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1}_q & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{1}_p & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \right)$$

It can be decomposed as follows:

$$\bigoplus \left| \begin{array}{l} (\mathbf{1}, \mathbf{1})^{\oplus p} \\ (\mathbf{1}, \boldsymbol{col})^{\oplus q} \\ (\boldsymbol{row}, \mathbf{1})^{\oplus r} \\ (\boldsymbol{col}, \boldsymbol{nul})^{\oplus n-p-q} \\ (\boldsymbol{row}, \boldsymbol{col})^{\oplus m-p-q-r} \\ (\boldsymbol{nul}, \boldsymbol{row})^{\oplus l-p-r} \end{array} \right.$$

By making substitutions as were done in the first problem, the following type of decomposition is obtained.

$$\bigoplus \left| \begin{array}{l} (\mathbf{1},\mathbf{1})^{\oplus a} \\ (\mathbf{1},\boldsymbol{col})^{\oplus b} \\ (\boldsymbol{row},\mathbf{1})^{\oplus c} \\ (\boldsymbol{col},\boldsymbol{nul})^{\oplus d} \\ (\boldsymbol{row},\boldsymbol{col})^{\oplus e} \\ (\boldsymbol{nul},\boldsymbol{row})^{\oplus f} \end{array} \right.$$

Proposition 3.8. *The complete list of indecomposables associated to the second matrix problem are* $(\mathbf{1},\mathbf{1})$*,* $(\mathbf{1},\boldsymbol{col})$*,* $(\boldsymbol{row},\mathbf{1})$*,* $(\boldsymbol{col},\boldsymbol{nul})$*,* $(\boldsymbol{row},\boldsymbol{col})$ *and* $(\boldsymbol{nul},\boldsymbol{row})$*.*

Proof:
The same techniques as in the proof of Proposition 3.6 can be applied. The details are left to the reader.

Since both of these problems have a finite list of indecomposables, a natural question is whether or not any problem of this type would have a finite list of indecomposables. The n:th relation that would the describe a problem of this type is;

$$(A_1, A_2, \ldots, A_n) \sim (A_1', A_2', \ldots, A_n') \Leftrightarrow \begin{cases} A_1' = {M_2}^{-1} A_1 M_1 \\ A_2' = {M_3}^{-1} A_2 M_2 \\ \quad \vdots \\ A_n' = {M_{n+1}}^{-1} A_n M_n \end{cases}$$

In fact the answer is yes to this question. This will not be proven. Instead it is time for the indroduction of quivers and the theory of representations.

4 Quivers and representations

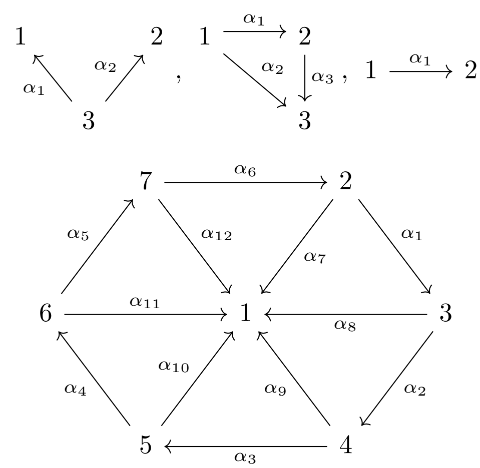
4.1 Definition: quiver

Quivers are directed graphs, presented in a workable way.

Definition 4.1. *A **quiver** Q is a quadrouple (Q_0, Q_1, s, t) where Q_0 is the set of vertices, Q_1 is the set of arrows and s, t are two maps $Q_1 \to Q_0$, assigning the starting vertex and the terminating vertex for each arrow. Terminology*

Terminology.
An arrow $\alpha \in Q_1$ starts in $s(\alpha)$ and ends in $t(\alpha)$.
A quiver Q is called finite if the sets Q_0 and Q_1 are finite.

Examples of quivers:



It is easy to see that there are an infinite number of different quivers and also an infinite number of different types of quivers. For the purpose of this document however, only the following type of quivers will be considered;

Definition 4.2. *A quiver Q, with enumerated vertices, that for each arrow α_i satifies $s(\alpha_i) = i$ and $t(\alpha_i) = i + 1$ and thus being on the form;*

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} \cdots \xrightarrow{\alpha_{n-2}} n-1 \xrightarrow{\alpha_{n-1}} n$$

is called $\mathbb{A}_n$.

Definition 4.3. *A **representation** V of a quiver Q is a pair;*

$$V = ((V_i)_{i \in Q_0}), (V_\alpha)_{\alpha \in Q_1})$$

where $(V_i)_{i \in Q_0}$ is a family of vectorspaces representing the vertices of Q, and $(V_\alpha)_{\alpha \in Q_1}$ is a family of linear maps representing the arrows of Q.

Example: Let Q be the quiver $Q_0 = \{1, 2\}, Q_1 = \{\alpha_1\}, s(\alpha_1) = 1$ and $t(\alpha_1) = 2$, such that $Q = \mathbb{A}_2$;

$$1 \xrightarrow{\alpha_1} 2$$

and let V be a representation of Q such that $V_1 = \mathbb{R}^3, V_2 = \mathbb{R}^2, V_{\alpha_1} = f$ where $f(x_1, x_2, x_3) = (x_1 + x_3, x_2)$. Then the representation can be written as both;

$$V_1 \xrightarrow{V_{\alpha_1}} V_2 \qquad \text{and} \qquad \mathbb{R}^3 \xrightarrow{f} \mathbb{R}^2 \ ,$$

where the left-hand diagram is used for genreal arguments about the representation V, whereas using the right-hand diagram denotes that the actual vectorspaces and the linear map are (or might be) of interest.

Definition 4.4. *The* ***zero representation*** *is a representation where all vertices are assigned the zero-dimensional vector space. It will be denoted* 0.

4.2 Morphisms of representations

Definition 4.5. *Let* V, W *be representations of a finite quiver* Q. *A* ***morphism*** *from* V *to* W *is a family of linear maps* $f = (f_i : V_i \to W_i)_{i \in Q_0}$ *such that for each arrow* $\alpha \in Q_1, \alpha : i \to j$ *the equation*

$$f_j V_\alpha = W_\alpha f_i$$

holds.

Terminology and notation:
$f : V \to W$ denotes that f is a morphism from V to W.
A morphism is called an **isomorphism** if $\forall i \in Q_0,\ f_i$ is invertible.
Two representations V and W are called **isomorphic** ($V \cong W$) if there exists an isomorphism from V to W.

5 Quivers of type $\mathbb{A}_n$

Example: Let V, W be representations of $Q = \mathbb{A}_2$ and let $f : V \to W$ be a morphism. Then by definition 4.5 the diagram below will commute.

$$\begin{array}{ccc} V_1 & \xrightarrow{V_{\alpha_1}} & V_2 \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} \\ W_1 & \xrightarrow{W_{\alpha_1}} & W_2 \end{array}$$

Example: Let V, W be representations of $Q = \mathbb{A}_3$ and let $g : V \to W$ be a morphism. Then by definition 4.5 the diagram below will commute.

$$\begin{array}{ccccc} V_1 & \xrightarrow{V_{\alpha_1}} & V_2 & \xrightarrow{V_{\alpha_2}} & V_3 \\ \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle g_3} \\ W_1 & \xrightarrow{W_{\alpha_1}} & W_2 & \xrightarrow{W_{\alpha_1}} & W_3 \end{array}$$

This is where the loose ends starts to get tied up. The key question is "What does it mean for the above diagrams to commute?". From linear algebra it is known that by assigning a basis (chosen arbitrarily) to the vector spaces V_i and W_j it is possible to take the matrices corresponding to V_{α_k} and W_{α_l} and consider if **inverible matrices** exists such that they correspond to f_m and g_n respectively and the diagrams commute. In other words, there is a strong connection between the matrix problems studied previously and the question whether representations are isomorphic or not.

Definition 5.1. *Let V be a representation of a quiver Q and let $(\mathcal{B}_i)_{i \in Q_0}$ be some (any) chosen family of bases for each vector space $(V_i)_{i \in Q_0}$. Then the representation will be denoted $V^{\mathcal{B}}$, the vector spaces will be denoted $V_i^{\mathcal{B}}$ and the matrices corrsponding to each linear map V_{α_j} will be denoted $[V_{\alpha_j}]_{\mathcal{B}}$. Whenever this notation is used, it is assumed that some (any) family of bases has already been chosen.*

Now consider the diagrams of the representations as above, having selected a family of bases $\mathcal{B}'_i$ for V and a family of bases $\mathcal{B}_j$ for W and replaced the linear maps with their corresponding matrices as follows: (For shorter notation $[V_{\alpha_i}]_{\mathcal{B}'}$ is replaced by A' and B', $[W_{\alpha_j}]_{\mathcal{B}}$ is replaced with A and B, $[f_k]$ with M_k and $[g_l]$ with N_l.)

$$\begin{array}{ccc} {V_1}^{\mathcal{B}'_1} & \xrightarrow{A'} & {V_2}^{\mathcal{B}'_2} \\ \downarrow{\scriptstyle M_1} & & \downarrow{\scriptstyle M_2} \\ {W_1}^{\mathcal{B}_1} & \xrightarrow{A} & {W_2}^{\mathcal{B}_2} \end{array}$$

$$\begin{array}{ccccc} {V_1}^{\mathcal{B}'_1} & \xrightarrow{A'} & {V_2}^{\mathcal{B}'_2} & \xrightarrow{B'} & {V_3}^{\mathcal{B}'_3} \\ \downarrow{\scriptstyle N_1} & & \downarrow{\scriptstyle N_2} & & \downarrow{\scriptstyle N_3} \\ {W_1}^{\mathcal{B}_1} & \xrightarrow{A} & {W_2}^{\mathcal{B}_2} & \xrightarrow{B} & {W_3}^{\mathcal{B}_3} \end{array}$$

For the $\mathbb{A}_2$ case the commutativity of the above diagrams would by definition 4.5 yield the equation;

$$f_1 V_{\alpha_1} = W_{\alpha_1} f_2$$

and similarly for the $\mathbb{A}_3$ case;

$$g_1 V_{\alpha_1} = W_{\alpha_1} g_2 \quad , \quad g_2 V_{\alpha_2} = W_{\alpha_2} g_3$$

Which in turn translates to the matrix equations;

$$M_2 A' = A M_1$$

and respectively;

$$N_2 A' = A N_1 \quad , \quad N_3 B' = B N_2$$

Furthermore, if for all i, j the matrices M_i, N_j are invertible, the equations obtained below should look very familiar. For $\mathbb{A}_2$;

$$A' = {M_2}^{-1} A M_1$$

and for $\mathbb{A}_3$;

$$A' = {N_2}^{-1} A N_1 \quad , \quad B' = {N_3}^{-1} B N_2$$

This is indeed the reqiurements for $A \sim A'$ in the $\mathbb{A}_2$ case, and $(A, B) \sim (A', B')$ in the $\mathbb{A}_3$ case. In other words, the information whether two representations are isomorphic or not is completely reflected in the information of whether the corresponding matrices to the representation's linear maps (for any arbitrairly chosen family of bases) are equivalent or not.

Proposition 5.2. *Let V and W be representations of a quiver $Q = \mathbb{A}_n$, also let $\mathcal{B}$ and $\mathcal{B}'$ be a chosen families of bases to each representation respectively. Then V and W are isomorphic if and only if $([V_{\alpha_1}]_{\mathcal{B}}, [V_{\alpha_2}]_{\mathcal{B}}, \ldots, [V_{\alpha_i}]_{\mathcal{B}}) \sim ([W_{\alpha_1}]_{\mathcal{B}'}, [W_{\alpha_2}]_{\mathcal{B}'}, \ldots, [W_{\alpha_i}]_{\mathcal{B}'})$*

This proposition can be proven as in the examples above.

5.1 Direct sums of representations

As will be shown in this chapter, there is a direct link between the matrix problems and representations of quivers regarding direct sums and indecomposable elements, all very similar to the case of isomorphism and equivalence. Define a direct sum of representations in the most straight forward way possible;

Definition 5.3. *The direct sum of two representations V and W of $Q = \mathbb{A}_n$ is;*

$$V \oplus W = U$$

where the vector space $U_i := V_i \oplus W_i$ is the direct sum of vector spaces and for any $(v, w) \in V_i \oplus W_i$ the linear map $U_{\alpha_j}(v, w) := \big(V_{\alpha_j}(v), W_{\alpha_j}(w)\big)$.

By chosing bases for each vector space in V and W the notation of direct sums of representaions can directly be connected to direct sums of matrices as follows;

Proposition 5.4. *Let $U = V \oplus W$ as in Definition 5.3 and let $\mathcal{B}$ and $\mathcal{B}'$ be bases to the vector spaces of V and W respectively, so that $(\mathcal{B}, \mathcal{B}')$ is a basis of U. Then $[V_{\alpha_i}]_{\mathcal{B}} \oplus [W_{\alpha_i}]_{\mathcal{B}'} = [U_{\alpha_i}]_{(\mathcal{B},\mathcal{B}')}$.*

Next follow the same idea as with matrices;

Definition 5.5. *A representation U is called indecomposable if $U \neq 0$ and*

$$U \cong V \oplus W \qquad \textit{implies that} \qquad V = 0 \qquad \textit{or} \qquad W = 0$$

Under the assumption that for $\mathbb{A}_2$ and $\mathbb{A}_3$ the list of indecomposables were finite, it would makes sense to check what representations correspond to each indecomposable matrix. Since all vectorspaces considered are over K, any one dimensional vector space must be isomorphic to K. Given the following representations;

$$V_1 \xrightarrow{\mathbf{1}} V_2 \ , \ W_1 \xrightarrow{\boldsymbol{col}} V_3 \ , \ V_4 \xrightarrow{\boldsymbol{row}} W_2$$

Observe that for all i, V_i has dimension 1 and hence $V_i = K$. Furthermore for all j, W_j has dimension 0 and hence $W_j = 0$. Obviously the linear maps that correspond to matrices ***col*** and ***row*** are zero maps.

Proposition 5.6. *A complete list of indecomposable representations of $Q = \mathbb{A}_2$ are;*

$$K \xrightarrow{\mathbf{1}} K \ , \ 0 \longrightarrow K \ , \ K \longrightarrow 0$$

Proof:
By taking any representation V and assuming that it is indecomposable but not in the list from Proposition 5.6 and then passing over to the corresponding matrix problem, the problem will be identical to that of proving Proposition 3.6.

The very same method can be used to show the same connection between matrix decomposability and the $\mathbb{A}_3$ case.

Proposition 5.7. *A complete list of indecomposable representations of $Q = \mathbb{A}_3$ are;*

$$K \xrightarrow{\mathbf{1}} K \xrightarrow{\mathbf{1}} K \ , \ K \xrightarrow{\mathbf{1}} K \longrightarrow 0 \ , \ 0 \longrightarrow K \xrightarrow{\mathbf{1}} K$$

$$K \longrightarrow 0 \longrightarrow 0 \ , \ 0 \longrightarrow K \longrightarrow 0 \ , \ 0 \longrightarrow 0 \longrightarrow K$$

6 Quivers of Dynkin type

Next follows two central theorems that can be proven whilst taking a course in representation theory. The first one is the Krull-Schmidt theorem concerning uniqueness of decomposition into indecomposables for *any* acyclic quiver.

For all quivers on the $\mathbb{A}_n$ form, there is in fact a theorem proven by Joseph Wedderbaum in 1909 knows as the Krull-Schmidt Theorem (from Wolfgang Krull and Otto Schmidt who originally worked on this theorem). It is often given in the context of representation theory of groups or of modules, but is the very same for representation theory of quivers. The theorem is called the Unique decomposition theorem in [1, I.4.10, p. 23] and can be paraphrased as follows;

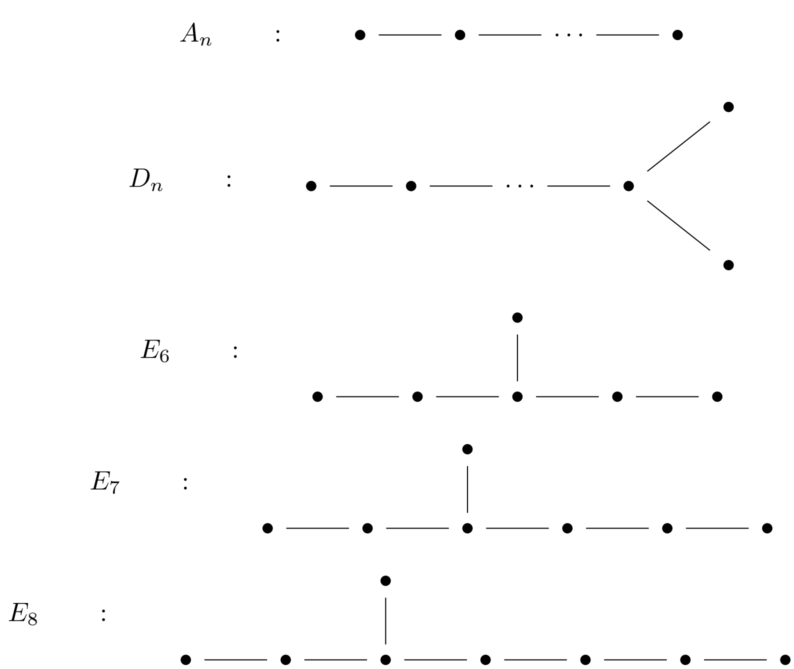
Proposition 6.1. ***Krull-Schmidt's Theorem*** - *Let V be a representation of an acyclic quiver Q. If*

$$V \cong D_1 \oplus D_2 \oplus \cdots \oplus D_i \qquad V \cong D'_1 \oplus D'_2 \oplus \cdots \oplus D'_j$$

are decompositions of V and D_k, D'_l are indecomposable then $i = j$ and there is a permutation σ of $1, 2, \ldots, i$ such that $D_k \cong D'_{\sigma(k)}$ for every $k \in \underline{i}$.

Next is Gabriel's theorem which describes what type of quivers that have a finite list of indecomposables. This theorem by itself proves that for all $\mathbb{A}_n$ quivers, there is in fact a finite list of indecomposables. This theorem is also taken from [1, VII.5.10, p. 291] and has here been reduced to what makes sense in regards to the topics covered in this paper.

Proposition 6.2. ***Gabriel's Theorem*** - *A connected acyclic quiver Q is of finite representation type if and only if Q is Dynkin (of type ADE). Q being of type ADE means that the underlying graph is A_n, D_n, E_6, E_7 or E_8 as given below.*



This is, as stated earlier, not something workable at the level of this document. The two theorems above merly served as a peak of what lies ahead, in the studies of representations theory of quivers.

6.1 Final notes

One last thing that can be said about larger groups of quivers is that for the $Q = A_n$ types of quivers, the list of all indecomposable represenations are these that are on the form;

$$0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow K \xrightarrow{1} \cdots \xrightarrow{1} K \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0$$

Taking this fact into consideration it is actually well known how many indecomposable represenatations it takes to make up an entire list of indecomposables for any given n, namely $\frac{n(n+1)}{2}$. Reconnecting this number to the idea of using the ranks of certain matrices as were done in the second matrix problem, it is possible to argue for the fact that *only* the ranks of certain matrices and their respective sizes may be all that is needed to finds the normal form of any given representation in some way. The following is how it looks for A_3 (the second matrix problem);

Given a representation R of a Quiver $Q = \mathbb{A}_3$, with a set of bases chosen for all vector spaces so that there is matrices corresponding to every linear map with given sizes and ranks, and the dimesions of any vector space V_i in R being d_i, it is possible to identify a unique normal form of R as follows;

$$\bigoplus \left| \begin{array}{l} (\mathbf{1},\mathbf{1})^{\oplus \mathrm{rank} BA} \\ (\mathbf{1},\boldsymbol{col})^{\oplus \mathrm{rank} A - \mathrm{rank} BA} \\ (\boldsymbol{row},\mathbf{1})^{\oplus \mathrm{rank} B - \mathrm{rank} BA} \\ (\boldsymbol{col},\boldsymbol{nul})^{\oplus d_1 - \mathrm{rank} A} \\ (\boldsymbol{row},\boldsymbol{col})^{\oplus d_2 - \mathrm{rank} A - \mathrm{rank} B + \mathrm{rank} BA} \\ (\boldsymbol{nul},\boldsymbol{row})^{\oplus d_3 - \mathrm{rank} B} \end{array} \right.$$

Most likely there is a general way to write any normal form given any $\mathbb{A}_n$ representation.

References

[1] Ibrahim Assem, Daniel Simson and Andrzej Skowronski. *Elements of the Representation Theory of Associative Algebras 1*, London mathematical society, student texts 65, 2006.

[2] Michael Barot. *Introduction to the representation theory of algebras*, Springer, 2015.