Pricing of Barrier Options
Using a Two-Volatility Model

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Abstract

The purpose of the current study is to assess and compare performance characteristics of a range of trinomial lattice and finite difference solvers for pricing barrier options using a two-volatility model. Emphasis is laid upon reducing the problem to a one-dimensional barrier option pricing problem with a time-independent barrier. Additionally, focus is placed upon handling discrete dividend payments and floating interest rates. Improvements such as the use of a non-uniform grid and the local refinement of the adaptive mesh model are also scrutinized. The results of this study reveal that the Crank-Nicolson finite difference scheme almost outperforms among all the considered methods. Only Ritchken’s tree may compete against it when a relatively fast response time is necessary.

Keywords: pricing, valuation, barrier option, two-volatility model, finite difference method, trinomial lattice method, non-uniform grid, adaptive mesh model, discrete dividends.
To the memory of my grandmother,
Evaggelia Mourti
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1. Introduction

Over the last decades, financial derivatives have become increasingly important in finance since they are traded on many exchanges throughout the world. A type of financial derivative is the barrier option whose payoff is contingent upon the price of the underlying asset reaching a barrier level during a certain period of time.

One of the most widely used models for option pricing is the Black-Scholes-Merton model. Although it has been subject to criticism levelled at its degree of correspondence with reality mainly because of the non-constant volatility, this model is analytically tractable and constitutes a foundation for more refine models.

Under the Black-Scholes-Merton model, a two-asset barrier option, where one of the underlying assets determines the payoff and the other one is linked to barrier hits, has previously been described in the literature and there is an analytical solution to this problem (Haug 2006). However, complications such as discrete dividend payments are not handled as delicately and thus, the use of numerical methods are necessary.

It is obvious that a two-volatility model can be obtained by making the two assets identical except for their volatilities. Because there is only a single driving Wiener process in the model, the pricing problem can be reduced to a one-dimensional barrier option pricing problem with a time-independent barrier. In this case, the study shows that lattice and finite difference methods for pricing this kind of barrier option can be employed in an effective and elegant way. Although the location of the barrier and the dividend payments give rise to some numerical errors, techniques are employed to mitigate them.

1.1. Barrier Options

A financial derivative instrument is a contract between parties whose value derives from one or more other financial instruments. An option is such a type of contract which gives the holder the right, but not the obligation, to exercise the contract. The option is often called a contingent claim because it is contingent on the evolution of an underlying asset. The contract function or payoff function is a real-valued function of the underlying asset which describes the value of the option when it is exercised.
Today, there are many different options being traded and they can be classified into various categories. A *call option* gives the buyer the right to buy the underlying financial asset at an agreed-upon price (*strike price*), whereas a *put option* gives the buyer the right to sell it. Besides the two different types of options, there are also two major styles of options. A *European option* can be exercised only at the expiration date (*maturity time*), whereas an *American option* can be exercised at any time before the expiration date. Finally, a similar classification holds if there are more than one underlying asset. All these regular options are known as *vanilla options*.

However, another important category is the *exotic options* that differ from standard call and put options. A *barrier option* is an exotic option where the payoff is contingent upon whether the underlying asset’s price reaches a certain level called barrier during a certain period of time. It is worth mentioning that barrier options are *path-dependent options* because the path of the underlying asset’s price determines the payoff.

Specifically, the barrier option becomes activated or extinguished only if the underlying asset’s price reaches the barrier during a certain period of time. A *knock-in option* is a barrier option that becomes activated, whereas a *knock-out option* is a barrier option that becomes void or null. According to the relative position of the barrier and the underlying asset’s price there are four main versions, whose payoffs are

for a down-and-out option:

$$X_{LO} = \begin{cases} \Phi(S_T), & \text{if } S_t > L \text{ for all } t \in [0, T] \\ 0, & \text{if } S_t \leq L \text{ for some } t \in [0, T] \end{cases}$$

for a down-and-in option:

$$X_{LI} = \begin{cases} \Phi(S_T), & \text{if } S_t \leq L \text{ for some } t \in [0, T] \\ 0, & \text{if } S_t > L \text{ for all } t \in [0, T] \end{cases}$$

for an up-and-out option:

$$X^{LO} = \begin{cases} \Phi(S_T), & \text{if } S_t < L \text{ for all } t \in [0, T] \\ 0, & \text{if } S_t \geq L \text{ for some } t \in [0, T] \end{cases}$$

and for an up-and-in option:

$$X^{LI} = \begin{cases} \Phi(S_T), & \text{if } S_t \geq L \text{ for some } t \in [0, T] \\ 0, & \text{if } S_t < L \text{ for all } t \in [0, T] \end{cases}$$

where $S_t$ is the underlying asset’s price at time $t$, $T$ is the maturity time, $\Phi$ is the pricing function and $X$ is the contingent claim of the form $X = \Phi(S_T)$. Regarding the notation, $L$ as a subscript or superscript indicates a down-type or up-type contract respectively, $O$ indicates that the contract is an “out” contract and $I$ indicates that the contract is an “in” contract.
Concerning the form of the pricing function, the contingent claim $\mathcal{X}$ may be a European call or put option and thus, the pricing functions are respectively:

$$\Phi_C(S_T) = \max\{S_T - K, 0\},$$
$$\Phi_P(S_T) = \max\{K - S_T, 0\},$$

where $K$ denotes the strike price.

It is readily proved that an in-out parity relation for barrier options holds because if someone holds a down-and-out version of $\mathcal{X}$ as well as a down-and-in version of $\mathcal{X}$, they will receive exactly $\mathcal{X}$ at expiration date.

For the sake of simplicity, the aforementioned formulas use only one underlying asset. Howbeit, there are multi-asset barrier options which are traded on a number of exchanges. One of them is a two-asset barrier option whose one of the underlying assets determines the payoff and the other one relates to barrier hits. For instance, the payoff of a two-asset barrier option might be:

$$\mathcal{Z}_{LO} = \begin{cases} 
\max\{S_T - K, 0\}, & \text{if } H_t > L \text{ for all } t \in [0, T] \\
0, & \text{if } H_t \leq L \text{ for some } t \in [0, T] 
\end{cases}$$

This is a down-and-out European call on the underlying asset $S$ that knocks out when the price of other underlying asset $H$ falls below the level of the barrier $L$. The same expression can be written using the indicator function.

$$\mathcal{Z}_{LO} = \max\{S_T - K, 0\} \mathbb{I}_{\{\min_{t\in[0,T]} H_t > L\}},$$

where $\mathbb{I}_\omega(S)$ ($\mathbb{I}_\omega$ in short) is the indicator function of the set $\omega$.

$$\mathbb{I}_\omega(S) = \begin{cases} 
1, & S \in \omega \\
0, & S \notin \omega 
\end{cases}$$

Likewise, all the other versions of the barrier option can be constructed using two underlying assets.

However, this study focuses on pricing standard barrier options using a two-volatility model. An underlying asset’s price $S$ determines the payoff while a stochastic process $H$, being identical to $S$ apart from their volatilities, is connected with barrier hits. Specifically, the option pricing model consists of one underlying asset, one riskless asset and the stochastic process $H$.

1.2. Model and Absence of Arbitrage

While the price of any exchange-traded asset is simply its market price, a model is required to value off-exchange derivatives, i.e. to determine a fair value for the premium.
A very essential concept in the option pricing is the *arbitrage* opportunity. A market is arbitrage-free when there are no arbitrage portfolios, i.e. there are no deterministic money-making machines.

Additionally, a very important principle in the option pricing is the *risk-neutral measure* or *equivalent martingale measure* which exists if and only if the market model is arbitrage-free. Under this probability measure, today’s asset prices are equal to the expected value of tomorrow’s asset prices discounted with the risk-free interest rate. Therefore, in the risk-neutral world where preferences are neither risk averse nor risk seeking, the price of an option can be valued by its discounted expected payoff. Although this does not hold in reality, the assumption tends to provide relatively good results.

Another fundamental concept is the market *completeness*. A market is complete when every derivative can be replicated, i.e. there exists a self-financing portfolio with the same properties (especially cash flows). Given the absence of arbitrage, the martingale measure is unique if and only if the market is complete.

Based on these fundamental concepts, the most prominent example of a model is the *Black-Scholes-Merton model* or *Black-Scholes model*. Although there are some assumptions on the assets and the market, this model is successful and widely accepted. By applying the mathematical theory to this model, two equivalent mathematical expressions arise and hence, the price can be estimated by either solving a partial differential equation (PDE) model or a conditional expectation under the risk-neutral measure. Because analytic solutions are not always available, applying numerical methods is inevitable.

### 1.3. Numerical Methods

Nowadays, financial derivatives have become more and more complex and thus, robust numerical methods are required. Even pricing standard options might need numerical procedures because the closed-form solutions may include logarithms or distribution functions. Although there are various numerical methods for pricing options, this study meticulously examines various *finite difference* and *trinomial lattice methods* for pricing barrier options.

The finite difference method approximates the value of a contingent claim by solving the partial differential equation that the option value satisfies. In general, the differential equation is transformed into a set of difference equations which are subsequently solved iteratively. The most common finite difference methods in quantitative finance are the explicit and implicit Euler method as well as the Crank-Nicolson method.

On the other hand, the trinomial model, being an extension of the binomial model for pricing options, constitutes a probabilistic approach. It can be shown that this approach is equivalent to the explicit finite difference method.
1.4. Objectives

The present study focuses on the valuation of barrier options using a two-volatility model with dividends and floating interest rates. Specifically, it reduces the option pricing problem to one-dimensional problem and presents a range of trinomial lattice and finite difference solvers. A performance comparison of them is scrutinized thoroughly and paves the way for further research.

1.5. Outline

The overall structure of this study takes the form of five chapters, including this introductory chapter. The second chapter lays out the theoretical dimensions of the research. The chapter three presents the methodology used for this study. The fourth chapter is concerned with the results of the research while the last chapter draws upon the entire thesis.
2. Theory

2.1. Stochastic Calculus

2.1.1. Stochastic Processes

Because the financial markets demonstrate erratic fluctuations, asset prices should be modelled as continuous time stochastic processes. A stochastic process or random process is a family of random variables $X_t$ defined for a set of parameters $t$. One of the most important families of stochastic processes is the Wiener process, which is also used extensively in finance. A simulated Wiener realization (or trajectory) is shown in Figure 1.

![Figure 1. A Wiener trajectory.](image)

A Wiener process $W = (W_t)_{t \geq 0}$ is a continuous-time stochastic process which has the following properties:

- $W(0) = 0$.
- $W$ has independent increments.
• For $s < t$ the stochastic variable $W(t) - W(s)$ is normally distributed with mean 0 and variance $t - s$.
• $W$ has continuous paths.

2.1.2. Stochastic Integrals

A key concept for any theory of stochastic calculus is the *stochastic integral*, which constitutes a stochastic generalization of the Riemann-Stieltjes integral. Let $W$ be a Wiener process and $f$ be another stochastic process. Under some kind of integrability conditions on $f$, the stochastic integral is defined by the formula

$$\int_{a}^{b} f(s) \, dW(s) = \lim_{n \to \infty} \sum_{i=0}^{n-1} f(t_i) \left( W(t_{i+1}) - W(t_i) \right),$$

where $t_i = a + \frac{b-a}{n} i$.

It is worth mentioning that the stochastic integral is defined by taking the forward increments of the Wiener process. In other words, the evaluation of the process $f$ does not know about the $W$-increment that multiplies it. This is natural in finance because there is no information about the future in current actions.

2.1.3. Itô Processes

Another very important family of stochastic processes is the *Itô processes* that can be expressed as the sum of an integral with respect to time and an integral with respect to Wiener process. Specifically, an Itô process is a stochastic process $X = (X_t)_{t \geq 0}$ which is defined by

$$X(t) = X(0) + \int_{0}^{t} \mu(s) \, ds + \int_{0}^{t} \sigma(s) \, dW(s),$$

where $\mu$ and $\sigma$ are adapted processes (non-anticipative processes) and $W$ is a Wiener process.

However, the last expression can be written in a more convenient way as a differential equation

$$dX(t) = \mu(t) \, dt + \sigma(t) \, dW(t).$$

This is also called the *dynamics* of the process $X$. In general, a differential equation that includes one or more stochastic processes is called *Stochastic Differential Equation* (SDE).
2.1.4. Itô’s Formula

The *Itô’s formula* being a powerful result in the theory of stochastic calculus has many important applications to quantitative finance.

Let \( f: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \) be twice differentiable and the dynamics of the Itô process \( X \) be given by

\[
dX(t) = \mu(t)dt + \sigma(t)dW(t),
\]

where \( \mu \) and \( \sigma \) are adapted processes. Then the process \( Y \) defined by \( Y(t) = f(t, X(t)) \) is also a process with dynamics given by

\[
df(t, X(t)) = \left\{ \frac{\partial f}{\partial t} + \mu \frac{\partial f}{\partial x} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial x^2} \right\} dt + \sigma \frac{\partial f}{\partial x} dW(t).
\]

For readability reasons, a lot of variables have been suppressed.

2.1.5. The Feynman-Kac Formula

The *Feynman-Kac formula* has built a bridge between stochastic processes and partial differential equations (PDEs).

Let \( u \) is a solution to the boundary value problem

\[
\frac{\partial u}{\partial t}(t, x) + \mu(t, x) \frac{\partial u}{\partial x}(t, x) + \frac{1}{2} \sigma^2(t, x) \frac{\partial^2 u}{\partial x^2}(t, x) + f(t, x) = q(t, x)u(t, x),
\]

\[
u(T, x) = \Phi(x),
\]

where \((t, x) \in [0, T] \times \mathbb{R}\).

Then the solution \( u \) can be written also as a conditional expectation

\[
u(t, x) = \mathbb{E}_\mathbb{P}\left[ e^{-\int_t^T q(s, X_s)ds} \Phi(X_T) + \int_t^T e^{-\int_u^T q(u, X_u)du} f(t, X_s)ds \mid X_t = x \right],
\]

where the process \( X \) satisfies the SDE

\[
dX_s = \mu(s, X_s)ds + \sigma(s, X_s)dW_s,
\]

\[
X_t = x.
\]

2.1.6. Geometric Brownian Motion

A *Geometric Brownian Motion* (GBM) is a fundamental building block for pricing options. It is well established in financial mathematics to model asset prices in the Black-Scholes model. A geometric Brownian motion is a continuous-time stochastic process \( X \) which is governed by the following stochastic differential equation

\[
dX(t) = \mu X(t)dt + \sigma X(t)dW(t).
\]
\[ X(0) = x_0. \]

The solution to the above equation is given by
\[ X(t) = x_0 e^{(\mu - \frac{\sigma^2}{2})t + \sigma W(t)} \]
and the expected value is given by
\[ \mathbb{E}[X(t)] = x_0 e^{\mu t}. \]

*Figure 2* presents two simulated trajectories of geometric Brownian motion with \( \mu = 1, \sigma = 0.4 \) and \( X(0) = 1 \). The smooth line in the figure is the graph of the expected value function \( \mathbb{E}[X(t)] \).

*Figure 2*. A geometric Brownian motion: \( \mu = 1, \sigma = 0.4 \).

### 2.2. The Black-Scholes-Merton Model

In the early 1970s, Fischer Black, Myron Scholes and Robert Merton proposed a model for pricing European stock options. This model constitutes a breakthrough in option pricing and it is widely used by option traders.

The Black-Scholes model assumes that the financial market consists of one riskless asset with price process \( B \) and one risky asset with price process \( S \) with dynamics given by
\[ dB(t) = rB(t)dt, \]
\[ dS(t) = \mu S(t)dt + \sigma S(t)d\overline{W}(t), \]

where \( r, \mu \) and \( \sigma \) are deterministic constants and \( \overline{W} \) is a Wiener process.

It is noteworthy that the price process \( B \) is also given by the equivalent expression

\[ B(t) = B(0)e^{rt}. \]

2.2.1. The Black-Scholes Equation

Given the Black-Scholes model and certain assumptions about the market such as no transaction cost and absence of arbitrage, the price of a contingent claim can be estimated by solving a partial differential equation.

Although the value of a derivative is a function of various parameters such as the strike price and the time to expiry, it mainly depends on the process of the underlying asset \( S \) and the time \( t \). Therefore, let the contingent claim of the form \( \mathcal{X} = \Phi(S_T) \) can be traded on a market described by the Black-Scholes model and \( F(t, S_t) \) be its value depending only on time \( t \) and the process \( S \).

By using the Ito's lemma, the dynamics of \( F \) are given by

\[ dF = \sigma S \frac{\partial F}{\partial S} d\overline{W} + \left( \frac{\partial F}{\partial t} + \mu S \frac{\partial F}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} \right) dt, \]

where many variables are suppressed for readability reasons.

Consider now a portfolio consisting of one derivative \( \mathcal{X} \) and a number \( -\Delta \) of the underlying asset. The value of this portfolio is

\[ \Pi = F - \Delta S. \]

Given that the dynamics of the processes \( S \) and \( F \) are known, the change in the value of this portfolio in one time-step is

\[ d\Pi = dF - \Delta dS. \]

It is quite clear that \( \Delta \) is held fixed during the time steps because otherwise \( d\Delta \) would be in the equation. Inserting \( dF \) into the last equation gives

\[ d\Pi = \sigma S \left( \frac{\partial F}{\partial S} - \Delta \right) d\overline{W} + \left( \frac{\partial F}{\partial t} + \mu S \frac{\partial F}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} - \mu \Delta S \right) dt. \]

Certainly, the random component can be eliminated by choosing

\[ \Delta = \frac{\partial F}{\partial S}. \]

This leads to a portfolio whose driving \( d\overline{W} \)-term vanishes completely.
\[
d II = \left(\frac{\partial F}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2}\right) dt.
\]

Because of the absence of arbitrage condition, the left-hand side should be equal to \( r II dt \). Alternatively, the arbitrager would make a riskless instantaneous profit. Therefore, the riskless portfolio should satisfy the equation

\[
r II dt = \left(\frac{\partial F}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2}\right) dt.
\]

Substituting the previous equations into the last one gives

\[
\frac{\partial F}{\partial t} + rS \frac{\partial F}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} - rF = 0.
\]

It can be shown that under very weak assumptions, \( S \) can take any value whatsoever. Therefore, the value \( F \) needs to satisfy the following partial differential equation (PDE) in the domain \([0, T] \times \mathbb{R}_+\)

\[
\frac{\partial F}{\partial t} + rS \frac{\partial F}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} - rF = 0,
\]

\[F(T, s) = \Phi(s),\]

which is the Black-Scholes partial differential equation.

Similarly, it is readily proved that if the instrument pays continuous dividend yield \( \delta(t) \) and the risk-free interest rate \( r \) is a function of time \( t \), the value \( F \) must satisfy the following PDE in the domain \([0, T] \times \mathbb{R}_+\)

\[
\frac{\partial F}{\partial t} + (r - \delta(s) \frac{\partial F}{\partial s} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial s^2} - rF = 0,
\]

\[F(T, s) = \Phi(s).\]

2.2.2. Risk-Neutral Valuation

Another way of pricing derivatives is the risk-neutral valuation. The main assumption is that investors are risk-neutral and do not care about risk. The Black-Scholes PDE can be transformed into a conditional expression under measure \( \mathbb{Q} \) by applying the Feynman-Kac formula. Specifically, the value of the option is given by

\[
F(t, s) = e^{-\int_t^T r(u)du} \mathbb{E}^\mathbb{Q}[\Phi(S_T)|S_t = s],
\]

where the process \( S \) is defined by the \( \mathbb{Q} \)-dynamics

\[
dS(u) = (r(u) - \delta(u))S(u)du + \sigma S(u)dW(u),
\]

\[S(t) = s,\]
2.3. Pricing Formulas of Standard Barrier Options

Barrier options are one of the most popular class of exotic options and they have been traded in over-the-counter markets since 1967. As was discussed in the introduction, a standard barrier option is a European option on an underlying asset whose existence depends on the underlying asset’s price reaching the barrier level $L$. They were first priced by Merton in 1973 and later Reiner and Rubinstein developed formulas for pricing standard barrier options.

The most well-known pricing formulas are the Black-Scholes-Merton formulas which estimate the theoretical prices of European call and put options. These formulas can be easily deduced from the homonymous model and are respectively given by

$$
C = S_0 e^{-\delta T} N(d_1) - Ke^{-r T} N(d_2),
$$

$$
P = Ke^{-r T} N(-d_2) - S_0 e^{-\delta T} N(-d_1),
$$

where

$$
d_1 = \frac{\ln(S_0/K) + (r - \delta + \sigma^2/2)T}{\sigma \sqrt{T}},
$$

$$
d_2 = \frac{\ln(S_0/K) + (r - \delta - \sigma^2/2)T}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T},
$$

and $r$ denotes the constant risk-free interest rate, $\delta$ the constant dividend yield, $\sigma$ the constant volatility of the asset’s price, $K$ the strike price, $S_0$ the spot price of the underlying asset at time zero and $T$ the time to maturity. Finally, $N(\cdot)$ is the cumulative distribution function of the standard normal distribution.

Options formulas for pricing standard barrier options are related to the Black-Scholes-Merton formulas. For the sake of simplicity, focus is given upon “out” barrier options. The price of “in” barrier options can be derived by the in-out parity. Under the Black-Scholes-Merton model, the analytical solutions for the different types of “out” barrier options are given by the following formulas.

A down-and-out call option ($S_0 > L$)

$$
C_{K \geq L} = C - S_0 e^{-\delta T} (L/S_0)^{2 \lambda} N(x)
+ Ke^{-r T} (L/S_0)^{2 \lambda - 2} N(x - \sigma \sqrt{T}),
$$
\[ C_{K<L} = S_0 e^{-\delta T} N(y_1) - Ke^{-rT} N(y_1 - \sigma \sqrt{T}) - S_0 e^{-\delta T} (L/S_0)^{2\lambda} N(y_2) + Ke^{-rT} (L/S_0)^{2\lambda-2} N(y_2 - \sigma \sqrt{T}), \]

An up-and-out call option \((S_0 < L)\)

\[ C_{K\geq L} = 0, \]

\[ C_{K<L} = S_0 e^{-\delta T} N(y_1) - Ke^{-rT} N(y_1 - \sigma \sqrt{T}) - S_0 e^{-\delta T} (L/S_0)^{2\lambda} (N(-x) - N(-y_2)) + Ke^{-rT} (L/S_0)^{2\lambda-2} \left( N(-x + \sigma \sqrt{T}) - N(-y_2 + \sigma \sqrt{T}) \right), \]

A down-and-out put option \((S_0 > L)\)

\[ P_{K>L} = P + S_0 e^{-\delta T} N(-y_1) - Ke^{-rT} N(-y_1 + \sigma \sqrt{T}) - S_0 e^{-\delta T} (L/S_0)^{2\lambda} N(y_2) + Ke^{-rT} (L/S_0)^{2\lambda-2} \left( N(x - \sigma \sqrt{T}) - N(y_2 + \sigma \sqrt{T}) \right), \]

\[ P_{K\leq L} = 0, \]

An up-and-out put option \((S_0 < L)\)

\[ P_{K>L} = -S_0 e^{-\delta T} N(-y_1) + Ke^{-rT} N(-y_1 + \sigma \sqrt{T}) + S_0 e^{-\delta T} (L/S_0)^{2\lambda} N(-y_2) - Ke^{-rT} (L/S_0)^{2\lambda-2} N(-y_2 + \sigma \sqrt{T}), \]

\[ P_{K\leq L} = P + S_0 e^{-\delta T} (L/S_0)^{2\lambda} N(-x) + Ke^{-rT} (L/S_0)^{2\lambda-2} N(-x + \sigma \sqrt{T}), \]

where

\[ \lambda = \frac{r - \delta + \sigma^2/2}{\sigma^2}, \]

\[ x = \frac{\ln\left( L^2/(S_0 K) \right)}{\sigma \sqrt{T}} + \lambda \sigma \sqrt{T}, \]

\[ y_1 = \frac{\ln(S_0/L)}{\sigma \sqrt{T}} + \lambda \sigma \sqrt{T}, \]

\[ y_2 = \frac{\ln(L/S_0)}{\sigma \sqrt{T}} + \lambda \sigma \sqrt{T}. \]
2.4. Trinomial Tree Models

A lattice model or tree model is a very efficient and popular technique for pricing derivatives. This is a diagram representing different possible paths that might be followed by the price of the underlying asset over the derivative's life. In fact, a lattice model provides a discrete-time approximation to a continuous-time model.

The simplest lattice model is the binomial options pricing model proposed by Cox, Ross and Rubinstein (CRR) in 1979. Each node in the tree represents a possible price of the underlying asset at a specific time and in each time step, the price of the underlying asset has a certain probability of moving down by a certain percentage amount and a certain probability of moving up by a certain percentage amount. The valuation is performed iteratively by estimating the value of the final nodes and then working towards the root of the tree. As the number of time-steps increases, the value converges on the true value.

A Trinomial tree model is an extension of the binomial options pricing model by adding one more possible path at each time step. Trinomial models are reckoned to produce more precise results since they can represent the possible paths of the underlying asset's price more accurately.

2.4.1. Standard Trinomial Tree Model

Let $S$ be the process of the underlying asset's price. The $\mathbb{Q}$-dynamics of $S$ are given by

$$dS(t) = (r(t) - \delta(t))S(t)dt + \sigma S(t)dW(t),$$

$$S(0) = s.$$

Defining the stochastic process $X = \ln S$ and applying Ito's lemma gives:

$$dX(t) = \left(r(t) - \delta(t) - \frac{\sigma^2}{2}\right)dt + \sigma dW(t),$$

$$X(0) = x = \ln s,$$

or, equivalently,

$$X(t) = x + \int_0^t \left(r(u) - \delta(u) - \frac{\sigma^2}{2}\right)du + \sigma W(t).$$

For convenience, assume a new stochastic process $X'$ defined by

$$X'(t) = X(t) - x - \int_0^t \left(r(u) - \delta(u) - \frac{\sigma^2}{2}\right)du = \sigma W(t).$$
The construction of the trinomial tree based on $X'$ is quite trivial. Let $h$ denote the size of an up and down move and $\Delta t = T/N$ be the length of a time step, where $T$ is the maturity time of the option and $N$ is the number of time steps. Moreover, let $p_u$, $p_d$ and $p_m$ denote the probability to move up, to move down and to remain unchanged respectively.

There are four constraints that should be satisfied:

\[
p_u + p_m + p_d = 1,
\]

\[
\mathbb{E}[X'(t + \Delta t) - X'(t)] = 0 = p_u h + p_m 0 + p_d (-h),
\]

\[
\mathbb{E} \left[ (X'(t + \Delta t) - X'(t))^2 \right] = \sigma^2 \Delta t = p_u h^2 + p_m 0 + p_d (-h)^2,
\]

\[
\mathbb{E} \left[ (X'(t + \Delta t) - X'(t))^4 \right] = 3\sigma^4 \Delta t^2 = p_u h^4 + p_m 0 + p_d (-h)^4,
\]

Solving the system of equations yields:

\[
p_u = p_d = \frac{1}{6},
\]

\[
p_m = \frac{2}{3}
\]

\[
h = \sigma \sqrt{3 \Delta t}.
\]

Hence, the trinomial tree which approximates the asset price distribution is given by:

\[
X'(t + \Delta t) - X'(t) = \begin{cases}
\sigma \sqrt{3 \Delta t}, & p_u = 1/6 \\
0, & p_m = 2/3 \\
-\sigma \sqrt{3 \Delta t}, & p_d = 1/6
\end{cases}
\]

returning to process $X$ gives

\[
X(t + \Delta t) - X(t) = \begin{cases}
\alpha(t) + \sigma \sqrt{3 \Delta t}, & p_u = 1/6 \\
\alpha(t), & p_m = 2/3 \\
\alpha(t) - \sigma \sqrt{3 \Delta t}, & p_d = 1/6
\end{cases}
\]

or, equivalently,

\[
S(t + \Delta t) = S(t)e^{\alpha(t)} \begin{cases}
e^{\sigma \sqrt{3 \Delta t}}, & p_u = 1/6 \\
1, & p_m = 2/3 \\
e^{-\sigma \sqrt{3 \Delta t}}, & p_d = 1/6
\end{cases}
\]

where

\[
\alpha(t) = \int_t^{t+\Delta t} \left( r(u) - \delta(u) - \frac{\sigma^2}{2} \right) du
\]

\[
\approx \left( r(t + \Delta t/2) - \delta(t + \Delta t/2) - \frac{\sigma^2}{2} \right) \Delta t.
\]
Finally, the value of the option \( F(t, S_t) \) can be estimated by

\[
F(t, S_t) = e^{-\beta(t)}.
\]

\[
(p_u F(S_{t+\Delta t}^{up}, t + \Delta t) + p_m V(S_{t+\Delta t}^{mid}, t + \Delta t) + p_d V(S_{t+\Delta t}^{down}, t + \Delta t)),
\]

where the superscripts show the three next nodes and

\[
\beta(t) = \int_t^{t+\Delta t} r(u) \, du \approx r(t + \Delta t/2) \cdot \Delta t.
\]

For \( t = T \), the value of the option is calculated by the payoff function

\[
F(T, S_T) = \Phi(S_T).
\]

2.4.2. Probability-Adjusted Trinomial Tree Model

Although the standard trinomial tree model is quite efficient for standard call and put options, when it is used to price a barrier option it exhibits large bias and low convergence rate. This fact is mainly derived from the position of the barrier relative to the discrete prices of the underlying asset. Figure 3 illustrates a trinomial lattice with two possible price paths representing possible underlying price realizations on the lattice. Both paths breach the barrier and hence, represent situations in which the option payoff should be set to zero in the case of a down-and-out option.

One solution to this problem is calculating the probability that the price of the underlying asset hits the barrier for any given step. Baldi, Caramellino and Iovino provide a series of approximations for the exit probability of a Brownian bridge. Specifically, the exit probability for a single constant barrier is given by

\[
\rho = \exp \left\{ -\frac{2}{\Delta t} \left( \ln \frac{S_t}{L} - \ln \frac{S_{t+\Delta t}}{L} \right) \right\},
\]

where \( L \) is the level of the barrier.

Therefore, the value of the option \( F(t, S_t) \) at a given node is estimated by

\[
F(t, S_t) = e^{-\beta(t)} \cdot \left\{ p_u (1 - \rho_u) F(S_{t+\Delta t}^{up}, t + \Delta t) 
+ p_m (1 - \rho_m) F(S_{t+\Delta t}^{mid}, t + \Delta t) 
+ p_d (1 - \rho_d) F(S_{t+\Delta t}^{down}, t + \Delta t) \right\}.
\]
2.4.3. Ritchken’s Trinomial Tree Model

Ritchken, trying to address the problem of barrier options, develops a method which repositions the tree nodes and exhibits good convergence rates. By using a stretch factor $\lambda$, the barrier lies exactly upon a given level of the lattice nodes. Therefore, regardless of the time partition, there always exists a layer of nodes coinciding with the barrier level.

Let $S$ be the price of the underlying asset as described in the previous model. It is readily apparent that

$$S(t + \Delta t) = S(t)e^{\alpha(t)+\sigma(W(t+\Delta t)-W(t))} = S(t)e^{\xi(t)},$$

where $\xi(t)$ is a normal random variable whose:

$$E[\xi(t)] = \alpha(t),$$

$$Var[\xi(t)] = \sigma^2 \Delta t,$$

Let a discrete random variable $\tilde{\xi}(t)$ be the approximating distribution for $\xi(t)$ over the period $[t, t + \Delta t]$.

$$\tilde{\xi}(t) = \begin{cases} 
\lambda \sigma \sqrt{\Delta t}, & \text{with probability } p_u \\
0, & \text{with probability } p_m \\
-\lambda \sigma \sqrt{\Delta t}, & \text{with probability } p_d 
\end{cases}$$
Where \( p_u, p_d \) and \( p_m \) denote the probability to move up, to move down and to remain unchanged respectively.

There are three constraints that should be satisfied:

\[
p_u + p_m + p_d = 1,
\]

\[
E[\xi(t)] = E[\tilde{\xi}(t)] = p_u \lambda \sigma \sqrt{\Delta t} + p_m 0 + p_d (-\lambda \sigma \sqrt{\Delta t}),
\]

\[
Var[\xi(t)] = Var[\tilde{\xi}(t)] = p_u (\lambda \sigma \sqrt{\Delta t})^2 + p_m 0 + p_d (-\lambda \sigma \sqrt{\Delta t})^2,
\]

Solving the system of equations yields:

\[
p_u = \frac{1}{2\lambda^2} + \frac{\alpha(t)}{2\lambda \sigma \sqrt{\Delta t}},
\]

\[
p_m = 1 - \frac{1}{\lambda^2},
\]

\[
p_d = \frac{1}{2\lambda^2} - \frac{\alpha(t)}{2\lambda \sigma \sqrt{\Delta t}}.
\]

Therefore, the Ritchken trinomial tree which approximates the asset price distribution is given by:

\[
S(t + \Delta t) = S(t) \cdot \begin{cases} 
 e^{\lambda \sigma \sqrt{\Delta t}}, & p_u = \frac{1}{2\lambda^2} + \frac{\alpha(t)}{2\lambda \sigma \sqrt{\Delta t}} \\
 1, & p_m = 1 - \frac{1}{\lambda^2} \\
 e^{-\lambda \sigma \sqrt{\Delta t}}, & p_d = \frac{1}{2\lambda^2} - \frac{\alpha(t)}{2\lambda \sigma \sqrt{\Delta t}}
\end{cases}
\]

Because the probabilities \( p_u, p_d \) and \( p_m \) should be in the interval \([0,1]\), the following condition on \( \lambda \) needs to be satisfied \( \lambda \geq 1 \). It should be noted that for \( \lambda = 1 \) the middle probability \( p_m = 0 \) and thus, the trinomial lattice is reduced to binomial lattice. In general, the stretch factor \( \lambda \) allows the underlying asset’s price partition and the time partition to be decoupled.

Ritchken provides an elegant procedure to determine the stretch parameter \( \lambda \). Let \( n \) be defined by

\[
n = \left\lfloor \frac{\ln(S(0)/L)}{\sigma \sqrt{\Delta t}} \right\rfloor,
\]

where \( L \) is the level of the constant barrier.

Then, the stretch parameter \( \lambda \) is given by

\[
\lambda = \frac{n}{\lfloor n \rfloor}.
\]

Figure 4 illustrates how the Ritchken’s procedure changes the trinomial tree model placing the nodes on the barrier.
However, for a study of this model for standard, non-barrier, options, see Kamrad and Ritchken. They show that choosing $\lambda = \sqrt{3}/2$ produces a rapid convergence.

*Figure 4.* Two trinomial tree models for a down-and-out option with $\lambda = 1$ and $\lambda$ estimated according to Ritchken’s procedure.

### 2.4.4. Adaptive Mesh Model

Many numerical solutions often use a uniform precision in the numerical grids. However, for many problems it would be preferable if specific areas which are needed precision could be refined. The Adaptive Mesh Model (AMM) which was introduced by Figlewski and Gao (1999) provides such an approach to efficient option pricing. In general, while a relatively coarse grid which is fast to calculate is applied to most of the lattice, finer mesh is constructed where greater accuracy matters. Therefore, the AMM can improve efficiency for a relatively small increase in computational effort.

For a barrier option, the AMM constructs high-resolution mesh in the region close to the barrier in order to decrease the nonlinearity error. This error arises because the price discreteness in the lattice interacts with the price bar-
rier. A construction of the AMM for barrier options is quite similar to Ritchken’s model. Consider again the stochastic process $X = \ln S$ which is given by

$$dX(t) = \left(r(t) - \delta(t) - \frac{\sigma^2}{2}\right) dt + \sigma dW(t),$$

$$X(0) = x = \ln s.$$  

The construction of the trinomial tree based on $X$ is quite trivial. Let $h$ denote the size of an up and down move and $k = \frac{T}{N}$ be the length of a time step, where $T$ is the maturity time of the option and $N$ is the number of time steps. Based on this notation, the change of $X$ is given by

$$X(t + k) - X(t) = a(t; k) + \sigma(W(t + k) - W(t)),$$

where

$$a(t; k) = \int_t^{t+k} (r(u) - \delta(u) - \frac{\sigma^2}{2}) du$$

$$\approx \left(r\left(t + \frac{k}{2}\right) - \delta\left(t + \frac{k}{2}\right) - \frac{\sigma^2}{2}\right) k.$$

Moreover, let $p_u$, $p_d$ and $p_m$ denote the probability that the process $X$ will move up, move down and remain unchanged respectively. There are three constraints that should be satisfied:

$$p_u + p_m + p_d = 1,$$

$$\mathbb{E}[X(t + k) - X(t)] = a(t; k) = p_u h + p_m 0 + p_d (-h),$$

$$\mathbb{E}\left[(X(t + k) - X(t))^2\right] = a(t; k)^2 + \sigma^2 k = p_u h^2 + p_m 0 + p_d (-h)^2,$$

Solving the system of equations yields:

$$p_u(h, k) = \frac{a(t; k)^2 + \sigma^2 k + h a(t; k)}{2h^2},$$

$$p_m(h, k) = 1 - \frac{a(t; k)^2 + \sigma^2 k}{h^2},$$

$$p_d(h, k) = \frac{a(t; k)^2 + \sigma^2 k - h a(t; k)}{2h^2}.$$

Therefore, the AMM tree which approximates the asset price distribution is given by:
\[
S(t + \Delta t) = S(t) \cdot \begin{cases} 
    e^h, & p_u(h, k) = \frac{a(t; k)^2 + \sigma^2 k + ha(t; k)}{2h^2} \\
    1, & p_m(h, k) = 1 - \frac{a(t; k)^2 + \sigma^2 k}{h^2} \\
    e^{-h}, & p_d(h, k) = \frac{a(t; k)^2 + \sigma^2 k - ha(t; k)}{2h^2}
\end{cases}
\]

Because the probabilities \(p_u, p_d\) and \(p_m\) should be in the interval \([0, 1]\), the following condition must be hold

\[
\frac{a(t; k)^2 + \sigma^2 k}{h^2} \approx \frac{\sigma^2 k}{h^2} < 1,
\]

or equivalently, \(\lambda > 1\) where

\[
\lambda = \frac{h}{\sigma \sqrt{k}}
\]

From the last equation, it is straightforward that when the price step \(h\) decreases by half, the time step \(k\) should be set to one-quarter of it. Hence, the relationship between the time and price step sizes will remain constant. The value \(\lambda\) can be chosen in the same way as described by Ritchken.

\[\text{Figure 5. An illustration of the adaptive mesh model for a down-and-out option.}\]

Without loss of generality, here is a brief overview of pricing a down-and-out barrier option. The first task is the construction of the coarse-mesh lattice and the fine-mesh lattice close to the barrier. Figure 5 presents an instance of an AMM and shows which nodes are needed to estimate the option value at each node. For instance, the value of a node in the fine-mesh lattice is given by
\[ F = e^{-\beta(t,k/4)} \{ p_u(h/2, k/4)F_u + p_m(h/2, k/4)F_m + p_d(h/2, k/4)F_d \}, \]

while the value of a node in coarse-mesh lattice is given by,

\[ F = e^{-\beta(t,k)} \{ p_u(h, k)F_u + p_m(h, k)F_m + p_d(h, k)F_d \}, \]

where

\[ \beta(t, k) = \int_t^{t+k} r(u) \, du \approx r(t + k/2) \cdot k. \]

### 2.5. Finite Difference Methods

**Finite differences** are one of the most popular numerical techniques for solving differential equations. They rely on the discretization of the derivatives in the PDE according to the Taylor expansion of some chosen order. Initially, the problem’s domain is discretized and the values at the boundary conditions are estimated. Then, the discretized PDE compute solutions for neighbouring points repeatedly until the PDE is solved for every point on the mesh.

As previously mentioned, let \( F \) be the value of an option on an underlying asset. Under the generalized Black-Scholes model, \( F \) is the solution of the following boundary value problem in the domain \([0, T] \times \mathbb{R}_+\)

\[
\frac{\partial F}{\partial t} + (r - \delta) s \frac{\partial F}{\partial s} + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 F}{\partial s^2} - rF = 0, \\
F(T, s) = \Phi(s),
\]

where \( r \) and \( \delta \) are functions of time \( t \) while many variables are suppressed for readability reasons.

Changing the variables to \( \tau = T - t \) gives

\[
- \frac{\partial \hat{F}}{\partial \tau} + (\hat{r} - \hat{\delta}) s \frac{\partial \hat{F}}{\partial s} + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 \hat{F}}{\partial s^2} - \hat{r} \hat{F} = 0, \\
\hat{F}(0, s) = \Phi(s),
\]

Let \( D = \{0 \leq \tau \leq T; s_{\text{min}} \leq s \leq s_{\text{max}}\} \) be the continuous domain of the problem. Without loss of generality, consider \( M \) and \( N \) equal subintervals of \([0, T]\) and \([s_{\text{min}}, s_{\text{max}}]\) respectively. Therefore, the mesh on \([0, T] \times [s_{\text{min}}, s_{\text{max}}]\) is given by:

\[
\bar{D} = \begin{cases} \\
\tau_k = k \Delta \tau, & \quad \Delta \tau = \frac{T}{M}, \quad k = 0, \ldots, M \\
s_j = s_{\text{min}} + j \Delta s, & \quad \Delta s = \frac{s_{\text{max}} - s_{\text{min}}}{N}, \quad j = 0, \ldots, N 
\end{cases}
\]
Let \( f_{k,j} \) denote the approximate value of \( \hat{F}(\tau_k, s_j) \) with \( r_k = \hat{r}(\tau_k) \) and \( \delta_k = \hat{\delta}(\tau_k) \).

Without loss of generality, the focus is placed upon the explicit, fully implicit and Crank-Nicolson method based on the aforementioned assumptions.

### 2.5.1. Explicit Discretization

The explicit discretization uses only a forward difference approximation in time. The solution to the PDE at each point on the current time slice is dependent only on points derived in the previous time period. Specifically, the forward approximation at the grid point \((\tau_k, s_j)\) yields

\[
\frac{\partial \hat{F}(\tau_k, s_j)}{\partial \tau} = \frac{f_{k+1,j} - f_{k,j}}{\Delta \tau} + O(\Delta \tau).
\]

The second-order central difference approximation yields

\[
\frac{\partial^2 \hat{F}(\tau_k, s_j)}{\partial s^2} = \frac{f_{k,j+1} - 2f_{k,j} + f_{k,j-1}}{(\Delta s)^2} + O(\Delta s^2).
\]

The central difference approximation yields

\[
\frac{\partial \hat{F}(\tau_k, s_j)}{\partial s} = \frac{f_{k,j+1} - f_{k,j-1}}{2\Delta s} + O(\Delta s^2).
\]

By dropping the error terms, the discrete equation at point \((\tau_k, s_j)\) is given by

\[
f_{k+1,j} = l_{k,j}f_{k,j-1} + d_{k,j}f_{k,j} + u_{k,j}f_{k,j+1}, \quad \text{for} \ j = 1, \ldots, N - 1
\]

where

\[
\alpha_j = \frac{\sigma_j^2 s_j^2}{2} \frac{\Delta \tau}{(\Delta s)^2},
\]

\[
\beta_{k,j} = (r_k - \delta_k)s_j \frac{\Delta \tau}{2\Delta s},
\]

\[
l_{k,j} = \alpha_j - \beta_{k,j},
\]

\[
d_{k,j} = 1 - 2\alpha_j - r_k \Delta \tau,
\]

\[
u_{k,j} = \alpha_j + \beta_{k,j},
\]

The same approximate equation can be written in matrix form as

\[
F_{k+1} = A_{k}^{\text{Explicit}} F_k + B_{k}^{\text{Explicit}},
\]

where
\[ F_k = \begin{pmatrix} f_{k,1} \\ f_{k,2} \\ \vdots \\ f_{k,N-1} \end{pmatrix}, \quad B_k^{Explicit} = \begin{pmatrix} l_{k,1}f_{k,0} \\ 0 \\ \vdots \\ 0 \\ u_{k,N-1}f_{k,N} \end{pmatrix}, \]

\[ A_k^{Explicit} = \begin{pmatrix} d_{k,1} & u_{k,1} \\ l_{k,2} & d_{k,2} & u_{k,2} \\ \vdots & \vdots & \vdots \\ l_{k,N-2} & d_{k,N-2} & u_{k,N-2} \\ l_{k,N-1} & d_{k,N-1} \end{pmatrix}. \]

It is worth mentioning that the stability of the explicit scheme depends on the relationship between the space and time discretization. A paradigm of this relationship is the so-called Courant-Friedrichs-Lewy condition given by

\[ 0 < \Delta \tau < \frac{(\Delta s)^2}{2K}, \]

for the heat equation

\[ u_{\tau} - ku_{ss} = 0. \]

2.5.2. Fully Implicit Discretization

The fully implicit discretization uses a backward difference approximation in time and two difference approximations within the current time slice. The solution to the PDE in the current time slice is given by solving all points in the time slice simultaneously. Specifically, the backward approximation at the grid point \((\tau_{k+1}, s_j)\) yields

\[ \frac{\partial \hat{F}(\tau_{k+1}, s_j)}{\partial \tau} = \frac{f_{k+1,j} - f_{k,j}}{\Delta \tau} + O(\Delta \tau). \]

The second-order central difference approximation yields

\[ \frac{\partial^2 \hat{F}(\tau_{k+1}, s_j)}{\partial s^2} = \frac{f_{k+1,j+1} - 2f_{k+1,j} + f_{k+1,j-1}}{(\Delta s)^2} + O(\Delta s^2). \]

The central difference approximation yields

\[ \frac{\partial \hat{F}(\tau_{k+1}, s_j)}{\partial s} = \frac{f_{k+1,j+1} - f_{k+1,j-1}}{2\Delta s} + O(\Delta s^2). \]
By dropping the error terms, the discrete equation at point \((\tau_{k+1}, s_j)\) is given by
\[
f_{k,j} = \hat{l}_{k+1,j} f_{k+1,j-1} + \hat{d}_{k+1,j} f_{k+1,j} + \hat{u}_{k+1,j} f_{k+1,j+1}, \quad \text{for } j = 1, \ldots, N - 1
\]
where
\[
\hat{l}_{k+1,j} = -l_{k+1,j}, \quad \hat{d}_{k+1,j} = 2 - d_{k+1,j}, \quad \hat{u}_{k+1,j} = -u_{k+1,j}.
\]
The same approximate equation can be written in matrix form as
\[
F_k = A_{k+1}^{Implicit} F_{k+1} + B_{k+1}^{Implicit}.
\]
where
\[
F_k = \begin{pmatrix} f_{k,1} \\ f_{k,2} \\ \vdots \\ f_{k,N-1} \end{pmatrix}, \quad B_{k+1}^{Implicit} = \begin{pmatrix} \hat{l}_{k+1,1} f_{k+1,0} \\ 0 \\ \vdots \\ 0 \end{pmatrix},
\]
\[
\hat{A}_{k+1}^{Implicit} = \begin{pmatrix} \hat{d}_{k+1,1} & \hat{u}_{k+1,1} \\ \hat{l}_{k+1,2} & \hat{d}_{k+1,2} & \hat{u}_{k+1,2} \\ & \ddots & \ddots & \ddots \\ & & \hat{l}_{k+1,N-2} & \hat{d}_{k+1,N-2} & \hat{u}_{k+1,N-2} \\ & & & \hat{l}_{k+1,N-1} & \hat{d}_{k+1,N-1} \end{pmatrix}.
\]

2.5.3. Crank-Nicolson Discretization

The Crank-Nicolson discretization scheme is a combination of the explicit and fully implicit method. Adding the explicit and fully implicit schemes gives
\[
\hat{l}_{k+1,j} F_{k+1,j-1} + (\hat{d}_{k+1,j} + 1) F_{k+1,j} + \hat{u}_{k+1,j} F_{k+1,j+1} = l_{k,j} F_{k,j-1} + (d_{k,j} + 1) F_{k,j} + u_{k,j} F_{k,j+1}, \quad \text{for } j = 1, \ldots, N - 1
\]
The same approximate equation can be written in matrix form as
\[
(A_{k}^{Explicit} + I) F_k + B_{k}^{Explicit} = (A_{k+1}^{Implicit} + I) F_{k+1} + B_{k+1}^{Implicit}.
\]
Although the Crank-Nicolson scheme is unconditional stable, the order of convergence may be less than the second order which is achievable for smooth initial data. Nevertheless, Rannacher suggests that a second-order convergence can be obtained by replacing the first Crank-Nicolson time step by four quarter time steps of fully implicit time integration. This solution is known as Rannacher time stepping.

2.5.4. Boundary Conditions

While in the most cases the initial condition of the boundary value problem is well known, the boundary condition is assumed to be explicitly or implicitly known. The choice of a boundary condition plays a decisive role in the accuracy of computed solutions.

The initial condition, which is the value of the option at $\tau = 0$ (or for $k = 0$), is given by

$$ F_{0,j} = \hat{F}(0, s_j) = \Phi(s_j), $$

where $\Phi$ is the payoff function of the derivative.

The boundary condition is the value of the derivative at $s_{\min}$ and $s_{\max}$ (or for $j = 0$ and $j = N$ respectively). Depending on the derivative, this value can be explicitly known. For instance, for a down-and-out option the value of the option at $s_{\min} = L$ at any time can be set to zero, as the option is worthless if the price of the underlying asset reaches the barrier level $L$. Likewise, for an up-and-out option the value of the option is zero at $s_{\max} = L$. Such boundary conditions that define the derivative value explicitly at the boundary points are Dirichlet boundary conditions.

On the other hand, Neumann boundary conditions are boundary conditions which determine the partial derivative of the option at the boundary. When the value of the option is unknown at the boundary, the second derivative of the option is employed because this is often known at extreme asset values. Typically, the boundary condition is given by

$$ \frac{\partial^2 F}{\partial s^2} (\tau, s_{\min}) = 0 $$

and/or

$$ \frac{\partial^2 F}{\partial s^2} (\tau, s_{\max}) = 0.$$ 

Using the central difference approximation gives

$$ \frac{\partial^2 F}{\partial s^2} (\tau_{k+1}, s_{\min} + \Delta s) = \frac{\partial^2 F}{\partial s^2} (\tau_{k+1}, s_1) $$
\[ F_{k+1.0} = \frac{F_{k+1,1} + 2F_{k+1,2}}{\Delta s} + O(\Delta s^2). \]

By dropping the error term, it yields
\[ F_{k+1,0} = 2F_{k+1,1} - F_{k+1,2}. \]

Likewise, the boundary condition at \( s_{max} \) is given by
\[ F_{k+1,N} = 2F_{k+1,N-1} - F_{k+1,N-2}. \]

2.5.5. Non-uniform Grid Points

Finite difference schemes that use uniform grids are the simplest and quite accurate. However, it would be better if there were finer grid points near critical prices such as barrier and strike prices, whereas coarser grid was at less important locations. This can be achieved by laying out the desired grid points and then discretizing the differential operator that would coincide with the grid points or by applying a coordinate transformation. While the former method is not described in detail, focus is given on the latter method.

As stated previously, let the Black-Scholes PDE be
\[
\frac{\partial F}{\partial t}(t, s) + (r(t) - \delta(t))s \frac{\partial F}{\partial s}(t, s) + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2}(t, s) - \tau(t)F(t, s) = 0.
\]

Consider the change of variables \((t, s) \rightarrow (\tau, x)\) defined by
\[ \tau = T - t, \quad s = s(x) \]
and let \( F(t, s) = \hat{F}(\tau, x) \).

Calculating the partial derivatives yields
\[ F_t(t, s) = -\hat{F}_\tau(\tau, x), \]
\[ F_s(t, s) = \hat{F}_x(\tau, x) \frac{1}{s_x(x)}, \]
\[ F_{ss}(t, s) = \hat{F}_{xx}(\tau, x) \left( \frac{1}{s_x(x)} \right)^2 + \hat{F}_x(\tau, x) \frac{1}{s_x(x)} \frac{\partial}{\partial x} \left( \frac{1}{s_x(x)} \right). \]

Then the initial PDE becomes
\[ -\hat{F}_\tau + \left( \frac{\hat{r} - \hat{\delta}}{s_x(x)} \right) s(x) + \frac{\sigma^2 s^2(x)}{2s_x(x)} \frac{\partial}{\partial x} \left( \frac{1}{s_x(x)} \right) \hat{F}_x + \frac{1}{2} \left( \frac{\sigma s(x)}{s_x(x)} \right)^2 \hat{F}_{xx} - \hat{r}\hat{F} = 0 \]
and the Crank-Nicolson discretization gives
\[
\hat{l}_{k+1} F_{k+1,j-1} + (\hat{d}_{k+1} + 1) F_{k+1,j} + \hat{u}_{k+1} F_{k+1,j+1} = l_k F_{k,j-1} + (d_k + 1) F_{k,j} + u_k F_{k,j+1} \quad \text{for } j = 1, \ldots, N - 1
\]
where
\[
\alpha_j = \frac{1}{2} \left( \frac{\sigma s(x_j)}{s_x(x_j)} \right)^2 \frac{\Delta \tau}{(\Delta x)^2},
\]

\[
\beta_{k,j} = \left( \frac{(r_k - \delta_k)s(x_j)}{s_x(x_j)} + \frac{\sigma^2 s^2(x_j)}{2s_x(x_j)} \frac{\partial}{\partial x} \left( \frac{1}{s_x(x_j)} \right) \right) \frac{\Delta \tau}{2 \Delta x},
\]

\[
l_{k,j} = \alpha_j - \beta_{k,j},
\]

\[
d_{k,j} = 1 - 2\alpha_j - r_k \Delta \tau,
\]

\[
u_{k,j} = \alpha_j + \beta_{k,j},
\]

\[
l_{k+1,j} = -l_{k+1,j},
\]

\[
d_{k+1,j} = 2 - d_{k+1,j},
\]

\[
u_{k+1,j} = -u_{k+1,j}.
\]

Although there are various transformations that map \(x\) to \(s\) with uniform grid points on \(x\) and non-uniform grid points on \(s\), emphasis is placed on an exponential and polynomial transformation.

### 2.5.5.1. Exponential Transformation

Consider the transformation \(s(x) = a + be^{cx}\), where \(a, b\) and \(c\) are estimated in order that there will be finer grid points close to the barrier. The parameters \(a\) and \(b\) are estimated as follows

\[
s(x = 0) = s_{min},
\]

\[
s(x = 1) = s_{max}
\]

and thus, the transformation becomes

\[
s(x) = s_{min} + (s_{max} - s_{min}) \frac{e^{cx} - 1}{e^c - 1}.
\]

It may easily be verified that for down-and-out barrier options \(c > 0\), whereas for up-and-out barrier options \(c < 0\). When \(c\) is close to zero the intervals become more equidistant. Figure 6 presents some graphs of \(s(x)\) for various values of \(c\) showing concentrations of evaluation points.
2.5.5.2. Polynomial Transformation

Another possible transformation is the quintic function

\[ s(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f, \]

where \(a, b, c, d, e\) and \(f\) are estimated in order that there will be finer grid points close to the barrier and the strike price. Without loss of generality, consider a down-and-out barrier option with strike price \(K\) greater than the barrier level \(L\). In this case, there are six conditions that should be satisfied.

\[
\begin{align*}
& s(x_K) = K, \quad s(x_L) = L, \\
& s_x(x_K) = s_x(x_L) = s_{xx}(x_K) = s_{xx}(x_L) = 0.
\end{align*}
\]

For simplicity, let \(x_L = 0\) and \(x_K = 1\), then

\[
s(x) = 6(K - L)x^5 - 15(K - L)x^4 + 10(K - L)x^3 + L.
\]

Regardless of the type of barrier option, there is a polynomial transformation which increases the concentration near the barrier and the strike price. Figure 7 confirms the result of the transformation when \(L = 10\) and \(K = 20\).

Figure 7. An example of a non-uniform grid via a polynomial transformation.
It is worth mentioning that if the strike price and the barrier coincide, a cubic function can be used instead.

2.5.6. Tridiagonal matrix algorithm

It is well known that the fully implicit and Crank-Nicolson methods require solving a tridiagonal system. *Thomas algorithm* can be used to solve such systems in an efficient way, given that it requires $O(n)$ operations, whereas Gaussian elimination requires $O(n^3)$.

Let a tridiagonal system with $n$ unknown be given by

$$a_i x_{i-1} + b_i x_i + c_i x_{i+1} = d_i,$$

where $a_1 = c_n = 0$. Expressing this system in matrix form yields

$$
\begin{pmatrix}
  b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
  \ddots & \ddots & \ddots \\
  a_{n-1} & b_{n-1} & c_{n-1} \\
  a_n & b_n & &
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_{n-1} \\
  x_n
\end{pmatrix}
= \begin{pmatrix}
  d_1 \\
  d_2 \\
  \vdots \\
  d_{n-1} \\
  d_n
\end{pmatrix}.
$$

The algorithm consists of two main sweeps, the forward and the backward sweep. The first sweep eliminates the subdiagonal elements $a_i$’s, while the second backward sweep yields the solution.

Specifically, during the first sweep the superdiagonal and right-hand side elements are modified as follows

$$c_i^{new} = \begin{cases} 
\frac{c_i}{b_i} & \text{for } i = 1 \\
\frac{c_i}{b_i - a_i c_{i-1}^{new}} & \text{for } i = 2, \ldots, n - 1
\end{cases}$$

and

$$d_i^{new} = \begin{cases} 
\frac{d_i}{b_i} & \text{for } i = 1 \\
\frac{d_i - a_i d_{i-1}^{new}}{b_i - a_i c_{i-1}^{new}} & \text{for } i = 2, \ldots, n - 1
\end{cases}$$

During the second sweep, the solution is obtained by back substitution

$$x_n = d_n^{new},$$

$$x_i = d_i^{new} - c_i^{new} x_{i+1} \text{ for } i = n - 1, n - 2, \ldots, 1.$$
It is proved that the Thomas algorithm is stable if the tridiagonal system is diagonal dominant. In other words, three conditions should be hold

\[ |b_i| \geq |a_i| + |c_i| \quad \text{for} \ i = 2, \ldots, n - 1, \]
\[ |b_1| \geq |c_1|, \]
\[ |b_n| \geq |a_n|. \]

2.6. Dividends

It is well known that the majority of contingent claims traded around the world are written on dividend-paying underlying assets. Therefore, there is a need of a model that will take into consideration the dividend payments at discrete points of time.

Let the process \( S \) be the underlying asset’s price and \( \delta \) be the dividend given by a deterministic continuous function

\[ \delta = \delta(S_{t-dt}) \]

where \( t \) is a dividend time. In other words, the size of the dividend at dividend time \( t \) is determined at \( t - dt \).

One of the fundamental assumptions is that there is no arbitrage opportunity. In order to guarantee the absence of arbitrage the following jump condition must hold at every dividend time \( t \)

\[ S_t = S_{t-dt} - \delta(S_{t-dt}) \]

That is, when the dividend \( \delta \) is paid out to the shareholders, the price of the underlying asset decreases by \( \delta \). Because of this condition, the value of the derivative at dividend time \( t \) must satisfy the following jump condition

\[ F(t - dt, S_{t-dt}) = F(t, S_t) = F(t, S_{t-dt} - \delta(S_{t-dt})) \]

Interpolation is one of the most prominent solutions to pricing options with discrete dividends. With the finite difference method, interpolation is employed between grid points to find an accurate option value before the dividend is paid. As shown in Figure 8, similar approach can also be used with the lattice methods. For instance, Vellekoop and Nieuwenhuis suggest using interpolation technique after each dividend date.

It is well known that for convex pricing functions the linear interpolation overestimates the value. Therefore, a cubic spline approximation or Lagrange interpolation is recommended for accurate results.
Figure 8. A dividend correction as described by Vellekoop and Nieuwenhuis.

2.7. Barycentric Lagrange Interpolation

The barycentric Lagrange interpolation constitutes a variant of the Lagrange polynomial interpolation.
Consider a set of $n + 1$ distinct data points $$(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n),$$
and let $p$ be the polynomial which interpolates $y_j$ at the point $x_j$. The interpolation polynomial in the Lagrange form is given by

$$p(x) = \sum_{j=0}^{n} y_j l_j(x), \quad l_j(x) = \prod_{k=0}^{n} \frac{x - x_k}{x_j - x_k}.$$ 

However, the Lagrange basis polynomials $l_j$ can be written as

$$l_j(x) = l(x) \frac{w_j}{x - x_j},$$

where

$$l(x) = \prod_{k=0}^{n} (x - x_k), \quad w_j = \prod_{k=0}^{n} \frac{1}{x_j - x_k}.$$ 

Therefore, the interpolation polynomial is given by
\[ p(x) = l(x) \sum_{j=0}^{n} y_j \frac{w_j}{x - x_j}. \]

Assume the interpolation of the constant function \( f(x) \equiv 1, \)
\[ f(x) = l(x) \sum_{j=0}^{n} \frac{w_j}{x - x_j} = 1. \]

Finally, the barycentric interpolation formula can be obtained by cancelling the common factor \( l(x) \)
\[ p(x) = \frac{\sum_{j=0}^{n} y_j \frac{w_j}{x - x_j}}{\sum_{j=0}^{n} \frac{w_j}{x - x_j}}. \]
3. Methodology

3.1. Problem

As stated in the introduction, this study focuses upon pricing standard barrier options using a two-volatility model. Specifically, an underlying asset’s price \( S \) determines the payoff while a stochastic process \( H \), being identical to \( S \) apart from the volatility, is linked to barrier hits. There are only one underlying asset and one riskless asset.

Because of the in-out parity relation, only “out” barrier options are scrutinized in this study. Specifically, there are four main versions of “out” barrier options.

A down-and-out call option with payoff:

\[
\Phi(S_T) = \max\{S_T - K, 0\} \mathbb{1}_{\min_{t \in [0, T]} H_t > L}
\]

A down-and-out put option with payoff:

\[
\Phi(S_T) = \max\{K - S_T, 0\} \mathbb{1}_{\min_{t \in [0, T]} H_t > L}
\]

An up-and-out call option with payoff:

\[
\Phi(S_T) = \max\{S_T - K, 0\} \mathbb{1}_{\max_{t \in [0, T]} H_t < L}
\]

An up-and-out put option with payoff:

\[
\Phi(S_T) = \max\{K - S_T, 0\} \mathbb{1}_{\max_{t \in [0, T]} H_t < L}
\]

3.2. Solution

The following sections present the derivations of the pricing PDE and the transformation of the problem to a time-independent barrier.

3.2.1. European Barrier Option

As previously mentioned, a European barrier option is a type of option whose payoff depends not only on the underlying asset’s price \( S \) at the expiration date \( T \), but also on whether the price \( S \) has reached or exceeded a predetermined price \( L \) during the option's lifetime. When the price \( L \) is a function of ...
time, the option is called time-dependent barrier option or moving barrier option.

Without loss of generality, consider the case of a down-and-out call option whose payoff is given by

$$\Phi(S_T) = \max\{S_T - K, 0\} \mathbb{1}_{\{\min_{t \in [0,T]} \{S_t - L_t\} > 0\}},$$

where $L_t$ is the barrier level at time $t$ and $K$ the strike price.

Under the generalized Black-Scholes model, the value of the option solves the following boundary value problem

$$\frac{\partial F}{\partial t}(t, s) + (r(t) - \delta(t))s \frac{\partial F}{\partial s}(t, s) + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 F}{\partial s^2}(t, s) - r(t)F(t, s) = 0,$$

$$F(T, s) = \max\{s - K, 0\},$$

$$F(t, L(t)) = 0,$$

where $\delta(t)$ is the continuous dividend yield, $r(t)$ is the risk-free interest rate and $\sigma_S$ is the volatility of the underlying asset’s price. Besides the terminal condition at $t = T$, there is also a boundary condition at $s = L(t)$ because the value of the option on the barrier is zero. Moreover, for “down” options the asset price must be above the barrier and thus, the domain $D$ of the PDE is

$$D = \{(t, s) \mid 0 \leq t \leq T, \quad L(t) \leq s < \infty\}$$

3.2.2. Stochastic Process $H$

According to the generalized Black-Scholes model, the price of the underlying asset under the risk-neutral measure is given by

$$dS(t) = (r(t) - \delta(t))S(t)dt + \sigma_S S(t)dW(t), \quad S(0) = s_0.$$ Let $H$ be a stochastic process with the following dynamics under the same measure.

$$dH(t) = (r(t) - \delta(t))H(t)dt + \sigma_H H(t)dW(t), \quad H(0) = s_0.$$ Applying the Ito’s formula to $\ln S(t)$ yields,

$$d(\ln S(t)) = (r(t) - \delta(t))dt - \frac{\sigma^2 S}{2} dt + \sigma_S dW(t)$$

and integrating the equation gives

$$\ln \frac{S(t)}{s_0} = a(t) - \frac{\sigma^2 S}{2} t + \sigma_S W(t), \quad \text{where} \quad a(t) = \int_0^t (r(u) - \delta(u)) du.$$ Hence,
\[ W(t) = \frac{\ln S(t) - a(t) + \frac{\sigma_S^2}{2} t}{\sigma_S}. \]

Similarly, the process \( \ln H(t) \) yields
\[ \ln \frac{H(t)}{s_0} = a(t) - \frac{\sigma_H^2}{2} t + \sigma_H W(t). \]

Substituting \( W(t) \) into the last equation gives
\[ \ln \frac{H(t)}{s_0} = a(t) - \frac{\sigma_H^2}{2} t + \sigma_H \ln \frac{S(t)}{s_0} - a(t) + \frac{\sigma_S^2}{2} t \]
or, equivalently,
\[ H(t) = S(t) \frac{\sigma_H}{\sigma_S} s_0^{1 - \frac{\sigma_H}{\sigma_S}} \exp \left\{ (\sigma_S - \sigma_H) \left( \frac{a(t)}{\sigma_S} + \frac{\sigma_H t}{2} \right) \right\}. \]

Therefore, the process \( H \) can be considered as a deterministic function of \( t \) and \( S \). Let \( L \) be a real number, then \( H(t) > L \) holds if and only if
\[ S(t) > L \frac{\sigma_S}{\sigma_H} s_0^{1 - \frac{\sigma_S}{\sigma_H}} \exp \left\{ (\sigma_S - \sigma_H) \left( \frac{a(t)}{\sigma_H} + \frac{\sigma_S t}{2} \right) \right\}. \]

This equivalence is useful for pricing barrier options using a two-volatility model.

3.2.3. Pricing of Barrier Options Using a Two-Volatility Model

Let the function of the time-dependent barrier \( L(t) \) be given by
\[ L(t) = L \frac{\sigma_S}{\sigma_H} s_0^{1 - \frac{\sigma_S}{\sigma_H}} \exp \left\{ (\sigma_H - \sigma_S) \left( \frac{a(t)}{\sigma_H} + \frac{\sigma_S t}{2} \right) \right\}. \]

Then, the aforementioned PDE holds for
\[ s > L \frac{\sigma_S}{\sigma_H} s_0^{1 - \frac{\sigma_S}{\sigma_H}} \exp \left\{ (\sigma_S - \sigma_H) \left( \frac{a(t)}{\sigma_S} + \frac{\sigma_S t}{2} \right) \right\} \quad \text{and} \quad t \in [0, T] \]
or, equivalently,
\[ \frac{\sigma_H}{\sigma_S} s s_0^{1 - \frac{\sigma_H}{\sigma_S}} \exp \left\{ (\sigma_S - \sigma_H) \left( \frac{a(t)}{\sigma_S} + \frac{\sigma_H t}{2} \right) \right\} > L \quad \text{and} \quad t \in [0, T]. \]

Substituting the process \( H \) in the barrier condition defines the barrier option in a two-volatility model and its payoff is given by
\[ \Phi(S_T) = \max\{S_T - K, 0\} \mathbb{1}_{\{\min_{t \in [0,T]} H_t > L\}}. \]

The pricing function of the option solves the following boundary value problem.
\[
\frac{\partial F}{\partial t}(t, s) + (r(t) - \delta(t))s \frac{\partial F}{\partial s}(t, s) + \frac{1}{2} \sigma_S^2 s^2 \frac{\partial^2 F}{\partial s^2}(t, s) - r(t)F(t, s) = 0,
\]

\[
F(T, s) = \max\{s - K, 0\},
\]

\[
F\left(t, L_s^\sigma H s_0 1 - \frac{\sigma_S}{\sigma_H} \exp\left\{ (\sigma_H - \sigma_S) \left( \frac{a(t)}{\sigma_H} + \frac{\sigma_S}{2} t \right) \right\}\right) = 0,
\]

which holds for

\[
0 \leq t \leq T \text{ and } L_s^\sigma H s_0 1 - \frac{\sigma_S}{\sigma_H} \exp\left\{ (\sigma_H - \sigma_S) \left( \frac{a(t)}{\sigma_H} + \frac{\sigma_S}{2} t \right) \right\} \leq s < \infty.
\]

It can readily be proved that also the rest of the barrier options can be valued by using a two-volatility model in a similar way.

### 3.2.4. Transformation to a Time-independent Barrier Option

The numerical solutions such as finite difference and lattice methods can readily be applied to pricing options with constant barriers. Therefore, a transformation of the problem is needed in order to make the barrier independent of time.

Consider the change of variables \((t, s) \rightarrow (\tau, x)\) defined by

\[
\tau = t, \quad x = s \frac{\sigma_H}{\sigma_S} s_0 1 - \frac{\sigma_H}{\sigma_S} e^{(\sigma_S - \sigma_H)(\frac{a(t)}{\sigma_S} + \frac{\sigma_H}{2} t)}
\]

and let \(F(t, s) = V(\tau, x)\). Calculating the partial derivatives yields

\[
F_t(t, s) = V_\tau(\tau, x) + V_x(\tau, x)x(\sigma_S - \sigma_H) \left( \frac{r(\tau) - \delta(\tau)}{\sigma_S} + \frac{\sigma_H}{2} \right),
\]

\[
F_s(t, s) = V_x(\tau, x) \frac{x\sigma_H}{s\sigma_S},
\]

\[
F_{ss}(t, s) = V_{xx}(\tau, x) \left( \frac{x\sigma_H}{s\sigma_S} \right)^2 + V_x(\tau, x) \frac{\sigma_H}{s} \left( \frac{\sigma_H}{s} - 1 \right) \frac{x}{s^2}.
\]

Hence, the initial PDE becomes

\[
V_\tau(\tau, x) + \left( r(\tau) - \delta(\tau) \right) x V_x(\tau, x) + \frac{1}{2} x^2 \sigma_H^2 V_{xx}(\tau, x) - r(\tau)V(\tau, x) = 0,
\]

\[
V(T, x) = \max \left\{ \frac{\sigma_S}{x\sigma_H x_0 1 - \frac{\sigma_S}{\sigma_H} e^{(\sigma_H - \sigma_S)(\frac{a(T)}{\sigma_H} + \frac{\sigma_S}{2} T)}} - K, 0 \right\},
\]

\[
F(\tau, L) = 0,
\]

which holds for \(\tau \in [0, T]\) and \(x > L\).

Therefore, the value of the barrier option can be estimated by solving a Black-Scholes PDE describing an ordinary barrier option on one underlying asset. In this case, the finite difference and lattice methods can readily be implemented and find numerical approximations to the solution of the PDE.
Nevertheless, a coordinate transformation alters the grid and thus, a uniform discretization of the domain \((\tau, x)\) implies a non-uniform grid of the domain \((t, s)\). Figure 9 demonstrates a possible modification of the grid due to the transformation.

![Figure 9](image)

*Figure 9. An example of a non-uniform grid via a coordinate transformation.*

Changing the concentration of evaluation points and mainly close to the barrier may increase or decrease the bias.

### 3.3. Implementation

Although there are analytical solutions to pricing standard barrier options, complications such as discrete dividend payments render the use of numerical methods inevitable. Lattice and finite difference solvers are implemented in C++ as they are described in the theory section and their performance characteristics are compared.

It is worthwhile noting that in both methods the time space is discretized according to the dividend payment dates in order that the value of the option can be estimated at these dates. Therefore, the time step may differ among the time intervals defined by the payment dividend dates. This method is chosen because nodes are always placed at these dates and the value of the option is calculated by interpolation. In the lattice methods, the next tree at the dividend date may have more nodes in order that the barycentric Lagrange interpolation can be employed.
4. Results

This section presents and analyses the results of convergence and performance tests of the implemented solvers. The programming language being used is C++14 which is a version of the standard for the programming language C++. Additionally, Google Test is used as the testing framework for the test-driven development process.

4.1. Test Specification

The hardware platform used for testing has the following characteristics

- Model: iMac (27-inch, Late 2013)
- Model Identifier: iMac14,2
- Processor: 3.2 GHz Intel Core i5
- Memory: 20GB 1600MHz DDR3
- Compiler: Apple LLVM version 8.1.0 (clang-802.0.42)

For the purpose of testing, consider the following barrier options:

A European down-and-out call option

- \( S_0 = [95,105] \) arb. units
- \( \delta(t) = 0.01t + 0.01 \)
- \( \sigma = 0.2 \)
- \( K = 100 \) arb. units
- \( L = 90 \) arb. units
- \( T = 1 \) arb. unit
- \( r(t) = 0.02t + 0.03 \)

A European up-and-out put option

- \( S_0 = [95,105] \) arb. units
- \( \delta(t) = 0.01t + 0.01 \)
- \( \sigma = 0.2 \)
- \( K = 100 \) arb. units
- \( L = 110 \) arb. units
- \( T = 1 \) arb. unit
- \( r(t) = 0.02t + 0.03 \)
where $S_0$ denotes the current spot price of the underlying asset, $\delta$ the continuous dividend yield, $\sigma$ the standard deviation of the asset's returns, $K$ the strike price, $L$ the barrier level, $T$ the time to maturity and $r$ the risk-free interest rate.

The discrete dividends for both options are given as follows:

<table>
<thead>
<tr>
<th>Dividend Times (arb. units)</th>
<th>Dividends as Fixed Amounts (arb. units)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1506</td>
<td>1.02</td>
</tr>
<tr>
<td>0.2307</td>
<td>3.23</td>
</tr>
<tr>
<td>0.6015</td>
<td>2.63</td>
</tr>
<tr>
<td>0.7023</td>
<td>3.46</td>
</tr>
<tr>
<td>0.8238</td>
<td>1.72</td>
</tr>
</tbody>
</table>

Dividends expressed as fixed amounts are chosen for testing, although the implementation takes into account also dividends expressed as percentages of the asset’s price.

In all convergence tests, the Crank-Nicolson discretization scheme on $3 \times 10^4 \times 3 \times 10^4$ grid is used as a reference. The accuracy of the solver is verified to some extent by unit tests and analytical formulas. However, Monte Carlo methods are needed to corroborate its validity and reliability. Stated errors are the maximum relative errors in pricing options for different current spot prices of the underlying asset.

Measuring application performance refers to the computational resources used by the application and the performance experienced by the user. However, the study examines only the response time of the implemented solvers which is the total amount of time that they estimate the value of an option. Each test is executed 100 times and the quickest response time is kept as an approximation of the runtime.

It should be noted that in the following figures the terms exponential tr. and polynomial tr. corresponds to non-uniform grids constructed by applying an exponential and polynomial transformation respectively.

4.2. Convergence Test

The convergence test verifies that the finite difference and lattice schemes are convergent since their solutions converge point-wise to the corresponding solutions as the number of stock nodes and time points approach infinity.

As can be illustrated in Figure 11, Figure 12, Figure 14 and Figure 15, the Crank-Nicolson method demonstrates second-order convergence both in
space and time regardless of whether the grid is uniform or not. On the other hand, the fully implicit method exhibits first-order convergence in time.

Figure 10 indicates that the Crank-Nicolson method using the polynomial transformation displays a peculiar trend of relative error after $10^3$ time points. This occurs because of the overflow and round-off errors by using double-precision floating-point format as a computer number format. This trend disappears and the results are more accurate and reliable if the precision increases from 64 bits to 128 bits. However, all performance tests are conducted under 64-bit precision which is supported directly by hardware.

Moreover, it is worth mentioning that the position of the grid nodes relative to the strike price at the maturity time can affect option pricing. This is illustrated in Figure 13 where fluctuations in relative error appear. However, the polynomial transformation renders these fluctuations imperceptible by placing more nodes close to the strike price. In contrast, Figure 14 and Figure 15 present the convergence in space when there is a grid node at or very close to the strike price.

Figure 10. Convergence of the finite difference methods for the down-and-out call option with 64-bit precision.
Figure 11. Convergence of the finite difference methods for the down-and-out call option with 128-bit precision.

Figure 12. Convergence of the finite difference methods for the up-and-out put option with 128-bit precision.
Figure 13. Convergence of the finite difference methods for the down-and-out call option.

Figure 14. Convergence of the finite difference methods for the down-and-out call option.
Figure 15. Convergence of the finite difference methods for the up-and-out put option.

On the other hand, as shown in Figure 16 and Figure 17, the trinomial tree models exhibit lower rates of convergence. Specifically, the standard trinomial tree and the probability-adjusted trinomial tree demonstrate convergence of order close to 0.5 whereas Ritchken’s trinomial tree and the adaptive mesh model display first-order convergence. Additionally, the trinomial trees display fluctuations in relative error.

The height of the finer grid in the adaptive mesh model is selected by trial and error and it is set equal to 1/10 of the number of time steps. Although this number is not the optimal one, it serves the purpose of testing the model.
Figure 16. Convergence of the lattice methods for the down-and-out call option.

Figure 17. Convergence of the lattice methods for the up-and-out put option.
4.3. Performance Test

In this section, the finite difference and lattice solvers are ranked in order of performance. The following figures shed light on it and reveal how well the solvers perform.

Although the performance test on the lattice methods can be done straightforwardly, the test on the finite difference methods is more complex. The relative error of the finite difference schemes is contingent on the number of stock nodes and time points. Therefore, an exhaustive search is required to estimate the time consumption and the relative error by examining possible combinations of numbers of stock nodes and time points. Nevertheless, a heuristic approach can be employed by selecting the number of stock nodes and time points in order that they will yield the same relative error. Figure 18 and Figure 19 indicate a relatively effective relationship between the number of stock nodes and time points which is derived from a linear regression of the convergence data. For the performance test the worst-case scenario, where both relative errors are in the same direction, is investigated.

The following figures highlight the performance of the finite difference and lattice solvers.

Figure 18. An effective relationship between the number of stock nodes and time points for the down-and-out call option.
Figure 19. An effective relationship between the number of stock nodes and time points for the up-and-out put option.

Figure 20. Performance of the finite difference solvers for the down-and-out call option.
Figure 21. Performance of the finite difference solvers for the up-and-out put option.

Figure 22. Performance of the lattice solvers for the down-and-out call option.
Figure 23. Performance of the lattice solvers for the up-and-out put option.

Figure 24. Performance of the finite difference and lattice solvers for the down-and-out call option.
As can be seen in Figure 24 and Figure 25, the Crank-Nicolson solvers seem to outperform the other solvers. It is worth mentioning that due to heuristic approach the performance of the finite difference solvers may be even better.

The Crank-Nicolson solver which uses a non-uniform grid derived by an exponential transformation is the most efficient solver for relative error less than $10^{-4}$. Although the Crank-Nicolson method on uniform grid exhibits similar results for the up-and-out put option, the same does not hold for the down-and-out call option.

On the other hand, only Ritchken’s trinomial tree may compete against the finite difference schemes because it displays low time consumption for relative error greater than $10^{-3}$. The adaptive mesh model which creates more nodes close to the barrier results in increasing the execution time. The probability-adjusted trinomial tree, albeit faster than the standard trinomial tree, is still inefficient.

4.4. Discussion

As displayed in the previous figures, the results of the convergence and performance tests indicate an advantage in favour of the Crank-Nicolson scheme. The choice of the non-uniform grid is a decisive factor in the performance of
the finite difference methods. This is proved by the fact that the finite difference scheme on a non-uniform grid constructed by the exponential transformation performs better than the scheme on uniform grid. Moreover, the benefits of the finite difference methods extend to calculating the Greeks and especially delta $\Delta$ and gamma $\Gamma$ which are the first and the second derivative of the option value with respect to the underlying asset’s price respectively.

Ritchken’s tree seems to achieve relatively good performance for a little time consumption. Nonetheless, this method encounters difficulty when the initial underlying price is very close to the barrier. When the maximum price step that permits one move before hitting the barrier is very small, the time steps can be extremely small and thus, the calculation may be impossible to perform practically. Instead, the adaptive mesh model can be employed constructing a fine-resolution lattice close to the barrier. However, still this method presents worse performance than the Crank-Nicolson scheme.
5. Conclusion

Although the Black-Scholes framework can yield analytical solutions to various pricing problems, it may reach a deadlock with complicated derivative instruments. Closed-form equations are impossible to be derived even though the principles of valuation based on no-arbitrage condition continue to hold. Lattice-based, finite difference and Monte Carlo methods can be employed for obtaining approximate solutions.

This study scrutinizes performance characteristics of a range of finite difference and trinomial solvers for pricing barrier options using a two-volatility model. Specifically, the study indicates how the problem can be transformed into a one-dimensional barrier option pricing problem with a time-independent barrier. Although this kind of option has an analytical solution, complications such as discrete dividend payments render the use of numerical methods indispensable. Consequently, various methods described in theory are implemented and evaluated based on rates of convergence and performance.

The results of this study corroborate that the Crank-Nicolson finite difference scheme exhibits quadratic convergence and better performance. Although enhancements to lattice models lead to some improvements in performance, the Crank-Nicolson finite difference scheme on a non-uniform grid constructed by the exponential transformation outperforms the rest. The present findings might help practitioners choose the most efficient method for pricing barrier options.

It is recommended that further research should be undertaken in local adaptive mesh refinement for the finite difference scheme to solve the untransformed problem with a time-dependent barrier. The construction of an appropriate grid can constitute an area of study.
6. Bibliography


