Option Pricing Model with Investor Sentiment

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ABSTRACT

Investor sentiment can affect option prices in real world markets. This paper aims to develop empirical option pricing formulas with investor sentiment for call and put options, such that they are more consistent with market prices than the Black-Scholes formula. Implemented with historical option data on AAPL, together with the VIX and ISEE Index values as sentiment input, the proposed sentiment pricing formulas generate option prices that have much lower pricing errors for most options in the dataset, with the exceptions being some at the money options with a shorter maturity (less than half a year). The sentiment input provides great potential for option pricing improvements.
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1. Introduction

Under the complete market assumption in the Black-Scholes framework, the option price for a non-dividend paying underlying asset can be uniquely determined through arbitrage-free arguments by five factors – current price of the underlying asset, time to maturity, strike price, risk-free interest rate, and volatility of returns of the underlying asset (Black and Scholes 1973). However, real world markets are incomplete due to various causes such as insufficiency of marketed assets, transaction costs, and short selling constraints (Hao 2008; Staum 2008). Options can no longer be perfectly hedged with replicating portfolios and thus become nonredundant assets to some extent (Figlewski 1989; Amin, Coval, and Seyhun 2004). Their prices are then subject to the demand and supply forces in the market and can deviate from their theoretical arbitrage-free prices. Hence other factors that do not appear in the standard option pricing model can affect option prices in real world markets.

One distinct factor that has received extensive attention in recent years is investor sentiment. There is a growing body of literature that examines the role of sentiment in the stock market and its effect on asset prices (e.g. De Long et al. 1990; Shleifer and Summers 1990; Lee, Jiang, and Indro 2002; Brown and Cliff 2004, 2005; Baker and Wurgler 2006, 2007; Kumar and Lee 2006; Lawrence, McCabe, and Prakash 2007; Joseph, Wintoki, and Zhang 2011; Statman 2011; Baker, Wurgler, and Yuan 2012; Xu and Green 2013; Bu and Pi 2014; Kim and Park 2015; Chau, Deesomsak, and Koutmos 2016; Smales 2017). The empirical findings provide evidence for investors’ irrational behaviors and psychological biases (e.g. representativeness, conservatism, and overconfidence), the correlation between sentiment proxies and stock market returns, the cross-sectional effects of sentiment, et cetera; and they suggest the incorporation of investor sentiment into asset pricing models.

Behavioral finance has emerged as an alternative theory in order to account for anomalies in the financial markets. The two tenets of behavioral finance theories are investor sentiment and limited arbitrage. As Shleifer and Summers outlined in their noise trader approach to finance (1990: 19-20):

First, some investors are not fully rational and their demand for risky assets is affected by their beliefs or sentiments that are not fully justified by fundamental news. Second, arbitrage – defined as trading by fully rational investors not subject to such sentiment – is risky and therefore limited. The two assumptions together imply that changes in investor sentiment are not fully countered by arbitrageurs and so affect security returns.

In line with the principles of behavioral finance, this paper explores investor sentiment in the option market. Empirical studies examining the effect of sentiment on option prices are mostly in consonance with research on stock prices. Stein (1989) uses data on S&P 100 index options and suggests investor overreaction to new information to be a cause of mispricing in the option market. Poteshman (2001) employs data on S&P 500 index options and the results confirm investors’ short-horizon underreaction and long-horizon overreaction, consistent with the investor sentiment model put forth by Barberis, Shleifer, and Vishny (1998) based on two well-established behavioral heuristics – conservatism and representativeness. Amin, Coval, and Seyhun (2004) investigate the relationship between index option prices and stock market momentum. They find that S&P 100 index calls are significantly overvalued relative to puts after large stock price increases, and the reverse is true after large stock price decreases. Buraschi and Jiltsov (2006) show that differences in investor beliefs are statistically significant to help explain both the cross section and the time series of option prices. Han (2007) examines whether sentiment drives variation in the index risk-neutral skewness and the results support that investor sentiment is an important determinant of index option prices. Mahani and Poteshman (2008) find that option traders overreact to news about the stocks that underlie equity options as well. Vlad and Musumeci (2008), Sheu and Wei (2011), Szu and Yang (2015), and Yang, Jhang, and Chang (2016) provide further evidence of the relationship between investor sentiment and option prices.
Since empirical research generally confirms that investor sentiment is correlated with option prices, it seems natural to consider the incorporation of a sentiment factor in the option pricing model. It is, however, not the same case as adding sentiment in a standard asset pricing model. There are some essential differences between classical asset pricing models (such as CAPM) and classical option pricing models (such as Black-Scholes). For instance, the Black-Scholes model, contrary to CAPM, does not assume rational investors, and there is no restriction on the risk preferences of investors (Björk 2009; Elton et al. 2014). While sentiment is excluded from the standard asset pricing model, it is however implicitly reflected in the theoretical option price obtained from a standard option pricing model such as the Black-Scholes, because the pricing formula takes the market stock price as input, and the market stock price already has sentiment in it. However, the sentiment of investors in the option market is not necessarily fully reflected in the market stock price. This means that although the Black-Scholes option price contains certain sentiment element brought in by the market stock price, there is still room for sentiment to affect option prices, which is not accounted for in the Black-Scholes pricing model. In other words, there is a part of investor sentiment that is reflected in the market option price, but not in the market stock price.

Based on these arguments, the present paper aims to incorporate investor sentiment in the option pricing model for call and put options. The remainder of the paper is organized as follows. Section 2 defines investor sentiment and provides a brief overview of some common sentiment indicators. Section 3 develops the sentiment pricing formulas for call and put options. Section 4 tests the formulas with historical option and sentiment data and presents the empirical results. Section 5 discusses limitations and potential problems. Section 6 concludes.

2. SENTIMENT INDICATORS

A natural question arises: What is sentiment? Researchers have defined sentiment in various ways. Brown and Cliff (2004) state that sentiment represents the expectations of market participants relative to a norm: a bullish (bearish) investor expects returns to be above (below) average, whatever “average” may be. Baker and Wurgler (2007) perceive sentiment as a belief about future cash flows and investment risks that is not justified by the facts at hand and mention another view of investor sentiment as simply optimism or pessimism about stocks in general. Han (2007) defines investor sentiment as the aggregate error in investor beliefs.

According to Cambridge dictionary (2017), sentiment is a thought, opinion, or idea based on a feeling about a situation, or a way of thinking about something. Investor sentiment thus refers to the general opinion and attitude of investors towards a particular security or financial market. Bullish (bearish) sentiment indicates an overall positive (negative) attitude, often accompanied by upward (downward) price movement. This attitude is based on both fundamental and technical factors, such as price development, business reports, market trends, and economic events.

In order to incorporate such a complex and abstract concept as sentiment into the option pricing model, the sentiment factor needs to be quantified. There is no universal or correct way to measure sentiment, and researchers have developed and employed a variety of sentiment proxies. Some of the common investor/market sentiment indicators are presented below.
CBOE Volatility Index (VIX)
The Chicago Board Options Exchange (CBOE) Volatility Index, with ticker symbol VIX, is a popular measure of implied volatility. The VIX Index is based on real-time prices of options on the S&P 500 Index and is designed to reflect investors' consensus view of future (30-day) expected stock market volatility. Often referred to as the market's "fear gauge", the VIX Index can indicate the level of investors' need for insurance, as it often increases when investors buy put options to insure their portfolios against potential losses. Spikes in the index usually imply a market overwhelmed with fear, and thus more bearish sentiment. More information can be found at their official website http://www.cboe.com/products/vix-index-volatility.

High/Low Index
The high/low indicator seeks to identify a market trend by comparing the daily number of stocks reaching new 52-week highs with the number reaching new 52-week lows. The 52-week high/low is considered an important factor by many traders and investors. When the market sentiment is bullish, we can expect to see more 52-week highs than lows, and vice versa.

Bullish Percent Index (BPI)
The Bullish Percent Index (BPI) is a breadth indicator based on the number of stocks on Point & Figure buy signals. It is calculated as the number of stocks on Point & Figure buy signals divided by the total number of stocks. The BPI indicates a bullish market sentiment when over 50% of the stocks are on a buy signal, and bearish when below 50%.

50-Day or 200-Day Moving Average
This is a measure of the percentage of stocks trading above their 50-day or 200-day moving average. When the market is bullish, we expect to see the percentage of stocks above their moving averages rising, and the opposite when the market is bearish.

Put/Call Ratio
The put/call ratio is another popular indicator of investor sentiment. It is calculated by dividing the trading volume for put options by the trading volume for call options. A value above 1 indicates a bearish sentiment in the market, and a value below 1 indicates a bullish sentiment.

ISE Sentiment Index (ISEE)
The International Securities Exchange (ISE) Sentiment Index is an indicator of investor sentiment that is computed using proprietary ISE option trade data. ISEE is calculated by dividing opening long call options bought by customers by opening long put options bought by customers.

\[
ISEE = \frac{\text{Customer Opening Long Calls}}{\text{Customer Opening Long Puts}} \times 100
\]

Unlike traditional put/call ratios, ISEE is based on opening long customer transactions only, meaning that trades by market makers and firms are excluded. The market maker's job is to provide liquidity, so they tend to be indifferent as to the direction of the underlying security and their trades are not likely to represent bullish or bearish sentiment. Firm trades tend to execute strategies that make it difficult to detect whether they are long/short or bullish/bearish on a particular security. Thus, market maker and firm trades are not considered representative of true market sentiment. On the other hand, a customer purchasing an opening long call (put) generally indicates a bullish (bearish) sentiment. Therefore, a higher (lower) ISEE value represents a more bullish (bearish) sentiment for the underlying security. More information can be found at http://www.ise.com/market-data/isee-index.
**Acertus Market Sentiment Indicator (AMSI)**

The Acertus Market Sentiment Indicator (AMSI) is a proprietary monthly sentiment indicator with values ranging from 0 to 100. The indicator views sentiment as a continuum with fear (<20), anxiety (20-40), complacency (60-80), and greed (>80). AMSI is constructed using five variables:

1. Price/Earnings Ratio, a measure of stock market valuations;
2. Price Momentum, a measure of market psychology;
3. Realized Volatility, a measure of recent historical risk;
4. High Yield Bond Returns, a measure of credit risk; and
5. The TED spread, a measure of systemic financial risk.

Each factor provides a unique perspective, and weighted together they may offer a more robust indicator of market sentiment. More can be found at [https://en.wikipedia.org/wiki/Acertus_Market_Sentiment_Indicator](https://en.wikipedia.org/wiki/Acertus_Market_Sentiment_Indicator).

**AAII Investor Sentiment Survey**

American Association of Individual Investors (AAII) has been conducting weekly sentiment surveys since 1987. A random sample of AAII members is polled each week and asked the same question: *Do you feel the direction of the market over the next six months will be up (bullish), no change (neutral), or down (bearish)?* The results show the percentages of individual investors that are bullish, neutral, and bearish on the stock market. More can be found at [http://www.aaii.com/sentimentsurvey/sent_results](http://www.aaii.com/sentimentsurvey/sent_results).

**Investors Intelligence Sentiment Survey**

Another survey-based sentiment measure is conducted by Investors Intelligence. It compiles weekly bull-bear spread by categorizing market newsletters as bullish, bearish, or neutral based on the expectation of future market movements. Since the authors of the newsletters are most likely market professionals, this sentiment survey is considered to reflect the sentiment of institutional investors (Brown and Cliff 2004). The AAII sentiment survey, on the other hand, is considered to reflect the sentiment of individual investors. These two sentiment measures differ from the ones above in that they are direct measures of investor sentiment. More can be found at [http://www.investorsintelligence.com](http://www.investorsintelligence.com).

Some other sentiment proxies employed in the literature are listed below (Brown and Cliff 2004; Baker and Wurgler 2006; Joseph, Wintoki, and Zhang 2011; Sheu and Wei 2011):

- Ratio of the number of advancing issues to declining issues
- Percent change in margin borrowing
- Ratio of short sales to total sales
- Closed-end fund discount
- Net purchases of mutual funds
- Proportion of fund assets held in cash
- Initial public offering first day returns
- Number of initial public offerings
- Share turnover
- Equity share in new issues
- Dividend premium
- Put-call open interest ratio
- Online ticker searches

Other than using these sentiment indicators individually, another way is to construct a composite index from a selection of sentiment proxies (e.g. Baker and Wurgler 2006).
3. SENTIMENT PRICING MODEL

The Black-Scholes model gives theoretical option prices in perfect markets, which are not always consistent with market option prices. Option mispricing with respect to the Black-Scholes model has been examined in many studies (e.g. Black and Scholes 1973; Black 1975; Merton 1976; Macbeth and Merville 1979; Gultekin, Rogalski, and Ticin 1982; Levy and Byun 1987; French and Martin 1988; Geske and Torous 1991; Bates 1995; Long and Officer 1997; Gençay and Salih 2003; Constantinides, Jackwerth, and Perrakis 2008). Since the classic Black-Scholes model has no room for an additional sentiment factor, the attempt here is to derive an option pricing model outside the Black-Scholes framework that can incorporate investor sentiment. The goal is to construct an option pricing model with investor sentiment for call and put options, such that it is more consistent with market prices.

We start with deriving a general option pricing model with only one assumption, that the price process of the underlying asset is lognormally distributed. Assume that we have a normal random variable \( X \sim N(\mu, \sigma^2) \) and \( Y = e^X \), i.e. \( Y \) has lognormal distribution, then we can derive the expected value and variance of \( Y \) as follows.

\[
E[Y] = E[e^X] = \int_{-\infty}^{\infty} e^x \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx
\]

\[
= \int_{-\infty}^{\infty} e^{\mu + y} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}} \, dy
\]

\[
= e^\mu \int_{-\infty}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{y^2}{2\sigma^2}} \, dy
\]

\[
= e^\mu \int_{-\infty}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(y-\sigma)^2}{2\sigma^2} - \frac{\sigma^2}{2}} \, dy
\]

\[
= e^{\mu + \frac{\sigma^2}{2}} \int_{-\infty}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(y-\sigma)^2}{2\sigma^2}} \, dy
\]

\[
= e^{\mu + \frac{\sigma^2}{2}},
\]

as \( \int_{-\infty}^{\infty} \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(y-\sigma)^2}{2\sigma^2}} \, dy = 1 \). The variance of \( Y \) is given by

\[
\text{Var}[Y] = E[Y^2] - (E[Y])^2 = E[e^{2X}] - (E[e^X])^2.
\]

Since \( X \sim N(\mu, \sigma^2) \) and \( E[e^X] = e^{\mu + \frac{\sigma^2}{2}} \), we get \( 2X \sim N(2\mu, 4\sigma^2) \) and \( E[e^{2X}] = e^{2\mu + 4\frac{\sigma^2}{2}} = e^{2\mu + 2\sigma^2} \). Thus

\[
\text{Var}[Y] = e^{2\mu + 2\sigma^2} - \left( e^{\mu + \frac{\sigma^2}{2}} \right)^2
\]

\[
= e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2}
\]

\[
= e^{2\mu + \sigma^2} (e^{\sigma^2} - 1).
\]

We can write \( X = \mu + \sigma Z \), where \( Z \sim N(0, 1) \), so \( Y = e^X = e^{\mu + \sigma Z} \). For a constant \( K > 0 \), we have the expected value of a payoff in the form of a call option as follows.
\[ E[\max(Y - K, 0)] = \int_{-\infty}^{\infty} \max(e^{\mu + \sigma z} - K, 0) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \quad \{ e^{\mu + \sigma z} - K > 0 \iff z > \frac{\ln K - \mu}{\sigma} \} \]

\[ = \int_{\frac{\ln K - \mu}{\sigma}}^{\infty} (e^{\mu + \sigma z} - K) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \]

\[ = \int_{\frac{\ln K - \mu}{\sigma}}^{\infty} e^{\mu + \sigma z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz - K \int_{-\infty}^{\frac{\ln K - \mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \]

\[ = e^{\mu} \int_{\frac{\ln K - \mu}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\sigma)^2}{2}} dz - K \int_{-\infty}^{\frac{\ln K - \mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \]

\[ = e^{\mu + \frac{\sigma^2}{2}} \int_{\frac{\ln K - \mu}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\sigma)^2}{2}} dy - K \int_{-\infty}^{\frac{\ln K - \mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \]

\[ = e^{\mu + \frac{\sigma^2}{2}} N\left(\sigma + \frac{\mu - \ln K}{\sigma}\right) - KN\left(\frac{\mu - \ln K}{\sigma}\right) \]

\[ = e^{\mu + \frac{\sigma^2}{2}} N(d_1) - KN(d_2) \quad (1) \]

where \( N(\cdot) \) denotes the cumulative distribution function of the standard normal distribution, and

\[ d_1 = \sigma + \frac{\mu - \ln K}{\sigma}, \]

\[ d_2 = \frac{\mu - \ln K}{\sigma} = d_1 - \sigma. \]

Analogously, for a payoff in the form of a put option, we have

\[ E[\max(K - Y, 0)] = \int_{-\infty}^{\infty} \max(K - e^{\mu + \sigma z}, 0) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \quad \{ K - e^{\mu + \sigma z} > 0 \iff z < \frac{\ln K - \mu}{\sigma} \} \]

\[ = \int_{\frac{\ln K - \mu}{\sigma}}^{-\infty} (K - e^{\mu + \sigma z}) \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \]

\[ = K \int_{-\infty}^{\frac{\ln K - \mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz - \int_{\frac{\ln K - \mu}{\sigma}}^{\infty} e^{\mu + \sigma z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \]

\[ = K \int_{-\infty}^{\frac{\ln K - \mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{\frac{z^2}{2}} dz - \mu \int_{-\infty}^{\frac{\ln K - \mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \frac{1}{\sqrt{2\pi}} e^{\frac{(z-\sigma)^2}{2}} dz \]

\[ = K \int_{-\infty}^{\frac{\ln K - \mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{\frac{z^2}{2}} dz - e^{\mu + \frac{\sigma^2}{2}} \int_{-\infty}^{\frac{\ln K - \mu}{\sigma} - \sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-\sigma)^2}{2}} dy \]

\[ = K \int_{-\infty}^{\frac{\ln K - \mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{\frac{z^2}{2}} dz - e^{\mu + \frac{\sigma^2}{2}} \int_{-\infty}^{\frac{\ln K - \mu}{\sigma} - \sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy \]

\[ = KN\left(\frac{\ln K - \mu}{\sigma}\right) - e^{\mu + \frac{\sigma^2}{2}} N\left(\frac{\ln K - \mu}{\sigma} - \sigma\right) \]

\[ = KN(-d_2) - e^{\mu + \frac{\sigma^2}{2}} N(-d_1) \quad (2) \]
Now, consider a stock price process $S_t$ that follows a geometric Brownian motion, i.e. it satisfies the stochastic differential equation

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where $\mu$ is the return rate of the underlying stock, $\sigma$ is the volatility of returns of the underlying stock, and $W_t$ is a Brownian motion (Wiener process). Both $\mu$ and $\sigma$ are constants. Applying Itô’s formula on the natural logarithm of $S_t$ gives

$$d(\ln S_t) = \frac{1}{S_t} dS_t + \frac{1}{2} \left( -\frac{1}{S_t^2} \right) (dS_t)^2$$

$$= \frac{1}{S_t} (\mu S_t dt + \sigma S_t dW_t) - \frac{1}{2S_t^2} \sigma^2 S_t^2 dt$$

$$= \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma dW_t.$$ 

Integrating both sides, we get

$$\ln S_t = \ln S_0 + \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W_t,$$

which leads to

$$S_t = S_0 e^{\left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W_t}.$$

In our context, it is convenient to write

$$S_T = S_t e^{\left( \mu - \frac{\sigma^2}{2} \right) (T-t) + \sigma (W_T - W_t)}.$$

As

$$\ln S_T = \ln S_t + \left( \mu - \frac{\sigma^2}{2} \right) (T - t) + \sigma \sqrt{T-t} Z,$$

we see that $S_T$ is lognormally distributed as $Y$ above, and $\ln S_T$ corresponds to $X$ above.

$$\ln S_T \sim N \left( \ln S_t + \left( \mu - \frac{\sigma^2}{2} \right) (T - t), \sigma^2 (T - t) \right).$$

Moreover, we can insert the mean and variance of $\ln S_T$ into the formulas (1) and (2) for expected option payoffs and get the following results immediately:

1. $E[\max(S_T - K, 0)] = S_t e^{\mu(T-t)} N(d_1) - K N(d_2)$
2. $E[\max(K - S_T, 0)] = KN(-d_2) - S_t e^{\mu(T-t)} N(-d_1)$

where

$$d_1 = \sigma \sqrt{T - t} + \ln S_t + \left( \mu - \frac{\sigma^2}{2} \right) (T - t)$$

$$= \frac{\ln(S_t/K) + \left( \mu - \frac{\sigma^2}{2} \right) (T - t)}{\sigma \sqrt{T - t}},$$

$$d_2 = d_1 - \sigma \sqrt{T - t} = \frac{\ln(S_t/K) + \left( \mu - \frac{\sigma^2}{2} \right) (T - t)}{\sigma \sqrt{T - t}}.$$

The essential idea of asset pricing is that the value of a financial asset should be equal to the discounted value of the expected future cash flows. Assume that the discount factor takes the simple form $e^{-d(T-t)}$, where $d$ is the discount rate. For the call option, the price $C(t, S_t)$ is the discount factor multiplied by the expected future payoff

$$C(t, S_t) = e^{-d(T-t)} E[\max(S_T - K, 0)] = S_t e^{(\mu - d)(T-t)} N(d_1) - e^{-d(T-t)} KN(d_2)$$

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And the price $P(t, S_t)$ of the put option is

$$
P(t, S_t) = e^{-d(T-t)} E[\max(K - S_T, 0)] = e^{-d(T-t)} K N(-d_2) - S_t e^{(\mu - d)(T-t)} N(-d_1)
$$

Before proceeding to discuss how sentiment comes into play, we note the resemblance of this pricing formula to the Black-Scholes formula. In the complete Black-Scholes market, options can be replicated and their values can be calculated as if we live in a risk neutral world (Björk 2009). More specifically, the risk-free interest rate $r$ will replace both the discount rate $d$ and the stock return rate $\mu$ in the formulas (5) and (6). In incomplete markets, however, these two rates are no longer simply $r$ and have to be determined. In particular, both rates can be affected by investor sentiment.

The effect of investor sentiment is multifaceted. Intuitively, bullish sentiment means that investors are optimistic about a particular security and expect its price to continue to rise, thus they tend to overvalue the asset and they are willing to pay more for the same asset. In other words, more bullish sentiment usually goes along with a higher expected return for the stock and more bearish sentiment with a lower expected return. The discount factor, on the other hand, reflects the time value of money and the riskiness of the asset, which can also be affected by investor sentiment. Positive sentiment induces investors to be more confident in their ability to evaluate investment options and more willing to take risks (Kuhnen and Knutson 2011). Bullish investors tend to perceive the risks as being lower than they actually are, which implies a lower discount rate. Furthermore, a bullish market usually leads to a higher demand for call options, bringing their prices upward, while a bearish market usually leads to a higher demand for put options as insurance policies in case of falling prices.

In order to implement the pricing formulas (5) and (6), we need to specify the discount rate $d$ and the stock return rate $\mu$. It is not simple to find the right parameters, since we need to adjust them with respect to investor sentiment, so that the price from this model corresponds to the market price. In Yang, Gao, and Yang’s approach (2016), the discount rate for call option is specified as $\mu_c e^{\alpha c S_c}$, where $\mu_c$ is the expected return rate of the call option, $S_c$ is the call option sentiment, and $\alpha_c$ is the constant coefficient of the call option sentiment; and analogously for put option. They specify the stock return rate as $\mu_u + \frac{\alpha_u S_u}{T-t}$, where $\mu_u$ is the expected return rate of the stock, $S_u$ is the stock sentiment, and $\alpha_u$ is the constant coefficient of the stock sentiment. In this way, the sentiment input acts as a factor that modifies the discount rate and stock return rate.

There are, however, some difficulties with implementing their approach empirically. Their model has two types of sentiment input – stock sentiment and option sentiment. In practice, it is difficult to separate investor sentiment and find proxies for only stock or only option sentiment. Moreover, after bringing in appropriate sentiment input, both the coefficient of the option sentiment and the coefficient of the stock sentiment have to be calibrated together with empirical data, which is not easy to achieve.

For these reasons, this paper attempts to find a pricing model that is more practical to implement empirically. We see that sentiment can lead to overpricing and underpricing of options in real world markets, compared to their theoretical prices in perfect markets. From this perspective, we can think of sentiment as a factor that inflates or deflates the Black-Scholes option price, contributing to the price discrepancies observed in the market. One mechanism that allows this upward or downward price modification is the option pricing model for dividend paying underlying assets. The intuitive connection is that investor sentiment can be regarded as a belief in future positive or negative dividends, thus deflating or inflating the option price. A stock paying positive dividends loses part of its value in a sense, and its return is expected to be lower than the case without dividends, leading to a lower call option price, which corresponds to the case of bearish sentiment where investors expect lower returns. A stock paying negative dividends, i.e. getting positive dividends, gains more value instead, and its return
is expected to be higher, leading to a higher call option price, which corresponds to the case of bullish sentiment where investors expect higher returns.

In this way, we are modifying the stock return rate with investor sentiment, as if it was dividends, and we use the risk-free interest rate $r$ as the discount rate. The proposed model is thus an adjustment to the standard risk neutral valuation formula in a formal sense and resembles the pricing model for European options on instruments paying continuous dividends.

A sentiment function can be defined as $f(\theta_t)$, where $\theta_t$ is the level of investor sentiment at time $t$, and $f(\cdot)$ depends on the specific form of the sentiment input $\theta_t$. For the sentiment data that will be used in this paper, $f(\theta_t)$ can take the simple form

$$f(\theta_t) = \frac{\alpha_t}{100},$$

where $\alpha_t$ denotes the coefficient that will be calibrated in the model implementation.

The proposed option pricing model with investor sentiment for call and put options has the following variables

- $S_t$: Price of the stock at time $t$
- $T$: Time to maturity
- $K$: Strike price
- $r$: Risk-free interest rate
- $\sigma$: Volatility of returns of the stock
- $\theta_t$: Level of investor sentiment at time $t$
- $\alpha_t$: Calibration coefficient

and the pricing formulas are

$$C(t, S_t) = S_t e^{f(\theta_t)(T-t)}N(d_1) - e^{-r(T-t)K}N(d_2)$$ \hspace{1cm} (7)

$$P(t, S_t) = e^{-r(T-t)K}N(-d_2) - S_t e^{f(\theta_t)(T-t)}N(-d_1)$$ \hspace{1cm} (8)

$$d_1 = \frac{1}{\sigma \sqrt{T-t}} \left[ \ln \left( \frac{S_t}{K} \right) + \left( r + f(\theta_t) + \frac{\sigma^2}{2} \right) (T-t) \right]$$

$$d_2 = d_1 - \sigma \sqrt{T-t}$$

where $N(\cdot)$ denotes the cumulative distribution function of the standard normal distribution.
4. Data and Empirical Results

With the proposed option pricing formulas (7) and (8) for call and put options, the next step is to test them with empirical data to see if there is any pricing improvement. Specifically, we want to know if the prices produced by the new formulas are closer to market option prices than the Black-Scholes model. This section starts with describing the option and sentiment data employed in the analysis and then presents the results from implementing the sentiment pricing model.

4.1 Option Data

Unlike data on historical stock prices, which can be easily downloaded from multiple online resources, one cannot simply gain access to data on historical option prices, except for some inadequate sample data. As a result, the option data used in this paper is limited to options on only one underlying stock – AAPL, for only two trading years 2015 and 2016. The data is purchased from OptionsDataMine.com and contains information including date, closing price of the underlying stock (ParentClose), strike price, option type (call/put), expiration date, opening/highest/lowest/closing price of the option, trading volume, open interest, and bid/ask price. Despite for only one underlying stock and two trading years, the dataset is rather large, with 287,004 entries for 2015 and 249,055 entries for 2016. Below is a snapshot of the beginning of the dataset.

<table>
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<tr>
<th>Date</th>
<th>ParentClose</th>
<th>Open</th>
<th>High</th>
<th>Low</th>
<th>Close</th>
<th>Volume</th>
<th>OpenInterest</th>
<th>bid</th>
<th>ask</th>
<th>strike</th>
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</table>
Each entry represents an option on AAPL with a different set of date, strike price, option type, and expiration date. It begins with call options that are available on Date 2015-01-02 with expDate 2015-01-02 for many different strike prices ranging from 80 to 170, then goes on to put options with the same expiration date for strike prices ranging from 85 to 160; then it shows call options with a different expiration date 2015-01-09 for various strike prices, and so on. For each trading day there are multiple possible expiration dates (thus different maturities), and for each expiration date there is a range of strike prices, for both call and put options, which explains the large size of the option data.

Notice that some of the options have zero trading volumes; see for example the last three rows in the table above. These entries are excluded from the analysis. Occasionally, there exist more than one entry for the same date, strike price, option type, and expiration date. In this case, the entry with the highest volume is selected and used in the analysis.

### 4.2 Sentiment Data

Brown and Cliff (2004) examine numerous sentiment proxies and find strong relations between many disparate measures of investor sentiment. Therefore, this paper makes no attempt at an extensive usage of sentiment proxies. Rather, two sentiment indicators are selected and employed due to their accessibility and availability as daily data.

The first sentiment input is the VIX Index, which is well established and commonly used in finance. It is a 30-day measure of the expected volatility of the S&P 500 Index. Historical data dating back to January 1990 can be readily downloaded at CBOE’s website: [http://www.cboe.com/products/vix-index-volatility/vix-options-and-futures/vix-index/vix-historical-data](http://www.cboe.com/products/vix-index-volatility/vix-options-and-futures/vix-index/vix-historical-data). The dataset contains Date, VIX Open, VIX High, VIX Low, and VIX Close. The daily changes of VIX Close values are calculated and supplemented in the dataset as VIX Diff to be used as potential input in the analysis. Below shows the data for January 2015. Note that the VIX is quoted in percentage points and as an annualized standard deviation. As the “fear index”, a higher VIX value implies a more bearish and volatile market.

<table>
<thead>
<tr>
<th>Date</th>
<th>VIX Open</th>
<th>VIX High</th>
<th>VIX Low</th>
<th>VIX Close</th>
<th>VIX Diff</th>
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<tbody>
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<td>17.05</td>
<td>17.79</td>
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<td>19.19</td>
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</tr>
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</tr>
<tr>
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<td>17.01</td>
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<tr>
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<td>18.02</td>
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The second sentiment input is the ISEE Index, as it offers a unique perspective on investor sentiment due to its specific construction. Historical ISEE data dating back to April 2002 can be directly downloaded at http://www.ise.com/market-data/isee-index. The dataset contains ISEE Index values for all securities, their 10-day, 20-day, and 50-day moving averages, ISEE Index values for all equity options only, and ISEE Index values for all index options and ETF (exchange-traded fund) options only. Below shows the data for January 2015. A higher ISEE value represents a more bullish sentiment, since long positions of call options are viewed as a bullish transaction.

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<th>20 Avg</th>
<th>50 Avg</th>
<th>Equity</th>
<th>Index</th>
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</tbody>
</table>

Direct measures of investor sentiment such as the AAII Investor Sentiment Survey and the Investors Intelligence Sentiment Survey are also attractive candidates for sentiment input in the analysis, but they are weekly data, whereas daily data is preferred here considering the limited 2-year data on options.

4.3 Empirical Results

The proposed sentiment pricing model has the following parameters: $S_t, T, K, r, \sigma, \theta_t, \alpha_t$, among which $S_t, T, K$ are explicitly known. Regarding the risk-free interest rate $r$, we follow the convention and use the 3-month US Treasury bill rate. Historical data on US Treasury bill rates can be downloaded for example at https://www.quandl.com/data/USTREASURY/BILLRATES-Treasury-Bill-Rates. The 3-month Treasury bill rates are 0.02% and 0.22% respectively for the beginning of year 2015 and 2016, so we apply $r = 0.0002$ and $r = 0.0022$ as the risk-free interest rates for these two years.
When it comes to the volatility parameter $\sigma$, it is no easy task to obtain a good estimate of a stock’s volatility. Many studies discuss the different methods of volatility estimation (e.g. Rogers, Satchell, and Yoon 1994; Elliott, Hunter, and Jamieson 1998; Curto, Pinto, and Tavares 2007; Alberg, Shalit, and Yosef 2008; Ho, Lee, and Marsden 2011; Rotkowski 2011; Dokuchaev 2014). Since volatility is not the main focus in this paper, we are content with the simplest approach of estimating volatility from historical stock price data (Björk 2009). That is, we first calculate the daily return rates of the stock prices with the formula (Closing price today – Closing price yesterday)/Closing price yesterday, and then calculate the standard deviation of these daily return rates, and finally annualize it by multiplying the daily standard deviation with $\sqrt{252}$ in order to get the annual volatility. With this method, the volatility estimates are 0.2167 for the AAPL stock prices in 2014 and 0.2677 for the AAPL stock prices in 2015.

The quantity $\theta_t$ is the sentiment input, which can take many forms, including VIX Close values, VIX Diff values, ISEE All Securities values, various ISEE Avg values, and other reasonable options, as well as transformations of these values. We will experiment with different forms of sentiment input and compare the results below.

The parameter $\alpha_t$ is determined in the calibration process and does not need to be specified beforehand. Using the pricing formulas (7) and (8) together with sentiment input, we can calculate the value of $\alpha_t$ that fits the market option price at time $t$ and utilize this value for price predictions at a later time. In particular, we use the 5-day moving averages of $\alpha_t$ to predict option prices 5 days ahead. The first moving average is calculated with the values of $\alpha_t$ for the first five trading days and is used to predict the option price for the 10th trading day, and so on.

We might expect the option data to be in such a way that we can pick out the options with the same strike price and maturity, but different underlying stock prices. However, we notice that for each trading day, there are certain expiration dates, which continue to apply for later trading days, until an expiration date is reached and potential new expiration dates enter. In other words, options starting at different dates have the same expiration dates, thus different maturities. So we are not able to analyze the prices of an option with a fixed strike price and maturity over consecutive days, but we can consider the options having the same expiration date to be the course of one option and look at the price development of this option.

Now that we have all the ingredients, we can do some experiments with empirical data, in order to test the sentiment pricing model. Since options that are deep in the money or deep out of the money might possess different properties than at the money options, we examine these three cases separately.

**Case I: At the Money Call and Put Options**

We start with call options that are at or near the money. Since AAPL’s closing price on 2015-01-02 is 109.33, we choose an arbitrary strike price of 110 and experiment first with a maturity of about half a year. The only expiration date around half a year apart is 2015-07-17. We extract entries with these properties from the option dataset and visualize the price development in Figure 1. The $x$-axis denotes the trading days, as in the first trading day during the period and so on. The $y$-axis denotes the price in US dollars. We see that the option price roughly follows the stock price movement, just as expected.
Figure 1 shows the price development of AAPL (in blue) and the call option on AAPL with strike price 110 and expiration date 2015-07-17 (in black), starting from 2015-01-02.

Let us first try with the daily ISEE values for all securities as sentiment input. Figure 2 plots the obtained option prices from the sentiment pricing model (in magenta) together with the Black-Scholes prices (in cyan), the market option prices (in black), the market stock prices (in blue), and the daily ISEE values (in orange).

Figure 2 shows the price development of AAPL (in blue) and the call option on AAPL with strike price 110 and expiration date 2015-07-17 (in black), the ISEE daily values (in orange), the option prices from the sentiment pricing model (in magenta), and the Black-Scholes prices (in cyan), starting from 2015-01-02.
We see that the daily ISEE values are very volatile and can vary a lot even after one day. Intuitively, investor sentiment, as investors’ attitude and beliefs, is more like a trend that develops with time and thus should not fluctuate too rapidly and heavily. In consideration of this aspect, it is reasonable to put the daily ISEE values aside and test the moving averages of the ISEE values, since they represent investors’ sentiment over a longer period. With the 10-day moving averages of ISEE as sentiment input, we get the results in Figure 3. A closer look at the option prices is given in Figure 4.

Figure 3 shows the price development of AAPL (in blue) and the call option on AAPL with strike price 110 and expiration date 2015-07-17 (in black), the ISEE 10-day moving averages (in orange), the option prices from the sentiment pricing model (in magenta), and the Black-Scholes prices (in cyan), starting from 2015-01-02.

Figure 4 shows the price development of the call option on AAPL with strike price 110 and expiration date 2015-07-17 (in black), the option prices from the sentiment pricing model with ISEE 10-day moving averages as sentiment input (in magenta), and the Black-Scholes prices (in cyan), starting from 2015-01-02.
We notice that many prices from the sentiment model lie closer to the market prices than the Black-Scholes model. In fact, 79.2% of the sentiment prices have lower pricing errors than the Black-Scholes prices. The root mean square error (RMSE), calculated with the formula

\[ \text{RMSE} = \sqrt{\frac{\sum (\text{Predicted Price} - \text{Market Price})^2}{\text{Number of Predictions}}} \]

is approximately 0.60 for the sentiment model and 0.96 for the Black-Scholes model. Analogously, we obtain 84.8% better predicted prices with a RMSE of 0.59 for ISEE 20-day moving averages, and 86.4% with a RMSE of 0.53 for ISEE 50-day moving averages. In other words, we attain obvious pricing improvements with the sentiment model. Note that the first nine trading days are excluded from the pricing error calculation, and the same applies throughout the analysis.

We have tried different forms of ISEE Index values as sentiment input; let us now apply VIX Close and VIX Diff values. VIX Close gives 85.6% better prices and a RMSE of 0.49, whereas VIX Diff generates worse results than the Black-Scholes model, possibly due to the frequent changes of signs.

Now that we see the sentiment pricing model is able to produce option prices that are more consistent with market prices, for this particular call option, a natural next-step question is whether the same applies to at the money call options with other maturities. After testing for many different expiration dates with the data at hand, we find that the sentiment model tends to perform better for at the money call options with longer maturities, usually more than half a year. We show one example with a call option that has strike 100 and expiration date 2016-04-15, starting from the first trading day in 2016, i.e. 2016-01-04. Figure 5 displays the pricing results with VIX Close values as sentiment input, and Figure 6 gives a closer look at the option prices. For this call option, only 47.5% of the prices from the sentiment model have lower pricing errors than the Black-Scholes model. The RMSE is 0.34, compared to 0.39 for the Black-Scholes model.

![Figure 5. Call options with strike 100 and expDate 2016-04-15 (VIX Close)](image)

Figure 5 shows the price development of AAPL (in blue) and the call option on AAPL with strike price 100 and expiration date 2016-04-15 (in black), the VIX Close values (in orange), the option prices from the sentiment pricing model (in magenta), and the Black-Scholes prices (in cyan), starting from 2016-01-04.
Figure 6 shows the price development of the call option on AAPL with strike price 100 and expiration date 2016-04-15 (in black), the option prices from the sentiment pricing model with VIX Close values as sentiment input (in magenta), and the Black-Scholes prices (in cyan), starting from 2016-01-04.

Let us now investigate how the sentiment pricing model performs for at the money put options. We select an at the money put option with strike price 110 and expiration date 2015-07-17 and implement the different types of sentiment input as above. VIX Close values give 100% better predicted prices with a RMSE of 0.26, while the RMSE of the Black-Scholes model is 1.21 for comparison. ISEE 10-day, 20-day, and 50-day moving averages all give 100% better prices as well, with RMSEs of 0.24, 0.19, and 0.20 respectively. Figure 7 presents the results for ISEE 20-day moving averages as sentiment input. We observe that the Black-Scholes prices are on a noticeably lower level than the market prices, but the sentiment model is able to produce prices that are generally consistent with market prices.

Again, we test the model on at the money put options with different maturities, and we generally see better pricing results for options with longer maturities, although the performance varies for different options. We show another example of a put option with strike 100 and expiration date 2016-10-21, starting from 2016-01-04. The pricing results with ISEE 50-day moving averages as sentiment input are found in Figure 8. We get 69.8% better predicted prices, with a RMSE of 0.39 compared to 0.77 for the Black-Scholes model. In this case, the pricing improvements are not as evident as in Figure 7, but the results are still very positive.
Figure 7 shows the price development of the put option on AAPL with strike price 110 and expiration date 2015-07-17 (in black), the option prices from the sentiment pricing model with ISEE 20-day moving averages as sentiment input (in magenta), and the Black-Scholes prices (in cyan), starting from 2015-01-02.

Figure 8 shows the price development of the put option on AAPL with strike price 100 and expiration date 2016-10-21 (in black), the option prices from the sentiment pricing model with ISEE 50-day moving averages as sentiment input (in magenta), and the Black-Scholes prices (in cyan), starting from 2016-01-04.
Case II: In the Money Call and Put Options

Now that we have seen promising pricing results with regard to at or near the money call and put options, we want to know whether the sentiment pricing model also works well with in the money and out of the money options. The situation is somewhat different for options that are not at or near the money, because systematic differences between the Black-Scholes prices and market prices have been documented through the years for these options. Black (1975) finds that deep in the money (out of the money) options generally have higher (lower) Black-Scholes prices than market prices. Merton (1976) states that practitioners observe Black-Scholes prices to be lower than market prices for both deep in the money and deep out of the money options. Macbeth and Merville (1979) report that the Black-Scholes model predicted prices are on average lower (higher) than market prices for in the money (out of the money) call options, which is the exact opposite of Black’s statement. With the limited option data at hand, we cannot draw any conclusions regarding the direction of the price discrepancy. Nonetheless, the data for AAPL call and put options in 2015 and 2016 is consistent with Macbeth and Merville’s results, i.e. the Black-Scholes prices are lower (higher) than market prices for deep in the money (out of the money) options.

We choose an arbitrary deep in the money call option with strike price 80 and expiration date 2016-10-21 starting from 2016-01-04. Following the same routine as above, we try with different sentiment input and document the results. We get 100% better prices with all types of sentiment input, with RMSEs of 4.52, 4.47, 4.08, and 3.90 respectively for VIX Close, ISEE 10-day, 20-day, and 50-day moving averages. The RMSE with the Black-Scholes model is 15.54 for comparison. Figure 9 plots the pricing results with ISEE 50-day moving averages as sentiment input.

![Figure 9. Call options with strike 80 and expDate 2016-10-21 (ISEE 50 Avg)](image)

Figure 9 shows the price development of AAPL (in blue) and the call option on AAPL with strike price 80 and expiration date 2016-10-21 (in black), the ISEE 50-day moving averages (in orange), the option prices from the sentiment pricing model (in magenta), and the Black-Scholes prices (in cyan), starting from 2016-01-04.

We notice immediately the striking differences between the Black-Scholes prices and the market prices throughout the course. There are many potential explanations for this phenomenon, but this is outside the purpose of this paper. We want to investigate how well the sentiment pricing model performs with
empirical data, and for this purpose we consider the pricing results from the sentiment model to be positive. The sentiment prices are, however, not as close to the market prices as the case for at the money call options, which can be partially due to suboptimal calibration method.

For in the money put options, we choose an arbitrary put option with strike price 125 and expiration date 2016-07-15, starting from 2016-01-04. All sentiment input gives 100% better pricing results, with RMSEs of 7.46, 7.19, 6.74, and 6.73 respectively for VIX Close, ISEE 10-day, 20-day, and 50-day moving averages. The RMSE with the Black-Scholes model is 21.41 for comparison. Figure 10 shows the results with ISEE 50-day moving averages as sentiment input.

Figure 10 shows the price development of AAPL (in blue) and the put option on AAPL with strike price 125 and expiration date 2016-07-15 (in black), the ISEE 50-day moving averages (in orange), the option prices from the sentiment pricing model (in magenta), and the Black-Scholes prices (in cyan), starting from 2016-01-04.

We see directly that the Black-Scholes prices are much lower than the market prices, which is the same as the case for deep in the money call options. Again, the sentiment pricing model does not perform as well as for at the money put options, but it offers certain pricing improvements. After testing for different maturities for both deep in the money call and deep in the money put options, we have the same conclusion as for at the money call and put options, i.e. the sentiment pricing model works better for options with longer maturities.

**Case III: Out of the Money Call and Put Options**

At last, we present the results for out of the money call and put options. We select an arbitrary out of the money call option with strike price 125 and expiration date 2016-07-15, starting from 2016-01-04. All sentiment input gives 100% better pricing results, with RMSEs of 0.49, 0.30, 0.31, and 0.25 respectively for VIX Close, ISEE 10-day, 20-day, and 50-day moving averages. The RMSE with the Black-Scholes model is 5.79 for comparison. Figure 11 displays the results with ISEE 50-day moving averages as sentiment input, and Figure 12 offers a closer look.
Figure 11 shows the price development of AAPL (in blue) and the call option on AAPL with strike price 125 and expiration date 2016-07-15 (in black), the ISEE 50-day moving averages (in orange), the option prices from the sentiment pricing model (in magenta), and the Black-Scholes prices (in cyan), starting from 2016-01-04.

Figure 12 shows the price development of the call option on AAPL with strike price 125 and expiration date 2016-07-15 (in black), the option prices from the sentiment pricing model with ISEE 50-day moving averages as sentiment input (in magenta), and the Black-Scholes prices (in cyan), starting from 2016-01-04.

We observe again the large differences between the Black-Scholes prices and the market prices for this out of the money call option, but the Black-Scholes prices are much higher instead. The sentiment pricing model is again able to generate option prices that are much more consistent with market prices.
Finally, we select an arbitrary out of the money put option with strike price 80 and expiration date 2016-07-15, starting from 2016-01-04. All sentiment input gives 100% better pricing results, with RMSEs of 0.81, 0.71, 0.64, and 0.67 respectively for VIX Close, ISEE 10-day, 20-day, and 50-day moving averages. The RMSE with the Black-Scholes model is 5.37 for comparison. Figure 13 presents the results with ISEE 20-day moving averages as sentiment input, and Figure 14 gives a closer look.

Figure 13 shows the price development of AAPL (in blue) and the put option on AAPL with strike price 80 and expiration date 2016-07-15 (in black), the ISEE 20-day moving averages (in orange), the option prices from the sentiment pricing model (in magenta), and the Black-Scholes prices (in cyan), starting from 2016-01-04.

Figure 14 shows the price development of the put option on AAPL with strike price 80 and expiration date 2016-07-15 (in black), the option prices from the sentiment pricing model with ISEE 20-day moving averages (in magenta), and the Black-Scholes prices (in cyan), starting from 2016-01-04.
We see that for both of these out of the money options, the market prices are much lower and less volatile than the Black-Scholes prices. The sentiment pricing model provides evident pricing improvements in both cases, and we find similar patterns again for out of the money options with other maturities, i.e. the sentiment pricing model performs better for options with longer maturities.

5. LIMITATIONS

Most of the traded options written on stocks, as compared to indices, are of the American style, and most stocks issued by large companies pay discrete dividends to the shareholders, including AAPL. For an American call option with dividends, it can be optimal to exercise immediately prior to an ex-dividend date (Hull 2012). Despite this difference between American options and European options, we have used the option prices from the Black-Scholes formula to compare with the sentiment prices throughout the analysis, because the Black-Scholes prices can still serve as a good reference.

AAPL pays quarterly dividends in the form of cash in February, May, August, and November each year. Historical records are listed at [http://investor.apple.com/dividends.cfm](http://investor.apple.com/dividends.cfm). Annual dividend yields can be found from multiple online resources. For instance, the yield rates are 1.6% on the first ex-dividend date February 5th in 2015 and 2.2% on February 4th in 2016, according to StreetInsider.com. These annual dividend yields can be used in the extended Black-Scholes formula for instruments paying dividends, and the resulting option prices will be slightly lower than the ones obtained in the analysis. This, however, does not affect the general results of this paper.

Although the option dataset contains over 536 000 different option entries, the data is still limited to only one stock and two trading years. Apple is the world’s largest company in terms of market capitalization, which stands at $806.76 billion as of May 24, 2017 according to Google Finance. Options on other underlying stocks might show different patterns than Apple, thus the sentiment pricing model might perform differently as well. Optimally, we would also like to have sentiment input for individual securities and specifically for AAPL in our case. But the available sentiment data is unfortunately not directed at particular securities.

In the empirical implementation, we have used the 5-day moving averages of \( \alpha \) to predict option prices 5 days ahead. This is an arbitrary choice, although 5 days correspond to one trading week. Since the goal is to find an option pricing model that works well with empirical data, we could try out different sets of parameters for predictions, i.e. to use the \( n \)-day moving averages of \( \alpha \) to predict option prices \( m \) days ahead, and compare the resulting prediction errors with the help of RMSE. This is, however, not carried out due to the scope of this paper.
6. CONCLUSION

Going from the complete Black-Scholes market to the incomplete real world market, options become nonredundant assets to some extent, and their prices are subject to the demand and supply forces as well as limited arbitrage considerations (Figlewski 1989; Amin, Coval, and Seyhun 2004). Factors like investor sentiment can thus come into play and affect option prices. Empirical research has also confirmed investors’ irrational behaviors in the financial market and the correlation between various sentiment proxies and option prices.

Behavioral finance is built on the two tenets of investor sentiment and limited arbitrage. This paper adopts the principles of behavioral finance and aims to incorporate investor sentiment in the option pricing model. The Black-Scholes model produces theoretical European option prices in perfect markets, which are not always consistent with market option prices. The goal of this paper is to develop option pricing formulas with investor sentiment for call and put options, such that they are more consistent with market prices.

The proposed sentiment pricing model is an adjustment to the standard risk neutral valuation formula in a formal sense and resembles the pricing model for European options on instruments paying continuous dividends. Investor sentiment can be regarded as a belief in future positive or negative dividends, thus deflating or inflating the option price. In this way, the stock return rate is modified with investor sentiment, as if it was dividends.

The sentiment pricing model is then tested empirically with historical data of all the options written on AAPL during the trading years 2015 and 2016, together with the VIX and ISEE Index values as sentiment input. Pricing improvements are achieved with the sentiment pricing formulas for most options in the dataset, with the exceptions being some at the money options with a shorter maturity (less than half a year). For deep in the money and deep out of the money call and put options, there exist striking systematic differences between the Black-Scholes prices and the market prices. But the sentiment pricing model is able to generate option prices that are more consistent with market prices for these options, although it does not perform as well as for at the money options. In general, the sentiment pricing model works better for options with longer maturities.

The sentiment input provides great potential for option pricing improvements. Investor sentiment plays an undeniable role in the financial market, and future research should further develop option pricing models incorporated with investor sentiment.
REFERENCES


