Removing cusps from Legendrian front projections

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Abstract

We show that it is possible to isotope certain Legendrian knots of rotation number zero inside the unit cotangent bundle of the plane, i.e. $\mathbb{R}^2 \times S^1$, so that the front projection becomes an immersion. The result is shown by hands-on techniques involving sequences of Reidemeister moves.
Contents

1 Introduction ........................................... 3
   1.1 History and applications .......................... 3
   1.2 Safe regular homotopy and Legendrian isotopy ......... 3
   1.3 Main result ......................................... 3

2 Contact Geometry and Legendrian knot theory preliminaries 5
   2.1 Contact geometry preliminaries ...................... 6
      2.1.1 Contact structures .............................. 6
      2.1.2 Contact structures on $\mathbb{R}^3, \mathbb{R}^2 \times S^1$ and $J^1(S^1)$ ......... 6
      2.1.3 Contactomorphisms ............................. 7
   2.2 Legendrian knot theory preliminaries ................. 7
      2.2.1 Legendrian knots ............................... 7
      2.2.2 Knot diagrams for Legendrian knots ............. 8
      2.2.3 The standard projections in $(\mathbb{R}^3, \xi)$ .......... 8
      2.2.4 The standard projections in $(\mathbb{R}^2 \times S^1, \xi)$ and $(J^1(S^1), \xi)$ ....... 9
   2.3 The projection $\rho : \mathbb{R}^2 \times S^1 \to \mathbb{R}^2$ .............. 10
   2.4 Drawing knot diagrams, Legendrian isotopy and invariants .. 11
      2.4.1 Legendrian isotopy and the Reidemeister moves .... 11
      2.4.2 The Reidemeister moves .......................... 12
      2.4.3 The additional move $R - II^*$ .................... 12
      2.4.4 The classical invariants ....................... 13

3 Obtaining immersions isotopic to front projections of Legendrian knots via $\mathbb{R}^2 \times S^1$ 15
   3.1 Obstructions to removing cusps in $\mathbb{R}^2 \times S^1$ .......... 16
      3.1.1 Proof of (M1) in Theorem 1 ...................... 16
      3.1.2 Proof of (M2) in Theorem 1 ...................... 17
      3.1.3 Proof of (M3) in Theorem 1 ...................... 18

4 Hands-on examples .................................... 22

5 Proof of Theorem 2 and more ................................ 23
1 Introduction

1.1 History and applications

Legendrian knot theory is a fascinating area of study, and still fairly poorly understood. The theory of Legendrian knots is in essence a synthesis of contact geometry and knot theory. Contact geometry has its origins in the 17th century with Huygens, and later Hamilton and Jacobi in the 19th century [6]. The synthesis of this work into the field of Legendrian knot theory, however, is a fairly recent development starting in the second half of the 20th century [2]. It has many applications in physics including optics, dynamics and cutting edge theoretical work. In the field of mathematics, the understanding of Legendrian knots is pertinent to current work on symplectic manifolds and the theory of smooth knots. Weinstein manifolds, an important example of open symplectic manifolds, can be presented as successive handle attachments along Legendrian spheres [9]. Legendrian knot theory relates to smooth knots in that the conormal lift of any submanifold is a Legendrian manifold of the unit cotangent bundle. Legendrian knot theory has also been shown to be a rich and interesting theory in its own right.

1.2 Safe regular homotopy and Legendrian isotopy

Some headway has of course been made: Arnold showed that safe regular homotopy is a notion related to Legendrian knots (up to Legendrian isotopy), as the conormal lift of a generic immersion is a Legendrian knot, and a safe homotopy lifts to Legendrian isotopy [10][3] - safe homotopy being regular homotopy with no "dangerous" self-tangencies. A dangerous self tangency is a self intersection where the tangents at the point of intersection point in the same direction. Work by Ng has used made use of modern Legendrian invariants to study classes of immersions [7]. The results in this paper can be taken as an indication that suggests that the connection between immersions and Legendrian knots are deeper than expected. It is, however, nevertheless important to draw a distinction between the relations of Legendrian isotopy and safe regular homotopy - they are not necessarily the same (see Figure 1).

The question is also meaningful and interesting in higher dimensions, even though we will restrict ourselves to 3-dimensional spaces in this paper. As the types of singularities increase, one might imagine the problem to become significantly more difficult. However, recent work shows that many singularities of higher codimension can in fact be removed by Legendrian isotopy[1][5].

1.3 Main result

In this paper, we will prove the main Theorem below, along with some consequent results and corollaries.

**Theorem 1.** For a Legendrian knot \( L \subset \mathbb{R}^2 \times S^1 \) there are Legendrian isotopies that realize the following local deformations of the canonical projection of the Legendrian knot to \( \mathbb{R}^2 \), i.e. its front projection.

- M-1: As specified in Lemma 2. A cusp adjacent to a loop may "move past" that loop. See Figure 2.
Figure 1: Turning a cuspless unknot inside out by a Legendrian isotopy inside $\mathbb{R}^2 \times S^1$. Note that the isotopy changes the orientation of the circle shown on the left compared to the same circle shown on the right. Even though there is a Legendrian isotopy, it must always go through Legendrians with cusps, since there is not even a regular homotopy due to the difference in tangent winding number of the circle with its two different orientations.

- **M-2**: As specified in Lemma 3. A solitary "outwards" facing cusp trapped by a writhing tangle with $n$ crossings can be isotoped to a free cusp and $n$ loops. See Figure 3

- **M-3**: As specified in Lemma 4. Two sister cusps connected by a segment $A$ such that segments intersecting $A$ does not allow the cusps to meet, can be isotoped with one that does allow it. See Figure 4

Figure 2: An example of the property (M1)

In section 3 we prove this result by realizing these deformations using sequences of Reidemeister moves.

We proceed to demonstrate an immediate application of the move M-2. The Legendrian unknot with $\text{rot}(L) = 0$ can be seen to be Legendrian isotopic inside $\mathbb{R}^2 \times S^1$ to Legendrian unknots with $\text{rot}(L) = 0$, and $\text{tb}(L) = -1 - 2k$. The top left part of Figure 30 illustrates the case where $k = 1$, the twice stabilized unknot. The general case has $2k + 1$ cusps to the left and to the right, but we shall use the Legendrian isotopic case where the knot has one cusp to each side and $2k$ crossings in between, as in the figure below.

Another family of Legendrian knots which can be Legendrian isotoped to have an immersed front in $\mathbb{R}^2 \times S^1$ is the standard $(2, n)$ torus knots as in Figure 28.
Note that the above examples all have rotation number zero. This is not a coincidence, but in fact a necessary condition for removing the cusps of a front projection by Legendrian isotopy.

**Theorem 2.** If a Legendrian knot \( L \subset \mathbb{R}^2 \times S^1 \) is Legendrian isotopic to a knot \( L' \subset \mathbb{R}^2 \times S^1 \) whose front projection \( \rho(L') \) is an immersion, then \( L \) has \( \text{rot}(L) = 0 \).

We will give the proof in section 5.

A natural further question is whether having \( \text{rot}(L) = 0 \) is not only necessary, but in fact sufficient for the existence of a Legendrian isotopy to a knot with an immersed front projection. Our result can be taken as partial evidence towards this.

### 2 Contact Geometry and Legendrian knot theory preliminaries

While in this paper, we will restrict ourselves to Legendrian one-dimensional knots inside three-dimensional contact manifolds, Legendrian knot theory is an interesting subject in any \( 2n + 1 \) dimensional contact manifold.
2.1 Contact geometry preliminaries

2.1.1 Contact structures

Let $M$ be an orientable 3-manifold, and $p \in M$ be a point. Denote by $\xi_p$ a tangent hyperplane of $M$ inside $T_pM$. A plane field $\xi$ can be defined as field of tangent hyperplanes, so that at every $p \in M$, $\xi = \xi_p$. We can visualize this as a "grid" of small plane elements, as in the example of Figure 2. Close to each $p$, $\xi$ can be described by the kernel of a locally defined 1-form $\alpha$ on $M$, so that $\xi_p = \ker(\alpha_p)$ and $\xi = \ker(\alpha)$.

A contact structure on $M$ is such a $\xi$, under the condition that $\alpha \wedge d\alpha \neq 0$ for any contact form $\alpha$ which describes $\xi$ in the prescribed way. The pair $(M, \xi)$ is then a contact manifold. Any given manifold is compatible with several different contact structures. In general, there is no natural choice of contact form.

2.1.2 Contact structures on $\mathbb{R}^3$, $\mathbb{R}^2 \times S^1$ and $J^1(S^1)$

We will mainly be working in the unit cotangent bundle of $\mathbb{R}^3$, i.e. $\mathbb{R}^2 \times S^1$, and $J^1(S^1)$ equipped with the standard contact structure (as well as the standard contact vector space $\mathbb{R}^3$). The latter two contact manifolds are in fact contactomorphic (see 2.1.3). We will occasionally include examples of $\mathbb{R}^3$ for context.

The contact structure $\xi_{\text{std}}$ on $\mathbb{R}^3$, called the standard contact structure, is given by $\ker(\alpha) = \text{span} \left\{ \frac{\partial}{\partial y}, \frac{\partial}{\partial x} + y \frac{\partial}{\partial z} \right\}$ where $\alpha = dz - ydx$.

The contact structure we will be using to form the contact manifold $(\mathbb{R}^2 \times S^1, \xi)$ is given by $\ker(\alpha) = \text{span} \left\{ \frac{\partial}{\partial \theta}, \sin(\theta) \frac{\partial}{\partial x} - \cos(\theta) \frac{\partial}{\partial y} \right\}$, where $\alpha = \cos(\theta)dx + \sin(\theta)dy$.

In the so-called 1-jet space $J^1(S^1)$, we shall make use of the contact form $\alpha = dz - p d\theta$, and the contact structure is thus given by $\ker(\alpha) = \text{span} \left\{ \frac{\partial}{\partial \theta}, p \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right\}$.

As we will see, there is a close relationship between $\xi_{J^1(S^1)}$ and $\xi_{\mathbb{R}^2 \times S^1}$. They are in fact the same up to contactomorphism (see 2.1.3).

**Note 1.** We will use $\xi$ to denote the contact structure for a number of manifolds. This does not mean it is the same contact structure, rather it is a way to make the notation easier to read. When I write e.g. $(\mathbb{R}^2 \times S^1, \xi)$, it is implicit that $\xi$ denotes the relevant contact structure as described in this section. Where confusion might happen, I have tried to include subscripts to clarify, e.g. $\xi_{\mathbb{R}^2 \times S^1}$.

![Figure 6: The standard contact structure $\xi_{\text{std}}$ on $\mathbb{R}^3$. (Released into the public domain by an anonymous source.)](image)
2.1.3 Contactomorphisms

**Definition 1.** A bijective differentiable map between two contact manifolds with a differentiable inverse which preserves the contact structure (and in particular, preserves the contact elements) is called a contactomorphism \( \Psi : (M^3_1, \xi_{M_1}) \to (M^3_2, \xi_{M_2}) \).

Intuitively, one might say it changes nothing intrinsic to the manifold or the contact structure, but merely shifts the way we describe the contact elements.

**A useful particular case** In particular, our work will make quite some use of the contactomorphism \( \Psi : \mathbb{R}^2 \times S^1 \to \mathcal{J}^1(S^1) : (x, y, \theta) \mapsto (\vartheta, p, z) \) given by

\[
\Psi((x, y, \theta)) = (\theta, -x \sin(\theta) + y \cos(\theta), x \cos(\theta) + y \sin(\theta))
\]

We will also be making implicit use of its inverse\(^1\) which is of course also a contactomorphism.

The contact form \( \alpha_{\mathcal{J}^1(S^1)} \) maps to \( \alpha_{\mathbb{R}^2 \times S^1} = \cos(\theta) dx + \sin(\theta) dy \) under \( \Psi^{-1} \), meaning that the contactomorphism is even contact form preserving, even though we will not need that fact.

2.2 Legendrian knot theory preliminaries

Here we present some background on Legendrian knots. We refer to [6] for more details. We begin by recalling the definition of a smooth knot.

**Definition 2.** A knot \( K \) is an embedding of \( S^1 \) into a 3-manifold \( M \). \( K \) is given an orientation by its parametrization, and so, we may choose one. We choose a parametrization \( \eta : S^1 \to \mathbb{R}^2 \times S^1 \) such that \( \eta' \neq 0 \) on \( S^1 \).

We will assume all knots to have an orientation from now on.

2.2.1 Legendrian knots

**Definition 3.** A Legendrian knot in a 3-dimensional contact manifold \( L \subset (M^3, \xi) \) is a knot \( L \) which is everywhere tangent to \( \xi \).

The following Lemma implies that the dimension of a submanifold which is everywhere tangent to the contact distribution is at most one.

**Lemma 1.** There exists no embedded surface \( \Sigma \subset M^3 \) that is everywhere tangent to a contact structure \( \xi \).\(^2\)

**Proof.** Assume there exists \( \Sigma \) a surface tangent to \( \xi \), and \( \varphi : M^2 \to M^3 \) an embedding of \( \Sigma \) in \( (M^3, \xi) \). Thus, \( \varphi \) must be of full rank. We have \( \text{im}(d\varphi) \subseteq \ker(\alpha) \) by assumption. Thus, making use of the pullback

\[
0 = d\varphi^*\alpha = \varphi^*d\alpha = d\alpha \circ d\varphi \Rightarrow d\alpha|_\xi \equiv 0
\]

\(^1\)As \( \Psi \) is bijective, it has a (rather messy) inverse

\[
\Psi^{-1}(\vartheta, p, z) = \left( \frac{z - p \tan(\vartheta)}{\cos(\vartheta) + \sin(\vartheta) \tan(\vartheta)}, \frac{p}{\cos(\vartheta)} + \frac{z - p \tan(\vartheta)}{\cos(\vartheta) + \sin(\vartheta) \tan(\vartheta)} \tan(\vartheta), \vartheta \right)
\]

which is included explicitly to spare interested readers the rather boring calculation.

\(^2\)This is a less general incarnation of the Frobenius theorem on integrability.
But since $\alpha \in \Omega^1(M^3)$ is a contact form, $d\alpha|_\xi$ must meet the non-degeneracy criterion of contact forms $d\alpha(x, y) = 0 \forall y \in \xi \Rightarrow x \equiv 0$. Thus $\text{rank}(d\alpha) \leq 1 < \text{rank}(d\varphi)$, which is a contradiction.

### 2.2.2 Knot diagrams for Legendrian knots

A knot diagram is a way to encode the knot into a two-dimensional picture. Typically, it arises as the image of the knot under a projection map, possibly with some extra information. In $\mathbb{R}^3$, that extra information is which strand is on top in a given crossing. In the manifolds under consideration in this paper, there are a number of natural projections, which capture the Legendrian characteristics of the knot. Those natural projections will be specified in the 2.2.4-6. Knot diagrams are useful, since they allow us to reduce the study of knots to combinatorics. Later we will see how to encode the relation of isotopy in the knot diagrams as Reidemeister moves.

There are several projections that are natural to consider in connection with Legendrian knots in the above contact manifolds. The projections themselves are not intrinsic to the contact structure; rather they are convenient, albeit arbitrary choices. However, there are two projections that are classical: the front projection $\Pi$ and the Lagrangian projection $\pi$.

For explanatory reasons, we will want in the following discussion to have a parametrization of a general Legendrian knot to refer to. To that end, let

$$\eta : S^1 \to \mathbb{R}^3 : \theta \mapsto (x(\theta), y(\theta), z(\theta))$$

be a Legendrian knot in the following.

### 2.2.3 The standard projections in $(\mathbb{R}^3, \xi)$

We begin by looking at the projections in $\mathbb{R}^3$ and some of their properties to gain some understanding.

#### The front projection in $\mathbb{R}^3$

Let $\Pi : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \mapsto (x, z)$. Then the image of $L$ under $\Pi$ is called the front projection of $L$. One of the more visually apparent properties of Legendrian knots under the front projection is the cusps. So how do these come about? As $L$ must be tangent to $\xi = \ker(\alpha)$, we know that along $L$, $z'(\theta) - y(\theta)x'(\theta) = 0$ must hold true. Hence, $x'(\theta) = 0 \Rightarrow z'(\theta) = 0$. While the Lagrangian projection (below) is always an immersion, the front projection certainly is not. Assuming $\eta : S^1 \to \mathbb{R}^3$ as above, and $L \subset (\mathbb{R}^3, \xi_{\text{std}})$, consider $\Pi \circ \eta$. An immersion of $S^1$ in $\mathbb{R}^2$ must have vertical tangencies. Now consider the contact form $\alpha$; as we traverse $L$, $z'(\theta) - y(\theta)x'(\theta) = 0$. Thus, $z'(\theta) = 0 \Leftrightarrow x'(\theta) = 0$ holds at a vertical tangency, which is prohibited: $\Pi \circ \eta$ cannot be an immersion at such a point. For the front projection, a useful feature is that the equation $\frac{z'(\theta)}{x'(\theta)} = y(\theta)$ is satisfied, which means we do not need the information of which strand is on top, we can always recover that information from the slope of the diagram. For a generic knot, the projection has cubical cusp singularities (see Figure 7).

#### The Lagrangian projection in $\mathbb{R}^3$

Define $\pi : \mathbb{R}^3 \to \mathbb{R}^2 : (x, y, z) \mapsto (x, y)$. The image of $L$ under $\pi$ is called the Lagrangian projection of $L$. Under this
projection, the obtained diagram is always an immersion of the knot. To recover
the z-coordinate from the projection, we will need to integrate:
\[
z(\theta) = \int_L y(\theta)x'(\theta)d\theta = \int_L ydx
\]
This of course means we lose some precision; we can only recover the z-coordinate
up to a constant, so we only recover the knot up to a translation of the z-
coordinate. By Stokes’ Theorem/Green’s theorem, the total signed area of \(\pi(L)\)
is always zero.

2.2.4 The standard projections in \((\mathbb{R}^2 \times S^1, \xi)\) and \((\mathcal{J}^1(S^1), \xi)\)

The standard projections in \((\mathbb{R}^2 \times S^1, \xi)\) and \((\mathcal{J}^1(S^1), \xi)\) are in some sense the
same, as the contact manifolds are related by contactomorphism. We will thus
state them together.

The front projection in \(\mathcal{J}^1(S^1)\) In \(\mathbb{R}^2 \times S^1\) the front projection is given by:
\[
(x, y, \theta) \mapsto (\theta, x \cos(\theta) + y \sin(\theta))
\]
Figure 9: The same segment of a Legendrian knot parametrized by \((x, y, z) = (t^2, 3/2t, t^3)\), but seen as if projected onto the \(xy\) and \(xz\) planes respectively.

Viewed in \((J^1(S^1), \xi)\), thus becomes \((\vartheta, p, z) \mapsto (\vartheta, z)\) under our contactomorphism \(\Psi\).

**The Lagrangian projection in \(J^1(S^1)\)** In \(\mathbb{R}^2 \times S^1\), the Lagrangian projection is given by:

\[(x, y, \theta) \mapsto (\theta, -x \sin(\theta) + y \cos(\theta))\]

or in terms of \((J^1(S^1), \xi)\), \((\vartheta, p, z) \mapsto (\vartheta, p)\)

**2.3 The projection** \(\rho : \mathbb{R}^2 \times S^1 \to \mathbb{R}^2\)

Any immersion in \(\mathbb{R}^2\) is the image of an immersed Legendrian knot under \(\rho\), uniquely determined by specifying a normal vector field along the immersion. In the diagram, we will draw this by adjoining small spikes to the knot. The Legendrian correlation to an immersion is called the *conormal lift*. Conversely, any generic Legendrian knot projects under \(\rho\) to a curve which is an immersion except for a finite number of cubical cusps. In particular, any Legendrian knot in \(\mathbb{R}^3\) corresponds to a Legendrian knot in \(\mathbb{R}^2 \times S^1\) since its front diagram in \(\mathbb{R}^2\) can be lifted using \(\rho\).

In \(\mathbb{R}^2 \times S^1\), there is another projection of interest, namely the one to the plane itself. It is interesting in that it allows tangencies of any direction, unlike the front projections on \(J^1(S^1)\), \(\mathbb{R}^2 \times S^1\) and \(\mathbb{R}^3\). Because of this, in \(\mathbb{R}^2 \times S^1\) there does not need to be a cusp for there to be a \(\text{R-II}\) type move, provided we account for the risk of double points. As shown in Figure 12 below, this can in fact be done.
In $\mathbb{R}^2$, an oriented immersion gives rise to a choice of a normal vector field, by taking the tangent vectors and subjecting them to a rotation by 90 degrees.

**Definition 4.** A self tangency of a plane curve is called a dangerous tangency if the directions of the tangents to the curve are the same at the point of tangency. Two plane curves are safely homotopic if they are homotopic through immersions, and no intermediate step results in a dangerous self-tangency.

**Note 2.** A safe homotopy of immersed plane curves corresponds to a Legendrian isotopy in $\mathbb{R}^2 \times S^1$. This is a result of the fact that an immersion $g$ of $S^1$ in $\mathbb{R}^2$ always has a Legendrian lift $f$ to $\mathbb{R}^2 \times S^1$. See the diagram below.

\[
\begin{array}{ccc}
\mathbb{R}^2 \times S^1 & \xrightarrow{\rho} & S^1 \\
\downarrow f & & \downarrow g \\
\mathbb{R}^2 & \rightarrow & \mathbb{R}^2 
\end{array}
\]

It is possible that there are points where the lift is not embedded, because it has a double point. As it turns out. In fact, a self-intersection of the lift is exactly a dangerous tangency.

Here we must introduce a bit of notation: I shall use arrows (as in Figure 13) and little line segments (as in Figure 12) to indicate the normals of the knots, depending on what I find to make the illustration the most readable. An important quirk of the arrow notation is that it is not visually apparent that the normals "switch" when we pass a cusp. See e.g. Figure 16, where all normals point to the top (or bottom) of the page - the arrows help us follow the line, and we must infer the normals ourselves from some initial choice.

### 2.4 Drawing knot diagrams, Legendrian isotopy and invariants

There are plenty of Legendrian knots. In particular, every smooth knot can be made Legendrian. Etnyre calls this process realization.

For the front diagram, one imagines the desired knot as a smooth knot diagram. Then, each place where the knot diagram has a vertical tangency with the "hump" pointing to the right, the Legendrian front receives a right-pointing cusp, and vice versa for left-pointing humps (See Figure 10).

For a more detailed discussion of this, I will refer to Etnyre[6]. There is also a discussion on moving from a front projection to a Lagrangian one there.

#### 2.4.1 Legendrian isotopy and the Reidemeister moves

**Definition 5.** We say that two knots are Legendrian isotopic if there is a smooth isotopy between them, and the knot stays Legendrian during the whole isotopy.

However, attempting to work in the space of isotopies is somewhat unwieldy - unless you have a very clear picture of what you want to do you it is hard to navigate the sheer amount of possible isotopies. Enter the Reidemeister moves. They are a set of operations on a knot diagram, guaranteed to preserve Legendrian isotopy, which allow us to think of isotopies in more combinatorial terms. In fact, any Legendrian isotopy can be turned into a sequence of Legendrian Reidemeister moves in the front projection, as shown by Świątkowski:
Figure 10: Realizing a knot diagram for a smooth knot as a Legendrian front. For the first row, imagine a corresponding right-pointing change. For the last row, there is a choice.

**Theorem 3.** *(The Legendrian Reidemeister Theorem [8])* Two front projections correspond to Legendrian isotopic Legendrian knots if and only if the projections can be connected by Legendrian planar isotopies and by Legendrian Reidemeister moves.

**R-I, R-II and R-III for Legendrian front projections**  The three Reidemeister moves R-I, R-II and R-III are can be applied to the image of a Legendrian knot under either Π or ρ.

### 2.4.2 The Reidemeister moves

- Reidemeister move I (R-I) is the equivalent of picking up a section of knot and twisting a loop into it - or indeed picking up a loop and untwisting it. As this move pertains to Legendrian knots, the loop gains two cusps.
- Reidemeister move II (R-II) consists of moving a cusp past a knot segment in either direction - "poking through" in a manner of speaking.
- Reidemeister move III (R-III) consists of moving one segment of knot past an intersection of two other segments, thus at some moment creating a triple point.

See Figure 11 for an illustration of the Legendrian Reidemeister moves.

### 2.4.3 The additional move $R - II^*$

The correspondence between Legendrian isotopy and safe homotopy may be used to construct a Reidemeister move of sorts. It will preserve Legendrian isotopy [7], but will only be applicable to the image of a Legendrian knot under
Figure 11: The Legendrian Reidemeister moves, from top to bottom: R-I, R-II and R-III.

Figure 12: Top: an example of a safe homotopy, without dangerous tangencies. Bottom: an example of a dangerous tangency, where the gray circle highlights the problematic point(s) of tangency - a self-intersection of the lift.

\( \rho \), as it relies upon the fact that under \( \rho \), normals can horizontal. Let us call this move \( R - II^{*} \) as it is an incarnation of the smooth Reidemeister-II move.

Figure 13: \( R - II^{*} \).

Definition 6. Let a legal sequence of Reidemeister moves applied in sequence to the same tangle be called a Reidemeister sequence.

2.4.4 The classical invariants

There are three so-called classical invariants of a given Legendrian knot L:

- The underlying topological knot type \( k(L) \)
• The Thurston-Bennequin invariant $tb(L)$
• The rotation number $rot(L)$

They are used to distinguish knots. While they are good tools, they are not very powerful. Their great limitation is that they cannot tell us whether knots are the same; only when they are not. (Two different knots may have the same, e.g. Thurston-Bennequin number or rotation number, but not the same underlying topological knot type, or vice versa.) There are however modern, more powerful invariants, e.g. Legendrian contact homology as defined by Chekanov [4] and Eliashberg-Givental-Hofer[11]. They are however neither used, nor further discussed in this paper.

**The underlying topological knot type:** Intuitively, one might think of $k(L)$ as a map that takes each knot to a set of its equivalent knots. Letting $K$ be a fixed knot, let $K'$ be the set of topological knots isotopic to $K$. Then $k(L) \in K'$.

**The Thurston-Bennequin invariant:** $tb(L)$ measures the integer “twisting” of $\xi$ around $L$. While we will not make use of $tb$ in this thesis, familiarity with it is preferable. Using the orientation of $L$, we can give each crossing of the knot either a positive or a negative sign as in to Figure 14.

![Figure 14: The maps that count positive and negative crossings are denoted $X^+$ (left) and $X^-$ (right) respectively.](image)

Thus, for the front projection, we may compute the writhe of $L$, $w(L) = X^+(\Pi(L)) - X^-(\Pi(L))$. Then $tb(L) = w(\Pi(L)) - \frac{1}{2}C$ where $C$ denotes the number of cusps in $\Pi(L)$.

**The rotation number:** Contemplate a Legendrian knot residing in a contact manifold. While $tb$ asks how much $\xi$ "twists" along the knot, we might also ask how much a given knot "turns" in some particular sense. If the contact planes of that contact manifold have a global trivialization, there is indeed a meaningful answer to that question - the rotation number. Asking how much an immersion $C$ of $S^1$ in the plane turns clearly makes sense, and we have a way to measure that in the turning number which is a quantity that measures how the tangent to a plane curve turns around itself.

**Definition 7.** For a smooth immersion $C \in \mathbb{R}^2$, the turning number $\gamma(C) \in \mathbb{Z}$ is defined in the following way: Imagine a map from the tangent space of $L$ to $\mathbb{R}^2$, where all tangent vectors emanate from the origin. The turning number then, is the number of times the curve that is traced out wraps around the origin in the counter-clockwise direction minus the number of times it wraps around in the clockwise direction.

3We have thus far not bothered with which strand crosses on top, but now we do. The top strand is the one with a smaller or more negative slope.
In $\mathbb{R}^2 \times S^1$, we can also define a turning number for $\rho(L) \in \mathbb{R}^2$, should we wish to, which will be well defined precisely when $\rho$ is an immersion.

**Definition 8.** For a Legendrian knot $L$ in $\mathcal{J}^1(S^1)$, $\mathbb{R}^2 \times S^1$ or $\mathbb{R}^3$, $\gamma(\pi(L)) = \text{rot}(L)$.

In $\mathcal{J}^1(S^1)$ (and hence in $\mathbb{R}^2 \times S^1$) both the turning number of the projection and the rotation number depend on an additional choice, namely the trivialization of the tangent planes of $S^1 \times \mathbb{R}$. As the Lagrangian projection here is an immersion, we will need to specify what we mean by ”the rotation of the tangent vector” of the knot if we are to make sense of its turning number. We will make the choice of the trivialization $(\partial \partial \theta, \partial \partial p)$. It is worth noting that $\text{rot}(L)$ is invariant under regular homotopies of Legendrian knots (i.e. where dangerous tangencies are permitted).

**Calculating the rotation number in $\mathbb{R}^3$** The rotation number gives a signed count of the number of full rotations of the tangent to $L$, projected onto the corresponding contact elements. As $L$ is always tangent to some contact element, we may count the rotations within a trivialization of $\xi$. Intuitively, one might visualize this as projecting each tangent vector (as we go once around the knot in the direction of its orientation) onto its corresponding contact element, stacking up every contact element, and counting the number of full, signed turns the tangents make as we traverse the stack from ”beginning” to ”end”. For $(\mathbb{R}^3, \xi_{std})$, the rotation number is in practice computed by $\text{rot}(L) = \gamma(\pi(L)) = \frac{1}{2}(D - U)$. Here, $D$ denotes the number of cusps traversed downwards, and $U$ the number of cusps traversed upwards, as given by the orientation of the knot after projection via $\Pi$. Thus, we can evaluate the rotation number of a knot by way of more reasonable computations.

**Calculating the rotation number in $\mathcal{J}^1(S^1)$** In $\mathcal{J}^1(S^1)$, the $\Pi$-formula for the rotation number takes into account not only whether cusps are ”down cusps” or ”up cusps”, but also whether they are ”left” or ”right” [7],

$$\text{rot}(L) = \frac{1}{2} \left( \sum_{\text{down}} - \sum_{\text{up}} \right)$$

Disregarding left-right orientation of the cusps, the formula simplifies into the familiar $\frac{1}{2}(D - U)$. An important point is that we will work extensively in $\mathbb{R}^2$, where this does not apply. To make use of this formula, we must lift and project back down via $\Pi$. Thankfully, this is not very hard for the knots in question. This may seem somewhat difficult to use as ”left” and ”right” become somewhat flimsy concepts in these spaces, but it will not cause us trouble in this paper.

### 3 Obtaining immersions isotopic to front projections of Legendrian knots via $\mathbb{R}^2 \times S^1$

We want to show that some Legendrian knots with zero rotation, and certain other properties, are Legendrian isotopic other knots whose fronts are immer-
Obstructions to removing cusps in $\mathbb{R}^2 \times S^1$

Choosing a starting point and walking along the knot, let us assume we come upon a left-facing cusp followed by a right-facing one - a prime candidate for cusp removal! - but there is some sort of interfering “mess” between them. In order to know what kind of mess to expect, and what we can do about it, it would be useful to have some kind of taxonomy. Assuming we have the cusp $c_1$ connected to the cusp $c_2$ by the segment $A$. Then we can have a few different problems:

- A loop or loops on $A$
- $c_1$ or $c_2$ or both trapped by a writhing segment
- One or more segments intersecting $A$
- Combinations of the above

While these problems are neither clearly defined in this form, nor really separate problems, they all boil down to one central problem which is easily stated clearly - $A$ is not embedded.

3.1.1 Proof of (M1) in Theorem 1

Here we formulate the property (M1) as a Lemma.

Lemma 2. Loops and cusps do not mind switching places (for either normal).

$\begin{align*}
\text{Proof.} & \text{ Moving a loop and cusp past each other when they have the same orientation. From left to right, the Reidemeister moves are: } R-I, R-II^*, R-II^*, R-I \\
\text{Moving a loop and cusp past each other when they have the different orientations. From left to right, the Reidemeister moves are: } R-II^*, R-I, R-II + R-II^* 
\end{align*}$

\[\square\]
We can also state a little corollary to Lemma 2:

**Corollary 1.** Loops can move freely past other loops.

**Proof.** Imagine two loops \( o_1 \) (to the left) and \( o_2 \) (to the right) as Figure 15 shows. Apply \( R - I \) to \( o_1 \), and use \( R - II^* \) to "pull apart" \( o_1 \). We are left with two cusps, each of which can pass \( o_2 \) freely, by Lemma 2.

### 3.1.2 Proof of (M2) in Theorem 1

We formulate the property (M2) as a separate Lemma. A knot segment featuring a single cusp trapped number of crossings is in fact isotopic to a cusp and a collection of loops 4.

**Lemma 3.** A solitary outwards facing cusp trapped by a writhing tangle consisting of \( n \) crossings can be isotoped to a free cusp and \( n \) loops.

**Proof.** We can use the case of a writhe with two crossings as a model to construct our proof, as in Figure 18. By the first move, an outwards turned cusp becomes vulnerable to \( R - II \). After pulling through, we end up at step 3 (first row, rightmost image) with one less crossing than before. If we can get the cusp trapped by a single crossing as in our starting state, we will be done. The two rightmost cusps move happily past the crossing, restoring the leftmost part of our diagram at step 5 (second row, middle image). The restored part has been made a little bolder for clarity. Once no more crossings remain, we can remove the added cusps beginning with the most recently added pair, moving

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4 The author humbly proposes the term "loop forest" for any such collection of loops.

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Figure 15
Figure 16: Examples where the cusp is trapped by 2 and 4 crossings respectively. The writhe trapping the cusp can be removed. Notice that the pattern of the resulting loop forest is subject to rearrangement as per Corollary 1.

Figure 17: Examples where the cusp is trapped by 1 and 3 crossings respectively. The writhe trapping the cusp can be removed. Notice that the pattern of the resulting loop forest is subject to rearrangement - up-up-down can become up-down-up without violating isotopy.

the residual loop if necessary, then the next most recently added pair, and so on until only the original cusp remains. As this procedure does not care how many crossings are to the right, the proof holds in generality. All that changes is whether the remaining cusp points "left" or "right" with respect to orientation in the end, and indeed the number of loops in the remaining forest.

Note 3. Should the trapped cusp be turned inwards at the start, it is no longer the simple case dealt with in this Lemma. A single inwards turned cusp trapped by $n$ crossings is in fact isotopic to a single outwards turned cusp trapped by $n + 1$ crossings, plus a loop next to the cusp.

3.1.3 Proof of (M3) in Theorem 1

We begin by establishing some terminology.

Definition 9. Two cusps separated by a knot segment, where both cusps point
Figure 18: Freeing a single cusp trapped by a single writhe segment. From left to right, the Reidemeister moves are: $R^{-1}$, $R^{-1}$, $R^{-1}$, $R^{-1}$, $R^{-1}$, $R^{-1}$, to the same side of the segment, with no other cusps on that segment, are called sister cusps. See Figure 19 and Figure 20 for examples.

![Figure 18: Freeing a single cusp trapped by a single writhe segment.](image)

Figure 19: To the left and in the middle are examples of sister cusps. To the right are cusps that are not sister cusps.

![Figure 19: To the left and in the middle are examples of sister cusps. To the right are cusps that are not sister cusps.](image)

**Definition 10.** Consider a knot segment containing two sister cusps. Let the part of between the cusps be locally embedded, and denote it $A_0$. Further, assume that $A_0$ is transversely intersected by $n$ knot segments $B_1, \ldots, B_n$. Consider Figure 20; the orientation of the segment "before" the left cusp is "up" and the orientation of the segment "after" the right cusp is "down" (let us denote them $a$ and $b$ respectively). We will call that state of $B := \{B_1, \ldots, B_n\}$ a restrictive arrangement of that set $B$ of segments if there is at any point along $A_0$ a pair of intersecting segments $B_iB_j$ such that $B_i$ is oriented "up" and $B_j$ is oriented down (in other words, matching the orientation of the ordered pair $ab$). We can also talk about a "restrictive arrangement over $A_0" when we wish to emphasize where it is happening.

The case of only one segment intersecting $A_0$ at one place (as in Figure 21) is not at all troublesome. In fact, the orientation of the intersecting segments always allows one of the two cusps to move past it.

We might imagine the case where a knot segment that separates two sister cusps is intersected by several other segments in a restrictive arrangement. One might imagine a "simple hard case" when two cusps are separated by three such crossing lines, two of which have right-pointing normals, and the third with left-pointing normals (see Figure 22).
Figure 20

Figure 21: Single intersecting segments always allow one of the cusps to move past.

Note 4. We will assume the segment separating the cusps to be locally embedded (see page 21 for definition).

Figure 22: Sister cusps kept apart by three crossing segments in a restrictive arrangement.

This problem reduces to creating a path along which all the segments with left-pointing normal components are to the one side, and all those with the right-pointing normal components are to the other, resulting in a non-restrictive arrangement.

Suppose we are being presented with the restrictive arrangement pictured in Figure 24, leftmost circle. Let us use the notation LLLRRR for the leftmost picture, indicating the arrangement of normal component directions of the knot.
Figure 23: Left: A restrictive arrangement of intersecting segments over a segment A. Middle: The grey area indicates a suitable area where a path may be created. Right: The grey area indicates where the two cusps can now meet, as it is bounded by a non-restrictive arrangement of intersecting segments.

Figure 24: LLLRRR ↔ RRRLLL over a segment A.

Figure 25: When two sister cusps are kept apart by a restrictive arrangement, one can always "clear a path" so that they can meet. The last step pictured consists of two instances of first $R - II^*$, then $R-II$ to bring the cusps to the inside region.

segments we would intersect if traversing a segment A from left to right. Suppose the arrangement we need is the opposite, or RRRLLL. By repeated and structured application of $R - II^*$, we can obtain such a non-restrictive arrangement (see Figure 24, rightmost circle).

In fact, it is a generalizable procedure, and we will state it as a Lemma.

**Definition 11.** Let a segment A between two cusps be called locally embedded if it is embedded when considered separately - that is, it may be intersected by other segments on the knot, but can have no self-intersections. See Figure 26.

![Figure 26](image)

Figure 26: A is both globally and locally embedded, i.e. embedded in the classical sense. B is locally embedded, but not globally embedded. C is neither.
Lemma 4. Any restrictive arrangement of intersecting segments over a locally embedded segment between sister cusps can be made non-restrictive.

Proof. Assume the worst case, where the desired arrangement is the exact opposite of our starting point. Using the above notation, let \( L_1, \ldots, L_n, R_1, \ldots, R_m \) denote the arrangement. Using R-II*, pull \( L_{n-k} \) past all \( R_i \) for \( k = 0, \ldots, n-1 \). After one pass, we yield \( L_1, \ldots, L_{n-1}, R_1, \ldots, R_m, L_n \). We then repeat this procedure until \( k = n-1 \). Note that for each \( k > 0 \), \( L_{n-k} \) cannot pass \( L_{n-k+1} \). Hence, our final state is \( R_1, \ldots, R_m, L_n, \ldots, L_1 \). This is non-restrictive: using R-II* followed by R-II (as in the last arrow of Figure 25) will allow us to remove the cusps.

We can in fact even say something about the maximum number of R-II*-moves needed to create such a "path" in a case like this (i.e. one with a restrictive arrangement over a locally embedded A is the only trouble). Assume, as earlier, the desired arrangement is the one with all L-segments to the right. Let \( n \) denote the total number of segments intersecting A, \( a \) be the number of L-segments and \( b \) the number of R-segments. The worst case consists of \( a = b = \frac{n}{2} \) if \( n \) is even, and \( a = \frac{n}{2} - \frac{1}{2} \) and \( b = \frac{n}{2} + \frac{1}{2} \) if \( n \) is odd, with the arrangement the reverse of the desired one. Then we can obtain a non-restrictive arrangement over A with \( \frac{n^2}{4} \) moves if \( n \) is even, and \( \frac{1}{4}(n^2-1) \) moves if \( n \) is odd, given that A is locally embedded.

4 Hands-on examples

In this section, we will work through two specific Legendrian knots, and one family of Legendrian knots, and isotope them to other knots with fronts that are immersions.

We begin by the very simplest case: the Legendrian unknot. The R-II* move affords us a way to obtain a safe homotopy to a "figure 8" knot (Figure 27).

![Figure 27: Removing cusps from the Legendrian unknot](image)

Obtaining immersions isotopic to \((2, n)\) torus knots A class of knots that lend themselves to the above common method of getting an immersion via \(\mathbb{R}^2 \times S^1\) is the \((2, n)\) torus knots \(^5\).

A good example case is that of the \((2, 3)\) torus knot, where use of R - II* enables us to remove the offending cusps. The procedure is detailed below.

Using this methodology, we may obtain immersions of any \((2, n)\) torus knots in the obvious way. (See Figure 29) The twice stabilized unknot is another illustrative example of clearing away cusps using R-II and R-II* (see Figure 30).

\(^5\)for \( n \in \{n \in \mathbb{Z} \mid \gcd(2, n) = 1\} \)
Figure 28: Legendrian (2, n) torus knots.

Figure 29: Isotoping (2, 3) to an immersion by repeated application of the sequence (R-II*, R-II*, R-I).

Figure 30: Finding an immersion Legendrian isotopic to the twice stabilized unknot.

5 Proof of Theorem 2 and more

We begin by stating and proving a Lemma we will need, before restating and proving Theorem 2.

**Lemma 5.** A Legendrian knot $L \subset \mathbb{R}^2 \times S^1$ can be Legendrian isotoped to $L'$ for which the number of clockwise and counterclockwise cusps of $\rho(L')$ are of the same number as the corresponding cusps of $\Pi(L')$.

**Proof.** Using the rescaling Legendrian isotopy $(q, p, z) \to (q, lp, lq)$ from $l = 1$ to $l = \epsilon$ for some small $\epsilon$, followed by the shifting Legendrian isotopy $(q, p, z) \to (q, p, z + m)$ from $m = 0$ to $m = 1$. The composition of these two Legendrian isotopies is in itself a Legendrian isotopy, expressed in the coordinates of $\mathcal{J}^1(S^1)$. Under the composition of these two Legendrian isotopies, $\rho$ looks like $\Pi$ in that
it lies close to the unit circle in $\mathbb{R}^2$, and the number of different (right/left versus clockwise/anti-clockwise) cusps are the same under both projections.

**Theorem 2.** If a Legendrian knot $L \subset \mathbb{R}^2 \times S^1$ is Legendrian isotopic to a knot $L' \subset \mathbb{R}^2 \times S^1$ whose front projection $\rho(L')$ is an immersion, then $L$ has $\text{rot}(L) = 0$.

**Proof.** It always produces cusps, and since $L$ is Legendrian isotopic to $L'$ by assumption, we know there are Reidemeister sequences to remove them. Cusps can only be removed in certain pairs:

$\prec \downarrow \succ \uparrow$ or $\prec \uparrow \downarrow \succ$ or $\succ \uparrow \downarrow \prec$ or $\succ \downarrow \prec \uparrow$

Each of these pairs contribute 0 to $\text{rot}(L)$. Making use of Lemma 5 and the formula for the rotation number in Section 2.4.4, we can conclude that $L$ must thus have $\text{rot}(L) = 0$.

We can also prove a property of all Legendrian knots in $\mathbb{R}^2 \times S^1$ that have rotation number zero:

**Lemma 6.** In a Legendrian knot $L \subset \mathbb{R}^2 \times S^1$ with $\text{rot}(L) = 0$, there is always at least one pair of sister cusps, or no cusps at all.

**Proof.** Assume $L$ has $2n$ cusps and $n > 0 \in \mathbb{N}$. Since the $\text{rot}(L) = 0$, we know $L$ has an equal number of left and right cusps. Since $L$ is a closed curve, all cusps have other cusps to both sides. Let us assume it is possible to arrange the cusps in such a way that there are no sister cusps. We have $n$ left and right cusps respectively. Starting with the right cusps, we place a right cusp $R_0$ somewhere, and surround it by two other right cusps. Hence, we get $\ldots |R_3, R_0, R_2|\ldots$. We cannot place a $L$, as it would create sister cusps. We continue placing $R$ cusps, and subsequently get $\ldots |R_{n-2}, \ldots, R_1, R_0, R_2, \ldots, R_{n-1}|\ldots$. At this point, we have no choice but to create sister cusps, as we still have $n$ left cusps to place.

Having Lemma 6, we can prove this little Theorem.

**Theorem 4.** If a Legendrian knot $L \subset \mathbb{R}^2 \times S^1$ has $\text{rot}(L) = 0$ and all segments between its sister cusps are globally embedded, then it is Legendrian isotopic to a knot $L'$ whose front projection $\rho(L')$ is an immersion.

**Proof.** Since $L$ has $\text{rot}(L) = 0$, we know that it has $2n$ cusps for some $n \in \mathbb{N}$, and $n$ right and left cusps respectively. Somewhere along $L$, there must by Lemma 6 be a pair of sister cusps. Since the segment connecting them is embedded, they can be removed. $L$ now has $2n_0 = 2(n - 1)$ cusps. When eliminating cusps, we may end up with a loop, but by the Theorem 1 property M1 (Lemma 2), cusps may move past them freely under $\rho$. Putting $n_0$ in the place of $n$, we remove two more cusps in the same manner, getting $2n_1 = 2(n - 2)$. By the $i$:th iteration of this process, $n_i = (n - (i + 1))$. Since $n < \infty$, we know that eventually $i = n - 1$, and hence $n_i = 0$.

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6Think of a clasp-less necklace - there is no endpoint where some bead has neighboring beads only to one side.
Using the Whitney-Graustein Theorem, we can state a corollary to Theorem 2. We start by stating Whitney-Graustein:

**Theorem 5** (The Whitney-Graustein Theorem [10]). Two regular, closed plane curves $C_0$ and $C_1$ are regularly homotopic iff $\text{rot}_\gamma(C_0) = \text{rot}_\gamma(C_1)$.

All immersions of closed curves have Legendrian lifts to the unit cotangent bundle. Making use of this fact, we get the following corollary.

**Corollary 2.** If a zero-homotopic Legendrian knot $L \subset \mathbb{R}^2 \times S^1$ has $\text{rot}(L) = 0$ and and all segments between its sister cusps are globally embedded, $\rho(L)$ is regularly homotopic to the figure 8 curve, and thus $L$ is Legendrian regular homotopic to the standard Legendrian unknot.

**Proof.** Let $L_0$ denote the Legendrian unknot, and $L$ another Legendrian knot as stated above. Then, by Theorem 2 there is a knot $L'$ whose front is an immersion under $\rho$, and which is Legendrian isotopic to $L$. An immersion is by definition a regular closed curve. Hence, by Whitney-Graustein, $L$ is is Legendrian regularly homotopic to $L_0$. 

□
References