All Order α' Expansion of One-Loop Open-String Integrals

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We present a new method to evaluate the α' expansion of genus-one integrals over open-string punctures and unravel the structure of the elliptic multiple zeta values in its coefficients. This is done by obtaining a simple differential equation of Knizhnik-Zamolodchikov-Bernard-type satisfied by generating functions of such integrals, and solving it via Picard iteration. The initial condition involves the generating functions at the cusp $\tau \to i\infty$ and can be reduced to genus-zero integrals.

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Introduction.—Elliptic analogs of polylogarithms [1,2] and multiple zeta values [3] have become a driving force in higher-order computations of scattering amplitudes in quantum field theories and string theories. The study of their differential equations and their connections with modular forms turned into a vibrant research area at the interface of particle phenomenology, string theory, and number theory. In the same way as a variety of Feynman integrals has been recently expressed in terms of elliptic polylogarithms and iterated integrals of modular forms [4,5], the low-energy expansion of one-loop open-string amplitudes introduces elliptic multiple zeta values (EMZVs) [6–8].

So far, the appearance of EMZVs in one-loop open-string amplitudes arose from direct integration over the punctures on a genus-one world sheet of cylinder or Möbius-strip topology. Although there is no conceptual bottleneck in extending the techniques of [6–8] to arbitrary multiplicities and orders in the inverse string tension α', in this Letter we will present a new method to evaluate these genus-one integrals.

Our method automatically generates the EMZVs in their minimal form [3,9] and reveals elegant structures in the α' expansions. It rests on new differential equations for genus-one integrals which open up connections with modern number-theoretic concepts: elliptic associators [10]—generating series of EMZVs—and Tsunogai's derivations dual to Eisenstein series [11], which govern the counting of independent EMZVs [9]. More details will be given in a longer companion paper [12].

Open-string integrals at genus one.—One-loop string amplitudes are described by correlation functions of vertex operators in a conformal field theory over a genus-one Riemann surface, the torus. The location of the vertex operator associated with the jth external string state is parametrized by the coordinates $z_j = u_j\tau + v_j$ with u_j, $v_j \in (0,1)$, where τ is the modulus with $\mathrm{Im}\tau > 0$, see Fig. 1, and we define $z_{ij} \equiv z_i - z_j$.

By suitable involutions of the torus [13], one obtains the surfaces describing the scattering of open-string states, the cylinder and the Möbius strip. The two boundaries of the cylinder will be parametrized by the A cycle $z_j \in (0,1)$ and its displacement $z_j \in (\tau/2) + (0,1)$ by half a B cycle, i.e., $u_j \in \{0, \frac{1}{2}\}$ and $dz_j = dv_j$. See Fig. 2.

The massless n-point one-loop amplitudes of the open superstring give rise to integrals of the form $(z_1 = 0)$ [6]

$$\int_{\mathcal{C}(*)} \left(\prod_{j=2}^{n} dz_j\right) f^{(k_1)}_{i_1 j_1} f^{(k_2)}_{i_2 j_2} \cdots \exp\left(\sum_{i<j}^{n} s_{ij}\mathcal{G}(z_{ij},\tau)\right), \qquad (1)$$

with differing integration domains $\mathcal{C}(*)$ for the cylinder and the Möbius strips. For planar cylinders, we set $* \to 1,2,\ldots,n$ and parametrize the domain as

$$\mathcal{C}(1,2,\ldots,n) = \{z_{j=2,\ldots,n} \in \mathbb{R},\, 0 < z_2 < \ldots < z_n < 1\}, \qquad (2)$$

see Fig. 2 and [12] for the nonplanar analog with $* \to {}^{r+1,\ldots,n}_{1,2,3,\ldots,r}$. Furthermore, in the integrand of (1), $f^{(k)}_{ij} \equiv f^{(k)}(z_{ij},\tau)$ denote the Laurent coefficients of the doubly periodic Kronecker-Eisenstein series defined by [2,14]

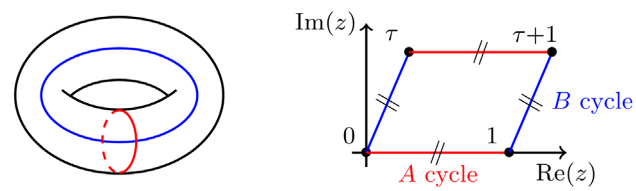

FIG. 1. We parametrize the torus through the lattice $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ with identifications $z \cong z+1 \cong z+\tau$ along the A and B cycles.

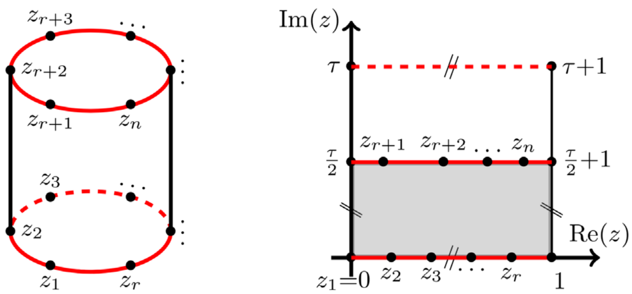

FIG. 2. The cylinder parametrization.

$$\Omega(z,\eta,\tau) = \exp\left(2\pi i\eta \frac{\mathrm{Im}z}{\mathrm{Im}\tau}\right)\frac{\theta_1'(0,\tau)\theta_1(z+\eta,\tau)}{\theta_1(z,\tau)\theta_1(\eta,\tau)}, \quad (3)$$

$$\Omega(z,\eta,\tau) = \sum_{k=0}^{\infty}\eta^{k-1}f^{(k)}(z,\tau), \quad (4)$$

which generates the integration kernels of multiple elliptic polylogarithms [2]. The simplest examples of the coefficient functions are $f^{(0)}(z,\tau)=1$ and $f^{(1)}(z,\tau)=\partial_z\log\theta_1(z,\tau)+2\pi i(\mathrm{Im}z/\mathrm{Im}\tau)$, and higher $f^{(k\geq 2)}(z,\tau)$ do not have any poles in z.

Finally, $\exp(\sum_{i<j}^{n}s_{ij}\mathcal{G}(z_{ij},\tau))$ in (1) is the Koba-Nielsen factor written in terms of dimensionless Mandelstam invariants $s_{ij}=-2\alpha' k_i k_j$ and Green functions $\mathcal{G}(z,\tau)$ subject to the universal differential equation

$$\partial_{v_i}\mathcal{G}(z_{ij},\tau) = -f^{(1)}(z_{ij},\tau),$$
$$2\pi i\partial_\tau\mathcal{G}(z_{ij},\tau) = -f^{(2)}(z_{ij},\tau)-2\zeta_2, \quad (5)$$

where ∂_{v_i} is the derivative along the cylinder boundary, and $\zeta_n=\sum_{k=1}^{\infty}(1/k^n)$ with $n\geq 2$ denote Riemann zeta values.

Generating functions: Instead of handling the α' expansion of the individual integrals (1) as in the method of [6–8], we will evaluate the following generating function of integrals (with $\eta_{23,\ldots,n}=\eta_2+\eta_3+\cdots+\eta_n$)

$$Z_{\vec\eta}^{\tau}(*|1,2,\ldots,n) = \int_{\mathcal{C}(*)}\prod_{j=2}^{n}dz_j\exp\left(\sum_{i<j}^{n}s_{ij}\mathcal{G}(z_{ij},\tau)\right)$$
$$\times\,\Omega(z_{12},\eta_{23,\ldots,n},\tau)\Omega(z_{23},\eta_{3,\ldots,n},\tau)\cdots$$
$$\Omega(z_{n-1,n},\eta_n,\tau). \quad (6)$$

The integrands $f_{i_1j_1}^{(k_1)}f_{i_2j_2}^{(k_2)}\cdots$ in (1) relevant to n-point open-superstring amplitudes have $k_1+k_2+\cdots=n-4$ and reside at the order of η_j^{-3} of (6). Moreover, ($n\geq 8$)-point integrands additionally involve holomorphic Eisenstein series $\mathrm{G}_{\ell\geq 4}(\tau)=-f^{(\ell)}(0,\tau)$ [6] multiplying (1) at $k_1+k_2+\cdots=n-4-\ell$ as seen at the $\eta_j^{-3-\ell}$ order of (6).

Although the cylinder contribution to one-loop open-string amplitudes is localized at purely imaginary τ as drawn in Fig. 2, we will define and evaluate the integrals (6) for generic τ in the upper half plane with $\mathrm{Re}\tau\neq 0$. In view of the parental torus, $Z_{\vec\eta}^{\tau}(1,2,\ldots,n|\cdot)$ and $Z_{\vec\eta}^{\tau}\left({r+1,\ldots,n \atop 1,2,3,\ldots,r}|\cdot\right)$ will be referred to as planar and nonplanar *A-cycle integrals*, respectively.

Möbius-strip integrals can be reconstructed by specializing planar A-cycle integrals to $\mathrm{Re}\tau=\frac{1}{2}$, and the cancellation of tadpole divergences from one-loop open-superstring amplitudes can be analyzed as in [15].

The A-cycle integrand (6) at n points involves $n-1$ factors of the Kronecker-Eisenstein series (4) at different arguments. The second entry $Z_{\vec\eta}^{\tau}(*|A)$ specifies permutations $A=a_1a_2\ldots a_n\in S_n$ of the arguments, and $\Omega(\ldots)$ at different z_{a_j}, η_{a_j} are related by the Fay identity [16]

$$\Omega(z_1,\eta_1,\tau)\Omega(z_2,\eta_2,\tau) = \Omega(z_1,\eta_1+\eta_2,\tau)\Omega(z_2-z_1,\eta_2,\tau)$$
$$+\Omega(z_2,\eta_1+\eta_2,\tau)\Omega(z_1-z_2,\eta_1,\tau), \quad (7)$$

which can be thought of as a doubly periodic generalization of the partial-fraction relation $1/(z_1z_2)=[1/(z_1-z_2)](1/z_2-1/z_1)$. Repeated use of (7) and imposing $\eta_1=-\sum_{j=2}^{n}\eta_j$ only leaves $(n-1)!$ independent permutations of the integrand in (6), and we will use a basis of $Z_{\vec\eta}^{\tau}(*|1,B)$ with permutations $B\in S_{n-1}$ acting on $2,3,\ldots,n$.

The differential equation: As will be derived in [12], the τ derivatives of (6) can be written as

$$2\pi i\partial_\tau Z_{\vec\eta}^{\tau}(A|1,B) = \sum_{C\in S_{n-1}}D_{\vec\eta}^{\tau}(B|C)Z_{\vec\eta}^{\tau}(A|1,C), \quad (8)$$

where the $(n-1)!\times(n-1)!$ matrix $D_{\vec\eta}^{\tau}$ is a differential operator with respect to η_j. Its detailed form will be exemplified in the next section and follows from the properties (5) of the Green function, the vanishing of boundary terms $\int dv_j\partial_{v_j}(\ldots)$ and the mixed heat equation (u, $v\in\mathbb{R}$)

$$2\pi i\partial_\tau\Omega(u\tau+v,\eta,\tau) = \partial_v\partial_\eta\Omega(u\tau+v,\eta,\tau). \quad (9)$$

Most importantly, the form of $D_{\vec\eta}^{\tau}(B|C)$ does not depend on the planar or nonplanar integration cycle A, and its entries are linear in the dimensionless Mandelstam invariants s_{ij} and therefore in α'.

Hence, the α' expansion of the A-cycle integrals $Z_{\vec\eta}^{\tau}$ follows from the solution of (8) via Picard iteration—an infinite iteration of the integrated version $Z_{\vec\eta}^{\tau}(A|1,B)=Z_{\vec\eta}^{i\infty}(A|1,B)+\int_{i\infty}^{\tau}(d\tau'/2\pi i)\sum_{C\in S_{n-1}}D_{\vec\eta}^{\tau'}(B|C)Z_{\vec\eta}^{\tau'}(A|1,C)$ of (8) such that

$$Z_{\vec\eta}^{\tau}(A|1,B) = \sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\right)^k\int_{i\infty}^{\tau}d\tau_1\int_{i\infty}^{\tau_1}d\tau_2\ldots\int_{i\infty}^{\tau_{k-1}}d\tau_k$$
$$\times\sum_{C\in S_{n-1}}(D_{\vec\eta}^{\tau_k}\cdot\ldots\cdot D_{\vec\eta}^{\tau_2}\cdot D_{\vec\eta}^{\tau_1})(B|C)Z_{\vec\eta}^{i\infty}(A|1,C) \quad (10)$$

with matrix products $D^{\tau_k}_{\vec\eta}\cdot\ldots\cdot D^{\tau_2}_{\vec\eta}D^{\tau_1}_{\vec\eta}$. As an initial value, the degeneration $Z^{i\infty}_{\vec\eta}$ at the cusp $\tau\to i\infty$ will be expressed in terms of disk integrals with two additional punctures from the pinching of the A cycle in Fig. 1.

As will be detailed in [12], the entire τ dependence of $D^\tau_{\vec\eta}$ is carried by Weierstrass functions (with $\mathrm{G}_0=-1$)

$$\wp(\eta,\tau)=-\frac{\mathrm{G}_0}{\eta^2}+\sum_{k=4}^{\infty}(k-1)\eta^{k-2}\mathrm{G}_k(\tau). \tag{11}$$

This allows us to decompose

$$D^\tau_{\vec\eta}=\sum_{k=0}^{\infty}(1-k)\mathrm{G}_k(\tau)r_{\vec\eta}(\epsilon_k), \tag{12}$$

where $r_{\vec\eta}(\epsilon_k)$ are $(n-1)!\times(n-1)!$ matrices whose entries are independent of τ, rational functions of η_j, linear in s_{ij} and may involve second derivatives $\partial_{\eta_i}\partial_{\eta_j}$. Note that $r_{\vec\eta}(\epsilon_2)=0$ and $r_{\vec\eta}(\epsilon_{2p-1})=0\;\forall\; p\in N$ by (11).

The main result: Based on (12), the open-string integrals (10) can be expressed in terms of iterated Eisenstein integrals

$$\gamma(k_1,k_2,\ldots,k_r|\tau)=\int_\tau^{i\infty}\frac{d\tau'}{2\pi i}\mathrm{G}_{k_r}(\tau')\gamma(k_1,\ldots,k_{r-1}|\tau'), \tag{13}$$

subject to $\gamma(\emptyset|\tau)=1$ and tangential-base-point regularization [17], e.g., $\gamma(0|\tau)=\tau/(2\pi i)$. As the main result of this work, we can therefore bring the open-string α' expansion into the following elegant form:

$$\begin{aligned}Z^\tau_{\vec\eta}(A|1,B)=&\sum_{r=0}^{\infty}\sum_{\substack{k_1,k_2,\ldots,k_r\\=0,4,6,8,\ldots}}\gamma(k_1,k_2,\ldots,k_r|\tau)\\&\times\prod_{j=1}^{r}(k_j-1)\sum_{C\in S_{n-1}}r_{\vec\eta}(\epsilon_{k_r}\ldots\epsilon_{k_2}\epsilon_{k_1})_B{}^C\\&\times Z^{i\infty}_{\vec\eta}(A|1,C),\end{aligned} \tag{14}$$

where $r_{\vec\eta}(\epsilon_{k_r}\ldots\epsilon_{k_2}\epsilon_{k_1})\equiv r_{\vec\eta}(\epsilon_{k_r})\cdots r_{\vec\eta}(\epsilon_{k_2})r_{\vec\eta}(\epsilon_{k_1})$. Since each order in α' is expressible in terms of EMZVs [6–8] but not all of the $\gamma(\ldots)$ are constructible from EMZVs, the $r_{\vec\eta}(\epsilon_k)$ must obey certain commutation relations. More specifically, the $r_{\vec\eta}(\epsilon_k)$ should preserve the commutation relations of Tsunogai's derivations ϵ_k dual to Eisenstein series [11] which select the $\gamma(\ldots)$ with a realization via EMZVs [9]. Hence, the $r_{\vec\eta}(\epsilon_k)$ are believed to furnish matrix representations of Tsunogai's derivations. In particular, (12) brings the differential equation (8) of $Z^\tau_{\vec\eta}$ into the same form as that of the elliptic Knizhnik-Zamolodchikov-Bernard associator whose τ derivative involves the derivations ϵ_k acting on its noncommutative arguments [10].

The decomposition of EMZVs into iterated Eisenstein integrals automatically incorporates all their relations over the rational numbers [9]. Moreover, the derivation of (14) does not rely on any relation among the Mandelstam invariants. The n point results of this Letter are valid for $\frac{1}{2}n(n-1)$ independent s_{ij}, and one can still impose momentum conservation when applying the α' expansion of $Z^\tau_{\vec\eta}$ to string amplitudes.

Examples for differential operators.—In this section, we present ($n\leq 3$)-point examples of the matrix-valued differential operators $D^\tau_{\vec\eta}$ in (8). All-multiplicity expressions as well as detailed derivations of the differential equations can be found in [12] (see, e.g., Section IV. B in the reference for the four-point case).

Two points allow for a single planar and nonplanar A-cycle integral (6) each,

$$\begin{aligned}Z^\tau_{\eta_2}(1,2|1,2)&=\int_0^1 dv_2\Omega(v_{12},\eta_2,\tau)e^{s_{12}\mathcal{G}(v_{12},\tau)},\\Z^\tau_{\eta_2}\left(\begin{matrix}2\\1\end{matrix}\middle|1,2\right)&=\int_0^1 dv_2\Omega\left(v_{12}+\frac{\tau}{2},\eta_2,\tau\right)e^{s_{12}\mathcal{G}(v_{12}+\tau/2,\tau)}.\end{aligned} \tag{15}$$

Their τ derivatives resulting from (5), (9) and integration by parts with respect to v_2 take the universal form

$$2\pi i\partial_\tau Z^\tau_{\eta_2}(*|1,2)=s_{12}\left(\frac{1}{2}\partial^2_{\eta_2}-\wp(\eta_2,\tau)-2\zeta_2\right)Z^\tau_{\eta_2}(*|1,2), \tag{16}$$

so one can read off the scalar differential operator in (8) and the resulting representation of the derivations,

$$\begin{aligned}D^\tau_{\eta_2}(2|2)&=s_{12}\left(\frac{1}{2}\partial^2_{\eta_2}-\wp(\eta_2,\tau)-2\zeta_2\right),\\r_{\eta_2}(\epsilon_0)&=s_{12}\left(\frac{1}{\eta_2^2}+2\zeta_2-\frac{1}{2}\partial^2_{\eta_2}\right),\qquad r_{\eta_2}(\epsilon_{k\geq4})=s_{12}\eta_2^{k-2}.\end{aligned} \tag{17}$$

Note that various combinations of iterated Eisenstein integrals drop out from the two-point instance of (14) since commutators $[r_{\eta_2}(\epsilon_{k_1}),r_{\eta_2}(\epsilon_{k_2})]$ with $k_1,k_2\geq4$ vanish.

Three points give rise to A-cycle integrals

$$\begin{aligned}Z^\tau_{\eta_2,\eta_3}(*|1,2,3)=&\int_{\mathcal{C}(*)}dz_2dz_3\Omega(z_{12},\eta_2+\eta_3,\tau)\\&\times\Omega(z_{23},\eta_3,\tau)e^{s_{12}\mathcal{G}(z_{12},\tau)+s_{13}\mathcal{G}(z_{13},\tau)+s_{23}\mathcal{G}(z_{23},\tau)}\end{aligned} \tag{18}$$

that mix under τ derivatives ($s_{12\ldots p}\equiv\sum_{1\leq i<j}^{p}s_{ij}$),

$$\begin{aligned}&2\pi i\partial_\tau Z^\tau_{\eta_2,\eta_3}(*|1,2,3)\\&=\Bigg(-2\zeta_2 s_{123}+s_{12}\left[\frac{1}{2}\partial^2_{\eta_2}-\wp(\eta_2+\eta_3,\tau)\right]\\&\quad+s_{13}\left[\frac{1}{2}\partial^2_{\eta_3}-\wp(\eta_3,\tau)\right]\\&\quad+s_{23}\left[\frac{1}{2}(\partial_{\eta_2}-\partial_{\eta_3})^2-\wp(\eta_3,\tau)\right]\Bigg)Z^\tau_{\eta_2,\eta_3}(*|1,2,3)\\&\quad+s_{13}[\wp(\eta_2+\eta_3,\tau)-\wp(\eta_3,\tau)]Z^\tau_{\eta_2,\eta_3}(*|1,3,2).\end{aligned} \tag{19}$$

The resulting matrix entries of the 2×2 differential operator in (8) read

$$D^\tau_{\eta_2,\eta_3}(2,3|2,3) = -2\zeta_2 s_{123} + s_{12}\left[\frac{1}{2}\partial^2_{\eta_2} - \wp(\eta_2+\eta_3,\tau)\right] + s_{23}\left[\frac{1}{2}(\partial_{\eta_2}-\partial_{\eta_3})^2 - \wp(\eta_3,\tau)\right] + s_{13}\left[\frac{1}{2}\partial^2_{\eta_3} - \wp(\eta_3,\tau)\right]$$
$$D^\tau_{\eta_2,\eta_3}(2,3|3,2) = s_{13}[\wp(\eta_2+\eta_3,\tau) - \wp(\eta_3,\tau)], \quad (20)$$

and the first row is always sufficient to generate the remaining entries via permutations of s_{ij} and η_j, e.g.,

$$D^\tau_{\eta_2,\eta_3}(3,2|3,2) = D^\tau_{\eta_2,\eta_3}(2,3|2,3)|^{s_{12}\leftrightarrow s_{13}}_{\eta_2\leftrightarrow\eta_3},$$
$$D^\tau_{\eta_2,\eta_3}(3,2|2,3) = D^\tau_{\eta_2,\eta_3}(2,3|3,2)|^{s_{12}\leftrightarrow s_{13}}_{\eta_2\leftrightarrow\eta_3}. \quad (21)$$

One can read off the 2×2 matrix representations of the derivations ($k\neq 2$),

$$r_{\eta_2,\eta_3}(\epsilon_k) = \delta_{k,0}\left(2\zeta_2 s_{123} - \frac{1}{2}s_{23}(\partial_{\eta_2}-\partial_{\eta_3})^2 - \frac{1}{2}s_{12}\partial^2_{\eta_2} - \frac{1}{2}s_{13}\partial^2_{\eta_3}\right)1_{2\times 2} + \eta_{23}^{k-2}\begin{pmatrix} s_{12} & -s_{13} \\ -s_{12} & s_{13}\end{pmatrix} + \eta_2^{k-2}\begin{pmatrix} 0 & 0 \\ s_{12} & s_{12}+s_{23}\end{pmatrix} + \eta_3^{k-2}\begin{pmatrix} s_{13}+s_{23} & s_{13} \\ 0 & 0\end{pmatrix}, \quad (22)$$

where $[r_{\eta_2,\eta_3}(\epsilon_{k_1\geq 4}), r_{\eta_2,\eta_3}(\epsilon_{k_2\geq 4})]$ no longer vanish individually, and relations in the derivation algebra [9,11,18] hold nontrivially.

Examples for initial values.—This section is dedicated to the degeneration of A-cycle integrals (6) at the cusp $\tau\to i\infty$ which enters the α' expansion (14) as an initial value.

Generalities: The behavior of A-cycle integrals at the cusp is most conveniently studied in the variables

$$\sigma_j = e^{2\pi i z_j}, \qquad dz_j = \frac{d\sigma_j}{2\pi i\sigma_j}, \qquad G_{ij} = i\pi\frac{\sigma_i+\sigma_j}{\sigma_i-\sigma_j}, \quad (23)$$

where the planar Green function and Kronecker-Eisenstein series degenerate to ($\sigma_{ji}\equiv\sigma_j-\sigma_i$)

$$\lim_{\tau\to i\infty}\Omega(v_{ij},\eta,\tau) = \pi\cot(\pi\eta) + G_{ij},$$
$$\lim_{\tau\to i\infty}\mathcal{G}(v_{ij},\tau) = \frac{1}{2}\log(\sigma_i) + \frac{1}{2}\log(\sigma_j) - \log(\sigma_{ji}). \quad (24)$$

Their nonplanar analogs take an even simpler form,

$$\lim_{\tau\to i\infty}\Omega\left(v_{ij}+\frac{\tau}{2},\eta,\tau\right) = \frac{\pi}{\sin(\pi\eta)}, \qquad \lim_{\tau\to i\infty}\mathcal{G}\left(v_{ij}+\frac{\tau}{2},\tau\right) = 0. \quad (25)$$

Since string-theory applications of (14) involve the coefficients with respect to η_j, we will need the expansions

$$\pi\cot(\pi\eta) = \frac{1}{\eta} - 2\sum_{k=1}^\infty \zeta_{2k}\eta^{2k-1},$$
$$\frac{\pi}{\sin(\pi\eta)} = \frac{1}{\eta} + \sum_{k=1}^\infty \frac{2^{2k-1}-1}{2^{2k-2}}\zeta_{2k}\eta^{2k-1}. \quad (26)$$

As will be detailed in [12], the σ_j integration in n-point $Z^{i\infty}_{\vec\eta}$ lines up with explicitly known combinations of $N=(n+2)$-point disk integrals [19]

$$Z^{\rm tree}(a_1,a_2,\ldots,a_N|1,2,\ldots,N) = \int_{-\infty<\sigma_{a_1}<\sigma_{a_2}<\ldots<\sigma_{a_N}<\infty} \times \frac{d\sigma_1 d\sigma_2\ldots d\sigma_N}{{\rm vol}{\rm SL}_2(\mathbb{R})}\frac{\prod_{i<j}^N|\sigma_{ij}|^{-s_{ij}}}{\sigma_{12}\sigma_{23}\ldots\sigma_{N-1,N}\sigma_{N,1}}. \quad (27)$$

The two extra punctures $n+1\to +$ and $n+2\to -$ are associated with Mandelstam invariants

$$s_{j+} = s_{j-} = -\frac{1}{2}\sum_{1\leq i\neq j}^n s_{ij}, \qquad s_{+,-} = \sum_{1\leq i<j}^n s_{ij}. \quad (28)$$

The α' expansion of (27) and therefore $Z^{i\infty}_{\vec\eta}$ involves multiple zeta values (MZVs) with $n_j\in\mathbb{N}$,

$$\zeta_{n_1,n_2,\ldots,n_r} = \sum_{0<k_1<k_2<\ldots<k_r}^\infty k_1^{-n_1}k_2^{-n_2}\cdots k_r^{-n_r}, \quad n_r\geq 2, \quad (29)$$

which can be systematically generated from the all-multiplicity methods of [20,21]. While the four-point expansions are captured by Riemann zeta values at $r=1$,

$$Z^{\rm tree}(1,2,+,-|1,2,-,+) = -\frac{1}{s_{12}}\exp\left(\sum_{k=2}^\infty\frac{\zeta_k}{k}[s_{12}^k + s_{2+}^k - (s_{12}+s_{2+})^k]\right), \quad (30)$$

disk integrals (27) at $N\geq 5$ points additionally involve MZVs (29) at higher depth $r\geq 2$ [22].

Two points: Planar initial values at two points descend from four-point tree-level integrals,

$$Z^{i\infty}_{\eta_2}(1,2|1,2) = \pi\cot(\pi\eta_2)2i\sin\left(\frac{\pi s_{12}}{2}\right) \times\int_0^1\frac{d\sigma_2}{2\pi i\sigma_2}\sigma_2^{s_{12}/2}(1-\sigma_2)^{-s_{12}} = \pi\cot(\pi\eta_2)\frac{\Gamma(1-s_{12})}{[\Gamma(1-\frac{s_{12}}{2})]^2}. \quad (31)$$

The factor of $2i\sin(\pi s_{12}/2)$ and similar trigonometric functions below stem from contour deformations detailed in [12]. The gamma functions with standard α' expansion

$$\frac{\Gamma(1-s_{12})}{[\Gamma(1-\frac{s_{12}}{2})]^2} = \exp\left(\sum_{k=2}^{\infty}\frac{\zeta_k}{k}(1-2^{1-k})s_{12}^k\right) = 1 + \frac{1}{4}s_{12}^2\zeta_2 + \frac{1}{4}s_{12}^3\zeta_3 + \frac{19}{160}s_{12}^4\zeta_2^2 + \mathcal{O}(\alpha'^5) \tag{32}$$

arise from the kinematic limit (28) of (30) and do not appear in the nonplanar counterpart of (31)

$$Z_{\eta_2}^{i\infty}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\Big|1,2\Big) = \frac{\pi}{\sin(\pi\eta_2)}. \tag{33}$$

Three points: Degenerate A-cycle integrals at three points introduce five-point disk integrals,

$$\begin{aligned} &Z_{\eta_2,\eta_3}^{i\infty}(1,a_2,a_3|1,2,3) \\ &= \pi^2\left(\cot(\pi\eta_{23})\cot(\pi\eta_3) + \frac{s_{13}}{s_{123}}\right)I^{\text{tree}}(1,a_2,a_3|1) \\ &\quad + \pi\left(\cot(\pi\eta_{23}) + \frac{s_{23}}{s_{12}}\cot(\pi\eta_3)\right)I^{\text{tree}}(1,a_2,a_3|G_{23}), \end{aligned} \tag{34}$$

where

$$\begin{aligned} I^{\text{tree}}(1,a_2,a_3|1) = -\frac{1}{2\pi^2}\Bigg[&\sin\left(\frac{\pi}{2}(s_{1a_2}+s_{23})\right)\sin\left(\frac{\pi}{2}s_{1a_3}\right) \\ &\times(Z^{\text{tree}}(+,a_2,a_3,1,-|+,2,3,-,1) \\ &+ Z^{\text{tree}}(+,a_2,a_3,1,-|+,3,2,-,1)) \\ &+ (2\leftrightarrow 3)\Bigg], \\ I^{\text{tree}}(1,a_2,a_3|G_{23}) = \frac{1}{2\pi}\Bigg[&\sin\left(\frac{\pi}{2}(s_{1a_2}+s_{23})\right)\cos\left(\frac{\pi}{2}s_{1a_3}\right) \\ &\times(Z^{\text{tree}}(+,a_2,a_3,1,-|+,2,3,-,1) \\ &- Z^{\text{tree}}(+,a_2,a_3,1,-|+,3,2,-,1)) \\ &+ (2\leftrightarrow 3)\Bigg]. \end{aligned} \tag{35}$$

Their leading low-energy orders read [12]

$$\begin{aligned} I^{\text{tree}}(1,2,3|1) &= \frac{1}{2} + \frac{\zeta_2}{8}(s_{12}^2 + s_{13}^2 + s_{23}^2) + \mathcal{O}(\alpha'^3), \\ I^{\text{tree}}(1,2,3|G_{23}) &= \frac{1}{s_{23}} + \frac{\zeta_2}{4s_{23}}(s_{12}+s_{13}+s_{23})^2 + \mathcal{O}(\alpha'^2), \end{aligned} \tag{36}$$

and exemplify that integrals over k factors of G_{ij} in (23) may have up to k kinematic poles.

Non-planar three-point initial values in turn boil down to four-point disk integrals with α' expansions in (32),

$$\begin{aligned} Z_{\eta_2,\eta_3}^{i\infty}\begin{pmatrix} 3 \\ 1,2 \end{pmatrix}\Big|1,2,3\Big) &= \frac{\pi^2\cot(\pi\eta_{23})}{\sin(\pi\eta_3)}\frac{\Gamma(1-s_{12})}{[\Gamma(1-\frac{s_{12}}{2})]^2}, \\ Z_{\eta_2,\eta_3}^{i\infty}\begin{pmatrix} 3 \\ 1,2 \end{pmatrix}\Big|1,3,2\Big) &= \frac{\pi^2}{\sin(\pi\eta_{23})\sin(\pi\eta_2)}\frac{\Gamma(1-s_{12})}{[\Gamma(1-\frac{s_{12}}{2})]^2}. \end{aligned} \tag{37}$$

Conclusions and further directions.—In this Letter we presented a method to expand a generating series of genus-one integrals (6) relevant to one-loop open-string amplitudes. At each order in the inverse string tension α', our main result (14) pinpoints the accompanying EMZVs in their minimal and canonical representation via iterated Eisenstein integrals.

Genus-zero integrals relevant to open-string tree amplitudes obey Knizhnik-Zamolodchikov equations with a characteristic linear factor of α' on their right-hand side [20]. This structure is analogous to the ε form of differential equations among Feynman integrals with dimensional-regularization parameter ε [5,23], suggesting a correspondence between α' and ε. By the linearity of the differential operators $D_{\vec\eta}^{\tau}$ in $s_{ij} = -2\alpha' k_i \cdot k_j$, the Knizhnik-Zamolodchikov-Bernard-type equation (8) also becomes linear in α'. So our results generalize this intriguing correspondence to genus one and provide the string-theory analog of the ε form for differential equations of elliptic Feynman integrals [5].

The generating functions $Z_{\vec\eta}^{\tau}$ are expected to comprise any moduli-space integral in massless one-loop amplitudes of open bosonic strings and superstrings upon expansion in η_j. Accordingly, they are proposed to generalize the universal disk integrals (27) that appear in the double-copy representation of string tree-level amplitudes [19,24]. Hence, the study of the genus-one integrals $Z_{\vec\eta}^{\tau}$ is an essential step towards universal double-copy structures in one-loop amplitudes of different string theories that generalize those of the superstring [25].

The generating functions $Z_{\vec\eta}^{\tau}$ can be adapted to a closed-string context, encoding the integrals over torus punctures in one-loop amplitudes of type-II, heterotic, and closed bosonic string theories. Closed-string analogs of $Z_{\vec\eta}^{\tau}$ will be shown [26] to obey similar differential equations and to shed new light on the properties of modular graph forms [27] including their relation with open-string amplitudes [28].

Moreover, the method of this work to infer moduli-space integrals from differential equations should be applicable at higher loops. In the same way as disk integrals were used as the initial value for our one-loop results, higher-genus integrals in string amplitudes are expected to obey differential equations with respect to complex-structure moduli such that their separating and nonseparating degenerations set the initial conditions. It would be interesting to explore a differential-equation approach of this type to the higher-genus modular graph functions of [29].

In summary, our new approach to one-loop open-string amplitudes via differential equations connects with state-of-the-art techniques in particle phenomenology and provides explicit matrix representations of profound number-theoretic structures. As will be elaborated in [12], our results manifest important formal properties of string amplitudes such as uniform transcendentality, coaction formulae and the dropout of twisted EMZVs from non-planar open-string amplitudes.

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