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Residual Bell Nonlocality

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Abstract

This report provides a new theoretical measure for the nonlocality of an arbitrary three-qubit pure state system similar to the method used to describe tripartite entanglement, resulting in a concept referred to as residual nonlocality, η. This report also investigates the special cases that can be encountered when using η. This method assigns a numerical value between 0 and 1 in order to indicate the degree of nonlocality between three-qubits. It was discovered that η has the characteristic of being consistently larger or equal to the value found for the residual entanglement which can provide further insights regarding the relation between nonlocality and entanglement.

Abstract

I rapporten föreslås och analyseras ett nytt teoretisk mått för ickelokalitet hos tre-kvantbitsystem på ett liknande sätt till metoden som används för tredelad sammanflätningar. Detta ger en koncept som vi har valt att benämna residual ickelokalitet η. Rapporten undersöker också specialfall som kan påträffas när man använder η. Metoden som läggs fram i rapporten ger ett numeriskt värde mellan 0 och 1 för att visa graden av ickelokalitet mellan kvantbitarna. Vår undersökning visar att η kommer under alla sammanhang vara större eller lika med den graden av tredelad sammanflätning i samma system vilket kan ge en bättre förståelse av relationen mellan sammanflätning och ickelokalitet.

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1 Introduction

Entanglement has had a persistent place at the center of discussions regarding the limitations and rules of quantum mechanics since Einstein, Podolosky, Rosen (EPR) [1] released their paper questioning the completeness of quantum mechanics. This led to John Bell's paper in 1964 and the introduction of further concepts such as nonlocality and entanglement [2].

Let us first begin by describing what entanglement and nonlocality respectively are. Entanglement comes about when two quantum systems are created from the same source and allows for the two quantum systems to in a way correlate to each other. Nonlocality is a result of a violation in a set of relations referred to as Bell's inequality and explains how these two entangled quantum systems can influence each other in a direct manner while not having any direct contact.

Until 1989 entanglement was believed to be the same as the violation of the Bell inequalities and by proxy nonlocality. It was proven that the violation of the Bell inequalities is in no way a requirement for the entanglement to exist [3]. Meaning that entanglement by itself is a much broader and larger concept than that of nonlocality and violations of the Bell inequality, making nonlocality more of a subset of entanglement, where all systems that do violate the Bell inequality are in fact entangled but not all entangled systems violate the Bell inequality.

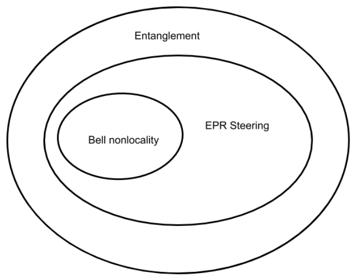


Figure 1: The following figure showes the relation between entanglement and nonlocallity as general concepts in the general case [8].

Currently both quantum nonlocality and quantum entanglement are central concepts used in modern quantum information. Entanglement is used in many applications of quantum information due to the fact that it is a quantifiable resource for communication and allows for phenomena such as teleportation of quantum states [4]. In the span of time between 1989 until now further discoveries have been made regarding the nature of few-qubit systems which could lead to a better understanding of the larger systems as well. Most of the research done regarding entanglement is centered around two-qubit systems.

The focus on two-qubit entanglement has been in large due to the relation between entanglement and nonlocality in a two-qubit pure state system, where, as proven by Gisin in such systems, entanglement and nonlocality are basically equivalent to one another in terms of resources they provide [5]. As previously mentioned there are many ways to measure the entanglement for a two-qubit system and to assign the degree of entanglement a quantifiable value. One such method that is going to be mainly used in this project is concurrence [7], which provides a distinct value between 0 and 1 for the degree of entanglement in two-qubit systems.

There has been further progress regarding larger qubit systems such as the tripartite system that will be discussed in this report. The equivalency proven by Gisin was only applicable to two-qubit pure states, thus it cannot be applied to the tripartite system meaning that nonlocality and concurrence are not equivalent. For entanglement in such systems the measurement method is the concept of residual entanglement [9], while for nonlocality the answer is less clear. Recently a paper [12] has made interesting progress on the subject. Here, we attempt to answer the question somewhat differently than in [12] by adapting the concept of residual entanglement to nonlocality between three qubits split among Alice, Bob and Charlie, yielding a concept that may be called 'residual nonlocality' η. Explicitly, for a pure three-qubit state $\left|\Psi^{abc}\right\rangle$, we define η as

$$\eta = C^2(\Psi^{a(bc)}) - B^2(\rho^{ab}) - B^2(\rho^{ac}),$$

where C and B are concurrence [7] and degree of nonlocality [13], respectively.
Using this approach, this report aims to examine the following properties:

(i) Is η the same under permutation of a, b, and c?

(ii) Is $\eta \geq 0$ and if so is the inequality strict (i.e., $\eta > 0$) for non-product states?

Both these properties are consistently found for the tripartite entanglement measure and due to the similarity between the two concepts of nonlocality and entanglement these properties are to be further studied.

2 Background

2.1 Nonlocality

This section aims to provide some historical background and fundamental theory regarding entanglement and nonlocality with a main focus on the EPR papers followed by the theory section which directly discusses the algebra and the formulae used to solve the problems.
Let us begin by considering a two electron system in a spin singlet state such that the total spin of the system would be zero. Let us continue further and assign each of the two resulting electrons to either Alice or Bob, such that they each have access to only one of the two electrons which are entangled. Assume that Alice and Bob have a relatively large distance between them. This system can be expressed as:

$$\left|\psi\right\rangle_z = \frac{1}{\sqrt{2}}(\left|\uparrow_{zA}\downarrow_{zB}\right\rangle - \left|\downarrow_{zA}\uparrow_{zB}\right\rangle), \tag{1}$$

where $\left|\psi\right\rangle_z$ is the expected measurements along the z axis and A and B stand for Alice and Bob. Now assume Alice takes the electron that she has access to and makes a measurement of it. There is a 50-50 chance for either up and down, by measuring Alice finds out what state her electron is in. If Bob chooses to make a measurement along the z axis after Alice, his result is already decided to be the opposite of Alice's, while if he chooses to make a measurement along any other axis such as the x axis the results would not be effected in by Alice's measurement at all. For this case we know that the probability in the x direction would be identical to that of the z direction since spin singlet states have no preference regarding any distinct direction, making the singlet state along the x axis to be:

$$|\psi\rangle_x = \frac{1}{\sqrt{2}}(|\downarrow_{xA}\uparrow_{xB}\rangle - |\uparrow_{xA}\downarrow_{xB}\rangle). \tag{2}$$

Given this we now know all possible outcomes of this experiment between Alice and Bob and we see how Alice can affect Bob's results from miles away.

Table 1: All possible results for Bob. Note that if Alice meassures along the x axis the results would be reversed instead and that x,y and z are interchangeable in regards to the relation.

Possible Alice measurments	Possible Bob measurements
$\uparrow_z$	$\downarrow_z$
$\uparrow_z$	$\downarrow_x$
$\uparrow_z$	$\uparrow_x$
$\downarrow_z$	$\uparrow_z$
$\downarrow_z$	$\downarrow_x$
$\downarrow_z$	$\uparrow_x$

It was exactly this interaction and the fact that one particle could alter the probability distribution of another that made Einstein uncomfortable. Due to the distance in between them and the speed of the interaction, the system was surpassing the speed of light given the way the information was exchanged between the two electrons after the measurement. At the time there where two general approaches regarding the explanation behind this phenomena.

The first approach assumes that there exists a hidden variable that decides the states of all parties even before the observation and dictates all the possible setups. Meaning that the states in a setup are already decided and by observing them we simply get to see what has been there this whole time, implying that the information never travels between Alice and Bob but instead it just is confirmed by them. This argument was originally made in the paper "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?" By Einstein Podolsky and Rosen (EPR) [1].

The second approach is that the quantum objects are by nature probabilistic and that their distinct positions are not exclusive to their local placements and hence they are "non-local", which is the current and generally accepted method.

Given that Einstein had a more "mechanistic" view he pushed for the notion of a hidden variable while people like Niels Bohr pushed for the more probabilistic variant at the time. There was no decisive answer to this problem until 1964 where John Bell released a paper disproving the notion of a hidden variable theory and its implications.

He started by making several logical arguments.

1) That in case the hidden variable theory was true then it should apply to all cases and all circumstances since it is a general concept.

2) That assuming the existence of a hidden variable in a system of many particles, there should exist a finite number of cases for each possible combinations of states along however many chosen axes. While the given states still cancel each other to give a total spin zero as it is observed/required.

These assumptions are rather self explanatory since the first one is required for the theory to be accepted as a general "law" of nature and the second is a direct implication of assuming everything precedes even if we could only observe one unit vector at a time. He followed this up by considering 3 unit vectors that are not necessarily mutually orthogonal to each other as opposed to ($\hat{x}$,$\hat{y}$,$\hat{z}$), let us call these 3 unit vectors ($\hat{\alpha}$,$\hat{\beta}$,$\hat{\gamma}$). So now take a system of N particles split into pairs among Alice and Bob and measured along the three aforementioned axes. It is worth stressing that there is no possible way to measure or ensure that a particle belongs to one exact state experimentally but given that they can and should exist if there is a hidden variable allows for the to the following table and description while it would be impossible in practice.

Table 2

Population of state	Possible Alice measurments	Possible Bob measurements
N_1	$(\uparrow_\alpha, \uparrow_\beta, \uparrow_\gamma)$	$(\downarrow_\alpha, \downarrow_\beta, \downarrow_\gamma)$
N_2	$(\uparrow_\alpha, \uparrow_\beta, \downarrow_\gamma)$	$(\downarrow_\alpha, \downarrow_\beta, \uparrow_\gamma)$
N_3	$(\uparrow_\alpha, \downarrow_\beta, \uparrow_\gamma)$	$(\downarrow_\alpha, \uparrow_\beta, \downarrow_\gamma)$
N_4	$(\downarrow_\alpha, \uparrow_\beta, \uparrow_\gamma)$	$(\uparrow_\alpha, \downarrow_\beta, \downarrow_\gamma)$
N_5	$(\uparrow_\alpha, \downarrow_\beta, \downarrow_\gamma)$	$(\downarrow_\alpha, \uparrow_\beta, \uparrow_\gamma)$
N_6	$(\downarrow_\alpha, \uparrow_\beta, \downarrow_\gamma)$	$(\uparrow_\alpha, \downarrow_\beta, \uparrow_\gamma)$
N_7	$(\downarrow_\alpha, \downarrow_\beta, \uparrow_\gamma)$	$(\uparrow_\alpha, \uparrow_\beta, \downarrow_\gamma)$
N_8	$(\downarrow_\alpha, \downarrow_\beta, \downarrow_\gamma)$	$(\uparrow_\alpha, \uparrow_\beta, \uparrow_\gamma)$

Now let us assume Alice makes a measurement and find $\uparrow_\alpha$ and Bob follows that up and makes a measurement and finds $\uparrow_\beta$ then given the table above it is easy to assume that the results would be part of the group 3 or 5 which means that the sum of total possibilities would be $N_3 + N_5$ and logically given assumption number two we can make the following statement:

$$N_3 + N_5 < (N_3 + N_7) + (N_5 + N_2), \tag{3}$$

which is also equivalent to

$$\frac{N_3 + N_5}{\sum_i^8 N_i} < \frac{(N_3 + N_7) + (N_5 + N_2)}{\sum_i^8 N_i},$$

which is the definition of classical probability and allows for the rewriting of this statement as:

$$\begin{aligned} P(\uparrow_\alpha, \uparrow_\beta) &= \frac{N_3 + N_5}{\sum_i^8 N_i}, \\ P(\uparrow_\alpha, \uparrow_\gamma) &= \frac{N_2 + N_5}{\sum_i^8 N_i}, \\ P(\uparrow_\gamma, \uparrow_\beta) &= \frac{N_3 + N_7}{\sum_i^8 N_i}. \end{aligned} \tag{4}$$

Inserting equation (4) into equation (3) we get a general rule that needs to allways apply to all systems regarding the probability of our setups.

$$P(\uparrow_\alpha, \uparrow_\beta) < P(\uparrow_\alpha, \uparrow_\gamma) + P(\uparrow_\gamma, \uparrow_\beta). \tag{5}$$

Notice that all these factors are still dependent on the chosen axes and in this case they are the arbitrary axes ($\hat{\alpha}$,$\hat{\beta}$,$\hat{\gamma}$) which are as stated not necessarily orthogonal to one another. This explains how the probability of a case happening such as above would depend on the relation between the two axis i.e the angle between them allowing for the probability to be written as [6]

$$P(\uparrow_\alpha, \uparrow_\beta) = \frac{1}{2}\sin^2\frac{\theta_{\alpha\beta}}{2}, \tag{6}$$

where $\theta_{\alpha B}$ is the angle between the two axes and this allows equation (5) to be expressed as:

$$\sin^2\frac{\theta_{\alpha\beta}}{2} < \sin^2\frac{\theta_{\alpha\gamma}}{2} + \sin^2\frac{\theta_{\gamma\beta}}{2}. \tag{7}$$

This condition that is built up from the base assumptions of the hidden variable theory is not always fulfilled take for example cases where $\theta_{\alpha B} = \frac{\pi}{2}$ while $\theta_{\alpha\gamma} = \theta_{\gamma B} = \frac{\pi}{4}$. This system would lead to the statement that.

$$0.5 < 0.292,$$

which is simply not true, showing that the notion of a hidden variable theory is not consistent in all cases and cannot be treated as a general solution. As a result causing that the only viable answer and option left is the idea of nonlocality and a truly probabilistic system that has no physical properties until it is observed. There are also other explanations and experimental methods used for observing nonlocality such as the CHSH method[16].

The idea of nonlocality by itself was considered revolutionary and extremely mind boggling but it allowed for large strides to be made in the line of quantum information such as teleportation of quantum states [4].

2.2 Quantum teleportation

Let us discuss teleportation in slightly more detail in order to better understand what nonlocality and entanglement allows for within a system consisting of entangled qubits. But before that let us take a moment to discuss the notation used in the report beyond this point.

As opposed to the case above this project uses $|0\rangle$ and $|1\rangle$ instead of $|\uparrow\rangle$ and $|\downarrow\rangle$ respectively. Using this notation one can express the entangled states without the use of the tensor product as $|010\rangle$ for example where the order of the statements is Alice, Bob and Charlie meaning that:

$$|\uparrow_A\rangle \otimes |\downarrow_B\rangle \otimes |\uparrow_C\rangle \Longleftrightarrow |010\rangle\,.$$

Now let us consider the case where Alice and Bob each have a qubit and their qubits are entangled with one another in such state:

$$\left|\psi^-_{AB}\right\rangle = \sqrt{\frac{1}{2}}(|0\rangle_A \otimes |1\rangle_B - |1\rangle_A \otimes |0\rangle_B). \tag{8}$$

Now consider that Alice's qubit is entangled with another qubit referred to as $|\epsilon\rangle$ which is in a superposition between the up and down states.

$$|\epsilon\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,,$$

where α and β are the probabilities of each state in the system and they sum to 1 as the total probability. Given this, Alice wants to move the superposition of α and β to Bob exactly as it is. At first glance this would seem impossible but given nonlocality it is doable. Let us first write out the total system and what it would be given the consideration of $|\epsilon\rangle$:

$$\begin{aligned}|\epsilon\rangle \otimes \left|\psi_{AB}^{-}\right\rangle = &\frac{\alpha}{\sqrt{2}}(|0\rangle_\epsilon \otimes |0\rangle_A \otimes |1\rangle_B - |0\rangle_\epsilon \otimes |1\rangle_A \otimes |0\rangle_B) \\ &+ \frac{\beta}{\sqrt{2}}(|1\rangle_\epsilon \otimes |0\rangle_A \otimes |1\rangle_B - |1\rangle_\epsilon \otimes |1\rangle_A \otimes |0\rangle_B).\end{aligned} \tag{9}$$

The equation above can be daunting at first but it can be simplified much further using a series of predefined superposition states known as Bell states.

$$\begin{aligned}\left|\Phi^{+}\right\rangle &= \sqrt{\frac{1}{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle), \\ \left|\Phi^{-}\right\rangle &= \sqrt{\frac{1}{2}}(|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle), \\ \left|\Psi^{+}\right\rangle &= \sqrt{\frac{1}{2}}(|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle), \\ \left|\Psi^{-}\right\rangle &= \sqrt{\frac{1}{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle),\end{aligned} \tag{10}$$

which then allows for the ordinary up and down states to be redefined as well:

$$\begin{aligned}|0\rangle \otimes |0\rangle &= \sqrt{\frac{1}{2}}(\left|\Phi^{+}\right\rangle + \left|\Phi^{-}\right\rangle), \\ |1\rangle \otimes |1\rangle &= \sqrt{\frac{1}{2}}(\left|\Phi^{+}\right\rangle - \left|\Phi^{-}\right\rangle), \\ |0\rangle \otimes |1\rangle &= \sqrt{\frac{1}{2}}(\left|\Psi^{+}\right\rangle + \left|\Psi^{-}\right\rangle), \\ |1\rangle \otimes |0\rangle &= \sqrt{\frac{1}{2}}(\left|\Psi^{+}\right\rangle - \left|\Psi^{-}\right\rangle).\end{aligned} \tag{11}$$

Now it is possible to use these definitions to describe the relation between the two qubits Alice has access to:

$$\begin{aligned}|\epsilon\rangle \otimes |\psi_{AB}\rangle = &\frac{\alpha}{2}((\left|\Psi_{\epsilon A}^{+}\right\rangle + \left|\Psi_{\epsilon A}^{-}\right\rangle) \otimes |1\rangle_B - (\left|\Phi_{\epsilon A}^{+}\right\rangle + \left|\Phi_{\epsilon A}^{-}\right\rangle) \otimes |0\rangle_B)) \\ &+ \frac{\beta}{2}((\left|\Psi_{\epsilon A}^{+}\right\rangle - \left|\Psi_{\epsilon A}^{-}\right\rangle) \otimes |1\rangle_B - (\left|\Phi_{\epsilon A}^{+}\right\rangle - \left|\Phi_{\epsilon A}^{-}\right\rangle) \otimes |0\rangle_B)).\end{aligned} \tag{12}$$

This can be further simplified to:

$$
\begin{aligned}
|\epsilon\rangle \otimes |\psi_{AB}\rangle = &\frac{1}{2}(|\Phi^{+}_{\epsilon A}\rangle) \otimes (\alpha |1\rangle_B - \beta |0\rangle_B) + \frac{1}{2}(|\Phi^{-}_{\epsilon A}\rangle) \otimes (\alpha |1\rangle_B + \beta |0\rangle_B) \\
&+ \frac{1}{2}(|\Psi^{+}_{\epsilon A}\rangle) \otimes (-\alpha |0\rangle_B + \beta |1\rangle_B) + \frac{1}{2}(|\Psi^{-}_{\epsilon A}\rangle) \otimes (-\alpha |0\rangle_B - \beta |1\rangle_B).
\end{aligned} \tag{13}
$$

So if all the qubits that Alice has access to become observed as an entagled state Bob's qubit becomes a combination of the original $|\epsilon\rangle$ state and depending on which Bell state Alice observes Bob can alter the phase of his qubits after Alice tells him about her results, effectively teleporting the original state. What is more interesting about this is that by technicality the speed of this teleportation can't be faster than light since it requires classical communication between Alice and Bob which is definitely slower than the speed of light.

2.3 Density Operators

The last bit of technical background needed in order to better understand this project is a small introduction to the notation systems and conventions generally applied in the study of quantum information and the allowed operations. This section is rather important since it is going to introduce the main linear-algebraic method that will be used for all topics after this point.

For most calculations within quantum mechanics the notation system of density vectors would be enough to explain many phenomena, but in quantum information an alternative tool referred to as density matrices or density operators generally noted with ρ is used. This allows one to describe a system with states that are not completely known. Take for example a system that consist of states $\Phi = \sum_i p_i \phi_i$ this system can then be expressed in the following manner as a density matrix.Due to the fact that this project mainly focuses on the pure state calculations only the density matrices for the aforementioned states will be discussed.

$$
\rho = \sum_i P_i |\phi_i\rangle \langle\phi_i| = \sum_i p_i p_i^* |\phi_i\rangle \langle\phi_i| = |\Phi\rangle \langle\Phi| .
$$

Here, P_i is the probability of each state and is the result of multiplying a vector based state function with its own complex conjugate and $\sum_i P_i = 1$. Using this system one can calculate the expectation value and even apply unitary operators to the ϕ_i states which is an unknown ensemble of pure states [11]. This notation system allows for many possibilities but there are only few that are required to understand this project.

The first of which is a distinction between the different types of density operators and how to effectively differentiate the types. In general density operator states are referred to as either mixed or pure states. In general, pure states are cases where the exact information about the quantum system are known. The mixed state is the combination of probabilities of the information about the quantum state. It is worth noting that different distributions of pure states can generate equivalent mixed states. This project chooses to focus on the dynamic of the pure states specifically which allows for each of the possible combination of Alice, Bob and Charlie's qubits as shown in table 8 to have their own distinct

probability of existing upon being observed.
There is a distinct requirement that has to be fulfilled for a system to be considered one made up of pure states. This requirement is for the trace operator of the square of system to be equal to 1, i.e., $tr(\rho^2) = 1$ note that this largely differs from the requirement of $tr(\rho) = 1$ which is a general requirement for a system to be considered a density operator without concern for whether it is pure or mixed.
This requirement implies that in order for a system to be considered a pure state it is required that $\sum_i^n P_i^2 = 1$ and the only way that such system would be possible is that a single P_i is equal to one while all other values for the probability in the ensemble are zero .
Now that the general requirement for the system is defined it is possible to discuss the possibilities this notation system allows for. One of the more useful factors in this notations is that in the case of entangled ensembles it is possible to reduce the ensembles and further simplify them through the use of the partial trace operator, meaning that in the case of an entangled ensemble the following rule applies to the density operators:

$$\rho^a = tr_B(\rho^{ab}),$$

$$\rho^b = tr_A(\rho^{ab}).$$

The final concept worth considering is the idea of the Schmidt decomposition which states that in the case of an entangled ensemble making a density operator both Alice and Bob would have the same exact probabilities in their singular density operators caused by the relation with the partial trace operator above, meaning that ρ^b and ρ^a have identical eigenstates to each other. These density operators can also be written as matrices even for an entangled states in such a manner:

$$\begin{bmatrix} |0\rangle_A \langle 0|_A \begin{Bmatrix} |0\rangle_B \langle 0|_B & |0\rangle_B \langle 1|_B \\ |1\rangle_B \langle 0|_B & |1\rangle_B \langle 1|_B \end{Bmatrix} & |0\rangle_A \langle 1|_A \begin{Bmatrix} |0\rangle_B \langle 0|_B & |0\rangle_B \langle 1|_B \\ |1\rangle_B \langle 0|_B & |1\rangle_B \langle 1|_B \end{Bmatrix} \\ |1\rangle_A \langle 0|_A \begin{Bmatrix} |0\rangle_B \langle 0|_B & |0\rangle_B \langle 1|_B \\ |1\rangle_B \langle 0|_B & |1\rangle_B \langle 1|_B \end{Bmatrix} & |1\rangle_A \langle 1|_A \begin{Bmatrix} |0\rangle_B \langle 0|_B & |0\rangle_B \langle 1|_B \\ |1\rangle_B \langle 0|_B & |1\rangle_B \langle 1|_B \end{Bmatrix} \end{bmatrix}. \tag{14}$$

This system continues in the same manner for tripartite systems and goes in the order of Alice, Bob, Charlie allowing for the calculations to be more flexible.

3 Theory

In order to explain any quantum system one needs to first establish a wave function or an arbitrary superposition of the states just to be able to explain any number of states and their setups. This project is no different, one could simply establish a setup containing all possible entanglements in a tripartite systems and setting the requirement of normalized probability on it in such a manner:

$$|\psi\rangle = \lambda_0 |000\rangle + \lambda_1 |001\rangle + \lambda_2 |010\rangle + \lambda_3 |100\rangle + \lambda_4 |011\rangle + \lambda_5 |101\rangle + \lambda_6 |110\rangle + \lambda_7 |111\rangle . \tag{15}$$

Here $\sum \lambda_i^2 = 1$. This system would very likely work but it would make the procedure of formulaic calculations more complicated. Another alternative which was used for this process is to use a more simplified variant of equation (15). It is possible to rephrase the equation above using the Schmidt decomposition while containing all possible state compositions [14], which results in a much simpler system that only has 6 unknown factors as opposed to the 8 unknowns in equation (15).

$$|\psi\rangle = \lambda_0 |000\rangle + \lambda_1 e^{i\phi} |100\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle + \lambda_4 |111\rangle . \tag{16}$$

Here the idea is that the three cases of $|100\rangle$, $|010\rangle$, $|001\rangle$ can be explained as the same ket through the addition of a phase factor which is the exponential value seen in equation while the $|\langle\rangle\, 011)$ can be simplified away since it is equivalent to the spin reversed version of $|100\rangle$ (16). Note that all values for λ_i are real and that the only imaginary part comes from the exponential factor. Now given this arbitrary explanation of the states it is rather easy to find the arbitrary density matrix that would cover all possible combinations a tripartite system could have. This density matrix is found by taking the multiple of $|\psi\rangle$ with its complex conjugate and results in the density operator of the total system.

$$\begin{gathered}
\rho_{abc} = |\psi\rangle \langle\psi| = \lambda_0^2 |000\rangle \langle 000| + \lambda_0\lambda_1 e^{-i\phi} |000\rangle \langle 100| + \lambda_0\lambda_2 |000\rangle \langle 101| + \lambda_0\lambda_3 |000\rangle \langle 110| + \lambda_0\lambda_4 |000\rangle \langle 111| \\
+ \lambda_1\lambda_0 e^{i\phi} |100\rangle \langle 000| + \lambda_1^2 |100\rangle \langle 100| + \lambda_1\lambda_2 e^{i\phi} |100\rangle \langle 101| + \lambda_1\lambda_3 e^{i\phi} |100\rangle \langle 110| + \lambda_1\lambda_4 e^{i\phi} |100\rangle \langle 111| \\
+ \lambda_2\lambda_0 |101\rangle \langle 000| + \lambda_2\lambda_1 e^{-i\phi} |101\rangle \langle 100| + \lambda_2^2 |101\rangle \langle 101| + \lambda_2\lambda_3 |101\rangle \langle 110| + \lambda_2\lambda_4 |101\rangle \langle 111| + \\
\lambda_3\lambda_0 |110\rangle \langle 000| + \lambda_3\lambda_1 e^{-i\phi} |110\rangle \langle 100| + \lambda_3\lambda_2 |110\rangle \langle 101| + \lambda_3^2 |110\rangle \langle 110| + \lambda_3\lambda_4 |110\rangle \langle 111| \\
+ \lambda_4\lambda_0 |111\rangle \langle 000| + \lambda_4\lambda_1 e^{-i\phi} |111\rangle \langle 100| + \lambda_4\lambda_2 |111\rangle \langle 101| + \lambda_4\lambda_3 |111\rangle \langle 110| + \lambda_4^2 |111\rangle \langle 111| .
\end{gathered}$$

This resulting density operator applies to the abc case which tells us very little. It is possible to use the fact that density operators are partial reducible to find the arbitrary density operators for Alice, Bob, Charlie and the density operators for Alice relative Bob and Charlie or reversed.

The following equations are defined using the partial trace function on the density operator above:

$$\begin{aligned}
\rho^{ab} = tr_C(\rho_{ABC}) &= \lambda_0^2 |00\rangle \langle 00| + \lambda_0\lambda_1 e^{-i\phi} |00\rangle \langle 10| + \lambda_0\lambda_3 |00\rangle \langle 11| \\
&+ \lambda_1\lambda_0 e^{i\phi} |10\rangle \langle 00| + (\lambda_1^2 + \lambda_2^2) |10\rangle \langle 10| + (\lambda_1\lambda_3 e^{i\phi} + \lambda_2\lambda_4) |10\rangle \langle 11| \\
&+ \lambda_3\lambda_0 |11\rangle \langle 00| + (\lambda_3\lambda_1 e^{-i\phi} + \lambda_4\lambda_2) |11\rangle \langle 10| + (\lambda_3^2 + \lambda_4^2) |11\rangle \langle 11| .
\end{aligned} \tag{17}$$

The same operation is repeated again but it is instead calculated with the trace of Bob instead.

$$\begin{aligned}
\rho^{ac} = tr_B(\rho_{ABC}) &= \lambda_0^2 |00\rangle \langle 00| + \lambda_0\lambda_1 e^{-i\phi} |00\rangle \langle 10| + \lambda_0\lambda_2 |00\rangle \langle 11| \\
&+ \lambda_1\lambda_0 e^{i\phi} |10\rangle \langle 00| + (\lambda_1^2 + \lambda_3^2) |10\rangle \langle 10| + (\lambda_1\lambda_2 e^{i\phi} + \lambda_3\lambda_4) |10\rangle \langle 11| \\
&+ \lambda_2\lambda_0 |11\rangle \langle 00| + (\lambda_2\lambda_1 e^{-i\phi} + \lambda_4\lambda_3) |11\rangle \langle 10| + (\lambda_2^2 + \lambda_4^2) |11\rangle \langle 11| .
\end{aligned} \tag{18}$$

The final case is the trace of Alice which gives both the largest results and the most pattern like variant of the results.

$$\begin{aligned}\rho^{bc} = tr_A(\rho_{ABC}) = (\lambda_0^2+\lambda_1^2)\left|00\right\rangle\left\langle 00\right| + \lambda_1\lambda_2 e^{i\phi}\left|00\right\rangle\left\langle 01\right| + \lambda_1\lambda_3 e^{i\phi}\left|00\right\rangle\left\langle 10\right| \\ + \lambda_1\lambda_4 e^{i\phi}\left|00\right\rangle\left\langle 11\right| + \lambda_1\lambda_2 e^{-i\phi}\left|01\right\rangle\left\langle 00\right| + \lambda_2^2\left|01\right\rangle\left\langle 01\right| + \lambda_2\lambda_3\left|01\right\rangle\left\langle 10\right| \\ + \lambda_2\lambda_4\left|01\right\rangle\left\langle 11\right| + \lambda_1\lambda_3 e^{-i\phi}\left|10\right\rangle\left\langle 00\right| + \lambda_3\lambda_2\left|10\right\rangle\left\langle 01\right| + \lambda_3^2\left|10\right\rangle\left\langle 10\right| \\ + \lambda_3\lambda_4\left|10\right\rangle\left\langle 11\right| + \lambda_1\lambda_4 e^{-i\phi}\left|11\right\rangle\left\langle 00\right| + \lambda_4\lambda_2\left|11\right\rangle\left\langle 01\right| + \lambda_4\lambda_3\left|11\right\rangle\left\langle 10\right| + \lambda_4^2\left|11\right\rangle\left\langle 11\right|.\end{aligned} \tag{19}$$

Now using equations (17),(18) and (19) it is possible to calculate the density operators for Alice,Bob and charlie with the possibility of cross checking the results to be sure.

$$\begin{aligned}\rho^a = tr_c(\rho^{ac}) = tr_B(\rho^{ab}) = \lambda_0^2\left|0\right\rangle\left\langle 0\right| + \lambda_0\lambda_1 e^{-i\phi}\left|0\right\rangle\left\langle 1\right| \\ + \lambda_0\lambda_1 e^{i\phi}\left|1\right\rangle\left\langle 0\right| + (\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2)\left|1\right\rangle\left\langle 1\right|.\end{aligned}$$

$$\begin{aligned}\rho^b = tr_A(\rho^{ab}) = tr_C(\rho^{bc}) = (\lambda_0^2+\lambda_1^2+\lambda_2^2)\left|0\right\rangle\left\langle 0\right| + (\lambda_3\lambda_1 e^{i\phi}+\lambda_2\lambda_4)\left|0\right\rangle\left\langle 1\right| \\ + (\lambda_3\lambda_1 e^{-i\phi}+\lambda_2\lambda_4)\left|1\right\rangle\left\langle 0\right| + (\lambda_3^2+\lambda_4^2)\left|1\right\rangle\left\langle 1\right|.\end{aligned} \tag{20}$$

$$\begin{aligned}\rho^c = tr_A(\rho^{ac}) = tr_B(\rho^{bc}) = (\lambda_0^2+\lambda_1^2+\lambda_3^2)\left|0\right\rangle\left\langle 0\right| + (\lambda_2\lambda_1 e^{i\phi}+\lambda_3\lambda_4)\left|0\right\rangle\left\langle 1\right| \\ + (\lambda_2\lambda_1 e^{-i\phi}+\lambda_3\lambda_4)\left|1\right\rangle\left\langle 0\right| + (\lambda_2^2+\lambda_4^2)\left|1\right\rangle\left\langle 1\right|.\end{aligned}$$

Now we have a basis to operate on for this project.It is here that the fact that density operators can be written as matrices comes into effect allowing for equations (17)-(20) to be rewritten as matrices.

$$\rho^{ab} = \begin{bmatrix} \lambda_0^2 & 0 & \lambda_0\lambda_1 e^{-i\phi} & \lambda_0\lambda_3 \\ 0 & 0 & 0 & 0 \\ \lambda_0\lambda_1 e^{i\phi} & 0 & \lambda_1^2+\lambda_2^2 & (\lambda_1\lambda_3 e^{i\phi}+\lambda_2\lambda_4) \\ \lambda_3\lambda_0 & 0 & (\lambda_1\lambda_3 e^{-i\phi}+\lambda_2\lambda_4) & \lambda_3^2+\lambda_4^2 \end{bmatrix}. \tag{21}$$

$$\rho^{ac} = \begin{bmatrix} \lambda_0^2 & 0 & \lambda_0\lambda_1 e^{-i\phi} & \lambda_0\lambda_2 \\ 0 & 0 & 0 & 0 \\ \lambda_0\lambda_1 e^{i\phi} & 0 & \lambda_1^2+\lambda_3^2 & (\lambda_1\lambda_2 e^{i\phi}+\lambda_3\lambda_4) \\ \lambda_2\lambda_0 & 0 & (\lambda_1\lambda_2 e^{-i\phi}+\lambda_3\lambda_4) & \lambda_2^2+\lambda_4^2 \end{bmatrix}. \tag{22}$$

$$\rho^{bc} = \begin{bmatrix} \lambda_0^2+\lambda_1^2 & \lambda_1\lambda_2 e^{i\phi} & \lambda_1\lambda_3 e^{i\phi} & \lambda_1\lambda_4 e^{i\phi} \\ \lambda_2\lambda_1 e^{-i\phi} & \lambda_2^2 & \lambda_2\lambda_3 & \lambda_2\lambda_4 \\ \lambda_3\lambda_1 e^{-i\phi} & \lambda_3\lambda_2 & \lambda_3^2 & \lambda_3\lambda_4 \\ \lambda_4\lambda_1 e^{-i\phi} & \lambda_4\lambda_2 & \lambda_4\lambda_3 & \lambda_2^2+\lambda_4^2 \end{bmatrix}. \tag{23}$$

$$\rho^a = \begin{bmatrix} \lambda_0^2 & \lambda_0\lambda_1 e^{-i\phi} \\ \lambda_0\lambda_1 e^{i\phi} & (\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2) \end{bmatrix}.$$

$$\rho^b = \begin{bmatrix} (\lambda_0^2+\lambda_1^2+\lambda_2^2) & (\lambda_3\lambda_1 e^{i\phi}+\lambda_2\lambda_4) \\ (\lambda_3\lambda_1 e^{-i\phi}+\lambda_2\lambda_4) & (\lambda_3^2+\lambda_4^2) \end{bmatrix}. \tag{24}$$

$$\rho^c = \begin{bmatrix} (\lambda_0^2+\lambda_1^2+\lambda_3^2) & (\lambda_2\lambda_1 e^{i\phi}+\lambda_3\lambda_4) \\ (\lambda_2\lambda_1 e^{-i\phi}+\lambda_3\lambda_4) & (\lambda_2^2+\lambda_4^2) \end{bmatrix}.$$

3.1 The suggested formula

The goal of this project is to provide a means for the measurement of nonlocality, and in order to do so inspiration has been taken from the method used to measure the entanglement of a tripartite system. In the case of the tripartite entanglement, measurements are generally done using residual entanglement[9].

$$\tau_{abc} = C^2(\Psi^{a(bc)}) - C^2(\rho^{ab}) - C^2(\rho^{ac}). \tag{25}$$

Here $C^2(\Psi^{a(bc)})$ is the total entanglement experienced by Alice's qubit and can simply be calculated as $4det(\rho^a)$ and serves as a consistent measurement for all entanglement in the system relative each of the 3 individuals with qubits. The other variables that are used here are $C^2(\rho^{ab})$ and $C^2(\rho^{ac})$ which are generally referred to as tangle. The tangle operation is simply a squared version of the concurrence of two bits. Concurrence is a measurement of the entanglement between two qubits and provides a normalized value between 0-1.

Assume that there is an arbitrary density operator between the two qubits of Alice and Bob denoted as ρ^{ab}. In order to calculate the concurrence for this arbitrary system one needs to first define $\tilde{\rho}^{ab}$, this factor is defined using the Pauli spin matrix along the y axis [7]:

$$\tilde{\rho}^{ab} = (\sigma_y \otimes \sigma_y)\rho^{ab*}(\sigma_y \otimes \sigma_y). \tag{26}$$

After calculating $\tilde{\rho}^{ab}$ it is possible to establish a non hermitian matrix with only positive and real eigenvalues by multiplying $\tilde{\rho}^{ab}$ and ρ^{ab}. By finding the eigenvalues we are left with four different factors referred to as γ. To find the concurrence one needs to take the square root of all four γ and choose largest of the four $\sqrt{\gamma}$ values and subtract the other three factors from it. If the result of the subtraction is positive the concurrence is equal to the subtraction. Otherwise the concurrence is automatically equal to 0. This procedure can be simplified to:

$$\gamma_4 < \gamma_3 < \gamma_2 < \gamma_1$$

$$max(0, \sqrt{\gamma_1} - \sqrt{\gamma_2} - \sqrt{\gamma_3} - \sqrt{\gamma_4}). \tag{27}$$

The tripartite measurement method using concurrence is specially interesting for two reasons. Firstly the fact that τ_{abc} is larger than zero suggests that the entanglement experienced by Alice's qubit in a tripartite system is larger than the individual entanglement between Alice and Bob plus that between Alice and Charlie and that there is a shared entanglement between all three happening simultaneously. Secondly that the results equation (25) is the same between Alice Bob and Charlie, i.e.

$$C^2(\Psi^{a(bc)}) - C^2(\rho^{ab}) - C^2(\rho^{ac}) = C^2(\Psi^{b(ac)}) - C^2(\rho^{ab}) - C^2(\rho^{bc}) = C^2(\Psi^{c(ab)}) - C^2(\rho^{bc}) - C^2(\rho^{ac}).$$

further proving the first point of a common entanglement that is between all three simultaneously and allowing for a consistent measuring tool for tripartite systems. It is worth mentioning that this common entanglement is generally referred to as residual entanglement. Modeling this approach led to the following formula for the tripartite nonlocality:

$$\eta = C^2(\Psi^{a(bc)}) - B^2(\rho^{ab}) - B^2(\rho^{ac}). \tag{28}$$

Here the total entanglement factor is held onto. The change to equation (25) comes in the form of the switching of concurrence values for that of Bell nonlocality between two particles since although Gisin proved that entanglement and nonlocality serve the same purpose at the two bit level [5], they still have different numerical values and formulae assigned to them. The method of measuring the nonlocality used here is the two qubit Bells inequality equation discovered by Clauser, Horne, Shimony and Holt (CHSH) [13], this method is also commonly noted as B_{CHSH}. This method provides numerical values for the nonlocality of two qubits that are normalised and lie between 0 and 1.
To understand this process one needs to be introduced to the concept of the correlation matrix. The correlation matrix generally denoted by T is a large matrix consisting of the trace of the density operation in combination with different Pauli spin matrices. Let us consider the case for the qubits of Alice and Bob with a density operator ρ^{ab}:

$$\begin{bmatrix} Tr(\rho^{ab}\sigma_x \otimes \sigma_x) & Tr(\rho^{ab}\sigma_x \otimes \sigma_y) & Tr(\rho^{ab}\sigma_x \otimes \sigma_z) \\ Tr(\rho^{ab}\sigma_y \otimes \sigma_x) & Tr(\rho^{ab}\sigma_y \otimes \sigma_y) & Tr(\rho^{ab}\sigma_y \otimes \sigma_z) \\ Tr(\rho^{ab}\sigma_z \otimes \sigma_x) & Tr(\rho^{ab}\sigma_z \otimes \sigma_y) & Tr(\rho^{ab}\sigma_z \otimes \sigma_z) \end{bmatrix}.$$

Using the matrix above times its own transpose it is possible to define a matrix with four fully real eigenvalues. By summing the two largest of these eigenvalues we define the variable $M(\rho) = \lambda_1 + \lambda_2$, using the definition of M one can apply the following condition that if $m(\rho) - 1$ is positive then it is equal to the numerical value for the nonlocality otherwise the value is simply zero.
Now given the specifications of the residual entanglement it is likely that the reasoning behind the main questions noted in the introduction section has become more clear. As far as the theory goes (i) will be answered by making a calculation while in the case of (ii) it is possible to provide a theoretical answer

to what is expected. The second question asked if the values coming from this method are consistently larger or equal to zero. Here the answer to the question is yes. Recall the following requirement for the tripartite concurrence [9].

$$0 \leq C^2(\Psi^{a(bc)}) - C^2(\rho^{ab}) - C^2(\rho^{ac}) \leq 1. \tag{29}$$

Now combine that with the knowledge that nonlocality is a part of entanglement, meaning that it is impossible to have bipartite bell nonlocality that surpasses the value of the bipartite concurrence. We know that $0 \leq B$ but it can be mathematically argued that [15]

$$B(\rho^{ab}) \leq C(\rho^{ab}) \Longrightarrow B_{max}(\rho^{ab}) = C(\rho^{ab}). \tag{30}$$

Using the information above it can be observed that $0 \leq B \leq c$. This means that $\tau_{abc} \leq \eta_{abc}$ consistently since the values for nonlocality can be less than that of concurrence and thus subtracting less from the total value. This by proxy means that $0 \leq \tau_{abc} \leq \eta_{abc} \to 0 \leq \eta_{abc}$. Although this conclusion answers one of the main questions of the project it also brings up the potential that this result could lead to values that are larger than one as the total value.

It is possible to theoretically show that the value of η is less than or equal to 1 by continuing with the same lane of tought. The lowest values B can possibly take is 0 which yields the largest possible η. Looking at the formula it can than be stated that $\eta = 4det(\rho_i)$ where "i" indicates Alice, Bob or Charlie. From a purely mathematical standpoint the value $4det(\rho_i)$ is equivalent to $1 - r^2$ where r is the radius of the Bloch sphere and as it applies $0 \leq r^2 \leq 1$ meaning that the largest possible value for η is in fact 1. This allows for the possible prediction of the expected limits for η which would be

$$\tau_{abc} \leq \eta_{abc} \leq 1. \tag{31}$$

Given that η now has its limits established it would be smart to consider the specific cases that may apply. Given the limitations above it is clear that η is definitely equal to 1 when τ is equal to 1 meaning that the same arbitrary superpositions would apply in both cases for their maximum. Since under those circumstances equation (31) can be rewritten as.

$$1 \leq \eta_{abc} \leq 1 \to \eta_{abc} = 1. \tag{32}$$

On the other hand the only rule that applies when looking at the minimum case is that τ must equal 0 in order for it to even be possible for η to be 0.

3.2 Special States

To further study this possibility another concept needs to be introduced. In the tripartite case entanglement is slightly different than the bipartite case. In the bipartite case there is only one form of entanglement, generally referred to as the Bell entanglement while the entanglements between tripartite systems are generally split into two cases[10].

Namely either a $|GHZ\rangle$ or $|W\rangle$ state. A $|GHZ\rangle$ state is the more general case and equation (16) all subsequent matrices belong to the $|GHZ\rangle$ variant, while the $|W\rangle$ states are more specific and are indicated by the fact that all the states kets availabe in a $|W\rangle$ have only even or only odd number of zeros present. Meaning that equation (16) can be rewritten as a $|W\rangle$ state in two forms.

$$|W\rangle_{odd} = \lambda_0 |000\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle \,,$$

$$|W\rangle_{even} = \lambda_1 e^{i\phi} |100\rangle + \lambda_4 |111\rangle \,.$$

What differentiates $|GHZ\rangle$ states from $|W\rangle$ in practice is in fact the value for τ_{abc}. All $|GHZ\rangle$ states have a residual entanglement value that is larger than zero while $|W\rangle$ states have a residual entanglement factor that is always zero independent of the numbers involved. That is why in order for η to ever be equal to 0 a $|W\rangle$ state must be involved since $\leq \eta$. This idea is one that can be theoretically tested and with it even answer question (i) as well.

Let us begin with showing why the $|W\rangle$ state has a τ that is consistently zero for both the even and odd variant. Note that all factors that are not present in both even and odd variant of $|W\rangle$ are zero. Let us start with the odd variant of $|W\rangle$. For the odd variant λ_1 and λ_4 are set to zero in the matrices (23)-(28). The residual entanglement factor is only properly proven relative Alice as to demonstrate the general purpose and to allow for further focus to be put on η

$$\tau_{A(BC)} = 4det(\rho^a) - C_{AB}^2 - C_{AB}^2. \tag{33}$$

The first section of this calculation is relatively straight forward.

$$4det(\rho^a) = 4det \begin{bmatrix} |\lambda_0|^2 & 0 \\ 0 & |\lambda_2|^2 + |\lambda_3|^2 \end{bmatrix} = 4\lambda_0^2(\lambda_2^2 + \lambda_3^2). \tag{34}$$

Given that all that is left to calculate is the concurrence of $\rho^{ab}$and ρ^{ac}. The procedure of these calculations will be detailed for ρ^{ab} for pedagogical reasons. To begin the process we first need to define $\tilde{\rho}_{AB}$

$$\tilde{\rho}^{ab} = (\sigma_y \otimes \sigma_y)\rho^{ab*}(\sigma_y \otimes \sigma_y) = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_0^2 & 0 & 0 & \lambda_0\lambda_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2^2 & 0 \\ \lambda_3\lambda_0 & 0 & 0 & \lambda_3^2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \tag{35}$$

$$\tilde{\rho}^{ab} = \begin{bmatrix} \lambda_3^2 & 0 & 0 & \lambda_0\lambda_3 \\ 0 & \lambda_2^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \lambda_3\lambda_0 & 0 & 0 & \lambda_0^2 \end{bmatrix}.$$

Now by multiplying $\tilde{\rho}^{ab}$ with the original value and finding its eigenvalues one can find the total concurrence.

$$\tilde{\rho}^{ab}\rho^{ab} = \begin{bmatrix} 2\lambda_0^2\lambda_3^2 & 0 & 0 & 2\lambda_0^3\lambda_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2\lambda_0\lambda_3^3 & 0 & 0 & 2\lambda_0^2\lambda_3^2 \end{bmatrix}. \tag{36}$$

The matrix above gives the eigenvalues $\gamma_1 = 4\lambda_0^2\lambda_3^2$, $\gamma_2 = 0$, $\gamma_3 = 0$, $\gamma_4 = 0$ inserting the square root of these eigenvalues it is now possible to define the concurrence between Alice and Bob.

$$C^{ab} = max(0, \sqrt{\gamma_1} - \sqrt{\gamma_2} - \sqrt{\gamma_3} - \sqrt{\gamma_4}) = 2\lambda_0\lambda_3. \tag{37}$$

Following the same procedure it would be observed that.

$$C^{ac} = 2\lambda_0\lambda_2. \tag{38}$$

Which makes the equation (33) be equal to 0 under all circumstances and all values for the arbitrary values λ and the same logic would apply to the cases relative Bob and Charlie as well. Now that the fact that the results regarding τ^{abc} are proven it is possible to move on to η^{abc} observing how it behaves under $|W\rangle$ states and whether it is also consistently equal to zero for nontrivial values for λ. Let us begin by considering Alice first.

$$\eta = 4det(\rho^a) - B^2(\rho^{ab}) - B^2(\rho^{ac}). \tag{39}$$

Here $4det(\rho^a)$ is the same as equation (34) so it is best to just begin with calculating the CHSH nonlocality. To do that one must first define the correlations matrix T.

$$\begin{aligned} T^{ab} &= \begin{bmatrix} Tr(\rho^{ab}\sigma_x \otimes \sigma_x) & Tr(\rho^{ab}\sigma_x \otimes \sigma_y) & Tr(\rho^{ab}\sigma_x \otimes \sigma_z) \\ Tr(\rho^{ab}\sigma_y \otimes \sigma_x) & Tr(\rho^{ab}\sigma_y \otimes \sigma_y) & Tr(\rho^{ab}\sigma_y \otimes \sigma_z) \\ Tr(\rho^{ab}\sigma_z \otimes \sigma_x) & Tr(\rho^{ab}\sigma_z \otimes \sigma_y) & Tr(\rho^{ab}\sigma_z \otimes \sigma_z) \end{bmatrix} \\ &= \begin{bmatrix} 2\lambda_0\lambda_3 & 0 & 0 \\ 0 & -2\lambda_0\lambda_3 & 0 \\ 0 & 0 & \lambda_0^2 + \lambda_3^2 - \lambda_2^2 \end{bmatrix}. \end{aligned} \tag{40}$$

The next step is to multiply this correlation matrix with its transpose and calculate its eigenvalues.

$$TT^T = \begin{bmatrix} 4\lambda_0^2\lambda_3^2 & 0 & 0 \\ 0 & 4\lambda_0^2\lambda_3^2 & 0 \\ 0 & 0 & (\lambda_0^2 + \lambda_3^2 - \lambda_2^2)^2 \end{bmatrix}. \tag{41}$$

The eigenvalues of the matrix above are $\gamma_1 = 4\lambda_0^2\lambda_3^2$, $\gamma_2 = 4\lambda_0^2\lambda_3^2$, $\gamma_3 = (|\lambda_0|^2 + |\lambda_3|^2 - |\lambda_2|^2)^2$. The next step in order to find the CHSH Bell nonlocality is to choose the two largest eigenvalues and take the sum of the two together to define the variable M. Given that these eigenvalues are arbitrary there is no way to establish one as larger than the other. Given this one can only have two possible combinations of the eigenvalues above which can be referred to as α and β here.

$$M_{ab}^{\alpha} = \gamma_1 + \gamma_2 = 8\lambda_0^2\lambda_3^2. \tag{42}$$

$$M_{ab}^{\beta} = \gamma_1 + \gamma_3 = \gamma_2 + \gamma_3 = 4\lambda_0^2\lambda_3^2 + (\lambda_0^2 + \lambda_3^2 - \lambda_2^2)^2. \tag{43}$$

These definitions of M_{ab} will each lead to their own possible value for the nonlocality both of which should be considered since by definition

$$B = \sqrt{max(0, M-1)}. \tag{44}$$

and in order to have a non trivial solution it would be required that $B = \sqrt{M-1}$ giving the two possible results of:

$$B_{ab}^{\alpha} = \sqrt{M_{ab}^{\alpha} - 1} = \sqrt{8\lambda_0^2\lambda_3^2 - 1}. \tag{45}$$

$$B_{ab}^{\beta} = \sqrt{M_{ab}^{\beta} - 1} = \sqrt{4\lambda_0^2\lambda_3^2 + (\lambda_0^2 + \lambda_3^2 - \lambda_2^2)^2 - 1}. \tag{46}$$

Following the same procedure for ρ^{ac} it is observed that similarly there are two cases of nonlocality possible there as well.

$$B_{ac}^{\alpha} = \sqrt{M_{ab}^{\alpha} - 1} = \sqrt{8\lambda_0^2\lambda_2^2 - 1}, \tag{47}$$

$$B_{ac}^{\beta} = \sqrt{M_{ab}^{\beta} - 1} = \sqrt{4\lambda_0^2\lambda_3^2 + (\lambda_0^2 + \lambda_2^2 - \lambda_3^2)^2 - 1}. \tag{48}$$

Meaning that for any arbitrary $|W\rangle$ state there are 4 different possible ways for $\eta_{a(bc)}$ to be defined. Let us observe these 4 cases separately as well.
Let us begin with the case that both B_{ab} and B_{ac} are expressed in the α form.

$$\begin{aligned} \eta_{a(bc)}^{\alpha\alpha} &= 4\lambda_0^2(\lambda_2^2 + \lambda_3^2) - (8\lambda_0^2\lambda_3^2 - 1) - (8\lambda_0^2\lambda_2^2 - 1) \\ &= 2 - 4\lambda_0^2(\lambda_2^2 + \lambda_3^2) = 2 - 4det(\rho^a). \end{aligned} \tag{49}$$

Now given that the limits of the $4det(\rho^a)$ is $0 \leq 4det(\rho^a) \leq 1$ and that $0 \leq \eta \leq 1$ it becomes clear that the $\alpha\alpha$-case is only possible when $4det(\rho^a) = 1$ and even when this law applies the resulting η would equal 1 and not 0.

Now let us consider the $\alpha\beta$ case where B_{ab} is in its α form and B_{ac} is in its β form.

$$\begin{aligned}\eta^{\alpha\beta}_{a(bc)} &= 4\lambda_0^2(\lambda_2^2+\lambda_3^2)-(8\lambda_0^2\lambda_3^2-1)-(4\lambda_0^2\lambda_3^2+(\lambda_0^2+\lambda_2^2-\lambda_3^2)^2-1)\\ &= 2-4\lambda_0^2\lambda_3^2-(\lambda_0^2+\lambda_2^2-\lambda_3^2)^2 = 2-4\lambda_0^2\lambda_3^2-(1-2\lambda_3^2)^2\\ &= 2-4\lambda_0^2\lambda_3^2-1+4\lambda_3^2-4\lambda_3^4 = 1-4\lambda_3^2(1-\lambda_0^2-\lambda_3^2) = 1-4\lambda_3^2(1+(\lambda_2^2-1))\\ &= 1-4\lambda_2^2\lambda_3^2.\end{aligned} \tag{50}$$

Now given the equation above there is no doubt that there are nontrivial cases where the results is 0 but it is clear that the answer is by no means always equal to zero. Additionally since the answer to the $\alpha\beta$ case is undoubtedly within the limit that where established for η it would not be surprising if the majority of the cases fell under the $\alpha\beta$ model due to to the fact that it has a larger variety of cases where it is possible. Another intresting fact is that the answer to the $\beta\alpha$ case where B_{ab} is in its β form and B_{ac} is in its α form was also identical to the results above, for that reason it will not be fully proven more so the same logic as $\alpha\beta$ applies as more cases are likely to belong to $\alpha\beta$ or $\beta\alpha$ in the $|W\rangle_{odd}$ state frame.

$$\eta^{\beta\alpha}_{a(bc)} = 1-4\lambda_3^2\lambda_2^2. \tag{51}$$

The final case left to consider is the $\beta\beta$ case

$$\begin{aligned}\eta^{\beta\beta}_{a(bc)} &= 4\lambda_0^2(\lambda_2^2+\lambda_3^2)-4\lambda_0^2\lambda_3^2+(\lambda_0^2+\lambda_3^2-\lambda_2^2)^2-1)\\ &-(4\lambda_0^2\lambda_3^2+(\lambda_0^2+\lambda_2^2-\lambda_3^2)^2-1)\\ &= 2-(\lambda_0^2+\lambda_2^2-\lambda_3^2)^2-(\lambda_0^2+\lambda_3^2-\lambda_2^2)^2\\ &= 2-(1-2\lambda_2^2)^2-(1-2\lambda_2^2)^2 = 2-2+4(\lambda_2^2+\lambda_3^2)-4(\lambda_2^4+\lambda_3^4)\\ &= 4(\lambda_2^2+\lambda_3^2)-4(\lambda_2^4+\lambda_3^4).\end{aligned} \tag{52}$$

As it can be seen the only way the equation above is equal to zero is if both λ_2 and λ_3 are equal to 1 which is impossible given the normalization meaning that unlike the case of residual entanglement the nonlocality can be nonzero for $|W\rangle_{odd}$ states. The cases relative Charlie and Bob will be provided in the table below but will not be discussed and proven as thoroughly due to their similarity in method and results. Giving rise to a clear pattern between the cases.

Table 3

$\eta_{b(ac)}$	$\alpha\alpha$	$2-4det(\rho^b)$
	$\alpha\beta$	$1-4\lambda_2^2\lambda_0^2$
	$\beta\alpha$	$1-4\lambda_0^2\lambda_2^2$
	$\beta\beta$	$4(\lambda_0^2+\lambda_2^2)-4(\lambda_0^4+\lambda_2^4)$
$\eta_{c(ab)}$	$\alpha\alpha$	$2-4det(\rho^c)$
	$\alpha\beta$	$1-4\lambda_3^2\lambda_0^2$
	$\beta\alpha$	$1-4\lambda_0^2\lambda_3^2$
	$\beta\beta$	$4(\lambda_0^2+\lambda_3^2)-4(\lambda_0^4+\lambda_3^4)$

Now that the odd case is fully considered and checked let us discuss the even case which is much more straight forward since due to its set up it can be considered trivial and is 0 in all cases this can be logically concluded in a relatively easy manner through some logical deduction. Calculating in the system of $|W\rangle_{even}$ the results observed for $det(\rho^a)$ is equal to zero meaning that due to established limitations that require η to be always positive $B(\rho^{ab})$ and $B(\rho^{ac})$ must be equal to zero in addition to the fact that $det(\rho^c) = det(\rho^b) = B^2(\rho^{bc})$ making the total results zero. It can be argued that this case gives the result zero and not due to the fact that it is a $|W\rangle$ state. This in a way could be considered a short coming of the simplification used in equation (16) and that the simplification does not allow for $|W\rangle_{even}$ states but nontheless the results discovered regarding the odd case should suffice and give enough of an understanding of the interactions in the $|W\rangle$ states.

There are several conclusions that can directly be drawn regarding the method from the calculations above. The first of these conclusions is that unlike the tripartite entanglement measure the value for η is not zero for all $|W\rangle$ cases. The second conclusion is that The result of η varies based on the person the nonlocality is measured relative to, i.e.,

$$\eta_{a(bc)} \neq \eta_{b(ac)} \neq \eta_{c(ab)}. \tag{53}$$

This answers one of the initial questions of the project but in the following sections further measures are discussed in order to unify and apply the method to receive a singular measurement.

4 Method

Since this project is very much a theoretical measurement tool there are no direct experimental ways it can be tested in a lab. Thus the best way of making observations regarding this method is to do several calculations of different systems in order to find a potential pattern or limit between all the results of the equation. Given the nature of the calculations and that calculating several cases would be tedious, inefficient and extremely prone to human error it seemed clear that the only way to test this theory would be with an iterative randomized program that would create random arbitrary pure state wave functions and than calculate their η value.

The coding language used for this process was Matlab due to personal familiarity with the language and the preexisting mathematical tools it provides. Since Matlab can not use the bra and ket notation the matrices in equation (21)-(24) are to be used again.

These matrices can now be used in Matlab and receive numerical values using Matlabs random number generator. For this project both concurrence and the CHSH Bell nonlocality where defined as their own separate functions in order to make the process much easier.

As previously proven the result for η varies depending on if the measurement is made relative Alice, Bob or Charlie. This means that the idea of a factor similar to residual entanglement is no longer an option and instead the goal should be to create a system that can provide a good numerical value for the nonlocality using the η formula.

The first method that came to mind to use the three varying cases to get a singular value was to take the mean value of all three cases which would solve a lot of the issues, but as a result the formula for the exact value for the total nonlocality of the system would be written as:

$$\eta^{abc}_{mean} = \frac{\eta^{a(bc)} + \eta^{b(ac)} + \eta^{c(ab)}}{3}. \tag{54}$$

Another alternative method is to simply take the smallest of the three possible values for η as the total nonlocality value. Making the final formula resemble the following case

$$\eta^{abc}_{min} = min(\eta^{a(bc)}, \eta^{b(ac)}, \eta^{c(ab)}). \tag{55}$$

The final approach was to do the opposite of the previous method and simply consider the largest value of the three possible cases.

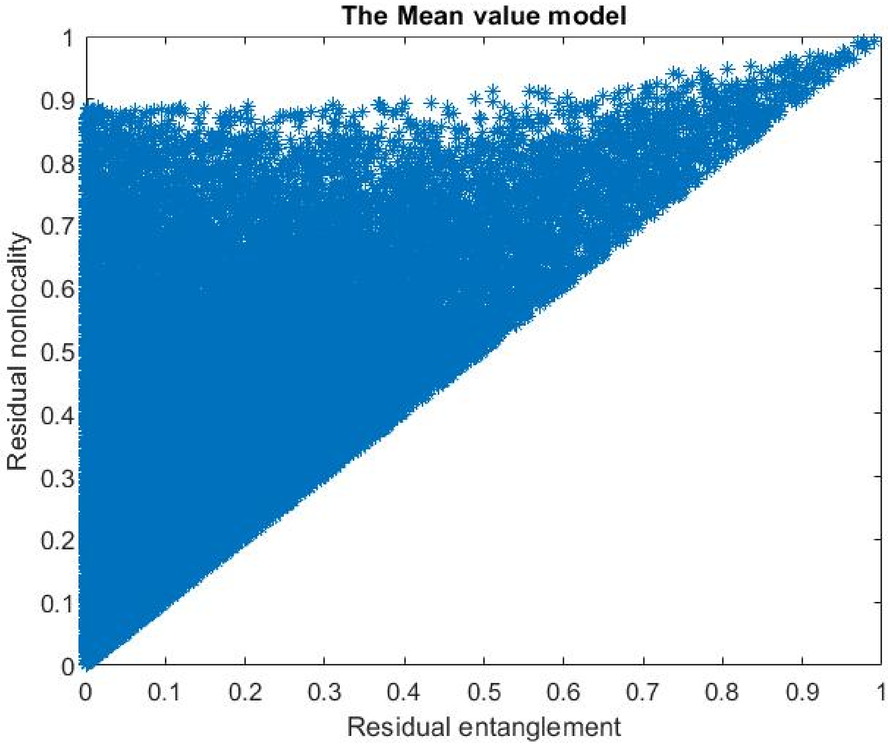
$$\eta^{abc}_{max} = max(\eta^{a(bc)}, \eta^{b(ac)}, \eta^{c(ab)}). \tag{56}$$

All these approaches were promptly considered in the code. Given that the final steps to consider was to decide on the number of iterations required to provide valid results and a valid way to portray the said results. The agreed upon number was 30000. Additionally the method chosen to portray the results was to create a graph showing the nonlocality value of each case relative its tripartite concurrence to see the relation between the two as well.
The most useful method amongst the three suggested above could potentially vary dependant on the use but analysis of the results could lead to a better idea of the relation of the η value relative to the value used for concurrence in addition to proving or disproving the expectations and limits that are expected to apply from the theory section [1]

[1]The full code is available in the appendix

5 Results

Here are the graphical results of the same Data set using the different suggested methods.



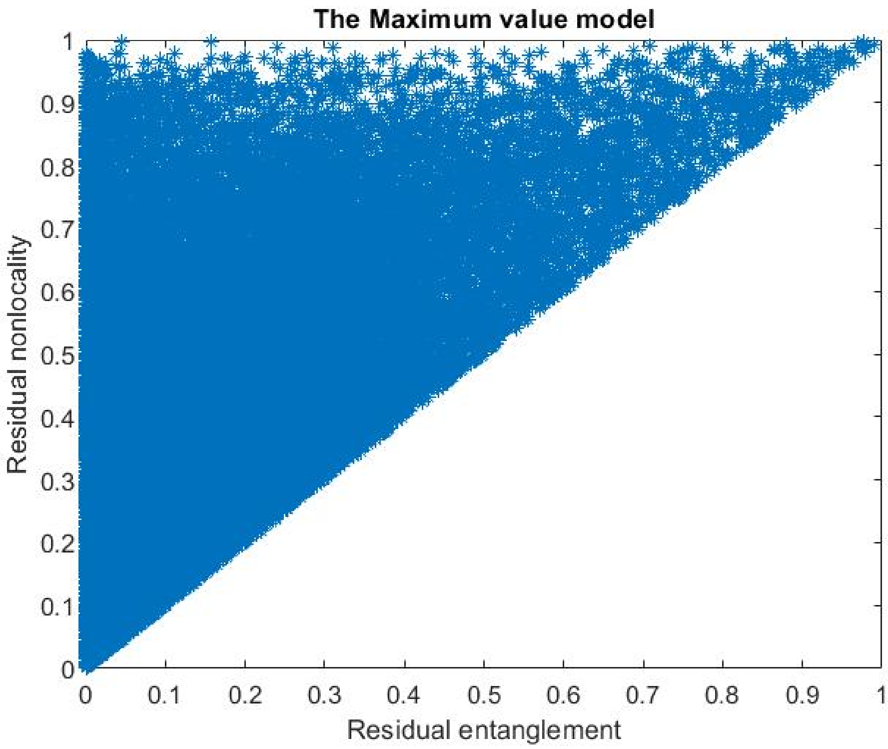


Figure 2: s

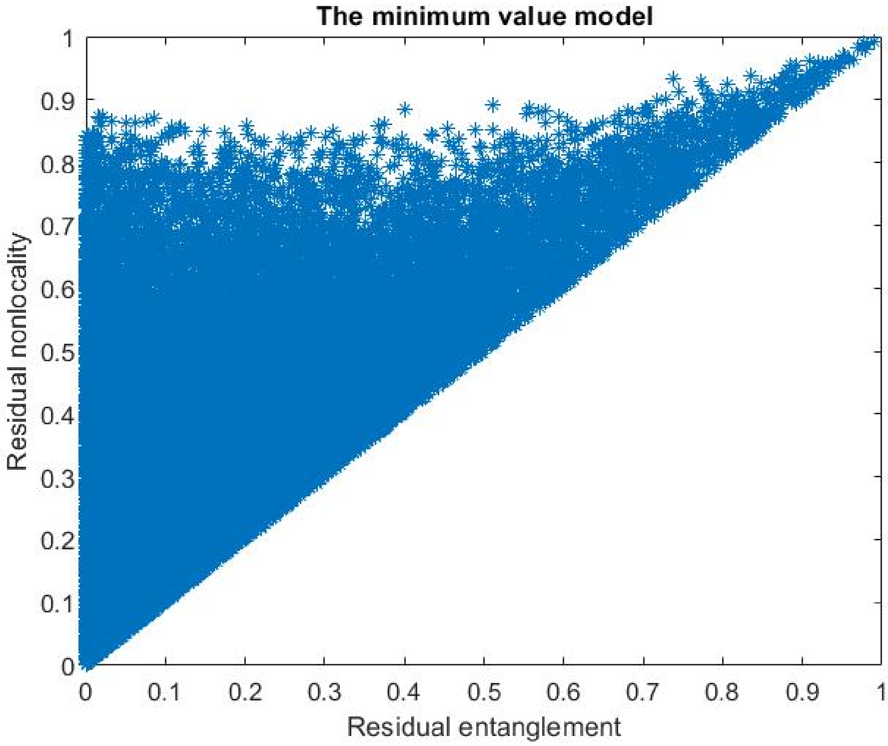


Figure 3

6 Conclusions

There are several possible takeaways from the results provided above, first of which would be the confirmation of the theoretical expectations such that none of the three approaches is ever larger than 1 or smaller than 0. In addition it can also be observed the value for η is never smaller than that of the residual entanglement and the two create the almost linear border between the two cases which confirms equation (31).

Interestingly another potential implication of the observation that $\tau \leq \eta$ at all times could be that the general notion of nonlocality being a sub part of entanglement as indicated by figure 1 which was proven for the bipartite case may not fully apply to the tripartite system. This idea requires further research and calculations and hence it can not be fully proven.

There are also many ways in which the formula for η can be tested or further improved such as the inclusion of $|GHZ\rangle$ states where one of the variables can be zero which is very unlikely given the random number generator in Matlab. The addition of such a system could shed further light on the limits of the method and help establish limit cases for the results that could consistently result in an exact value such as zero or cases where η would be the same value relative all three of Alice, Bob and Charlie.

Lastly the final step that could potentially further confirm and test the results above would be a proposal for an experimental method to put the formula to the test and maybe even help dictate which of the three normalization methods or even potentially another unknown method should be applied to receive values that are the most effective and correct.

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Appendix

The Code

The concurrence function

```
function c=concurrence(rho)
 y=[0,-1i;1i,0];
 R=rho*kron(y,y)*conj(rho)*kron(y,y);
 s=real(sqrt(eig(R)));
 e=-sort(-s);
 c= max(e(1)-e(2)-e(3)-e(4),0);
end
```

The nonlocality code

```
function D=nonlocality(rho)
 x=[0,1;1,0];
 y=[0,-1i;1i,0];
 z=[1,0;0,-1];
 T=[trace(rho*kron(x,x)),trace(rho*kron(x,y)),trace(rho*kron(x,z));
    trace(rho*kron(y,x)),trace(rho*kron(y,y)),trace(rho*kron(y,z));
    trace(rho*kron(z,x)),trace(rho*kron(z,y)),trace(rho*kron(z,z))];

 p= T*transpose(T);

 a=eig(p);

 a=-sort(-real(a));

 M=a(1)+a(2);

 D=real(sqrt(max(M-1,0)));
end
```

The data collection loop

```
clc
n=1;
length=30000;
Data=zeros(length,12);
analysis1=[];
analysis2=[];
analysis3=[];
```

```
analysis4=[];
maxi=[0];
mini=[1];
maxin=[0];
minin=[0];
while n<length

r=rand(1,5);
l=[(1/(sqrt(sum(r.^2))))*r,pi*rand];

A=[l(1)^2,l(1)*l(2)*exp(-1i*l(6));
    l(1)*l(2)*exp(1i*l(6)),l(2)^(2)+l(3)^(2)+l(4)^(2)+l(5)^(2)];

B=[l(1)^(2)+l(2)^(2)+l(3)^(2),l(2)*l(4)*exp(1i*l(6))+l(3)*l(5);
    l(4)*l(2)*exp(-1i*l(6))+l(5)*l(3),l(4)^(2)+l(5)^(2)];

C=[l(1)^(2)+l(2)^(2)+l(4)^(2),l(2)*l(3)*exp(1i*l(6))+l(4)*l(5);
    l(3)*l(2)*exp(-1i*l(6))+l(5)*l(4),l(3)^(2)+l(5)^(2)];

AB=[l(1)^(2),0,l(1)*l(2)*exp(-1i*l(6)),l(1)*l(4);
    0,0,0,0;
    l(1)*l(2)*exp(1i*l(6)),0,l(2)^(2)+l(3)^(2),l(2)*l(4)*exp(1i*l(6))+l(3)*l(5);
    l(4)*l(1),0,l(4)*l(2)*exp(-1i*l(6))+l(5)*l(3),l(4)^(2)+l(5)^(2)];



AC=[l(1)^(2),0,l(1)*l(2)*exp(-1i*l(6)),l(1)*l(3);
    0,0,0,0;
    l(1)*l(2)*exp(1i*l(6)),0,l(2)^(2)+l(4)^(2),(l(2)*l(3)*exp(1i*l(6))+l(4)*l(5));
    l(1)*l(3),0,l(3)*l(2)*exp(-1i*l(6))+l(4)*l(5),l(5)^(2)+l(3)^(2)];

BC=[l(1)^(2)+l(2)^(2),l(2)*l(3)*exp(1i*l(6)),l(2)*l(4)*exp(1i*l(6)),l(2)*l(5)*exp(1i*l(6));
l(3)*l(2)*exp(-1i*l(6)),l(3)^(2),l(3)*l(4),l(3)*l(5);
l(4)*l(2)*exp(-1i*l(6)),l(4)*l(3),l(4)^(2),l(4)*l(5);
l(5)*l(2)*exp(-1i*l(6)),l(5)*l(3),l(5)*l(4),l(5)^(2)];

conA=4*det(A)-abs(concurrence(AC))^(2)-abs(concurrence(AB))^(2);
conB=4*det(B)-abs(concurrence(BC))^(2)-abs(concurrence(AB))^(2);
conC=4*det(C)-abs(concurrence(AC))^(2)-abs(concurrence(BC))^(2);

nonlocA=(4*det(A))-nonlocality(AC)^(2)-nonlocality(AB)^(2);
nonlocB=(4*det(B))-nonlocality(BC)^(2)-nonlocality(AB)^(2);
nonlocC=(4*det(C))-nonlocality(AC)^(2)-nonlocality(BC)^(2);
```

```
nonloc=[nonlocA,nonlocB,nonlocC];

res1=mean(nonloc);
res2=min(nonloc);

Data(n,1:6)=1;
Data(n,7)=conA;
Data(n,8:10)=nonloc;
Data(n,11)=res1;
Data(n,12)=res2;


    if (real(nonlocA)==real(nonlocB)&& real(nonlocB)==real(nonlocC))
    analysis1=[analysis1,n];
    end

    if(real(nonlocA)>1)
        analysis2=[analysis2,n];
    end

      if(real(nonlocB)>1)
        analysis2=[analysis2,n];
      end

        if(real(nonlocC)>1)
        analysis2=[analysis2,n];
        end

     if(real(res1)==real(conA))
     analysis3=[analysis3,n];
     end

   if(real(res2)==real(conA))
     analysis4=[analysis4,n];
   end

    if(res1>=maxi(size(maxin)))
        maxi= [maxi,res1];
        maxin=[maxin,n];
    end
```

```
      if(res2<=mini(size(minin)))
        mini= [mini,res1];
        minin=[minin,n];
      end
    n=n+1;
end
figure(1)
plot(Data(:,7),Data(:,11),'*');
figure(2)
plot(Data(:,7),Data(:,12),'*');
```