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Results in Localization for Supersymmetric Gauge Theories

ANASTASIOS GORANTIS

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Abstract

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The strong coupling dynamics of Quantum Field Theories with gauge symmetries constitutes a profound problem in Theoretical Physics. Supersymmetric theories offer rare instances where this elusive problem is tractable and can be a valuable source of information and intuition. One of the most powerful approaches available for this purpose is supersymmetric localization. In this thesis, we employ this technique to study many facets of supersymmetric theories.

After a brief theoretical introduction to the basic concepts that make an appearance in this thesis, we begin with a localization computation for supersymmetric gauges theories on spheres of variable dimension, with eight and four supersymmetries, proving an earlier conjecture. We also analytically continue our results to get a partition function for the four-dimensional N=1 theory, and perform various consistency checks. Then, we consider a special case of these theories, namely two-dimensional Yang-Mills gauge theory, and we study the matrix model that arises from its localization. We analyze the theory in the limit of large gauge group rank using numerical and analytical tools and connect it to previous related results from the literature. Next, we turn our attention to N=2 gauge theories with matter. We construct the theories on a general class of four-dimensional manifolds, ensuring that they are globally well-defined. Then, we reformulate them using twisted variables and perform localization on the resulting cohomological theory. Lastly, we study N=2 theories on another four-dimensional manifold, namely a product of a three-dimensional hyperbolic space and a circle, with the goal of finding infinite-dimensional chiral algebras. To pursue this goal, we once again employ a localization technique and find evidence of this structure in two cases.

Keywords: Supersymmetric localization, Matrix models, Supersymmetric field theories, Cohomological field theories, Chiral algebras

Anastasios Gorantis, Department of Physics and Astronomy, Theoretical Physics, Box 516, Uppsala University, SE-751 20 Uppsala, Sweden.

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List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I Anastasios Gorantis, Joseph A. Minahan, and Usman Naseer. "Analytic Continuation of Dimensions in Supersymmetric Localization." Journal of High Energy Physics 2018, no. 2 (February 2018). https://doi.org/10.1007/JHEP02(2018)070. arXiv: 1711.05669

II Guido Festuccia, Anastasios Gorantis, Antonio Pittelli, Konstantina Polydorou, and Lorenzo Ruggeri. "Cohomological Localization of $\mathcal{N} = 2$ Gauge Theories with Matter". Submitted to the Journal of High Energy Physics. arXiv: 2005.12944

Contents

1. Introduction

Quantum field theory could be described as a theoretical framework for the study of physical systems. It is a collection of techniques, notions and tools that allow us to calculate physical quantities of interest. Despite its name, quantum field theory is not precisely a theory, but only a framework that needs to be supplemented by a model in order to form a complete physical theory. A celebrated example of such a theory is the Standard Model of particle physics. Quantum field theory (in conjunction with suitable models) has proved extraordinarily successful during the previous decades. For instance, quantum electrodynamics boasts one of the most accurate predictions of any natural science for the calculation of the anomalous magnetic dipole moment (see for example [Hanneke, Hoogerheide, & Gabrielse, 2011]). Another case of a resounding success was the prediction of the existence of the Higgs particle ([Englert & Brout, 1964], [Higgs, 1964] and [Guralnik, Hagen, & Kibble, 1964]), whose experimental confirmation came almost fifty years later ([Aad et al., 2012] and [Chatrchyan et al., 2012]). Furthermore, the usefulness of quantum field theory extends to other domains, such as condensed matter physics and cosmology.

Despite its success, the framework is surrounded by a wealth of open problems. For instance, its full rigorous mathematical formulation still evades us. The problem of the existence of the Yang–Mills theory and its mass gap is included in the "Millennium Problems" list of the most important open mathematical problems of the Clay Mathematics Institute [Witten & Jaffe, n.d.]. The strong coupling dynamics of gauge quantum field theories remain largely intractable. When one starts considering also the models that get combined with quantum field theory to build physical theories, even more issues arise. The hierarchy problem, the cosmological constant problem, and the strong CP problem are just a few of the major open issues. Perhaps the greatest challenge of all arises when one attempts to incorporate gravity in the framework of quantum field theory and reconcile it with general relativity to formulate a theory of quantum gravity, a theory necessary for the microscopic description of black holes. This brief overview of open problems is far from exhaustive and it is only indicative of the opportunities for further research.

One of the most promising ideas proposed to resolve some of the problems of quantum field theory is that of *supersymmetry*, first introduced in the papers [Gervais & Sakita, 1971], [Golfand & Likhtman, 1971] and [Volkov & Akulov, 1973]. Supersymmetry is a spacetime symmetry that connects bosonic and fermionic fields. It manages to evade the Coleman–Mandula no-go theorem [Coleman & Mandula, 1967] by introducing fermionic symmetry generators and thus enlarges the spacetime symmetry group to the SuperPoincaré group. According to the Haag–Lopuszanski–Sohnius theorem [Haag, Lopuszanski, & Sohnius, 1975], this extension of the Poincaré group is unique. The canonical reference work for supersymmetry is [Wess & Bagger, 1992], while a more pedagogic introduction is offered in the lecture notes by Bertolini [Bertolini, n.d.].

Regardless of whether a quantum field theory has supersymmetry or not, one can distinguish between two different approaches in studying it: the perturbative approach, where one tries to find a small parameter in the theory in order to set up a perturbative expansion, and the exact approach, where one seeks exact solutions, sometimes in regimes that perturbation theory cannot reach. While both techniques have their merits and are used by researchers in a complementary fashion, here we shall focus on exact results.

In this thesis we employ various techniques to extract exact results for supersymmetric quantum field theories. One might have objections for the study of theories with supersymmetry, as despite their numerous theoretical appeals, the predictions of supersymmetry have yet to be experimentally verified. However, in this thesis we take an agnostic view towards this point. Regardless of whether supersymmetry is physically realized or accessible at energy scales we can probe with our experiments, it has been an essential ingredient in the vast majority of exact computations performed both in this thesis and in theoretical physics in general. Our motivation for studying supersymmetric quantum field theories in this thesis is not its possible relevance in nature, but the fact that it allows us to improve our understanding of the complicated subject of quantum field theory. Supersymmetric theories offer a rare opportunity to obtain *exact* results even in cases of interacting theories. This can provide valuable insights about the subject, regardless of the realization of this symmetry in nature (although if indeed realized, arguably the conclusions drawn from such calculations will be much more relevant). In any case, a great number of exact results in supersymmetry have been obtained for theories with $\mathcal{N} = 2$ or more supersymmetry, which cannot be models for a theory of nature (due to the loss of the chirality property).

The basic technique that is utilized in this thesis is that of supersymmetric localization, a technique that allows us to obtain one-loop exact results. Supersymmetric localization is a particular case of equivariant localization, first used in theoretical physics in [Witten, 1982] and more recently in a seminal paper by Pestun [Pestun, 2012], where it was employed to derive the partition function and a circular Wilson loop expectation value in $\mathcal{N} = 2$ super Yang–Mills on the four-sphere. Using this technique, one is able to reduce an infinite-dimensional path integral to one of lower dimensionality. In favorable cases, one could end up even with a zero-dimensional field theory, i.e. a matrix model.

In addition to supersymmetric localization, another important notion that appears in the thesis is that of topological twisting, whose first appearance was in a fundamental paper by Witten [Witten, 1988a]. In this paper, Witten studied $\mathcal{N} = 2$ super Yang–Mills and revealed a deep relation between quantum field theory and geometry. He demonstrated that correlation functions of this theory compute Donaldson invariants [Donaldson, 1990] for the manifolds, offering a paradigmatic example of the interplay between theoretical physics and mathematics. While the equivariant generalization of this twisting procedure seems a priori unrelated to Pestun's localization, it was shown in [Festuccia, Qiu, Winding, & Zabzine, 2020] that the two classes of theories studied using these techniques can be examined with a unified approach.

Lastly, another powerful concept we employ that constrains the dynamics of theories, is that of infinite-dimensional chiral algebras. This concept is so powerful that, when present, allows us to investigate even cases that lack a Lagrangian description. Normally such algebras are present only in two-dimensional conformal field theories, but it was shown in [Beem, Lemos, et al., 2015] that they can also be found in $\mathcal{N} = 2$ superconformal field theories in four dimensions, by passing to the cohomology of a special supercharge.

The thesis at hand is composed of four works on the topic of exact results in supersymmetric gauge theories and it is split in two parts. The first part presents a short introduction to the techniques employed in these four works, and the second reviews (in the case of the published works) or describes (in the case of unpublished ones) the developments made in our research efforts.

The rest of this thesis is structured as follows. In Chapter 2, we begin with an introduction to the technique of supersymmetric localization. We explain how to construct a supersymmetric theory on a curved manifold, and present the localization argument. Then, in Chapter 3, we continue with an exposition of a few basic tools for the study of matrix models, which often arise from localization. In particular, we concentrate on the analysis of the saddle-point equation of the matrix models. In Chapter 4, we conclude our introductory material in Part I with a presentation of the main ideas of topological twisting in supersymmetric theories. We define the notion of a topological quantum field theory, and then proceed to explain the twisting procedure for $\mathcal{N} = 2$ theories.

We begin Part II, where novel developments are presented, with a review of Paper I in Chapter 5. In this work, we employ localization to calculate the partition function for certain gauge theories in spheres of various (not necessarily integer) dimensions d, proving a conjecture formulated in [Minahan, 2016]. After defining the theory on the spheres (with 8 supercharges on $d \leq 5$ and 4 on $d \leq 3$), we set up the localization computation. Then, we embark on computing the super-determinants by computing the spectra of the relevant operators. Lastly, we perform an analytic continuation of the theory with 4 supercharges for the case of the four-sphere and conduct checks for the result.

Then, we proceed with Chapter 6, where we present unpublished work on two-dimensional maximally supersymmetric Yang–Mills. This effort leverages the results of Paper I to investigate the matrix model arising from the localization of the aforementioned theory. We study the theory using analytical and numerical approaches and find evidence of a phase transition. We conclude the chapter with a discussion of our preliminary results and sketch the next steps for this project.

Next, in Chapter 7, we review Paper II, where we take on a slightly different research direction and execute a cohomological twist of a class of $\mathcal{N} = 2$ gauge theories with matter, extending the work of [Festuccia et al., 2020]. After defining the theory on a general class of four-manifolds, we introduce a novel projector acting on spinors (in the spirit of the "flipping projectors" of [Festuccia et al., 2020]) and perform the twisting. Having translated the theory in the cohomological language, we proceed to localize the resulting theory using an index computation.

Finally, in Chapter 8, we present unpublished work on another approach to $\mathcal{N} = 2$ supersymmetric theories, where we search for chiral algebras analogous to those of [Beem, Lemos, et al., 2015] in $H^3 \times S^1$. The choice of this particular space is explained in the chapter and has to do with its isometry group. We begin by motivating the work and reviewing the notion of a chiral algebra in conformal field theory. We also recall the construction of the chiral algebra in $\mathbb{R}^4$ in [Beem, Lemos, et al., 2015]. Then, we start the exposition of our preliminary results on the topic. We define a supersymmetric theory on $H^3 \times S^1$, we examine the superalgebra of the theory, and proceed to define the twisted superalgebra. We subsequently employ a localization technique, appearing also in [Dedushenko, Pufu, & Yacoby, 2018] and [Bonetti & Rastelli, 2018], that allows us to recover chiral algebras via a technique different from the original work [Beem, Lemos, et al., 2015]. We finish the chapter with an examination of the line operators that could be incorporated in our framework, and a discussion of the project and its future directions.

Part I:
Introductory material

2. An brief introduction to supersymmetric localization

In this chapter, we present a short general introduction to the technique of supersymmetric localization, which we will use throughout the thesis. We begin in Section 2.1 by briefly reviewing two common approaches to define a supersymmetric field theory on a curved manifold. Then, we proceed in Section 2.2 with a presentation of the arguments behind supersymmetric localization and a discussion of the main steps in a localization computation. The most complete source of information on the subject is probably the recent review [Pestun et al., 2017], while pedagogical introductions to the subject can be found in the lecture notes [Benini, 2016], [Cremonesi, 2013] and [Marino, 2011].

2.1 Rigid supersymmetric field theories in curved spacetime

Before starting to discuss supersymmetric localization, one needs to possess a supersymmetric theory on a curved manifold, as most localization computations are performed on compact curved spaces (the curvature acts as a cut-off for infrared divergences). This can be done by transferring a field theory known in flat space. To begin this process, we can insert the curved metric wherever we encounter the flat metric in the original theory and covariantize all derivatives. For a supersymmetric theory, this should be done for both the Lagrangian and the field variations. A fundamental requirement for a supersymmetric field theory is that the variation of its Lagrangian with respect to the supercharges is a total derivative:

$$\delta^{(f)}\mathcal{L}^{(f)} = \partial_\mu \left(\ldots\right)^\mu . \tag{2.1}$$

where the superscript (f) denotes flat space quantities. However, when one performs these changes to transfer the theory to a curved manifold, one ends up with a field theory that fails to be supersymmetric:

$$\delta^{(c)}\mathcal{L}^{(c)} \neq \nabla_\mu \left(\ldots\right)^\mu , \tag{2.2}$$

where the superscript (c) indicates the curved space analogs of the flat space quantities obtained by replacing the metric and covariantizing the derivatives.

We will present two approaches to solving this issue. The first one seeks to remedy the problem, by treating the flat space variation with the metric and derivative replacements, henceforth denoted by $\delta^{(c)}$, as the zeroth order contribution in an expansion of the curvature scale r of the manifold:

$$\delta = \delta^{(c)} + \sum_{n\geq 1} \frac{1}{r^n}\delta^{(n)} \tag{2.3}$$

and similarly for the Lagrangian:

$$\mathcal{L} = \mathcal{L}^{(c)} + \sum_{n\geq 1} \frac{1}{r^n}\mathcal{L}^{(n)}, \tag{2.4}$$

where $\mathcal{L}^{(c)}$ is the flat-space Lagrangian with its metric and derivatives replaced. Following this method, one has to check the closure of the supersymmetry algebra and ensure that the Lagrangian is supersymmetric order by order. The careful reader will notice that the upper limit of the sums in the two equations above was omitted. The sum has to terminate at some finite order, because r has dimension $[m]^{-1}$ and at some point one runs out of relevant operators from the theory for building $\mathcal{L}^{(n)}$.

Of course this procedure is not guaranteed to succeed. If one exhausts all the relevant operators of the theory and the supersymmetry algebra does not close or the Lagrangian is not supersymmetric, then the manifold does not admit a supersymmetric field theory with this field content.

This approach, while straightforward, has some disadvantages. First of all, it has to be repeated for each manifold. Furthermore, in all known examples where this method was successful, it sufficed to use first and second order corrections in the expansions for the variations (2.3) and Lagrangian (2.4) respectively. The perturbative approach described above does not seem to be able to provide a justification for this observation.

Some of the shortcomings of this method get addressed in another approach, put forward in [Festuccia & Seiberg, 2011]. The authors of this paper suggest that to build a supersymmetric theory on a curved spacetime, one should couple the original theory to some supergravity theory and take the so-called "rigid limit". This corresponds to:

- taking the limit of vanishing Newton's constant G_N,
- freezing the metric to that of the desired space, and
- fixing the auxiliary fields to appropriate background values (depending on the number of supercharges that we want to preserve).

The condition we need to impose is that all fermionic fields of the gravity multiplet, as well as their variations, should be equal to zero. This condition gives rise to the *generalized Killing spinor equations*, which are a set of first order partial differential equations that contain the bosonic fields of the gravity multiplet. The supersymmetric theory can be put on a specific curved manifold only if we can find non-vanishing spinors that satisfy the generalized Killing spinor equations. The solutions to the generalized Killing spinor equations span a vector space of dimension equal to the number of supercharges that the background admits.

If we solve the generalized Killing spinor equations and derive the background gravity multiplet fields that satisfy them, we can get the desired supersymmetric theory by the rigid limit of the combined supergravity and original theory. The resulting theory will contain the background values of the supergravity fields and the supersymmetry variation of its Lagrangian will satisfy:

$$\delta\mathcal{L} = \nabla_\mu \left(\ldots\right)^\mu . \tag{2.5}$$

Its supersymmetry transformations form a subalgebra of the local supersymmetry algebra and thus the supersymmetry algebra will close without any additional effort.

Note that some of the formulas derived using this method can be reused for other backgrounds that satisfy the same Killing spinor equations, in contrast to the first technique, where one needs to repeat the analysis for every new case.

Using the rigid supergravity approach of [Festuccia & Seiberg, 2011], we can also solve the riddle of why the expansions (2.3) and (2.4) terminate at first and second order in $1/r$ respectively. The expansion can be thought of as an expansion in the gravity multiplet auxiliary fields. These fields enter the supersymmetry transformations only linearly and the Lagrangian only quadratically, hence the truncation of the expansions (2.3) and (2.4).

2.2 The supersymmetric localization argument

2.2.1 Introductory remarks

Many problems in quantum field theory amount to computing expectation values of one or more operators. This is often stated in terms of an infinite-dimensional path integral:

$$\langle \mathcal{O}_1 \ldots \mathcal{O}_n \rangle = \frac{1}{Z} \int [D\phi]\, \mathcal{O}_1 \ldots \mathcal{O}_n e^{-S[\{\phi\}]}, \tag{2.6}$$

where $\mathcal{O}_i$ are operators in the theory, ϕ denote collectively the fields in the theory, $[D\phi]$ is some integration measure and Z is the partition function of the theory:

$$Z = \int [D\phi]\, e^{-S[\{\phi\}]}. \tag{2.7}$$

The path integral is an object that is notoriously difficult to formalize mathematically and also very difficult to compute, especially in interacting field theories. Over the years, various techniques have been developed in order to enable and facilitate path integral computations. These techniques range from perturbative/diagrammatic to exact, the latter being the focus of this thesis.

One of the most powerful exact computational techniques for path integrals is that of supersymmetric localization. This method is a type of an equivariant localization (the interested reader can consult the mathematical literature on the subject, for instance [Berline & Vergne, 1982], [Duistermaat &

Heckman, 1982] and [Atiyah & Bott, 1984]). The first early signs of the usefulness of localization in the Theoretical Physics literature appeared in 1982, when Edward Witten first used it in his paper on supersymmetric quantum mechanics [Witten, 1982]. Witten continued utilizing similar concepts in his papers on the two-dimensional topological sigma model [Witten, 1988b], four-dimensional topological gauge theories [Witten, 1988a] and later mirror symmetry [Witten, 1991] and two-dimensional gauge theories [Witten, 1992]. Another important milestone in the evolution of the subject is the use of this technique by Nikita Nekrasov in the computation of the instanton partition function [N. A. Nekrasov, 2003], building on prior work in [Losev, Moore, Nekrasov, & Shatashvili, 1996], [Losev, Nekrasov, & Shatashvili, 1998], [Lossev, Nekrasov, & Shatashvili, 1999], and [Moore, Nekrasov, & Shatashvili, 2000b]. Finally, the paper that opened the floodgates for the field was [Pestun, 2012], where Vasily Pestun used the technique to compute the partition function and the circular Wilson loop expectation value for $\mathcal{N} = 2$ super Yang–Mills on S^4 and showed that the technique can be used beyond topological field theories.

2.2.2 How to perform a localization computation

Let us now turn to explaining how one can use the technique. Consider a quantum field theory that possesses a supercharge Q (or more generally a Grassmann-odd charge), whose square satisfies:

$$Q^2 = B, \tag{2.8}$$

where B is some bosonic charge. It can be a combination of gauge, global and spacetime symmetries. We will restrict our attention to the set of gauge invariant operators that satisfy the following property:

$$Q\mathcal{O}_{\text{BPS}} = 0. \tag{2.9}$$

These operators $\mathcal{O}_{\text{BPS}}$ are called *BPS operators*. It is the expectation values of exactly these operators that one can compute using supersymmetric localization:

$$\langle \mathcal{O}_{\text{BPS}} \rangle = \frac{1}{Z} \int [D\phi] \, \mathcal{O}_{\text{BPS}} \, e^{-S[\{\phi\}]}. \tag{2.10}$$

Note that from now on, we will omit the normalization of the path integral by the partition function to avoid visual clutter. The argument below is identical when one restores the factor $1/Z$ to the expressions.

Now, let us deform the action in the path integral by a Q-exact term:

$$\langle \mathcal{O}_{\text{BPS}} \rangle_t = \int [D\phi] \, \mathcal{O}_{\text{BPS}} \, e^{-S[\{\phi\}]-tQV}. \tag{2.11}$$

The expression QV is commonly referred to as the *localizing action*. Then, one can show that $\langle \mathcal{O}_{\text{BPS}} \rangle_t$ in fact does not depend on t:

$$\begin{aligned}\frac{d}{dt}\langle \mathcal{O}_{\text{BPS}} \rangle_t &= -\int [D\phi]\, \mathcal{O}_{\text{BPS}}\, (QV) e^{-S[\{\phi\}]-tQV} \\ &= -\int [D\phi]\, Q \left(\mathcal{O}_{\text{BPS}} V e^{-S[\{\phi\}]-tQV} \right) \\ &= 0. \end{aligned} \tag{2.12}$$

At this point we have assumed that the symmetry Q is not anomalous, and hence the integration measure $D\phi$ is invariant under Q. We have also assumed there are no boundary terms. Additionally, one can now understand why we focused our efforts on BPS operators: $Q\mathcal{O} = 0$ is crucial for our argument.

So, since $\langle \mathcal{O}_{\text{BPS}} \rangle_t$ is t-independent, we can evaluate the path integral for a value of t of our choice. We usually choose to take the limit $t \to +\infty$ and demand that the bosonic part of QV is greater or equal to zero. This means that main contribution for the integral will come from the saddle points of QV, commonly referred to as the *localization locus*.

The canonical choice for the localizing action is:

$$\mathcal{L}_{\text{loc}} = Q \sum_{\{\lambda\}} \left[(Q\lambda)^\dagger \lambda + \lambda^\dagger (Q\lambda^\dagger)^\dagger \right], \tag{2.13}$$

where we sum over the fermionic field content of the theory. The bosonic part of the localizing action is then

$$\sum_{\{\lambda\}} \left(|Q\lambda|^2 + |Q\lambda^\dagger|^2 \right) \tag{2.14}$$

and it is positive semi-definite. With this choice of localizing action, the localization locus consists of the following configurations:

$$\lambda = 0, \qquad \lambda^\dagger = 0, \qquad Q\lambda = 0, \qquad Q\lambda^\dagger = 0. \tag{2.15}$$

The next step is to expand our fields around the localization locus:

$$\phi = \phi_0 + \frac{1}{\sqrt{t}} \phi', \tag{2.16}$$

with ϕ' denoting the fluctuations. In the limit $t \to +\infty$ the only terms that survive from the argument of the exponential in the path integral (2.11), are those that are not suppressed by powers of $1/\sqrt{t}$. So, we end up with an exact semi-classical one-loop computation (with respect to the parameter t):

$$\begin{aligned}\langle \mathcal{O}_{\text{BPS}} \rangle &= \int_{\{\phi_0\}} [D\phi_0] \mathcal{O}_{\text{BPS}}\big|_{\{\phi_0\}} e^{-S[\phi_0]} Z_{\text{1-loop}}[\{\phi_0\}] \\ &= \int_{\{\phi_0\}} [D\phi_0] \mathcal{O}_{\text{BPS}}\big|_{\{\phi_0\}} e^{-S[\phi_0]} \frac{1}{\text{SDet}\left(\frac{\delta^2 S_{\text{loc}}[\{\phi_0\}]}{\delta \phi_0^2} \right)}, \end{aligned} \tag{2.17}$$

where we integrate over the localization locus and SDet denotes the super-determinant of the fluctuations. We have witnessed a substantial reduction in the integration space: from the original infinite-dimensional space of fields to the BPS locus. The size of this space is determined by the spacetime dependence of the fields that comprise it. This can be a quantum field theory of lower dimensionality or in the most dramatic cases, a zero-dimensional matrix model, corresponding to ordinary finite-dimensional integrals. In the latter case, one is able to use the tools we will introduce in Chapter 3 to study the theory.

The last step to arrive to an explicit expression in the evaluation of the path integral is the computation of the super-determinant in (2.17). There exist two basic approaches to accomplish this. The first, which we employ in Paper I, is to compute the spectra of the operators that appear in the fluctuation expressions for the space at hand. The second, exploited in Paper II, entails invoking the Atiyah–Singer index theorem (assuming the action of an elliptic operator, or a generalization of the theorem for a transversally elliptic one). Both approaches have their merits and their weaknesses. When using the first approach, we often observe many cancellations. In that sense, the second approach is more efficient, since we concentrate solely to the modes that are not going to cancel instead of the full spectra. On the other hand, the index method is dimension-dependent[1] and is not without subtleties.

Finally, as the reader might notice, in the procedure presented above, there have been a few points where we have a freedom of choice. More specifically, we can choose any of the supercharges of the theory (assuming of course that the theory has multiple) to localize, and we can choose different localizing actions instead of (2.13). This freedom is sometimes referred to as using different *localization schemes*. Calculations performed with different schemes might yield results that appear to disagree with each other. However, a more careful study will reveal that the different results agree with each other up to path integrals of Q-exact expressions, which should be equal to zero (due to a supersymmetric analog of the Stokes' theorem).

[1] Dimension-independence is a sine qua non for the computation in Paper I.

3. A brief introduction to matrix models

In this chapter, we are going to provide a short introduction to the analysis of matrix models. While matrix models are a very interesting topic and have numerous applications, we are going to focus on the study of matrix models for their application in the supersymmetric localization literature. Matrix models arise in certain cases where the spacetime dependence of the localization locus field configurations is trivial. The interested reader can find a more detailed pedagogical introduction in [Marino, 2004]. We will begin in Section 3.1 by explaining the basic setup of a matrix model, and then we will proceed in Section 3.2 to examining its saddle-point equation using the resolvent technique.

3.1 Basic setup

Matrix models present an opportunity to study gauge theories in a simpler setting as they are zero-dimensional field theories (i.e. they have no spacetime dependence). Their partition function has the following form:

$$Z = \int [DM]\, e^{-S[M]}, \tag{3.1}$$

where M is an $N \times N$ Hermitian matrix, $[DM]$ is an appropriate integration measure and $S[M]$ is the action. Expectation values of operators $\mathcal{O}(M)$ can be computed in the usual way:

$$\langle \mathcal{O}(M) \rangle = \frac{1}{Z} \int [DM]\, e^{-S[M]} \mathcal{O}(M). \tag{3.2}$$

The integration measure commonly employed is:

$$[DM] = \prod_{i=1}^{N} dM_{ii} \prod_{i<j} d(\mathrm{Re} M_{ij}) \prod_{i<j} d(\mathrm{Im} M_{ij}), \tag{3.3}$$

while the action is generally of the form:

$$S[M] = \frac{1}{2g^2} \mathrm{Tr}\, M^2 + \frac{1}{g^2} V(M), \tag{3.4}$$

with a potential $V(M)$:

$$V(M) = \sum_{k \geq 3} \frac{g_k}{k} \mathrm{Tr}\, M^k. \tag{3.5}$$

The case where $V(M) = 0$ is called the *Gaussian matrix model*.

The presence of the trace in the action endows us with a gauge symmetry:

$$M \to UMU^{-1}, \tag{3.6}$$

where U is any $N \times N$ unitary matrix. This means that we can take advantage of this symmetry to reduce the N^2 degrees of freedom using a Faddeev–Popov procedure. In essence we are seeking to diagonalize M via a unitary transformation (3.6), into:

$$\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N), \tag{3.7}$$

reducing our model to N degrees of freedom. We begin by inserting a unit:

$$\int D\Lambda DU \delta^{(N^2)}\left(UMU^{-1} - \Lambda\right) \Delta^2(\Lambda) = 1 \tag{3.8}$$

in the path integral for the partition function, which becomes:

$$Z = \int \prod_{i=1}^{N} d\lambda_i \, \Delta^2(\Lambda) e^{-\sum_i S(\{\lambda_i\})}, \tag{3.9}$$

where we have omitted an overall constant factor, equal to the volume of the unitary group, as it simplifies in the computation of expectation values. Next, we need to provide an explicit expression for $\Delta^2(\Lambda)$ using its defining equation (3.8). For a fixed M, the main contribution in the integral comes from terms that very close to the unitary matrix U that diagonalizes it. This prompts us to perform the following changes of variables:

$$U = (1 + R)U' \tag{3.10}$$

with an infinitesimal, anti-hermitian R. Then, the delta function becomes:

$$\begin{aligned} \delta^{(N^2)}(UMU^{-1} - \Lambda) &= \delta^{(N^2)}\left[(1+R)U'MU'^{-1}(1-R) - \Lambda\right] \\ &= \delta^{(N^2)}(R\Lambda' - \Lambda' R + \Lambda' - \Lambda), \end{aligned} \tag{3.11}$$

where $\Lambda' = U'MU'^{-1}$. The matrices Λ and Λ' are diagonal, and so is their difference. On the other hand, the commutator $[R, \Lambda']$ has solely off-diagonal components. Thus, we can split the delta function into two:

$$\delta^{(N^2)}(R\Lambda' - \Lambda' R + \Lambda' - \Lambda)) = \delta^{(N^2-N))}([R, \Lambda'])\delta^{(N)}(\Lambda' - \Lambda). \tag{3.12}$$

Then, by writing the commutator as:

$$[R, \Lambda'] = R_{ij}(\lambda'_j - \lambda'_i), \tag{3.13}$$

we can rewrite the condition (3.8) as follows:

$$1 = \int \prod_{i<j} d(\mathrm{Re}\, R_{ij}) \, d(\mathrm{Im}\, R_{ij}) \, \Delta^2(\Lambda') \, \delta^{(N^2-N)}\left(R_{ij}(\lambda'_j - \lambda'_i)\right) \tag{3.14}$$

and finally get an expression for $\Delta(\{\lambda_i\})$:

$$\Delta(\{\lambda_i\}) = \prod_{i<j}(\lambda_j - \lambda_i), \tag{3.15}$$

which we recognize as the Vandermonde determinant.

So, after the dust has settled, we are left with the following partition function:

$$Z = \int \prod_i d\lambda_i \prod_{i<j}(\lambda_j - \lambda_i)^2 e^{-S(\{\lambda_i\})} = \int \prod_i d\lambda_i \, e^{-S_{\text{eff}}(\{\lambda_i\})}, \tag{3.16}$$

where we have introduced the effective action:

$$S_{\text{eff}}(\{\lambda_i\}) = \frac{1}{2g^2}\sum_i \lambda_i^2 + \frac{1}{g^2}V(\{\lambda_i\}) - 2\sum_{i<j}\log(\lambda_i - \lambda_j). \tag{3.17}$$

While matrix models are the simplest gauge theories we can study, the problems posed by their analysis are still quite difficult to tackle in their full generality. So, we will take a hint from the $O(N^2)$ scaling behavior of the effective action (3.17) and try to analyze the theory in the *'t Hooft limit* where $N \to +\infty$ and $\lambda = g^2 N =$ fixed ['t Hooft, 1974]. In the 't Hooft limit, the path integral is dominated by the saddle points of the effective action. Hence, the equation we need to study is:

$$\frac{\partial S_{\text{eff}}}{\partial \lambda_i} = 0, \tag{3.18}$$

or

$$\frac{1}{g^2}\lambda_i = 2\sum_{j\neq i}\frac{1}{\lambda_i - \lambda_j} - \frac{1}{g^2}\frac{\partial V}{\partial \lambda_i}. \tag{3.19}$$

Note that the saddle-point equation, while actually describing the behavior of the eigenvalues of the matrix model, can also be interpreted in the following way. One can think of each eigenvalue λ_i as a particle, under the influence of a potential V and a logarithmic interaction between the eigenvalues. Depending on the interplay and the relative magnitude of these two factors, we can end up with different configurations for the eigenvalues/particles. In particular, as we will discuss later, in the case where the potential V has more than one minimum, the eigenvalues can split into groups.

3.2 Investigating the saddle-point equation

The starting point for the study of the saddle-point equation is the introduction of the eigenvalue density distribution $\rho(z)$:

$$\rho(z) := \frac{1}{N}\sum_{i=1}^{N}\delta(z - z_i) \tag{3.20}$$

where z_i are the eigenvalues that satisfy the saddle-point equation.[1] In the 't Hooft limit, we can replace the sum with an integral over a contour C:

$$\frac{1}{N}\sum_{i=1}^{N} \to \int_C dz. \tag{3.21}$$

We will also demand that the distribution function is normalized:

$$\int_C dz\, \rho(z) = 1. \tag{3.22}$$

The integration contour C is taken over the intervals of finite support of $\rho(z)$. If we only have one such interval, we are discussing a *one-cut solution*; otherwise we are discussing a *multi-cut solution*.

By employing equation (3.21), we rewrite the saddle-point equation as follows:

$$\frac{1}{\lambda}z + \frac{1}{\lambda}\frac{\partial V}{\partial z} = 2 \fint dz' \frac{\rho(z')}{z - z'}. \tag{3.23}$$

We will now proceed to search for a solution of the saddle-point equation, first by assuming that all the eigenvalues z_i lie on a single interval, and then generalize to the case where the eigenvalues lie on multiple intervals.

3.2.1 One-cut solution

We begin by introducing the *resolvent*:

$$\omega(z) := \frac{1}{N}\sum_i \frac{1}{z_i - z}, \tag{3.24}$$

or in the continuous limit:

$$\omega(z) = \int_{\mathbb{R}} dz' \frac{\rho(z')}{z' - z}. \tag{3.25}$$

This is a function which is analytic everywhere in the complex plane except for the location of the eigenvalues on an interval $\mathcal{C}$. We can also extract the asymptotic behavior of the resolvent using the normalization condition of the eigenvalue density distribution $\rho(z)$ (3.22):

$$\omega(z) \underset{z \to +\infty}{\longrightarrow} -\frac{1}{z}. \tag{3.26}$$

[1] We rename our eigenvalues to z_i avoid confusion with the 't Hooft coupling in the continuous limit.

Let's now see how to the resolvent enables us to write down a solution for the saddle-point equation (3.23). If we evaluate the resolvent at $z \pm i0$, we get:

$$
\begin{aligned}
\omega(z \pm i0) &= \int dz' \frac{\rho(z')}{z' - z \mp i0} \\
&= \fint dz' \frac{\rho(z')}{z' - z} \pm \pi i \int dz \rho(z') \delta(z' - z) \\
&= \fint dz' \frac{\rho(z')}{z' - z} \pm \pi i \rho(z). \qquad (3.27)
\end{aligned}
$$

Then, by using the saddle-point equation (3.23):

$$\omega(z + i0) + \omega(z - i0) = -\frac{1}{\lambda} z - \frac{1}{\lambda} V'(z), \qquad (3.28a)$$

$$\omega(z + i0) - \omega(z - i0) = 2\pi i \rho(z). \qquad (3.28b)$$

We now have a Riemann–Hilbert problem at our hands. Since we are looking for a one-cut solution, the singular part of the resolvent $\omega_s(z)$should have the following form:

$$\omega_s(z) \sim \sqrt{(z - a_1)(z - a_2)}, \qquad (3.29)$$

where $a_{1,2}$ are the endpoints of the interval $\mathcal{C}$. Then, by replacing this expression in (3.28a), we get:

$$\omega(z) = \frac{\sqrt{(z - a_1)(z - a_2)}}{2\pi\lambda} \int_{a_1}^{a_2} dx \, \frac{V'(x) + x}{x - z} \frac{1}{\sqrt{(a_2 - x)(x - a_1)}}. \qquad (3.30)$$

We can then fix the endpoints $a_{1,2}$ by inspecting the asymptotic behavior of the resolvent:

$$\omega(z) = \frac{z \left(1 - \frac{a_1 + a_2}{z}\right)}{2\pi\lambda} \int_{a_1}^{a_2} dx \, \frac{V'(x) + x}{\sqrt{(a_2 - x)(x - a_1)}} \left(-\frac{1}{z} - \frac{x}{z^2}\right), \qquad (3.31)$$

and so for each power of z we get:

$$\frac{1}{2\pi\lambda} \int_{a_1}^{a_2} dx \, \frac{V'(x) + x}{\sqrt{(a_2 - x)(x - a_1)}} = 0, \qquad (3.32)$$

and

$$\frac{1}{2\pi\lambda} \int_{a_1}^{a_2} dx \, \frac{x \left(V'(x) + x\right)}{\sqrt{(a_2 - x)(x - a_1)}} = 1. \qquad (3.33)$$

3.2.2 Multi-cut solution

The discussion of the preceding subsection can be generalized for the case of a solution with multiple cuts. More precisely, the most general solution has n cuts, where n is smaller or equal to the number of minima of the potential V. In this case, the eigenvalues are distributed in n (disjoint) intervals $\mathcal{C}_i$, $i \in \{1, \dots, n\}$. We now require that the resolvent ω has $2n$ branch points, which fixes its form into:

$$\omega(z) := \frac{1}{2\pi\lambda}\sqrt{\prod_{k=1}^{2n}(z-a_k)}\int dx\,\frac{(V'(x)+x)}{x-z}\frac{1}{\sqrt{\prod_{k=1}^{2n}(x-a_k)}}, \qquad (3.34)$$

where a_k are the endpoints of the intervals $\mathcal{C}_i$ and the integration should be performed on all $\mathcal{C}_i$'s. The endpoints are subject to the following $n-1$ conditions:

$$\frac{1}{2\pi\lambda}\int dx\frac{x^m(V'(x)+x)}{\sqrt{\prod_{k=1}^{2n}(x-a_k)}} = \delta_{mn}, \qquad (3.35)$$

where $m = 1, \dots, n$. This means that we are left with $2n-(n-1) = n+1$ of the interval endpoints remaining to be fixed. The position of these points will have to be fixed using some additional condition (such as putting the cuts at equipotential lines). It is also common practice to introduce the *filling fractions* f_i:

$$f_i = \int_{\mathcal{C}_i} d\lambda\,\rho(\lambda), \qquad (3.36)$$

such that:

$$\sum_{i=1}^{n} f_i = 1. \qquad (3.37)$$

The filling fractions indicate the number of eigenvalues contained in each cut $\mathcal{C}_i$.

4. A brief introduction to topological twisting

In this chapter we will make a brief introduction to the topic of topological quantum field theories (TQFTs) and their construction from $\mathcal{N} = 2$ supersymmetric theories via the procedure of *twisting*. The study of TQFTs has been on the most important developments of the last decades in mathematical physics, since it has revealed deep connections between quantum field theory, geometry and topology, and had significant insights to offer to both subjects.

We begin this chapter by introducing the concept of a topological quantum field theory in Section 4.1. Since our purpose is to offer an introduction to work conduced in Paper II, we will limit our presentation to the case four-dimensional theories. Then, in Section 4.2, we turn to the twisting procedure for $\mathcal{N} = 2$ supersymmetric theories. A great deal of additional information and references can be found in the book [Labastida & Marino, 2005].

4.1 Topological quantum field theories

Consider a Riemannian manifold $\mathcal{M}$ with a metric $g_{\mu\nu}$. If we construct a quantum field theory on the manifold, then the theory will generally depend on the spacetime metric $g_{\mu\nu}$. However, there exist cases of theories that possess a sector with observables that are independent of the metric:

$$\frac{\delta}{\delta g_{\mu\nu}} \langle \mathcal{O} \rangle = 0. \tag{4.1}$$

These theories are called *topological quantum field theories* (TQFTs). We can distinguish two types of TQFT's:

- *Schwarz type* TQFTs: these are field theories where all the ingredients are explicitly independent of the spacetime metric. A celebrated example that belongs in this category is Chern–Simons theory, first presented by Witten in [Witten, 1988a].

- *Witten type* or *cohomological* TQFTs: these are field theories where the action and some operators, seem to depend on the metric. However, they possess a symmetry of the action δ:

  $$\delta S\left[\{\phi_i\}\right] = 0. \tag{4.2}$$

  This symmetry ensures that the expectation values of a set of operators in the theory do not depend on the metric and hence are topological. This can be seen as follows. The energy-momentum tensor is defined as the variation of the action with respect to the metric:

$$T_{\mu\nu} = \frac{\delta S[\{\phi_i\}]}{\delta g^{\mu\nu}}. \tag{4.3}$$

If $T_{\mu\nu}$ can be rendered in a δ-exact form:

$$T_{\mu\nu} = -i\delta R_{\mu\nu}, \tag{4.4}$$

where $R_{\mu\nu}$ is some arbitrary tensor, then the expectation value of an operator $\mathcal{O}$ that is invariant under the action of the symmetry, becomes:

$$\frac{\delta\langle\mathcal{O}\rangle}{\delta g^{\mu\nu}} = \langle\mathcal{O}\, T_{\mu\nu}\rangle = -i\langle\mathcal{O}\,\delta R_{\mu\nu}\rangle = \pm i\langle\delta(\mathcal{O}\, R_{\mu\nu})\rangle = 0, \tag{4.5}$$

where we have assumed that the symmetry has no anomaly and that there are no boundary term contributions when performing the field-space integration by parts. Such a symmetry is sometimes referred to as *topological*.

In the following, we will concentrate on Witten type TQFTs due to their relevance for Paper II.

For a Witten type theory, the observables we are interested in are those which are invariant under the symmetry δ (so that the argument of equation (4.5) holds), but cannot be written themselves as the δ-variation of another operator (in which case their expectation value would be identically zero). Restated in a more formal manner, we concentrate on operators that belong to the cohomology of the symmetry operator δ: $\mathrm{Ker}\,\delta/\mathrm{Im}\,\delta$ (hence the alternative name *cohomological* TQFT for this type of theory). Indeed, the symmetry operator δ is Grassmannian in all cases found so far, but it is not always nilpotent ($\delta^2 \neq 0$). In fact, in many theories δ squares to a combination of gauge and global symmetries:

$$\delta^2 = B, \tag{4.6}$$

leading us to study operators that are invariant under the action of B (*equivariant* cohomology).

4.2 $\mathcal{N} = 2$ topological twisting

In this section, we will describe shortly an important technique for obtaining a cohomological TQFT from a four-dimensional $\mathcal{N} = 2$ supersymmetric field theory, introduced by Witten in [Witten, 1988a], referred to as *twisting*.

The theory possesses an $SU(2)_+ \times SU(2)_-$ rotation symmetry group and an $SU(2)_I \times U(1)$ internal symmetry group. The left- and right-handed supersymmetry generators are $Q_{\alpha i}$ and $\bar{Q}_{\dot{\alpha} i}$ respectively. They satisfy the following commutation relations:

$$\begin{aligned}
&[M_{\alpha\beta}, Q_{\delta i}] = \varepsilon_{\delta(\alpha} Q_{\beta)i}, && [M_{\alpha\beta}, \bar{Q}_{\dot{\delta}i}] = 0, \\
&[\bar{M}_{\dot{\alpha}\dot{\beta}}, Q_{\delta i}] = 0, && [\bar{M}_{\dot{\alpha}\dot{\beta}}, \bar{Q}_{\dot{\delta}i}] = \varepsilon_{\dot{\delta}(\dot{\alpha}} \bar{Q}_{\dot{\beta})i}, \\
&[B^{ij}, Q^k_{\alpha}] = \varepsilon^{k(i} Q^{j)}_{\alpha}, && [B^{ij}, \bar{Q}^k_{\dot{\alpha}}] = -\varepsilon^{k(i} \bar{Q}^{j)}_{\dot{\alpha}},
\end{aligned} \tag{4.7}$$

where $M_{\alpha\beta}$ ($\bar{M}_{\dot{\alpha}\dot{\beta}}$) are the $SU(2)_-$ ($SU(2)_+$) generators, while B^{ij} are the $SU(2)_I$ generators. The Latin indices $i, j = 1, 2$ are $SU(2)_I$ indices, while the Greek indices α, β ($\dot{\alpha}, \dot{\beta}$) are $SU(2)_-$ ($SU(2)_+$) indices. The fully antisymmetric symbols ε_{ij} are defined so that $\varepsilon^{12} = -\varepsilon_{12} = +1$ and similarly for their dotted counterparts. Symmetric index combinations are denoted by indices in parentheses pairs.

To perform the twisting, we identify the internal $SU(2)_I$ group indices i, j with the $SU(2)_+$ indices $\dot{\alpha}, \dot{\beta}$. We also introduce a new generator $M'_{\dot{\alpha}\dot{\beta}}$:

$$M'_{\dot{\alpha}\dot{\beta}} = M_{\dot{\alpha}\dot{\beta}} - B_{\dot{\alpha}\dot{\beta}}. \tag{4.8}$$

This generator belongs to the new rotation group of the theory, $SU'(2)_+ \times SU(2)_-$. The supercharges of the $\mathcal{N} = 2$ algebra become:

$$Q_{\alpha i} \to Q_{\alpha\dot{\beta}}, \qquad \bar{Q}_{\dot{\alpha} i} \to \bar{Q}_{\dot{\alpha}\dot{\beta}} \tag{4.9}$$

Now we can isolate and define a special supercharge:

$$\bar{\mathcal{Q}} := \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{Q}_{\dot{\alpha}\dot{\beta}}, \tag{4.10}$$

called the *topological supercharge*. This supercharge is nilpotent, invariant under the new rotation group $SU'(2)_+ \times SU(2)_-$ and corresponds precisely to the δ symmetry operator that renders the theory topological.

If one follows this procedure for the four-dimensional $\mathcal{N} = 2$ super Yang–Mills (as in [Witten, 1988a]), one obtains the celebrated topological theory called Donaldson–Witten theory, whose correlation functions provide Donaldson invariants [Donaldson, 1990]. The theory has additional remarkable properties. First of all, since the twisting procedure eliminates all spinor fields from the theory (converting them to differential forms), the resulting theory can be defined on any orientable Riemannian four-manifold. What is more, one can show that the action of the theory is $\bar{\mathcal{Q}}$-exact (up to a topological term) and that the expectation values of products of $\bar{\mathcal{Q}}$-invariant operators are independent of the coupling constant. This means that we can send the coupling constant to infinity and obtain the correlation function just by computing the classical configuration and the quadratic fluctuations. Hence, we have another example of an exact semi-classical approximation (which historically preceded the other famous case of Pestun's supersymmetric localization [Pestun, 2012]).

The same technique can be applied to the four-dimensional hypermultiplet. Here the twisting cannot remove the spinor objects from the theory, a fact which limits the generality of the resulting theory to Spin manifolds. The obstruction is the global construction of a spinor field on the manifold, quantified by the second Stiefel–Whitney class, which needs to vanish for the spinor field to be globally well-defined. However, this issue can be remedied if one replaces the Spin structure with a Spin_c structure, which exists for any orientable Riemannian four-manifold (see [Hyun, Park, & Park, 1995] and [Labastida & Marino, 1997]). This problem can be also circumvented in the case of a hypermultiplet transforming in the fundamental of an $SO(3)$ gauge group [Moore & Witten, 1997].

Part II: Developments

5. Localization on S^d

In this chapter we discuss the developments made in Paper I on the localization of supersymmetric gauge theories on d-dimensional spheres. We begin in Section 5.1 by motivating and discussing the work, and then we proceed by providing a more in-depth view. In Section 5.2, we present the construction of a supersymmetric theory on a S^d, for the case of 16, 8 and 4 supercharges, including the off-shell formulation required for localization. Then, in Section 5.3, we describe the localization process, and in Section 5.4, we briefly review the results from the calculation of the super-determinants. Finally, in Section 5.5, we perform an analytical continuation of the partition function for the case of 4 supercharges to the four-dimensional sphere and conduct a few consistency checks.

5.1 Introduction

After the foundational work of Pestun [Pestun, 2012], a substantial effort was made to calculate observables in supersymmetric field theories on curved spacetimes of various shapes, dimensions, and with different numbers of supercharges. In the majority of these works, the theories are placed on compact manifolds, so that the theory is not plagued by infrared divergences, as the curvature of the manifold plays the role of an infrared regulator. A natural choice for such a compact manifold is the sphere, a simple, maximally symmetric space that is conformally flat.

Motivated by the results of computations with 8 supercharges on S^3 [Kapustin, Willett, & Yaakov, 2010], S^4 [Pestun, 2012] and S^5 [Källén & Zabzine, 2012], Minahan formulated a conjecture in [Minahan, 2016][1], on the general form of the one-loop determinants that result from localization, as functions of the spacetime dimension d of the spheres, where $3 \leq d \leq 7$. The relevant localizing action for arbitrary dimensions had been written in an earlier paper [Minahan & Zabzine, 2015]. An important observation was that the computation did not assume that the parameter d is necessarily integer.

The proposal is consistent with the results for S^3, S^4 and S^5 for 8 supercharges, as well as the results for maximally supersymmetric field theory on S^6

[1]Note also that the conjecture was exploring the case of zero instanton contributions, which is sufficient for the case of large-N limit studies, but will need to be extended if one needs to move beyond it.

and S^7 in [Minahan & Zabzine, 2015]. The conjecture was also subjected to one-loop tests in [Minahan & Naseer, 2017] and was found to be consistent with the appropriate flat space limits.

The conjecture of [Minahan, 2016] was proved in Paper I for the case of d-dimensional spheres S^d, with $d \leq 5$, since we were able to compute the partition function for supersymmetric gauge theories with 8 supercharges. Furthermore, in the same work, we employed similar techniques to derive the partition function for the case of theories with 4 supercharges on spheres of dimension $d \leq 3$.

Browsing the localization literature, an attentive reader might notice that there exist two significant missing cases: $\mathcal{N} = 1$ on S^4 and $\mathcal{N} = 1$ on S^6. While a supersymmetric field theory with 4 supercharges has been written for the four-sphere [Festuccia & Seiberg, 2011], it is not known how to write a semi-positive-definite localizing action with supercharges that close the algebra up to a symmetry of the Lagrangian. The case of the six-sphere is perhaps more challenging, as there seems to be no superalgebra with 8 supercharges that is compatible with the manifold (but see [Naseer, 2019] for an alternative approach based on a spacetime-dependent coupling constant).

In an effort to address the elusive case of $\mathcal{N} = 1$ on S^4, in Paper I, we also performed analytic continuation from three to four dimensions to obtain a partition function. Our proposal passes the test of comparison with the partition function of a vector multiplet with $U(1)$ gauge group and that of a free chiral multiplet with zero mass. Furthermore, the one-loop β function obtained from our formulas match that of a general $\mathcal{N} = 1$ theory. In addition, we performed analytic continuation for a mass deformation of $\mathcal{N} = 4$ super Yang–Mills, that preserves $\mathcal{N} = 1$ supersymmetry, in order to make contact with holographic results for $\mathcal{N} = 1^*$ from [Bobev, Elvang, Kol, Olson, & Pufu, 2016]. In particular, we found a match in the general structure of the real part of the free energy F between the two sides of the correspondence (but see Subsection 5.5.3 for a subtlety regarding the masses).

The main challenge in proving the conjecture of [Minahan, 2016], was the computation of the super-determinants of the fluctuations away from the localization locus. In general, this calculation can be performed in two ways. The first method is using an index theorem, as was done in [Pestun, 2012]. The second entails calculating the fluctuations of the bosons and the fermions separately, by means of diagonalizing quadratic forms and computing eigenvalues along with their degeneracies. In some sense the index method is more efficient, as in the second method we witness large cancellations between the fermionic and the bosonic contribution. The nature of the computation of Paper I, however, demanded generality in the dimension d. This means that the index method was not an option, since it is dimension-dependent. In particular, the cases of even and odd dimensions are quite different, as in the case of the odd dimension, there exists a vector field that does not vanish anywhere (this is related to the so-called "hairy ball theorem" in algebraic topology). The computation we performed is a generalization of the one done in [Kapustin et al., 2010] and in [H.-C. Kim & Kim, 2013]. Once again, we observed large cancellations between the bosons and the fermions, a hint that perhaps there might be a non-integer version of the index theorems.

5.2 Defining a supersymmetric theory on S^d

Our first task in the road to localization is to write down a supersymmetric field theory on a curved manifold, in this case a d-dimensional sphere. In general, this can be done using the methods described in Chapter 2, but here we will follow the approach of Pestun in [Pestun, 2012], as was done in [Minahan & Zabzine, 2015], and we will construct the theory via a dimensional reduction (Scherk–Schwarz reduction) from the ten-dimensional $\mathcal{N} = 1$ super Yang–Mills (16 supercharges). This process was done in [Blau, 2000] for on-shell supersymmetry and in [Berkovits, 1993] and [Fujitsuka, Honda, & Yoshida, 2013] for the off-shell version. The following material constitutes a revision and extension of results that appeared in [Minahan & Zabzine, 2015].

The Lagrangian of $\mathcal{N} = 1$ SYM is:

$$\mathcal{L} = -\frac{1}{g_{10d}^2}\mathrm{Tr}\left(\frac{1}{2}F_{MN}F^{MN} - \Psi \not{D}\Psi\right), \tag{5.1}$$

where M and N are spacetime indices ($M, N = 0, \ldots 9$) and Ψ^α ($\alpha = 1, \ldots, 16$) is a Majorana–Weyl spinor, living in the adjoint representation. The matrices $\Gamma^{M\alpha\beta}$ and $\tilde{\Gamma}^M_{\alpha\beta}$ are the ten-dimensional real and symmetric Dirac matrices. The Lagrangian is invariant under the supersymmetry variations:

$$\delta_\epsilon A_M = \epsilon\Gamma_M\Psi, \tag{5.2a}$$

$$\delta_\epsilon \Psi = \frac{1}{2}\Gamma^{MN}F_{MN}\epsilon, \tag{5.2b}$$

with ϵ being a arbitrary constant bosonic real spinor.

Now, we perform the dimensional reduction down to (Euclidean) d dimensions: we split the indices 10d spacetime indices M, N into d indices denoted by Greek lowercase letters $\mu = 1, \ldots, d$, with gauge fields A_μ, while the rest of the 10d indices will be denoted by I, J, with $I = 0, d+1, \ldots, 9$, providing the scalars ϕ_I. The field strength F_{MN} will be split into:

$$F_{MN} \longrightarrow \begin{cases} F_{\mu\nu}, \\ F_{\mu I} = [D_\mu, \phi_I], \\ F_{IJ} = [\phi_I, \phi_J], \end{cases} \tag{5.3}$$

while derivatives along the compactified directions are considered to vanish. The coupling constant of the resulting theory will be given by the relation:

$$g_{\mathrm{YM}}^2 = \frac{g_{10d}^2}{V_{10-d}}, \tag{5.4}$$

where the ten-dimensional coupling constant, g_{10d}, is divided by the volume of the compactified space, V_{10-d}. Let's now focus on the case where the compactified dimensions have the shape of a d-dimensional sphere.

As in [Pestun, 2012], we take the metric of the S^d sphere to be:

$$ds^2 = \frac{1}{(1+\beta^2 x^2)^2} dx_\mu dx^\mu, \tag{5.5}$$

with $\beta = 1/2r$. The constants ϵ that parametrize the supersymmetry transformations (5.2) are converted to conformal Killing spinors that are given by the differential equations:

$$\nabla_\mu \epsilon = \tilde{\Gamma}_\mu \tilde{\epsilon}, \tag{5.6a}$$

$$\nabla_\mu \tilde{\epsilon} = -\beta^2 \Gamma_\mu \epsilon. \tag{5.6b}$$

With the intention of preserving 16 supercharges, we can introduce the following additional condition:

$$\nabla_\mu \epsilon = \beta \tilde{\Gamma}_\mu \Lambda \epsilon. \tag{5.7}$$

A choice for Λ that is compatible with the consistency conditions stemming from equations (5.6) is:

$$\Lambda = \Gamma^0 \tilde{\Gamma}^8 \Gamma^9. \tag{5.8}$$

We can now solve the resulting system of differential equations to get:

$$\epsilon = \frac{1}{(1+\beta^2 x^2)^{1/2}} \left(1 + \beta\, x_\mu \tilde{\Gamma}^\mu \Lambda\right) \epsilon_s, \tag{5.9}$$

where ϵ_s is a constant spinor. Now, depending on the number of supercharges we want to preserve, we need to perform different modifications to supersymmetry transformations and the Lagrangian. Let us review each case separately below.

16 supercharges (on-shell)

For the case of 16 supercharges, we need to modify only the transformation rule for the fermions into:

$$\delta_\epsilon \Psi = \frac{1}{2} \Gamma^{MN} F_{MN} \epsilon + \sum_I \frac{\alpha_I}{2} \Gamma^{\mu I} \phi_I \nabla_\mu \epsilon, \tag{5.10}$$

where α_I are constants of the following form:

$$\alpha_I = \begin{cases} \dfrac{4(d-3)}{d}, & I = 8, 9, 0, \\[2ex] \dfrac{4}{d}, & I = d+1, \ldots, 7. \end{cases} \tag{5.11}$$

The Lagrangian needs to be augmented with the following terms:

$$\mathcal{L}_{\Psi\Psi} = -\frac{1}{g_{\text{YM}}^2}\text{Tr}\,\frac{d-4}{2r}\Psi\Lambda\Psi, \tag{5.12a}$$

$$\mathcal{L}_{\phi\phi} = -\frac{1}{g_{\text{YM}}^2}\left(\frac{d\Delta_I}{2r^2}\text{Tr}\,\phi_I\phi^I\right), \tag{5.12b}$$

$$\mathcal{L}_{\phi\phi\phi} = \frac{1}{g_{\text{YM}}^2}\frac{2(d-4)}{3r}\varepsilon_{ABC}\text{Tr}\left([\phi^A,\phi^B],\phi^C\right). \tag{5.12c}$$

where we have split the scalars into two groups, labeled by indices A, B (with $A, B = 0, 8, 9$) and i, j (with $i, j = d+1, \ldots, 7$). The constants Δ_I are given by:

$$\Delta_I = \begin{cases} \alpha_I, & I = 8, 9, 0, \\ \dfrac{2(d-2)}{d}, & I = d+1, \ldots, 7. \end{cases} \tag{5.13}$$

16 supercharges (off-shell)
For the off-shell formulation of the theory, we introduce the auxiliary fields K_m and the bosonic pure spinors ν_m where $m = 1, \ldots, 7$. The bosonic spinors need to satisfy certain properties which can be found in the Appendix A of Paper I. The Lagrangian needs to be augmented by the term:

$$\mathcal{L}_{\text{aux}} = \frac{1}{g_{\text{YM}}^2}\text{Tr}K^m K_m. \tag{5.14}$$

The supersymmetry variations now become:

$$\delta_\epsilon A_M = \epsilon\Gamma_M\Psi, \tag{5.15a}$$

$$\delta_\epsilon\Psi = \frac{1}{2}\Gamma^{MN}F_{MN}\epsilon + \sum_I \frac{\alpha_I}{2}\Gamma^{\mu I}\phi_I\nabla_\mu\epsilon + K^m\nu_m, \tag{5.15b}$$

$$\delta_\epsilon K^m = -\nu^m \not{D}\Psi + \beta(d-4)\nu^m\Lambda\Psi. \tag{5.15c}$$

The second variation of the fields is given by a sum of the Lie derivative along a vector field v^M (some components of which are along the sphere), a gauge transformation and perhaps an R-symmetry transformation.

8 supercharges (on-shell)
For the case of 8 supercharges in $d \leq 5$, we should impose the following additional constraint on the Killing spinor:

$$\Gamma\epsilon = +\epsilon, \tag{5.16}$$

where we have introduced Γ:

$$\Gamma := \Gamma^{6789}. \tag{5.17}$$

We also split the field Ψ into vector multiplet and hypermultiplet components:

$$\Psi = \psi + \chi, \tag{5.18}$$

which satisfy the chirality conditions:

$$\begin{aligned} \Gamma\psi &= +\psi, \\ \Gamma\chi &= -\chi. \end{aligned} \tag{5.19}$$

The scalars ϕ^I with $I = 6, \ldots, 9$ are distributed in the hypermultiplet, while the rest are included in the vector multiplet. The constants α_I for $I = 6, \ldots, 9$ are modified to:

$$\alpha_I = \frac{2(d-2)}{d} + \frac{4i\sigma_I mr}{d}, \tag{5.20}$$

where m is the hypermultiplet mass and the σ_I are defined as below:

$$\sigma_{6,7} \coloneqq +1, \qquad \sigma_{8,9} \coloneqq -1. \tag{5.21}$$

The Lagrangian needs to be supplemented by the terms:

$$\mathcal{L}_{\psi\psi} = -\frac{1}{g_{\mathrm{YM}}^2}\mathrm{Tr}\frac{d-4}{2r}\psi\Lambda\psi, \tag{5.22a}$$

$$\mathcal{L}_{\phi\phi} = -\frac{1}{g_{\mathrm{YM}}^2}\left(\frac{d\Delta_I}{2r^2}\mathrm{Tr}\,\phi_I\phi^I\right), \tag{5.22b}$$

$$\mathcal{L}_{\chi\chi} = -\frac{1}{g_{\mathrm{YM}}^2}\left(-im\,\mathrm{Tr}\,\chi\Lambda\chi\right). \tag{5.22c}$$

$$\begin{aligned} \mathcal{L}_{\phi\phi\phi} = -\frac{4}{g_{\mathrm{YM}}^2}\big[\,&(\beta(d-4)+im)\,\mathrm{Tr}(\phi^0[\phi^6,\phi^7]) \\ &-(\beta(d-4)-im)\,\mathrm{Tr}(\phi^0[\phi^8,\phi^9])\big], \end{aligned} \tag{5.22d}$$

where now Δ_I is given by:

$$\Delta_I = \frac{2}{d}\left(mr(mr+i\sigma_I) + \frac{d(d-2)}{4}\right), \qquad I = 6, \ldots, 9. \tag{5.23}$$

8 supercharges (off-shell)

For the off-shell formulation of a theory with 8 supercharges, the pure spinors needs to satisfy the relations:

$$\Gamma\nu_m = +\nu_m, \qquad \text{for} \quad m = 1, 2, 3, \tag{5.24a}$$

$$\Gamma\nu_m = -\nu_m, \qquad \text{for} \quad m = 4, \ldots, 7 \tag{5.24b}$$

and the auxiliary fields K^m split up accordingly and get distributed in the vector multiplet and the hypermultiplet. Their variations are:

$$\delta_\epsilon K^m = \begin{cases} -\nu^m \not{D}\psi + \beta(d-4)\nu^m\Lambda\psi, & \text{for} \quad m = 1, 2, 3, \\ -\nu^m \not{D}\chi - 2imr\beta\nu^m\Lambda\chi, & \text{for} \quad m = 4, \ldots, 7. \end{cases} \tag{5.25}$$

4 supercharges (on-shell)
For the case of 4 supercharges in $d \leq 3$, we should impose the following additional constraint on the Killing spinor:

$$\Gamma' \epsilon = +\epsilon, \tag{5.26}$$

where we have introduced Γ':

$$\Gamma' \coloneqq \Gamma^{4589}. \tag{5.27}$$

The situation in the case of 4 supercharges is somewhat more complex, as is often the case when moving to theories with fewer supersymmetries. The spinor Ψ splits into a vector multiplet fermionic field (ψ) and three chiral multiplet fermionic fields (χ_l):

$$\Psi = \psi + \sum_{l=1}^{3} \chi_l. \tag{5.28}$$

The chirality conditions now become:

$$\Gamma \chi_l = (-1)^{\beta_1(l)} \chi_l, \tag{5.29a}$$
$$\Gamma' \chi_l = (-1)^{\beta_2(l)} \chi_l, \tag{5.29b}$$
$$\Gamma' \psi = \Gamma \psi = +\psi, \tag{5.29c}$$

where we have written l in a binary form using the binary digits for l, $\beta_s(l)$, in the following form:

$$l = 2\beta_2(l) + \beta_1(l). \tag{5.30}$$

The scalar fields are distributed into a vector multiplet (ϕ^0 and ϕ^i, with $i = d+1, \ldots, 3$) and three chiral multiplets, containing two scalar fields ϕ_{I_l}, with $I_l = 2l+2$ or $2l+3$. The constants α_I now are:

$$\alpha_{(l)} \coloneqq \alpha_{I_l} = \frac{2(d-2)}{d} + \frac{4i\sigma_{I_l} m_l r}{d}, \tag{5.31}$$

with

$$\sigma_{(l)} \coloneqq \sigma_{I_l} = (-1)^{\beta_2(l)\beta_1(l)}. \tag{5.32}$$

The supersymmetry variations for the fermionic fields now become:

$$\delta_\epsilon \psi = \frac{1}{2} F_{M'N'} \Gamma^{M'N'} \epsilon + \frac{1}{2} \sum_{l=1}^{3} [\phi_{I_l}, \phi_{J_l}] \Gamma^{I_l J_l} \epsilon + \sum_a \frac{\alpha_a}{2} \Gamma^{\mu a} \phi_a \nabla_\mu \epsilon, \tag{5.33a}$$

$$\delta_\epsilon \chi_l = D_\mu \phi_{I_l} \Gamma^{\mu I_l} \epsilon + [\phi_a, \phi_{I_l}] \Gamma^{a I_l} \epsilon + \frac{1}{2} \epsilon^{lmn} [\phi_{I_m}, \phi_{J_n}] \Gamma^{I_m J_n} \epsilon + \sum_{I_l} \frac{\alpha_{(l)}}{2} \Gamma^{\mu I_l} \phi_{I_l} \nabla_\mu \epsilon, \tag{5.33b}$$

where $M', N' = 0, \ldots, 3$ and $a = 0, d+1, \ldots, 3$. Note that the existence of fields outside each multiplet in the variations is related to the on-shell formulation supersymmetry we are using.

The Lagrangian needs to be supplemented by the terms:

$$\mathcal{L}_{\chi\chi} = -\frac{1}{g_{\mathrm{YM}}^2} \sum_{l=1}^{3} \left(-im_l \operatorname{Tr} \chi_l \Lambda \chi_l\right), \tag{5.34a}$$

$$\mathcal{L}_{\phi\phi} = -\frac{1}{g_{\mathrm{YM}}^2} \sum_{l=1}^{3} \left(\frac{d\Delta_{(l)}}{2r^2} \operatorname{Tr} \phi_{I_l} \phi^{I_l}\right), \tag{5.34b}$$

$$\mathcal{L}_{\phi\phi\phi} = -\frac{4}{g_{\mathrm{YM}}^2} \sum_{l=1}^{3} \left[(im_l + \beta\sigma_{(l)}(d-4)) \operatorname{Tr}\left(\phi^0[\phi_{2l+2}, \phi_{2l+3}]\right)\right], \tag{5.34c}$$

where

$$\Delta_{(l)} \coloneqq \Delta_{I_l} = \frac{2}{d}\left[m_l r(m_l r + i\sigma_{(l)}) + \frac{d(d-2)}{4}\right]. \tag{5.35}$$

The requirement of invariance of the Lagrangian under a supersymmetry variation now yields the following condition on the masses:

$$\beta(d-4) + i\sum_{l=1}^{3} \sigma_{(l)} m_{(l)} = 0. \tag{5.36}$$

4 supercharges (off-shell)

For the case of the off-shell formulation of the theory with 4 supercharges, we impose an additional condition on the pure spinors:

$$\Gamma' \nu_m = +\nu_m, \qquad \text{for} \quad m = 1, 4, 5, \tag{5.37a}$$

$$\Gamma' \nu_m = -\nu_m, \qquad \text{for} \quad m = 2, 3, 6, 7. \tag{5.37b}$$

The supersymmetry variations of the auxiliary fields become:

$$\delta_\epsilon K^m = \begin{cases} -\nu^m \not{D}\psi + \beta(d-4)\nu^m \Lambda \Psi, & \text{for } m = 1, \\ -\nu^m \not{D}\chi_1 - 2i\mu_1 \beta \nu^m \Lambda \chi_1, & \text{for } m = 2, 3, \\ -\nu^m \not{D}\chi_2 - 2i\mu_2 \beta \nu^m \Lambda \chi_2, & \text{for } m = 4, 5, \\ -\nu^m \not{D}\chi_3 - 2i\mu_3 \beta \nu^m \Lambda \chi_3, & \text{for } m = 6, 7, \end{cases} \tag{5.38}$$

where we have introduced the dimensionless parameters $\mu_l \coloneqq m_l r$.

5.3 Localization preliminaries

Having defined the supersymmetric field theories we would like to investigate on the d-dimensional spheres, we now move on to the task of localization. The potential out of which we will build our localizing action is:

$$V=\int d^dx\sqrt{g}\,\mathrm{Tr}'\left(\Psi\overline{\delta_\epsilon\Psi}\right), \tag{5.39}$$

where we have introduced a positive-definite inner product Tr′ on the Lie algebra, which could be distinct from the one used in the original Lagrangian. For the sake of avoiding visual clutter, we omit the trace from all the rest of this chapter. Then, the localizing action will be:

$$QV=\int d^dx\,\sqrt{g}\,\delta_\varepsilon\Psi\overline{\delta_\varepsilon\Psi}-\int d^dx\,\sqrt{g}\,\Psi\,\delta_\varepsilon\left(\overline{\delta_\varepsilon\Psi}\right). \tag{5.40}$$

The factor $\overline{\delta_\epsilon\Psi}$ will be given by:

$$\overline{\delta_\epsilon\Psi}=\frac{1}{2}\tilde{\Gamma}^{MN}F_{MN}\Gamma^0\epsilon+\frac{\alpha_I}{2}\,\tilde{\Gamma}^{\mu I}\phi_I\Gamma^0\nabla_\mu\epsilon-K^m\Gamma^0\nu_m. \tag{5.41}$$

It will also prove useful to split the action into a fermionic part $\mathcal{L}^{\mathrm{f}}$ and a bosonic part $\mathcal{L}^{\mathrm{b}}$.

To identify the localization locus, we turn to the bosonic part of the action:

$$\begin{aligned}\mathcal{L}^{\mathrm{b}}=&\frac{1}{2}F_{MN}F^{MN}-\frac{1}{4}F_{MN}F_{M'N'}\left(\epsilon\Gamma^{MNM'N'0}\epsilon\right)\\&+\frac{\beta d\alpha_I}{4}\,F_{MN}\phi_I\left(\epsilon\Lambda(\tilde{\Gamma}^I\tilde{\Gamma}^{MN}\Gamma^0-\tilde{\Gamma}^0\Gamma^I\Gamma^{MN})\epsilon\right)\\&-K^mK_mv^0-\beta d\alpha_0\phi_0K^m\left(\nu_m\Lambda\epsilon\right)+\frac{\beta^2d^2}{4}\sum_I(\alpha_I)^2\phi_I\phi^Iv^0.\end{aligned} \tag{5.42}$$

By choosing $v^0=1$, $v^{8,9}=0$ and confining ourselves to the case of no instanton contribution, the locus will be given by the condition:

$$\nabla_\mu\phi_I\nabla^\mu\phi^I-\left(K^m+2\beta(d-3)\phi_0\left(\nu_m\Lambda\epsilon\right)\right)^2+\frac{\beta^2d^2}{4}\sum_{I\neq 0}(\alpha_I)^2\phi_I\phi^I=0. \tag{5.43}$$

Finally, choosing K^m and ϕ_0 to be purely imaginary, we obtain the following locus:

$$K_m=-2\beta(d-3)\phi_0\left(\nu_m\Lambda\epsilon\right), \tag{5.44a}$$

$$\phi_0=\phi_0^{\mathrm{cl.}}=\mathrm{const.}, \tag{5.44b}$$

$$\phi_{J\neq 0}=0. \tag{5.44c}$$

The classical action evaluated on this field configuration, will be:

$$S_{\mathrm{cl.}}=\frac{8\pi^{\frac{d+1}{2}}r^{d-4}}{g_{\mathrm{YM}}^2\Gamma\left(\frac{d-3}{2}\right)}\mathrm{Tr}\,\sigma^2, \tag{5.45}$$

where we have introduced the dimensionless Lie-algebra-valued variable σ:

$$\sigma = r\phi_0. \tag{5.46}$$

Let us comment in passing, that for $d = 3$, the classical action (5.45) vanishes, but we can always add a Chern-Simons term.

As explained in Chapter 2, the next step is to examine the contribution of the quadratic fluctuations for the bosonic and fermionic part of each multiplet. To that end, we introduce a new type of index: $\tilde{M}$:

$$\tilde{M} \ = \ \{\mu, i\}, \qquad \mu = 1, \ldots, d, \qquad i = d+1, \ldots, D, \tag{5.47}$$

where $D = 5$ for the cases of 8 supersymmetries and $D = 3$ for the cases of 4 supersymmetries. We will also denote the regular vector fields by A_μ and the scalars of the vector multiplet by A_i (except for ϕ_0).

Since this is a lengthy computation, so we will restrict ourselves to presenting a summary of the final results. The interested reader can refer to Appendix B of Paper I.

Vector multiplet

$$\begin{aligned}\mathcal{L}^{\mathrm{b}}_{\mathrm{v.m.}} \ &= \ A^{\tilde{M}}\,\mathcal{O}_{\tilde{M}}{}^{\tilde{N}}\,A_{\tilde{N}} \ - [A_{\tilde{M}}, \phi_0^{\mathrm{cl}}][A^{\tilde{M}}, \phi_0^{\mathrm{cl}}] - K^m K_m \\ &\quad - 4\beta(d-3)\phi_0 K^m(\nu_m \Lambda\epsilon) - \phi_0\left(-\nabla^2 + 4\beta^2(d-3)^2\right)\phi_0,\end{aligned} \tag{5.48}$$

$$\begin{aligned}\mathcal{L}^{\mathrm{f}}_{\mathrm{v.m.}} &= \left(\psi \not{\nabla}\psi\right) + \left(\psi\Gamma^0\left[\phi_0^{\mathrm{cl}}, \psi\right]\right) - \frac{1}{2}(d-3)\beta v^{\tilde{M}}\left(\psi\Gamma^0\tilde{\Gamma}_{\tilde{M}}\Lambda\psi\right) \\ &\quad - \frac{1}{4}(d-3)\beta\left(\epsilon\tilde{\Gamma}^{\tilde{M}\tilde{N}}\Lambda\epsilon\right)\left(\psi\Gamma^0\Gamma_{\tilde{M}\tilde{N}}\psi\right) + m_\psi\left(\psi\Lambda\psi\right),\end{aligned} \tag{5.49}$$

where

$$\mathcal{O}_{\tilde{M}}{}^{\tilde{N}} = -\delta_{\tilde{M}}{}^{\tilde{N}}\nabla^2 + \alpha_{\tilde{M}}{}^{\tilde{N}} - 2\beta(d-3)\epsilon\Gamma_{\tilde{M}}{}^{\nu\tilde{N}89}\epsilon\nabla_\nu, \tag{5.50}$$

$$\alpha_{\tilde{M}}{}^{\tilde{N}} \ = \ 4\beta^2\begin{pmatrix}(d-1)\delta_\mu^\nu & 0 \\ 0 & \delta_i^j\end{pmatrix}, \tag{5.51}$$

$$m_\psi = \begin{cases}\dfrac{d-1}{2}, & \text{for 8 supercharges,} \\ d-2, & \text{for 4 supercharges.}\end{cases} \tag{5.52}$$

Hypermultiplet

$$\mathcal{L}^{\text{b}}_{\text{h.m.}} = \sum_{i=6}^{9} \left[\phi_i \left(-\nabla^2 + \beta^2(d-2+2i\sigma_i\mu)^2\right)\phi^i - [\phi_0^{\text{cl}}, \phi_i][\phi_0^{\text{cl}}, \phi^i]\right] \\ + 4\beta\,(2i\mu-1)\,\phi_6 v^\mu \nabla_\mu \phi_7 + 4\beta\,(2i\mu+1)\,\phi_8 v^\mu \nabla_\mu \phi_9, \tag{5.53}$$

$$\mathcal{L}^{\text{f}}_{\text{h.m.}} = \left(\chi \not\nabla \chi\right) + \left(\chi \Gamma^0 [\phi_0^{\text{cl}}, \chi]\right) - \frac{1}{2}\beta\left(\epsilon \tilde{\Gamma}^{\tilde{M}\tilde{N}} \Lambda \epsilon\right)\left(\chi \Gamma^0 \Gamma_{\tilde{M}\tilde{N}} \chi\right) \\ + 2i\mu\beta v^{\tilde{N}} \left(\chi \Gamma^0 \tilde{\Gamma}_{\tilde{N}} \Lambda \chi\right). \tag{5.54}$$

Chiral multiplet

$$\mathcal{L}^{\text{b}}_{\text{ch.m.}} = \sum_{l=1}^{3} \left[\phi_{I_l}\left(-\nabla^2 + \beta^2(d-2+2i\sigma_{(l)}\mu_l)^2\right)\phi^{I_l} - [\phi_0^{\text{cl}}, \phi_{I_l}][\phi_0^{\text{cl}}, \phi^{I_l}] \right. \\ \left. +4\beta\left(2i\mu_l - \sigma_{(l)}\right)\phi_{2l+2} v^\mu \nabla_\mu \phi_{2l+3}\right], \tag{5.55}$$

$$\mathcal{L}^{\text{f}}_{\text{ch.m.}} = \sum_{l=1}^{3} \Bigg[\left(\chi_l \not\nabla \chi_l\right) + \left(\chi_l \Gamma^0 [\phi_0^{\text{cl}}, \chi_l]\right) - \frac{1}{2}\beta\left(\epsilon \tilde{\Gamma}^{\tilde{M}\tilde{N}} \Lambda \epsilon\right)\left(\chi_l \Gamma^0 \Gamma_{\tilde{M}\tilde{N}} \chi_l\right) \\ + \sigma_{(l)}\beta\left(2i\mu_l v^{\tilde{N}}\left(\chi_i \Gamma^0 \tilde{\Gamma}_{\tilde{N}} \Lambda \chi_l\right) + \chi_i \Lambda \chi_l\right)\Bigg]. \tag{5.56}$$

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The final step before the computation of the super-determinants is the process of gauge fixing. This amounts to adding to the Lagrangian the term:

$$S_{\text{g.f.}} = -\int d^d x \sqrt{g}\,\mathrm{Tr}\left(b \nabla_\mu A'^\mu - \bar{c}\,\nabla^2 c\right), \tag{5.57}$$

which includes the Fadeev-Popov ghosts $c, \bar{c}$ and Lagrange multiplier b. We have also employed A'^μ to split the gauge field into a divergence-less part and a pure divergence part:

$$A'_\mu = A_\mu + \nabla_\mu \phi. \tag{5.58}$$

Next, we perform the path integration for the ghosts b, c and $\bar{c}$, the field ϕ (which encodes the pure divergence part of the gauge field), the auxiliary fields K^m and the scalar ϕ_0. The contributions from all these fields cancel each other, and we end up the following path integral computation to derive the partition function:

$$Z = \int d\sigma e^{-S_{\text{f.p.}}(\sigma)} \int \mathcal{D}A_\mu \mathcal{D}\phi_{I\neq 0} \mathcal{D}\Psi e^{-S_{\text{quad}}(\phi_0 = 2\beta\sigma)}. \tag{5.59}$$

We can also reduce the integration over the Lie algebra to an integration over only the Cartan subalgebra, at the cost of a Vandermonde determinant, by taking advantage of the invariance under the gauge group adjoint action. Thus, we get:

$$Z = \int [d\sigma]_{\text{Cartan}} e^{-S_{\text{f.p.}}(\sigma)} \prod_\alpha i\langle\alpha,\sigma\rangle \int \mathcal{D}A_\mu \mathcal{D}\phi_{I\neq 0}\mathcal{D}\Psi e^{-S_{\text{quad}}(\phi_0=2\beta\sigma)}. \tag{5.60}$$

Let us also recast the quadratic fluctuations in their final form before computing the super-determinants. The quadratic fluctuations are of the following schematic form:

$$\mathcal{L}^b = \text{Tr}'\left(\Phi\cdot\mathcal{O}^b\cdot\Phi - [\Phi,\phi_0^{\text{cl}}][\Phi,\phi_0^{\text{cl}}]\right), \tag{5.61a}$$

$$\mathcal{L}^f = \text{Tr}'\left(\Psi\Gamma^0\mathcal{O}^f\Psi + \Psi\Gamma^0[\phi_0^{\text{cl}},\Psi]\right). \tag{5.61b}$$

By expanding the fields in the Cartan-Weyl basis, ignoring an overall constant stemming from ϕ_0^{cl} and writing the fields using the Lie algebra root vectors E_α:

$$\Phi = \sum_\alpha \Phi^\alpha E_\alpha, \qquad \Psi = \sum_\alpha \Psi^\alpha E_\alpha, \tag{5.62}$$

we can rewrite the Lagrangian as:

$$\mathcal{L}^b = \sum_\alpha \Phi^{-\alpha}\left(\mathcal{O}^b + 4\beta^2\langle\alpha,\sigma\rangle^2\right)\Phi^\alpha, \tag{5.63a}$$

$$\mathcal{L}^f = \sum_\alpha \Psi^{-\alpha}\Gamma^0\left(\mathcal{O}^f + 2\beta\langle\alpha,\sigma\rangle\right)\Psi^\alpha, \tag{5.63b}$$

where we have used $[\sigma, E_\alpha] = \langle\alpha,\sigma\rangle E_\alpha$. So, at the end of the day, the integration over the fields Φ and Ψ will give:

$$\int \mathcal{D}\Phi\mathcal{D}\Psi \exp\left[-\int d^dx\sqrt{g}\left(\mathcal{L}^b+\mathcal{L}^f\right)\right] = \prod_\alpha \frac{\det\left(\mathcal{O}^f + 2\beta\langle\alpha,\sigma\rangle\right)_\Psi}{\sqrt{\det\left(\mathcal{O}^b + 4\beta^2\langle\alpha,\sigma\rangle^2\right)_\Phi}}. \tag{5.64}$$

5.4 Super-determinants

In this section we will summarize the results of the calculation of the 1-loop contributions to the partition function from the fluctuations around the fixed point locus. Since the computations are long and technical we will restrict ourselves to outlining the basic methodology and refer the interested reader to Paper I.

First, we need to find a complete basis for the space of fields. To this end, we define a set of spinors, which we combine with spherical harmonics to construct the desired basis. Then, we show that the basis we built is complete. The next step is to compute the eigenvalues of the operators included in $\mathcal{O}^b$ and $\mathcal{O}^f$ of (5.63) for the case of a d-dimensional sphere, along with their degeneracies and diagonalize the action of the quadratic operators $\mathcal{O}^b$ and $\mathcal{O}^f$ on the basis. The computation has to be performed for the bosons and the fermions of each multiplet separately. Then, when we combine the results from the bosonic and the fermionic calculation, we get large cancellations and finally reach a relatively compact expression for the 1-loop contribution. The computation has to be repeated for each multiplet.

We performed the computation first for the case of 8 supercharges and then for the case of 4 supercharges. The computations are largely similar, except for a subtle issue with the completeness of the basis we constructed, which needed to be supplemented. For the case of 8 supercharges the vector field bilinear v^μ leaves an S^{d-4} invariant (i.e. free action on S^5, two fixed points on S^4, fixed S^1 on S^3, etc), while for the case of 4 supercharges, the vector field leaves an S^{d-2} invariant.

All the results for 8 supercharges match with the predictions in [Minahan, 2016], confirming the conjectures in the paper, as well as with the prior literature on the subject.

8 supercharges

$$Z^{\text{v.m.}}_{\text{1-loop}} \prod_\alpha i\langle\alpha,\sigma\rangle = \prod_\alpha \prod_{k=0}^{\infty} \left[(k+i\langle\alpha,\sigma\rangle)(k+d-2+i\langle\alpha,\sigma\rangle)\right]^{N_{k,d}}, \tag{5.65a}$$

$$\begin{aligned} Z^{\text{h.m.}}_{\text{1-loop}} = \prod_\alpha \prod_{k=0}^{k=\infty} \Bigg[&\left(k+\frac{d-2}{2}+i\mu+i\langle\alpha,\sigma\rangle\right) \\ &\times \left(k+\frac{d-2}{2}-i\mu-i\langle\alpha,\sigma\rangle\right)\Bigg]^{-N_{k,d}}, \end{aligned} \tag{5.65b}$$

where

$$N_{k,d} = \frac{\Gamma(k+d-2)}{\Gamma(k+1)\Gamma(d-2)}. \tag{5.66}$$

4 supercharges

$$Z^{\text{v.m.}}_{\text{1-loop}} \prod_\alpha i\langle\alpha,\sigma\rangle = \prod_\alpha \prod_{k=0}^{k=\infty} \left[\frac{(k+i\langle\alpha,\sigma\rangle)}{(k+d-1-i\langle\alpha,\sigma\rangle)}\right]^{n_{k,d}}, \tag{5.67a}$$

$$Z^{\text{ch.m.}}_{\text{1-loop}}(\mu_j) = \prod_\alpha \prod_{k=0}^{k=\infty} \left[\frac{k+\frac{d}{2}-i\mu_j-i\langle\alpha,\sigma\rangle}{k+\frac{d-2}{2}+i\mu_j+i\langle\alpha,\sigma\rangle}\right]^{n_{k,d}}, \tag{5.67b}$$

where $j = 1, 2, 3$ and

$$n_{k,d} = \frac{\Gamma(k+d-1)}{\Gamma(k+1)\Gamma(d-1)}. \tag{5.68}$$

The complete one-loop contribution from the chiral multiplet is given by the product:

$$Z^{\text{ch.m.}}_{\text{1-loop}}(\mu_1, \mu_2, \mu_3) \;=\; Z^{\text{ch.m.}}_{\text{1-loop}}(\mu_1) Z^{\text{ch.m.}}_{\text{1-loop}}(\mu_2) Z^{\text{ch.m.}}_{\text{1-loop}}(-\mu_3). \tag{5.69}$$

5.5 Analytic continuation to S^4 for 4 supercharges

Having derived the partition function for a supersymmetric field theory with 4 supercharges on $S^{d\leq 3}$, we can attempt to analytically continue in the dimension and try to address the elusive case of S^4. Before we embark on this endeavor, let us mention a subtle point. It has been pointed out by Siegel in [Siegel, 1979] and [Siegel, 1980], that dimensional regularization in this context is not mathematically consistent (at least when a superspace formalism is employed). However, for low loop levels this can be circumvented by utilizing a component formulation (see [Avdeev, Chochia, & Vladimirov, 1981]). This argument should be enough to cover the case of a supersymmetric localization computation which is a 1-loop exact calculation.

In this section we engage in consistency checks of our analytic continuation proposal. We compare our proposal with the partition function of a free $U(1)$ theory on S^4 and compute the one-loop beta function that results from the analytic continuation of our formulas for an arbitrary gauge supersymmetric theory with 4 supercharges and we find that everything is consistent. Lastly, we make contact with a holographic computation for $\mathcal{N} = 1^*$ super Yang–Mills in [Bobev et al., 2016] and juxtapose the universal parts of the free energy. We find that at least their real parts match.

5.5.1 Check against $U(1)$ gauge theory

Our first check is to compare the results from the analytic continuation of our result for 4 supercharges for the case of a $U(1)$ gauge theory with matter in the adjoint representation. This is a free, conformal theory, so it will be straightforward to write it on S^4 and compute its partition function.

The action of the aforementioned theory will be:

$$S^{\text{ch.m.}}_{U(1)} = \int d^4x\sqrt{g}\left[\frac{1}{2}\phi_1\left(-\nabla^2 + 8\beta^2\right)\phi_1 \right.$$
$$\left. + \frac{1}{2}\phi_2\left(-\nabla^2 + 8\beta^2\right)\phi_2 - \psi\not\nabla\psi\right], \tag{5.70}$$

where $\phi_{1,2}$ are two (real) scalar fields and ψ is a Weyl fermion field with two components. The partition function for the chiral multiplet is:

$$Z^{\text{ch.m.}}_{U(1)} = \frac{\det \not{\nabla}}{\det(-\nabla^2 + 8\beta^2)}. \tag{5.71}$$

Using the expressions for the eigenvalues and their degeneracies found in the Appendix C of the Paper I, and after a shift in the product variable we get:

$$Z^{\text{ch.m.}}_{U(1)} = \prod_{k=0}^{\infty} \left(\frac{k+2}{k+1}\right)^{\frac{(k+1)(k+2)}{2}}. \tag{5.72}$$

The action for the vector multiplet, written in a suggestive form, is:

$$S^{\text{v.m.}}_{U(1)} = \int d^4x \sqrt{g} \Big(A'^{\nu} \left[\delta_{\nu}{}^{\mu}(-\nabla^2 + 12\beta^2) + \nabla_{\nu}\nabla^{\mu}\right] A'_{\mu} - \psi \not{\nabla} \psi \\ + b\nabla_{\mu}A'^{\mu} - \bar{c}\,\nabla^{\mu}\partial_{\mu}c \Big), \tag{5.73}$$

where A'^{μ} is a gauge field, ψ is a Weyl fermion field with two components, c, $\bar{c}$ are the Fadeev-Popov ghosts and b is the Lagrange multiplier that enforces the Lorenz gauge condition. We split the gauge field into a divergence-less part A_{μ} and a pure-divergence part $\nabla_{\mu}\phi$ and perform the path integration for the field ϕ and ghosts. Their contributions cancel, and we are left with the following expression to compute:

$$Z^{\text{v.m.}}_{U(1)} = \frac{\sqrt{\det'(-\nabla^2)}\det(\not{\nabla})}{\sqrt{\det(-\nabla^2 + 12\beta^2)}}, \tag{5.74}$$

where the expression on the denominator concerns only the divergence-less part of the vector field. Using the formulas for the eigenvalues and their degeneracies from Appendix C of Paper I, we finally get:

$$Z^{\text{v.m.}}_{U(1)} = \frac{1}{\sqrt{3}} \prod_{k=0}^{\infty} (k+1)^{3(k+1)}. \tag{5.75}$$

Let's now compare with the analytically continued result for 4 supercharges on S^4:

$$Z^{\text{ch.m.}}_{\text{1-loop}} = \prod_{k=0}^{\infty} \left(\frac{k+2}{k+1}\right)^{\frac{(k+1)(k+2)}{2}}, \tag{5.76a}$$

$$Z^{\text{v.m.}}_{\text{1-loop}} = \prod_{k=0}^{\infty} (k+1)^{3(k+1)}. \tag{5.76b}$$

We observe that the two results match, up to an unimportant multiplicative constant.

5.5.2 Beta function check

In this subsection we will compute the beta function for a supersymmetric theory with 4 supercharges, for the case of one vector multiplet and N_c chiral multiplets living in the R_c representation of the gauge group. We will follow [Minahan & Naseer, 2017]. We replace σ with $t\sigma$ to discern the $\mathcal{O}(\sigma^2)$ terms in the one-loop determinant. We begin with vector multiplet:

$$\frac{d \log Z^{\text{v.m.}}_{\text{1-loop}}}{dt^2} + \sum_{\alpha>0} \frac{1}{t^2} = \sum_{\alpha>0} \langle \alpha, \sigma \rangle^2 \Big[\mathcal{F}(d-1,\, 0,\, t\langle \alpha, \sigma \rangle) + \mathcal{F}(d-1,\, d-1,\, t\langle \alpha, \sigma \rangle) \Big], \tag{5.77}$$

where $\mathcal{F}$ is defined as:

$$\mathcal{F}(x,y,z) := \sum_{n=0}^{\infty} \frac{\Gamma(n+x)}{\Gamma(n+1)\Gamma(x)} \frac{1}{(n+y)^2 + z^2}. \tag{5.78}$$

Then, we set $d = 4 - \epsilon$ and expand the right hand side of equation (5.77) in the variables t and ϵ, while discarding any sub-leading terms, to finally get:

$$\log Z^{\text{v.m.}}_{\text{1-loop}} = \frac{3}{\epsilon} C_2(\text{Adj})\, \sigma^2 + \dots, \tag{5.79}$$

where C_2 is the quadratic Casimir operator. Proceeding in a similar fashion for the chiral multiplet, we get:

$$\log Z^{\text{ch.m.}}_{\text{1-loop}} = -\frac{1}{\epsilon} C_2(R_c)\, \sigma^2 + \dots . \tag{5.80}$$

Taking also into account the $\mathcal{O}(\sigma^2)$ contributions from the fixed point action (5.45), we obtain:

$$\frac{8\pi^2}{g^2(\Lambda)} = \left(\frac{8\pi^2}{g_0^2} - \frac{3}{\epsilon} C_2(\text{Adj}) + \frac{1}{\epsilon} N_c C_2(R_c) \right) \frac{1}{\Lambda^\epsilon}, \tag{5.81}$$

where g_0 is the bare coupling constant and Λ is a renormalization scale. Finally, taking the derivative with respect to the logarithm of the renormalization scale we find:

$$\beta(g) = -\frac{g^3}{16\pi^2} \left(3C_2(\text{Adj}) - N_c C_2(R_c) \right). \tag{5.82}$$

Hence, our proposal passes also this test.

5.5.3 Free energy of an $\mathcal{N} = 1^*$ theory

We will now turn our attention to the comparison between our results from the analytic continuation of the formulas for the case of 4 supercharges to the corresponding case studied in [Bobev et al., 2016]. The theory we will discuss is commonly referred to as $\mathcal{N} = 1^*$ super Yang–Mills and one can construct it as follows. Starting from $\mathcal{N} = 4$ super Yang–Mills, one can break the supersymmetry in many ways. From the $\mathcal{N} = 1$ perspective, $\mathcal{N} = 4$ super Yang–Mills comprises one vector multiplet and three massless ($m^{(1)} = m^{(2)} = m^{(3)} = 0$) chiral multiplets in the adjoint representation of the gauge group. The different ways to break $\mathcal{N} = 4$ correspond to the various choices we have for assigning masses to the chiral multiplets. If we keep one chiral multiplet as massless and assign the same nonzero mass to the other two, we obtain an $\mathcal{N} = 2$ theory. We can further break $\mathcal{N} = 2$ into $\mathcal{N} = 1$ by changing one of the previously equal masses or by introducing a nonzero mass for the initially massless multiplet. If this process is performed so that superpotential remains unaffected, the theory is referred to as $\mathcal{N} = 1^*$. As with the usual $\mathcal{N} = 1$ on S^4, the theory has not been successfully localized yet in a direct manner. A caveat that should be stressed here is that, as found in [Gerchkovitz, Gomis, & Komargodski, 2014], superconformal field theories with 4 supercharges on the four-sphere have a scheme dependence, but the fourth derivatives with respect to the mass parameter of the free energy of the theory are universal [Bobev et al., 2016].

The correspondence between the results in [Bobev et al., 2016] and our analytical continuation is not straightforward. Starting from an $\mathcal{N} = 1^*$ theory on four dimensions and compactifying down to a three-dimensional one, we end up with complex masses. However, the theory we will use for analytical continuation involves real masses. Having stated that, we will proceed with the analytical continuation and the derivation of the free energy for the $\mathcal{N} = 1^*$ theory.

The three-dimensional masses are:

$$\mu_j^{(3)} = i\Delta_j + rm_j^R, \tag{5.83}$$

where Δ_j is a flavor symmetry charge and m_j^R is the real three-dimensional mass. Upon analytical continuation, μ_j should be converted to $\sigma_{(j)}\mu_j$, where μ_j is the complex four-dimensional mass times r, and $\sigma_{(j)}$ is given in equation (5.32). Then, the analytically continued partition function will be:

$$Z = \int d\sigma_i \, e^{-\frac{8\pi^2}{g_{\mathrm{YM}}^2}\mathrm{Tr}\,\sigma^2} \prod_\alpha \prod_{k=0}^{\infty} \left[\frac{k - i\langle\alpha,\sigma\rangle}{k + i\langle\alpha,\sigma\rangle + 3} \times \prod_{j=1}^{3} \frac{k - i\langle\alpha,\sigma\rangle - i\sigma_{(j)}\mu_j + 2}{k + i\langle\alpha,\sigma\rangle + i\sigma_{(j)}\mu_j + 1} \right]^{\frac{(k+1)(k+2)}{2}}. \tag{5.84}$$

Then, using the identity:

$$\prod_{k=0}^{\infty} \left[\frac{(k + i\langle\alpha,\sigma\rangle)(k + i\langle\alpha,\sigma\rangle + 2)^3}{(k + i\langle\alpha,\sigma\rangle + 3)(k + i\langle\alpha,\sigma\rangle + 1)^3} \right]^{\frac{(k+1)(k+2)}{2}} = i\langle\alpha,\sigma\rangle, \tag{5.85}$$

we get:

$$Z = \int d\sigma_i \, e^{-\frac{8\pi^2}{g_{\rm YM}^2}\operatorname{Tr}\sigma^2} \prod_\alpha i\langle\alpha,\sigma\rangle Z_{\rm mass}, \tag{5.86}$$

where we have introduced $Z_{\rm mass}$:

$$Z_{\rm mass} = \prod_\alpha \prod_{k=0}^{\infty} \prod_{j=1}^{3} \left[\frac{(k - i\langle\alpha,\sigma\rangle - i\sigma_{(j)}\mu_j + 2)(k + i\langle\alpha,\sigma\rangle + 1)}{(k + i\langle\alpha,\sigma\rangle + i\sigma_{(j)}\mu_j + 1)(k - i\langle\alpha,\sigma\rangle + 2)}\right]^{\frac{(k+1)(k+2)}{2}}. \tag{5.87}$$

What we have now is a matrix model, which we can proceed to analyze with the usual methods (see Chapter 3). However, before doing that, we need to treat the divergence in the expression (5.87). By defining:

$$Z_k(\sigma-\sigma',\mu) := \left[\frac{(k - i(\sigma-\sigma') - i\mu + 2)(k + i(\sigma-\sigma') + 1)}{(k + i(\sigma-\sigma') + i\mu + 1)(k - i(\sigma-\sigma') + 2)}\right]^{\frac{(k+1)(k+2)}{2}} \tag{5.88}$$

and expanding $\log Z_k$ in $1/k$:

$$\log Z_k(\sigma-\sigma',\mu) = -i\left(k + \frac{1}{2} + \frac{(\sigma-\sigma')^2}{k}\right)\mu - \frac{1}{2k}\mu^2 + \frac{i}{3k}\mu^3 + \mathcal{O}\left(\frac{1}{k^2}\right), \tag{5.89}$$

we see that in the limit $k \gg 1$ we have a divergence of cubic order in terms of the masses. The condition for the masses:

$$\mu_1 + \mu_2 - \mu_3 = 0 \tag{5.90}$$

takes care of the first order in μ term, while the rest of the divergences can be eliminated by the addition of constant local counterterms.

To make contact with the results in [Bobev et al., 2016], we will analyze the resulting matrix model in the large-N limit at the strong coupling regime. For this purpose, it is enough to engage in a saddle-point analysis. The saddle-point equation for our theory is:

$$\frac{16\pi^2}{\lambda}\sigma \simeq 2 \fint d\sigma' \rho(\sigma') \frac{1 + \frac{1}{2}\left(\mu_1^2 + \mu_2^2 + \mu_3^2\right) + i\mu_1\mu_2\mu_3}{\sigma - \sigma'}. \tag{5.91}$$

This is a Gaussian matrix model saddle-point equation, and its solution is the well-known Wigner semi-circle distribution:

$$\rho(\sigma) = \frac{2}{\pi A^2}\sqrt{A^2 - \sigma^2}, \tag{5.92}$$

where A is given by:

$$A^2 = \frac{\lambda\left[1 + \frac{1}{2}\left(\mu_1^2 + \mu_2^2 + \mu_3^2\right) + i\mu_1\mu_2\mu_3\right]}{8\pi^2}. \tag{5.93}$$

Thus, we can now derive an expression for the free energy:

$$\begin{aligned}
F &\simeq -\frac{N^2}{2}\int d\sigma d\sigma' \log(\sigma-\sigma')^2 \\
&\simeq -\frac{N^2}{2}\left[1+\frac{1}{2}\left(\mu_1^2+\mu_2^2+\mu_3^2\right)+i\mu_1\mu_2\mu_3\right] \\
&\quad \times \log\left[\lambda\left(1+\frac{1}{2}\left(\mu_1^2+\mu_2^2+\mu_3^2\right)+i\mu_1\mu_2\mu_3\right)\right]. \qquad (5.94)
\end{aligned}$$

Finally, we expand this expression for small μ_i and keep only terms above third order that are universal (see [Gerchkovitz et al., 2014]) to get:

$$\begin{aligned}
F \simeq -N^2 \bigg[&\frac{1}{16}(\mu_1^2+\mu_2^2+\mu_3^2)^2+\frac{i}{4}(\mu_1^2+\mu_2^2+\mu_3^2)^2\mu_1\mu_2\mu_3 \\
&-\frac{1}{96}(\mu_1^2+\mu_2^2+\mu_3^2)^3-\frac{1}{4}(\mu_1\mu_2\mu_3)^2+\mathcal{O}(\mu^7)\bigg]. \qquad (5.95)
\end{aligned}$$

The expectation from [Bobev et al., 2016] for the form of the free energy is:

$$\begin{aligned}
F = -N^2\big[&A_1(\mu_1^4+\mu_2^4+\mu_3^4)+A_2(\mu_1^2+\mu_2^2+\mu_3^2)^2 \\
&+iB_1(\mu_1^2+\mu_2^2+\mu_3^2)\mu_1\mu_2\mu_3-C_1(\mu_1^6+\mu_2^6+\mu_3^6) \\
&-C_2(\mu_1^2+\mu_2^2+\mu_3^2)^3-C_3(\mu_1\mu_2\mu_3)^2+\mathcal{O}(\mu^7)\big]. \qquad (5.96)
\end{aligned}$$

By comparing this general form with our result (5.95), while also taking into account equation (5.90), we get:

$$A_1+2A_2=\frac{1}{8}, \quad B_1=-\frac{1}{4}, \quad C_1+C_2=\frac{1}{24}, \quad -12C_2+C_3=\frac{1}{8}. \qquad (5.97)$$

Comparing with [Bobev et al., 2016], we see that the first and the third relation are in agreement with their results. The fourth relation is a new result that does not have a counterpart in [Bobev et al., 2016]. This prediction however seems to be in disagreement with a subsequent investigation performed in [N. Kim & Kim, 2019], which finds $-12C_2+C_3 \simeq -0.58$. Furthermore, the result for B_1 does not match with the holographic analysis of [Bobev et al., 2016]. Their free energy is real, while the one we obtained is complex. In principle, since we are dealing with a non reflection-positive theory, it is not guaranteed that the free energy should be real. At this point it is not clear what the resolution to this issue is. Another related open problem is to understand from the analytic continuation perspective the appearance of a gaugino condensate that emerges in the holographic study.

6. A phase transition in 2d maximally super Yang–Mills

In this chapter, we present preliminary results based on a currently unpublished work in collaboration with Joseph Minahan and Anton Nedelin, on the study of a matrix model resulting from localizing the two-dimensional Yang–Mills theory. We begin in Section 6.1 with an introduction to this work. Then, in Section 6.2 we write down the matrix model, first in general dimensions and then in two dimensions, as was done in [Bobev, Bomans, Gautason, Minahan, & Nedelin, 2020]. Then, in Section 6.3, we review and extend the results of [Kazakov, Kostov, & Nekrasov, 1999], where the authors analyzed the same matrix model, but with an opposite sign for the 't Hooft coupling. We analytically continue these results in Section 6.4 for positive values of the 't Hooft coupling. Next, we engage in numerical and analytical studies of the matrix model in Sections 6.5 and 6.6 respectively. We conclude in Section 6.7 with a discussion of the open problems of our work.

6.1 Introduction

In the previous chapter, we constructed and localized a class of supersymmetric gauge theories on d-dimensional spheres. Such localization computations can provide valuable information on quantum field theories, especially in the challenging strong coupling limit. Furthermore, they present a rare opportunity to perform highly non-trivial tests of the holographic correspondence. These tests are even more interesting when the quantum field theory side is not conformal. A test of this kind was recently performed in [Bobev et al., 2020]. In this work, the authors employed the matrix models that arise from the localization of maximally supersymmetric Yang–Mills theories, derived in Paper I. By examining the planar limit of these models, they were able to compute the free energies and (BPS) Wilson loop expectation values for $2 \leq d \leq 7$ in the strong coupling regime. Then, they made use of the spherical brane solutions found in [Bobev, Bomans, & Gautason, 2018], to derive the same results from the supergravity side, and found the two sides indeed match.

The quantum field theory side of the study of [Bobev et al., 2020], entailed the analysis of the saddle-point equation of the relevant matrix models. As demonstrated in the previous chapter, the results of the localization of the theories on S^d were written in Paper I in a general form as functions of the dimension. Thus, the authors of [Bobev et al., 2020] were able to engage in a general study of the resulting saddle-point equations, but due to the structure of the quantities involved, this was feasible only for $3 < d < 6$. The two-dimensional case had to be studied using analytical continuation. Despite this obstacle, they were able to obtain a result that matches the supergravity calculation.

While the treatment of the two-dimensional case in [Bobev et al., 2020] yielded the results that were expected from the supergravity side, a careful analysis reveals that they were obtained under assumptions that no longer hold for $d = 2$. Both the interaction between the eigenvalues and the potential in the saddle-point equation are repulsive for $d = 2$, implying that a finite-width distribution on the real line, such as the one found for $3 < d < 6$, cannot occur. This presents the starting point of the research project whose preliminary results are reported in this chapter. Our main goal is to study in detail the matrix model that arises from the two-dimensional maximally supersymmetric Yang–Mills and derive an expression for its free energy.

A very similar saddle-point equation to the one that results from our two-dimensional theory, was studied before in [Kazakov et al., 1999], in an unrelated context (it described the motion of D-particles). The equation is almost identical, with the only difference being a sign, which from our perspective would correspond to a negative 't Hooft coupling. Thus, we can use their findings to aid our study by analytically continuing to positive 't Hooft coupling. Pursuing this approach, we find evidence that the system undergoes a third order phase transition. This can be also understood both from a numerical investigation of the analytically continued solution, and more intuitively from the "collision" of two cuts of the matrix model. This is similar to what happens, for instance, in [Russo & Zarembo, 2013], [Anderson & Zarembo, 2014], or [Nedelin, 2015].

Equipped with the results from the analytic continuation of [Kazakov et al., 1999], we proceed to study the saddle-point equation numerically. We find that there are two classes of solutions: a one-cut solution and a bifurcating solution, the latter of which, is not commonly encountered in the literature. By analyzing the free energy of these solutions, we find that the bifurcating solution is energetically more favorable, and hence dominates. The numerical study is augmented with an analytical study of the problem. We are able to derive analytically the angle of the one-cut solutions in the complex plane, and get an expression for the eigenvalue density. Our numerical and analytical results are in very good agreement. We are also able to reproduce the scaling of the analytically continued two-point function of [Kazakov et al., 1999]. Lastly, we study analytically a special bifurcating solution that displays a reflection symmetry relative to the imaginary axis, and show that the resulting free energy is real and that it is dominating with respect to the other solutions.

6.2 Matrix model

6.2.1 General dimension

We will begin with a short review of the results of [Bobev et al., 2020] regarding the matrix model arising from the localization of maximally supersymmetric Yang–Mills on d-dimensional spheres, for $3 < d < 6$, which are based in turn on [Minahan & Zabzine, 2015], [Minahan, 2016] and Paper I. Then we will concentrate on the two-dimensional case. Since we have discussed the construction of maximally supersymmetric Yang–Mills on d-dimensional spheres in some detail in Chapter 5, here we will start from the resulting partition function and matrix model after having performed localization. In particular, the partition function on S^d, in the no-instanton sector, takes the form:[1]

$$Z = \int\limits_{\text{Cartan}} d\sigma \, \exp\left(-\frac{4\pi^{(d+1)/2}\mathcal{R}^{d-4}}{g_{\text{YM}}^2 \Gamma(\frac{d-3}{2})} \operatorname{Tr}\sigma^2 \right) Z_{\text{1-loop}}(\sigma) \tag{6.1}$$

where σ is a Hermitian $N \times N$ matrix and $\mathcal{R}$ is the radius of the sphere. The integration is performed over the Cartan of the gauge group and so we can replace the matrices σ with their eigenvalues σ_i. The one-loop contribution from the fluctuations around the localization locus is:

$$Z_{\text{1-loop}}(\sigma) \prod_{\alpha>0} \langle\alpha,\sigma\rangle^2 = \prod_{\alpha>0} \prod_{n=0}^{\infty} \left(\frac{n^2 + \langle\alpha,\sigma\rangle^2}{(n+d-3)^2 + \langle\alpha,\sigma\rangle^2} \right)^{\frac{\Gamma(n+d-3)}{\Gamma(n+1)\Gamma(d-3)}}, \tag{6.2}$$

where we have included the Vandermonde determinant, and α are the positive roots of the gauge group. This expression converges for $d < 6$, so for our purposes we do not have to worry about regularization.

While the expression (6.1) for the partition function generically needs to be supplemented with the appropriate instanton contribution, we will concentrate on the large-N limit, and hence we will be able to safely neglect instantons. In this limit, we can make use of the saddle-point approximation, with the saddle-point equation being:

$$\frac{C_1 N}{\lambda}\sigma_i = \sum_{j\neq i} G_{16}(\sigma_{ij}), \tag{6.3}$$

where $\sigma_{ij} \coloneqq \sigma_i - \sigma_j$ and λ is the dimensionless version of the 't Hooft coupling constant:

$$\lambda \coloneqq \mathcal{R}^{4-d} g_{\text{YM}}^2 N. \tag{6.4}$$

[1] Note that in [Bobev et al., 2020], the coupling constant g_{YM}^2 was modified to $2g_{\text{YM}}^2$ with respect to the conventions of Paper I to comply with the supergravity literature convention, and in this chapter we will do the same.

The constant C_1 is given by:

$$C_1 \coloneqq \frac{8\pi^{(d+1)/2}}{\Gamma(\frac{d-3}{2})}, \tag{6.5}$$

while the kernel of the matrix model $G_{16}(\sigma)$ is:

$$\frac{iG_{16}(\sigma)}{\Gamma(4-d)} = \frac{\Gamma(-i\sigma)}{\Gamma(4-d-i\sigma)} - \frac{\Gamma(i\sigma)}{\Gamma(4-d+i\sigma)} - \frac{\Gamma(d-3-i\sigma)}{\Gamma(1-i\sigma)} + \frac{\Gamma(d-3+i\sigma)}{\Gamma(1+i\sigma)}. \tag{6.6}$$

For large separations of the eigenvalues $|\sigma_{ij}| \gg 1$, the kernel $G_{16}(\sigma_{ij})$ can be expanded to:

$$G_{16}(\sigma_{ij}) \simeq C_2 |\sigma_{ij}|^{d-5} \mathrm{sign}(\sigma_{ij}), \tag{6.7}$$

where the constant C_2 is defined as:

$$C_2 \coloneqq 2(d-3)\Gamma(d-5)\sin\frac{\pi(d-3)}{2}. \tag{6.8}$$

An analysis of the saddle point equation (6.3) for general d is hindered by the fact that $C_2 = 0$ for $d = 3$, effectively limiting the validity of such a study to $3 < d$.

By replacing the kernel with its expansion (6.7) to the saddle-point equation, we get:

$$\frac{C_1}{\lambda} N\sigma_i = C_2 \sum_{j\neq i} |\sigma_i - \sigma_j|^{d-5} \mathrm{sign}(\sigma_i - \sigma_j). \tag{6.9}$$

We proceed by introducing the eigenvalue density $\rho(\sigma)$:

$$\rho(\sigma) \coloneqq \frac{1}{N} \sum_{i=1}^{N} \delta(\sigma - \sigma_i), \tag{6.10}$$

which can be inserted in the strong-coupling form of the saddle-point equation, to give:

$$\frac{C_1}{\lambda}\sigma = C_2 ⨍_{-b}^{b} d\sigma' \rho(\sigma') |\sigma - \sigma'|^{d-5} \mathrm{sign}(\sigma - \sigma'), \tag{6.11}$$

where b denotes the endpoints of the distribution of the matrix model eigenvalues. The solution of this saddle-point equation is [Bobev et al., 2020]:

$$\rho(\sigma) = \frac{2\pi^{(d+1)/2}}{\pi\lambda\Gamma(6-d)\Gamma(\frac{d-1}{2})(b^2-\sigma^2)^{(d-5)/2}}. \tag{6.12}$$

The constant b can be fixed by imposing the normalization condition to the eigenvalue density $\rho(\sigma)$. This gives:

$$b = (4\pi)^{\frac{d+1}{2(d-6)}} \left[32\lambda\Gamma\left(\frac{8-d}{2}\right)\Gamma\left(\frac{6-d}{2}\right)\Gamma\left(\frac{d-1}{2}\right)\right]^{\frac{1}{6-d}}. \tag{6.13}$$

The free energy of the theory in the strong coupling, large-N limit can be found by evaluating the expression:

$$\begin{aligned} F = N^2 \Bigg[& \frac{C_1}{2\lambda}\int_{-b}^{b} d\sigma \rho(\sigma)\sigma^2 \\ & - \frac{C_2}{2(d-4)}\int_{-b}^{b} d\sigma\rho(\sigma)\int_{-b}^{b} d\sigma'\rho(\sigma')|\sigma-\sigma'|^{d-4}\Bigg]. \end{aligned} \tag{6.14}$$

Performing the integrals (see [Bobev et al., 2020] for more details), we get:

$$\begin{aligned} \frac{F}{N^2} &= -\frac{C_1}{2\lambda}\frac{6-d}{(8-d)(d-4)}b^2 \\ &= -\frac{16\pi^{\frac{(d+1)(4-d)}{2(6-d)}}(6-d)}{\lambda\Gamma(\frac{d-3}{2})(8-d)(4-d)}\left[\frac{\lambda}{4}\Gamma\left(\frac{8-d}{2}\right)\Gamma\left(\frac{6-d}{2}\right)\Gamma\left(\frac{d-1}{2}\right)\right]^{\frac{2}{6-d}}. \end{aligned} \tag{6.15}$$

6.2.2 Two-dimensional case

Let us now focus on the two-dimensional case, the main topic of this chapter. The kernel of the matrix model for $d = 2$ becomes:

$$G_{16}(\sigma) = \frac{4}{\sigma + \sigma^3}. \tag{6.16}$$

The saddle-point equation now is:

$$-\frac{\pi N}{\lambda}\sigma_i = \sum_{j\neq i}\frac{1}{\sigma_{ij}+\sigma_{ij}^3}, \tag{6.17}$$

or expressed in the continuum limit:

$$-\frac{\pi}{\lambda}\sigma = \fint_{-b}^{b} d\sigma' \frac{\rho(\sigma')}{(\sigma-\sigma')^3}. \tag{6.18}$$

One possible way to proceed now, is to perform analytical continuation of the results of the previous subsection to $d = 2$, as was done in [Bobev et al., 2020]. Then the eigenvalue distribution is given by:

$$\rho(\sigma) = \frac{1}{3\lambda}(b^2-\sigma^2)^{3/2}, \tag{6.19}$$

and its endpoint is:

$$b_2 = \left(\frac{8\lambda}{\pi}\right)^{1/4}. \tag{6.20}$$

Inserting this expression for b_2 in equation (6.15), the free energy becomes:

$$F_2 = -\frac{4(2\pi)^{1/2}}{3\lambda^{1/2}}N^2. \tag{6.21}$$

This means that free energy scales as:

$$F_2 \sim N^{3/2}. \tag{6.22}$$

This result agrees with the holographic study that was performed in the same paper.

The results for the eigenvalue density (6.19) and the free energy (6.21) were derived under a large separation assumption. This was a valid assumption for the range $d \in (3, 6)$, where, in the strong coupling regime, the central potential is weakly attractive and the interactions between the eigenvalues are repulsive at wide separations. However, this assumption is not warranted for $d = 2$, as both the central potential and the interaction between the eigenvalues are repulsive (for positive 't Hooft coupling). This implies that a distribution on the real line with finite width, such as the one above, is not attainable.

A clue to our problem is provided by [Kazakov et al., 1999], where the same matrix model was studied. The matrix model was obtained under a completely different motivation, as the authors were studying correlation functions of certain operators arising from zero-dimensional reduction of a matrix model that captures the motion of D-particles. However, in their case, the left-hand side of the saddle-point equation (6.18) comes with an opposite sign, which from our perspective corresponds to a negative 't Hooft coupling (hence the tile of the next section). In the following section, we will review the relevant parts of [Kazakov et al., 1999] and in particular the analysis of the matrix model and supplement the presentation with a few results that do not appear in the paper, but follow readily from the expressions therein.

6.3 Study of the matrix model under negative 't Hooft coupling

To make contact with the conventions of [Kazakov et al., 1999], we will introduce the coupling constant:

$$g^2 = -\frac{\lambda}{\pi}, \tag{6.23}$$

Note that the authors of this work, study the matrix model for $g^2 > 0$, which corresponds a negative 't Hooft coupling constant. With the purpose of solving the saddle-point equation (6.18), they write the resolvent:

$$W(\sigma) = \int_{-a}^{a} d\sigma' \frac{\rho(\sigma')}{\sigma - \sigma'}, \tag{6.24}$$

which for positive g^2 has a cut along the real axis. Then, the saddle-point equation becomes:

$$\frac{2x}{g^2} = W(x + i0) + W(x - i0) - W(x + i) - W(x - i), \tag{6.25}$$

and the eigenvalue density $\rho(\sigma)$ can be obtained by the equation:

$$-2\pi i \rho(x) = W(x + i0) - W(x - i0). \tag{6.26}$$

Following [Kazakov et al., 1999], we introduce a second resolvent $G(z)$:

$$G(z) = \frac{z^2}{g^2} + i \left[W\left(z + \frac{i}{2}\right) - W\left(z - \frac{i}{2}\right) \right], \tag{6.27}$$

which allows us to recast the saddle-point equation in the following simple form:

$$G\left(x + \frac{i}{2}\right) = G\left(x - \frac{i}{2}\right), \tag{6.28}$$

where $x \in (-a, +a)$. The cuts of the second resolvent $G(z)$ are located at $\left(\pm\frac{i}{2} - a, \pm\frac{i}{2} + a\right)$ (see Figure 6.1), and the function is real when $z \in \mathbb{R}$, $i\mathbb{R}$ and $\left(\pm\frac{i}{2} - a, \pm\frac{i}{2} + a\right)$. It would be useful to think of the function $G(z)$ as a holomorphic map of a region delineated by the positive real axis $\mathbb{R}_+$, the positive imaginary axis $i\mathbb{R}_+$ and the sides of the cut $(\frac{i}{2}, \frac{i}{2} + a)$ (see Figure 6.1). This inverse of this map is $G(z) = \zeta$, with:

$$z = A \int_{x_1}^{\zeta} \frac{dt(t - x_3)}{\sqrt{(t - x_1)(t - x_2)(t - x_4)}}. \tag{6.29}$$

We assume that the we have ordered the points x_i so that $x_1 > x_2 > x_3 > x_4$, and we impose the following conditions on the map:

ζ	z
$+\infty$	$+\infty$
x_1	0
x_2	$i/2$
x_3	$a + i/2$
x_4	$i/2$
$-\infty$	$+i\infty$

(6.30)

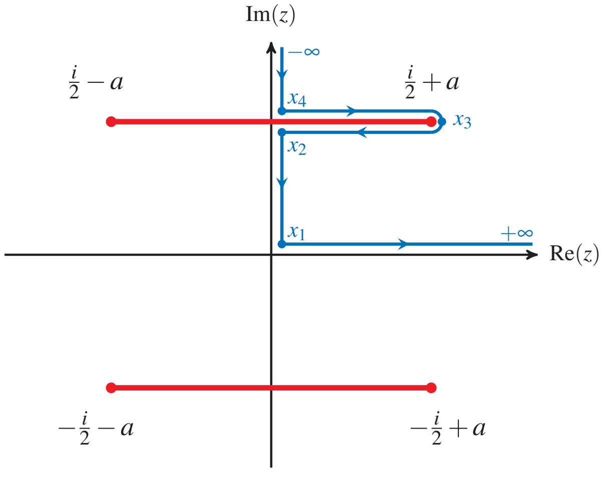


Figure 6.1. The structure of the resolvent $G(z)$ of the matrix model in the complex plane. The cuts of the function are presented in red. The region that is mapped under the holomorphic map $G(z) = \zeta$ is delineated by the contour in blue.

which gives us the equations below:

$$
\begin{aligned}
\frac{1}{2} &= A \int_{x_2}^{x_1} \frac{dt(t - x_3)}{\sqrt{(x_1 - t)(t - x_2)(t - x_4)}},\\
a &= A \int_{x_3}^{x_2} \frac{dt(x_3 - t)}{\sqrt{(x_1 - t)(x_2 - t)(t - x_4)}},\\
-a &= A \int_{x_4}^{x_3} \frac{dt(x_3 - t)}{\sqrt{(x_1 - t)(x_2 - t)(t - x_4)}}.
\end{aligned} \tag{6.31}
$$

Analyzing the asymptotic behavior of the function $G(z)$ [Kazakov et al., 1999], we get also the additional equations:

$$
\begin{aligned}
A &= \frac{g}{2},\\
x_1 + x_2 + x_4 &= 2x_3,\\
x_1^2 + x_2^2 - 2x_3^2 + x_4^2 &= \frac{6}{g^2}.
\end{aligned} \tag{6.32}
$$

These relations, in conjunction with the conditions (6.31) specify the mapping completely. Introducing the parameter:

$$
m = \frac{x_2 - x_4}{x_1 - x_4}, \tag{6.33}
$$

with $0 < m < 1$, we can express the points x_i, the coupling constant g^2 and the endpoint of the eigenvalue distribution a, as a function of the parameter m using complete elliptic integral functions. The points x_i are given by:

$$x_1 = \frac{3\pi^2}{K^2(m)} \frac{2\vartheta(m) + m - 2}{3\vartheta^2(m) + 2(m-2)\vartheta(m) + 1 - m}, \tag{6.34a}$$

$$x_2 = \frac{3\pi^2}{K^2(m)} \frac{2\vartheta(m) - 1}{3\vartheta^2(m) + 2(m-2)\vartheta(m) + 1 - m}, \tag{6.34b}$$

$$x_3 = \frac{3\pi^2}{K^2(m)} \frac{3\vartheta(m) + m - 2}{3\vartheta^2(m) + 2(m-2)\vartheta(m) + 1 - m}, \tag{6.34c}$$

$$x_4 = \frac{3\pi^2}{K^2(m)} \frac{2\vartheta(m) + m - 1}{3\vartheta^2(m) + 2(m-2)\vartheta(m) + 1 - m}, \tag{6.34d}$$

where $K(m)$ and $E(m)$ are the complete elliptic integral functions of the first and the second kind respectively, defined as follows:

$$K(m) \coloneqq \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m\sin^2\theta}}, \tag{6.35a}$$

$$E(m) \coloneqq \int_0^{\pi/2} d\theta \sqrt{1 - m\sin^2\theta}, \tag{6.35b}$$

and we have also introduced the symbol $\vartheta(m)$ for the ratio of the two elliptic integral functions:

$$\vartheta(m) \coloneqq \frac{E(m)}{K(m)}. \tag{6.36}$$

The coupling constant g^2 then becomes:

$$g^2(m) = \frac{K^4(m)}{3\pi^4} \left[-3\vartheta^2(m) + 2(2-m)\vartheta(m) - (1-m).\right] \tag{6.37}$$

We can also derive the endpoints of the eigenvalue distribution a:

$$a(m) = -g\sqrt{x_1 - x_2}\, E\left(\phi, \frac{m}{m-1}\right) + \frac{g(x_1 - x_2 - x_4)}{2\sqrt{x_1 - x_2}}\, F\left(\phi, \frac{m}{m-1}\right), \tag{6.38}$$

where $F(\phi, x)$ and $E(\phi, x)$ are the incomplete elliptic integral functions of the first and second kind respectively, which are defined as:

$$F(\phi, x) \coloneqq \int_0^{\sin\phi} dt\, \frac{1}{\sqrt{(1-t^2)(1-x\,t^2)}}, \tag{6.39a}$$

$$E(\phi, x) \coloneqq \int_0^{\sin\phi} dt\, \sqrt{\frac{1-x\,t^2}{1-t^2}}, \tag{6.39b}$$

and we have also introduced the quantity $\phi(m)$ that is implicitly defined via the relation:

$$\sin^2\phi(m) = \frac{x_2(m) - x_3(m)}{x_2(m) - x_4(m)}. \tag{6.40}$$

Lastly, in [Kazakov et al., 1999], we can find an expression for the correlation function $\nu(m) = \langle \mathrm{Tr}\,\sigma^2 \rangle$:

$$\begin{aligned}\nu(m) &= \frac{g^4}{2N^2}\frac{\partial F(N,g)}{\partial g^2}\\ &= \frac{1}{12} - \frac{K^2(m)}{5\pi^2}\frac{1}{3\vartheta(m)^2 + 2(m-2)\vartheta(m) + 1 - m}\times\\ &\times \left[10\vartheta(m)^2(\vartheta(m) + m - 2) + 2\vartheta(m)(6 - 6m + m^2) + (1-m)(m-2)\right].\end{aligned} \tag{6.41}$$

6.4 Analytical continuation of the solution of [Kazakov et al., 1999]

6.4.1 Locus of positive ’t Hooft coupling in the m-plane

As mentioned above, the study of the matrix model with the saddle-point equation (6.18) in [Kazakov et al., 1999] was done under the assumption of negative ’t Hooft coupling or $g^2 > 0$ (for $g^2 = -\lambda/\pi$), at $0 < m < 1$. However, for our interests, we need to study the case of negative g^2. To that end, we will analytically continue the results summarized in the previous section, and perform a numerical study.

Since we have an expression for the coupling constant g^2 as a function of m (equation (6.37)), we can search for the locus where it is negative. This can be accomplished by drawing a contour diagram of the real part of g^2 and search for areas where the imaginary part is vanishing and the real part is negative. The resulting plot is shown in Figure 6.2. It appears that the locus of points where $g^2(m) < 0$ is a (semi)-circle.

The semi-circle intersects the real axis at $m = 2$ (and $m = 0$). However, the functions $E(m)$ and $K(m)$ have a cut along the line $(+1, \infty)$, which is inherited by g^2. Thus, every time we cross a cut we need to jump to another sheet of the elliptic integral functions. At the n-th sheet, the elliptic integral functions $K(m)$ and $E(m)$ become:

$$K_n(m) \coloneqq K(m) + 2inK(1-m), \tag{6.42a}$$

$$E_n(m) \coloneqq E(m) + 2in\left[K(1-m) - E(1-m)\right]. \tag{6.42b}$$

Using these expression in the coupling constant function $g^2(m)$ and moving to subsequent sheets, we find something interesting. As one can see in Figure 6.3

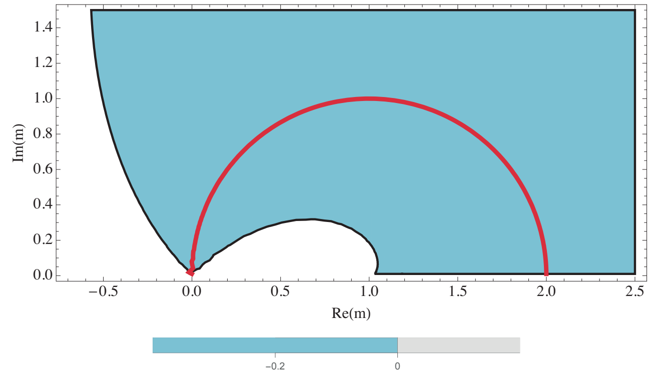


Figure 6.2. Contour diagram of the real part of the coupling constant $g^2(m)$ in the m complex plane. The red curve is the locus of points where $\operatorname{Im} g^2(m) = 0$ and hence $g^2(m) < 0$.

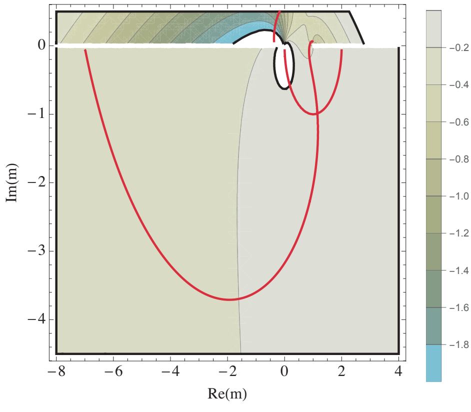
(a), the red locus curve trifurcates. The part of the curve that leads to the most rich behavior is the one that seems to move towards the point $m = 1$. In particular, if one zooms in the region around $m = 1$ (Figure 6.4), one can see that the curve moves above the real axis and turns downwards again, before it gets interrupted by the cut $(+1, \infty)$. By continuing to the next sheets, one can see that curve winds helically around $m = 1$ and approaches it asymptotically for $n \to \infty$. As the curve gets closer to $m = 1$, g^2 moves towards more negative values.

In order to get more information on the branch of the curve that is approaching $m = 1$, we can use an iterative method, such as Newton–Raphson, to solve the nonlinear equation $\operatorname{Im}(g^2(m)) = 0$, using an initial estimate close to this branch. Doing so, we can also compute the correlation function $\nu(m)$ using equation (6.41) and its derivatives (with a finite differences formula). The results are shown in Figure 6.5. From Figure 6.5, we see clear evidence that at $g^2 \simeq -0.1$, which is the exact point where the red curve of Figure 6.3 (a) trifurcates, there exists a discontinuity in $\partial^2\nu/\partial(g^2)^2$. Thus, it appears that the system undergoes a third order phase transition at $g^2 \simeq -0.1$.

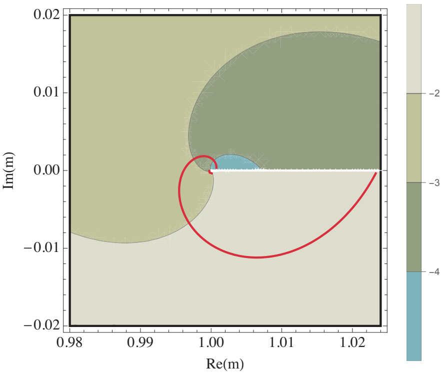
After the phase transition, the numerical solution for the correlation function $\nu(g^2)$ displays a non-zero imaginary part. In principle, this issue could be a numerical artifact. However further numerical studies with greater accuracy did not show signs of a decrease in the imaginary part.

6.4.2 Eigenvalue distribution endpoints

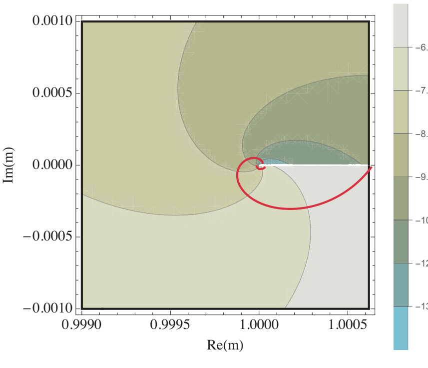
We will now try to understand the phase transition hinted by Figure 6.5 by looking at the behavior of the cuts of the resolvent $G(z)$ (see Figure 6.1 for the case of $g^2 > 0$). In particular, we will investigate what happens to the function $a(m)$ (6.38), which describes the position of the endpoints of the cuts,



(a) Second sheet



(b) Third sheet



(c) Fourth sheet

Figure 6.3. Contour diagram of the real part of the coupling constant $g^2(m)$ in the complex m plane, where we have jumped to the second, third and fourth sheet of the complete elliptic integral functions $E(m)$ and $K(m)$. The red curve is once again the locus of points where $g^2(m) < 0$.

when we move to a negative coupling $g^2 < 0$. This amounts to navigating the convoluted cut structure of the function and carefully selecting the appropriate sheets.

The first cut that we need to address is the one that is related to the presence of $g = \sqrt{g^2}$ in (6.38). The circular locus in the m-plane where $g^2(m) < 0$, that is the focus our study, is precisely the position of the cut that affects the function $g(m) = \sqrt{g^2(m)}$. The correct choice is to remain in the internal side of the cut, otherwise we would have $\mathrm{Re}[a(0)] \neq 0$, which would result in a discontinuity at $g^2(m) = 0$. Having chosen the inner side of the circle, we have place the endpoints $a(m)$ of the resolvent cuts on the imaginary axis (see Figure 6.6 (b)).

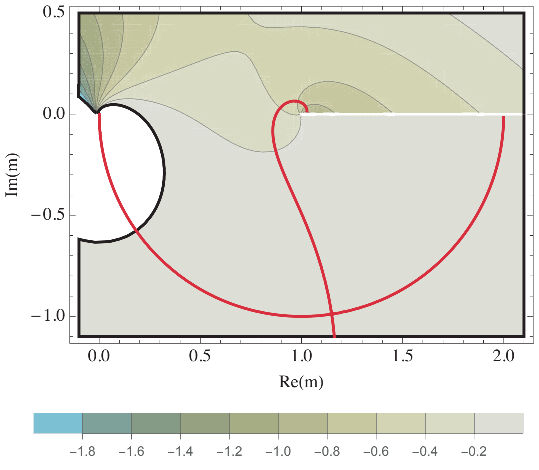
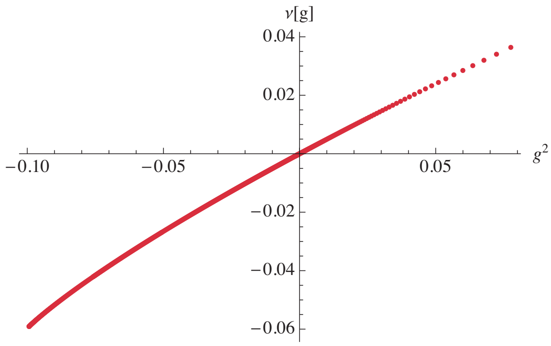
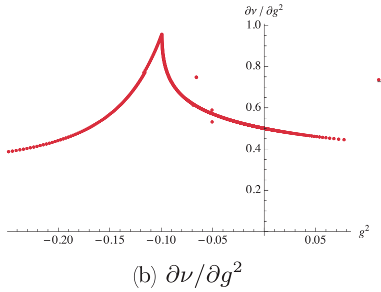


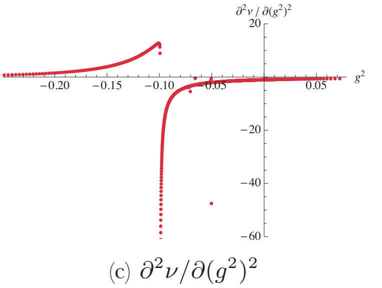
Figure 6.4. Contour diagram of the real part of the coupling constant $g^2(m)$ in the m complex plane at the second sheet of $E(m)$ and $K(m)$.



(a) $\nu(g^2)$



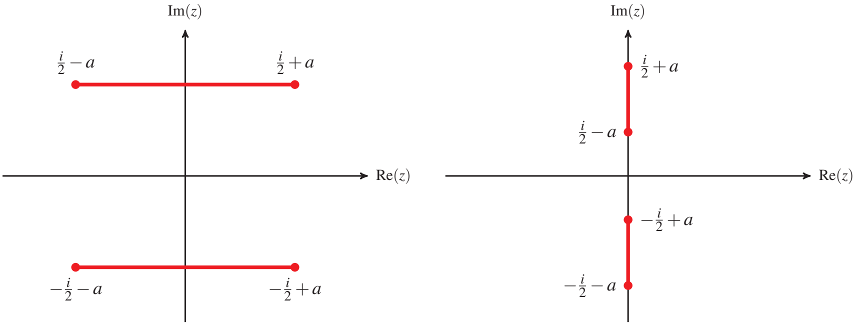
(b) $\partial\nu/\partial g^2$



(c) $\partial^2\nu/\partial(g^2)^2$

Figure 6.5. Plots of the correlation function $\nu(g^2)$ and its derivatives on the locus of $g^2(m) < 0$.

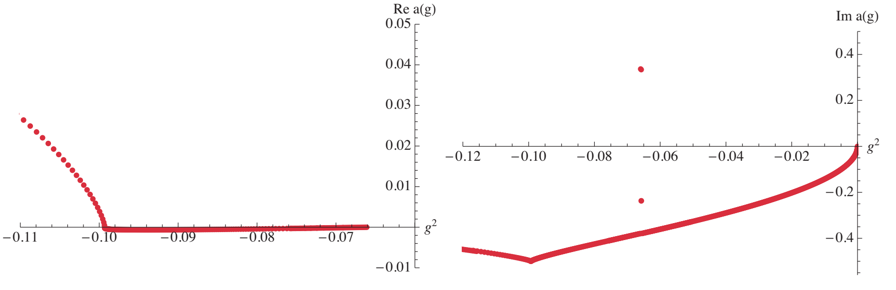
Continuing further down the spiral, the imaginary part of the endpoints $a(m)$ gets more negative while the real part remains zero. At some point the spiral reaches the cut at $(+1, \infty)$ and we need to change to another sheet for



(a) $g^2 > 0$ (b) $g^2 < 0$

Figure 6.6. The cuts of the matrix model resolvent for positive (a) and negative (b) coupling constant $g^2(m)$.

the complete elliptic functions $E(m)$ and $K(m)$ using (6.42), while for the incomplete elliptic functions $E(\phi, m/(m-1))$ and $F(\phi, m/(m-1))$, we do not need to do anything, since they are continuous on that line. We present the results from the analytic continuation in Figure 6.7. We observe that when crossing the cut $(+1, \infty)$ at $g \simeq -0.065$ and $a \simeq -0.38i$, a remains continuous. As the imaginary part of a keeps getting smaller, at some point it reaches the value $a \simeq -0.5i$. However, as we can see at Figure 6.6, at that value, the two cuts collide and merge into one—a common occurrence for third-order phase transitions in the matrix model literature (see for instance [Gross & Witten, 1980] and [Wadia, 1980]).



(a) $\mathrm{Re}(a)$ (b) $\mathrm{Im}(a)$

Figure 6.7. The real and imaginary parts of endpoints of the cuts of the resolvent on the spiral curve where $g^2(m) < 0$.

6.5 Numerical study

Apart from the analytical continuation of the solution of [Kazakov et al., 1999], another avenue of exploration is perform a numerical study of the saddle-point equation (6.17). To accomplish this, we will follow a technique introduced in [Herzog, Klebanov, Pufu, & Tesileanu, 2011]. Firstly, let us remember that the saddle-point equation (6.17) that we need to solve was derived from the following expression:

$$\frac{\partial F}{\partial \sigma_i} = 0. \tag{6.43}$$

The essential idea behind the technique of [Herzog et al., 2011], is to introduce a time dependence to the eigenvalues $\sigma_i \to \sigma_i(t)$ and describe their dynamics via a heat equation:

$$\tau_\sigma \frac{d\sigma_i}{dt} = -\frac{\partial F}{\partial \sigma_i}, \tag{6.44}$$

where $\tau_\sigma \in \mathbb{C}$ is a parameter of our choice. If we choose τ_σ appropriately, then the equilibrium solution of equation (6.44), in the limit $t \to \infty$, will satisfy the original saddle-point equation (6.43), or (6.17). Thus, to solve numerically the original equation (6.17), we need to solve numerically the partial differential equation (6.44) (using standard techniques) for long enough time intervals to reach equilibrium.

First, we consider the case below the phase transition point $g^2 \simeq -0.1$. In Figure 6.8 we present the position of the cuts on the complex plane. In Figure 6.9 we show the eigenvalue density on the imaginary axis using the definition:

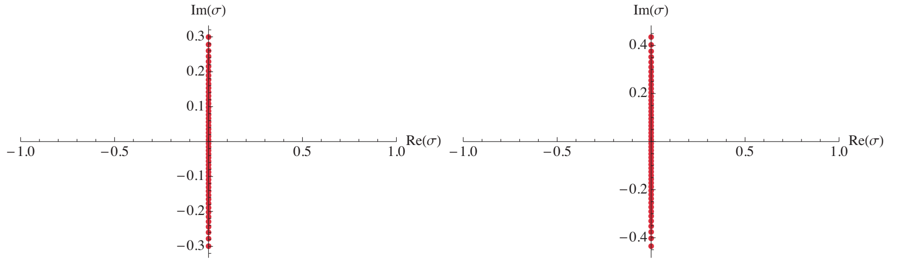
$$\rho(x) := \frac{1}{N} \frac{dx}{d\mathrm{Im}(\sigma(x))}. \tag{6.45}$$

On the same figure, we plot the eigenvalue density using the expression for the endpoints derived from the solution by [Kazakov et al., 1999]:

$$\rho(x) = \frac{2}{a^2 \pi} \sqrt{a^2 - x^2}. \tag{6.46}$$

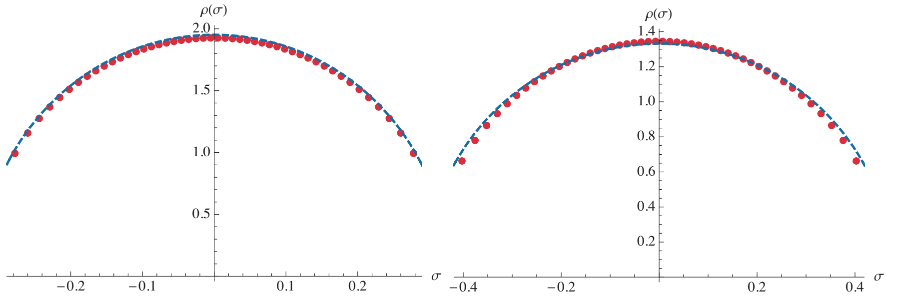
We observe that the results from the numerical solution match exactly the results obtained from the analytic continuation, both qualitatively and quantitatively.

Next, we turn to what happens after the phase transition. Here, the scene is changing dramatically. In each sub-figure of Figure 6.10, we have plotted three solutions. The two that form a cross are complex conjugates of each other. The third one, which splits in two on its two ends is the most remarkable. In fact, this solution is representative of an entire class of solutions of similar shape, which can be obtained by trying various initial conditions, although they are harder to obtain than the other type of solution. These bifurcating solutions are not very frequent in the literature. A case with quiver gauge theories where something similar occurs, can be found in [Amariti, Fazzi, Mekareeya, & Nedelin, 2019].



(a) $g^2 = -0.05$ (b) $g^2 = -0.095$

Figure 6.8. The position of the cuts of the matrix model, below the phase transition point.



(a) $g^2 = -0.05$ (b) $g^2 = -0.095$

Figure 6.9. The eigenvalue density of the matrix model before the phase transition point ($g^2 > g^2_{\mathrm{cr}}$). The dashed line corresponds to the solution (6.46), while the points are derived from (6.45) and the numerical solution.

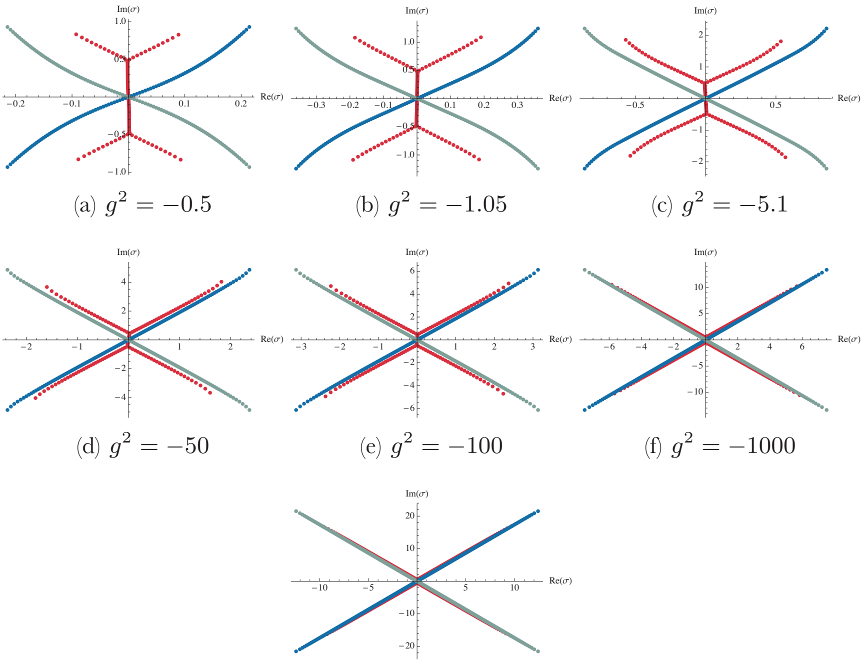
To find which of the three solutions is the physically dominant one, we can turn to computing the free energy for each solution using the formula:

$$F = -\log Z = \frac{2N}{g^2} \sum_i \phi_i^2 - 2 \sum_{i<j} \left[\log(\phi_i - \phi_j)^2 - \log(1 + (\phi_i - \phi_j)^2) \right] . \tag{6.47}$$

The results for the three solutions are plotted in Figure 6.11. We observe that the bifurcating solution dominates, since it has the lowest real free energy.

6.6 Analytical study

In this section we will perform an analytical study of the various types of solutions of the saddle-point equation (6.17) we found in the previous section. We will also study the free energies for these solutions.

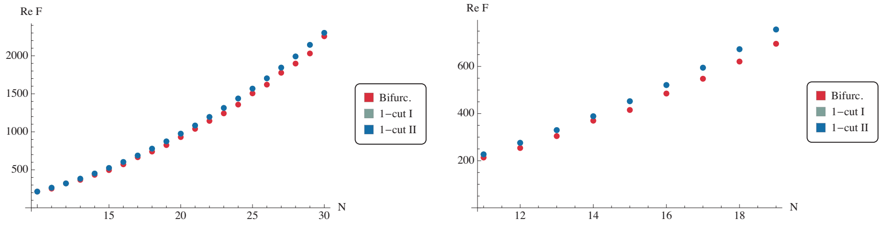


(a) $g^2 = -0.5$ (b) $g^2 = -1.05$ (c) $g^2 = -5.1$

(d) $g^2 = -50$ (e) $g^2 = -100$ (f) $g^2 = -1000$

(g) $g^2 = -4000$

Figure 6.10. The position of the cuts of the matrix model, after the phase transition point ($g^2 < g^2_{\mathrm{cr}}$).



(a) $g^2 = -0.35$ (b) $g^2 = -0.5$

Figure 6.11. The free energies of the three solutions presented in the previous figure.

6.6.1 One-cut solutions

We begin by introducing a new variable τ_i in the saddle-point equation (6.17), by scaling σ_i with the 't Hooft coupling as follows:

$$\tau_i := \epsilon \sigma_i, \tag{6.48}$$

where $\epsilon = \lambda^{-1/4}$. Then, the saddle-point equation (6.17) becomes:

$$-2\pi N \tau_i = 2 \sum_{j \neq i} \frac{1}{\tau_{ij}(\epsilon^2 + \tau_{ij}^2)}. \tag{6.49}$$

In the large-N limit, this equation becomes:

$$\begin{aligned}
-2\pi\tau &= 2 \fint_{-b\epsilon}^{b\epsilon} d\tau' \frac{\rho(\tau')}{\epsilon^2(\tau-\tau') + (\tau-\tau')^3} \\
&= \fint_{-b\epsilon}^{b\epsilon} d\tau' \frac{\rho(\tau')}{\epsilon^2} \left(\frac{2}{\tau-\tau'} - \frac{1}{\tau-\tau'+i\epsilon} - \frac{1}{\tau-\tau'-i\epsilon} \right). \qquad (6.50)
\end{aligned}$$

If we assume that the integration contour is on the real axis and passes through the origin, then we can replace the principle value integrals in the last two terms with ordinary integrals. If we also expand equation (6.50) (operating in the small-ϵ limit), we finally get:

$$\begin{aligned}
-2\pi\tau &= \fint_{-b\epsilon}^{b\epsilon} d\tau' \frac{1}{\epsilon^2} \left(\frac{2\rho(\tau') - \rho(\tau'+i\epsilon) - \rho(\tau'-i\epsilon)}{\tau-\tau'} \right) \\
&\qquad + \frac{\pi i}{\epsilon^2} \left[\rho(\tau+i\epsilon) - \rho(\tau-i\epsilon) \right] + \mathcal{O}(\epsilon) \\
&= \fint_{-b\epsilon}^{b\epsilon} d\tau' \frac{\rho''(\tau')}{\tau-\tau'} - \frac{2\pi}{\epsilon} \rho'(\tau) + \mathcal{O}(\epsilon). \qquad (6.51)
\end{aligned}$$

The appearance of the divergent $1/\epsilon$ term in the expression above is troubling. It is not entirely clear how one should deal with this issue. We will examine the consequences of the presence or absence of this term.

The $1/\epsilon$ term will not be present if the principle value integral does not contain a region between $\tau \pm k\epsilon$, with $k \gg 1$. With the divergent term out of the way, equation (6.51) is reminiscent of a Gaussian matrix model integral equation with $-\rho''(\tau)$ in place of $\rho(\tau)$, so it can be solved to give (see [Russo & Zarembo, 2012]):

$$\rho''(\tau) = -2\sqrt{b^2\epsilon^2 - \tau^2} + \frac{C}{\sqrt{b^2\epsilon^2 - \tau^2}}. \qquad (6.52)$$

Note that the second term does not contribute to the integral. By choosing $C = b^2\epsilon^2$, we get:

$$\rho(\tau) = \frac{1}{3} \left(b^2\epsilon^2 - \tau^2 \right)^{3/2}, \qquad (6.53)$$

which reproduces equation (6.19). The constant b can be fixed by the normalization of the density:

$$b = \left(\frac{8\lambda}{\pi} \right)^{1/4}, \qquad (6.54)$$

which, as expected, agrees with (6.20).

Let us now turn to the case where the $1/\epsilon$ term is present. Then, if we expand for small ϵ, we get:

$$\rho'(\tau) = \epsilon\tau + \mathcal{O}(\epsilon^2), \qquad (6.55)$$

and so $\rho(\tau)$ will be:

$$\rho(\tau) = \frac{\epsilon}{2}\left(\tau^2 - k^2\right). \tag{6.56}$$

If we also assume that the contour passes through zero, we get:

$$\int_{-k}^{k} d\tau \frac{\epsilon}{2}\left(\tau^2 - k^2\right) = -\frac{\epsilon}{3}k^3 = 1, \tag{6.57}$$

or

$$k = \left(-\frac{3\lambda^{1/4}}{2}\right)^{1/3}. \tag{6.58}$$

The roots of this equation that are compatible with contour flow are:

$$k = \exp\left(\pm\frac{i\pi}{3}\right)\left(-\frac{3\lambda^{1/4}}{2}\right)^{1/3}. \tag{6.59}$$

In the original parameters σ_i, this means that endpoints of the one-cut solution are given by $\exp(\pm i\pi/3)(3\lambda/2)^{1/3}$. In Figure 6.12, we draw the two numerical one-cut solutions, as well the lines with slope $\pm\pi/3$, and we find a perfect match in the angle.

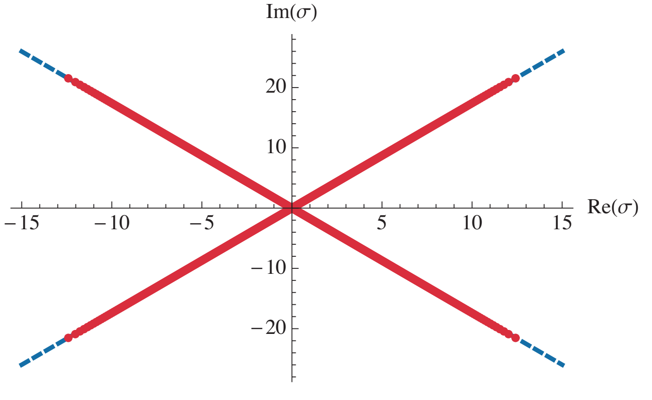
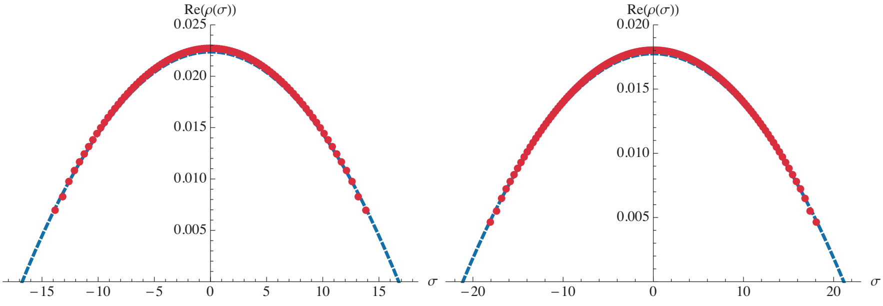


Figure 6.12. The two numerical one-cut solutions of the saddle-point equation for $g^2 = -4000$ (red points) along with the lines of slope $\pm\pi/3$ (dashed lines).

In addition to plot of the cuts in the complex plane, we can compare the eigenvalue density (6.56), with that derived from the numerical study. We present the results in Figure 6.13. Once again, the numerical and analytical results match.

Another non-trivial check we can perform for our analytic solution is to reproduce the scaling behavior of the correlation function ν:

$$\nu = \langle \mathrm{Tr}\,\sigma^2\rangle = \frac{g^4}{2N^2}\frac{\partial F(N,g)}{\partial g^2} = -\frac{2}{15}\exp\left(\mp\frac{\pi i}{3}\right)\left(\frac{3}{2}\right)^{5/3}\lambda^{2/3}, \tag{6.60}$$



(a) $g^2 = -1000$ (b) $g^2 = -2000$

Figure 6.13. Comparison of the analytic eigenvalue density (equation (6.56)) and the results of the numerical study.

using the results of the analytic continuation of [Kazakov et al., 1999]. To this end, we will investigate the scaling of the intersection points of the spiral towards $m = 1$ with the real axis in the limit of $g^2 \ll 0$. Using the expressions for a (6.38), g^2 (6.37), ν (6.41) and the expressions for the elliptic integral functions in the n-th sheet (6.42), we get the following results for the first few intersection points:

n	$m-1$
1	1
2	0.02351
3	0.0006419
4	0.000017325

(6.61)

We observe that the distance between $m = 1$ and the intersection point shrinks exponentially fast, and so we have the following scaling behavior:

$$m = 1 + ab^n, \qquad a \simeq 42.52, \qquad b \simeq 0.0235. \tag{6.62}$$

The values of the parameters a and b were obtained via a numerical fit, but their precise values are not important for our argument. Using (6.62) in the expressions for g^2 (6.37) and ν (6.41), we get:

$$g^2 \sim n^3, \qquad \nu \sim n^2. \tag{6.63}$$

Thus, the scaling of ν is:

$$\nu \sim g^{4/3} \qquad \text{or} \qquad \nu \sim \lambda^{2/3}, \tag{6.64}$$

which agrees with (6.60). This is the same scaling behavior one obtains at large positive coupling.

6.6.2 Bifurcating solutions

We will now turn to the analysis of the class of the bifurcating solutions. We will begin with considering the density that corresponds to the one-cut solutions, which in the large-λ limit becomes:

$$\rho(\sigma) = \frac{1}{2\lambda}(\sigma^2 - k^2), \tag{6.65}$$

with

$$k = e^{i\pi/3}\left(\frac{3\lambda}{2}\right)^{1/3}. \tag{6.66}$$

In order to study the bifurcating solutions, we will start pulling eigenvalues away from the one-cut solution line. The $\mathbb{Z}_2$ symmetry of the system implies that we need to pull eigenvalues in pairs. Of course, this process is bound to have a back-reaction to the distribution of the rest of the eigenvalues, but if we remove a small number of them (say $2M \ll N$), we can assume that that in the large N limit, the effect is suppressed.

We will now pull two eigenvalues $\pm\tilde{\sigma}$ from the distribution (6.65). Then, the equation of motion for $\tilde{\sigma}$ will be:

$$-\frac{2\pi}{\lambda}\tilde{\sigma} = \frac{1}{N}\frac{2}{(2\tilde{\sigma})^3 + 2\tilde{\sigma}} + 2\int_{-k}^{k}\frac{d\sigma'\rho(\sigma')}{(\tilde{\sigma}-\sigma')^3 + (\tilde{\sigma}-\sigma')}. \tag{6.67}$$

If the first term in the right hand side of this equation is not to be suppressed in the large-N limit, then $2\tilde{\sigma}$ should be close to $\pm i$. Thus, we will parameterize $\tilde{\sigma}$ as follows:

$$\tilde{\sigma} = \frac{i}{2} + z, \tag{6.68}$$

and assume that z is small. So, the equation of motion (6.67) will be (to leading order):

$$\begin{aligned}\frac{1}{2zN} &\simeq \frac{i\pi}{\lambda} + \frac{2\pi z}{\lambda}\\ &\quad + \int_{-k}^{k} d\sigma'\rho(\sigma')\left[\frac{2}{i/2+z-\sigma'} - \frac{1}{3i/2+z-\sigma'} - \frac{1}{-i/2+z-\sigma'}\right]\\ &\simeq \frac{i\pi k^2}{\lambda} + \frac{\pi z}{\lambda},\end{aligned} \tag{6.69}$$

which gives:

$$z \simeq \frac{e^{5i\pi/6}}{\pi N}\left(\frac{\lambda}{18}\right)^{1/3}. \tag{6.70}$$

Next, we will pull $2M$ eigenvalues (with $2M \ll N$) off the one-cut solution. Similarly to the case of one eigenvalue pair, we will parameterize them with the expression:

$$\tilde{\sigma}_i = \pm\left(\frac{i}{2} + z_i\right), \tag{6.71}$$

with small z_i. The fact that the eigenvalues which are removed from the one-cut line are centered around $\pm i/2$, seems to be supported by the numeric solutions (see Figure 6.10). Then, to leading approximation, from the equation of motion, we get:

$$-\frac{i\pi k^2}{\lambda}-\frac{\pi z_i}{\lambda}=\frac{1}{N}\sum_{j\neq i}^{M}\frac{2}{z_i-z_j}-\frac{1}{N}\sum_{j}^{M}\frac{1}{z_i+z_j}. \tag{6.72}$$

We will also make the assumption that every z_i comes with a z_j such that $z_j = -z_i + 2\delta z$, with δz given by (6.70). In case M is odd, we take one j to be equal to i. The reason for this assumption is that δz annihilates the first term in the left-hand side. Doing so, we get:

$$-\frac{2\pi N}{\lambda M}z_i=\sum_{j\neq i}^{M}\frac{2}{z_i-z_j}. \tag{6.73}$$

We recognize this as the saddle-point equation of an $SU(M)$ matrix model. It has effective 't Hooft coupling constant $\lambda_{\text{eff}} = M\lambda/N$ and its potential is negative and quadratic. The solution of this matrix model is well-known: the eigenvalues z_i will be distributed on the Wigner semi-circle along the imaginary line. This will be true as long as z_i are small (and so M needs to be smaller than N/λ) and δz is much smaller than the distance between neighboring eigenvalues (which translates to the condition that $\lambda \gg M^3/N^3$).

6.6.3 Symmetric bifurcating solutions

We will now study the case where the bifurcating solutions analyzed above have a reflection symmetry with respect to the imaginary axis. Our ansatz for the eigenvalue distribution will consist of the following expressions for the five branches:

$$\rho(\sigma)=\begin{cases}+C_1\left[e^{2\pi i/3}B^2-(\sigma-i/2)^2\right], & \sigma=i/2+e^{\pi i/3}x,\\ -C_1\left[e^{4\pi i/3}B^2-(\sigma-i/2)^2\right], & \sigma=i/2+e^{2\pi i/3}x,\\ +C_1\left[e^{-2\pi i/3}B^2-(\sigma+i/2)^2\right], & \sigma=-i/2+e^{-\pi i/3}x,\\ -C_1\left[e^{-4\pi i/3}B^2-(\sigma+i/2)^2\right], & \sigma=-i/2+e^{-2\pi i/3}x,\\ -iC_2, & -i/2\le\sigma\le i/2,\end{cases} \tag{6.74}$$

with $x, B \in \mathbb{R}$ and $0 \le x \le B$. Using this distribution in the saddle-point equation with $-i/2 \le \sigma \le i/2$:

$$-\frac{2\pi}{\lambda}\sigma=⨍ d\sigma'\rho(\sigma')\left(\frac{2}{\sigma-\sigma'}-\frac{1}{\sigma-\sigma'+i}-\frac{1}{\sigma-\sigma'-i}\right). \tag{6.75}$$

In order to achieve a cancellation of terms like $\log(i/2 \pm \sigma)$ and $\log(3i/2 \pm \sigma)$, we need to set:

$$C_2 = -\sqrt{3} B^2 C_1. \tag{6.76}$$

Then, equation (6.75) in the limit of large B becomes:

$$-\frac{2\pi}{\lambda}\sigma = 4\pi C_1 \sigma + \mathcal{O}(B^{-1}), \tag{6.77}$$

or

$$C_1 \simeq= -\frac{1}{2\lambda}, \tag{6.78}$$

which is the same as the overall normalization constant of the eigenvalue density $\rho(\sigma)$ for the bifurcating solutions (6.65). Imposing the normalization of the eigenvalue density and using the large B limit, we get:

$$B = \left(\frac{3\lambda}{4}\right)^{1/3}. \tag{6.79}$$

This reduces the size of its branch by a factor of $2^{-1/3}$ with respect to the respective one for the one-cut solution.

6.6.4 On the free energy

We will now turn to the study of the free energy of the theory in the large-λ limit. We will use the eigenvalue density for the one-cut solution:

$$\rho(\sigma) = \frac{1}{2\lambda}(\sigma^2 - k^2). \tag{6.80}$$

Then, the expression for the free energy will be:

$$\begin{aligned} F = &-\frac{2\pi}{\lambda} N^2 \int_{-k}^{k} d\sigma \rho(\sigma) \sigma^2 \\ &- N^2 \int_{-k}^{k} d\sigma d\sigma' \rho(\sigma)\rho(\sigma') \left[\log(\sigma - \sigma')^2 - \log[(\sigma - \sigma')^2 + 1]\right]. \end{aligned} \tag{6.81}$$

We will also introduce the anti-derivative $C(\sigma)$ of the density $\rho(\sigma)$, such that $C(k) = 1$ and $C(-k) = 0$. Next, we integrate by parts the second integral in (6.81) and make use of the saddle-point equation to get:

$$
\begin{aligned}
F &= -\frac{2\pi}{\lambda}N^2\int_{-k}^{k} d\sigma\rho(\sigma)\sigma^2 - \frac{2\pi}{\lambda}N^2\int_{-k}^{k} d\sigma\, C(\sigma)\sigma \\
&\quad - N^2\int d\sigma\rho(\sigma)\log\frac{(k-\sigma)^2}{(k-\sigma)^2+1} \\
&= -\frac{\pi}{\lambda}N^2\int_{-k}^{k} d\sigma\rho(\sigma)\sigma^2 - \frac{\pi}{\lambda}N^2k^2 - N^2\int d\sigma\rho(\sigma)\log\frac{(k-\sigma)^2}{(k-\sigma)^2+1} \\
&= -\frac{6\pi}{5\lambda}N^2k^2 - N^2\int d\sigma\rho(\sigma)\log\frac{(k-\sigma)^2}{(k-\sigma)^2+1}.
\end{aligned}
\tag{6.82}
$$

The last term in this expression is $\mathcal{O}(N^2\frac{k}{\lambda}\log k)$ and thus can be omitted in the limit we are studying.

One can also work in a similar fashion for the case of the symmetric bifurcating solution, to reach an expression of the following form:

$$
F = -\frac{1}{2}\frac{6\pi}{5\lambda}N^2\left[\left(\frac{k}{2^{1/3}}\right)^2 + \left(\frac{k^*}{2^{1/3}}\right)^2\right]. \tag{6.83}
$$

The terms in this formula can be understood as follows. The four outer legs of the solution provide the dominant part of the expression, which is similar to the one provided by the two one-cut solutions. The overall factor of $1/2$ can be attributed to the fact these legs contain half as many eigenvalues as the one-cut solutions, while the factor $1/2^{1/3}$ is related to the reduced length of the legs with respect to the one-cut solutions (see Subsection 6.6.3). What is most important here is that the free energy for the symmetric bifurcating solution turns out to be real (up to a λ-independent constant) and that it is smaller than the real part of the free energy of the one-cut solutions, making it the leading solution.

6.7 Future directions and open problems

In this chapter we presented results from the analysis of the matrix model that arises from the localization of maximally supersymmetric two-dimensional super Yang–Mills and discovered evidence of a phase transition. While we were able to perform both numerical and analytical studies and find them in good agreement, and while we have obtained a number of interesting results, there exists a number of open questions.

The most important of the challenges ahead is to understand the discrepancy between the scaling of the free energy obtained from the holographic study in [Bobev et al., 2020] and the solutions of our matrix model. Of course the analytic continuation of the general-d study of the matrix model in [Bobev et al., 2020] is capable of reproducing the predicted scaling, but as we explained in this chapter, its assumptions are not warranted in the two-dimensional case.

Another significant point to understand is how to deal with the presence of the divergent $1/\epsilon$ term in (6.51). If we remove the region $\tau \pm k\epsilon$, with $k \gg 1$, as in the previous section, we are effectively in the $\lambda \gg N^{1/4}$ regime. Doing so, we reproduce the results from the original analysis of the matrix model in [Bobev et al., 2020], which agreed with the supergravity calculation. However, this puts us in the dual string territory, with the gauge theory being equivalent to a free orbifold conformal field theory (see [Itzhaki, Maldacena, Sonnenschein, & Yankielowicz, 1998] and [Peet & Polchinski, 1999]). It is not clear how the holographic results can match the field theory calculation under these conditions.

One possible approach to eliminate the $1/\epsilon$ divergence in (6.51), is to see it as an extra term in the potential of the matrix model and try to remove it by the addition of an appropriate counter-term. Indeed, if we recast equation (6.51) in the original variables:

$$-\frac{2\pi}{\lambda}\sigma = \fint_{-b}^{b} d\sigma' \frac{\rho''(\sigma')}{\sigma - \sigma'} - 2\pi\rho'(\sigma) + \mathcal{O}(1/\lambda), \tag{6.84}$$

we see that the divergent term can be canceled if the potential is altered to:

$$V(\sigma) = -\frac{2\pi N}{\lambda}\sum_i \sigma_i^2 - 4\pi N \sum_i \rho(\sigma_i), \tag{6.85}$$

and choose

$$\rho(\sigma) = \frac{1}{3\lambda}(b^2 - \sigma^2)^{3/2}. \tag{6.86}$$

However, before accepting this possibility, we need to ensure that the matrix model potential (6.85) can be derived from a supersymmetric field theory.

Finally, another possible task would be to try repeating the derivation of the behavior of the correlation function ν (6.60) from the analytic continuation of the solution of [Kazakov et al., 1999] in order to reproduce the full expression and not just the scaling. However, this is not an easy task, since once needs to derive an explicit expression for the analytical continuation curve. Alternatively, an expression for the intersection points of the curve with the real axis would be sufficient, but also challenging.

7. Cohomological twisting and localization of $\mathcal{N} = 2$ gauge theories with matter

In this chapter we discuss Paper II, where we extend the work of [Festuccia et al., 2020] to include matter in the form of a gauged hypermultiplet. We rewrite the theory in terms of cohomological variables and we proceed to localize it.

The chapter is structured as follows. We begin by offering an introduction and motivation to the work in Section 7.1. Then, we describe how to define a supersymmetric theory in Section 7.2, reviewing material that originates from [Festuccia et al., 2020], as well as new results from Paper II. To do this, we review how the supergravity auxiliary background fields were written in terms of Killing spinor bilinears in [Festuccia et al., 2020] (Subsection 7.2.1). Then, in Subsection 7.2.2 we augment the demonstration from [Festuccia et al., 2020] that supersymmetry is globally well-defined by showing that the auxiliary Killing spinors, necessary for the off-shell closure of the hypermultiplet algebra, are also globally well-defined. Next, in Subsections 7.2.3 and 7.2.4 we present the vector multiplet ([Festuccia et al., 2020]) and the hypermultiplet (Paper II) respectively. We proceed in Subsection 7.3.1 with the extension of the "flipping projectors" of [Festuccia et al., 2020] to the case of spinors, established in Paper II, before reviewing how one can rewrite the vector multiplet ([Festuccia et al., 2020]) and the hypermultiplet (Paper II) in terms of twisted variables in Subsections 7.3.2 and 7.3.3. We conclude with presenting the localization of the resulting cohomological theory in 7.4 from Paper II.

7.1 Introduction

Supersymmetric localization has been one of the major themes in this thesis. Supersymmetric theories amenable to localization, such as $\mathcal{N} = 2$ on S^4 in the original work of Pestun [Pestun, 2012], provide precious information on quantum field theories and their strong coupling behavior. Another interesting perspective in the study of quantum field theories has been given by the topological twisting of supersymmetric theories. Since the original work of Witten [Witten, 1988a] on $\mathcal{N} = 2$ super Yang–Mills, topological twisting has been a very fruitful technique that offered highly non-trivial results to both Theoretical Physics and Mathematics, and formed yet another bridge between the two subjects. A prime example of connection is the extraction of Donaldson invariants [Donaldson, 1990] via the computation of correlators in $\mathcal{N} = 2$ super Yang–Mills in Witten's aforementioned work. As was the case with Pestun's work, the topological twisting procedure was extended and applied in many different settings. A particular instance of such an extension, especially relevant in this chapter, is that of *equivariant* twisting, applicable when the theory is defined on a manifold which allows a torus action. This possibility was examined in [Losev et al., 1998],[Lossev et al., 1999], [Moore et al., 2000b] and [Moore, Nekrasov, & Shatashvili, 2000a], and led to the computation of the celebrated Nekrasov partition function in [N. A. Nekrasov, 2003] and [N. Nekrasov & Okounkov, 2006].

The connection between the supersymmetrically localizable theories of Pestun and the (equivariant) topologically twisted theories in four dimensions was not very clear until the advent of [Festuccia et al., 2020], where the authors were able to connect the two classes of theories under a unified theoretical framework. This framework describes a wide set of $\mathcal{N} = 2$ super Yang–Mills theories that can be defined on any four-dimensional manifold with a Killing vector field possessing only isolated fixed points.

The construction of [Festuccia et al., 2020] entails decomposing the squared norm of a Killing vector field into a product of two semi-positive-definite scalar functions: $||v||^2 = s\tilde{s}$. The fixed points of this vector field are subsequently split into two groups: those where $\tilde{s}$ vanishes (dubbed *plus fixed points*), and those where s vanishes (dubbed *minus fixed points*). Then, the authors introduced a novel notion of self-duality, where the two-forms can range from being self-dual in the neighborhood of plus fixed points to being anti-self dual in the neighborhood of minus fixed points in a smooth manner. When the time comes to write down the partition function, one assigns an instanton contribution to the plus fixed points and an anti-instanton contribution to the minus ones. For instance, the equivariant Donaldson–Witten theory can be retrieved by distributing only instanton partition functions to all the fixed points.

The authors of [Festuccia et al., 2020] affirm that any arbitrary allocation of pluses and minuses on the fixed points of the Killing vector field produces a supersymmetric theory. They also show that the theory is globally well-defined on the class of manifolds they consider. Eventually, they reformulate the theory using cohomological (twisted) variables and write down a conjecture for the full partition function of an $\mathcal{N} = 2$ super Yang–Mills theory on any (orientable) four-dimensional manifold. More mathematical aspects of this construction were elaborated further in [Festuccia, Qiu, Winding, & Zabzine, 2019].

The work of [Festuccia et al., 2020] presents the starting point for Paper II, as the former covered only the case of the $\mathcal{N} = 2$ vector multiplet. Hence, in Paper II, we extend the framework to incorporate matter coupled to radiation, via the inclusion of a gauged hypermultiplet.

The twisting procedures for the vector multiplet and the hypermultiplet seem to move largely in parallel ways, with the vector multiplet one being underlied by a decomposition of the two-forms bundle ([Festuccia et al., 2020]) and the hypermultiplet one by the decomposition of the spinor bundle (Paper II). The notion of a "flipping projector" that acts on two-forms in [Festuccia et al., 2020] is extended in Paper II for the case of spinors and the connection between the two is examined.

Despite the similarities, there exist a few key differences between the two cases. The first difference has to do with the presence or absence of spinors. For the twisted vector multiplet, the fields are two-forms and hence can be defined on any orientable manifold. Both the $SU(2)_R$ dependence and the spinorial nature of the fields disappear. However, in the case of the hypermultiplet, this is not possible, and the resulting theory demands a spin structure to be defined on the manifold. At first glance, this limits the applicability of the hypermultiplet case construction to spin manifolds. However the analysis of this issue is quite subtle, and needs to be performed case by case, as certain flavor symmetries can render the theory well-defined on spin^c manifolds, that is, all orientable manifolds [Teichner & Vogt, n.d.].[1] Another difference with the vector multiplet case of [Festuccia et al., 2020] is that for the off-shell formulation of the theory, we need to introduce a pair of auxiliary Killing spinors, subject to certain constraints. We make a choice for the auxiliary spinors and show them to be globally well-defined (as for the ordinary Killing spinors).

Having established that the hypermultiplet theory is globally well-defined, we begin writing down the relevant Lagrangian and supersymmetry transformations. What is more, we provide a novel expression of the hypermultiplet Lagrangian as a δ-exact term. This generalizes a result found in [Hama & Hosomichi, 2012].

Next, we rewrite the theory in cohomological variables using an invertible map. This map requires the explicit solutions of the supergravity background fields as functions of the Killing spinor bilinears, obtained in [Festuccia et al., 2020] (also reviewed in Appendix B of Paper II).

Our work on the rewriting of the hypermultiplet theory can be reduced to the usual topological twisting (for a review, see for instance [Labastida & Marino, 2005]) by removing one set of ordinary and auxiliary Killing spinors (one can choose $(\bar{\chi}_{i\dot{\alpha}}, \check{\zeta}_{i\alpha})$ or $(\zeta_{i\alpha}, \check{\bar{\chi}}_{i\dot{\alpha}})$) and making one of the Killing spinor bilinears vanish, thus effectively removing the equivariance ($v^\mu = 0$).

Having written the theory in cohomological variables, we proceed to localize the theory. We compute the resulting one-loop determinant using index techniques for the relevant transversally elliptic operator. The result depends on a parameter a_0, which takes values in the Cartan of the gauge group. If the hypermultiplet is massive, its masses will result in a shift of the parameter a_0. Note also that the result needs to be modified when the manifold allows fluxes, and one needs to add the relevant magnetic flux $\mathfrak{m}_i$ to the parameter a_0 (see for instance the $S^2 \times S^2$ example in Paper II). We also include a discussion of the various regularization options for the resulting expression. An exposition eluci-

[1] The mathematical requirement is that the second Stiefel–Whitney class should be vanishing.

dating in some detail and generality which regularization should be employed, would deserve its own dedicated work.

Finally, the paper concludes with an illustration of our results with two examples: a squashed S^4, where our answer coincides with that of [Hama & Hosomichi, 2012] and an $S^2 \times S^2$, where the answer becomes trivial when there are no fluxes.

7.2 Defining an $\mathcal{N} = 2$ supersymmetric theory on four-manifolds

The first task we are presented with is that of defining an $\mathcal{N} = 2$ supersymmetric field theory on the general class of four-manifolds we are about to study. Part of this procedure was accomplished in [Festuccia et al., 2020], and we will briefly review it in the following subsection. Then, we will demonstrate that the supersymmetric theories we want to formulate are globally well-defined, and write down the $\mathcal{N} = 2$ vector multiplet and hypermultiplet along with their Lagrangians in the original variables. The vector multiplet was studied in [Festuccia et al., 2020], while our results from Paper II refer to the hypermultiplet.

7.2.1 Killing spinor equations and their solutions

As demonstrated in [Festuccia & Seiberg, 2011], a supersymmetric theory can be defined on a curved manifold in a systematic manner, by coupling the theory to a supergravity background (and subsequently freezing the gravitational degrees of freedom). For the case of $\mathcal{N} = 2$ theories with a conserved $SU(2)_R$ current this was done in [Festuccia et al., 2020], where the theory was coupled to the $\mathcal{N} = 2$ Poincaré supergravity.

We consider a four-dimensional Riemannian manifold endowed with a spin structure. The setup also contains an $SU(2)_R$ connection (${V_\mu}^i{}_j$ where $i, j, \ldots$ are $SU(2)_R$ indices). The supergravity multiplet includes the following list of auxiliary fields: a scalar (N), a one-form (G_μ) and a two-form ($W_{\mu\nu}$), another closed two-form ($F_{\mu\nu}$) and a scalar $SU(2)_R$ triplet (S_{ij}).

The supersymmetry transformation parameters are ζ^i_α and $\bar{\chi}^{\dot{\alpha}}_i$, which are left- and right-handed spinors respectively, both in the $SU(2)_R$ fundamental, with i being an $SU(2)_R$ index and $\alpha, \dot{\alpha}$ being spinor indices. Their reality conditions are taken to be symplectic Majorana conditions:

$$(\zeta_{i\alpha})^* = \zeta^{i\alpha}, \qquad (\bar{\chi}^{\dot{\alpha}}_i)^* = \bar{\chi}^i_{\dot{\alpha}}. \tag{7.1}$$

The spinors ζ^i_α and $\bar{\chi}^{\dot{\alpha}}_i$ need to satisfy the so-called generalized Killing spinor equations, which are derived by demanding that the variations of certain auxiliary supergravity fields vanish. The two sets of these partial differential equations are:

$$
\begin{aligned}
(D_\mu - iG_\mu)\zeta_i \; - \frac{i}{2}W^+_{\mu\rho}\sigma^\rho\bar\chi_i - \frac{i}{2}\sigma_\mu\bar\eta_i = 0\;, \\
(D_\mu + iG_\mu)\bar\chi^i + \frac{i}{2}W^-_{\mu\rho}\bar\sigma^\rho\zeta^i - \frac{i}{2}\bar\sigma_\mu\eta^i = 0\;,
\end{aligned}
\tag{7.2}
$$

and

$$
\begin{aligned}
\Big(N - \frac{1}{6}R\Big)\bar\chi^i &= 4i\partial_\mu G_\nu\bar\sigma^{\mu\nu}\bar\chi^i + i\big(\nabla^\mu + 2iG^\mu\big)W^-_{\mu\nu}\bar\sigma^\nu\zeta^i \\
&\qquad + i\bar\sigma^\mu\big(D_\mu + iG_\mu\big)\eta^i\;, \qquad (7.3\text{a}) \\
\Big(N - \frac{1}{6}R\Big)\zeta_i &= -4i\partial_\mu G_\nu\bar\sigma^{\mu\nu}\zeta_i - i\big(\nabla^\mu - 2iG^\mu\big)W^+_{\mu\nu}\sigma^\nu\bar\chi_i \\
&\qquad + i\sigma^\mu\big(D_\mu - iG_\mu\big)\bar\eta_i\;, \qquad (7.3\text{b})
\end{aligned}
$$

where in the equations above, the covariant derivative D_μ includes the $SU(2)_R$ connection ${V_\mu}^i{}_j$ and R is the Ricci scalar. The spinors η^i and $\bar\eta^i$ introduced above are defined as follows:

$$
\begin{aligned}
\eta_i &= (\mathcal{F}^+ - W^+)\zeta_i - 2G_\mu\sigma^\mu\bar\chi_i - S_{ij}\zeta^j\;, \\
\bar\eta^i &= -(\mathcal{F}^- - W^-)\bar\chi^i + 2G_\mu\bar\sigma^\mu\zeta^i - S^{ij}\bar\chi_j\;.
\end{aligned}
\tag{7.4}
$$

where $W^+ := \frac{1}{2}W_{\mu\nu}\sigma^{\mu\nu}$ and $W^- := \frac{1}{2}W_{\mu\nu}\bar\sigma^{\mu\nu}$ (with analogous expressions for $\mathcal{F}$). For additional information regarding our conventions, the interested reader can refer to Appendix A of Paper II.

We will also introduce the following scalars:

$$
s = 2\zeta^i\zeta_i, \qquad \tilde s = 2\bar\chi^i\bar\chi_i \tag{7.5}
$$

and the vector field:

$$
v^\mu = 2\bar\chi^i\bar\sigma^\mu\zeta_i, \tag{7.6}
$$

made of Killing spinors ζ^i_α and $\bar\chi^{\dot\alpha}_i$. By employing the reality conditions (7.1), we deduce that the vector field v^μ and the scalars $s, \tilde s$ are real and that $s, \tilde s \geq 0$. Furthermore, the norm of v^μ is given by:

$$
||v||^2 = s\tilde s. \tag{7.7}
$$

In [Festuccia et al., 2020], the Killing spinor equations written above were solved and the auxiliary supergravity fields were written as functions of bilinears of the Killing spinors ζ^i_α and $\bar\chi^{\dot\alpha}_i$. To accomplish this, first one uses equations (7.2) and by demanding that v^μ is a Killing vector and that the bilinears s and $\tilde s$ are invariant along v, one gets a solution for $W_{\mu\nu}$ and $(V_\mu)^{ij}$. These solutions contain also a one-form b^μ that satisfies $v^\mu b_\mu = 0$ and acts as a parameter for them. Then, one can use equations (7.3) to obtain N, and (7.4) to find $\mathcal{F}_{\mu\nu}$ (parameterized by a constant K) and S_{ij}. We will refrain from reproducing the explicit solutions for the supergravity fields here, as some of them are quite lengthy, but they can be readily found in [Festuccia et al., 2020] or in Appendix B of Paper II. Note also that these solutions are not unique.

7.2.2 On the global properties of the Killing spinors

A globally well-defined supersymmetric field theory on a curved manifold relies crucially on the Killing spinors ζ^i_α and $\bar{\chi}^{\dot{\alpha}}_i$ to be also globally well-defined. What is more, since we are interested in the case of theories with a hypermultiplet in its off-shell formulation, we will need to examine that the auxiliary Killing spinors $\check{\zeta}_{\check{\imath}}$ and $\check{\bar{\chi}}_{\check{\imath}}$ also satisfy the same properties.

According to the construction of [Festuccia et al., 2020], the fixed points of the Killing vector field v^μ are separated into two kinds, referred to as "plus" and "minus" fixed points. These fixed points can be distinguished by which of the two scalars s and $\tilde{s}$ vanish on them (see equation (7.7)). Wherever $\tilde{s}$ vanishes ("plus" fixed points), s tends to some positive constant K (and vice versa). Wherever $\tilde{s}$ is non-vanishing ("minus" fixed points), s approaches zero like $||v||^2/K$ (and vice versa).

Killing spinors

We will show that one can construct globally well-defined Killing spinors ζ^i_α and $\bar{\chi}^{\dot{\alpha}}_i$ on the manifolds we are considering. To do so, we can create a covering of the manifold using a collection of charts, so that each chart contains at most one fixed point of the Killing vector field v^μ. Then, we can build the desired Killing spinors using a non-negative scalar s and a vector field v^μ as follows:

$$\zeta^i_\alpha = \frac{\sqrt{s}}{2}\delta^i_\alpha, \qquad \bar{\chi}^{\dot{\alpha}}_i = \frac{1}{s}v^\mu(\bar{\sigma}_\mu\zeta_i)^{\dot{\alpha}}. \tag{7.8}$$

However, in the minus fixed point charts ($s = 0$), these are not well-defined, so they have to be modified. This can be accomplished by beginning from a plus fixed point chart and transforming it appropriately. Performing an $SU(2)_R$ transformation:

$$U_i{}^j = i\frac{v^\mu}{||v||}\sigma_{\mu\, i}{}^j, \tag{7.9}$$

we get for the minus fixed points:

$$\bar{\chi}^{\dot{\alpha}}_i = -i\frac{\sqrt{\tilde{s}}}{2}\delta^{\dot{\alpha}}_i, \qquad \zeta^i_\alpha = -\frac{1}{\tilde{s}}v^\mu(\sigma_\mu\bar{\chi}^i)_\alpha. \tag{7.10}$$

Thus, we have defined a pair of Killing spinors that are globally well-defined.

Auxiliary Killing spinors

The off-shell closure of the hypermultiplet requires the addition of an auxiliary field as well as that of a pair of auxiliary spinors $\check{\zeta}_{\check{\imath}}$ and $\check{\bar{\chi}}_{\check{\imath}}$, transforming under $SU(2)_l \times_{\mathbb{Z}_2} SU(2)_{\check{R}}$ ($\check{\zeta}_{\check{\imath}}$) and $SU(2)_r \times_{\mathbb{Z}_2} SU(2)_{\check{R}}$ ($\check{\bar{\chi}}_{\check{\imath}}$). These spinors need to satisfy the following consistency conditions with the Killing spinors:

$$\begin{aligned} \zeta_i\check{\zeta}_{\check{\jmath}} - \bar{\chi}_i\check{\bar{\chi}}_{\check{\jmath}} &= 0, & \check{\zeta}_{\check{\imath}}\check{\zeta}^{\check{\imath}} &= \bar{\chi}^i\bar{\chi}_i, \\ \check{\bar{\chi}}^{\check{\imath}}\bar{\sigma}^\mu\check{\zeta}_{\check{\imath}} + \bar{\chi}^i\bar{\sigma}^\mu\zeta_i &= 0, & \check{\bar{\chi}}_{\check{\imath}}\check{\bar{\chi}}^{\check{\imath}} &= \zeta^i\zeta_i, \end{aligned} \tag{7.11}$$

which except for local $SU(2)_{\check{R}}$ transformations, fix them uniquely. The auxiliary Killing spinors shall be taken to have symplectic Majorana reality conditions:

$$(\check{\zeta}^{\alpha}_{\check{\imath}})^* = \check{\zeta}^{\check{\imath}}_{\alpha}, \qquad\qquad (\check{\bar{\chi}}^{\check{\imath}\dot{\alpha}})^* = \check{\bar{\chi}}_{\check{\imath}\dot{\alpha}}. \tag{7.12}$$

We proceed in a manner analogous to that for the Killing spinors, making sure that the constraints (7.11) are satisfied, and find that for charts that do not contain minus fixed points:

$$\check{\bar{\chi}}^{\dot{\alpha}}_{\check{\imath}} = \frac{\sqrt{s}}{2}\delta^{\dot{\alpha}}_{\check{\imath}}, \qquad\qquad \check{\zeta}_{\check{\imath}\,\alpha} = -\frac{1}{s}v^{\mu}(\sigma_{\mu}\check{\bar{\chi}}_{\check{\imath}})_{\alpha}. \tag{7.13}$$

For a chart possessing a minus fixed point, we need to use an $SU(2)_{\check{R}}$ transformation:

$$U_{\check{\imath}}{}^{\check{\jmath}} = i\frac{v^{\mu}}{||v||}(\sigma_{\mu})_{\check{\imath}}{}^{\check{\jmath}}, \tag{7.14}$$

to get:

$$\check{\zeta}^{\check{\imath}}_{\alpha} = i\frac{\sqrt{\tilde{s}}}{2}\delta^{\check{\imath}}_{\alpha}, \qquad\qquad \check{\bar{\chi}}^{\dot{\alpha}}_{\check{\imath}} = \frac{1}{\tilde{s}}v^{\mu}(\bar{\sigma}_{\mu}\check{\zeta}_{\check{\imath}})^{\dot{\alpha}}. \tag{7.15}$$

It is worth noting that this is distinct from the commonly employed practice of identifying the $SU(2)_{\check{R}}$ bundle with the $SU(2)_R$ one using the solution:

$$\check{\zeta}^{\check{\imath}}_{\alpha} = i\sqrt{\frac{\tilde{s}}{s}}\,\delta^{\check{\imath}}_{i}\,\zeta^{i}_{\alpha}, \qquad\qquad \check{\bar{\chi}}^{\dot{\alpha}}_{\check{\imath}} = i\sqrt{\frac{s}{\tilde{s}}}\,\delta^{i}_{\check{\imath}}\,\bar{\chi}^{\dot{\alpha}}_{i}, \tag{7.16}$$

which is not globally well-defined.

7.2.3 Vector multiplet

The case of the $\mathcal{N} = 2$ vector multiplet was examined in [Festuccia et al., 2020]. We present a brief review for the sake of completeness, and because some of the fields make an appearance in the gauged hypermultiplet expressions.

The $\mathcal{N} = 2$ vector multiplet consists of a complex scalar (X), a gauge field (A^{μ}) and two fermionic fields ($\lambda_{i\alpha}$ and $\tilde{\lambda}^{i}_{\dot{\alpha}}$) called gauginos, that live in the fundamental representation of $SU(2)_R$. Finally, it also includes an auxiliary real scalar field (D_{ij}), that transforms as an $SU(2)_R$ triplet. All the vector multiplet fields except for the vector field A^{μ} itself, live in the adjoint representation of the gauge group.

The supersymmetry variations of the vector multiplet fields are:

$$
\begin{aligned}
&\delta\bar{X} = \bar{\chi}^i\bar{\lambda}_i\ , \qquad \delta X = -\zeta_i\lambda^i\ ,\\
&\delta A_\mu = i\zeta_i\sigma_\mu\bar{\lambda}^i + i\bar{\chi}^i\bar{\sigma}_\mu\lambda_i\ ,\\
&\delta D_{ij} = i\zeta_i\sigma^\mu\big(D_\mu + iG_\mu\big)\bar{\lambda}_j - i\bar{\chi}_i\bar{\sigma}^\mu\big(D_\mu - iG_\mu\big)\lambda_j\\
&\qquad\quad + 2i[X,\bar{\chi}_i\bar{\lambda}_j] + 2i[\bar{X},\zeta_i\lambda_j] + (i\leftrightarrow j)\ ,\\
&\delta\lambda_i = -2i(D_\mu - 2iG_\mu)X\sigma^\mu\bar{\chi}_i + 2\big(F^+ - \bar{X}\,W^+\big)\zeta_i\\
&\qquad\quad + D_{ij}\zeta^j + 2i[X,\bar{X}]\zeta_i - 2X\eta_i\ ,\\
&\delta\bar{\lambda}^i = 2i(D_\mu + 2iG_\mu)\bar{X}\bar{\sigma}^\mu\zeta^i + 2\big(F^- - X\,W^-\big)\bar{\chi}^i\\
&\qquad\quad - D^{ij}\bar{\chi}_j - 2i[X,\bar{X}]\bar{\chi}^i + 2\bar{X}\bar{\eta}^i\ .
\end{aligned}
\tag{7.17}
$$

where $F_{\mu\nu}$ is the field strength two-form corresponding to A^μ, and as in the previous subsection, we made use of the notation $F^+ := \frac{1}{2}F_{\mu\nu}\sigma^{\mu\nu}$ and $F^- := \frac{1}{2}F_{\mu\nu}\bar{\sigma}^{\mu\nu}$.

Applying a supersymmetry transformation twice on a vector multiplet field Ψ, we get:

$$
\delta^2\Psi = i\mathcal{L}_v\Psi + iv^\mu V_\mu\circ\Psi + i\Lambda^{(R)}\circ\Psi - i[\Phi,\Psi], \tag{7.18}
$$

where $\mathcal{L}_v$ is the Lie derivative along the Killing vector v and the symbol "$\circ$" indicates that we need to act on the field according to the $SU(2)_R$ representation under which it transforms. In equation (7.18), $\Lambda^{(R)}$ is an $SU(2)_R$ parameter:

$$
\Lambda^{(R)}_{ij} = \bar{\chi}_i\bar{\sigma}^\mu(D_\mu - iG_\mu)\zeta_j - \zeta_i\sigma^\mu(D_\mu + iG_\mu)\bar{\chi}_j + (i\leftrightarrow j), \tag{7.19}
$$

and Φ is a gauge transformation parameter:

$$
\Phi = iv^\mu A_\mu + s\bar{X} + \tilde{s}X. \tag{7.20}
$$

Thus, a supersymmetric transformation of a vector multiplet field squares to a translation along the Killing vector field plus an $SU(2)_R$ and a gauge transformation.

7.2.4 Hypermultiplet

We will now turn to the case of the (gauged) hypermultiplet, which we study in Paper II. We will consider the gauge group as embedded in $Sp(k)$, with the hypermultiplet transforming in its fundamental representation. The field content of the hypermultiplet includes a scalar field q_{ni}, two spinorial fields $\psi_{\alpha n}$ and $\bar{\psi}_{\dot{\alpha} n}$ and, since we will study the multiplet in its off-shell formulation, an auxiliary field $F_{n\check{\imath}}$. The indices $n, m = 1,\ldots,2k$ are $Sp(k)$ indices, while i

and $\check{\imath}$ are $SU(2)_R$ and $SU(2)_{\check{R}}$ indices respectively, transforming in the fundamental representation of the corresponding group.

The reality conditions that we will use for this work are:

$$(q_{ni})^* = q^{ni}, \qquad\qquad (F_{n\check{\imath}})^* = F^{n\check{\imath}}. \tag{7.21}$$

The hypermultiplet fields transform under supersymmetry as follows:

$$\begin{aligned}
\delta q_{ni} &= \zeta_i \psi_n + \bar{\chi}_i \bar{\psi}_n, \\
\delta \psi_n &= 2i(D_\mu q_{ni})\sigma^\mu \bar{\chi}_i + i q_{ni} \sigma^\mu \left(D_\mu + iG_\mu\right)\bar{\chi}_i + 4i \bar{X}_n{}^m q_{mi} \zeta^i + 2i F_{n\check{\imath}} \check{\zeta}^{\check{\imath}}, \\
\delta \bar{\psi}^n &= 2i(D_\mu q^{ni})\bar{\sigma}^\mu \zeta_i + i q^{ni} \bar{\sigma}^\mu \left(D_\mu - iG_\mu\right)\zeta_i + 4i X^n{}_m q^{mi} \bar{\chi}_i + 2i F^{n\check{\imath}} \check{\bar{\chi}}_{\check{\imath}}, \\
\delta F_{n\check{\imath}} &= \check{\zeta}_{\check{\imath}} \left[\sigma^\mu \left(D_\mu - iG_\mu\right)\bar{\psi}_n - 2X_n{}^m \psi_m + 2(\lambda^j)_n{}^m q_{mj} - iW^+ \psi_n\right] \\
&\qquad + \check{\bar{\chi}}_{\check{\imath}} \left[\bar{\sigma}^\mu \left(D_\mu + iG_\mu\right)\psi_n + 2\bar{X}_n{}^m \bar{\psi}_m - 2(\bar{\lambda}^j)_n{}^m q_{mj} + iW^- \bar{\psi}_n\right].
\end{aligned} \tag{7.22}$$

The derivate D_μ that appears in the expressions above is covariantized under the gauge and R symmetries. Note that wherever a vector multiplet field like $X_n{}^m$ appears in the variations above, it is understood that it is contracted with $t^\alpha{}_n{}^m$, e.g. $X_n{}^m = X^\alpha t^\alpha{}_n{}^m$. As stated above, the off-shell closure of the hypermultiplet requires the introduction of the auxiliary field $F_{n\check{\imath}}$ as well as a pair of auxiliary spinors $\check{\zeta}_{\check{\imath}}$ and $\check{\bar{\chi}}_{\check{\imath}}$ which transform in the fundamental of $SU(2)_{\check{R}}$.[2] These spinors need to satisfy certain consistency conditions (7.11) with the Killing spinors. Their reality conditions shall be taken to be symplectic Majorana ones (7.12).

Applying a supersymmetry transformation twice on a hypermultiplet field Ψ (other than the auxiliary field $F_{n\check{\imath}}$), we get:

$$\delta^2 \Psi = i\mathcal{L}_v \Psi + i v^\mu V_\mu \circ \Psi + i\Lambda^{(R)} \circ \Psi + \mathcal{G}_\Phi \diamond \Psi, \tag{7.23}$$

which, as in the case of the vector multiplet (7.18), consists of a Lie derivative along the Killing vector and a combination of gauge and $SU(2)_R$ transformations, always in accordance to the appropriate representation (the symbol "$\diamond$" indicates that we should act on the field according to the gauge group representation under which it transforms). The analogous expression for the auxiliary fields contains an $SU(2)_{\check{R}}$ transformation instead of an $SU(2)_R$ one:

$$\delta^2 \Psi = i\mathcal{L}_v \Psi + i v^\mu \check{V}_\mu \circ \Psi + i\Lambda^{(\check{R})} \circ \Psi + \mathcal{G}_\Phi \diamond \Psi, \tag{7.24}$$

[2]The reason that we use checked indices ($\check{\imath}$) for the auxiliary field and the auxiliary Killing spinors is to emphasize that the $SU(2)_{\check{R}}$ bundle is generally distinct from the usual $SU(2)_R$ bundle.

where $\check{V}_\mu$ and $\Lambda^{(\check{R})}$ are the $SU(2)_{\check{R}}$ background connection and transformation parameter respectively. The latter is equal to:

$$\Lambda^{(\check{R})}_{\check{\imath}\check{\jmath}} = 2\check{\zeta}_{\check{\imath}}\sigma^\mu\Big(\check{D}_\mu - iG_\mu\Big)\check{\bar{\chi}}_{\check{\jmath}} + 2i\check{\zeta}_{\check{\imath}}W^+\check{\zeta}_{\check{\jmath}} - 2\check{\bar{\chi}}_{\check{\imath}}\bar{\sigma}^\mu\Big(\check{D}_\mu + iG_\mu\Big)\check{\zeta}_{\check{\jmath}} + 2i\check{\bar{\chi}}_{\check{\imath}}W^-\check{\bar{\chi}}_{\check{\jmath}} + \big(\check{\imath} \leftrightarrow \check{\jmath}\big), \quad (7.25)$$

with $\check{D}_\mu$ being an $SU(2)_{\check{R}}$ covariantized derivative.

The dynamics of the hypermultiplet are governed by the following supersymmetric Lagrangian:

$$\begin{aligned}\mathcal{L}_B = &+\frac{1}{2}(D^\mu q^{ni})(D_\mu q_{ni}) - \frac{i}{2}q^n{}_i(D^{ij})_n{}^m q_{mj} + \frac{1}{2}F^{n\check{\imath}}F_{n\check{\imath}} \\ &- \left(\frac{R}{12} + \frac{N}{4}\right) q^{ni}q_{ni} + q^{ni}\{\bar{X}, X\}_n{}^m q_{mi}, \end{aligned} \quad (7.26a)$$

$$\begin{aligned}\mathcal{L}_F = &-\frac{i}{2}\psi^n\sigma^\mu\left(D_\mu - iG_\mu\right)\bar{\psi}_n + \frac{i}{2}\psi^n X_n{}^m\psi_m + \frac{i}{2}\bar{\psi}^n\bar{X}_n{}^m\bar{\psi}_m \\ &- i\psi^n(\lambda^i)_n{}^m q_{mi} - i\bar{\psi}^n(\bar{\lambda}^i)_n{}^m q_{mi} - \frac{1}{4}\left(\psi^n W^+\psi_n + \bar{\psi}^n W^-\bar{\psi}_n\right),\end{aligned} \quad (7.26b)$$

where we have split the Lagrangian into a bosonic ($\mathcal{L}_B$) and a fermionic part ($\mathcal{L}_F$). The gauged hypermultiplet Lagrangian $\mathcal{L}_B+\mathcal{L}_F$ turns out to be δ-exact:

$$\mathcal{L}_{\text{hyper}} = \mathcal{L}_B + \mathcal{L}_F = \delta V_G, \quad (7.27)$$

where, after some work, it can be shown that V_G is given by:

$$\begin{aligned}V_G = \frac{1}{2(s+\tilde{s})}\Big[&2i(D_\mu + iG_\mu)(q_{ni}\zeta^i)\sigma^\mu\bar{\psi}^n - 2i(D_\mu - iG_\mu)(q_{ni}\bar{\chi}^i)\bar{\sigma}^\mu\psi^n \\ &+ 2iF_{n\check{\imath}}(\check{\bar{\chi}}^{\check{\imath}}\bar{\psi}^n - \check{\zeta}^{\check{\imath}}\psi^n) - 4iq_{mi}(X^m{}_n\bar{\chi}^i\bar{\psi}^n + \bar{X}^m{}_n\zeta^i\psi^n) \\ &- 2q_{ni}(\bar{\chi}^i W^-\bar{\psi}^n + \zeta^i W^+\psi^n) \\ &- \frac{2}{s+\tilde{s}}v^\nu\mathcal{F}_{\mu\nu}q_{ni}(\bar{\chi}^i\bar{\sigma}^\mu\psi^n - \zeta^i\sigma^\mu\bar{\psi}^n) \\ &- 4iq^{ni}\left[(\lambda_i)_n{}^m\zeta_j + (\bar{\lambda}_i)_n{}^m\bar{\chi}_j\right]q_m{}^j\Big]. \end{aligned} \quad (7.28)$$

A similar result for the hypermultiplet Lagrangian as a δ-exact expression appeared in [Hama & Hosomichi, 2012], albeit in a somewhat less general form, as it is predicated on the fact that $s+\tilde{s}$ is constant and hence $\mathcal{F}_{\mu\nu} = 0$, while we make no such assumption here.

7.3 Twisted supersymmetry

We will begin with a short account that rewrites the vector multiplet in terms of cohomological variables, worked out in [Festuccia et al., 2020]. Then, we will proceed with the hypermultiplet case which was considered in Paper II.

7.3.1 Decomposition of two-forms and spinors

A crucial point for the argument of [Festuccia et al., 2020] was the construction of certain projectors, dubbed "flipping projectors". These projectors are defined such that when they act on a two-form they produce self-dual forms at plus fixed points of the Killing vector field, and anti-self-dual ones at the minus fixed points (hence the name). These projectors are given by:

$$P_{+} = \frac{1}{2(s^2 + \tilde{s}^2)} \left((s + \tilde{s})^2 \mathbb{1} + (s^2 - \tilde{s}^2) \star - 4\kappa \wedge \iota_v \right), \tag{7.29}$$

and

$$P_{-} = \mathbb{1} - P_{+}, \tag{7.30}$$

where $\mathbb{1}$ is the identity operator and κ is the one-form dual to v.

The analogous construction in the case of the hypermultiplet (Paper II), relies on similar projectors that act on Dirac spinors. Written in terms of its components, a Dirac spinor has the form:

$$\Psi = \begin{pmatrix} \psi_{\alpha} \\ \tilde{\psi}^{\dot{\alpha}} \end{pmatrix}. \tag{7.31}$$

Usually, one can isolate the left and right-handed components by employing the projectors:

$$L = \frac{1}{2} \left(\mathbb{1} + \gamma^5 \right), \qquad R = \frac{1}{2} \left(\mathbb{1} - \gamma^5 \right), \tag{7.32}$$

where $\gamma^5 = -\gamma_1\gamma_2\gamma_3\gamma_4$, with γ^{μ} being the Dirac matrices in the chiral representation (see Appendix A of Paper II for more information on the conventions we follow). However, for the purposes of our work, in the spirit of [Festuccia et al., 2020], we define another projector:

$$Z_{+} = \frac{1}{2} \left(\mathbb{1} + \frac{s - \tilde{s}}{s + \tilde{s}} \gamma_5 - \frac{2}{s + \tilde{s}} v^{\mu} \gamma_5 \gamma_{\mu} \right). \tag{7.33}$$

One can show that Z_{+} is idempotent (and linear) and is indeed a projector. In fact, one can obtain more projectors by simple variations on the theme:

$$Z_{-} = \mathbb{1} - Z_{+}, \qquad \tilde{Z}_{+} = \gamma_5 Z_{+} \gamma_5, \qquad \tilde{Z}_{-} = \mathbb{1} - \tilde{Z}_{+}. \tag{7.34}$$

Then, one can see that Z_{+} and $\tilde{Z}_{+}$, analogously to their two-form counterpart P_{+}, project spinors to left-handed ones on plus fixed points and to right-handed ones on minus fixed points. In fact, there seems to exist some relation between the self-dual two-forms and the left-handed spinors defined by P_{+} and Z_{+} respectively (as well as analogous relations for the other projectors). This can be seen by composing the following two-form:

$$\omega_{\mu\nu} = \bar{\Psi}_2 \gamma_{\mu\nu} \Psi_1, \tag{7.35}$$

where $\Psi_{1,2}$ are spinors that satisfy the projection relation $Z_+\Psi_{1,2} = \Psi_{1,2}$. It can be shown then that ω is self-dual:

$$P_+\omega = \omega. \tag{7.36}$$

7.3.2 Twisted vector multiplet

As part of [Festuccia et al., 2020], the authors wrote the vector multiplet in terms of new cohomological variables. The dictionary between the usual formulation and the cohomological one can be found in the aforementioned paper or in the Appendix C of Paper II. As some of the vector multiplet fields make an appearance in the gauged hypermultiplet expressions, we would be remiss not to summarize the vector multiplet theory in its twisted formulation.

In the cohomological language, the vector multiplet organizes naturally in three submultiplets, one long and two short ones. The long submultiplet contains a scalar field ϕ, a Grassmann one-form Ψ and a gauge field A (with field strength F). The scalar and the one-form transform in the adjoint representation. Their supersymmetry variations are:

$$\begin{aligned}
\delta\phi &= \imath_v\Psi, \\
\delta\Psi &= \imath_v F + i d_A\phi, \\
\delta A &= i\Psi,
\end{aligned} \tag{7.37}$$

where $\imath_v$ denotes the inner product with the Killing vector field v.

One of the two short submultiplets comprises a scalar φ and another Grassmann scalar η, both transforming in the adjoint representation, whose supersymmetry variations are:

$$\begin{aligned}
\delta\varphi &= i\eta, \\
\delta\eta &= \imath_v d_A\varphi - [\phi, \varphi].
\end{aligned} \tag{7.38}$$

The second short submultiplet contains a two-form H and another Grassmann two-form χ, both transforming in the adjoint representation, with the following supersymmetry variations:

$$\begin{aligned}
\delta\chi &= H, \\
\delta H &= i\mathcal{L}_v^A\chi - i[\phi, \chi],
\end{aligned} \tag{7.39}$$

where $\mathcal{L}_v^A$ denotes the Lie derivative along v that contains also the gauge field A. Both two-forms satisfy a projection relation: $P_+\chi = \chi$ (and similarly for H). The reality conditions of the twisted fields follow from those of the original vector multiplet theory.

7.3.3 Twisted hypermultiplet

The hypermultiplet expressed in the cohomological language, contains four fermionic fields $\mathfrak{q}_n, \mathfrak{h}_n$ (Grassmann-even) and $\mathfrak{b}_n, \mathfrak{c}_n$ (Grassmann-odd), all in the fundamental representation of the gauge group $Sp(k)$. For the purposes of our work, it is convenient to package the Killing spinors ζ_i and $\bar{\chi}_i$ into a Dirac spinor:

$$\mathfrak{z}_i = \begin{pmatrix} \zeta_i \\ \bar{\chi}_i \end{pmatrix}, \tag{7.40}$$

that enjoys the projection relation $Z_+ \mathfrak{z}_i = \mathfrak{z}_i$. We will do the same for the auxiliary Killing spinors $\check{\zeta}_{\check{\imath}}$ and $\check{\bar{\chi}}_{\check{\imath}}$:

$$\check{\mathfrak{z}}_{\check{\imath}} = \begin{pmatrix} \check{\zeta}_{\check{\imath}} \\ \check{\bar{\chi}}_{\check{\imath}} \end{pmatrix}, \tag{7.41}$$

which also enjoy a similar projection relation: $\tilde{Z}_- \check{\mathfrak{z}}_{\check{\imath}} = \check{\mathfrak{z}}_{\check{\imath}}$.

First, let us show how one can create these cohomological fields from the ordinary hypermultiplet fields presented in Subsection 7.2.4. The scalar field q_{ni} gets mapped to the field $\mathfrak{q}_n$:

$$\mathfrak{q}_n = \mathfrak{z}^i q_{ni} = \begin{pmatrix} \zeta^i q_{ni} \\ \bar{\chi}^i q_{ni} \end{pmatrix}. \tag{7.42}$$

The inverse of this mapping is given by:

$$q_{ni} = -\frac{4}{s+\tilde{s}} \bar{\mathfrak{z}}_i \, \mathfrak{q}_n. \tag{7.43}$$

The degrees of freedom from the two spinorial fields ψ_n and $\bar{\psi}_n$ get distributed to $\mathfrak{c}_n$:

$$\mathfrak{c}_n = -\frac{s+\tilde{s}}{4} Z_+ \begin{pmatrix} \psi_n \\ \bar{\psi}_n \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} s\psi_n - v^\mu \sigma_\mu \bar{\psi}_n \\ \tilde{s}\bar{\psi}_n + v^\mu \bar{\sigma}_\mu \psi_n \end{pmatrix}, \tag{7.44}$$

and $\mathfrak{b}_n$:

$$\mathfrak{b}_n = \frac{s+\tilde{s}}{4} \tilde{Z}_- \gamma_5 \begin{pmatrix} \psi_n \\ \bar{\psi}_n \end{pmatrix} = \frac{1}{4} \begin{pmatrix} \tilde{s}\psi_n + v^\mu \sigma_\mu \bar{\psi}_n \\ -s\bar{\psi}_n + v^\mu \bar{\sigma}_\mu \psi_n \end{pmatrix}. \tag{7.45}$$

Their inverses are:

$$\begin{pmatrix} \psi_n \\ \bar{\psi}_n \end{pmatrix} = \frac{4}{s+\tilde{s}} (\gamma_5 \mathfrak{b}_n - \mathfrak{c}_n). \tag{7.46}$$

Finally, the dictionary entry for the field $\mathfrak{h}_n$ in terms of the original hypermultiplet fields can be obtained by demanding that $\mathfrak{h}_n = -i\delta \mathfrak{b}_n$, giving us:

$$\mathfrak{h}_n = \frac{s+\tilde{s}}{2} \check{\mathfrak{z}}^{\check{\imath}} F_{n\check{\imath}} + \tilde{Z}_- \left[\frac{s+\tilde{s}}{2} \gamma^\mu (D_\mu + iT_\mu) \mathfrak{q}_n + i v^\mu G_\mu \mathfrak{q}_n - i \frac{(s+\tilde{s})}{2} \varphi_n{}^m \mathfrak{q}_m \right], \tag{7.47}$$

where T_μ was introduced to make the expression shorter, and is defined as:

$$T_\mu = \frac{s-\tilde{s}}{(s+\tilde{s})}G_\mu + \frac{s\tilde{s}}{(s+\tilde{s})^2}b_\mu + i\frac{\partial_\mu(s^2+\tilde{s}^2)}{2(s+\tilde{s})^2}, \tag{7.48}$$

where b^μ is a one-form that arises from the derivation of the supergravity background fields (see Subsection 7.2.1). To return to the auxiliary F_{ni} we can use the inverse mapping:

$$F_{n\check{\imath}} = \frac{8}{(s+\tilde{s})^2}\Bigg[\bar{\mathfrak{z}}_{\check{\imath}}\mathfrak{h}_n - \frac{s+\tilde{s}}{2}\bar{\mathfrak{z}}_{\check{\imath}}\gamma^\mu(D_\mu + iT_\mu)\mathfrak{q}_n - iv^\mu G_\mu(\bar{\mathfrak{z}}_{\check{\imath}}\mathfrak{q}_n) \\ + i\frac{(s+\tilde{s})}{2}\varphi_n{}^m(\bar{\mathfrak{z}}_{\check{\imath}}\mathfrak{q}_m)\Bigg]. \tag{7.49}$$

The cohomological fields also satisfy certain projection relations:

$$Z_+\mathfrak{q} = \mathfrak{q}, \qquad Z_+\mathfrak{c} = \mathfrak{c}, \qquad \tilde{Z}_-\mathfrak{b} = \mathfrak{b}, \qquad \tilde{Z}_-\mathfrak{h} = \mathfrak{h}. \tag{7.50}$$

Using the reality conditions of the original hypermultiplet fields (7.21) and the reality conditions of the background supergravity fields (see Appendix B of Paper II), we can deduce the reality conditions for the twisted fields:

$$\begin{aligned}(\mathfrak{h}_n)^* &= -\bar{\mathfrak{h}}^n - \Big[(s+\tilde{s})(D_\mu + iT_\mu)\bar{\mathfrak{q}}^n\gamma^\mu - 2iv^\mu G_\mu\bar{\mathfrak{q}}^n \\ &\qquad - i(s+\tilde{s})\bar{\mathfrak{q}}^m\varphi_m{}^n\Big]\tilde{Z}_-, \\ (\mathfrak{q}_n)^* &= \bar{\mathfrak{q}}^n.\end{aligned} \tag{7.51}$$

Now we can write down the supersymmetry transformation of the cohomological hypermultiplet fields. These are organized into two submultiplets:

$$\delta\mathfrak{q} = \mathfrak{c} \tag{7.52a}$$
$$\delta\mathfrak{c} = (i\mathcal{L}_v - \mathcal{G}_\Phi)\mathfrak{q} \tag{7.52b}$$

and

$$\delta\mathfrak{b} = i\mathfrak{h} \tag{7.53a}$$
$$\delta\mathfrak{h} = (\mathcal{L}_v + i\mathcal{G}_\Phi)\mathfrak{b} \tag{7.53b}$$

where $\mathcal{G}_\Phi$ is a gauge transformation with $\Phi = (i\iota_v A + \phi)$, acting differently on fields depending on their representation. Finally, the square of a supersymmetry transformation is

$$\delta^2 = i\mathcal{L}_v - \mathcal{G}_\Phi, \tag{7.54}$$

which a sum of a Lie derivative along the Killing vector field and a gauge transformation.

As we reported in Subsection 7.2.4, the hypermultiplet Lagrangian is δ-exact: $\mathcal{L} = \delta V_G$. Thus, to rewrite the hypermultiplet Lagrangian as a function of the twisted fields, we can first translate $V_G(q_{ni}, \psi_n, \bar{\psi}_n, F_{n\check{\imath}})$ to $V_G(\mathfrak{q}, \mathfrak{b}, \mathfrak{c}, \mathfrak{h})$ and then apply the supersymmetry transformation rules (7.52) and (7.53). Carrying out the translation of V_G, we get:

$$
\begin{aligned}
V_G = \frac{8}{(s+\tilde{s})^3} \Bigg\{ & i\bar{\mathfrak{c}}\mathcal{L}_v\mathfrak{q} + i\bar{\mathfrak{q}}\left[\phi + i(s-\tilde{s})\varphi\right]\mathfrak{c} - i(\partial_\mu v_\nu)\bar{\mathfrak{c}}\,\gamma^{\mu\nu}\mathfrak{q} \\
& - (s+\tilde{s})\left(G_\mu - \frac{s^2-\tilde{s}^2}{64}b_\mu\right)\bar{\mathfrak{c}}\gamma^\mu\mathfrak{q} \\
& + i(s+\tilde{s})\bar{\mathfrak{b}}\gamma^\mu(D_\mu + iT_\mu)\mathfrak{q} \\
& - (s+\tilde{s})\bar{\mathfrak{b}}\,\varphi\,\mathfrak{q} - 2\iota_v G\bar{\mathfrak{b}}\mathfrak{q} - \frac{i}{4}(s+\tilde{s})^2\bar{\mathfrak{q}}\chi\mathfrak{q} - i\bar{\mathfrak{h}}\mathfrak{b}\Bigg\},
\end{aligned}
\tag{7.55}
$$

where the $Sp(k)$ indices are implicitly contracted with the convention $\bar{\Psi}_1\Psi_2 = (\bar{\Psi}_1)_m\Psi_2^m$, and we also introduced the notation $\chi \coloneqq \frac{1}{2}\chi_{\mu\nu}\gamma^{\mu\nu}$. Taking the supersymmetric variation of V_G using (7.52) and (7.53), we derive the hypermultiplet Lagrangian in cohomological variables:

$$
\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F \tag{7.56}
$$

where $\mathcal{L}_B$ is the bosonic part given by:

$$
\begin{aligned}
\mathcal{L}_B = \frac{8}{(s+\tilde{s})^3} \Bigg\{ & - \mathcal{L}_v\bar{\mathfrak{q}}\mathcal{L}_v\mathfrak{q} - D_\mu v_\nu\bar{\mathfrak{q}}\gamma^{\mu\nu}\mathcal{L}_v\mathfrak{q} - i(s-\tilde{s})\bar{\mathfrak{q}}\,\varphi\,\mathcal{L}_v\mathfrak{q} \\
& + i(s+\tilde{s})\left(G_\mu - \frac{s^2-\tilde{s}^2}{64}b_\mu\right)\left(\bar{\mathfrak{q}}\gamma^\mu\mathcal{L}_v\mathfrak{q} + \bar{\mathfrak{q}}\gamma^\mu\phi\,\mathfrak{q}\right) \\
& - \partial_\mu v_\nu\bar{\mathfrak{q}}\gamma^{\mu\nu}\phi\,\mathfrak{q} \\
& - \frac{1}{2}\bar{\mathfrak{q}}\left[\frac{i}{2}(s+\tilde{s})^2H^+ + \{\phi,\phi\} + i(s-\tilde{s})\{\varphi,\phi\}\right]\mathfrak{q} \\
& - (s+\tilde{s})\bar{\mathfrak{h}}\gamma^\mu(D_\mu + iT_\mu)\mathfrak{q} - 2i\iota_v G\bar{\mathfrak{h}}\mathfrak{q} \\
& + i(s+\tilde{s})\bar{\mathfrak{q}}\,\varphi\,\mathfrak{h} + \bar{\mathfrak{h}}\mathfrak{h}\Bigg\},
\end{aligned}
\tag{7.57}
$$

and $\mathcal{L}_F$ is the fermionic part given by:

$$\begin{aligned}\mathcal{L}_F = \frac{8}{(s+\tilde{s})^3}\bigg\{ & -i\bar{\mathfrak{b}}\mathcal{L}_v\mathfrak{b} + i\bar{\mathfrak{b}}\phi\mathfrak{b} - i\bar{\mathfrak{c}}\mathcal{L}_v\mathfrak{c} - i(s+\tilde{s})\bar{\mathfrak{b}}\gamma^\mu(D_\mu + iT_\mu)\mathfrak{c} \\ & + 2\iota_v G\bar{\mathfrak{b}}\mathfrak{c} + (s+\tilde{s})\left(G_\mu - \frac{s^2-\tilde{s}^2}{64}b_\mu\right)\bar{\mathfrak{c}}\,\sigma^\mu\mathfrak{c} \\ & + i\bar{\mathfrak{c}}\left[\phi + i(s-\tilde{s})\varphi\right]\mathfrak{c} + i\partial_\mu v_\nu\bar{\mathfrak{c}}\,\sigma^{\mu\nu}\mathfrak{c} \\ & + (s+\tilde{s})\bar{\mathfrak{c}}\,\varphi\,\mathfrak{b} + i(s+\tilde{s})\bar{\mathfrak{q}}\gamma^\mu\Psi_\mu\mathfrak{b} + i(s+\tilde{s})\bar{\mathfrak{q}}\,\eta\,\mathfrak{b} \\ & + i\bar{\mathfrak{q}}\left[2\iota_v\Psi - (s-\tilde{s})\eta + \frac{1}{2}(s+\tilde{s})^2\chi\right]\mathfrak{c}\bigg\}. \end{aligned} \tag{7.58}$$

7.4 Cohomological Localization

In this section we will review the localization computation performed in Paper II.

7.4.1 Localization preliminaries

As explained in Chapter 2, in a localization computation the path integral localizes on the BPS configurations of the theory. For the case of the twisted hypermultiplet theory, this is given by vanishing fermionic fields and supersymmetric transformations thereof. The resulting equations are:

$$\begin{aligned} \mathfrak{h} &= 0, \\ (i\mathcal{L}_v - \mathcal{G}_\Phi)\mathfrak{q} &= 0. \end{aligned} \tag{7.59}$$

To solve the second of these equations, we may recall the reality conditions of the quantities involved (complex Φ, real v, and equation (7.51) for $\mathfrak{q}$), to get:

$$\mathfrak{q} = 0, \tag{7.60}$$

and hence our BPS locus is trivial.

The next step in the localization procedure, is to specify the localizing action, which for us will be given by the supersymmetry variation of the following expression:

$$V_{\text{loc}} = \frac{1}{4\bar{\mathfrak{z}}^i\mathfrak{z}_i}(\delta\Psi_n)^*\Psi_n = \frac{8}{(s+\tilde{s})^3}\left[(\delta\mathfrak{b}_n)^*\mathfrak{b}_n + (\delta\mathfrak{c}_n)^*\mathfrak{c}_n\right], \tag{7.61}$$

or

$$\begin{aligned} V_{\text{loc}} = \frac{8}{(s+\tilde{s})^3}\Big\{ & -\bar{\mathfrak{b}}\delta\mathfrak{b} + \bar{\mathfrak{b}}\left[i(s+\tilde{s})\gamma^\mu(D_\mu + iT_\mu) - 2\iota_v G - g(s+\tilde{s})\varphi\right]\mathfrak{q} \\ & + (\overline{\delta\mathfrak{q}})\left[i\mathcal{L}_v - i\Phi - i(2\phi + i(s-\tilde{s})\varphi)\right]\mathfrak{q}\Big\}. \end{aligned} \tag{7.62}$$

Our reality conditions (7.51) ensure that δV_{loc} is greater or equal to zero.

For the purposes of the one-loop determinant computation, we may rewrite equation (7.62) in a matrix form:

$$V_{\text{loc}} = \frac{8}{(s+\tilde{s})^3}\left(\overline{\delta\mathfrak{q}},\ \overline{\mathfrak{b}}\right)\begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11}\end{pmatrix}\begin{pmatrix}\mathfrak{q} \\ \delta\mathfrak{b}\end{pmatrix}, \tag{7.63}$$

where:

$$\begin{aligned} D_{00} &= i\mathcal{L}_v - i\Phi - i[2\phi + i(s-\tilde{s})\varphi], & D_{01} &= 0, \\ D_{10} &= i(s+\tilde{s})\gamma^\mu(D_\mu + iT_\mu) - 2\iota_v G - g(s+\tilde{s})\varphi, & D_{11} &= -\mathbb{1}. \end{aligned} \tag{7.64}$$

Note that the matrix entry D_{10} satisfies two important properties:

- $\tilde{Z}_- D_{10} Z_+ = D_{10}$ (by virtue of equation (7.50)),
- D_{10} is transversally elliptic (i.e. it is elliptic in the directions orthogonal to the orbits of a group action).

To see the latter, consider the symbol of D_{10}:

$$\sigma[D_{10}] = \frac{8p_\mu}{(s+\tilde{s})^2}\tilde{Z}_-\gamma^\mu Z_+ \tag{7.65}$$

which on plus/minus fixed points becomes:

$$\sigma\left[D_{10}\right]\big|_{\tilde{s}=0} = \frac{8\,p_\mu}{s^2}\,R\gamma^\mu L, \qquad \sigma\left[D_{10}\right]\big|_{s=0} = \frac{8\,p_\mu}{\tilde{s}^2}\,L\gamma^\mu R \tag{7.66}$$

respectively (remember that $s=0$ or $\tilde{s}=0$ implies that $v=0$). But these are precisely the symbols for chiral Dirac operators, which are well-known to be elliptic in Euclidean space. To establish that D_{10} is *transversally* elliptic, we can examine its behavior where $s=\tilde{s}$:

$$\begin{aligned} \sigma\left[D_{10}\right]\big|_{s=\tilde{s}} &= \frac{p_\mu}{2s^2}\left(1-\frac{v_\nu}{s}\gamma_5\gamma^\nu\right)\gamma^\mu\left(1-\frac{v_\rho}{s}\gamma_5\gamma^\rho\right) \\ &= \frac{4}{s^2}\,\tilde{Z}_-\gamma_5\gamma^{\mu\nu}p_\mu v_\nu. \end{aligned} \tag{7.67}$$

However, when $p^\mu = v^\mu \neq 0$:

$$\sigma\left[D_{10}\right]\big|_{s=\tilde{s}} = 0, \tag{7.68}$$

and thus ellipticity fails. Still, wherever p is non-zero and orthogonal to v, the symbol is invertible. So, we have demonstrated that D_{10} is transversally elliptic with respect to the Killing vector field and we can compute the one-loop determinant by employing index theorems for transversally elliptic operators:

$$\operatorname{ind}(D_{10})(t) = \sum_{x\,:\,\tilde{x}=x}\frac{\operatorname{Tr}_{\mathfrak{q}} e^{-it\mathcal{H}} - \operatorname{Tr}_{\mathfrak{b}} e^{-it\mathcal{H}}}{\det(1-\partial\tilde{x}/\partial x)}. \tag{7.69}$$

In this expression, $\mathcal{H}$ denotes the torus action:

$$\mathcal{H} = \delta^2 = i\mathcal{L}_v - \mathcal{G}_\Phi, \tag{7.70}$$

which maps x to $\widetilde{x}$. The sum in equation (7.69) runs over the fixed points of the aforementioned torus action, and t takes values in the real numbers. Let us now compute the various elements that appear in the index formula (7.69) for the case of plus and minus fixed points.

7.4.2 Index computation

Plus fixed points

First, let us switch coordinates to (z_1, z_2), where $z_i \in \mathbb{C}$. In the neighborhood of a plus fixed point, the Killing vectors are parameterized by a set of real parameters $(\epsilon_1^+, \epsilon_2^+)$:

$$v = i\epsilon_1^{(+)}(z_1\partial_{z_1} - \bar{z}_1\partial_{\bar{z}_1}) + i\epsilon_2^{(+)}(z_2\partial_{z_2} - \bar{z}_2\partial_{\bar{z}_2}). \tag{7.71}$$

In terms of the complex coordinates (z_1, z_2), the torus action acts as:

$$(z_1, z_2) \mapsto (\widetilde{z}_1, \widetilde{z}_2) = (q_1 z_1, q_2 z_2), \tag{7.72}$$

where

$$q_{1,2} = e^{i\epsilon_{1,2}^{(+)}t}, \qquad t \in \mathbb{R}, \tag{7.73}$$

i.e. as a $U(1) \times U(1)$ action. Now, we can write the determinant in the index formula (7.69) as:

$$\det\left(1 - \frac{\partial \widetilde{z}_i}{\partial z_j}\right) = (1 - q_1)(1 - \overline{q}_1)(1 - q_2)(1 - \overline{q}_2)\ , \tag{7.74}$$

where $\overline{q}_{1,2}$ denotes the complex conjugate of $q_{1,2}$ respectively. The last necessary ingredient is the action of $\mathcal{H}$ on the spinor fields. As explained in Paper II, this can be accomplished by embedding the $U(1) \times U(1)$ group in $Spin(4) = SU(2)_+ \times SU(2)_-$. We introduce $\mathbf{z}$:

$$\mathbf{z} = x_\mu \gamma^\mu = \begin{pmatrix} 0 & 0 & \overline{z}_2 & \overline{z}_1 \\ 0 & 0 & z_1 & -z_2 \\ -z_2 & -\overline{z}_1 & 0 & 0 \\ -z_1 & \overline{z}_2 & 0 & 0 \end{pmatrix}. \tag{7.75}$$

Then, a transformation g from the group $SU(2)_+ \times SU(2)_-$ will have the following form:

$$g = \mathrm{diag}\left(\sqrt{\overline{q}_1\overline{q}_2}, \sqrt{q_1 q_2}, \sqrt{\overline{q}_1 q_2}, \sqrt{q_1\overline{q}_2}\right), \tag{7.76}$$

and will act on the coordinates as:

$$\mathbf{z} \to g\mathbf{z}g^{-1}, \tag{7.77}$$

and on the fields as:

$$\Psi \to g^{-1}\Psi. \tag{7.78}$$

Taking into account that on the plus fixed points, the spinors $\mathfrak{q}$ and $\mathfrak{b}$ are left- and right-handed respectively, we obtain the following expressions for the influence of $\mathcal{H}$ on the spinor fields of (7.69):

$$\begin{aligned} \mathfrak{q}_+ &\to \sqrt{q_1 q_2}\,\mathfrak{q}_+, \\ \mathfrak{q}_- &\to \sqrt{\bar{q}_1 \bar{q}_2}\,\mathfrak{q}_-, \\ \tilde{\mathfrak{b}}^{\dot{+}} &\to \sqrt{q_1 \bar{q}_2}\,\tilde{\mathfrak{b}}^{\dot{+}}, \\ \tilde{\mathfrak{b}}^{\dot{-}} &\to \sqrt{\bar{q}_1 q_2}\,\tilde{\mathfrak{b}}^{\dot{-}}. \end{aligned} \tag{7.79}$$

Combining (7.74) and (7.79), we find that the index formula (7.69), evaluated on a plus fixed point, gives:

$$\operatorname{ind}(D_{10})|_{\text{plus point}} = \frac{\sqrt{q_1 q_2}}{(1-q_1)(1-q_2)} \sum_{\rho\in\mathcal{R}} e^{-t\,\rho(\Phi_0)}, \tag{7.80}$$

where we are summing over the weights ρ of the representation $\mathcal{R}$, and Φ_0 is a combination of a Coulomb branch modulus a_0 and a flux contribution.

Minus fixed points

For the case of minus fixed points, we follow a similar procedure. We introduce complex coordinates (z'_1, z'_2) and write the Killing vector field in terms of the real parameters $(\epsilon_1^-, \epsilon_2^-)$:

$$v = i\epsilon_1^{(-)}(z'_1\partial_{z'_1} - \bar{z}'_1\partial_{\bar{z}'_1}) + i\epsilon_2^{(-)}(z'_2\partial_{z'_2} - \bar{z}'_2\partial_{\bar{z}'_2}), \tag{7.81}$$

with the torus action acting on the coordinates as:

$$(z'_1, z'_2) \mapsto (\widetilde{z}'_1, \widetilde{z}'_2) = (q'_1 z'_1, q'_2 z'_2), \tag{7.82}$$

where

$$q'_{1,2} = e^{i\epsilon_{1,2}^{(-)} t}, \qquad t \in \mathbb{R}. \tag{7.83}$$

We can now compute the determinant in (7.69), as in the case of plus fixed points. For the spinors $\mathfrak{q}$ and $\mathfrak{b}$ we need to remember that the chirality switches with respect to the previous case, and so:

$$\begin{aligned} \mathfrak{b}_+ &\to \sqrt{q'_1 q'_2}\,\mathfrak{b}_+, \\ \mathfrak{b}_- &\to \sqrt{\bar{q}'_1 \bar{q}'_2}\,\mathfrak{b}_-, \\ \tilde{\mathfrak{q}}^{\dot{+}} &\to \sqrt{q'_1 \bar{q}'_2}\,\tilde{\mathfrak{q}}^{\dot{+}}, \\ \tilde{\mathfrak{q}}^{\dot{-}} &\to \sqrt{\bar{q}'_1 q'_2}\,\tilde{\mathfrak{q}}^{\dot{-}}\,. \end{aligned} \tag{7.84}$$

Therefore, we obtain the index formula (7.69) for a minus fixed point:

$$\operatorname{ind}(D_{10})|_{\text{minus point}} = -\frac{\sqrt{q'_1 q'_2}}{(1-q'_1)(1-q'_2)} \sum_{\rho\in\mathcal{R}} e^{-t\,\rho(\Phi'_0)}. \tag{7.85}$$

7.4.3 Regularization

The final step in the computation entails converting the expressions for the indices in equations (7.80) and (7.85) into an infinite product and performing a regularization. This is a subtle issue, which we do not treat in its generality in Paper II and would merit an independent work. In what follows, we restrict ourselves to presenting various ways to perform the regularization.

First, let's introduce some useful notation for the regularization:

$$\left[\frac{1}{1-q_i}\right]_+ := \sum_{n\geq 0} q_i^n, \tag{7.86}$$

and

$$\left[\frac{1}{1-q_i}\right]_- := -\sum_{n\leq -1} q_i^n = -\sum_{n\geq 0} q_i^{-n-1}. \tag{7.87}$$

We will also use the plus/minus subscripts to denote which regularization we employ for the expression, with the convention that the first subscript will refer to q_1 and the second one to q_2. Finally, for the purposes of the regularization of the infinite products, we will employ the gamma function:

$$\Gamma_N(\omega|\vec{a}) = \prod_{\vec{n}\in\mathbb{N}^N} (\omega + \vec{a}\cdot\vec{m})^{-1} = e^{\partial_s \zeta_N(s,\omega|\vec{a})|_{s=0}}, \tag{7.88}$$

where $\zeta_N(s,\omega|\vec{a})$ is the Barnes multiple zeta function:

$$\zeta_N(s,\omega|\vec{a}) = \sum_{\vec{n}\in\mathbb{N}^N} (\omega + \vec{a}\cdot\vec{m})^{-s}. \tag{7.89}$$

Let us now examine the regularization for the plus and minus fixed points.

Plus fixed points

• Plus-plus $(++)$ regularization:

$$\Big[\mathrm{ind}(D_{10})|_{\text{plus point}}\Big]_{++} = +\sum_{\rho\in\mathcal{R}} \sum_{n_1,n_2\in\mathbb{N}} q_1^{n_1+\frac{1}{2}} q_2^{n_2+\frac{1}{2}} e^{-t\,\rho(\Phi_0)}, \tag{7.90}$$

where in all the expressions for a plus fixed point, Φ_0 is given by:

$$\Phi_0 = a_0 + k_+(\epsilon_1^{(+)}, \epsilon_2^{(+)}), \tag{7.91}$$

with $k_+(\epsilon_1^{(+)}, \epsilon_2^{(+)})$ corresponding to the flux contribution, as explained below equation (7.80). The corresponding expression for the one-loop determinant is:

$$\Big[Z^{\mathrm{HM}}_{\epsilon_1^{(+)},\epsilon_2^{(+)}}(a_0,k_+)\Big]_{++} = \prod_{\rho\in\mathcal{R}} \Gamma_2\big(i\,\rho(\Phi_0) + \frac{\epsilon_1^{(+)}+\epsilon_2^{(+)}}{2} | \epsilon_1^{(+)}, \epsilon_2^{(+)}\big). \tag{7.92}$$

• Plus-minus $(+-)$ regularization:

$$\Big[\mathrm{ind}(D_{10})|_{\text{plus point}}\Big]_{+-} = -\sum_{\rho\in\mathcal{R}}\sum_{n_1,n_2\in\mathbb{N}} q_1^{n_1+\frac{1}{2}} q_2^{-n_2-\frac{1}{2}} e^{-t\,\rho(\Phi_0)} \tag{7.93}$$

and the one-loop determinant is:

$$\Big[Z^{\text{HM}}_{\epsilon_1^{(+)},\epsilon_2^{(+)}}(a_0,k_+)\Big]_{+-} = \prod_{\rho\in\mathcal{R}}\left[\Gamma_2\big(i\,\rho(\Phi_0)+\frac{\epsilon_1^{(+)}-\epsilon_2^{(+)}}{2}|\epsilon_1^{(+)},-\epsilon_2^{(+)}\big)\right]^{-1}. \tag{7.94}$$

• Minus-plus $(-+)$ regularization:

$$\Big[\mathrm{ind}(D_{10})|_{\text{plus point}}\Big]_{-+} = -\sum_{\rho\in\mathcal{R}}\sum_{n_1,n_2\in\mathbb{N}} q_1^{-n_1-\frac{1}{2}} q_2^{n_2+\frac{1}{2}} e^{-t\,\rho(\Phi_0)} \tag{7.95}$$

and the one-loop determinant is:

$$\Big[Z^{\text{HM}}_{\epsilon_1^{(+)},\epsilon_2^{(+)}}(a_0,k_+)\Big]_{-+} = \prod_{\rho\in\mathcal{R}}\left[\Gamma_2\big(i\,\rho(\Phi_0)+\frac{-\epsilon_1^{(+)}+\epsilon_2^{(+)}}{2}|-\epsilon_1^{(+)},\epsilon_2^{(+)}\big)\right]^{-1}. \tag{7.96}$$

• Minus-minus $(--)$ regularization:

$$\Big[\mathrm{ind}(D_{10})|_{\text{plus point}}\Big]_{--} = +\sum_{\rho\in\mathcal{R}}\sum_{n_1,n_2\in\mathbb{N}} q_1^{-n_1-\frac{1}{2}} q_2^{-n_2-\frac{1}{2}} e^{-t\,\rho(\Phi_0)} \tag{7.97}$$

and the one-loop determinant is:

$$\Big[Z^{\text{HM}}_{\epsilon_1^{(+)},\epsilon_2^{(+)}}(a_0,k_+)\Big]_{--} = \prod_{\rho\in\mathcal{R}}\Gamma_2\big(i\,\rho(\Phi_0)-\frac{\epsilon_1^{(+)}+\epsilon_2^{(+)}}{2}|-\epsilon_1^{(+)},-\epsilon_2^{(+)}\big). \tag{7.98}$$

Minus fixed points

• Plus-plus $(++)$ regularization:

$$\Big[\mathrm{ind}(D_{10})|_{\text{minus point}}\Big]_{++} = -\sum_{\rho\in\mathcal{R}}\sum_{n_1,n_2\in\mathbb{N}} (q_1')^{n_1+\frac{1}{2}} (q_2')^{n_2+\frac{1}{2}} e^{-t\,\rho(\Phi_0')} \tag{7.99}$$

where in all the expressions for a minus fixed point, Φ_0' is given by:

$$\Phi_0' = a_0' + k_-(\epsilon_1^{(-)},\epsilon_2^{(-)}), \tag{7.100}$$

with $k_-(\epsilon_1^{(-)}, \epsilon_2^{(-)})$ corresponding to the flux contribution. The resulting expression for the one-loop determinant is:

$$\left[Z^{\text{HM}}_{\epsilon_1^{(-)},\epsilon_2^{(-)}}(a_0', k_-)\right]_{++} = \prod_{\rho\in\mathcal{R}} \left[\Gamma_2\big(i\,\rho(\Phi_0') + \frac{\epsilon_1^{(-)} + \epsilon_2^{(-)}}{2}\big)|\epsilon_1^{(-)}, \epsilon_2^{(-)}\big)\right]^{-1}. \tag{7.101}$$

• Plus-minus $(+-)$ regularization:

$$\Big[\text{ind}(D_{10})|_{\text{minus point}}\Big]_{+-} = +\sum_{\rho\in\mathcal{R}} \sum_{n_1,n_2\in\mathbb{N}} (q_1')^{n_1+\frac{1}{2}} (q_2')^{-n_2-\frac{1}{2}} e^{-t\,\rho(\Phi_0')} \tag{7.102}$$

and the one-loop determinant is:

$$\left[Z^{\text{HM}}_{\epsilon_1^{(-)},\epsilon_2^{(-)}}(a_0', k_-)\right]_{+-} = \prod_{\rho\in\mathcal{R}} \left[\Gamma_2\big(i\,\rho(\Phi_0') + \frac{\epsilon_1^{(-)} - \epsilon_2^{(-)}}{2}|\epsilon_1^{(-)}, -\epsilon_2^{(-)}\big)\right]. \tag{7.103}$$

• Minus-plus $(-+)$ regularization:

$$\Big[\text{ind}(D_{10})|_{\text{minus point}}\Big]_{-+} = +\sum_{\rho\in\mathcal{R}} \sum_{n_1,n_2\in\mathbb{N}} (q_1')^{-n_1-\frac{1}{2}} (q_2')^{n_2+\frac{1}{2}} e^{-t\,\rho(\Phi_0')} \tag{7.104}$$

and the one-loop determinant is:

$$\left[Z^{\text{HM}}_{\epsilon_1^{(-)},\epsilon_2^{(-)}}(a_0', k_-)\right]_{-+} = \prod_{\rho\in\mathcal{R}} \left[\Gamma_2\big(i\,\rho(\Phi_0') + \frac{-\epsilon_1^{(-)} + \epsilon_2^{(-)}}{2}| - \epsilon_1^{(-)}, \epsilon_2^{(-)}\big)\right]. \tag{7.105}$$

• Minus-minus $(--)$ regularization:

$$\Big[\text{ind}(D_{10})|_{\text{minus point}}\Big]_{--} = -\sum_{\rho\in\mathcal{R}} \sum_{n_1,n_2\in\mathbb{N}} (q_1')^{-n_1-\frac{1}{2}} (q_2')^{-n_2-\frac{1}{2}} e^{-t\,\rho(\Phi_0')} \tag{7.106}$$

and the one-loop determinant is:

$$\left[Z^{\text{HM}}_{\epsilon_1^{(-)},\epsilon_2^{(-)}}(a_0', k_-)\right]_{--} = \prod_{\rho\in\mathcal{R}} \left[\Gamma_2\big(i\,\rho(\Phi_0') - \frac{\epsilon_1^{(-)} + \epsilon_2^{(-)}}{2}| - \epsilon_1^{(-)}, -\epsilon_2^{(-)}\big)\right]^{-1}. \tag{7.107}$$

8. Chiral algebras and $H^3 \times S^1$

In this chapter we report partial results from a project in collaboration with Thomas T. Dumitrescu, Guido Festuccia, Antonio Pittelli and Konstantina Polydorou. The basic idea of the project is to look for a structure similar to the chiral algebra found in $\mathbb{R}^4$ in [Beem, Lemos, et al., 2015], for the case of an $\mathcal{N} = 2$ supersymmetric field theory on $H^3 \times S^1$.

We begin in Section 8.1 with an introduction and motivation to the project. Then, in Sections 8.2 and 8.3 we review the basic ideas from [Beem, Lemos, et al., 2015], recalling the difficulties in obtaining a chiral algebra in conformal theories of dimension greater than two, and explaining how the authors of the aforementioned work were able to evade them in an $\mathcal{N} = 2$ superconformal theory. Then, in Section 8.4 we lay the groundwork for the supersymmetry setup of our work on $H^3 \times S^1$. In Section 8.5 we discuss the relevant aspects of the superalgebra we will employ. We continue with Section 8.6 where we use the idea of "boundary localization", also employed in [Dedushenko et al., 2018], for a free hypermultiplet and an Abelian vector multiplet, in order to extract chiral algebras on the conformal boundary ∂H_3. In Section 8.7 we examine the possibility of introducing defects to the theory. We finish in Section 8.8 with discussion of the work and its future directions.

8.1 Introduction

Exact results are rare and difficult to obtain in non-trivial quantum field theories. Supersymmetric localization, a technique employed in several chapters of this thesis, is an example of a method which allows us to extract exact answers for both free and interacting theories. Another such example is that of *conformal bootstrap*, first introduced in [Polyakov, 1974] and [Ferrara, Grillo, & Gatto, 1973][1]. This method does not require the knowledge of a Lagrangian, giving us access to non-Lagrangian theories which are very difficult to approach by other means. The problem with the conformal bootstrap though is that the resulting bootstrap equations are infinite-dimensional and challenging to solve in most cases, unless the theory is either a meromorphic rational conformal field theory (normally realizable only in two dimensions) or a topological field

[1]A description of the techniques of conformal bootstrap falls well beyond the scope of this thesis. The interested reader can find an accessible introduction in the lecture notes [Simmons-Duffin, 2017].

theory. In these two classes of theories, the crossing symmetry constraints can help us solve the theory.

The search for such a solvable truncation of the bootstrap equations was one of the motivations behind [Beem, Lemos, et al., 2015]. In this paper the authors were able to circumvent the obstructions for constructing a chiral algebra of operators with meromorphic correlators for the case of a four-dimensional $\mathcal{N} = 2$ (or greater) superconformal field theory. The essential ingredient in this construction is supersymmetry. If one attempts to search for a chiral algebra in a theory that is defined on a manifold of dimension greater than two, one will find that the only operator compatible with the requirements is the identity operator, and thus the chiral algebra is trivial. This was evaded in [Beem, Lemos, et al., 2015] by passing to the cohomology of a specific supercharge $\mathbb{Q}$, selecting a plane $\mathbb{R}^2$ in $\mathbb{R}^4$, and concentrating on a protected subsector of the theory. Then, they were able to show that the correlation functions of (appropriately twisted) operators from the protected subsector are meromorphic functions of their positions on the plane, up to $\mathbb{Q}$-exact terms. These operators form an infinite-dimensional chiral algebra, thus establishing a mapping between four-dimensional superconformal field theories and two-dimensional chiral algebras. Furthermore, the operators coincide with those that count for the Schur limit of the superconformal index (see [Kinney, Maldacena, Minwalla, & Raju, 2007], [Gadde, Rastelli, Razamat, & Yan, 2011] and [Gadde, Rastelli, Razamat, & Yan, 2013]) and hence the authors of [Beem, Lemos, et al., 2015] dubbed them *Schur operators*. One noteworthy aspect of this correspondence is the relation between the central charge of the two-dimensional theory, c_{2d}, and the conformal anomaly coefficient of the four-dimensional one, c_{4d}, which is $c_{2d} = -12c_{4d}$. Thus, for a unitary superconformal field theory, the resulting two-dimensional theory will be non-unitary.

This construction is not unique to four dimensions. Analogous results have been found among others on three-dimensional manifolds and S^3 with $\mathcal{N} \geq 4$ in [Chester, Lee, Pufu, & Yacoby, 2015] and [Dedushenko et al., 2018], and in $\mathcal{N} = (2,0)$ six-dimensional theories [Beem, Rastelli, & van Rees, 2015].

The basic idea behind the ongoing project reported in this chapter, is to search for a chiral algebra structure in an $\mathcal{N} = 2$ superconformal field theory defined on another four-dimensional manifold, $H^3 \times S^1$. This is a conformally flat manifold, albeit a non-compact one. The motivation behind picking this particular manifold was that the symmetries used in [Beem, Lemos, et al., 2015] for the construction of the chiral algebra are *spacetime symmetries* for $H^3 \times S^1$. This implies that there exists the possibility that this infinite-dimensional symmetry will persist even if one adds massive deformations and moves away from the superconformal fixed point where the initial theory lives. In fact this possibility was investigated and exploited in [Dedushenko et al., 2018] for the case of $\mathcal{N} = 4$ theories on S^3, where the authors were able to study non-conformal theories with real mass and Fayet–Iliopoulos paratemers. This paper extended earlier work [Chester et al., 2015] for three-dimensional

$\mathcal{N} = 8$ superconformal field theories, where it was shown that the correlators of certain half-BPS operators confined on a line give rise to a topological quantum mechanics.

To pursue our goals, we begin by describing our background and construct a supersymmetric theory on it. The procedure is nearly identical to that of Paper II, so we will be brief. Then, we discuss the superalgebra of our theory and move to the main tool we will employ for our study. This is a supersymmetric localization method previously utilized in various recent papers, such as [Dedushenko et al., 2018], [Bonetti & Rastelli, 2018] and [Pan & Peelaers, 2019]. This method is explained in detail in Section 8.6, but the basic idea is to reduce the path integral over the bulk into some integral that instead will involve only the values of the BPS fields/configurations evaluated at the boundary. We will dub this technique "boundary localization".

Once we possess the action of the boundary theory, we can use it to compute correlation functions. These will correspond to correlators of operators inserted at the boundary for the original theory, effectively allowing us to investigate the presence of a chiral algebra. This constitutes a different approach to the algebraic method followed in [Beem, Lemos, et al., 2015] and [Chester et al., 2015].

After performing the boundary localization computations for the free hypermultiplet and the Abelian vector multiplet, we also examine the possibility of inserting non-local operators to our framework. Finally, we conclude with a discussion of the future directions of the project.

8.2 Chiral symmetry and the lack thereof

We will begin with a short review of the presence or absence of chiral symmetry in conformal field theories of two and higher dimensions.[2] In the interest of space, we will not present any derivations, which can be readily found in the relevant literature. For instance, a comprehensive pedagogical review of conformal field theory can be found in the classic lecture notes by Ginsparg [Ginsparg, 1988].

In the case of two dimensions, the theory has a global $SL(2,\mathbb{C})$ symmetry, whose infinitesimal version can be encoded using the operators:

$$\begin{aligned} L_{-1} &= -\partial_z, \qquad & L_0 &= -z\partial_z, \qquad & L_{+1} &= -z^2\partial_z, \\ \overline{L}_{-1} &= -\partial_{\overline{z}}, \qquad & \overline{L}_0 &= -\overline{z}\partial_{\overline{z}}, \qquad & \overline{L}_{+1} &= -\overline{z}^2\partial_{\overline{z}}, \end{aligned} \tag{8.1}$$

[2] The notion of chirality in conformal field theories is different to that of $\mathcal{N} = 1$ supersymmetric field theories, where we call *chiral* those operators $\mathcal{O}(x)$ that satisfy the condition $\{\mathcal{Q}_\alpha, \mathcal{O}(x)\} = 0$, for a supercharge $\mathcal{Q}_\alpha$, where α could be $+$ or $-$.

that satisfy the following relations:

$$
\begin{aligned}
&[L_{+1}, L_{-1}] = 2L_0, && [L_0, L_{\pm 1}] = \mp L_{\pm 1}, \\
&[\overline{L}_{+1}, \overline{L}_{-1}] = 2\overline{L}_0, && [\overline{L}_0, \overline{L}_{\pm 1}] = \mp \overline{L}_{\pm 1}.
\end{aligned}
\tag{8.2}
$$

We observe that the algebra factorizes into $sl(2) \times \overline{sl}(2)$.

Chiral symmetry manifests itself in terms of operators $\mathcal{O}(z)$, that do not depend on the anti-holomorphic coordinate $\overline{z}$, and hence transform trivially under $\overline{sl}(2)$.[3] In the presence of such operators, one obtains an infinite number of conserved charges in the form of:

$$
\mathcal{O}_n \coloneqq \oint \frac{dz}{2\pi i} z^{n+h-1} \mathcal{O}(z),
\tag{8.3}
$$

where h is the scaling dimension of the operator, which we can take to be positive to avoid finite-dimensional representations of the $sl(2)$ algebra. The canonical examples in this setting are those of the energy-momentum tensor $T_{\mu\nu}$ and the conserved current J^A related to global symmetries of the theory.

The tracelessness and the conservation of the energy-momentum tensor imply that its only non-zero components, T_{zz} and $T_{\overline{z}\overline{z}}$, are meromorphic and anti-meromorphic respectively, and are hence commonly written as:

$$
T(z) \coloneqq T_{zz}(z), \qquad \overline{T}(\overline{z}) \coloneqq \overline{T}_{\overline{z}\overline{z}}(\overline{z}).
\tag{8.4}
$$

The conserved charges (8.3) that correspond to $T(z)$ are then:

$$
L_n \coloneqq \oint \frac{dz}{2\pi i} z^{n+1} T(z)
\tag{8.5}
$$

and satisfy the commutation relations:

$$
[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m+n,0},
\tag{8.6}
$$

where c is the central charge. We recognize this as the well-known Virasoro algebra. An analogous structure exists for the anti-meromorphic $\overline{T}(\overline{z})$.

Similarly, for the conserved currents $J^A(z)$ of the global symmetries we get the charges:

$$
J^A_n \coloneqq \oint \frac{dz}{2\pi i} z^n J^A(z),
\tag{8.7}
$$

with commutation relations:

$$
[J^A_m, J^B_n] = \sum_c i f^{ABC} J^C_{m+n} + m\, k\, \delta^{AB} \delta_{m+n,0},
\tag{8.8}
$$

which we recognize as a level-k affine Lie algebra.

[3]Such operators $\mathcal{O}(z)$ are often called "holomorphic" in the literature, instead of the more appropriate "meromorphic", in a slight abuse of terminology. We will adopt the latter term.

It is precisely the algebra of such meromorphic operators (or equivalently their conserved charges) that is called a *chiral algebra*. This infinite-dimensional algebra, places strong constraints in the form of correlation functions and operator product expansions and offers a powerful approach to computing quantities in two-dimensional conformal field theories.

Let us now turn to the case of higher-dimensional conformal field theories and attempt to search for similar chiral algebras. In the d-dimensional flat space $\mathbb{R}^d$, we can select a two-dimensional plane and study operators with support only on that surface. These operators will need to transform trivially under an $\overline{sl}(2)$ and non-trivially under another $sl(2)$, which should be parts of the $so(2+d)$ spacetime isometry of $\mathbb{R}^d$ ($sl(2)\times\overline{sl}(2)\subset so(d+2)$). This seems promising until one realizes that if the operators transform trivially under an $sl(2)$ in $so(d+2)$, they will transform trivially under the entire $so(d+2)$. Thus the only operator that will satisfy these criteria is the identity operator. It appears that one cannot construct a non-trivial chiral algebra in conformal field theories of dimension greater than two. In the next section, we will describe how [Beem, Lemos, et al., 2015] were able to circumvent this obstruction and build such an infinite-dimensional algebra in a four-dimensional $\mathcal{N}=2$ superconformal field theory.

8.3 Obtaining a chiral algebra in four dimensions

In this section, we are going to explain the procedure developed in [Beem, Lemos, et al., 2015] to obtain a chiral algebra in a four-dimensional superconformal field theory. The key ingredient in this construction is the possibility to pass to the cohomology class of a special supercharge $\mathbb{Q}$, where we can find non-trivial operators that transform in a chiral representation of $sl(2)\times\overline{sl}(2)$, forming the desired chiral algebra.

In order to accomplish this, we have to identify a special supercharge $\mathbb{Q}$ and a subalgebra $sl(2)\times\overline{sl}(2)$ of the theory's superconformal algebra (which for the case of the $\mathcal{N}=2$ theory we are discussing is $sl(4|2)$) that fulfill the following four conditions:

- nilpotency of the supercharge ($\mathbb{Q}^2=0$);
- the subalgebras $sl(2)$ and $\overline{sl}(2)$ generate holomorphic/anti-holomorphic Möbius transformations on a complex plane;
- $sl(2)$ is closed with respect to $\mathbb{Q}$;
- $\overline{sl}(2)$ is exact with respect to $\mathbb{Q}$.

A subalgebra of $sl(4|2)$ that seems promising to satisfy these conditions is $sl(2)\times sl(2|2)$, which corresponds to a two-dimensional $\mathcal{N}=(0,4)$ superconformal field theory. The next step is to understand how objects of the $\mathcal{N}=2$ theory transform under this subalgebra. For details on this topic we refer the interested reader to the original work [Beem, Lemos, et al., 2015] (especially Appendix A). Finally, we need to identify the special nilpotent supercharge $\mathbb{Q}$ that fits the description above. There exist two possibilities:

$$\mathbb{Q}_1 \coloneqq \mathcal{Q}^1 + \overline{\mathcal{S}}^2, \qquad\qquad \mathbb{Q}_2 \coloneqq \mathcal{S}_1 - \overline{\mathcal{Q}}_2. \tag{8.9}$$

where $\mathcal{Q}^I_\alpha, \overline{\mathcal{Q}}_{I\,\dot\alpha}$ are Poincaré supercharges and $\mathcal{S}^\alpha_I, \overline{\mathcal{S}}^{I\,\dot\alpha}$ are conformal supercharges. Both these supercharges meet the criteria and there doesn't seem to be a reason to discard either of them.

Thus, we can now turn to the analysis of the non-trivial operators in the cohomology classes of these supercharges. At the origin, these are $\mathbb{Q}_i$-closed but not $\mathbb{Q}_i$-exact operators. Using various commutation relations and a cohomological argument, one can show that the eigenvalues of these operators need to satisfy the following conditions:

$$\frac{1}{2}\left(E-(j_1+j_2)\right)-R=0 \qquad \text{and} \qquad r+(j_1-j_2)=0, \tag{8.10}$$

where E is the conformal dimension, j_1 and j_2 are $sl(2)_1$ and $sl(2)_2$ charges that correspond to the rotation generators $\mathcal{M}_+{}^+$ and $\mathcal{M}^{\dot+}{}_{\dot+}$ of the four-dimensional $\mathcal{N}=2$ superconformal algebra, R is the $sl(2)_R$ charge and r is the $U(1)_r$ charge. In fact, the conditions (8.10) describe precisely the operators that contribute to the so-called Schur limit of the four-dimensional superconformal index [Gadde et al., 2013], and thus the authors of [Beem, Lemos, et al., 2015] have dubbed them *Schur operators*.

As hinted before, the discussion above holds when the operators are examined at the origin of the complex plane in $\mathbb{R}^4$. However, because the nilpotent supercharges $\mathbb{Q}_i$ contain conformal supercharges $\mathcal{S}$, which do not commute with the displacement generators, the Schur operators need to be modified if they are to be moved away from the origin. The modification consists of a combination of twisting and translation (on the plane (x_3, x_4)):

$$\mathcal{O}(z,\bar z) = e^{zL_{-1}+\bar z\widehat{L}_{-1}}\mathcal{O}(0)e^{-zL_{-1}-\bar z\widehat{L}_{-1}}, \tag{8.11}$$

where we have introduced the usual holomorphic/anti-holomorphic coordinates $z = x_3 + ix_4$ and $\bar z = x_3 - ix_4$. The operator $\mathcal{O}(0)$ is a Schur operator inserted at the origin, and the symmetry generators that appear in the expression above can be written in terms of the four-dimensional generators as follows:

$$L_{-1} = \mathcal{P}_{+\dot+}, \qquad L_0 = \frac{1}{2}\left(\mathcal{H}+\mathcal{M}\right), \qquad L_{+1} = \mathcal{K}^{\dot++}, \tag{8.12}$$

with $\mathcal{P}, \mathcal{K}$ being the translation and special conformal transformation generators, and $\mathcal{H}$ being the dilation generator, and

$$\widehat{L}_{-1} \coloneqq \overline{L}_{-1} + \mathcal{R}^-, \qquad \widehat{L}_0 \coloneqq 2(\overline{L}_0 - \mathcal{R}), \qquad \widehat{L}_{+1} \coloneqq \overline{L}_{+1} - \mathcal{R}^+, \tag{8.13}$$

where $\mathcal{R}^-$, $\mathcal{R}$ and $\mathcal{R}^+$ are Chevalley basis generators for $sl(2)_R$, and $\widehat{L}_{-1}$, $\widehat{L}_0$ and $\widehat{L}_{+1}$ are given by the following generators of the $\mathcal{N}=2$ superconformal field theory algebra:

$$\overline{L}_{-1} = \mathcal{P}_{-\dot{-}}, \qquad \overline{L}_0 = \frac{1}{2}\left(\mathcal{H} - \mathcal{M}\right), \qquad \overline{L}_{+1} = \mathcal{K}^{\dot{-}-}. \tag{8.14}$$

While the twisted-translated Schur operator seems to depend on the anti-holomorphic coordinate $\overline{z}$, one can show that, by passing to the cohomology of one of the supercharges $\mathbb{Q}_i$, the operator $\mathcal{O}$ depends only on the holomorphic coordinate z:

$$\partial_{\overline{z}}[\mathcal{O}(z,\overline{z})]_{\mathbb{Q}_i} = 0 \tag{8.15}$$

up to $\mathbb{Q}_i$-exact terms.

Finally, let us write a Schur operator inserted at a point $(z,\overline{z})$ using an $sl(2)_R$ spin-k representation $\mathcal{O}^{\mathcal{I}_1\ldots\mathcal{I}_{2k}}$ with $\mathcal{I}_i = 1,2$. In the origin, it will be equal to $\mathcal{O}^{1\ldots1}(0,0)$, while at any other point of the plane it will become:

$$\mathcal{O}(z,\overline{z}) \coloneqq u_{\mathcal{I}_1}(\overline{z})\ldots u_{\mathcal{I}_{2k}}(\overline{z})\mathcal{O}^{\mathcal{I}_1\ldots\mathcal{I}_{2k}}(z,\overline{z}) \tag{8.16}$$

with $u_{\mathcal{I}} \coloneqq (1,\overline{z})$.

8.4 Supersymmetry on $H^3 \times S^1$

In this section we will write an $\mathcal{N}=2$ supersymmetric field theory on $H^3 \times S^1$. We will also state our conventions and set up the notation we will use for the results in the rest of the chapter.

8.4.1 Metric and Killing spinors

The metric we will use for the study of $H^3 \times S^1$ is:

$$ds^2 = \frac{L^2}{r^2}\left(dx^2 + dy^2 + dr^2\right) + L^2\beta^2 d\theta^2, \tag{8.17}$$

where L is the H^3 radius, β is the ratio of the S^1 and H^3 radii, and the angle variable θ is periodic: $\theta \in [0, 2\pi)$. The vielbein we will use is:

$$e^1 = \frac{L}{r}dx, \qquad e^2 = \frac{L}{r}dy, \qquad e^3 = \frac{L}{r}dr, \qquad e^4 = L\beta d\theta. \tag{8.18}$$

We will denote vielbein indices by Latin letters (a, b) and spacetime indices by Greek letters (μ, ν). Our conventions for the Killing spinors, the sigma matrices and the related quantities are those of Paper II unless otherwise noted. They are summarized in Appendix A of the aforementioned work.

We would like to study an $\mathcal{N}=2$ supersymmetric field theory on this background. To that end, we employ the technique introduced in [Festuccia & Seiberg, 2011]: couple the theory to supergravity and freeze the gravitational degrees of freedom. Since the procedure is identical to that of Paper II, where we also make use of an $\mathcal{N}=2$ vector multiplet and a hypermultiplet, we will be brief and restrict ourselves to the basic equations that will be most pertinent to our goals.

The supersymmetry transformations on $H^3 \times S^1$ will be parameterized by the Killing spinors, which are given by the solutions of the equations:

$$\mathcal{D}_\mu \zeta_I - i A^\nu \sigma_{\mu\nu} \zeta_I = 0, \tag{8.19a}$$

$$\mathcal{D}_\mu \widetilde{\chi}^I + i A^\nu \widetilde{\sigma}_{\mu\nu} \widetilde{\chi}^I = 0, \tag{8.19b}$$

where we have introduced the covariant derivative $\mathcal{D}_\mu$, containing the spin connection $\omega_{\mu\, ab} = e_{b\,\nu} \nabla e_a{}^\nu$ and the $SU(2)_R$ connection $V_\mu{}^J{}_I$:

$$\mathcal{D}_\mu \zeta_I = \partial_\mu \zeta_I + \frac{1}{2}\omega_{\mu\, ab}\, \sigma^{ab} \zeta_I - \frac{1}{2} V_\mu{}^J{}_I \zeta_J, \tag{8.20a}$$

$$\mathcal{D}_\mu \widetilde{\chi}^I = \partial_\mu \widetilde{\chi}^I + \frac{1}{2}\omega_{\mu\, ab}\, \widetilde{\sigma}^{ab} \widetilde{\chi}^I + \frac{1}{2} V_\mu{}^I{}_J \widetilde{\chi}^J. \tag{8.20b}$$

For $H^3 \times S^1$, with $A = -\beta d\theta$ and $V_\mu{}^I{}_J = 0$, we can solve the Killing spinor equations and get:

$$\zeta_{I\alpha} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{r}}(a_I - b_I \overline{w}) \\ \sqrt{r}\, b_I \end{pmatrix}_\alpha, \qquad \widetilde{\chi}^{I\dot{\alpha}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{r}}(c^I - d^I \overline{w}) \\ \sqrt{r}\, d^I \end{pmatrix}^{\dot{\alpha}}, \tag{8.21}$$

where we have once again introduced the holomorphic/anti-holomorphic coordinates $w = x + iy$ and $\overline{w} = x - iy$, and a_I, b_I, c_I, d_I are arbitrary parameters.

8.4.2 Vector multiplet

The $\mathcal{N}=2$ vector multiplet in its off-shell formulation consists of a complex scalar X, two gauginos of opposite chirality λ_I and $\widetilde{\lambda}^I$, a vector field a_μ and an auxiliary field D_{IJ}. Their supersymmetry transformations are:

$$\delta X = -\zeta_I \lambda^I, \tag{8.22a}$$

$$\delta \widetilde{X} = \widetilde{\chi}^I \widetilde{\lambda}_I, \tag{8.22b}$$

$$\delta a_\mu = i\zeta_I \sigma_\mu \widetilde{\lambda}^I + i\widetilde{\chi}^I \widetilde{\sigma}_\mu \lambda_I \tag{8.22c}$$

$$\delta \lambda_I = -2i\, \sigma^\mu \widetilde{\chi}_I \mathcal{D}_\mu X + F_{\mu\nu}\sigma^{\mu\nu}\zeta_I + D_{IJ}\zeta^J + 2ig[X, \widetilde{X}]\zeta_I \tag{8.22d}$$

$$\delta \widetilde{\lambda}^I = 2i\, \widetilde{\sigma}^\mu \zeta^I \mathcal{D}_\mu \widetilde{X} + F_{\mu\nu}\widetilde{\sigma}^{\mu\nu}\widetilde{\chi}^I - D^{IJ}\widetilde{\chi}_J - 2ig[X, \widetilde{X}]\widetilde{\chi}^I, \tag{8.22e}$$

$$\begin{aligned} \delta D_{IJ} = {} & 2i\, \zeta_{(I}\sigma^\mu \left(\mathcal{D}_\mu + \frac{i}{2}A_\mu\right)\widetilde{\lambda}_{J)} - 2i\, \widetilde{\chi}_{(I}\widetilde{\sigma}^\mu \left(\mathcal{D}_\mu - \frac{i}{2}A_\mu\right)\lambda_{J)} \\ & + 4ig\left[X, \widetilde{\chi}_{(I}\widetilde{\lambda}_{J)}\right] + 4ig\left[\widetilde{X}, \zeta_{(I}\lambda_{J)}\right]. \end{aligned} \tag{8.22f}$$

Using these fields, we can write the following supersymmetric Lagrangian:

$$\begin{aligned} \mathcal{L}_V = \operatorname{Tr}\Big\{ & -4(\mathcal{D}^\mu + iA^\mu)\widetilde{X}(\mathcal{D}^\mu - iA^\mu)X - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D^{IJ}D_{IJ} \\ & + \frac{2R}{3}X\widetilde{X} + 4g^2[X, \widetilde{X}]^2 - 2i\lambda_I\sigma^\mu\left(D_\mu + \frac{i}{2}A_\mu\right)\widetilde{\lambda}^I \\ & -2ig\lambda^I[\widetilde{X}, \lambda_I] - 2ig\widetilde{\lambda}^I[X, \widetilde{\lambda}_I]\Big\}. \end{aligned} \tag{8.23}$$

8.4.3 Hypermultiplet

As in Paper II, we will consider a hypermultiplet transforming in the fundamental representation of a gauge group, which we will embed in $Sp(k)$. The index that transforms under this $Sp(k)$ will be denoted by n or m and it will take values in the range $1, \ldots, 2k$. The index I or J will transform in the fundamental representation of $SU(2)_R$. The hypermultiplet consists of a scalar q_{nI}, two fermionic fields of opposite chiralities ψ_n and $\widetilde{\psi}^n$ and an auxiliary field F_{nI}. These fields transform under supersymmetry as follows:

$$\delta q_{nI} = \zeta_I \psi_n + \widetilde{\chi}_I \widetilde{\psi}_n, \tag{8.24a}$$

$$\begin{aligned}\delta\psi_n &= 2i\,\sigma^\mu \widetilde{\chi}^I \mathcal{D}_\mu q_{nI} + i q_{nI}\sigma^\mu \left(\mathcal{D}_\mu + \frac{i}{2}A_\mu\right)\widetilde{\chi}^I \\ &\qquad + 4ig\widetilde{X}_n{}^m q_{mI}\zeta^I - 2iF_{nI}\check{\zeta}^I,\end{aligned} \tag{8.24b}$$

$$\begin{aligned}\delta\widetilde{\psi}^n &= 2i\,\widetilde{\sigma}^\mu \zeta_I \mathcal{D}_\mu q^{nI} + i q^{nI}\widetilde{\sigma}^\mu \left(\mathcal{D}_\mu - \frac{i}{2}A_\mu\right)\zeta_I \\ &\qquad + 4igX^n{}_m q^{mI}\widetilde{\chi}_I - 2iF^{nI}\check{\widetilde{\chi}}_I,\end{aligned} \tag{8.24c}$$

$$\begin{aligned}\delta F_{nI} &= \check{\zeta}_I\left[\sigma^\mu\left(\mathcal{D}_\mu - \frac{i}{2}A_\mu\right)\widetilde{\psi}_n - 2gX_n{}^m\psi_m + 2g(\lambda^J)_n{}^m q_{mJ}\right] \\ &\quad + \check{\widetilde{\chi}}_I\left[\widetilde{\sigma}^\mu\left(\mathcal{D}_\mu + \frac{i}{2}A_\mu\right)\psi_n + 2g\widetilde{X}_n{}^m\psi_m - 2g(\widetilde{\lambda}^J)_n{}^m q_{mJ}\right],\end{aligned} \tag{8.24d}$$

where we have also introduced the auxiliary Killing spinors $\check{\zeta}_I$ and $\check{\widetilde{\chi}}_I$, necessary for the off-shell formulation of the hypermultiplet. As mentioned in Chapter 7, these spinors need to satisfy certain consistency relations with respect to the Killing spinors ζ_I and $\widetilde{\chi}^I$, but are otherwise arbitrary. We will follow [Hama & Hosomichi, 2012] and make the following choice (which is not the only possible one):

$$\check{\zeta}_I = \sqrt{\frac{\widetilde{s}}{s}}\,\zeta_I, \qquad \check{\widetilde{\chi}}^I = -\sqrt{\frac{s}{\widetilde{s}}}\,\widetilde{\chi}^I, \tag{8.25}$$

where we have introduced the following scalar combinations of the Killing spinors:

$$s = \zeta^I\zeta_I, \qquad \widetilde{s} = \widetilde{\chi}_I\widetilde{\chi}^I. \tag{8.26}$$

Note that the relation (8.25) between the Killing spinors and the auxiliary spinors is not valid for all supercharges (as both s and $\widetilde{s}$ need to be nonzero). Not all supercharges can be extended off-shell, but this does not pose a problem. What is important is that the supercharge used to perform localization should satisfy this property.

The Lagrangian for the hypermultiplet theory coupled to the vector multiplet is:

$$\begin{aligned}\mathcal{L}_H &= -\frac{1}{2}\mathcal{D}^\mu q^{nI}\mathcal{D}_\mu q_{nI} + \frac{i}{2}gq^n{}_I D^{IJ}{}_n{}^m q_{mJ} - \frac{1}{2}F^{nI}F_{nI} + \frac{R}{12}q^{nI}q_{nI} \\ &\quad - g^2 q^{nI}\{\widetilde{X}, X\}_n{}^m q_{mI} + \frac{i}{2}\psi^n\sigma^\mu\left(D_\mu - \frac{i}{2}A_\mu\right)\widetilde{\psi}_n \\ &\quad - \frac{i}{2}g\psi^n X_n{}^m\psi_m - \frac{i}{2}g\widetilde{\psi}^n\widetilde{X}_n{}^m\widetilde{\psi}_m \\ &\quad + ig\psi^n(\lambda^I)_n{}^m q_{mI} + ig\widetilde{\psi}^n(\widetilde{\lambda}^I)_n{}^m q_{mI}.\end{aligned} \tag{8.27}$$

8.5 Superalgebra

In this section we will expound on the algebra of the $\mathcal{N} = 2$ superconformal field theory on $H^3 \times S^1$ and present the supercharges that we will use for the cohomological construction.

8.5.1 Superalgebra and Killing spinors

The action of the spacetime symmetry generators J^i and the R-symmetry generators $R^{(i)}$ on a field Φ^I living in the fundamental representation of $SU(2)_R$ will be:

$$\left[J^{(i)}, \Phi^I\right] = -\mathcal{L}_{J^{(i)}} \Phi^I, \qquad \left[R^{(i)}, \Phi^I\right] = \mathcal{R}^{(i)I}{}_J \Phi^J, \tag{8.28}$$

where $\mathcal{R}^i$ is a 2×2 traceless matrix with the following entries:

$$\mathcal{R}^1{}_1 = -\mathcal{R}^2{}_2 = \frac{1}{2L}\tau^3, \quad \mathcal{R}^1{}_2 = \frac{1}{2L}(\tau^1 - i\tau^2), \quad \mathcal{R}^2{}_1 = \frac{1}{2L}\left(\tau^1 + i\tau^2\right), \tag{8.29}$$

where we denote the Pauli matrices by τ^i. Let us now present the generators of the conformal algebra. We begin with $\mathcal{P}$ (anti-holomorphic translations on the conformal boundary ∂H^3), $\mathcal{K}$ (special conformal transformations), $\mathcal{M}_\perp$ (rotations orthogonal to ∂H^3 or translations along S^1), $\mathcal{M}_{||}$ (rotations preserving ∂H^3), D (dilations), $\overline{\mathcal{P}}$ (holomorphic translations on ∂H^3) and $\overline{\mathcal{K}}$ (special conformal transformations on ∂H^3):

$$\begin{aligned}
\mathcal{L}_{\mathcal{K}} &= \frac{i}{L}\left(\overline{w}^2\partial_{\overline{w}} + \overline{w}r\partial_r - r^2\partial_w\right), & \mathcal{L}_{\mathcal{M}_{||}} &= \frac{1}{L}\left(w\partial_w - \overline{w}\partial_{\overline{w}}\right),\\
\mathcal{L}_D &= \frac{i}{L}\left(w\partial_w + \overline{w}\partial_{\overline{w}} + r\partial_r\right), & \mathcal{L}_{\mathcal{M}_\perp} &= -\frac{i}{L\beta}\partial_\theta,\\
\mathcal{L}_{\mathcal{P}} &= -\frac{i}{L}\partial_{\overline{w}}, & \mathcal{L}_{\overline{\mathcal{P}}} &= -\frac{i}{L}\partial_w,\\
\mathcal{L}_{\overline{\mathcal{K}}} &= \frac{i}{L}\left(w^2\partial_w + wr\partial_r - r^2\partial_{\overline{w}}\right).
\end{aligned} \tag{8.30}$$

Having introduced these generators, we can compute their commutation relations, the non-trivial of which are:

$$\begin{aligned}
[D, \mathcal{P}] &= -\frac{i}{L}\mathcal{P}, & [D, \mathcal{K}] &= \frac{i}{L}\mathcal{K},\\
[\mathcal{M}_{||}, \mathcal{P}] &= \frac{1}{L}\mathcal{P}, & [\mathcal{M}_{||}, \mathcal{K}] &= -\frac{1}{L}\mathcal{K},\\
[\mathcal{P}, \mathcal{K}] &= -\frac{1}{L}\left(iD + \mathcal{M}_{||}\right), & [\mathcal{R}^I{}_J, \mathcal{R}^K{}_L] &= \frac{1}{L}\left(\delta^K{}_J\mathcal{R}^I{}_L - \delta^I{}_L\mathcal{R}^K{}_J\right).
\end{aligned} \tag{8.31}$$

Note in particular that $\mathcal{M}_\perp$ commutes with all other bosonic generators. We can summarize the basic facts about the algebras with the following statements:

$$\begin{aligned}
&\text{span}\{\mathcal{M}_\perp\} = u(1)_{\mathcal{M}_\perp},\\
&\text{span}\{\mathcal{P},\, \mathcal{K},\, D + i\mathcal{M}_{||}\} = \overline{sl}(2),\\
&\text{span}\{R^I{}_J\} = su(2)_R.
\end{aligned} \tag{8.32}$$

The $\overline{sl}(2)$ above is the anti-holomorphic subalgebra of $so(1,3)$ ($so(1,3) \simeq sl(2) \oplus \overline{sl}(2)$). It is worth noting that this algebra is the isometry algebra of the hyperbolic space H^3.

Before moving on to the supersymmetric part of the algebra, we need to clarify a point regarding the interpretation of some of the generators above on the conformal boundary. Looking at the expressions (8.30), we observe the presence of terms containing a differential operator with respect to the radial coordinate: ∂_r. Thus, we need to justify the claim that the generators act as boundary symmetry generators in the limit $r \to 0$. To do so, we will examine the action on a boundary scalar field $\phi(w, \overline{w}, r, \theta)$ with mass m, of the generators D and $\mathcal{K}$, which contain the offending radial coordinate. The periodicity along S^1:

$$\phi(w, \overline{w}, r, \theta + 2\pi) \sim \phi(w, \overline{w}, r, \theta) \tag{8.33}$$

allows us to perform a Fourier expansion:

$$\phi(w, \overline{w}, r, \theta) = \sum_k e^{ik\theta} \phi_k(w, \overline{w}, r). \tag{8.34}$$

This results in a tower of Kaluza–Klein modes $\phi_k(w, \overline{w}, r)$ on H^3, whose effective mass is:

$$m_k^2 = m^2 - \frac{1}{L^2} + \left(\frac{k}{L\beta}\right)^2. \tag{8.35}$$

Solving the equation of motion for the scalars, provides us a way to relate these modes to their value $\phi_{k,\Delta_k^-}(w, \overline{w})$ on the conformal boundary (this should be familiar from the AdS/CFT correspondence [Witten, 1998]) as follows:[4]

$$\lim_{r \to 0} \phi_k(w, \overline{w}, r) = r^{\Delta_k^-}\, \phi_{k,\Delta_k^-}(w, \overline{w}) + r^{\Delta_k^+}\, \phi_{k,\Delta_k^+}(w, \overline{w}), \tag{8.36}$$

where $\Delta_k^\pm$ are the scaling dimensions, which need to satisfy $\Delta_k^+ + \Delta_k^- = \dim \partial H^{d+1}$. For a free theory the scaling dimensions are given by the expres-

[4]Note that if $\Delta^+ = \Delta^- = d/2$, the BF bound is saturated, and the scaling on the conformal boundary becomes (see [Freedman, Mathur, Matusis, & Rastelli, 1999]):

$$\lim_{r \to 0} \phi_k(w, \overline{w}, r) = -r^{d/2} \log r\, \phi_{k,\Delta_k^-}(w, \overline{w}) + r^{d/2}\, \phi_{k,\Delta_k^+}(w, \overline{w}).$$

It can be shown that our argument continues to hold in that case as well.

sion:

$$\Delta_k^{\pm} = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + L^2 m_k^2}. \tag{8.37}$$

This asymptotic behavior was shown to persist when turning to interacting theories in [D'Hoker & Freedman, 2002].

Let us now examine the action of D on the Kaluza–Klein modes:

$$\begin{aligned}
\lim_{r \to 0} [D, \phi_k] &= r^{\Delta_k^-} \left[D, \phi_{k,\Delta_k^-} \right] + r^{\Delta_k^+} \left[D, \phi_{k,\Delta_k^+} \right] \\
&= -\frac{i}{L} r^{\Delta_k^-} \left(w\partial_w + \overline{w}\partial_{\overline{w}} + \Delta_k^- \right) \phi_{k,\Delta_k^-} \\
&\quad - \frac{i}{L} r^{\Delta_k^+} \left(w\partial_w + \overline{w}\partial_{\overline{w}} + \Delta_k^+ \right) \phi_{k,\Delta_k^+}.
\end{aligned} \tag{8.38}$$

Now, we only have to perform an r-power matching on the two sides of the equation above, to get:

$$\left[D, \phi_{k,\Delta_k^{\pm}} \right] = -\frac{i}{L} \left(w\partial_w + \overline{w}\partial_{\overline{w}} + \Delta_k^{\pm} \right) \phi_{k,\Delta_k^{\pm}}. \tag{8.39}$$

Thus, the operator D indeed generates dilations on the conformal boundary and the fields $\phi_{k,\Delta_k^{\pm}}$ have conformal dimension $\Delta_k^{\pm}$. Doing the same for the action of $\mathcal{K}$:

$$\left[\mathcal{K}, \phi_{k,\Delta_k^{\pm}} \right] = -\frac{i}{L} \left(\overline{w}^2 \partial_{\overline{w}} + \Delta_k^{\pm} \overline{w} \right) \phi_{k,\Delta_k^{\pm}}, \tag{8.40}$$

which implies that $\mathcal{K}$ indeed acts as a generator of anti-holomorphic special conformal transformations on ∂H^3.

***

We will now turn to the supersymmetric part of the algebra. A field Φ^I transforms under the action of $\{\delta_\zeta, \delta_{\overline{\chi}}\}$ as follows:

$$\{\delta_\zeta, \delta_{\overline{\chi}}\} \Phi^I = 2i \mathcal{L}_K \Phi^I + i K^\mu V_\mu{}^I{}_J \Phi^J - i\, \Theta^I{}_J \Phi^J, \tag{8.41}$$

where $\mathcal{L}_K$ is the Lie derivative on the direction of the Killing vector K^μ:

$$K^\mu := \widetilde{\chi}^I \widetilde{\sigma}^\mu \zeta_I, \tag{8.42}$$

and we also introduced the Killing spinor bilinear $\Theta^I{}_J$:

$$\Theta^I{}_J := 4i A_\mu \left(\widetilde{\chi}^I \widetilde{\sigma}^\mu \zeta_J - \frac{1}{2} \delta^I{}_J K^\mu \right). \tag{8.43}$$

The Killing spinors in equation (8.21) correspond to eight supercharges depending on our choice of the parameters a_I, b_I, c^I and d^I. Using a chiral representation and indicating the generators as $\delta_{a_1 a_2 b_1 b_2}$ and $\widetilde{\delta}_{c_1 c_2 d_1 d_2}$, we can establish the following correspondence:

$$\begin{aligned} &\mathcal{Q}^1 \to \delta_{1000}, && \mathcal{Q}^2 \to \delta_{0100}, && \widetilde{\mathcal{S}}^1 \to \delta_{0010}, && \widetilde{\mathcal{S}}^2 \to \delta_{0001} \\ &\widetilde{\mathcal{Q}}^1 \to \widetilde{\delta}_{1000}, && \widetilde{\mathcal{Q}}^2 \to \widetilde{\delta}_{0100}, && \mathcal{S}_1 \to \widetilde{\delta}_{0010}, && \mathcal{S}_2 \to \widetilde{\delta}_{0001}, \end{aligned} \tag{8.44}$$

where $\mathcal{Q}^I_\alpha$, $\widetilde{\mathcal{Q}}_{I\,\dot\alpha}$ are Poincaré supercharges and $\mathcal{S}^\alpha_I$, $\widetilde{\mathcal{S}}^{I\,\dot\alpha}$ are conformal supercharges. These supercharges satisfy the following anti-commutation relations:

$$\left\{\mathcal{Q}^I, \widetilde{\mathcal{Q}}_J\right\} = -2\delta^I{}_J \mathcal{P}, \tag{8.45a}$$

$$\left\{\widetilde{\mathcal{S}}^I, \mathcal{S}_J\right\} = +2\delta^I{}_J \mathcal{K}, \tag{8.45b}$$

$$\left\{\mathcal{Q}^I, \mathcal{S}_J\right\} = \delta^I{}_J \left(-D + i\mathcal{M}_{||} + i\mathcal{M}_\perp\right) + 2iR^I{}_J, \tag{8.45c}$$

$$\left\{\widetilde{\mathcal{S}}^I, \widetilde{\mathcal{Q}}_J\right\} = \delta^I{}_J \left(-D + i\mathcal{M}_{||} - i\mathcal{M}_\perp\right) - 2iR^I{}_J, \tag{8.45d}$$

where we have included only the non-trivial relations. The commutation relations (8.31) along with the anti-commutation relations (8.45) form the superalgebra $\overline{sl}(2|2)$, which is the anti-holomorphic part of the superconformal algebra $sl(2) \times \overline{sl}(2|2)$ of the $\mathcal{N} = (0, 4)$ theory in two dimensions.

8.5.2 Twisted superalgebra

Let us now define the special nilpotent supercharges that we will use. As in the case of [Beem, Lemos, et al., 2015], we find two possible supercharges:

$$\begin{aligned} &\mathbb{Q}_1 := \mathcal{Q}^1 - \widetilde{\mathcal{S}}^2, && \mathbb{Q}_2 := \mathcal{S}_1 + \widetilde{\mathcal{Q}}_2, \\ &\widetilde{\mathbb{Q}}_1 := \mathcal{S}_1 - \widetilde{\mathcal{Q}}_2, && \widetilde{\mathbb{Q}}_2 := \mathcal{Q}^1 + \widetilde{\mathcal{S}}^2. \end{aligned} \tag{8.46}$$

These combinations of supercharges anticommute with the central element $\widehat{\mathcal{Z}} = L\mathcal{M}_\perp$. They can also be used to generate twisted translations:

$$\widehat{L}_- := -\frac{iL}{2}\{\mathbb{Q}_1, \widetilde{\mathcal{Q}}_1\} = -\frac{iL}{2}\{\mathbb{Q}_2, \mathcal{Q}^2\} = -iL\mathcal{P}_{\overline{w}} + L\mathcal{R}^2{}_1, \tag{8.47a}$$

$$\widehat{L}_0 := \frac{iL}{4}\{\mathbb{Q}_1, \widetilde{\mathbb{Q}}_1\} = \frac{iL}{4}\{\mathbb{Q}_2, \widetilde{\mathbb{Q}}_2\} = \frac{L}{2}(\mathcal{M}_{||} - iD) - L\mathcal{R}^1{}_1, \tag{8.47b}$$

$$\widehat{L}_+ := \frac{iL}{2}\{\mathbb{Q}_1, \mathcal{S}_2\} = +\frac{iL}{2}\{\mathbb{Q}_2, \widetilde{\mathcal{S}}^1\} = -iL\mathcal{K}_{\overline{w}} - L\mathcal{R}^1{}_2, \tag{8.47c}$$

where we have used:

$$\{\mathbb{Q}_1, \mathbb{Q}_2\} = -\frac{2i}{L}\widehat{\mathcal{Z}}. \tag{8.48}$$

These generators $\widehat{L}_-$, $\widehat{L}_0$ and $\widehat{L}_+$ generate an $su(2)_R$ twisted $\overline{sl}(2)$, henceforth denoted by $\widehat{sl}(2)$. Their commutation relations are:

$$[\widehat{L}_+, \widehat{L}_-] = 2\widehat{L}_0, \qquad [\widehat{L}_0, \widehat{L}_\pm] = \mp\widehat{L}_\pm. \tag{8.49}$$

The action of these generators on $su(2)_R$ singlets living on the conformal boundary ($r \to 0$) is:

$$\widehat{L}_- = -\partial_{\overline{w}}, \qquad \widehat{L}_0 = -\overline{w}^2 \partial_{\overline{w}}, \qquad \widehat{L}_+ = -\overline{w}\partial_{\overline{w}}, \tag{8.50}$$

as in the case of a two-dimensional conformal field theory (8.1). This is very similar to the $su(2)_R$ twist performed in [Beem, Lemos, et al., 2015] to get the Schur operators. In terms of the generators we have introduced above, the Schur operators are those $su(2)_R$ highest-weight states that fulfill the additional conditions at the origin of the conformal boundary of H^3:

$$[\widehat{L}_0, \mathcal{O}(0)] = 0, \qquad [\widehat{\mathcal{Z}}, \mathcal{O}(0)] = 0. \tag{8.51}$$

These commutation relations give rise to the condition (8.10) written in terms of their eigenvalues.

8.6 Boundary localization

In this section, we will follow a localization technique previously employed in [Dedushenko et al., 2018], [Bonetti & Rastelli, 2018] and [Pan & Peelaers, 2019] to extract a chiral algebra at what would correspond to the conformal boundary for ∂H^3 for our setup. To illustrate the procedure, we will outline how it works for the case of a hypermultiplet. Consider a hypermultiplet scalar Φ, which we separate in three parts:

$$\Phi = \Phi_0 + \widehat{\Phi} + \delta\Phi, \tag{8.52}$$

where $\Phi_0, \widehat{\Phi}$ comprise the BPS part and $\delta\Phi$ represents the fluctuations around it. The difference between Φ_0 and $\widehat{\Phi}$ is that the former satisfies the equations of motion, while the latter does not. The fluctuations $\delta\Phi$ also do not satisfy the equations of motion. Replacing equation (8.52) in the path integral partition function, we get the following schematic expression:

$$Z_{\text{HM}}[\Phi_0, \mathfrak{m}] = \exp\left(-S^{\partial}_{\text{BPS}}[\Phi_0, \mathfrak{m}]\right) \int D\widehat{\Phi} \exp\left(-S^{\partial}_{\text{eff}}[\Phi_0, \mathfrak{m}]\right) \Delta'_{\text{1-loop}}[\mathfrak{m}], \tag{8.53}$$

where $\mathfrak{m}$ is a collection of moduli, $\Delta'_{\text{1-loop}}$ is the one-loop determinant of the fluctuations $\delta\Phi$, and the actions $S^{\partial}_{\text{BPS}}$ and $S^{\partial}_{\text{eff}}$ are given by:

$$\begin{aligned} S^{\partial}_{\text{BPS}} &= \int_{\partial H^3} d^2 z_1 d^2 z_2 \, \frac{\overline{\Phi}_0(z_1)\Phi_0(z_2)}{z_1 z_2} + \dots, \\ S^{\partial}_{\text{eff}} &= \int_{\partial H^3} d^2 z \, \overline{\widehat{\Phi}}(z) \partial_{\overline{z}} \widehat{\Phi}_0(z) + \dots . \end{aligned} \tag{8.54}$$

The one-loop determinant can be thought as an integration measure for the path integral, as in [Dedushenko et al., 2018]. The non-local boundary action

$S^{\partial}_{\text{BPS}}$ is found by the evaluation of the initial action using classical BPS solutions of boundary value Φ_0. The local boundary action $S^{\partial}_{\text{eff}}$ can be computed by replacing in the initial action the auxiliary field F as a function $\widehat{\Phi}$, obtained via the BPS equations. Note that the derivation of both boundary actions demands that we know the boundary behavior of the fields Φ_0 and $\widehat{\Phi}$. The field Φ_0 is non-normalizable, thus inducing a divergence which needs to be treated using the well-known prescriptions of holographic renormalization [Bianchi, Freedman, & Skenderis, 2002], [Skenderis, 2002], to get the finite action $S^{\partial}_{\text{BPS}}$. The derivation of the action $S^{\partial}_{\text{eff}}$ however does not entail such a complication, as $\widehat{\Phi}$ is normalizable.

In the rest of this section, we will examine a free hypermultiplet and an Abelian vector multiplet, but first we will investigate the scaling behavior of various of the fields of these multiplets in anticipation of our work in Subsections 8.6.2 and 8.6.3.

8.6.1 Analysis of scaling behaviors of the fields

In preparation for the boundary localization of the hypermultiplet and the vector multiplet, we will study the scaling behavior of their fields.

Hypermultiplet

For the hypermultiplet, the BPS equations are $\delta\psi = \delta\widetilde{\psi} = 0$, which for the special nilpotent supercharges $\mathbb{Q}_1$ and $\mathbb{Q}_2$ become:

$$L\widehat{f} + r(\partial_r \mathcal{G} + 2r\partial_w \mathcal{F}) = 0, \qquad r\partial_r \mathcal{F} - 2\partial_{\overline{w}} \mathcal{G} - L f_0 = 0, \tag{8.55}$$

where L is the H^3 radius and we have introduced the following quantities:

$$\begin{aligned}
\mathcal{G} &= \frac{1}{r}\left(q^1 + \overline{w} q^2\right) = \frac{1}{r}\left(q_2 - \overline{w} q_1\right), \\
\mathcal{F} &= \frac{1}{r} q_1 = -\frac{1}{r} q^2, \\
\widehat{f} &= \frac{1}{r}\left(F^1 + \overline{w} F^2\right) = \frac{1}{r}\left(F_2 - \overline{w} F_1\right), \\
f_0 &= \frac{1}{r} F_1 = -\frac{1}{r} F^2.
\end{aligned} \tag{8.56}$$

Next, we diagonalize the BPS equations (8.55) without the presence of auxiliary fields:

$$r\partial_r\left(r^{-1}\partial_r \mathcal{G}\right) + 4\partial_{\overline{w}}\partial_w \mathcal{G} = 0, \qquad r^{-1}\partial_r\left(r\partial_r \mathcal{F}\right) + 4\partial_{\overline{w}}\partial_w \mathcal{F} = 0. \tag{8.57}$$

To solve these partial differential equations, we can use a Fourier transformation for $\mathcal{F}$ and $\mathcal{G}$ for the coordinates x and y:

$$
\begin{aligned}
\mathcal{F} &= \int \frac{d^2p}{(2\pi)^2} e^{i(p_x x + p_y y)} \widetilde{\mathcal{F}}(p_x, p_y, r), \\
\mathcal{G} &= \int \frac{d^2p}{(2\pi)^2} e^{i(p_x x + p_y y)} \widetilde{\mathcal{G}}(p_x, p_y, r).
\end{aligned}
\tag{8.58}
$$

Demanding that $\mathcal{F}$ and $\mathcal{G}$ solve the BPS equations (8.57), and that they remain finite in the bulk of H^3, we find:

$$
\begin{aligned}
\widetilde{\mathcal{F}}(p_x, p_y, r) &= K_0(r|p|) \widetilde{\mathcal{F}}^{(0)}(p_x, p_y), \\
\widetilde{\mathcal{G}}(p_x, p_y, r) &= \frac{r(ip_x + p_y)}{|p|} K_1(r|p|) \widetilde{\mathcal{F}}^{(0)}(p_x, p_y),
\end{aligned}
\tag{8.59}
$$

where $K_n(z)$ are Bessel functions of the second kind. Here we also introduced the modulus of p: $|p| = \sqrt{p_x^2 + p_y^2}$. We can translate these results for the Fourier transforms $\widetilde{q}_1$ and $\widetilde{q}_2$ of the hypermultiplet scalars q_1 and q_2, to get:

$$
\begin{aligned}
\widetilde{q}_1(p_x, p_y, r) &= rK_0(r|p|) \widetilde{\mathcal{F}}^{(0)}(p_x, p_y), \\
\widetilde{q}_2(p_x, p_y, r) &= rK_0(r|p|) \left(i\partial_{p_x} + \partial_{p_y} \right) \widetilde{\mathcal{F}}^{(0)}(p_x, p_y).
\end{aligned}
\tag{8.60}
$$

Thus, the boundary behavior of our fields we will be:

$$
\begin{aligned}
&\lim_{r\to 0} \widetilde{q}_1 \to r \log r \widetilde{\mathcal{F}}^{(0)}(p_x, p_y), \\
&\lim_{r\to 0} \widetilde{q}_2 \to r \log r \left(i\partial_{p_x} + \partial_{p_y} \right) \widetilde{\mathcal{F}}^{(0)}(p_x, p_y), \\
&\lim_{r\to 0} (\widetilde{q}^1 + \overline{w} q^2) \to r \lim_{r\to 0} \widetilde{\mathcal{G}}(p_x, p_y, r), \\
&\lim_{r\to 0} \widetilde{\mathcal{G}} \to \left[\frac{i}{p_x + ip_y} + \frac{r^2}{2}(ip_x + p_y) \left(\log r + \gamma \right.\right. \\
&\qquad\qquad \left.\left. - \frac{1}{2} + \log \frac{|p|}{4} \right) + \ldots \right] \widetilde{\mathcal{F}}^{(0)}(p_x, p_y).
\end{aligned}
\tag{8.61}
$$

Vector multiplet

The procedure for the vector multiplet moves in a manner analogous to that of the hypermultiplet. First, let's begin by examining the following BPS equations:

$$
\delta D^{IJ} = 0. \tag{8.62}
$$

In momentum space, these equations are satisfied by the following gaugino expressions:

$$\begin{aligned}
\lambda_+^I(p_x,p_y,r) &= r^{3/2}(ip_x+p_y)K_0(r|p|)\lambda^{(0)\,I}(p_x,p_y),\\
\lambda_-^I(p_x,p_y,r) &= r^{3/2}|p|K_1(r|p|)\lambda^{(0)\,I}(p_x,p_y),\\
\widetilde{\lambda}_I^{\dot{+}}(p_x,p_y,r) &= r^{3/2}(ip_x+p_y)K_0(r|p|)\overline{\lambda}_I^{(0)}(p_x,p_y),\\
\widetilde{\lambda}_I^{\dot{-}}(p_x,p_y,r) &= r^{3/2}|p|K_1(r|p|)\overline{\lambda}_I^{(0)}(p_x,p_y),
\end{aligned} \tag{8.63}$$

where $\lambda^{(0)\,I}(p_x,p_y)$ and $\overline{\lambda}^{(0)\,I}(p_x,p_y)$ are the gaugino boundary configurations. The rest of the BPS equations:

$$\delta X = 0, \qquad (\delta+\delta_{\mathrm{BRST}})a_\mu = 0 \tag{8.64}$$

constrain further equations (8.63). By imposing them, we get for λ:

$$\begin{aligned}
[\mathbb{Q}_2,a_x] &= -\frac{iL}{r^{3/2}\sqrt{2}}\left(\lambda_+^1+\overline{w}\lambda_+^2+r\lambda_-^2\right) = -\partial_X c(x,y,r),\\
[\mathbb{Q}_2,a_y] &= \frac{L}{r^{3/2}\sqrt{2}}\left(\lambda_+^1+\overline{w}\lambda_+^2-r\lambda_-^2\right) = -\partial_y c(x,y,r),\\
[\mathbb{Q}_2,a_r] &= \frac{iL}{r^{3/2}\sqrt{2}}\left(\lambda_-^1+\overline{w}\lambda_-^2-r\lambda_+^2\right) = -\partial_r c(x,y,r),\\
[\mathbb{Q}_2,a_\theta] &= [\mathbb{Q}_1,X] = 0 \qquad \to \qquad \lambda_-^1+\overline{w}\lambda_-^2+r\lambda_+^2 = 0,
\end{aligned} \tag{8.65}$$

where $c(x,y,r)$ is the bulk ghost field, with analogous relations holding for $\widetilde{\lambda}$. We can also use the BPS equations to get an expression for the Fourier transform $\widetilde{c}(p_x,p_y,r)$ of the ghost field $c(x,y,r)$:

$$\widetilde{c}(p_x,p_y,r) = r\frac{(p_x-ip_y)}{|p|}K_1(r|p|)\lambda^{(0)\,2}(p_x,p_y). \tag{8.66}$$

8.6.2 Hypermultiplet boundary localization

We will begin by considering the case of the free hypermultiplet. We are going to evaluate its Lagrangian on the BPS locus. For the free hypermultiplet, the Lagrangian (8.27) reduces to:

$$\mathcal{L}_H = -\mathcal{D}^\mu q^{nI}\mathcal{D}_\mu q_{nI} + \frac{R}{6}q^{nI}q_{nI} - F^{nI}F_{nI} + \text{fermions}. \tag{8.67}$$

We now need to express F_{nI} as a function of q_{nI} using the BPS equations $\delta\psi = \delta\widetilde{\psi} = 0$. To do so, we contract the BPS equations with the Killing spinors $\zeta_I, \widetilde{\chi}^I$:

$$2i\mathcal{L}_K q^{nI} - i\Theta^I{}_J q^{nJ} = 0, \tag{8.68}$$

as well as with the auxiliary Killing spinors $\check{\zeta}_I, \check{\widetilde{\chi}}^I$, which give us:

$$\begin{aligned} F_{nJ} &= -\frac{1}{\widetilde{s}}\left[2\mathcal{D}_\mu q_{nI}\check{\zeta}_J\sigma^\mu\widetilde{\chi}^I + q_{nI}\check{\zeta}_J\sigma^\mu\left(\mathcal{D}_\mu + \frac{i}{2}A_\mu\right)\widetilde{\chi}^I\right], \\ F_{nJ} &= \frac{1}{s}\left[2\mathcal{D}_\mu q_{nI}\check{\widetilde{\chi}}_J\widetilde{\sigma}^\mu\zeta^I + q_{nI}\check{\widetilde{\chi}}_J\widetilde{\sigma}^\mu\left(\mathcal{D}_\mu - \frac{i}{2}A_\mu\right)\zeta^I\right]. \end{aligned} \tag{8.69}$$

If we concentrate on the special supercharges $\mathbb{Q}_i$, the Killing vector K^μ becomes parallel to A^μ, implying $\Theta^{IJ} = 0$. Then:

$$\{\mathbb{Q}_1, \mathbb{Q}_2\}q_{nI} = 0 \qquad \Longleftrightarrow \qquad \mathcal{L}_K q_{nI} \propto \partial_\theta q_{nI} = 0, \tag{8.70}$$

so the hypermultiplet scalars q_{nI} do not depend on the S^1 coordinate on the BPS locus. Using equations (8.69), $F^{nI}F_{nI}$, evaluated on the BPS locus, becomes:

$$(F_{nI}F^{nI})\big|_{\text{BPS}} = -\mathcal{D}^\mu q^{nI}\mathcal{D}_\mu q_{nI} + \frac{R}{6}q^{nI}q_{nI} + 4\nabla_\mu\left[\frac{1}{\widetilde{s}}(\widetilde{\chi}_I\widetilde{\sigma}^{\mu\nu}\widetilde{\chi}_J)q_n{}^I\mathcal{D}_\nu q^{nJ}\right]. \tag{8.71}$$

We can now replace this expression to the free hypermultiplet action (8.67) to obtain the action on the BPS locus:

$$\mathcal{S}_H^{\text{BPS}} = 2\pi\beta L\int_{\partial H^3} d^2x\left[\frac{4}{\widetilde{s}}\sqrt{g}(\widetilde{\chi}_I\widetilde{\sigma}^{\perp\widehat{\mu}}\widetilde{\chi}_J)q_n{}^I\mathcal{D}_{\widehat{\mu}}q^{nJ}\right]_{\partial H^3}, \tag{8.72}$$

where $\widehat{\mu} \neq \perp, \theta$ denote the boundary coordinates. We would like to rewrite this expression for the action in a cleaner form. First, we will use the identity:

$$\widetilde{\chi}_I\widetilde{\sigma}^{\mu\nu}\nabla_\nu\widetilde{\chi}_J = iA_\nu\widetilde{\chi}_I\widetilde{\sigma}^{\mu\nu}\widetilde{\chi}_J + \frac{3i}{8}\widetilde{s}A^\mu\varepsilon_{IJ}, \tag{8.73}$$

which can be obtained by contracting one of the Killing spinor equations (8.19) with $\widetilde{\chi}_I\widetilde{\sigma}^{\mu\nu}$, as well as the fact that $V^\perp = 0$. We will also define the following spinor field living in the fundamental representation of $Sp(k)$:

$$\mathfrak{q}^{n\dot{\alpha}} = -\widetilde{\chi}^{I\dot{a}}q^n{}_I = \frac{1}{\sqrt{2}}\begin{pmatrix} r^{-1/2}\mathcal{Q}^n \\ -r^{-1/2}q^{n\,2}\end{pmatrix}^{\dot{\alpha}}, \tag{8.74}$$

where

$$\mathcal{Q}^n = u_I(\overline{w})q^{nI} = q^{n\,1} + \overline{w}q^{n\,2}. \tag{8.75}$$

Now our action becomes:

$$\mathcal{S}_H^{\text{BPS}} = -8\pi\beta L^4\int_{\partial H^3} d^2x\left[r^{-3}\mathfrak{q}_n\widetilde{\sigma}^{\perp\widehat{\mu}}\mathcal{D}_{\widehat{\mu}}\mathfrak{q}^n\right]_{r=0}. \tag{8.76}$$

Notice that the field $\mathcal{Q}^n$ has exactly the form of the twisted-translated Schur operators (8.16) of [Beem, Lemos, et al., 2015] and that it is a BPS operator with respect to both of our special supercharges $\mathbb{Q}_i$. This suggests that we should investigate the behavior of this field on the conformal boundary to establish whether $\mathcal{Q}^n$ is indeed a Schur operator there. To that end, we employ the results for the asymptotic behavior of the fields found in (8.61), which lead to the following boundary BPS action:

$$\mathcal{S}_H^{\text{BPS CT}} = 2\pi\beta L^2 \int_{\partial H^3} d^2w \Big[2\mathcal{G}^{(0)n}(w,\overline{w})\partial_{\overline{w}}\mathcal{G}_n^{(0)}(w,\overline{w}) \\ + \log r\mathcal{G}^{(0)n}(w,\overline{w})\mathcal{F}_n^{(0)}(w,\overline{w})\Big]_{r=0}, \quad (8.77)$$

where $\mathcal{G}^{(0)n} = \lim_{r\to 0}\mathcal{G}^n$. The part that contains $\mathcal{F}_n^{(0)}$ in the expression above is diverging and needs to be subtracted using appropriate (boundary) counterterms:

$$\mathcal{S}_H^{\text{CT}} = -2\pi\beta L^2 \int_{\partial H^3} dw d\overline{w}\left[r^{-2}q_{nI}q^{nI}\right], \quad (8.78)$$

resulting in the regularized action $\widehat{S}_H$:

$$\widehat{\mathcal{S}}_H^{\text{BPS}} = \mathcal{S}_H^{\text{BPS}} + \mathcal{S}_H^{\text{BPS CT}}. \quad (8.79)$$

Doing so, we finally get:

$$\widehat{\mathcal{S}}_H^{\text{BPS}}[\widehat{\mathcal{Q}}] = g_H \int_{\partial H^3} dw d\overline{w}\, \widehat{\mathcal{Q}}^n(w,\overline{w})\partial_{\overline{w}}\widehat{\mathcal{Q}}_n(w,\overline{w}), \quad (8.80)$$

where $\widehat{\mathcal{Q}}^n = \mathcal{G}^{(0)\,n}$ and $g_H = 4\pi\beta L^2$. We recognize this action as the action of a symplectic boson, as was observed for the case of the hypermultiplet in [Beem, Lemos, et al., 2015] and [Pan & Peelaers, 2018].

Using the action $\widehat{\mathcal{S}}_H^{\text{BPS}}$, it is now straightforward to compute two-point correlation functions for the operators $\widehat{\mathcal{Q}}^n$:

$$\widehat{\mathcal{Q}}_m(z,\overline{z})\widehat{\mathcal{Q}}_n(w,\overline{w}) \sim \frac{1}{\pi g_H}\frac{\Omega_{mn}}{z-w}, \quad (8.81)$$

where Ω_{mn} is the $Sp(k)$ invariant form that satisfies $\Omega^{mn'}\Omega_{n'n} = \delta^m{}_n$. We observe that indeed the two-point function is meromorphic, as expected when dealing with a chiral algebra.

We will now proceed to utilize the BPS boundary action $\widehat{\mathcal{S}}_H^{\text{BPS}}$ (8.80), to compute operator product expansions for the energy-momentum tensor and the global symmetry currents and extract valuable information for the boundary theory.

Energy-momentum tensor

Let us now consider the canonical energy-momentum tensor corresponding to the boundary BPS action $\widehat{\mathcal{S}}_H^{\text{BPS}}$, which we will denote by $\vartheta_{\widehat{\mu}\widehat{\nu}}$. Its components are:

$$\vartheta_{\widehat{\mu}\widehat{\nu}}(w,\overline{w}) = \frac{g_H}{2}\begin{pmatrix} \widehat{\mathcal{Q}}^n(w,\overline{w})\,\partial_w\widehat{\mathcal{Q}}^n(w,\overline{w}) & 0 \\ -\widehat{\mathcal{Q}}^n(w,\overline{w})\,\partial_{\overline{w}}\widehat{\mathcal{Q}}_n(w,\overline{w}) & 0 \end{pmatrix}. \tag{8.82}$$

Note that these expressions are to be understood as normal ordered. All components of $\vartheta_{\widehat{\mu}\widehat{\nu}}$ are equal to zero on shell. Now, we can build a traceless and symmetric energy-momentum tensor $T(w,\overline{w})$ out of $\vartheta_{\widehat{\mu}\widehat{\nu}}(w,\overline{w})$, which will be equal to:

$$T(w,\overline{w}) = \frac{g_H}{2}\widehat{\mathcal{Q}}^n(w,\overline{w})\,\partial_w\widehat{\mathcal{Q}}^n(w,\overline{w}). \tag{8.83}$$

We can compute operator product expansions involving the energy-momentum tensor $T(w,\overline{w})$. To make the expressions cleaner, we will omit factors of πg_H, which can be easily restored via dimensional analysis. Doing so, we get:

$$T(z)\widehat{\mathcal{Q}}_n(w) \sim \frac{(1/2)\widehat{\mathcal{Q}}_n(w)}{(z-w)^2} + \frac{\partial_w\widehat{\mathcal{Q}}_n(w)}{z-w} \tag{8.84}$$

and

$$T(z)T(w) \sim \frac{-k/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w}. \tag{8.85}$$

We can now read off the two-dimensional theory central charge $c_{2d} = -k$ and the conformal dimension $h = 1/2$ of the operator $\widehat{\mathcal{Q}}_n$. As is the case in [Beem, Lemos, et al., 2015], the two-dimensional theory has a negative central charge and is thus non-unitary.

Currents

Consider the following current:

$$J_{mn}(w,\overline{w}) = \widehat{\mathcal{Q}}_m(w,\overline{w})\widehat{\mathcal{Q}}_n(w,\overline{w}). \tag{8.86}$$

We can once again compute operator product expansions involving these currents:

$$J_{mn}(z,\overline{z})\widehat{\mathcal{Q}}_k(w,\overline{w}) \sim \frac{1}{z-w}\Big(\Omega_{mk}\widehat{\mathcal{Q}}_n(w,\overline{w}) + \Omega_{nk}\widehat{\mathcal{Q}}_m(w,\overline{w})\Big) \tag{8.87}$$

and finally:

$$\begin{aligned} J_{kl}(z,\overline{z})J_{mn}(w,\overline{w}) \sim & \frac{1}{(z-w)^2}\left(\Omega_{kn}\Omega_{lm} + \Omega_{km}\Omega_{nl}\right) \\ & + \frac{1}{z-w}\left(\Omega_{ln}J_{km} + \Omega_{km}J_{ln} + \Omega_{kn}J_{lm} + \Omega_{lm}J_{kn}\right). \end{aligned} \tag{8.88}$$

The presence of a second order pole implies that $J_{mn}(w)$ is an affine Lie algebra current with a central extension. The generators of this algebra are the coefficients of a Laurent expansion of $J_{mn}(w)$.

8.6.3 Vector multiplet boundary localization

We now turn to the case of an Abelian vector multiplet. The procedure we will follow will be very similar to the one for the free hypermultiplet explained in the previous section. We begin by contracting the BPS equations $\delta\lambda^I = 0$ and $\delta\widetilde{\lambda}^I = 0$ with the Killing spinors and the auxiliary Killing spinors, to get:

$$\mathcal{L}_K X + gs\left[X, \widetilde{X}\right] = 0, \qquad \mathcal{L}_K \widetilde{X} + g\widetilde{s}\left[X, \widetilde{X}\right] = 0, \tag{8.89}$$

and

$$\begin{aligned} D_{IJ} &= \frac{2}{s}\left[\zeta_{(I}\sigma^{\mu\nu}\zeta_{J)}F_{\mu\nu} - 2i\widetilde{\chi}_{(I}\widetilde{\sigma}^{\mu}\zeta_{J)}\mathcal{D}_\mu X\right], \\ D^{IJ} &= -\frac{2}{\widetilde{s}}\left[\widetilde{\chi}^{(I}\widetilde{\sigma}^{\mu\nu}\zeta^{J)}F_{\mu\nu} + 2i\widetilde{\chi}^{(I}\widetilde{\sigma}^{\mu}\zeta^{J)}\mathcal{D}_\mu \widetilde{X}\right]. \end{aligned} \tag{8.90}$$

Then $D^{IJ}D_{IJ}$ on the BPS locus becomes:

$$\begin{aligned} D^{IJ}D_{IJ}\big|_{\text{BPS}} = {} & F_{\mu\nu}F^{\mu\nu} + 8\mathcal{D}_\mu\widetilde{X}\,\mathcal{D}^\mu X + 8g^2\left[X, \widetilde{X}\right]^2 \\ & + 8\left(A_\mu A^\mu - R/6\right)\widetilde{X}X \\ & + \frac{8i}{s\widetilde{s}}A^\mu\left(s\widetilde{X} + \widetilde{s}X\right)\mathcal{D}_\mu\left(\widetilde{s}X - s\widetilde{X}\right) \\ & - \frac{2i}{s\widetilde{s}}\nabla_\rho\left[\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}K_\sigma\left(s\widetilde{X} + \widetilde{s}X\right)\right]. \end{aligned} \tag{8.91}$$

Replacing this expression in the bosonic part of the vector multiplet action, we get:

$$\mathcal{S}^{\text{BPS}}_{V,\text{bos}} = -4\pi i\beta L\int_{\partial H^3} d^2x\left[\sqrt{g}\varepsilon^{r\widehat{\mu}\widehat{\nu}\theta}F_{\widehat{\mu}\widehat{\nu}}K_\theta\left(s\widetilde{X} + \widetilde{s}X\right)\right]_{r=0}. \tag{8.92}$$

Using $\sqrt{g} = L^4\beta/r^3$, $K_\theta = iL\beta$ and the boundary behavior of the fields $F_{\widehat{\mu}\widehat{\nu}}, X, \widetilde{X}$:

$$\begin{aligned} \varepsilon^{r\widehat{\mu}\widehat{\nu}\theta}F_{\widehat{\mu}\widehat{\nu}} &\to \frac{r^2}{L^2\beta}\varepsilon^{ij}\widehat{F}_{ij}, \\ \left(s\widetilde{X} + \widetilde{s}X\right) &\to r^2\left(s\widehat{\widetilde{X}} + \widetilde{s}\widehat{X}\right), \end{aligned} \tag{8.93}$$

where i, j are flat boundary indices ranging in $i = 1, 2$, we get the action:

$$\mathcal{S}^{\text{BPS}}_{V,\text{bos}} = 4\pi L^4\beta^2\lim_{r\to 0} r\int_{\partial H^3} d^2x\,\varepsilon^{ij}\widehat{F}_{ij}\left(s\widehat{\widetilde{X}} + \widetilde{s}\widehat{X}\right) = 0. \tag{8.94}$$

Thus, the bosonic part of the BPS vector multiplet action vanishes on the conformal boundary. Next, let us investigate its fermionic counterpart $\mathcal{S}^{\text{BPS}}_{V,\text{ferm}}$. In order to avoid having a trivial BPS action, we will not consider the usual BPS locus, where all fermions vanish. We will also introduce the ghost fields $c, \widetilde{c}$ and b, as well as the bosonic and fermionic zero modes denoted by $(a_0, \widetilde{a}_0, b_0; c_0, \widetilde{c}_0)$. We also define the following cohomological fields:

$$\begin{aligned}
\sigma &= -2iK^\mu a_\mu + 2\left(\widetilde{s}X - s\widetilde{X}\right), \\
\Lambda_\mu &= i\zeta_I\sigma_\mu\widetilde{\lambda}^I + i\widetilde{\chi}^I\widetilde{\sigma}_\mu\lambda_I, \\
\Sigma_{IJ} &= \frac{1}{s\widetilde{s}}\left(s\widetilde{\chi}_{(I}\widetilde{\lambda}_{J)} - \widetilde{s}\zeta_{(I}\lambda_{J)}\right), \\
\Delta_{IJ} &= D_{IJ} - 2i\,Y^\mu_{IJ}\mathcal{D}_\mu\left(s\widetilde{X} + \widetilde{s}X\right) + 2s\widetilde{s}\,Y^\mu{}_{K(I}Y^\nu{}_{J)}{}^K F_{\mu\nu},
\end{aligned} \tag{8.95}$$

where $Y^\mu{}_{IJ}$ is a Killing spinor bilinear defined as:

$$Y^\mu{}_{IJ} = \frac{1}{s\widetilde{s}}\widetilde{\chi}_{(I}\widetilde{\sigma}^\mu\zeta_{J)}. \tag{8.96}$$

These fields form the cohomological complex described below:

$$\begin{aligned}
&\widehat{\delta}a = \Lambda + dc, && \widehat{\delta}\Lambda = 2i\mathcal{L}_K a + \partial_\mu\sigma, && \widehat{\delta}^2\Lambda = 2i\mathcal{L}_K\Lambda, \\
&\widehat{\delta}X = \Lambda_0, && \widehat{\delta}\Lambda_0 = 2i\mathcal{L}_K X, && \widehat{\delta}^2\Lambda_0 = 2i\mathcal{L}_K\Lambda_0, \\
&\widehat{\delta}\widetilde{X} = \widetilde{\Lambda}_0, && \widehat{\delta}\,\widetilde{\Lambda}_0 = 2i\mathcal{L}_K\widetilde{X}, && \widehat{\delta}^2\widetilde{\Lambda}_0 = 2i\mathcal{L}_K\widetilde{\Lambda}_0, \\
&\widehat{\delta}\Sigma_{IJ} = \Delta_{IJ}, && \widehat{\delta}\sigma = -2i\mathcal{L}_K c, && \widehat{\delta}^2\Delta_{IJ} = 2i\mathcal{L}_K\Sigma_{IJ},
\end{aligned} \tag{8.97}$$

and for the ghost fields and the zero modes:

$$\begin{aligned}
&\widehat{\delta}\sigma = -\sigma + a_0, && \widehat{\delta}a_0 = 0, && \widehat{\delta}\,\widetilde{c} = b, && \widehat{\delta}b = 2i\mathcal{L}_K b, \\
&\widehat{\delta}\,\widetilde{a}_0 = \widetilde{c}_0, && \widehat{\delta}\,\widetilde{c}_0 = 0, && \widehat{\delta}b_0 = c_0, && \widehat{\delta}c_0 = 0.
\end{aligned} \tag{8.98}$$

In the expressions above, $\widehat{\delta}$ is a sum of the usual supersymmetry variations δ and the BRST variations δ_{BRST}.

Now, the fermionic part of the vector multiplet action on the BPS locus will be:

$$\mathcal{S}^{\mathrm{BPS/BRST}}_{\text{V, ferm}} = 4\pi\beta L\int_{H^3} d^3x\sqrt{g}\,\nabla_\mu\left[\widetilde{s}sY^{\mu\,J}{}_K Y^{\nu\,IK}\Sigma_{IJ}\partial_\nu c\right]. \tag{8.99}$$

Note that the presence of $\partial_\mu c$ in the action originates from the equation $\widehat{\delta}a = 0$ which is equivalent to $\Lambda_\mu = -\partial_\mu c$. The action stated above is the integral of a total derivative on H^3, which corresponds to an action on the conformal boundary ∂H^3. To obtain an explicit expression for this action, we need to evaluate Y^μ_{IJ} for our special supercharges $\mathbb{Q}_i$ and use the boundary scaling behavior for the fields Σ_{IJ} and c. The boundary scaling of Σ_{IJ} descends from that of λ^I and $\widetilde{\lambda}^I$ in equation (8.63), while for the ghost c we can deduce it by noticing that its kinetic term is identical to that of a massless scalar, yielding:

$$\lim_{r\to 0} c(w,\overline{w},r) \to -r\log r\, c_0(w,\overline{w}) + r\widehat{c}(w,\overline{w}). \tag{8.100}$$

Finally, the convergent part of the vector multiplet action evaluated on the BPS locus involving only fluctuations is:

$$\mathcal{S}^{\text{BPS/BRST}}_{\text{V, ferm}} = g_V \int_{\partial H^3} d^2w \left(\widehat{\Lambda}(w,\overline{w}) - \widehat{\widetilde{\Lambda}}(w,\overline{w})\right) \partial_{\overline{w}}\, \widehat{c}(w,\overline{w}), \tag{8.101}$$

where $g_V = -\sqrt{2}\pi\beta^2 L^3$ and $\widehat{\Lambda}(w,\overline{w})$, $\widehat{\widetilde{\Lambda}}(w,\overline{w})$ are given by:

$$\widehat{\Lambda}(w,\overline{w}) = \widehat{\lambda}^1_+ + \overline{w}\widehat{\lambda}^2_+, \qquad \widehat{\widetilde{\Lambda}}(w,\overline{w}) = \widehat{\widetilde{\lambda}}^{1\dot{+}} + \overline{w}\widehat{\widetilde{\lambda}}^{2\dot{+}}. \tag{8.102}$$

Now, if we set:

$$b(w,\overline{w}) \coloneqq \widehat{\Lambda}(w,\overline{w}) - \widehat{\widetilde{\Lambda}}(w,\overline{w}), \tag{8.103}$$

we observe that the action (8.101) coincides with the (b,c) ghost system action. This is in agreement with the results of [Beem, Lemos, et al., 2015] for the case of a free vector multiplet. Using the action $\mathcal{S}^{\text{BPS/BRST}}_{\text{V, ferm}}$, We can now compute the usual operator product expansions, such as (we omit factors of πg_V):

$$\begin{aligned} b(z,\overline{z})c(w,\overline{w}) &\sim \frac{1}{z-w}, \\ b(z,\overline{z})\partial c(w,\overline{w}) &\sim \frac{1}{(z-w)^2}, \end{aligned} \tag{8.104}$$

which turn out to be meromorphic as expected.

8.7 Line operators

In this section we examine the possibility of including non-local operators to our setup, for both the case of the hypermultiplet and vector multiplet.

8.7.1 Hypermultiplet

For the case of the hypermultiplet, a line operator will have the form:

$$W = \mathrm{Tr}_R\, \mathcal{P} \exp \oint_\gamma d\theta\, t_{nI} q^{nI}, \tag{8.105}$$

where the loop γ is taken to be on the boundary and along the S^1 direction. For the loop to preserve supersymmetry, the following condition must be met:

$$\mathbb{Q}\left(t_{nI}q^{nI}\right) = \psi^n(t_{nI}\zeta^I) + \widetilde{\psi}^n(t_{nI}\widetilde{\chi}^I) = 0. \tag{8.106}$$

However, this implies that t_{nI} must be vanishing and so there can not be such non-local operators in the $\mathbb{Q}_i$ cohomology.

8.7.2 Vector multiplet

For the case of the vector multiplet, a Wilson loop transforming in a representation R of the gauge group should be of the following form:

$$W = \mathrm{Tr}_R \, \mathcal{P} \exp \oint_\gamma dt \left[a_\mu \dot{x}^\mu + |\dot{x}| \left(nX + \widetilde{n}\widetilde{X} \right) \right], \tag{8.107}$$

where γ is a closed loop. Let's consider the most general case where this Wilson loop is invariant under a general linear combination of the special supercharges $\mathbb{Q}_i$, that is $\mathbb{Q} = C_1\mathbb{Q}_1 + C_2\mathbb{Q}_2$. By demanding that the Wilson loop is invariant under the supercharge $\mathbb{Q}$, we get:

$$\mathbb{Q}\left[a_\mu \dot{x}^\mu + |\dot{x}| \left(nX + \widetilde{n}\widetilde{X} \right) \right] = \Lambda_\mu \dot{x}^\mu + |\dot{x}| \left(n\Lambda_0 + \widetilde{n}\widetilde{\Lambda}_0 \right) = 0, \tag{8.108}$$

where we have assumed no gauge fixing. We can now expand Λ_μ along K_μ and ${Y_\mu}^{IJ}$ (one can show that they form a full basis):

$$\Lambda_\mu = \Lambda K_\mu + \Lambda_{IJ} {Y_\mu}^{IJ}, \tag{8.109}$$

where Λ is such that it satisfies the condition $iK^\mu \Lambda_\mu = \widetilde{s}\Lambda_0 - s\widetilde{\Lambda}_0$, and hence is given by the equation:

$$\Lambda = i \left(\frac{\Lambda_0}{s} - \frac{\widetilde{\Lambda}_0}{\widetilde{s}} \right). \tag{8.110}$$

Then, the variation equation (8.108) becomes:

$$\Lambda_{IJ} {Y_\mu}^{IJ} \dot{x}^\mu + \Lambda_0 \left(\frac{i}{s} K_\mu \dot{x}^\mu + |\dot{x}| n \right) + \widetilde{\Lambda}_0 \left(-\frac{i}{\widetilde{s}} K_\mu \dot{x}^\mu + \widetilde{n}|\dot{x}| \right) = 0. \tag{8.111}$$

This equation implies that:

$${Y_\mu}^{IJ} \dot{x}^\mu = 0, \qquad \frac{i}{s} K_\mu \dot{x}^\mu + |\dot{x}| n = 0, \qquad \frac{i}{\widetilde{s}} K_\mu \dot{x}^\mu - |\dot{x}| \widetilde{n} = 0. \tag{8.112}$$

Thus, the Wilson loop should be on the S^1 direction and $n, \widetilde{n}$ should be given by:

$$n = -\frac{i}{s} K_\theta, \qquad \widetilde{n} = \frac{i}{\widetilde{s}} K_\theta. \tag{8.113}$$

8.8 Discussion

In this chapter we reported partial results of a search for chiral algebras in $\mathcal{N} = 2$ supersymmetric field theory on $H^3 \times S^1$. What has been obtained so far seems encouraging and invites further investigation.

First of all, the technique of boundary localization developed in Subsection 8.6, depends crucially on solving BPS equations. For any boundary condition, there needs to exist a BPS solution that fills the interior uniquely and regularly. In the case of [Dedushenko et al., 2018], one can always find such a solution, corresponding to any boundary condition on a great circle of the sphere. For our case this turns out to be a subtle issue, and the answer depends on the choice of the auxiliary Killing spinors. We would like to show in some rigor that our BPS equations do possess regular, normalizable solutions on $H^3 \times S^1$, for some choice of auxiliary Killing spinors (and perhaps after the introduction of some other appropriate background field). We would also like to understand in more generality when one can find a non-singular solution to the BPS equations that fill the bulk, as a function of the boundary condition.

Furthermore, it would be very interesting to investigate the implications of the AdS/CFT correspondence for our setup. This investigation would not be without complications. For example, for the case of the hypermultiplet, the list of Schur operators includes currents, which for our case are localized on the boundary. However, according to the holographic dictionary, boundary currents correspond to propagating fields in the bulk. For the massless hypermultiplet there are no such propagating gauge fields in the bulk. The resolution of this issue is not clear at the moment.

Finally, we would like to apply our framework to non-free theories and eventually examine the possibility that a chiral algebra exists even for non-conformal theories.

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10. Svensk sammanfattning

Kvantfältteori, framförallt gaugeteori, har spelat en viktigt roll i vår förståelse av universum. Gaugeteori har varit en grundläggande ingrediens i beskrivningen av bland annat partikelfysikens standardmodell, fasta tillståndets fysik och kosmologi. Trots dess oerhörda framgång och nytta finns det fortfarande en lång lista med olösta problem av både matematisk och fysikalisk natur. Ett primärt exempel av ett sådant problem är beteendet hos en starkt kopplad gaugeteori. Denna regim är extremt svår att studera med befintliga analytiska- och störningstekniker utan att införa ytterligare antaganden.

En lovande väg mot ökad förståelse av starkt kopplade gaugeteorier, och ett flertal andra problem, är att studera supersymmetri. Supersymmetri är en rumtids-symmetri som relaterar fermioniska och bosoniska fält och är, enligt Haag–Lopuszanski–Sohnius teoremet, en unik utvidgning av Poincaresymmetri. Oavsett om supersymmetri återfinns i naturen, och då realiseras inom de energiskalor vi har tillgång till med våra begränsade experimentella resurser, förser den oss med sällsynta möjligheter att erhålla exakta resultat i komplicerade gaugeteorier.

En av de viktigaste och mest användbara teknikerna från supersymmetri är lokalisering. Lokalisering gör det möjligt att utföra exakta en-loops beräkningar även i komplicerade växelverkande teorier. I de mest gynnsamma fallen kan en oändligdimensionell vägintegral i den ursprungliga modellen reduceras till nolldimensionell fältteori, en så kallad matrismodell.

I denna doktorsavhandling applicerar vi lokaliseringstekniken i olika sammanhang med syfte att erhålla exakta resultat i supersymmetriska gaugeteorier. Vi undersöker också resulterande matrismodeller och olika aspekter av dessa teorier. Avhandlingen är uppdelad i två delar. I den första delen ger vi en kort genomgång av koncept samt tekniker som används i avhandlingen, nämligen lokalisering, matrismodeller och topologisk vridning. Vi fokuserar endast på grundläggande aspekter av dessa omfattande ämnen och begränsar diskussionen till vad vi behöver för våra ändamål.

I den andra delen presenterar vi de nya resultaten från vår forskning. Dessa är grupperade i fyra distinkta kapitel. Först diskuterar vi lokalisering av en samling av supersymmetriska gaugeteorier med åtta eller fyra superladdningar på en d-dimensionell sfär. Resultaten från denna diskussion håller i godtyckliga dimensioner i ett särskilt intervall. Vi bygger teorierna och beräknar därefter associerade en-loop determinanter. Till sist utför vi en analytisk fortsättning av våra resultat för en teori med fyra superladdningar på en fyrdimensionell sfär och gör en rimlighetsanalys.

Efter det fortsätter vi med en analys av matrismodeller som uppkommer i det tidigare nämnda arbetet; specifikt vid lokalisering av maximalt supersymmetrisk tvådimensionell Yang–Mills gaugeteori. Vi undersöker med analytiska samt numeriska metoder sadelpunktsekvationen och jämför våra resultat mot litteraturen där samma ekvation har analyserats men med annat förtecken på en parameter.

Vi flyttar sedan fokus till en fyrdimensionell $\mathcal{N} = 2$ supersymmetrisk gaugeteori med materia och generaliserar tidigare arbeten i litteraturen. Vi konstruerar teorin, kontrollerar att den är globalt definierad och vrider den. Slutligen lokaliserar vi den vridna teorin.

Till sist fokuserar vi på $\mathcal{N} = 2$ supersymmetriska gaugeteorier på en annan fyrdimensionell mångfald: en produkt av ett tredimensionellt hyperboliskt rum och en cirkel. Vårt syfte är att finna om oändligdimensionella kirala algebror kan existera i detta scenario. Efter att vi konstruerat teorierna och studerat symmetrierna använder vi en lokaliseringsteknik för att studera möjligheten till kirala algebror. Vi avslutar med att undersöka linjeoperatorer som passar in i ramverket.

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