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Supersymmetric Localization

A Journey from Seven to Three Dimensions

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Abstract

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Quantum Field Theory has been a dominating framework in elementary particle physics during the last century. Within this framework, supersymmetric theories have attracted a lot of attention due to their mathematical structure, simplicity and insight into the problems of unification, dark matter and hierarchy. Even though the boundaries of our perturbative understanding of supersymmetric theories have been pushed far, there is generically no systematic way to obtain exact results. Especially for strongly coupled theories, where perturbative techniques cannot be applied, methods for exact computations are crucial. A powerful technique to obtain exact results of partition functions and correlators of certain protected operators is *supersymmetric localization*. This thesis studies physical and geometrical properties of localization results in different supersymmetric theories.

The first model considered is maximally supersymmetric Yang-Mills placed on a 7-dimensional Sasaki-Einstein manifold. After redefining the fields to differential forms, the cohomological complex is formed and a localization computation of the partition function is performed using data from the moment map cone. This procedure and the factorization properties reveal a strong structural dependence of the result on the geometry. A second part discusses $N=4$ matter-multiplets in 3d. We identify BPS operators supported on a 1-dimensional submanifold. Applying the localization formula, the partition function simplifies to a partition function of a dual topological quantum mechanics. Lastly, we perform an *equivariant twisting* on $N=2$ gauge theories with matter on 4-dimensional manifolds with a torus action. This is achieved by a global field redefinition leading to differential forms or spinors that are defined on a large class of manifolds. The resulting cohomological theory admits two kinds of fixed points which are treated differently in a localization computation.

Keywords: Supersymmetry, Localization, Equivariant Cohomology, BPS Operators, Topological Field Theory, Equivariant Twisting

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To my family

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

- I K. Polydorou, A. Rocén and M. Zabzine, *7D supersymmetric Yang-Mills on curved manifolds*, JHEP **12** (2017) 152, arXiv:1710.09653 [hep-th].
- II R. Panerai, A. Pittelli, and K. Polydorou,, *Topological Correlators and Surface Defects from Equivariant Cohomology*, JHEP 09 (2020) 185, arXiv:2006.06692 [hep-th].
- III G. Festuccia, A. Gorantis, A. Pittelli, K. Polydorou, and L. Ruggeri, *Cohomological Localization of $\mathcal{N} = 2$ Gauge Theories with Matter*, JHEP 09 (2020) 133, arXiv:2005.12944 [hep-th].

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1. Introduction

Quantum field theory has been an important framework in modern particle physics during the last century. It provides fundamental tools for any theoretical physicist since it combines quantum mechanics, classical field theory, and special relativity. Its most successful realization is the so called Standard Model which is as yet the most accurate fundamental model of nature. An example is the prediction of the Higgs particle, that was experimentally confirmed using the Large Hadron Collider at CERN [1] 50 years after its existence was originally proposed in [2]. Despite the success of quantum field theory, it still poses serious challenges on many fronts. There is a number of open problems such as the existence of a mass gap for Yang-Mills theory [3] or the lack of a rigorous mathematical definition of path integrals, formally an infinite dimensional integral.

Independently of the explicit model under consideration, there exist two fundamentally different approaches for studying partition functions or correlators. A widely used technique is perturbation theory, building upon the idea of computing quantities order by order in a small parameter, for example the coupling constant. Obviously, this approach is problematic for strongly coupled theories, where the coupling constant is of order one, preventing the perturbative expansion from converging.

On the other hand, the calculation of certain observables in some models can be performed exactly. Most often symmetry arguments have been an important factor needed to obtain these results. One example is the category of theories that possess conformal symmetry (CFT). Correlation functions can be analytically or numerically *bootstrapped*, see for example [4]. The basic idea of Conformal Bootstrap is to use the conformal symmetry of the theory to constrain the operator product expansion (OPE) data, namely the anomalous dimensions and three point couplings. In turn, this data can be used to reconstruct correlation functions, even for theories lacking a Lagrangian formulation.

A different class of theories featuring exact results consists of theories enhanced with supersymmetry. Supersymmetry was originally introduced in [5, 6] in order to resolve problems in the Standard Model. One of them is the hierarchy problem which is based on the guiding principle of naturalness. The principle, formulated in [7], in short says that the dimensionless ratio between physical constants appearing in a physical theory should be of order one. Specifically, the 17 order of magnitude

difference between the Higgs, which defines the weak scale, and the Planck mass, the natural short-distance scale, would be considered unnatural.

In supersymmetric theories every particle has a partner of opposite statistics. I.e. supersymmetry relates two superpartners, a boson and a fermion. It is important to note that there is no experimental indication of the existence of superpartners for Standard Model particles. However, this can be explained by supersymmetry being *spontaneously broken* in a higher than the currently reachable energy band. This furthermore provides an appealing argument to solve the hierarchy problem. As Witten pointed out in [8], if supersymmetry is broken by small non-perturbative effects (dynamically broken), the scale of the supersymmetry breaking is naturally much smaller than the Planck mass.

An invaluable computational tool for supersymmetric quantum field theories is called *supersymmetric localization*, a particular case of *equivariant localization*. The latter was first introduced by Atiyah-Bott [9] and Berline-Vergne [10] in independent studies. They proved that for certain integrals the saddle point approximation provides the exact result. Supersymmetric localization uses the same technique to compute path integrals in supersymmetric quantum field theories.

The first application of localization was presented by Witten in the contexts of supersymmetric quantum mechanics [11] and later in the context of topological field theories [12]. An important development was put forward by Nekrasov using localization to compute the instanton partition function on the Omega background in [13], a continuation of the work [14–17].

More recently Pestun placed an $\mathcal{N} = 2$ gauge theory on a round sphere in four dimensions [18]. His computation opened the door to a new exciting research direction, see [19] for a review on the matter. His work was generalized to the squashed sphere [20, 21]. Later studies have used supersymmetric localization in different dimensions and manifolds, for example [22–31]. However, this list is far from exhaustive.

The way of approaching a localization computation sometimes differs from even to odd dimensions. Additionally, the structure of the answer is different. For odd dimensions one can use geometrical data, such as the moment map cone to obtain the result. Higher dimensional theories living on e.g. compact 7-dimensional manifold are especially interesting since there exist a whole variety of geometries one can study, such as Sasaki-Einstein, 3-Sasaki, or G_2 -manifolds. All of them induce a different structure of the partition function manifesting the important role of geometry in the calculation of observables [32, 33]. We are going to elaborate on this matter in Part III.

Localization is also a valuable tool to show duality relations between different theories when considering some specific sector of local operators, see for example [34, 35]. For submanifold with a fixed locus of dimen-

sion higher than zero the partition function simplifies to one of a lower dimensional theory. That, however, is not exclusive to partition function but extends to some smooth protected operator supported on a submanifold [36]. A detailed discussion thereof is given in Part IV.

Another important concept associated with localization is *topological twisting*. Introduced in [12] for $N = 2$ super Yang-Mills, it was demonstrated (using saddle point approximation) that the correlation function and correlators compute Donaldson invariants [37]. These invariants are of mathematical importance. As we will explain in Part V, the topological twisting was generalized in an equivariant version [38] and Paper III. This not only enables us to define $\mathcal{N} = 2$ SQCD on a large class of manifolds with a torus action, but also creates a natural regime for the localization calculation.

1.1 Thesis outline

The thesis is composed in five parts. The first two contain introductory material and the rest are devoted to the separate works of Papers I-III.

Part I is an introduction to geometrical concepts needed throughout the thesis. Specifically in Chapter 2 we are reviewing aspects of symplectic geometry which is exhibited in even dimension, focusing on the case of a torus group action. The analogue of symplectic geometry for odd dimensional spaces is called contact geometry which is reviewed in Chapter 3, with focus on Sasaki-Einstein manifolds which are used in Part III. In Chapter 4, we explain the basic concepts of equivariant cohomology in generic dimensions, which is crucial for the understanding of the proof of equivariant localization.

Part II contains a pedagogical introduction to localization, the main concept of the thesis. We start in Chapter 5 with a proof of the Atiyah-Bott formula of equivariant localization for a fixed point locus. This concept is then extended for any higher dimensional fixed locus. Another generalization of the result for the case of supersymmetric localization is discussed in Chapter 6.

We then proceed in Part III with the review of Paper I which studies maximally supersymmetric Yang-Mills on a 7-dimensional Sasaki-Einstein manifold. Chapter 7 introduces the concept of contact instantons in parallel with the well known instantons in four dimension. In Chapter 8, we motivate our choice of underlying geometry by studying 7-dimensional manifolds that admit at least one Killing spinor required by our formulation of supersymmetry. In Chapter 9, we focus on Sasaki-Einstein manifolds and review $\mathcal{N} = 2$ super Yang-Mills in its original and later in its cohomological formulation. The latter is then used in the localization procedure. The localization locus is analysed and the com-

putation of the 1-loop determinant is motivated. Finally, for the case of the 7-sphere we comment on two different factorization properties of its partition function.

The work of Paper II is reviewed in Part IV. Our main goal is to explain the definition of certain protected operators in a co-dimension 2 submanifold after a twisting of the supersymmetric algebra using the R -symmetry in 4 and 3 dimensions. Chapter 10 contains an introduction of such operators for $\mathcal{N} = 2$ in 4d and $\mathcal{N} = 4$ in 3d in superconformal field theories (SCFTs). Focusing on arguments using the superconformal algebra, the twisting procedure as well as the natural existence of such operators is demonstrated. The next Chapter 11 contains a discussion of the non-conformal case of $S^2 \times S^1$, where we present arguments for the existence of such operators using the supersymmetric algebra. In Chapter 12 a more field-theoretic point of view is adopted. Performing a localization computation we obtain a partition function of a topological quantum mechanics supported only on a 1-dimensional submanifold.

The final Part V is dedicated to summarize Paper III. Chapter 13 contains the definition of a topological field theory followed by the procedure of topological twisting in $\mathcal{N} = 2$ theories in 4d. The resulting theories are topological and can be defined on a large class of manifolds. The technique of equivariant twisting is studied in the final Chapter 14. We define killing vectors and spinors that have global solutions on general spin manifolds accompanied with a torus action along the killing vectors. These manifolds admit two different categories of fixed points of the torus action. It is possible to perform a twist on the fields in such a way that they become all singlets under the R -symmetry. The resulting vector multiplet fields are differential forms whereas the hypermultiplet ones are spinors. We find the cohomological complex and the localization locus. In the end a localization computation is performed by adding contributions around the different fixed points.

Part I: Review of geometry

This part serves as an introduction to the mathematical framework used throughout the thesis. The main technique, localization, reviewed in Part II, heavily uses notions from equivariant cohomology. Depending which dimension is considered, symplectic or contact geometry are needed to perform this technique.

We start by recalling basic concepts in geometry for even (symplectic) and odd dimensions (contact) with a compact Lie group action. Special attention is given to Sasaki-Einstein manifolds needed in Part III. Finally we introduce the notion of equivariant cohomology in general dimensions.

We assume the reader is familiar with basic geometric notions such as complex manifolds, differential forms and de Rham cohomology. [39] is a thorough introduction on these notions. For symplectic or contact geometry see also [40] and [41, 42] respectively.

2. Symplectic and toric geometry

Let \mathcal{M} be an even dimensional ($2n$) smooth manifold. This manifold is said to be symplectic if there exists a closed non-degenerate 2-form on \mathcal{M} , $\omega \in \Omega^2(\mathcal{M})$. Since ω is non-degenerate, $\omega^n \neq 0$, ω^n is a top form and it calculates the volume of \mathcal{M} , thus it is considered a volume form. Then ω is called symplectic form. Let us review some basic concepts related to symplectic geometry. For a thorough review see e.g. [40].

A vector field X_H is called Hamiltonian if there exists a function H such that

$$\iota_{X_H}\omega = dH, \quad (2.1)$$

where ι_{X_H} is the interior product with X_H and d the differential. The function H is also called Hamiltonian function.

A central result is the Darboux theorem which states that any two symplectic manifolds of the same dimension are locally symplectomorphic to each other. Its importance for our discussion is reflected in the corollary that every symplectic manifold locally looks the same as flat space. Thus, one can use the Darboux coordinates in local patches of \mathcal{M} that bring the symplectic form into its standard form

$$\omega = \sum_{i=1}^n dq^i \wedge dp^i, \quad (2.2)$$

where q^i and p^i are the coordinates in flat space. As a result the symplectic form does not encode any global information.

Furthermore a Kähler manifold is a symplectic manifold with a compatible almost complex structure J , meaning

$$g(X, Y) = \omega(X, JY). \quad (2.3)$$

Note that $J^2 = -1$. If there exists such a structure the symplectic form is also called Kähler form. One can complexify the tangent bundle of the manifold using its decomposition into a holomorphic and anti-holomorphic part, $T\mathcal{M} = T\mathcal{M}^+ \oplus T\mathcal{M}^-$. It is possible to decompose a differential form α accordingly. To clarify, let us look for example at a space with complex coordinates (z_i, \bar{z}_i) . The base for $T\mathcal{M}^+$ is dz_i and for $T\mathcal{M}^-$ $d\bar{z}_i$. The differential form α can be expressed as

$$\alpha = \frac{1}{p!q!} \alpha_{\mu_1 \dots \mu_p \nu_1 \dots \nu_q} dz^{\mu_1} \wedge \dots \wedge dz^{\mu_p} \wedge d\bar{z}^{\nu_1} \wedge \dots \wedge d\bar{z}^{\nu_q}. \quad (2.4)$$

That form is called a (p, q) -form, $\alpha \in \Omega^{(p,q)}(\mathcal{M})$. More generally, the space of k -forms can be decomposed as

$$\Omega^k(\mathcal{M}) = \sum_{p+q=k} \Omega^{(p,q)}(\mathcal{M}) \quad (2.5)$$

due to the complex structure on \mathcal{M} . It follows that one can separate the differential into two Dolbeault operators $d = \partial + \bar{\partial}$, defined as maps:

$$\partial : \Omega^{(p,q)}(\mathcal{M}) \rightarrow \Omega^{(p+1,q)}(\mathcal{M}) \text{ and } \bar{\partial} : \Omega^{(p,q)}(\mathcal{M}) \rightarrow \Omega^{(p,q+1)}(\mathcal{M}). \quad (2.6)$$

Using the Dolbeault operators one defines holomorphic k -forms $\alpha \in \Omega^{(k,0)}(\mathcal{M})$ satisfying $\bar{\partial}\alpha = 0$.

Now let us use these notions to introduce a group action.

Let G be a Lie group with a left group action on \mathcal{M} , $\sigma : G \times \mathcal{M} \rightarrow \mathcal{M}$. We are going to restrict ourselves to compact Lie groups since the relevant cases for this thesis are toric Lie groups T^n . Also, let \mathfrak{g} be the algebra of G and X one of its elements $X \in \mathfrak{g}$. The vector field X^\sharp generated by X ($\{e^{tX} | t \in \mathbb{R}\}$) is called *fundamental vector field* and it generates a flow in the direction of X . We also denote by \mathfrak{g}^* the dual Lie algebra and with $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g}^* \rightarrow \mathbb{R}$ the pairing between them.

We call the left Lie group action Hamiltonian if there exist a map $\mu : \mathcal{M} \rightarrow \mathfrak{g}^*$ such that

$$d\langle \mu, X \rangle = \iota_{X^\sharp} \omega \quad (2.7)$$

and it is equivariant with respect to the action σ . Such a map μ is called *moment map*, and the group action G is Hamiltonian. For Abelian groups, which are the ones we are interested in, equivariant means $\mu \circ \sigma = \mu$. Furthermore, by comparison with (2.1), the Hamiltonian function takes the form $H = \langle \mu, X \rangle$.

The moment map μ is helpful for analyzing the geometry. An especially useful tool is the *moment map cone*. According to [43], a connected symplectic manifold (\mathcal{M}, ω) with torus Hamiltonian action T^n has a moment map μ whose image is a convex hull. The convex hull represents the images of the fixed points of the action. This polytope is called moment map cone.

Focusing on symplectic quotients, the moment map can be used to reduce the original manifold \mathcal{M} to a lower dimensional one. More specifically, if G acts freely on $\mu^{-1}(0)$ the manifold

$$\mathcal{M}/G := \mu^{-1}(0)/G \quad (2.8)$$

is a symplectic manifold that is also referred to as the symplectic quotient of \mathcal{M} . If the original manifold is also Kähler, the quotient is called Kähler quotient and resulting manifold also inherits the complex structure, i.e. it is also Kähler. It is worth mentioning that, even if the original space is flat,

the resulting manifold, after quotienting out G , can be highly non-trivial. An example are Sasaki-Einstein manifolds that we discuss in Section 3.1. They can be viewed as Kähler quotients. A 7-dimensional example is given in [44].

Finally, a Calabi-Yau manifold is Kähler manifold that has vanishing Ricci tension, $R_{\mu\nu} = 0$. An example of such a manifold is flat space \mathbb{C}^n . For a more detailed review of toric geometry we refer the reader to [45].

Example

Let us go through an example to absorb these notions. We consider \mathbb{C}^n with a $G = U(1)$ left action, $U(1) \times \mathbb{C}^n \rightarrow \mathbb{C}^n$.

The standard symplectic form is

$$\omega = \frac{1}{2} \sum_n dz_i \wedge d\bar{z}_i. \quad (2.9)$$

The group action $U(1) = S^1$ can be parametrized by $e^{i\theta}$ where θ has the interpretation of an angle. Then the group acts via the map $\sigma : S^1 \times \mathbb{C}^n \rightarrow \mathbb{C}^n$ as a rotation on the coordinates

$$(e^{i\theta}, (z_i, \bar{z}_i)) \mapsto (e^{i\theta} z_i, e^{-i\theta} \bar{z}_i). \quad (2.10)$$

The angle θ is an element of the Lie algebra $\mathfrak{g} \simeq \mathbb{R}$. If we work with the standard basis vectors of \mathbb{C}^n , the fundamental vector field takes the form

$$X^\sharp = \sum_{i=1}^n (z_i \partial_{z_i} - \bar{z}_i \partial_{\bar{z}_i}). \quad (2.11)$$

The moment map can be found using (2.7). Since the algebra is \mathbb{R} we can use its standard basis, $\{e\}$, such that $\langle \mu, X \rangle = \langle \mu, e \rangle = \mu_e$ where $\mu : \mathbb{C} \rightarrow R^* \simeq R$. Then

$$d\mu_e = \iota_{X^\sharp} \omega = \frac{1}{2} \sum_i (z_i d\bar{z}_i + \bar{z}_i dz_i) = d \left(\frac{1}{2} \sum_i |z_i|^2 \right). \quad (2.12)$$

As a result, there exists a family of solutions for the moment map

$$\mu(z) = \frac{1}{2} \sum_i |z_i|^2 + c, \quad (2.13)$$

where c is an arbitrary constant. Note that the moment map is indeed equivariant since $\mu(e^{i\theta}) = \mu(z)$.

The image of μ is just $\mathbb{R}_{\geq 0}$ which itself is the moment map cone. Now let us assume that $c = 1/2$. In order to find the symplectic quotient we have to solve $\mu = 0$ according to (2.8):

$$\sum_i |z_i|^2 = 1 \implies \mu^{-1}(0) = S^{2n-1}, \quad (2.14)$$

corresponding to the $(2n - 1)$ -dimensional unit sphere. The group S^1 acts freely on the unit sphere such that the quotient S^{2n-1}/S^1 is also a symplectic manifold and is equivalent to $\mathbb{C}P^{n-1}$. In other words, what we just described is a principal $U(1)$ -bundle

$$\begin{array}{ccc} S^{2n-1} & \leftarrow & S^1 \\ \downarrow & & \\ \mathbb{C}P^{n-1} & & \end{array} \quad (2.15)$$

which is also called a Hopf fibration.

Let us consider a generalization of this example where we pick the torus T^n as the action group. The Lie algebra and its dual are both \mathbb{R}^n . One can pick the standard basis on \mathbb{R}^n , $\{e_a\}$, such that the Hamiltonian function is $\mu_a = \langle \mu, e_a \rangle$ and the moment map $\mu : \mathbb{C}^n \rightarrow \mathbb{R}^n$. Similarly to the case above, both acquire a form depending on the specific action of the group.

The realization of a torus action $T^k \times \mathbb{C}^n \rightarrow \mathbb{C}^n$, $k \leq n$, is

$$z_i \rightarrow e^{iQ_i^a \theta_a} z_i, \quad a = 1, \dots, k. \quad (2.16)$$

The Q_i^a are numbers which represent the charges of the torus action. For the case where $k = n$ and all $Q_i^a = 1$, the moment map cone is of the form

$$\begin{aligned} \mu : \mathbb{C}^n &\rightarrow \mathbb{R}^n, \\ \mu &= \frac{1}{2}(|z_1|^2, \dots, |z_n|^2), \end{aligned} \quad (2.17)$$

where the image of the moment map is the cone $\mathbb{R}_{\geq 0}^n$. The moment map cone is non-compact since \mathbb{C}^n is not compact. There is only one fixed point, the corner of the polytope ($\mu = 0$) which according to (2.17) is the origin of \mathbb{C}^n ($z_i = 0$). In the case of a compact manifold we would end up with the polygon as a moment map cone. To visualize see the case of odd-dimensional spheres in Figure 3.1.

3. Contact geometry

We review the most important concepts of contact geometry, the analogue of symplectic geometry for odd dimensions. Specifically we focus on Sasaki-Einstein manifolds with a toric action, since they allow for supersymmetry. A more apprehended review of the matter can be found in the textbooks [46, 47], or the lecture notes [48].

A contact manifold is a $(2n+1)$ -dimensional manifold \mathcal{M} equipped with a contact form κ . A contact form is a 1-form that satisfies $\kappa \wedge (d\kappa)^n \neq 0$, i.e. it provides the volume form on \mathcal{M} . There exists a unique vector field R associated with κ , such that

$$\iota_R d\kappa = 0 \quad \text{and} \quad \iota_R \kappa = 1. \quad (3.1)$$

This vector field R is called the *Reeb vector*.

The tangent bundle of \mathcal{M} can be decomposed into the line tangential to R or *Reeb direction* and the hyperplane that is defined by the kernel of κ , also called *horizontal space*. It can be noted that $d\kappa$, when restricted on the horizontal space, defines a symplectic form on it. This is a connection between contact manifolds and symplectic manifolds in one dimension lower.

If \mathcal{M} is also Riemannian, i.e. it is equipped with a metric g , we define a K-contact manifold by imposing that:

- The Reeb vector is also a Killing vector, i.e. $\mathcal{L}_R g = 0$.
- There exists a $(1,1)$ -tensor J satisfying

$$\begin{aligned} J^2 &= -1 + \kappa \otimes R, \\ g(X, Y) &= d\kappa(X, JX), \\ g(JX, JY) &= g(X, Y) - \kappa(X)\kappa(Y). \end{aligned} \quad (3.2)$$

In that case the volume form of the manifold is given by

$$\text{Vol} = \frac{(-1)^n}{2^n n!} \kappa \wedge d\kappa. \quad (3.3)$$

Similarly to the decomposition of the tangent bundle we discussed above, there is a decomposition of differential forms that we make use of in Part III. Using the contact form one defines the projectors

$$P_V = \kappa \wedge \iota_R \quad \text{and} \quad P_H = 1 - P_V, \quad (3.4)$$

with the projection relation $P_{V/H}^2 = P_{V/H}$.

Such projectors decompose forms into a *vertical* and a *horizontal* part

$$\Omega^r = \Omega_V^r \oplus \Omega_H^r. \quad (3.5)$$

It is important to note that, since the horizontal space inherits the complex structure it can be decomposed into (p, q) -forms where Dolbeault operators act according to (2.6). As a result (3.5) can be further decomposed into

$$\Omega^r = \bigoplus_{p+q=r} \Omega_H^{(p,q)} \oplus \Omega_H^{(r-1)} \kappa. \quad (3.6)$$

One may also want to decompose the form once more and extract $d\kappa$ from the horizontal forms This is showcased later on in Table 9.1.

3.1 Toric Sasaki-Einstein manifolds

An interesting subset of contact manifolds are Sasaki-Einstein manifolds that admit a torus action. A comprehensive introduction can be found for example in [40, 42].

We start with the definition of a Sasaki manifold. A Sasaki manifold \mathcal{M} is a manifold with a metric cone $C(\mathcal{M})$ that is a Kähler manifold. This means that there exists a Kähler manifold $C(\mathcal{M})$ which is of the form $C(\mathcal{M}) = \mathcal{M} \times \mathbb{R}_{\geq 0}$ with metric

$$ds_C^2 = dr^2 + r^2 ds_{\mathcal{M}}^2. \quad (3.7)$$

Here, r is the coordinate of $\mathbb{R}_{\geq 0}$ and \mathcal{M} is the base of the cone. The geometric data of the two manifolds are related as follows. The symplectic form on $C(\mathcal{M})$ is $\omega_C = d(r^2 \kappa)$, with κ a contact form on \mathcal{M} with associated Reeb vector R . The almost complex structure on $C(\mathcal{M})$ is defined using the base contact manifold as $J_C(X) = J(X) - \kappa(X)r\partial_r$, where $J_C(r\partial_r) = R$.

A Sasaki-Einstein manifold is a manifold whose cone is Calabi-Yau. Equivalently, this means that a Sasaki-Einstein manifold is a Sasaki manifold that admits an Einstein metric.¹ I.e. the Ricci tensor is proportional to the metric

$$R_{\mu\nu} = \lambda g_{\mu\nu}, \quad (3.8)$$

for a cosmological constant λ . For a Sasaki-Einstein manifold of $(2n - 1)$ dimension, the cosmological constant is $\lambda = 2n - 2$.

As mentioned, the relevant manifolds for this thesis are the Sasaki-Einstein manifolds. This means that the cone $C(\mathcal{M})$ admits a Hamiltonian action that respects its complex structure. Furthermore, the Reeb

¹It satisfies the Einstein vacuum equation.

vector needs to lie on the algebra of the T^n action. In that case, there exist a moment map and a non-compact moment map cone. The basis of this cone is a compact $(n - 1)$ -dimensional polytope called Delzant polytope and is associated to the Sasaki-Einstein manifold. Since the Reeb vector is part of the toric algebra, it can be represented by a vector on the convex cone. As in the case of the moment map cone the Delzant polytope gives important information about the manifold and its action. In particular, the Sasaki-Einstein manifold is a T^n fibration over the Delzant polytope. At each face a single S^1 of the action degenerates. Since the polytope is $(n - 1)$ and the action is n -dimensional, a combination of circles never degenerates at the vertices giving rise to a Hopf fibration,

There are multiple ways to extract geometric data from the moment polytope, see e.g. [49]. One way is by specifying its inward pointing normals v_1, \dots, v_m so that the moment map cone can be defined as

$$C_\mu = \{y \in \mathbb{R}^n | y \cdot v_i \geq 0, i = 1, \dots, m\} \subset \mathbb{R}^n \quad (3.9)$$

We assume that $v_i \in \mathbb{Z}^n$ and that their set is minimal. An additional assumption is that the cone considered is *good*, meaning every subset $\{v_{i_1} \dots v_{i_k}\}$ that defines a codimension k face of the cone can be completed into an $SL(n, \mathbb{Z})$ matrix.²

The inward normals are related to the charges of the torus actions in e.g. (2.16) by

$$\sum_{i=1}^m v_i Q_i^a, \forall a. \quad (3.10)$$

In the case where the metric cone is Calabi-Yau we have $\sum_i Q_i^a = 0$ which simplifies the above equation. In particular, it means that there exists a vector $\xi \in \mathbb{Z}^n$, such that

$$\xi \cdot v_i = 1, \quad \forall i. \quad (3.11)$$

This is known as the *1-Gorenstein condition* [42], and it implies that all the normal vectors are coplanar.

Example

A simple example of a Sasaki-Einstein manifold is the sphere S^{2n-1} . Its cone is \mathbb{C}^n . We will consider a T^n action as in the example in Section 2. The torus acts according to (2.16). As a result the moment map of the cone $C(\mathcal{M})$ is of the form (2.17). We recall that the image of the moment map is $\mathbb{R}_{\geq 0}^n$, where the axes are $|z_i|^2$, see Figure 3.1. Since the Reeb vector lies on the Lie algebra of the Torus action, we can represent it with the vector $R = (1, \dots, 1)$.

²For a rigorous definition see Appendix B of [50].

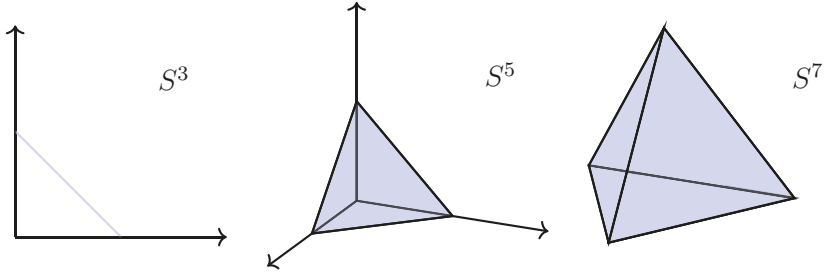


Figure 3.1. Examples of moment map cones together with the polytopes of the odd-spheres (blue filling).

The image of the moment map of the Sasaki-Einstein manifold S^{2n-1} can be viewed as a hypersurface of the cone $C(\mathcal{M})$ when $r = 1$.

In the case of S^3 the cone is \mathbb{C}^2 with T^2 action. The polytope is a simple interval as shown in Figure 3.1. Therefore one can think of S^3 as a T^2 fibration over the interval where, as we go closer to the boundaries, one of the circles degenerates. At the neighbourhood of the boundaries the local geometry is $S^1 \times \mathbb{C}$. Also, there exists a linear combination of the two circles which does not degenerate at any point on the interval. This gives rise to the usual Hopf fibration $S^1 \rightarrow S^3 \rightarrow \mathbb{C}P^1$.

For more interesting examples the polytope's shape distorts but there exists tools to read out geometric data from these polytopes. We refer the interested reader to [49].

4. Equivariant cohomology

We consider a smooth n -dimensional manifold \mathcal{M} with action G . The idea of *equivariant cohomology* is to take into consideration the group action on the de Rham cohomology.

In the case of a free action G the quotient space \mathcal{M}/G is a smooth manifold and we can define a de Rham cohomology $H^*(\mathcal{M}/G)$. If the action is not free and there exist fixed points the quotient is not a smooth manifold and the situation is more complicated, see e.g. [51].

In order to understand this better we continue using the Cartan model [52].¹ We follow the review [53].

Let us consider a Lie algebra \mathfrak{g} with elements $\phi = \phi^a T_a$, where ϕ^a can be thought of an element of the dual algebra \mathfrak{g}^* and T_a are generators of the algebra. For the Cartan model we need the symmetric algebra of \mathfrak{g}^* denoted as $S(\mathfrak{g}^*) \approx \mathbb{R}[\mathfrak{g}]$, where we called $\mathbb{R}[\mathfrak{g}]$ the commutative ring of polynomial functions on the vector space underlying \mathfrak{g} . Its elements are polynomials of $\{\phi^a\}$.

We introduce a form $\alpha \in \Omega^\bullet(\mathcal{M}) \otimes \mathbb{R}[\mathfrak{g}]$ which is invariant under the G -action. This form is not homogeneous with respect to the degree of the form but rather it is a sum of forms of various degrees

$$\alpha = \sum_{k=0}^n \alpha_k, \quad (4.1)$$

where α_k is a k -form. For every generator T_a , the group action on \mathcal{M} induces a vector field v_a on \mathcal{M} , which points in the direction of the flow of that group element. The vector field $v = \phi^a \varepsilon_a$ is the fundamental vector field that points in the direction of the flow of the full group.

The following Cartan differential can be defined which acts on α

$$d_v : \Omega^\bullet(\mathcal{M}) \otimes \mathbb{R}[\mathfrak{g}] \rightarrow \Omega^\bullet(\mathcal{M}) \otimes \mathbb{R}[\mathfrak{g}], \\ d_v = d + \iota_v, \quad (4.2)$$

where $d : \Omega^\bullet(\mathcal{M}) \rightarrow \Omega^{\bullet+1}(\mathcal{M})$ is the de Rham differential and $\iota_v : \Omega^\bullet(\mathcal{M}) \rightarrow \Omega^{\bullet-1}(\mathcal{M})$ is the contraction of the differential form and the fundamental vector field. We refer to this differential as the equivariant

¹It should also be noted that there exist other models, such as the Weil model. We will not consider those in this manuscript.

differential along v . An important property of the equivariant differential is that it squares to the Lie derivative along the vector field v

$$d_v^2 = \mathcal{L}_v. \quad (4.3)$$

As it is argued in [53], if the form α is G -invariant it follows that $d_v^2\alpha = \mathcal{L}_v\alpha = 0$. This defines the Cartan model of equivariant cohomology $H_G^\bullet(\mathcal{M}) = H((\Omega^\bullet(\mathcal{M}) \otimes \mathbb{R}[\mathfrak{g}])^G, d_v)$. The invariant form α is called *equivariantly closed*, i.e. $d_v^2\alpha = 0$. In the case of $\alpha = d_v\lambda$, where $\lambda \in \Omega^\bullet(\mathcal{M}) \otimes \mathbb{R}[\mathfrak{g}]$, α is called *equivariantly exact*.

As mentioned in (4.1) α is actually a polyform. This means that the constraint that it is equivariantly closed is actually a relation between the different degrees of α

$$d_v^2\alpha = 0 \implies d\alpha_{k-2} + \iota_v\alpha_k = 0 \quad \forall k \in [0, n]. \quad (4.4)$$

A final remark is that when we integrate a polyform α over the full manifold only the top-form gives a contribution

$$\int_{\mathcal{M}} \alpha = \int_{\mathcal{M}} \alpha_n. \quad (4.5)$$

Supersymmetric example

In order to make contact with Part II we will use the Cartan model in the context of supergeometry.

Let x^μ be the coordinates on \mathcal{M} . We also introduce the Grassmann coordinates ψ^μ . These coordinates are anti-commuting, i.e. $\psi^\mu\psi^\nu = -\psi^\nu\psi^\mu$. They are fermionic degrees of freedom, whereas the even coordinates x^μ represent the bosonic degrees of freedom.

The fermionic coordinates have the following integration rules

$$I = \int d^n\psi \left(\prod_{\mu=1}^k \psi^\mu \right) = 1, \quad \text{if } k = n; \quad \text{otherwise } I = 0, \quad (4.6)$$

which mimics the way the equivariant form is manipulated.

In the specific case the super-manifold is the odd tangent bundle $\Pi T\mathcal{M}$, the odd coordinates correspond to the 1-forms dx^μ . Multiplication corresponds to the wedge product between forms. The equivariant differential is represented as a transformation on the coordinates

$$\begin{aligned} d_v x^\mu &= dx^\mu = \psi^\mu, \\ d_v \psi^\mu &= v^\mu. \end{aligned} \quad (4.7)$$

We will use the notation common in physics literature in the following parts and denote the transformation by δ .

Part II:

Localization technique

Localization is a way to compute partition functions and certain protected correlators. In a mathematical language: it helps computing integrals of equivariant closed forms in terms of the fixed locus of the group action. We will first derive the mathematical formula of Berline-Verne-Atiyah-Bott by performing equivariant localization on finite dimensional integrals. Then, we look at the supersymmetric localization technique in a generic supersymmetric field theory. The difference there is that we apply the localization technique on path integrals which are defined on an infinite dimensional space.

A comprehensive review about the subject is [19], another pedagogical introduction can for example be found in [54].

5. Finite dimensional integrals

For simplicity we restrict ourselves to the case of a $U(1)$ action. However, the method can be generalized to more complicated group actions.

Let \mathcal{M} be a compact n -dimensional manifold with a $U(1)$ action that has isolated fixed points. Therefore, let v be the vector field associated with this $U(1)$.

Consider an equivariant closed polyform α , i.e. $(d + \iota_v)\alpha = 0$. A polyform is a form of mixed degrees $\alpha = \alpha_n + \alpha_{n-1} + \dots + \alpha_0$ as introduced in (4.1). The Berlin-Vergne-Atiyah-Bott formula states

$$\int_{\mathcal{M}} \alpha = \sum_i (2\pi)^n \frac{\alpha_0(x_i)}{\sqrt{\det(\partial_\mu v^\nu(x_i))}}, \quad (5.1)$$

where the sum goes over all the fixed points of $U(1)$. In the following, we will review the derivation of this formula. For a thorough review we refer the reader to [55].

We consider the odd tangent bundle $\Pi T\mathcal{M}$ with coordinates (x^μ, ψ^μ) , as it was introduced in Section 4. Using a supergeometric approach, x^μ are coordinates on \mathcal{M} and ψ^μ are Grassmann coordinates on the fibre. We define the following supersymmetric-like transformations, similar to (4.7)

$$d_v x^\mu = \psi^\mu, \quad (5.2)$$

$$d_v \psi^\mu = v^\mu. \quad (5.3)$$

The polyform α is a polynomial in the Grassmann coordinates. In particular, monomials of the same degree k in ψ^μ correspond to the same degree of the polyform α_k . The integral of α over \mathcal{M} becomes

$$Z = \int_{\Pi T\mathcal{M}} d^n x d^n \psi \alpha(x, \psi). \quad (5.4)$$

Note that the equivariant differential acts the same way as a supersymmetric transformation: it squares to $d_v^2 = \mathcal{L}_v$, and since α is equivariantly closed it can be interpreted as a supersymmetric observable, for example the exponential of a supersymmetric action.

We consider the deformation of (5.4) with respect to a real parameter, $t \in \mathbb{R}$,

$$Z(t) = \int_{\Pi T\mathcal{M}} d^n x d^n \psi \alpha(x, \psi) e^{-td_v W(x, \psi)}, \quad (5.5)$$

where $W(x, \psi)$ is a function such that $d_v^2 W = \mathcal{L}_v W = 0$. It follows that $Z(t)$ is independent of t

$$\frac{d}{dt} Z(t) = \int_{\Pi T \mathcal{M}} d^n x d^n \psi [d_v W(x, \psi)] \alpha(x, \psi) e^{-t d_v W(x, \psi)} \quad (5.6)$$

$$= \int_{\Pi T \mathcal{M}} d^n x d^n \psi d_v [W(x, \psi) \alpha(x, \psi) e^{-t d_v W(x, \psi)}] = 0, \quad (5.7)$$

where in the second line we have integrated by parts and used that α and $d_v W$ are equivariantly closed. Finally, since \mathcal{M} is a compact manifold, we use Stokes theorem to conclude that the integral vanishes. This conveniently will allow us to compute the original integral in (5.4) by a clever choice of t in (5.5). Consider the limit $t \rightarrow \infty$. From the saddle point approximation the main contribution will arise from $d_v W = 0$.

Let us investigate this limit using the following convenient choice of W :

$$W = g_{\mu\nu} \psi^\mu (d_v \psi)^\nu, \quad (5.8)$$

with g being the metric on \mathcal{M} which is invariant under the group action, i.e $\mathcal{L}_v g = 0$. Here, v is a Killing vector. For this choice of W , we have

$$d_v W = |v|^2 + \partial_\lambda (g_{\mu\nu} v^\mu) \psi^\nu \psi^\lambda. \quad (5.9)$$

Notice that the first term is a zero form and it is semi-positive definite. As a result the exponential in (5.5) is dominated at $t \rightarrow \infty$ by the fixed points x_i of the $U(1)$ action, $v(x_i) = 0$.

Let us concentrate on the contribution of a single isolated fixed point $x_{f.p.}$ and rescale the coordinates

$$x \rightarrow \tilde{x} = \sqrt{t} x, \quad \psi \rightarrow \tilde{\psi} = \sqrt{t} \psi, \quad (5.10)$$

so that the measure of the integral (5.5) stays invariant. The exponent becomes after rescaling

$$t d_v W = H_{\mu\nu} \delta \tilde{x}^\mu \delta \tilde{x}^\nu + S_{\mu\nu} \tilde{\psi}^\mu \tilde{\psi}^\nu + \mathcal{O}\left(\frac{1}{t}\right), \quad (5.11)$$

where $\delta \tilde{x} = \tilde{x} - x_{f.p.}$ denotes the distance from the fixed point. The functions H and S are of the form

$$H_{\mu\nu} = g_{\lambda\rho} (\partial_\mu v^\lambda \partial_\nu v^\rho)|_{x_{f.p.}}, \quad (5.12)$$

$$S_{\mu\nu} = g_{\mu\lambda} \partial_\nu v^\lambda|_{x_{f.p.}}. \quad (5.13)$$

Hence, the exponent is quadratic and does not depend on t . The integral is only supported on the fixed point because all the other points

give exponentially suppressed contributions. In particular we have

$$\begin{aligned} Z[0] &= \lim_{t \rightarrow \infty} Z[t] = \lim_{t \rightarrow \infty} \int d^n x d^n \psi \alpha \left(\frac{\tilde{x}}{\sqrt{t}}, \frac{\tilde{\psi}}{\sqrt{t}} \right) e^{-H_{\mu\nu} \delta \tilde{x}^\mu \delta \tilde{x}^\nu - S_{\mu\nu} \tilde{\psi}^\mu \tilde{\psi}^\nu + \mathcal{O}(\frac{1}{t})} \\ &= (2\pi)^n \alpha(0, 0) \frac{\text{Pf}(S)}{\sqrt{\det(H)}}, \end{aligned} \quad (5.14)$$

where the integral separates into two Gaussian integrals, one with real and one with Grassmann integration parameters. In the last step we used that the higher forms (ψ monomials) of α will come with factors of $1/\sqrt{t}$. Thus, only the its 0-form part will contribute and the Grassmann integration is only the Gaussian integral. The form of the integral is

$$Z[0] = (2\pi)^n \alpha_0|_{x_{f.p.}} \frac{\text{Pf}(S)}{\sqrt{\det(H)}}. \quad (5.15)$$

By substituting S and H and summing over all the fixed points we get the Berlin-Vergne-Atiyah-Bott formula (5.1).

One can follow the same steps and find a similar formula for a fixed submanifold locus. If $F \subset \mathcal{M}$ is the submanifold fixed locus it is easy to guess that the integral will have only contributions of the top component from the restriction of α on F . Indeed, the formula reads [9, 18]

$$\int_{\mathcal{M}} \alpha = \int_{\mathcal{F}} \frac{\iota_F^* \alpha}{e(\mathcal{N})}, \quad (5.16)$$

where $\iota_F^* \alpha$ is the pullback of α onto F and \mathcal{N} is the normal bundle.

6. Infinite dimensional integrals

Let us explain next how the argument is modified in the case of QFTs.

Consider a supersymmetric quantum field theory with supersymmetry transformation δ (usually Grassmann odd). The goal is to calculate observables of the theory such as the partition function or correlators of local operators.

We will start with the path integral representation of a partition function

$$Z = \int [D\phi] e^{-S[\phi]}, \quad (6.1)$$

where we represent all the fields of the theory by ϕ . The difference to the previous case is that we are integrating over the space of field-configurations, which is infinite dimensional.

The square of the supersymmetry transformations on the fields ϕ consists of symmetries of the theory. In the case where a torus action is acting on the manifold the usual operator takes a form like (14.41) used in a specific example discussed later in the thesis. This supersymmetry transformations will play the role of the equivariant differential d_v .

Let us follow the same steps as before and deform the action using a δ -exact term

$$Z_t = \int [D\phi] e^{-S[\phi] - t\delta V}. \quad (6.2)$$

As before, Z_t is independent of the parameter t .

The canonical choice of the deformation term V is

$$V = \sum_{\Psi} (\delta\Psi)^\dagger \Psi, \quad (6.3)$$

where Ψ refers to fermionic terms. The bosonic part of the deformed action is

$$(\delta V)_{\text{bos}} = \sum_{\Psi} |\delta\Psi|^2, \quad (6.4)$$

which is positive semi-definite. The main contribution to the partition function is given by $(\delta V)_{\text{bos}} = 0$, which results in a localization locus

$$\delta\Psi = 0. \quad (6.5)$$

Note that for the case where the number of fermionic terms summed over is less than the number of fermionic degrees of freedom, the localization locus might be higher dimensional.

As before, one expands the fields around their localization locus value ϕ_0 as $\phi \rightarrow \phi_0 + t^{-1/2}\phi'$, where ϕ' is the fluctuation of the field.

In the limit where $t \rightarrow \infty$ the only terms surviving will be the classical part, containing only ϕ_0 terms, and a one-loop contribution

$$Z = \int [D\phi_0] e^{-S[\phi_0]} \frac{1}{\text{sdet}(\delta^2)}, \quad (6.6)$$

where the superdeterminant, sdet , is to be interpreted such that the contribution of the bosonic fields are in the denominator and the contribution of the fermions in the numerator.

Apart from the partition function, localization allows the exact computation of correlation functions of certain gauge invariant operators \mathcal{O} . In order for the above argument to go through, these operators need to be δ -closed, i.e $\delta\mathcal{O} = 0$. Such operators are called *protected operators* and therefore are interesting objects to study since their correlators can be exactly computed.

A challenging part of finding an explicit form for (6.6) is the calculation of the 1-loop contribution. There are many ways of performing the superdeterminant as we will hint to in the following chapters. One way is by explicitly finding the spectrum of the fluctuations under the δ^2 action. Examples of this approach in three, five, and generic dimensions are given in [23], [28] and [56] respectively.

Another way is using an index theorem [9], whereof an finite dimensional example was shown in the previous section. For some specific manifold in odd dimensions one can prove that the index is associated to counting points on the moment map cone of your manifold. We will discuss this in Section 9.1 in the context of 7-dimensional Sasaki-Einstein manifolds. This additional simplification does not exist for all cases. Related work in the 5-dimensional case can be found in [50]. Another way is to explicitly calculate the index under the torus action as we explain in Section 14.1.4.

Finally, we want to stress some points. Firstly, for gauge theories the action needs to be gauge fixed. We can then consider as the appropriate operator the supersymmetric variation together with the BRST-transformation. Another consideration is that there is freedom in the choice of the localizing supercharge and the deformation term used. Different deformation terms result in different localization schemes. Results obtained by different schemes need to be the same after algebraic manipulations. Finally, there is the need for regularization. There exist many different regularization schemes as we will also briefly discuss in Section 9.1. The choice of the regularization scheme is not an easy task. For all these reasons, there exist many different representations of the partition function and correlators of the same theory.

Part III:

$N = 1$ super Yang-Mills in 7d on Sasaki-Einstein manifolds

There is a variety of exact results for partition functions in the literature considering many different dimensions and geometries. In the following three sections we are going to summarize a systematic way of deriving the partition function for certain odd dimensional manifolds using the example of $N = 1$ super Yang-Mills in 7d on Sasaki-Einstein manifolds from Paper I, which was first considered in [22]. This example is an excellent toy model since it showcases how the geometry is encoded in the structure of the partition function.

We will start by looking at instanton contributions we are going to stumble over during the localization computation. A discussion of *contact instantons* in 7d is given with a simultaneous introduction of the more common 4d instantons. We are going to give a motivation for supersymmetric theories on Sasaki-Einstein manifolds by studying the set of 7d manifolds that allow killing spinors. Afterwards, we introduce $\mathcal{N} = 1$ super Yang-Mills in 7d on Sasaki-Einstein manifolds. We will use the localization technique described in Section 5 to calculate its partition function. Finally, we are going to discuss the geometrical structure of the result and its interpretation using fibrations.

7. Instantons

7.1 4d case

Let us first discuss the case of Yang-Mills on a 4-dimensional manifold \mathcal{M} , introduced in [57], before we generalize it. In this discussion we will use some of the concepts of Section 2.¹

The Yang-Mills action is

$$S = \int F \wedge *F = \frac{1}{2} \int d^4x F_{\mu\nu} F^{\mu\nu}, \quad (7.1)$$

where $*F$ is the Hodge dual form of the Lie algebra valued 2-form F . The Euler-Lagrange equation and the Bianchi identity take the form

$$D_\mu F^{\mu\nu} = 0 \rightarrow d_A *F = 0, \quad (7.2)$$

$$D_{[\mu} F_{\nu\rho]} = 0 \rightarrow d_A F = 0, \quad (7.3)$$

with $D_\mu = \partial_\mu + [A_\mu, \cdot]$ the covariant derivative with respect to the gauge field A_μ . This covariant derivative can be rewritten as the de Rham differential twisted by the gauge field d_A acting on a form $\omega \in \Omega^k(\mathcal{M})$ as $d_A \omega = d\omega + [A, \omega]$.

In 4-dimensions one defines the projectors P^\pm that act on 2-forms

$$P^\pm = \frac{1}{2}(1 \pm *). \quad (7.4)$$

They satisfy $(P^\pm)^2 = P^\pm$ and $P^- = 1 - P^+$. Using these projectors 2-forms can be decomposed as $\Omega^2(\mathcal{M}) = \Omega^{2+}(\mathcal{M}) \oplus \Omega^{2-}(\mathcal{M})$. For example, the field strength becomes

$$F = F^+ + F^-, \quad \text{where} \quad *F^\pm = \pm F^\pm. \quad (7.5)$$

These are the self-dual and anti-self-dual components of the field strength.

We can use this decomposition in the action (7.1) to rewrite it as

$$S = \int F^+ \wedge *F^+ + \int F^- \wedge *F^-. \quad (7.6)$$

Thus, we have the following bound

$$S \geq \left| \int F^+ \wedge *F^+ - \int F^- \wedge *F^- \right| = \left| \int F \wedge F \right|, \quad (7.7)$$

¹We assume a general semi-simple group G but we not use it explicitly. We will omit to write explicitly the trace over the Lie algebra \mathfrak{g} in the integrals.

where we substituted the Hodge-dual fields according to (7.5). This lower bound is saturated only if F is self- or anti-self-dual, i.e $F = F^+$ or $F = F^-$. These solutions can be summarized as

$$F = \pm * F. \quad (7.8)$$

These equations are called instanton equations and their solutions, depending on the sign, are called *instantons* and *anti-instantons* respectively. The bound is also interesting since it involves a topological invariant. The integral

$$c_2 = -\frac{1}{8\pi^2} \int F \wedge F \quad (7.9)$$

is the *instanton number*. It furthermore contains the second Chern class which is a topological invariant quantity [39].

7.2 7d case

We are going to discuss the 7d Yang-Mills action in the context of contact geometry, which we introduced in Section 3.

We start with the same action as before on a smooth 7-dimensional manifold \mathcal{M}

$$S = \int F \wedge *F, \quad (7.10)$$

where the fields are Lie algebra valued which we omit here for convenience. The field strength F is a 2-form. As it was discussed in Section 3 one can utilize the contact form κ and the Reeb vector R associated to it to define the projectors (3.4) that decompose forms into a vertical and a horizontal part.

We further decompose the horizontal part by restricting ourselves to 2-forms for the purpose of the instanton equation. As it was explained in Paper I, the projector

$$\check{P}F = \frac{1}{12}[* (F \wedge *d\kappa)]d\kappa \quad (7.11)$$

extracts the part which is proportional to $d\kappa$. Then, the 2-form F becomes

$$F = \hat{F} + \check{F} = \hat{F} + \frac{1}{24}\tilde{F}d\kappa, \quad (7.12)$$

where $\hat{F}_{\mu\nu}(d\kappa)^{\mu\nu} = 0$ and $\tilde{F} = F_{\mu\nu}(d\kappa)^{\mu\nu}$. Similar to the case before, the horizontal part of \hat{F} can be decomposed using the projector

$$P^\pm F = \frac{1}{2}(F \pm \frac{1}{2}\iota_R * (d\kappa \wedge F)). \quad (7.13)$$

The decomposition of F becomes

$$F = F_V + F_H = F_V + \hat{F}_H^+ + \hat{F}_H^- + \check{F}_H, \quad (7.14)$$

where the last term also reads $\check{F}_H = \frac{1}{24}\tilde{F}d\kappa$.

The Yang-Mills action is finally decomposed into

$$S = \int F_V \wedge *F_V + \int \hat{F}_H^+ \wedge *\hat{F}_H^- + \int \hat{F}_H^- \wedge *\hat{F}_H^- + \int \check{F}_H \wedge *\check{F}_H. \quad (7.15)$$

A bound on the action is obtained as

$$S \geq \int \hat{F}_H^+ \wedge *\hat{F}_H^- + \int \hat{F}_H^- \wedge *\hat{F}_H^- \implies \quad (7.16)$$

$$S \geq \left| \int \hat{F}_H^+ \wedge *\hat{F}_H^- - \int \hat{F}_H^- \wedge *\hat{F}_H^- \right| = \frac{1}{2} \left| \int \kappa \wedge d\kappa \wedge \hat{F} \wedge \hat{F} \right|. \quad (7.17)$$

The first inequality is satisfied whenever $F_V = 0$ and $\check{F}_H = 0$ and the second one if $\hat{F}_H^+ = 0$ or $\hat{F}_H^- = 0$. One can combine these restrictions into one equation which is referred to as *contact instanton* equation in 7 dimensions,

$$*F = \pm \frac{1}{2} \kappa \wedge d\kappa \wedge F. \quad (7.18)$$

It is also important to note that contact instantons and anti-instantons are automatically solutions of the Yang-Mills equation since

$$d_A * F = \pm \frac{1}{2} d\kappa \wedge d\kappa \wedge F = 0, \quad (7.19)$$

due to the orthogonality of \hat{F}_H and $d\kappa$.

It is worth mentioning that the instanton equation was first written down for the case of five dimensions in [27] and further studied in [58].

8. Killing spinors

From now on we will restrict ourselves to the case of 7d manifolds. The idea of this Section is to motivate the restriction to Sasaki-Einstein manifolds following Paper I.

We will revise some facts about Killing spinors and refer the reader to [59–61] for further reading. Let us start from a n -dimensional Riemannian manifold (\mathcal{M}, g) which admits a spin structure. A spinor ζ is *Killing* if there exists a constant $\alpha \in \mathbb{C}$ such that for all tangent vectors X ,

$$\nabla_X \zeta = \alpha X \cdot \zeta \quad (8.1)$$

is satisfied. Here, ∇ denotes the covariantized derivative with respect to the spin connection and $X \cdot \zeta$ the Clifford multiplication. In the case of $\alpha \in \mathbb{R}$, ζ is called *real Killing spinor*.

If there exists a Killing spinor on \mathcal{M} the manifold is Einstein with Ricci scalar

$$R = -4n(n-1)\alpha^2. \quad (8.2)$$

If we restrict our attention to positive curvature then for a given manifold there are two sign possibilities for α . Putting these back into the Killing spinor equation (8.1) there are several independent solutions for the Killing spinors ζ . For the familiar formulation of supersymmetric theories (and for localization) we require at least one Killing spinor.

Seven dimensional manifolds with positive curvature admitting Killing spinors were classified by [61]. The 7-dimensional complete simply-connected Riemannian spin manifolds with positive curvature that admit a non-trivial Killing spinor are

- the 7-sphere S^7 with 16 Killing spinors,
- 3-Sasaki manifolds with 3 Killing spinors,
- Sasaki-Einstein manifolds with 2 Killing spinors,
- and proper G_2 -manifolds with 1 Killing spinor,

as was also explained in Paper I. In this thesis we are going to mainly focus on the first and the third case. In Paper I there is also a short discussion about the 3-Sasaki manifolds.

The second and last cases are out of the scope of this thesis. However, let us say a few words about these manifolds.

A 3-Sasaki manifold is a Sasakian manifold that admits an $su(2)$ triplet of Sasaki structures $\{R_a, \kappa_a, J_a\}$, such as

$$\iota_{R_a} \kappa_b = \delta_{ab}, \quad (8.3)$$

$$[R_a, R_b] = \epsilon_{abc} R^c. \quad (8.4)$$

It is important to note that the cone over a 3-Sasaki manifold is a hyperkähler manifold which has three complex structures. Localization results for the 3-Sasaki manifolds can be found partially in Paper I but for a more in depth work one should look into [32] and [33].

The proper G_2 manifold admits a 3-form Φ associated to the G_2 structure, which satisfies $d\Phi = -8\lambda(*\Phi)$ for some $\lambda \neq 0$. For more information about such manifolds we refer the reader to [62]. There have been developments adding supersymmetric theories onto them, for example [63, 64]. It is interesting that one can find a minimization of the action, called G_2 -instanton. It is different that the contact instanton in (7.18) since these manifolds do not have a contact structure. The instanton equation then is $*F = \Phi \wedge F$. These manifolds have not yet been discussed in the context of localization due to the lack of contact structure on which the formalism heavily relies.

9. Localization in 7 dimensions

We proceed by placing supersymmetric theories on 7-dimensional Sasaki-Einstein manifolds following mainly Paper I and [22].

The starting point is $\mathcal{N} = 1$ super Yang-Mills on ten-dimensional $\mathbb{R}^{9,1}$. The theory contains a gauge field A_M , $M = 0, \dots, 9$, with its field strength and a Majorana-Weyl fermion Ψ_α , with $\alpha = 1, \dots, 16$, transforming under the adjoint representation of the gauge group G . Finally, Γ_M denote the 10-dimensional Dirac matrices. The 10-dimensional action is [65]

$$S_{10} = \frac{1}{g_{10}^2} \int d^{10}x \text{Tr} \left(\frac{1}{2} F^{MN} F_{MN} - \Psi \Gamma^M D_M \Psi \right), \quad (9.1)$$

where D_M is the 10 dimensional covariant derivative with respect to the gauge field. This action is invariant under the supersymmetry transformations

$$\delta A_M = \epsilon \Gamma_M \Psi, \quad (9.2)$$

$$\delta \Psi = \frac{1}{2} \Gamma^{MN} F_{MN} \epsilon. \quad (9.3)$$

where the parameter ϵ is a constant Majorana-Weyl spinor.

We are going to dimensionally reduce in order to get a supersymmetric theory in a lower dimensional curved manifold. This reduction is called Scherk-Schwartz reduction and it was performed first in [59] for on-shell supersymmetry and in [66, 67] for the off-shell version.

We separate the gauge field into the 7-dimensional one A_μ , $\mu = 1 \dots 7$ and scalars ϕ_A , $A = 0, 8, 9$, arising from the compactified directions. The derivative in the compactified dimensions vanishes.

In order to have consistent supersymmetry on the 7-dimensional manifold, the 10-dimensional Majorana-Weyl spinor needs to satisfy a generalized Killing spinor equation,

$$\nabla_\mu \epsilon = \frac{1}{2r} \tilde{\Gamma}_\mu \Lambda \epsilon. \quad (9.4)$$

The matrix $\Lambda = \Gamma^8 \tilde{\Gamma}^9 \Gamma^0$ and r is a dimensionful parameter corresponding to the size of the manifold. We will drop r for convenience. It can be restored at any point using dimensional analysis. It follows that such a 7-dimensional manifold has to be in the set discussed in Section 8.

In order to perform localization we need off-shell supersymmetry. Thus, we also need to introduce an auxiliary field K^m , $m = 1 \dots 7$ and a bosonic pure spinor ν_m associated to it. We are going to briefly review the theory however we refer the reader to Paper I for more details.

Following [18,22] and Paper I, the pure spinors ν_m satisfy the relations

$$\epsilon \Gamma^M \nu_m = 0, \quad (9.5)$$

$$\nu_m \Gamma^M \nu_n = \delta_{mn} v^M, \quad (9.6)$$

$$\nu_\alpha^m \nu_\beta^m + \epsilon_\alpha \epsilon_\beta = \frac{1}{2} v^M \tilde{\Gamma}_{M\alpha\beta}.$$

Note that these relations only determine the ν 's up to an internal $so(7)$ symmetry.

Here v^M denotes the vector field

$$v^M = \epsilon \Gamma^M \epsilon, \quad (9.7)$$

which is a Killing vector since ϵ is a Killing spinor. Note that we are free to choose $v^0 = 1$ and $v^8 = v^9 = 0$ resulting in $v^\mu v_\mu = 1$.¹

The off-shell supersymmetric transformations read

$$\begin{aligned} \delta A_M &= \epsilon \Gamma_M \Psi, \\ \delta \Psi &= \frac{1}{2} F_{MN} \Gamma^{MN} \epsilon + \frac{8}{7} \Gamma^{\mu B} \phi_B \nabla_\mu \epsilon + K^m \nu_m, \\ \delta K^m &= -\nu^m \Gamma^M D_M \Psi + \frac{3}{2} \nu^m \Lambda \Psi. \end{aligned} \quad (9.8)$$

It can be shown, see [18,22] or Paper I, that this off-shell supersymmetry transformation squares to bosonic symmetries of the theory. Schematically this reads

$$\delta^2 = -\mathcal{L} - G - R - S, \quad (9.9)$$

where \mathcal{L} is a Lie derivative along the vector field v , G is a gauge transformation, R is the R -symmetry, and S are the $so(7)$ rotations of the auxiliary fields K^m .

The reduced action becomes

$$\begin{aligned} S = \frac{1}{g_D^2} \int d^7 x \sqrt{-g} \text{Tr} & \left(\frac{1}{2} F^{MN} F_{MN} - \Psi \Gamma^M D_M \Psi + 8 \phi^A \phi_A + \frac{3}{2} \Psi \Lambda \Psi \right. \\ & \left. - 2[\phi^A, \phi^B] \phi^C \varepsilon_{ABC} - K^m K_m \right). \end{aligned} \quad (9.10)$$

Nonetheless one can simplify the transformations by a redefinition of fields. This leads to a clever geometric interpretation. Thus, we are going

¹For useful identities containing the Γ matrices we refer the reader to Appendix A of Paper I.

to map these fields to differential forms. One starts by identifying as Reeb vector $R^\mu = v^\mu$ from (9.7) and as contact form $\kappa_\mu = g_{\mu\nu}R^\nu$. Note that the condition $v^\mu v_\mu = 1$ corresponds to the definition of the Reeb vector (3.1).²

We make the following redefinitions of the fermionic degrees of freedom

$$\psi_\mu = \epsilon \Gamma_\mu \Psi, \quad (9.11)$$

$$\Upsilon_{\mu\nu} = (\nu_m \Gamma^0 \Psi)(\nu^\mu \Gamma_{\mu\nu 0} \epsilon), \quad (9.12)$$

and of the bosonic fields

$$\Phi_{\mu\nu\lambda} = \frac{1}{2} \phi_A (\epsilon \Gamma_{\mu\nu\lambda} \Gamma^{A0} \epsilon), \quad (9.13)$$

$$\phi_0 = \sigma. \quad (9.14)$$

It is important to note that ϕ_0 is special since it comes from compactifying the time-like direction. All these maps are invertible.

Additional to these redefinitions we introduce $\eta = \frac{1}{2} \epsilon \Gamma_A \Psi (\epsilon \Gamma_{\mu\nu\lambda} \Gamma^{A0} \epsilon)$, the superpartner of Φ , and $H_\mu = (\nu^m \Gamma_{\mu\nu 0} \epsilon)(K_m + \dots)$, the superpartner of Υ .

The result is the cohomological complex

$$\begin{aligned} \delta A &= \psi, \\ \delta \psi &= -\iota_R F + i G_\sigma A, \\ \delta \sigma &= i \iota_R \psi, \\ \delta \Phi &= \eta, \\ \delta \eta &= -\mathcal{L}_R^A \Phi + i G_\sigma \Phi, \\ \delta \Upsilon &= H, \\ \delta H &= -\mathcal{L}_R^A \Upsilon + i G_\sigma \Upsilon, \end{aligned} \quad (9.15)$$

where d_A is the de Rham differential coupled to the connection A and \mathcal{L}_R^A the corresponding Lie derivative along the Reeb vector field covariantized with respect to the connection A , $\mathcal{L}_R^A = \iota_R d_A + d_A \iota_R$. The gauge transformation G_σ is given by $G_\sigma A = d_A \sigma$ on the gauge field and $G_\sigma = -[\sigma, \cdot]$ on all other fields.

It is important to point out that the fields above are now differential forms and can also be decomposed using projectors. In Table 9.1 we list the fields in the cohomological complex together with their superpartners. Note that we have decomposed the 2-forms using the decomposition (3.6).

²In Paper I all the components of the contact structure are identified.

Bosons	Fermions
A connection	$\psi \in \Omega^1$
$H \in \Omega_H^{(2,0)} \oplus \Omega_H^{(0,2)} \oplus \Omega^0 d\kappa$	$\Upsilon \in \Omega_H^{(2,0)} \oplus \Omega_H^{(0,2)} \oplus \Omega^0 d\kappa$
$\Phi \in \Omega_H^{(3,0)} \oplus \Omega_H^{(0,3)}$	$\eta \in \Omega_H^{(3,0)} \oplus \Omega_H^{(0,3)}$
$\sigma \in \Omega^0$	

Table 9.1. *The bosonic and fermionic fields of the cohomological complex are presented. (X, X') -pairs of bosons and fermions appearing in the transformations (9.16) are written on the same line. Note that we have suppressed the Lie algebra dependence.*

Redefining the field $\sigma \rightarrow -\sigma + i\iota_R A$, the above transformations can be written in the compact form

$$\begin{aligned}
\delta X &= X', \\
\delta X' &= -\mathcal{L}_R X - iG_\sigma X, \\
\delta \sigma &= 0,
\end{aligned} \tag{9.16}$$

where the (X, X') pairs are given by (A, ψ) , (H, Υ) , and (Φ, η) . We see that $\delta^2 = -\mathcal{L}_R - iG_\sigma$ are once more symmetries of the theory.

Off-shellness is not the only requirement in order to get a correct localization result. It is necessary to gauge fix to theory by introducing Faddeev-Popov ghosts c, \bar{c} , a Lagrange multiplier b , and zero modes (a_0, \bar{a}_0, b_0) and (c_0, \bar{c}_0) which are bosonic and fermionic respectively. We also use the standard BRST transformation δ_B [18] which we combine with supersymmetry into a new transformation $Q = \delta + \delta_B$. It squares to $Q^2 = -\mathcal{L}_R + iG_{a_0}$. The field transformations have the same form as (9.16) with the replacement $\sigma \rightarrow a_0$.

9.1 Localization technique

In order to perform localization one requires the localization locus. In [68] and Paper I it was found to take the form

$$\begin{aligned}
\iota_R F &= 0, \\
\iota_R d_A \Phi &= 0, \\
d_A \sigma &= 0, \\
\hat{F}_H^- &= -d_A^\dagger \Phi, \\
\check{F}_H \wedge d\kappa \wedge d\kappa &= 4[\Phi^-, \Phi^+].
\end{aligned} \tag{9.17}$$

In the case where $\Phi = 0$ ³ the localization locus is equivalent to the contact instanton \hat{F}^+ we discussed in Chapter 7.2.

The action (9.10) on the BPS locus takes the form

$$S_{f.p.} = \frac{1}{g_7^2} \left[\int 24 V_7 \text{Tr}(\sigma^2) + \frac{1}{2} \int \text{Tr}(\kappa \wedge d\kappa \wedge F \wedge F) \right], \quad (9.18)$$

where V_7 denotes the volume form with respect to the metric g . The second term gives a contribution from the contact instanton. In practice, the instanton contribution is complicated. We will continue our discussion by concentrating on contributions from flat connections following [22] and Paper I.

The localization term that is added is of the usual form (6.6). This results in the following full perturbative partition function

$$\int_g d\sigma e^{-\frac{24}{g_7^2} V_7 \text{Tr}(\sigma^2)} \frac{\overbrace{\sqrt{\det_{\Omega_H^{(2,0)}}(S)} \det_{\Omega_H^{(0,2)}}(S) \det_{\Omega^0}(S)}^{\Upsilon} \overbrace{\sqrt{\det_{\Omega^0}(S)}}^c \overbrace{\sqrt{\det_{\Omega^0}(S)}}^{\bar{c}}}{\underbrace{\sqrt{\det_{\Omega^1}(S)}}_A \underbrace{\sqrt{\det_{\Omega_H^{(3,0)}}(S)} \det_{\Omega_H^{(0,3)}}(S)}_{\Phi} \underbrace{\sqrt{\det_{H^0}(S)}}_{b_0} \underbrace{\sqrt{\det_{H^0}(S)}}_{\bar{a}_0}} \quad (9.19)$$

where we have replaced $Q^2 = S$. H^0 refers to harmonic 0-forms. Since they are constant on a compact manifold we can safely discard them (using that they stay unchanged under Q^2). A discussion of Paper I analyzes individual contributions of each field. By ignoring phases and decomposing all forms the result after cancellations is found:

$$Z = \int_g d\sigma e^{-\frac{24}{g_7^2} V_7 \text{Tr}(\sigma^2)} \det'_{\text{adj}} \text{sdet}_{\Omega_H^{(0,\bullet)}}(-\mathcal{L}_R + iG_{a_0}), \quad (9.20)$$

where \det'_{adj} is the determinant over the adjoint representation of the Lie algebra. Note that the superdeterminant means that even forms appear in the numerator and odd forms in the denominator. The next step is to calculate this superdeterminant.

For Sasaki-Einstein manifolds with metric cone $C(X)$, it was first pointed out by Schmude in [41] that the superdeterminant can be obtained by counting holomorphic function on the metric cone, see also [69, 70]. On the horizontal space, as described in [42], due to the complex structure we can define Dolbeault operators; most importantly the operator $\bar{\partial}_H$ which gives rise to the Kohn-Rossi cohomology groups, $H_{KR}^{p,q}$ [71].

More specifically, consider the part of the determinant that does not contain the gauge transformation. Since the Lie derivative \mathcal{L}_R commutes

³One can also set $\sigma = 0$. However, that is not strictly necessary since they are decoupled.

with $\bar{\partial}_H$ the superdeterminant can be evaluated over $H_{KR}^{0,\bullet}$ instead. In a similar fashion as [50] there is a pairing between $H_{KR}^{0,1}$ and $H_{KR}^{0,2}$. But since the latter is zero from simple connectness both vanish. Additionally, there is a pairing between the (0,0)-forms and the (0,3)-forms. As it was argued in [72] the same exists for the Kohn-Rossi cohomology. As a result, the only factor remaining is of the form $H_{KR}^{0,0} \equiv H^0(C(M))$. I.e. the computation reduces to counting holomorphic functions on the moment map cone which we discussed in Chapter 3.1.

Ignoring the Lie algebra part for the moment and using the 1-Gorenstein condition, $\vec{\xi} \cdot \vec{u}_i = 1$ [42] or (3.11), we write our superdeterminant in terms of the moment map cone:

$$\text{sdet}_{\Omega_H^{(0,\bullet)}}(-\mathcal{L}_R + x) = \frac{\prod_{\vec{n} \in C_\mu(X) \cap \mathbb{Z}^4 \setminus \{\vec{0}\}} (\vec{n} \cdot \vec{R} + x)}{\prod_{\vec{n} \in C_\mu^\circ(X) \cap \mathbb{Z}^4} (\vec{n} \cdot \vec{R} - x)}, \quad (9.21)$$

where $C_\mu^\circ(X)$ denotes the interior of the moment map cone. We absorbed the minus signs into the denominator since they do not matter after regularization.

As a result the final action takes the form

$$\begin{aligned} Z_X^{\text{pert}} &= \int_t d\sigma e^{-\frac{24}{g_7^2} V_7 \text{Tr}(\sigma^2)} \prod_\beta i\langle \sigma, \beta \rangle \frac{\prod_{\vec{n} \in C_\mu(X) \cap \mathbb{Z}^4 \setminus \{\vec{0}\}} (\vec{n} \cdot \vec{R} + i\langle \sigma, \beta \rangle)}{\prod_{\vec{n} \in C_\mu^\circ(X) \cap \mathbb{Z}^4} (\vec{n} \cdot \vec{R} - i\langle \sigma, \beta \rangle)} \\ &= \int_t d\sigma e^{-\frac{24}{g_7^2} V_7 \text{Tr}(\sigma^2)} \prod_\beta S_4^{C_\mu(X)}(i\langle \sigma, \beta \rangle | \vec{R}), \end{aligned} \quad (9.22)$$

where

$$S_4^{C_\mu(X)}(x | \vec{R}) = \frac{\prod_{\vec{n} \in C_\mu(X) \cap \mathbb{Z}^4} (\vec{n} \cdot \vec{R} + x)}{\prod_{\vec{n} \in C_\mu^\circ(X) \cap \mathbb{Z}^4} (\vec{n} \cdot \vec{R} - x)}. \quad (9.23)$$

The function $S_4^{C_\mu(X)}$ is known as the generalized quadruple sine function associated to the cone $C_\mu(X)$ [49, 73]. See Paper I in Appendix C for more information on the multiple sine functions. Note that β is the non-zero root of the Lie algebra \mathfrak{g} and t the Cartan subalgebra.

9.2 S^7 and factorization

Let us decode the partition function (9.22) for the example of S^7 . Let T^4 be the torus action that acts according to (2.16). The cone over S^7 is simply \mathbb{C}^4 . The moment map cone will have a form similar to (2.17)

$$\mu : \mathbb{C}^4 \rightarrow \mathbb{R}_{\geq 0}^4 \quad (9.24)$$

$$\mu = \frac{1}{2} \left(|z_1|^2, |z_2|^2, |z_3|^2, |z_4|^2 \right). \quad (9.25)$$

The polytope restricted to $\sum_i |z_i|^2 = 1$ is a pyramid as shown in Figures 9.1 and 9.2. The holomorphic functions on \mathbb{C}^4 correspond to the lattice points in the moment map.

We will denote the lattice points $\vec{n} = (n_1, n_2, n_3, n_4)$ where $n_i \in \mathbb{Z}_{\geq 0}$ since the moment map cone is $C_\mu = \mathbb{R}$. Its interior is $C_\mu^\circ = \mathbb{R}_{>0}$. If we set e^i to be the basis vector associated to each $U(1)$ the associated Reeb vector is of the form $R = \omega_i e^i$ on the moment map cone. In the case of the round sphere $\omega_i = 1$. However we are mostly interested in the squashed sphere in what follows. We have $\vec{n} \cdot \vec{R} = \omega_i n^i$ which is a factor that enters the superdeterminant.

This means that the generalized quadruple sine takes the form

$$S_4(x|\vec{R}) = \frac{\prod_{n_1, n_2, n_3, n_4 \geq 0} (n_1 \omega_1 + n_2 \omega_2 + n_3 \omega_3 + n_4 \omega_4 + x)}{\prod_{n_1, n_2, n_3, n_4 \geq 1} (n_1 \omega_1 + n_2 \omega_2 + n_3 \omega_3 + n_4 \omega_4 - x)}, \quad (9.26)$$

which is an ordinary quadruple sine function.

This result encodes part of the geometry. We will illustrate this following Paper I. The S^7 has two different Hopf fibrations depicted in Figures 9.1 and 9.2.

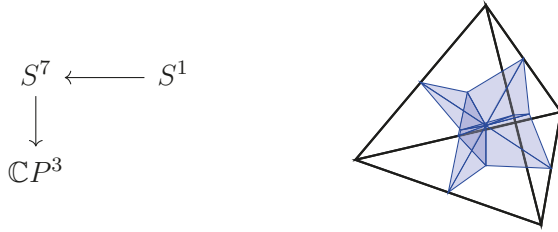


Figure 9.1. This figure presents the S^1 fibration of S^7 which can be visualized by performing a cut on the polytope that gives four pieces of $S^1 \times \mathbb{C}^3$.

Let us start with the first S^1 fibration. The result factorizes according to

$$S_4(x|\vec{R}) \approx \prod_{k=1}^4 (z_k | q_k)_\infty, \quad (9.27)$$

where $z_k = e^{2\pi i \frac{x}{\omega_k}}$ and $q_k = \left(e^{2\pi i \frac{\omega_1}{\omega_k}}, \dots, e^{2\pi i \frac{\omega_{k-1}}{\omega_k}}, e^{2\pi i \frac{\omega_{k+1}}{\omega_k}}, \dots, e^{2\pi i \frac{\omega_4}{\omega_k}} \right)$.

The individual functions are called q -shifted factorials [73]. The proportionality factor is associated to Bernoulli polynomials and out of the scope of this discussion. For further details refer to Paper I and [73].

It is worth mentioning that the q -shifted factorial corresponds to a perturbative Nekrasov partition function on $S^1 \times_{\epsilon} \mathbb{C}^3$ [74]. This hints to the Hopf decomposition of S^7 into four pieces around the closed Reeb orbits, similar to 5d [50, 75]. Locally, in the neighbourhoods of the Reeb orbits, the geometry is $S^1 \times_{\epsilon} \mathbb{C}^3$. We have imposed some twisted periodic boundary condition on S^1 corresponding to \mathbb{C}^3 .

We argue that the full partition function on S^7 can be factorized into four Nekrasov partition functions corresponding to the four closed Reeb orbits

$$Z_{S^7}^{\text{full}} = \prod_{i=1}^4 Z_{S^1 \times \mathbb{C}^3}^{\text{full}}(\beta_i, \epsilon_{1i}, \epsilon_{2i}, \epsilon_{3i}), \quad (9.28)$$

This can be also seen from the polytope in Figure 9.1. Each vertex corresponds only to one Reeb orbit being non-degenerate, as discussed in Section 3.1. The local geometry is $S^1 \times \mathbb{C}^3$.

Also note that every Sasaki-Einstein manifold admits a similar factorization which can be found in Paper I.

S^7 allows for a second Hopf fibration, associated to the fact that, in contrast to S^5 , the 7-sphere is also a 3-Sasaki manifold. This $SU(2)$ -Hopf fibration does not pick only one complex structure but it preserves the properties of the $SU(2) \approx S^3$ complex structure.⁴

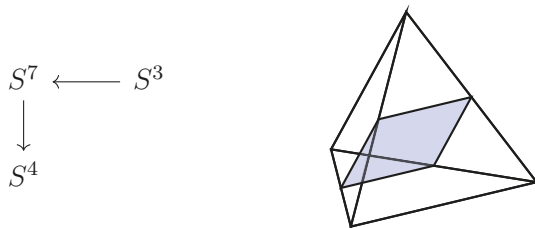


Figure 9.2. This figure shows the S^1 fibration of S^7 visualized by performing a cut on the polytope that locally gives two copies of $S^3 \times \mathbb{C}^2$.

Following the same thought process as before we find a second factorization of the quadruple sine, see Paper I,

$$S_4(x|\vec{\omega}) = \prod_{j_i=0}^{\infty} S_2(x+j_1\omega_3+j_2\omega_4|\omega_1, \omega_2) S_2(x-(j_1+1)\omega_1-(j_2+1)\omega_2|\omega_3, \omega_4). \quad (9.29)$$

It is argued in Paper I that the double sine in the above factorization is the perturbative part of the partition function on $S^3 \times \mathbb{C}^2$. Hence, the

⁴For a short introduction on 3-Sasaki manifolds see Section 8 and the references within.

perturbative part of the full partition function is

$$Z_{S^7}^{\text{pert}} = \prod_{i=1}^2 Z_{S^3 \times \mathbb{C}^2}^{\text{pert}}(a_{i1}, a_{i2}, \epsilon_{i1}, \epsilon_{i2}). \quad (9.30)$$

A relation exists between the parameters a_1, a_2 , which are associated with the S^3 and ϵ_1, ϵ_2 which are associated with the rotations in \mathbb{C}^2 as it is clear from (9.29).

This factorization can be also visualized in the polytope, see Figure 9.2. Each of the two parts after the cut has a $S^3 \times \mathbb{C}^2$ local geometry.

Part IV:

N=4 in 3d mapped to 1d TQM

Localization is a powerful tool for the computation of exact partition functions and correlators of certain local operators in supersymmetric theories. These operators have to be δ -closed, or in other words, in a protected sector with respect to the localizing supercharge, see Section 6. This is one of the reasons why [36] has attracted a lot of interest. This work studied $\mathcal{N} = 2$ SCFT in 4d, and defined new operators that are supported only on a 2-dimensional plane and lie in the cohomology of a specific supercharge. Most importantly, the correlation functions of such operators define an associated chiral algebra. This means that these unconventional operators create a map between $\mathcal{N} = 2$ SCFT in 4d and 2d chiral algebras.

In this part we are going to briefly review the work of [36], focusing mostly on the algebra. We will then turn our attention to three dimensions where the same trick was applied for the first time in flat space [76]. We extend the formulation to $S^2 \times S^1$ following Paper II. We also consider a field theoretic approach, initially introduced by [34]. Finally we will make some comments on the importance of the results of Paper II.

10. SCFT

Before discussing explicit cases we would like to explain the general strategy put forward by the authors in [36]. It can be summarized by the following steps:

- First pick a hyperplane of codimension two.
- Consider the subalgebra that acts on the hyperplane.
- Choose a nilpotent supercharge \mathcal{Q} with certain desirable properties.
- If the subalgebra on the plane is not commuting with \mathcal{Q} , a twisted algebra is derived whose generators either commute with \mathcal{Q} or are \mathcal{Q} -commutators.
- Consider an operator in the \mathcal{Q} -cohomology that is supported only on a point.
- Use the subalgebra to (twist-)translate the operator on the hyperplane.

10.1 4d $\mathcal{N} = 2$

The superconformal algebra of $\mathcal{N} = 2$ on 4-dimensional flat space is $sl(4|2)$. The authors of [36] identified a subalgebra $sl(2) \oplus \widehat{sl(2)} \subset sl(4|2)$, where a specific nilpotent supercharge \mathcal{Q} , $\mathcal{Q}^2 = 0$, acts in a specific way on a bosonic subalgebra. This is performed as follows:

- A plane $\mathbb{C} \subset \mathbb{R}^4$ is chosen.
- The bosonic subalgebra on \mathbb{C} is $sl(2) \oplus sl(2)$, with holomorphic and anti-holomorphic generators is identified.
- A nilpotent supercharge \mathcal{Q} , which commutes with the holomorphic part of the subalgebra, is picked.
- The R -symmetry generators are used in order to twist the anti-holomorphic $sl(2) \rightarrow \widehat{sl(2)}$ to become \mathcal{Q} -exact.
- The operator in the \mathcal{Q} -cohomology at the origin of \mathbb{C} is identified using constraints provided by the algebra.
- The operator is translated along \mathbb{C} using the twisted algebra $sl(2) \oplus \widehat{sl(2)}$ which preserve the cohomology. These are *twisted-translated operators* that are supported on \mathbb{C} and live in the \mathcal{Q} -cohomology.

Let us now expand this strategy and identify the algebra. The $\mathcal{N} = 2$ superconformal algebra in 4d has as a maximal bosonic subalgebra

$so(6) \oplus sl(2)_R \oplus U(1)$.¹ The $so(6)$ is the conformal algebra in four dimensions with generators: the translations P , Lorentz transformation M , dilatation D , and special conformal transformations K . The R -symmetry generators consists of an Abelian factor r and the generators R^\pm and R in the Chevalley basis [77]. Finally, there are 16 fermionic generators, 8 Poincaré $Q^I_\alpha, \tilde{Q}_{I\dot{\alpha}}$, and 8 conformal supercharges $S_I^\alpha, \tilde{S}^{I\dot{\alpha}}$. The Greek letters $\alpha, \dot{\alpha}$ are left and right handed spinor indices respectively; I, J are R -symmetry indices. We are not going to write out the algebra. Its explicit form is e.g. given in Appendix A1 of [36].

Let us concentrate our attention on a specific plane $\mathbb{R}^2 \subset \mathbb{R}^4$. There are two relevant rotation generators: M^\parallel acting within the plane and M^\perp rotating the vector orthogonal to the plane. We define complex coordinates (z, \bar{z}) on \mathbb{R}^2 . The holomorphic generators of $sl(2)$ building up the holomorphic part of the algebra action on the plane, can be expressed as

$$L_- = -\partial_z, \quad L_+ = -z^2\partial_z, \quad L_0 = -z\partial_z. \quad (10.1)$$

These generators can also be identified with part of the conformal algebra in 4d: $L_- = P_{+\dot{+}}$, $L_+ = K^{\dot{+}+}$ and $2L_0 = M^\parallel + D$. The translation and special conformal transformation are appropriately contracted with sigma matrices.

The anti-holomorphic generators on the plane, together with a subset of the supercharges form $sl(2|2)$. The anti-holomorphic part of the conformal algebra on the plane consists of

$$\bar{L}_- = P_{-\dot{-}} = -\partial_{\bar{z}}, \quad \bar{L}_+ = K^{\dot{-}-} = -\bar{z}^2\partial_{\bar{z}}, \quad \bar{L}_0 = \frac{1}{2}(D - M^\parallel) = -\bar{z}\partial_{\bar{z}}. \quad (10.2)$$

The fermionic generators are

$$Q^I = Q^I_-, \quad S_I = S_I^-, \quad \tilde{Q}_I = \tilde{Q}_{I\dot{-}}, \quad \tilde{S} = \tilde{S}^{I\dot{-}}. \quad (10.3)$$

Finally, the algebra has a central element $Z = M^\perp + r$.

The nilpotent supercharges

$$\mathcal{Q}_1 = Q^1 + \tilde{S}^2 \quad \text{and} \quad \mathcal{Q}_2 = S_1 - \tilde{Q}_2, \quad (10.4)$$

commute with the holomorphic $sl(2)$.

Having identified the supercharges, one can find the R -twisted $\widehat{sl(2)}$, which is $\mathcal{Q}_{1,2}$ -exact. Its generators are

$$\hat{L}_+ = \bar{L}_+ - R^+ = \{\mathcal{Q}_1, \tilde{Q}_1\} = -\{\mathcal{Q}_2, Q^2\}, \quad (10.5)$$

$$\hat{L}_- = \bar{L}_- + R^- = \{\mathcal{Q}_1, S_2\} = \{\mathcal{Q}_2, \tilde{S}^1\}, \quad (10.6)$$

$$\hat{L}_0 = \bar{L}_0 - R = \{\mathcal{Q}_1, Q_1^\dagger\} = \{\mathcal{Q}_2, Q_2^\dagger\}. \quad (10.7)$$

¹In this chapter we are going to assume that all algebras are defined over complex numbers.

Furthermore, the central extension

$$\mathcal{Z} = -\{\mathcal{Q}_1, \mathcal{Q}_2\}, \quad (10.8)$$

is also $\mathcal{Q}_{1,2}$ -exact. This twisted algebra still acts anti-holomorphically on the plane.

Having identified the algebra, we can look into the local operators. Let us assume that the desired operator is inserted at the center of \mathbb{R}^4 . We would like it to be in a non-trivial cohomology of both supercharges $\mathcal{Q}_{1,2}$. For simplicity we define a new family of supercharges $\mathcal{Q} = \mathcal{Q}_1 + \xi \mathcal{Q}_2$ for an arbitrary parameter ξ . The operator that is inserted at zero, $\mathcal{O}(0)$, is a representative of the non-trivial \mathcal{Q} -cohomology if

$$\{\mathcal{Q}, \mathcal{O}(0)\} = 0 \quad \text{and} \quad \mathcal{O}(0) \neq \{\mathcal{Q}, \mathcal{O}'(0)\}. \quad (10.9)$$

The appropriate commutator/anti-commutator is implicitly used depending on the nature of the operator.

One can identify these operators by their quantum numbers. Since \hat{L}_0 and Z are both \mathcal{Q} -exact and they commute with \mathcal{Q} , the operator lives in the zero-eigenspace of both. This translates to the following conditions for these operators.

$$\frac{1}{2}(E - j_1 - j_2) - R = 0 \quad \text{and} \quad r + j_1 - j_2 = 0, \quad (10.10)$$

where E is the eigenvalue of D , (the conformal dimension), $j_{1,2}$ are Lorentz quantum numbers and R is the R -charge. These operators are called Schur operators since they coincide with operators surviving in the Schur limit of the superconformal index [78–80].

Until now, we identified the operator at the origin of \mathbb{R}^2 . We can twist-translate this operator on the plane using the above algebra:

$$\mathcal{O}(z, \bar{z}) = e^{zL_- + \bar{z}\hat{L}_-} \mathcal{O}(0) e^{-zL_- - \bar{z}\hat{L}_-}. \quad (10.11)$$

We used that $\{\mathcal{Q}, \mathcal{O}(z, \bar{z})\} = 0$, since L_- is \mathcal{Q} -closed and \hat{L}_- is \mathcal{Q} -exact with \mathcal{Q} nilpotent. Besides, since $\mathcal{O}(z, \bar{z}) = \{\mathcal{Q}, \mathcal{O}'(z, \bar{z})\}$ would imply that $\mathcal{O}(0) = \{\mathcal{Q}, \mathcal{O}'(0)\}$ – which is not the case – it is clear that $\mathcal{O}(z, \bar{z})$ defines a \mathcal{Q} cohomology class on the plane.

Let us consider an operator of the form $\mathcal{O}^{I_1 \dots I_{2k}}$, with $I_i = 1, 2$, transforming under the spin representation k of $sl(2)_R$. In [36] it was proven that the Schur operators at the origin are the highest weight states $\mathcal{O}^{1 \dots 1}$ and the twisted-translated operators at any point are defined as

$$\mathcal{O}(z, \bar{z}) = u_{I_1} \dots u_{I_{2k}} \mathcal{O}^{I_1 \dots I_{2k}}, \quad (10.12)$$

where the dressing factor $u_I = (1, \bar{z})$.

In [36] it was furthermore shown that these operators have meromorphic correlators up to a \mathcal{Q} -exact term. The operators form an infinite-dimensional chiral algebra which establishes a map between 4d CFTs and chiral algebras in two dimensions.

An illustrative example is the free hypermultiplet in four dimensions. Using (10.10), we find that the Schur operators at the origin are the scalars Q and \tilde{Q} . The corresponding twisted-translated operators become

$$q_1 := Q(z, \bar{z}) + \bar{z}\tilde{Q}^*(z, \bar{z}) \quad \text{and} \quad q_2 := \tilde{Q}(z, \bar{z}) - \bar{z}Q^*(z, \bar{z}). \quad (10.13)$$

Their OPE is worked out to be

$$q_i(z)q_j(w) \approx \frac{\epsilon_{ij}}{z-w} + \text{regular terms}. \quad (10.14)$$

In this case, the correlators are indeed meromorphic and there is no need for extraction of a \mathcal{Q} -exact term.

10.2 3d $\mathcal{N} = 4$

We now turn our attention to three dimensions and work in \mathbb{R}^3 with coordinates $x = (x_1, x_2, x_3)$. We will review the work [81]. The total superconformal algebra is $osp(4|4)$. It contains as a subalgebra $su(2|2)$ which is the superconformal algebra of a 1d SCFT with 8 real supercharges.

Let us briefly analyze the superconformal algebra in three dimensions. Its maximal bosonic subalgebra is

$$osp(4|4) \supset so(3, 2) \oplus su(2)_C \oplus su(2)_H. \quad (10.15)$$

The 3d conformal algebra consists of P_μ , $M_{\mu\nu}$, D , and K_μ . These generate translations, Lorentz transformations, dilatation, and superconformal transformations respectively ($\mu, \nu = 1, 2, 3$). The generators of the left and right R -symmetries, $su(2)_C$ and $su(2)_H$, are R^a_b and $\bar{R}^{\dot{a}}_{\dot{b}}$, where the undotted indices are $su(2)_C$ (Coulomb) indices and the dotted ones are $su(2)_H$ (Higgs) indices.² Finally, the super-generators of $osp(4|4)$ consists of eight Poincaré supercharges $Q_{\alpha a \dot{a}}$ and eight conformal supercharges $S^{\alpha}_{a \dot{a}}$, where $\alpha = 1, 2$ is the spinor index.

Consider the subalgebra $su(2|2)$. We can separate \mathbb{R}^3 into $\mathbb{R} \times \mathbb{R}^2$ where we assume $(x_2, x_3) \in \mathbb{R}^2$. The $su(2|2)$ is created by 1d conformal transformations $sl(2)$. These are the generators that act on the line: $P \equiv P_1$, $K \equiv K_1$, D and an $su(2)_R$ R -symmetry, which we can choose to identify with $su(2)_C$. By looking at the $osp(4|4)$ superalgebra³ one can

²The right and left R -symmetries are also called Coulomb and Higgs respectively.

³We are not rewriting the algebra here. See [76] equations (B.31)–(B.37), where one needs to look for which supercharges the algebra with only P , K , D and R closes. This can be done up to an $su(2)_R$ rotation.

consider, as fermionic generators of the $su(2|2)$, the $Q_{1a\dot{2}}, Q_{2a\dot{1}}, S^1_{a\dot{1}}$ and $S^2_{a\dot{2}}$. Finally, there exists a central extension $Z \equiv iM^\perp - R^1_{\dot{1}}$ where M^\perp is the Lorentz transformation whose fixed locus is the line.

The strategy is similar as in 4d:

- First, we will pick a line and explain the superconformal algebra acting on it.
- Then we find nilpotent supercharges \mathcal{Q} i.e. $\mathcal{Q}^2 = 0$. There are several choices for defining them, e.g.

$$\mathcal{Q}_1 = Q_1^{1\dot{2}} - S^2_{1\dot{2}} \quad \text{and} \quad \mathcal{Q}_2 = S^1_{1\dot{1}} + Q_2^{1\dot{2}}, \quad (10.16)$$

following [81]. It should be noted that again the commutator of the two supercharges gives the central charge

$$\{\mathcal{Q}_1, \mathcal{Q}_2\} = 2Z, \quad (10.17)$$

as in (10.8).

- We twist the bosonic $sl(2)$ in such a way that the twisted operators are $\mathcal{Q}_{1,2}$ -exact. The generators are

$$\hat{L}_- = P_{+-} + R^2_{\dot{1}} = \frac{1}{2}\{\mathcal{Q}_1, Q_2^{2\dot{2}}\} = -\frac{1}{2}\{\mathcal{Q}_2, Q_1^{2\dot{1}}\}, \quad (10.18)$$

$$\hat{L}_+ = K^{+-} - R^1_{\dot{2}} = -\frac{1}{4}\{\mathcal{Q}_1, S^1_{1\dot{1}}\} = \frac{1}{4}\{\mathcal{Q}_2, S^2_{1\dot{2}}\}, \quad (10.19)$$

$$\hat{L}_0 = D - R^1_{\dot{1}} = \frac{1}{4}\{\mathcal{Q}_1, \mathcal{Q}_1^\dagger\} = \frac{1}{4}\{\mathcal{Q}_2, \mathcal{Q}_2^\dagger\}. \quad (10.20)$$

They close in an $sl(2)$ subalgebra, $[\hat{L}_0, \hat{L}_\pm] = \pm \hat{L}_\pm$ and $[\hat{L}_+, \hat{L}_-] = -2\hat{L}_0$. This algebra is $su(2)_R$ twisted with respect to the original one and as such it is referred to as $\widehat{sl(2)}$.

From now on we are going to consider the one parameter family $\mathcal{Q} = \mathcal{Q}_1 + \xi \mathcal{Q}_2$, with parameter ξ .

- The operators at the origin of the plane $\mathcal{O}(0)$ are identified, which satisfy some non-trivial \mathcal{Q} -cohomology as in (10.9).
- One can use \hat{L}_- to translate them along the line

$$\begin{aligned} \mathcal{O}(x_1) &= e^{-ix_1 \hat{L}_-} \mathcal{O}(0) e^{ix_1 \hat{L}_-} \\ &= u^{I_1}(x_1) \dots u^{I_k}(x_1) \mathcal{O}_{I_1 \dots I_k}, \end{aligned} \quad (10.21)$$

with $u^I \equiv (1, x_1)$ similarly to the 4d spin representation (10.12). These are the desired operators. Note that, since the submanifold is one dimensional, the only generator needed for translating the operators is the one contained in the twisted $\widehat{sl(2)}$.

Since the twisted translated operators are \mathcal{Q} -exact, the operator defined above is independent of x_1 at the level of the cohomology. The only

relevant information is the ordering of operators along the line, since the operators cannot change order without leaving the kernel of \mathcal{Q} .

The OPEs and the correlators of these operators are well-defined at the level of the \mathcal{Q} -cohomology. In fact, since the original SCFT OPE algebra is associative, the resulting OPE remains associative, as one can first compute the full OPE and pass over to the cohomology in the end. The resulting algebra of operators is associative and depends only on the ordering of the operators on the line. Finally, there exists an evaluation map: one can take the expectation values of these operators by evaluating the correlation functions in the full SCFT. Operator algebras with these characteristics are called *topological*.

As an example, let us consider the case of the free hypermultiplet which contains a scalar q^a and a fermion $\psi^{\dot{a}}$. When restricting ourselves to the origin of \mathbb{R} , using the quantum numbers of the fields, we find that the desired operator is $q^1(0)$. The twisted translated operators are of the form

$$q(x_1) := q^1(x_1) + x_1 q^2. \quad (10.22)$$

Using the original OPE

$$q^a(x)q^b(y) = \frac{\epsilon^{ab}}{|x-y|} + \dots \quad (10.23)$$

where the ellipsis refers to regular terms, the OPE of the operators in the \mathcal{Q} -cohomology reads

$$q(0)q(x) \approx \text{sgn}(x) + \dots \quad (10.24)$$

In fact, one can introduce an operation $*$ which denotes the multiplication of \mathcal{Q} -cohomology class of local operators ordered from left to right on the line, i.e.

$$\mathcal{O}_1 * \mathcal{O}_2 := \mathcal{O}_1(x) * \mathcal{O}_2(y), \quad x < y. \quad (10.25)$$

For the above example we have

$$q(0) * q(x) \approx 1 + \dots \quad (10.26)$$

which is, indeed, topological since it does not depend on the coordinates.⁴

⁴See also Section 13 for the formal definition of topological theories.

11. $S^2 \times S^1$ Hypermultiplet

The goal of this section is to apply the same procedure to theories without superconformal symmetry. A first example, namely S^3 , was presented in [34] using $\mathcal{N} = 4$ hypermultiplets.

In this section we are looking into a supersymmetric theory on $S^2 \times S^1$. In particular, we use the coordinates

$$ds^2 = r^2 \beta^2 dt^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (11.1)$$

where t is the coordinate on S^1 and β is its radius. The other two coordinates are parametrizing the S^2 as depicted in Figure 11.1. Notice that in these coordinates the S^2 is written as a fibre of S^1 over an interval.

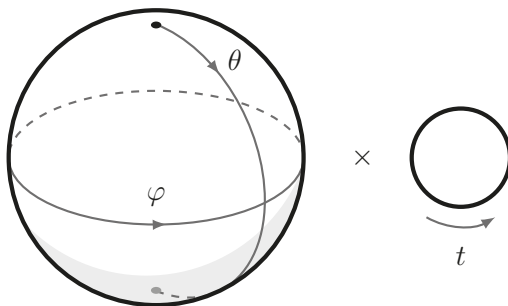


Figure 11.1. The choice of coordinates on $S^2 \times S^1$, extracted from Paper II.

The same construction as before can be applied here since the subalgebra used to translate the specific operators did not contain any conformal transformations. In particular, the algebra is realized via spacetime isometries and not as part of a superconformal algebra. We have eight supergenerators and the bosonic algebra is $su(2) \oplus u(1) \oplus su(2)_R$.

The way to realize the R -symmetry is via a background R -symmetry connection A . Starting from the conformal case of $S^2 \times \mathbb{R}$ and then compactifying the \mathbb{R} dimension, one needs to turn on a background R -symmetry connection. Our choice of R -connection is $(A_H)^a_b \neq 0$ (breaking the $su(2)_H$). This leads to the bosonic algebra $su(2) \oplus u(1) \oplus su(2)_C$.¹ The generators are $R_{\dot{a}\dot{b}}$ for the R -symmetry, z for the $u(1)$ on the S^1 , and j_3, j_{\pm} for the $su(2)$ on the S^2 .

¹Equivalently, one can choose to break the $su(2)_C$ using a background R -symmetry connection $(A_C)^{\dot{a}}_{\dot{b}} \neq 0$. The fixed locus will also be different in that case.

The supersymmetric part of the algebra is

$$\{Q_{11\dot{a}}, Q_{12\dot{c}}\} = -r^{-1}\epsilon_{\dot{a}\dot{c}}J_+ \quad (11.2)$$

$$\{Q_{21\dot{a}}, Q_{22\dot{c}}\} = -r^{-1}\epsilon_{\dot{a}\dot{c}}J_- \quad (11.3)$$

$$\{Q_{11\dot{a}}, Q_{22\dot{c}}\} = -r^{-1}\epsilon_{\dot{a}\dot{c}}(iJ_3 + \mathcal{Z}) - ir^{-1}R_{\dot{a}\dot{c}} \quad (11.4)$$

$$\{Q_{21\dot{a}}, Q_{12\dot{c}}\} = -r^{-1}\epsilon_{\dot{a}\dot{c}}(iJ_3 - \mathcal{Z}) - ir^{-1}R_{\dot{a}\dot{c}} \quad (11.5)$$

where we defined

$$J_* = -\mathcal{L}_{j_*}, \quad \text{and} \quad \mathcal{Z} = -\mathcal{L}_z \quad (11.6)$$

following Paper II. The precise definition of R can be found in (4.17) of Paper II.

We will now focus on the twisted subalgebra of

$$\mathcal{Q} = \frac{i}{2}(\mathcal{Q}_1 + \mathcal{Q}_2), \quad (11.7)$$

where $\mathcal{Q}_1 = Q_{11\dot{2}} + Q_{12\dot{1}}$ and $\mathcal{Q}_2 = Q_{21\dot{2}} + Q_{22\dot{1}}$. It turns out that both, \mathcal{Q}_1 and \mathcal{Q}_2 , are nilpotent. Any combination thereof is equivalent.

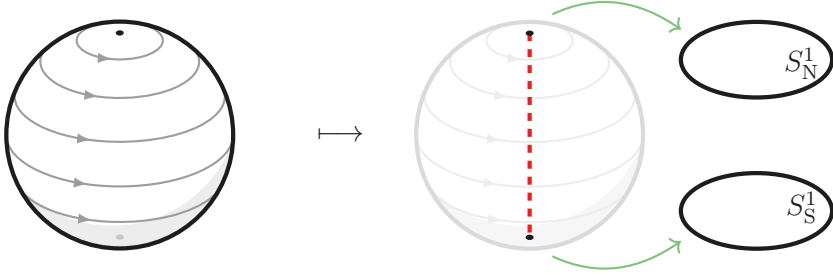


Figure 11.2. The fixed point locus $S^2 \times S^1$ is the disjoint union of two copies of S^1 located at the poles of the two-sphere. The figure is part of Paper II

As in the previous case one has to find twisted operators that live in the fixed locus of $\mathcal{Q}^2 \approx J_3 - R_{1\dot{2}}$. If these operators are singlets under $su(2)_C$ the operators live in the fixed locus of $J_3 \approx \mathcal{L}_{\partial_\phi}$. This contains the fixed locus on S^2 which consists of two points, the north pole (NP) and the south pole (SP). The total locus on $S^2 \times S^1$ is $\{\text{NP}\} \times S^1$ and $\{\text{SP}\} \times S^1$, see Figure 11.2.

For an $\mathcal{N} = 4$ hypermultiplet the operators are the scalars q^a similar to in (10.22). These are indeed singlets under $su(2)_C$ and the above fixed locus discussion goes trivially through.

It was shown in Paper II that, on the two disjoint parts of the fixed locus, the operators are

$$q^\pm = u_a^\pm q^a = q^1 \pm q^2, \quad (11.8)$$

where plus and minus indicate north and south pole submanifolds respectively. Notice that the dressing factor takes the form $u_a^\pm = (1, \pm 1)$.

12. Partial localization

Another way of realizing this procedure is using a localization technique. It was originally introduced in [34] and additionally studied in [35, 82–85]. In Paper II it was generalized for a large class of manifolds for hypermultiplets.

The general strategy is as follows. Let us consider a closed 3d manifold \mathcal{M} with metric g . Take a 3d theory with an $\mathcal{N} = 4$ hypermultiplet $(q_a, \tilde{q}_a, \psi_{\dot{a}}, \tilde{\psi}_{\dot{a}})$, where the scalars q, \tilde{q} belong to the (2,1) and the fermions to the (1,2) representation of $su(2)_H \oplus su(2)_C$. The R -symmetry currents are coupled to the background flat connections A_H and A_C . The theory is coupled to a background vector multiplet with a connection A . We will pack all these connections into a covariant derivative $D = \nabla - i(A + A_H + A_C)$.

The supercharge $Q_{a\dot{a}}$ is associated with a spinor $\xi^{a\dot{a}}$. From now on we are going to omit the explicit dependence on the spinor index α and treat objects as matrices. The spinor $\xi^{a\dot{a}}$ satisfies the conformal killing spinor equation for an auxiliary spinor $\tilde{\xi}^{a\dot{a}}$. The supersymmetric transformations are of the form

$$\delta q^a = \xi^{a\dot{a}} \psi_{\dot{a}}, \quad \delta \tilde{q}^a = \xi^{a\dot{a}} \tilde{\psi}_{\dot{a}}, \quad (12.1)$$

$$\delta \psi_{\dot{a}} = i\gamma^\mu \xi_{a\dot{a}} D_\mu q^a + i\tilde{\xi}_{a\dot{a}} q^a - i\xi_{a\dot{c}} \Phi_{\dot{a}}^{\dot{c}} q^a, -i\nu_{a\dot{a}}, \quad (12.2)$$

$$\delta \tilde{\psi}_{\dot{a}} = i\gamma^\mu \xi_{a\dot{a}} D_\mu \tilde{q}^a + i\tilde{\xi}_{a\dot{a}} \tilde{q}^a - i\xi_{a\dot{c}} \tilde{q}^a \Phi_{\dot{a}}^{\dot{c}} - i\nu_{a\dot{a}} \tilde{G}^a, \quad (12.3)$$

$$\delta G^a = \nu^{a\dot{a}} (\gamma^\mu D_\mu \psi_{\dot{a}} - \Phi_{\dot{a}\dot{c}} \psi^{\dot{c}}), \quad (12.4)$$

$$\delta \tilde{G}^a = \nu^{a\dot{a}} (\gamma^\mu D_\mu \tilde{\psi}_{\dot{a}} - \Phi_{\dot{a}\dot{c}} \tilde{\psi}^{\dot{c}}), \quad (12.5)$$

where we have introduced two auxiliary fields G, \tilde{G} together with their associated auxiliary spinors $\nu^{a\dot{a}}$. This ensures off-shell closure of the supersymmetry transformations.¹

Finally, the action is

$$S = \int_{\mathcal{M}} \star \mathcal{L}, \quad \text{where} \\ \mathcal{L} = D^\mu \tilde{q}^a D_\mu q_a - i\tilde{\psi}^{\dot{a}} \gamma^\mu D_\mu \psi_{\dot{a}} + R/8 \tilde{q}^a q_a + \tilde{G}^a G_a \\ - \tilde{q}^a (iD_{ac} + 1/2 \epsilon_{ac} \Phi^{\dot{a}\dot{c}} \Phi_{\dot{a}\dot{c}}) q^c + i\tilde{\psi}_{\dot{a}} \Phi^{\dot{a}\dot{c}} \psi_{\dot{c}}. \quad (12.6)$$

¹The auxiliary spinors satisfy some conditions mentioned in (2.15) of Paper II.

The supersymmetry transformations square to the \mathcal{L}_v , where v is the killing vector

$$v^\mu = i\xi^{a\dot{a}}\gamma^\mu\xi_{a\dot{a}}. \quad (12.7)$$

In particular v satisfies $\mathcal{L}_v g = 0$.

Let M_v be the fixed locus manifold of v . In this manuscript we study the Abelian case for simplicity, the non-Abelian version can be found in Paper II.

We are going to apply the localization argument from Section 5. Hence we introduce the equivariant differential $d_v = d - \iota_v$ and an equivariant closed polyform Ω such that $d_v\Omega = 0$. This polyform is given by

$$\Omega = *\mathcal{L} + \alpha_1, \quad (12.8)$$

where α_1 is a 1-form and $*\mathcal{L}$ is a 3-form or a top form in three dimensions. This means we are allowed to replace

$$\int_{\mathcal{M}} *\mathcal{L} \rightarrow \int_{\mathcal{M}} \Omega = \int_{\mathcal{M}_v} \frac{\iota^*\alpha_1}{e(N\mathcal{M}_v)}. \quad (12.9)$$

In the last step we used the localization formula from (5.16).

The only remaining task is to identify the 1-form α_1 . This was done in Paper II using that Ω is equivariantly closed:

$$d_v(*\mathcal{L} + \alpha_1) = 0 \rightarrow \iota_v(*\mathcal{L}) = d\alpha_1 \quad \text{and} \quad \iota_v\alpha_1 = 0, \quad (12.10)$$

so that

$$\alpha_{1\mu} = X_{ac}(\tilde{q}^a D_\mu q^c - D_\mu \tilde{q}^a q^c) + \tilde{q}^a q_a w_\mu + 2(\Lambda_{ac})_\mu \tilde{q}^a q^c. \quad (12.11)$$

The newly introduced matrices that are functions of the killing spinor:

$$X_{ac} = \xi_a^{\dot{a}} \xi_{c\dot{a}}, \quad w_\mu = \xi^{a\dot{a}} \gamma_\mu \tilde{\xi}_{a\dot{a}}, \quad (12.12)$$

$$(\Lambda_{ac})_\mu = (\xi_{a\dot{a}} \gamma_\mu \xi_{c\dot{c}}) \Phi^{\dot{a}\dot{c}}. \quad (12.13)$$

12.1 $S^2 \times S^1$

Let us apply the prescription discussed in Chapter 11 to $S^2 \times S^1$.

As already noted the choice of the supercharge (11.7) means that the killing vector is along the equator of S^2 , $v = -r^{-1}\partial_\phi$. The localizing locus is the disjoint union of $\mathcal{M}_v = (\{\text{NP}\} \times S^1) \cup (\{\text{SP}\} \times S^1)$, which is shown in Figure 11.2. By using the localization procedure described above one finds that on the north pole the localizing integral (12.9) simplifies to

$$S_N = 2\pi r \oint_{S^1} dt \tilde{q}^+ (\partial_t - i\zeta) q^+, \quad (12.14)$$

having used the definition of α_1 and restricting to $\{NP\} \times S^1$. Here, ζ is a BPS combination of the vector multiplet fields: $\zeta = a - i\beta\sigma$, where a and σ are the BPS connection and scalar field of the vector multiplet respectively.

Similarly, for the south pole we have

$$S_S = -2\pi r \oint_{S^1} dt \tilde{q}^- (\partial_t - i\zeta^*) q^- . \quad (12.15)$$

It is important to notice that we arrived exactly at the BPS operators (11.8). One can combine $\mathcal{D}_N = \partial_t - i\zeta$ and $\mathcal{D}_S = \partial_t - i\zeta^*$, which can be seen as twisted covariant derivatives. The final representation of the action is

$$S = 2\pi r \oint_{\{NP\} \times S^1} dt \tilde{q}^+ \mathcal{D}_N q^+ - 2\pi r \oint_{\{SP\} \times S^1} dt \tilde{q}^- \mathcal{D}_S q^- . \quad (12.16)$$

This is the form of two copies of quantum mechanics.

The field configurations are found to be

$$\begin{aligned} q^+(\theta, t) &= \sum_{k \in \mathbb{Z}} e^{-ikt} [u_{+,k} \cosh(\omega_k \cos \theta) + i v_{+,k} \sinh(\omega_k \cos \theta)] , \\ q^-(\theta, t) &= \sum_{k \in \mathbb{Z}} e^{-ikt} [u_{-,k} \cosh(\omega_k \cos \theta) - i v_{-,k} \sinh(\omega_k \cos \theta)] , \\ \tilde{q}^+(\theta, t) &= \sum_{k \in \mathbb{Z}} e^{+ikt} [u_{+,k}^* \cosh(\omega_k \cos \theta) + i v_{+,k}^* \sinh(\omega_k \cos \theta)] , \\ \tilde{q}^-(\theta, t) &= \sum_{k \in \mathbb{Z}} e^{+ikt} [-u_{+,k}^* \cosh(\omega_k \cos \theta) + i v_{+,k}^* \sinh(\omega_k \cos \theta)] , \end{aligned} \quad (12.17)$$

by imposing the BPS equation $\delta\psi_{\dot{\alpha}} = \delta\tilde{\psi}_{\dot{\alpha}} = 0$ and the reality conditions $\tilde{q}^a = (q_a)^*$ and $\tilde{G}^a = (G_a)^*$. We have that

$$\omega_k = \beta^{-1} |k + \zeta| , \quad (12.18)$$

and constant u 's and v 's such that

$$\begin{aligned} u_{-,k} &= \frac{k + \zeta}{|k + \zeta|} v_{+,k} , \\ v_{-,k} &= \frac{k + \zeta}{|k + \zeta|} u_{+,k} . \end{aligned} \quad (12.19)$$

These configurations are evaluated on the north and south pole accordingly. The one-dimensional action will only contain \tilde{q}_N^+ , q_N^+ , \tilde{q}_S^- , and q_S^- . The action in (12.16) corresponds to a quantum mechanical system with path integral

$$Z = \int_{\text{BPS}} [d\tilde{q}_N^+] [dq_N^+] [d\tilde{q}_S^-] [dq_S^-] e^{-S_{\text{1d}}} . \quad (12.20)$$

For simplicity, we will drop the subscripts N/S as the superscript \pm is sufficient to resolve the ambiguity.

By plugging the solutions (12.17) into the one-dimensional action, one finds

$$S_{1d} = 4\pi^2 r \sum_{k \in \mathbb{Z}} |k + \zeta| \left[(|u_k|^2 + |v_k|^2) \sinh(2\omega_k) - i(u_k v_k^* + u_k^* v_k) \right], \quad (12.21)$$

whose real part is manifestly positive semi-definite. Note that we have used (12.19) to eliminate $v_{-,k}$ and $u_{-,k}$. We have also dropped the sign subscript of u and v .

We can rewrite the partition function using

$$\mathbf{z}_k = \begin{pmatrix} u_k \\ v_k \end{pmatrix}, \quad \mathbf{M}_k = 4\pi^2 r |k + \zeta| \begin{pmatrix} \sinh(2\omega_k) & -i \\ -i & \sinh(2\omega_k) \end{pmatrix}. \quad (12.22)$$

and performing the Gaussian integration over each Fourier mode:

$$\begin{aligned} Z_{1d}(\zeta, \zeta^*) &= \prod_{k \in \mathbb{Z}} \int_{\mathbb{C}^2} du_k du_k^* dv_k dv_k^* \cosh^2(2\omega_k) e^{-\mathbf{z}_k^\dagger \mathbf{M}_k \mathbf{z}_k} \\ &= \prod_{k \in \mathbb{Z}} \frac{1}{4\pi^2 r^2 |k + \zeta|^2}. \end{aligned} \quad (12.23)$$

Zeta-function regularization can be used to simplify the partition function. Specifically, using

$$\prod_{k \in \mathbb{Z}} |ak + b|^2 = 4 |\sin(\pi b/a)|^2, \quad (12.24)$$

if $a > 0$ and $b \in \mathbb{C}$. The partition function takes the form

$$Z(\zeta, \zeta^*) = \frac{1}{4 |\sin(\pi \zeta)|^2}. \quad (12.25)$$

This partition function was generalized in Paper II for the case of multiple hypermultiplets. Also, it is important to note that this partition function matches the one obtained by performing a direct localization computation with the three-dimensional hypermultiplet as was shown in Paper II. Hence, a mapping between 3d $\mathcal{N} = 4$ theories and 1d quantum mechanics of protected operators is established.

One can extract the correlators of the Fourier modes using the inverse of the matrix \mathbf{M}_k :

$$\begin{aligned} \langle u_k^* u_{k'} \rangle &= \langle v_k^* v_{k'} \rangle = \delta_{k,k'} \frac{\sinh(2\omega_k)}{4\pi^2 r |k + \zeta| \cosh^2(2\omega_k)}, \\ \langle u_k^* v_{k'} \rangle &= \langle v_k^* u_{k'} \rangle = \delta_{k,k'} \frac{i}{4\pi^2 r |k + \zeta| \cosh^2(2\omega_k)}, \end{aligned} \quad (12.26)$$

and combine them according to (12.17) into

$$\begin{aligned}\langle \tilde{q}^+(t_2) q^+(t_1) \rangle &= +\frac{i}{4\pi^2 r} \sum_{k \in \mathbb{Z}} \frac{1}{k + \zeta} e^{ik(t_2 - t_1)} , \\ \langle \tilde{q}^-(t_2) q^-(t_1) \rangle &= -\frac{i}{4\pi^2 r} \sum_{k \in \mathbb{Z}} \frac{1}{k + \zeta^*} e^{ik(t_2 - t_1)} .\end{aligned}\tag{12.27}$$

Performing the summation, the correlators take the form

$$\langle \tilde{q}^\pm(t_2) q^\pm(t_1) \rangle = \mathcal{G}_\pm(t_2 - t_1) ,\tag{12.28}$$

where

$$\begin{aligned}\mathcal{G}_+(t) &= -\frac{\text{sign}(t) - i \cot(\pi \zeta)}{4\pi r} e^{-i\zeta t} , \\ \mathcal{G}_-(t) &= +\frac{\text{sign}(t) - i \cot(\pi \zeta^*)}{4\pi r} e^{-i\zeta^* t} .\end{aligned}\tag{12.29}$$

It is important to note that these correlators are only valid in the range $t_1, t_2 \in [0, 2\pi)$, or $t \in (-2\pi, 2\pi)$ for $t = t_2 - t_1$. Outside this domain the following periodicity condition is enforced

$$\mathcal{G}_\pm(t) = \mathcal{G}_\pm(t + 2\pi) .\tag{12.30}$$

For the free hypermultiplet, i.e. $\zeta = 0$, these correlators are topological as the ones found in Section 10.2. It was argued in [34] that the resulting one-dimensional quantum mechanical system of the gauge hypermultiplet can be viewed as a *gauged topological quantum mechanics*, where ζ plays the role of the gauge field.

In Paper II we studied the path integral of such a quantum mechanical system by addressing the problem of identifying the correct integration cycle along the lines of [86]. Indeed, it was found that using the cycle (12.17) leads to the same expression for the partition function. Additionally, Paper II studies the algebra for the flavour currents obtained via the operators \tilde{q}^\pm and q^\pm resulting in an affine algebra. Finally, we also looked at the theory in the presence of surface defects supported on S^2 . The two-dimensional theory (which includes vector multiplets) can be localized with the same supercharge used for the three-dimensional hypermultiplet. Furthermore, it is immediate to extend our computation to include insertions of appropriate BPS operators coming from the multiplets on S^2 , e.g [87, 88].

Part V:

$N = 2$ in 4d and equivariant twisting

This part introduces topological twisting in $\mathcal{N} = 2$ supersymmetric field theories in four dimensions. This procedure was first explored by Witten in [12], and has induced important developments in mathematical physics. We are first going to review Witten's work and explain the concept of *twisting*. We will continue by explaining the key concepts of Paper III and the connection of the twisting procedure with localization.

13. Topological twisting

We begin with a review of topological field theories and their importance for the discussion afterwards. We are going to follow [89] for most of the key concepts, thus we refer the reader there for a more thorough discussion.

13.1 Review of topological field theories

Let us consider a 4d Riemannian manifold (\mathcal{M}, g) together with a quantum field theory defined thereon. In general, the observables of this theory will depend on the metric g . In the case of a topological quantum field theory (TQFT) there is a sector of operators whose observables do not depend on the metric. These observables are called *topological observables*. If we denote by $\{\mathcal{O}_i\}$ the set of topological operators the independence of the metric is equivalent to

$$\frac{\delta}{\delta g_{\mu\nu}} \langle \mathcal{O}_{i_1} \dots \mathcal{O}_{i_m} \rangle = 0. \quad (13.1)$$

Topological quantum field theories are categorized into two types, the Schwarz and the Witten/cohomological type. Here we recap their definition:

- Schwarz type: theories where the action and the observables are manifestly independent of the metric. A well-known example is Chern-Simons theory [12].
- Cohomological type: more subtle TQFTs with an action and operators depending on the metric. However the observables do not. This is possible in the following way: Let us consider a scalar, non-anomalous symmetry δ which acts on the fields of the theory $\{\Phi_i\}$, such that the action is invariant,

$$\delta S[\Phi_i] = 0. \quad (13.2)$$

Operators with correlators whose variation with respect to the metric are δ -closed are called *topological*. An illustrative example is the energy momentum tensor [89]

$$T_{\mu\nu} = \frac{\delta S}{\delta g_{\mu\nu}}. \quad (13.3)$$

If the energy momentum tensor is δ -exact, i.e.

$$T = \delta\alpha, \quad (13.4)$$

for some tensor α , the theory is topological since for a set of operators \mathcal{O}

$$\frac{\delta}{\delta g_{\mu\nu}} \langle \mathcal{O} \rangle \approx \langle \mathcal{O} T_{\mu\nu} \rangle = \pm \langle \delta(\mathcal{O}\alpha) \rangle = 0, \quad (13.5)$$

where we assumed the vacuum is invariant under δ . The \pm comes from the property of the scalar symmetry which can be either even or odd. The symmetry that we are going to discuss, i.e the supersymmetry variation, is odd.

We are going to restrict ourselves to the latter type of theories.

It is important to note that for a nilpotent δ , i.e. $\delta^2 = 0$, operators that are δ -exact have vanishing correlators as can be easily see from (13.5). Thus, we are focusing only on operators in the cohomology of δ .

In the generic case of $L := \delta^2 \neq 0$ we will additionally need to restrict to operators that are invariant under the action of L . These operators live in the equivariant cohomology of δ as discussed in Section 4. As such, the language of equivariant cohomology will continue being at the heart of our formalism.

13.2 Witten's $\mathcal{N} = 2$ twisting

In this section we describes the work of Witten [12], where he obtained topological field theories of the cohomological type. It is based in a procedure called twisting which we summarize in the following paragraphs.

We start with an $\mathcal{N} = 2$ supersymmetric field theory in four dimensional flat space. The superconformal algebra was introduced in Section 10.1. We will not need the full superconformal algebra but rather the super-subalgebra which contains only the supersymmetric generators Q^I_α , $\tilde{Q}_{I\dot{\alpha}}$, the rotation group $su(2)_l \oplus su(2)_r$, and generators $M_{\alpha\beta}$ and $M_{\dot{\alpha}\dot{\beta}}$ respectively. The R -symmetry remains the same and we have $su(2)_R$ with generators R_{IJ} .

The relevant commutation relations of the algebra are

$$\begin{aligned} [M_{\alpha\beta}, Q_{I\gamma}] &= \epsilon_{\gamma(\alpha} Q_{I\beta)}, & [M_{\alpha\beta}, \tilde{Q}_{I\dot{\gamma}}] &= 0, \\ [\tilde{M}_{\dot{\alpha}\dot{\beta}}, Q_{I\gamma}] &= 0, & [\tilde{M}_{\dot{\alpha}\dot{\beta}}, \tilde{Q}_{I\dot{\gamma}}] &= \epsilon_{\dot{\gamma}(\dot{\alpha}} \tilde{Q}_{I\dot{\beta})}, \\ [R_{IJ}, Q_{K\alpha}] &= \epsilon_{K(I} Q_{J)\alpha}, & [R_{IJ}, \tilde{Q}_{K\dot{\alpha}}] &= -\epsilon_{K(I} \tilde{Q}_{J)\dot{\alpha}}, \end{aligned} \quad (13.6)$$

not including the relations for the Lorentz transformations. These can be found in Appendix A1 of [36].

We continue with the twisting procedure. We identify the R -symmetry algebra with the $su(2)_r$. The corresponding indices are identified accordingly $(I, J) \rightarrow (\dot{\alpha}, \dot{\beta})$. We can rotate the Lorentz generator with respect to the R -symmetry group

$$\hat{M}_{\dot{\alpha}\dot{\beta}} = \tilde{M}_{\dot{\alpha}\dot{\beta}} - R_{\dot{\alpha}\dot{\beta}}. \quad (13.7)$$

These generators define a new rotation group $\widehat{su(2)}_l \oplus su(2)_r$.¹ The supercharges become $Q_{I\alpha} \rightarrow Q_{\dot{\alpha}\alpha}$ and $\tilde{Q}_{I\dot{\beta}} \rightarrow \tilde{Q}_{\dot{\alpha}\dot{\beta}}$. This allows us to define a topological supercharge

$$\mathcal{Q} := \epsilon^{\dot{\alpha}\dot{\beta}} \tilde{Q}_{\dot{\alpha}\dot{\beta}}. \quad (13.8)$$

It commutes with the generators of the new rotation algebra. This supercharge is a symmetry operator that we will need to define the topological sector.

Furthermore, we define $\alpha_{0\mu} := \alpha_\mu \approx (\sigma_\mu)^{\dot{a}a} Q_{\dot{a}a}$ from (13.4) so that

$$\{\mathcal{Q}, \alpha_\mu\} = \partial_\mu. \quad (13.9)$$

This ensures the \mathcal{Q} -exactness of the energy momentum tensor such that (13.5) holds. Finally, notice that in this twisted algebra \mathcal{Q} is nilpotent. Thus, one can use it to define a cohomology of twisted topological operators. This result was important historically since for the case of the $\mathcal{N} = 2$ super Yang-Mills theory the resulting topological theory is the Donaldson Witten theory [12], whose correlators are the Donaldson invariants [37]. This links the topological field theories to mathematical objects.

This work is not only important because of its mathematical implications but also because of its use of semi classical approximation from which localization originates. Specifically, for this theory one can show that the action is \mathcal{Q} -exact up to a topological term. As a consequence, the expectation values do not depend on the coupling constant of super Yang-Mills. Hence it is justified to use the saddle point approximation that we discussed in Part II.

A interesting consequence of the twisting is that all spinors are eliminated in the process and become differential forms. Thus, the resulting theory can be defined in any orientable manifold. The orientability is important in order for the Hodge star to be well defined.

¹Notice that the redefinition performed in Section 10.1 is also some kind of twisting with the R -symmetry. This is the reason that we called the operators in Part IV twisted operators. We also see that, in 3d, we end up with operators that are topological in the resulting sub-manifold but not in the original manifold. Indeed, in Paper II we explore the topological correlators in the 1d theory.

14. Pestunization

We will now focus on the case considered in Paper III and its predecessor [38]. We are going to perform a twisting that is called *equivariant twisting* which is relevant for manifolds that allow for a torus action. An example of such an equivariant twisting was [90], which resulted in the Nekrasov partition function [91]. The connection between equivariant twisting and supersymmetric localization was first made in [38] for the case of vector multiplets. The work of Paper III extended this discussion to the case of hypermultiplets. Our main focus will lie on the latter.

The difference between equivariant twisting and the topological twisting before is that we do not identify $SU(2)_L$ with $SU(2)_R$. Instead we use supersymmetric tools, such as Killing spinors, in order to redefine fields and make them singlets under $SU(2)_R$. That way, the fields become differential forms, or spinors that can be defined on a large class of manifolds. Hence, we will have a similar cohomological complex as in the case of seven dimensions, see e.g. (9.16). The supersymmetry transformations play the role of an equivariant differential and square to the Lie derivative along some Killing vector plus gauge transformations. This is the usual cohomological complex used within the localization technique.

First we are going to define an $\mathcal{N} = 2$ supersymmetric field theory on a four dimensional manifold. In order to define the theory on a general background we will argue the existence for a global solution of the spinors. Then we are going to present the twisting before discussing the final result.

14.1 $\mathcal{N} = 2$ supersymmetric theories

Let (\mathcal{M}, g) be a Riemannian manifold with spin structure. As explained in [92] one can define a supersymmetric theory on \mathcal{M} by coupling it to an off-shell background supergravity. When one sets the supergravity variations to zero, the generalized Killing spinor equation emerges. The resulting theory is supersymmetric if there exists a non-vanishing Killing spinor satisfying that equation. Then, the supergravity degrees of freedom become background fields. In this thesis we are going to consider $\mathcal{N} = 2$ supersymmetric theories with a conserved $SU(2)$ R -current.

The supergravity background fields are: a 1-form G_μ , a closed 2-form $\mathcal{F}_{\mu\nu}$, a scalar N , a 2-form $W_{\mu\nu}$, and a scalar S_{ij} transforming as a triplet under $SU(2)_R$.¹

¹For more details on the background we refer to [38] and Paper III.

The supersymmetric transformations are associated with the Killing spinors ζ_α^I and $\bar{\chi}_I^{\dot{\alpha}}$ which obey the symplectic Majorana conditions

$$(\zeta_{I\alpha})^* = \zeta^{I\alpha} \quad \text{and} \quad (\bar{\chi}_I^{\dot{\alpha}})^* = \bar{\chi}_{\dot{\alpha}}^I. \quad (14.1)$$

Requiring that the supergravity variations vanish leads to generalized Killing spinor equations. We refer the reader to Paper III for the explicit form of the equations and additional information about the field configuration of the supergravity background.

One can construct a bilinear using the Killing spinors

$$s = 2\zeta^I \zeta_I, \quad \tilde{s} = 2\bar{\chi}^I \bar{\chi}_I \quad (14.2)$$

and

$$v_\mu = 2\bar{\chi}^I \bar{\sigma}_\mu \zeta_I, \quad \text{where} \quad ||v||^2 = s\tilde{s}. \quad (14.3)$$

Using the reality condition we find that $s, \tilde{s} \geq 0$. Also using the Killing spinor equations, one can prove that v is a Killing vector and $D_\mu s = 0$, $D_\mu \tilde{s} = 0$. One should bear in mind that the covariant derivative D_μ also contains the background $SU(2)_R$ connection.

The fixed points of the Killing vector, i.e. $||v||^2 = 0$, can be separated into two families according to (14.3). The fixed points where $s = 0$ and the ones where $\tilde{s} = 0$ are called *plus* and *minus* fixed points respectively [38].²

14.1.1 Global Spinors

The spinors associated with supersymmetry transformations are the Killing spinors. However there is the need of additional auxiliary spinors to be introduced in order to allow for off-shell closure of the supersymmetric algebra (without using the equations of motion). These auxiliary spinors are associated to auxiliary fields as it is discussed in Section 14.1.3. In order to perform the twisting one has to ensure that the Killing and auxiliary spinors are globally defined. [38] contains a longer discussion about the Killing spinors that is also mentioned in Paper III together with the construction of globally defined auxiliary spinors.

The Killing spinors transform under $SU(2)_l \times_{\mathbb{Z}_2} SU(2)_R$ for ζ_I and $SU(2)_r \times_{\mathbb{Z}_2} SU(2)_R$ for $\bar{\chi}_I$. The auxiliary spinors on the other hand transform under $SU(2)_l \times_{\mathbb{Z}_2} SU(2)_{\check{R}}$ for $\check{\zeta}_I$ and $SU(2)_l \times_{\mathbb{Z}_2} SU(2)_{\check{R}}$ for $\check{\bar{\chi}}^{\check{I}}$. Notice that, in principle, the external symmetry $SU(2)_{\check{R}}$ is different from the R -symmetry group $SU(2)_R$. In order to obtain off-shell closure, the auxiliary spinors need to satisfy the consistency conditions

$$\begin{aligned} \zeta_I \check{\zeta}_{\check{J}} - \bar{\chi}_I \check{\bar{\chi}}_{\check{J}} &= 0, & \check{\zeta}_{\check{I}} \check{\bar{\chi}}^{\check{I}} &= \frac{\tilde{s}}{2}, \\ \check{\bar{\chi}}^{\check{I}} \bar{\sigma}^\mu \zeta_I &= -\frac{1}{2} v^\mu, & \check{\bar{\chi}}_{\check{I}} \check{\bar{\chi}}^{\check{I}} &= \frac{s}{2}. \end{aligned} \quad (14.4)$$

²They are also called instanton and anti-instanton fixed points respectively.

Using the reality conditions for the Killing spinors (14.1), the auxiliary spinors fulfil their own set of reality conditions

$$(\check{\zeta}_{\check{I}\alpha})^* = -\check{\zeta}^{\check{I}\alpha} \quad \text{and} \quad (\check{\chi}_{\check{I}}^{\check{\alpha}})^* = -\check{\chi}_{\check{\alpha}}^{\check{I}}. \quad (14.5)$$

The general strategy is: We will cover \mathcal{M} with charts U_k in such a way that each chart contains only one fixed point. For each chart, we will make a choice of the vielbein e^a_k . We will start with a chart with at most one plus fixed point ($\tilde{s} = 0$) and find a regular solution of the spinors. Then, we will construct a transition function to a chart with a minus fixed point ($s = 0$) and find the regular solution for the spinors in that chart. Altogether, we have constructed globally defined spinors.

Let us assume that we are in a chart U_k with $s \neq 0$. Thereon define the Killing spinors

$$\zeta_\alpha^I = \frac{\sqrt{s}}{2} \delta_\alpha^I, \quad \bar{\chi}_I = \frac{1}{s} v^\mu \bar{\sigma}_\mu \zeta_I \quad (14.6)$$

and the auxiliary spinors

$$\check{\chi}_{\check{I}}^{\check{\alpha}} = \frac{\sqrt{s}}{2} \delta_{\check{I}}^{\check{\alpha}}, \quad \check{\zeta}_{\check{I}\alpha} = -\frac{1}{s} v^\mu \left(\sigma_\mu \check{\chi}_{\check{I}} \right)_\alpha. \quad (14.7)$$

These spinors are regular and satisfy the bilinear and reality conditions (14.4), (14.1), and (14.5).

The spinorial transition function for the spinors from U_k to a new chart $U_{k'}$ containing a point $s = 0$ is of the form

$$U_I^J = i \frac{v^\mu}{||v||} \sigma_{\mu I}^J \quad (14.8)$$

for the Killing spinors and

$$U_{\check{I}}^{\check{J}} = i \frac{v^\mu}{||v||} (\sigma_\mu)_{\check{I}}^{\check{J}}. \quad (14.9)$$

for the auxiliary ones. Therefore, the spinors in the new chart read

$$\bar{\chi}_I^{\check{\alpha}} = -i \frac{\sqrt{\tilde{s}}}{2} \delta_I^{\check{\alpha}}, \quad \zeta_I = -\frac{1}{\tilde{s}} v^\mu \sigma_\mu \bar{\chi}_I \quad (14.10)$$

and

$$\check{\zeta}_{\check{\alpha}}^{\check{I}} = i \frac{\sqrt{\tilde{s}}}{2} \delta_{\check{\alpha}}^{\check{I}}, \quad \check{\chi}_{\check{I}}^{\check{\alpha}} = \frac{1}{\tilde{s}} v^\mu \left(\bar{\sigma}_\mu \check{\zeta}_{\check{I}} \right)^{\check{\alpha}}. \quad (14.11)$$

These spinors are regular in $U_{k'}$. The two solutions in U_k and $U_{k'}$ together with their transition function ensure that the spinors are globally well defined.

14.1.2 Vector Multiplet

The twisting of the vector multiplet was discussed in length in [38]. In Paper III we were interested in a hypermultiplet coupled to a vector multiplet. We are going to briefly review the twisting map in the case of the vector and hypermultiplet.

Original multiplet

The form of $\mathcal{N} = 2$ vector multiplet consists of a gauge field A_μ , a complex scalar X , two fermions $\lambda_{I\alpha}$ and $\bar{\lambda}_\alpha^I$, and an auxiliary scalar D_{IJ} . All fields, except from the gauge field, transform in the adjoint of the gauge group. Furthermore the fermions transform in the fundamental of $SU(2)_R$ and the auxiliary field transform as an $SU(2)_R$ triplet.

The supersymmetric variations are of the form

$$\begin{aligned}
\delta \bar{X} &= \bar{\chi}^I \bar{\lambda}_I, & \delta X &= -\zeta_I \lambda^I, \\
\delta A_\mu &= i\zeta_i \sigma_\mu \bar{\lambda}^I + i\bar{\chi}^I \bar{\sigma}_\mu \lambda_I, \\
\delta D_{IJ} &= i\zeta_I \sigma^\mu (D_\mu + iG_\mu) \bar{\lambda}_J - i\bar{\chi}_I \bar{\sigma}^\mu (D_\mu - iG_\mu) \lambda_J \\
&\quad + 2i[X, \bar{\chi}_I \bar{\lambda}_J] + 2i[\bar{X}, \zeta_I \lambda_J] + (i \leftrightarrow J), \\
\delta \lambda_I &= -2i(D_\mu - 2iG_\mu) X \sigma^\mu \bar{\chi}_I + 2(F^+ - \bar{X} W^+) \zeta_I \\
&\quad + D_{IJ} \zeta^J + 2i[X, \bar{X}] \zeta_I - 2X \eta_I, \\
\delta \bar{\lambda}^I &= 2i(D_\mu + 2iG_\mu) \bar{X} \bar{\sigma}^\mu \zeta^I + 2(F^- - X W^-) \bar{\chi}^I \\
&\quad - D^{IJ} \bar{\chi}_J - 2i[X, \bar{X}] \bar{\chi}^I + 2\bar{X} \bar{\eta}^I.
\end{aligned} \tag{14.12}$$

Here, $F_{\mu\nu}$ is the field strength of A_μ and η is a spinor defined by the supergravity fields.³ In general we will use the notation $B^+ := \frac{1}{2} B_{\mu\nu} \sigma^{\mu\nu}$ and $B^- := \frac{1}{2} B_{\mu\nu} \bar{\sigma}^{\mu\nu}$ for any 2-form field B .

The derivative D_μ is covariantized with respect to the $SU(2)_R$ connection and the gauge field.

The supersymmetric transformations square to

$$\delta^2 = i\mathcal{L}_v + iv^\mu V_\mu \circ + i\Lambda^{(R)} \circ - i[\Phi, \cdot], \tag{14.13}$$

where \circ denotes the action corresponding to the field's $SU(2)_R$ representation. The gauge transformation parameter reads

$$\Phi = iv^\mu A_\mu + s\bar{X} + \tilde{s}X \tag{14.14}$$

and $\Lambda^{(R)}$ is the $SU(2)_R$ parameter

$$\Lambda_{IJ}^{(R)} = \bar{\chi}_I \bar{\sigma}^\mu (D_\mu - iG_\mu) \zeta_J - \zeta_I \sigma^\mu (D_\mu + iG_\mu) \bar{\chi}_J + (I \leftrightarrow J). \tag{14.15}$$

³One can find its exact definition in (2.4) of Paper III.

Vector Twisting

We notice that the fermionic fields transform under $SU(2)_R$. As a result one can start by twisting them using the Killing spinors into a scalar, η^I , a 1-form Ψ_μ and a 2-form $\chi_{\mu\nu}$.⁴

Let us define *flipping* projectors acting on 2-forms

$$P_+ = \frac{1}{2(s^2 + \tilde{s}^2)} \left((s + \tilde{s})^2 \mathbb{I} + (s^2 - \tilde{s}^2) \star -4\kappa \wedge \iota_v \right) , \quad (14.16)$$

$$P_- = 1 - P_+ .$$

They are called flipping since P_+ projects 2-forms into their self-dual parts at the plus fixed points and to the anti-self-dual parts at the minus fixed points. For example the fermionic 2-form $\chi_{\mu\nu}$ is such a form. It satisfies $P_+ \chi = \chi$. Furthermore, we also twist the auxiliary field. The resulting field is $H_{\mu\nu}$ which satisfies $P_+ H = H$.

Twisting preserves the number of degrees of freedom of the vector multiplet. We started from eight complex fermions. The resulting multiplet has the following fermionic counting: one degree of freedom from the scalar η , four from the 1-form Ψ_μ and originally six degrees of freedom from the 2-form χ but the projection equation shows that half of them only remain, which sum again to eight.

If one also redefines the two scalars X and \bar{X} into ϕ and φ , the counting of the bosons is simpler since the gauge field remains unchanged: There are two complex scalars in the starting and final multiplet. The supersymmetric transformations simplify into three submultiplets. The long one contains (A, ϕ, Ψ) with

$$\delta A = i \Psi , \quad (14.17)$$

$$\delta \Psi = \iota_v F + i d_A \phi , \quad (14.18)$$

$$\delta \phi = \iota_v \Psi . \quad (14.19)$$

The first short multiplet contains a scalar and a fermion (φ, η)

$$\begin{aligned} \delta \varphi &= i \eta , \\ \delta \eta &= \iota_v d_A \varphi - [\phi, \varphi] . \end{aligned} \quad (14.20)$$

And finally, the second short multiplet is built from the bosonic and fermionic 2-forms (χ, H)

$$\begin{aligned} \delta \chi &= H , \\ \delta H &= i \mathcal{L}_v^A \chi - i[\phi, \chi] , \end{aligned} \quad (14.21)$$

where $\mathcal{L} = d_A \iota_v + \iota_v d_A$ is the Lie derivative that also contains A , and $d_A = d + [A, \cdot]$. The gauge transformation parameter can be rewritten as

⁴The interested reader can find the complete map in Section 3.3.3 of [38] or Paper III.

$$\Phi = i\nu_v A + \phi. \quad (14.22)$$

One can write the square of the supersymmetric transformations as

$$\delta^2 = i\mathcal{L}_v - \mathcal{G}_\Phi \quad (14.23)$$

where with \mathcal{G}_Φ we indicate the action of Φ on the fields according to their gauge representation.

Note that as we discussed in the start of the chapter, the original square of the supersymmetry (14.13) is now twisted into one resembling the equivariant differential.

14.1.3 Hypermultiplet

Let us continue with a hyper- coupled to a vector multiplet. We embed the gauge group in $Sp(k)$. Thus, the hypermultiplet is transforming under the fundamental of $Sp(k)$ with indices $n = 1, \dots, 2k$. The hypermultiplet's physical degrees of freedom contain a scalar, q_{nI} , and a pair of spinors, $\psi_{\alpha n}$ and $\bar{\psi}_{\dot{\alpha} n}$. To ensure off-shell closure of the supersymmetric algebra we add auxiliary fields $F_{n\check{I}}$ accompanied by the auxiliary spinors $\check{\zeta}^{\check{I}}$ and $\check{\bar{\chi}}_{\check{I}}$ we already discussed in Section 14.1.1. The checked indices transform under the fundamental of an $SU(2)_{\check{R}}$.

The supersymmetric transformations read

$$\begin{aligned} \delta q_{nI} &= \zeta_I \psi_n + \bar{\chi}_I \bar{\psi}_n, \\ \delta \psi_n &= 2i(D_\mu q_{nI})\sigma^\mu \bar{\chi}_I + iq_{nI}\sigma^\mu (D_\mu + iG_\mu) \bar{\chi}_I + 4i\bar{X}_n^m q_{mI}\zeta^I + 2iF_{n\check{I}}\check{\zeta}^{\check{I}}, \\ \delta \bar{\psi}^n &= 2i(D_\mu q^{nI})\bar{\sigma}^\mu \zeta_I + iq^{nI}\bar{\sigma}^\mu (D_\mu - iG_\mu) \zeta_I + 4iX_n^m q^{mI}\bar{\chi}_I + 2iF^{n\check{I}}\check{\bar{\chi}}_{\check{I}}, \\ \delta F_{n\check{I}} &= \check{\zeta}_{\check{I}} \left[\sigma^\mu (D_\mu - iG_\mu) \bar{\psi}_n - 2X_n^m \psi_m + 2(\lambda^J)_n^m q_{mJ} - iW^+ \psi_n \right] \\ &\quad + \check{\bar{\chi}}_{\check{I}} \left[\bar{\sigma}^\mu (D_\mu + iG_\mu) \psi_n + 2\bar{X}_n^m \bar{\psi}_m - 2(\bar{\lambda}^J)_n^m q_{mJ} + iW^- \bar{\psi}_n \right]. \end{aligned} \quad (14.24)$$

For the vector multiplet fields we have, for example, $X_n^m = X_a t^a{}_n{}^m$ using a gauge group generator t^a .

Furthermore, the supersymmetric algebra, restricted to the physical degrees of freedom, squares to

$$\delta^2 = i\mathcal{L}_v + iv^\mu V_\mu \circ + i\Lambda^{(R)} \circ + \mathcal{G}_\Phi \diamond, \quad (14.25)$$

where we used the same notation as in (14.13). The new term, $\mathcal{G}_\Phi \diamond$, indicates the action of the gauge transformation parameter Φ on the fields according to their gauge group representation. The squared supersymmetry variation on the auxiliary field is similarly given by

$$\delta^2 = i\mathcal{L}_v + iv^\mu \check{V}_\mu \circ + i\Lambda^{(\check{R})} \circ + \mathcal{G}_\Phi \diamond, \quad (14.26)$$

where the auxiliary spinors transform under the $SU(2)_{\check{R}}$ instead. The field \check{V}_μ is the background $SU(2)_{\check{R}}$ connection. We have a similar transformation parameter for $SU(2)_{\check{R}}$

$$\Lambda_{\check{I}\check{J}}^{(\check{R})} = 2\check{\zeta}_{\check{I}}\sigma^\mu\left(\check{D}_\mu - iG_\mu\right)\check{\chi}_{\check{J}} + 2i\check{\zeta}_{\check{I}}W^+\check{\zeta}_{\check{J}} \quad (14.27)$$

$$- 2\check{\chi}_{\check{I}}\bar{\sigma}^\mu\left(\check{D}_\mu + iG_\mu\right)\check{\zeta}_{\check{J}} + 2i\check{\chi}_{\check{I}}W^-\check{\chi}_{\check{J}} + (\check{I} \leftrightarrow \check{J}) , \quad (14.28)$$

as in (14.15). Here, \check{D}_μ denotes the covariantized derivative with respect to the $SU(2)_{\check{R}}$ connection.

We will not give the supersymmetric Lagrangian in this thesis but we refer the interested reader to equation (2.18) of Paper III. However, an interesting aspect of it is that the Lagrangian is (up to total derivatives) δ -exact, i.e. $\mathcal{L} = \delta V_G$.

Twisting projectors

The twisting procedure in the case of the hypermultiplet is more subtle than in the case of a vector multiplet. The fields transforming under $SU(2)_R$ are not the fermions but the scalars. This means that, after the twisting, the resulting cohomological fields will not be differential forms but spinors instead. That restricts the set of manifolds since a spin structure is required.

A convenient way to perform the redefinition is in the language of Dirac spinors. Firstly, let us define a convenient projector acting on Dirac spinors. Using the decomposition of a Dirac spinor

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \tilde{\psi}^{\dot{\alpha}} \end{pmatrix} , \quad (14.29)$$

one can define the projector

$$Z_+ = \frac{1}{2} \left(\mathbb{I} + \frac{s - \tilde{s}}{s + \tilde{s}} \gamma_5 - \frac{2}{s + \tilde{s}} v^\mu \gamma_5 \gamma_\mu \right) , \quad (14.30)$$

with $\gamma_5 = -\gamma_1\gamma_2\gamma_3\gamma_4$ and γ_μ the Dirac matrices in four dimensions. One obtains additional useful projectors

$$Z_- = \mathbb{I} - Z_+ , \quad \tilde{Z}_+ = \gamma_5 Z_+ \gamma_5 , \quad \tilde{Z}_- = \mathbb{I} - \tilde{Z}_+ . \quad (14.31)$$

Analogously to P_+ , the new operators, Z_+ and \tilde{Z}_+ , project spinors into their left-handed part at the plus fixed points and right-handed part at the minus fixed points.

For the Dirac Killing spinor

$$\mathfrak{z}_i = \begin{pmatrix} \zeta_I \\ \bar{\chi}_I \end{pmatrix} , \quad (14.32)$$

we have $Z_+\mathfrak{z}_I = \mathfrak{z}_I$. For its auxiliary spinor analogue we have

$$\check{\mathfrak{z}}_{\check{I}} = \begin{pmatrix} \check{\zeta}_{\check{I}} \\ \check{\chi}_{\check{I}} \end{pmatrix}, \quad (14.33)$$

with $\tilde{Z}_-\check{\mathfrak{z}}_{\check{I}} = \check{\mathfrak{z}}_{\check{I}}$.

The scalars can be redefined as

$$\mathfrak{q}_n = \mathfrak{z}^I q_{nI} = \begin{pmatrix} \zeta^I q_{nI} \\ \bar{\chi}^I q_{nI} \end{pmatrix}. \quad (14.34)$$

Thus the new field is well defined on the whole manifold \mathcal{M} .

In order to arrive at a similar cohomological complex as in (9.16), we also redefine the fermionic degrees of freedom according to

$$\mathfrak{c}_n = -\frac{s+\tilde{s}}{4}Z_+ \begin{pmatrix} \psi_n \\ \bar{\psi}_n \end{pmatrix} = -\frac{1}{4} \begin{pmatrix} s\psi_n - v^\mu \sigma_\mu \bar{\psi}_n \\ \tilde{s}\bar{\psi}_n + v^\mu \bar{\sigma}_\mu \psi_n \end{pmatrix}, \quad (14.35)$$

and

$$\mathfrak{b}_n = \frac{s+\tilde{s}}{4}\tilde{Z}_-\gamma_5 \begin{pmatrix} \psi_n \\ \bar{\psi}_n \end{pmatrix} = \frac{1}{4} \begin{pmatrix} \tilde{s}\psi_n + v^\mu \sigma_\mu \bar{\psi}_n \\ -s\bar{\psi}_n + v^\mu \bar{\sigma}_\mu \psi_n \end{pmatrix}. \quad (14.36)$$

Finally, the auxiliary spinor transforms under the $SU(2)_{\check{R}}$ so we will use the auxiliary Dirac spinor (14.33)

$$\begin{aligned} \mathfrak{h}_n = & \tilde{Z}_- \left(\frac{s+\tilde{s}}{2} \gamma^\mu (D_\mu + iT_\mu) \mathfrak{q}_n + iv^\mu G_\mu \mathfrak{q}_n \right) \\ & + \tilde{Z}_- \left(-i \frac{(s+\tilde{s})}{2} \varphi_n^m \mathfrak{q}_m \right) + \frac{s+\tilde{s}}{2} \check{\mathfrak{z}}^{\check{I}} F_{n\check{I}}. \end{aligned} \quad (14.37)$$

In this formula, T is a combination of supergravity background fields and derivatives of Killing spinor bilinears.

The new fields satisfy the projection relations

$$Z_+\mathfrak{q} = \mathfrak{q}, \quad Z_+\mathfrak{c} = \mathfrak{c}, \quad \tilde{Z}_-\mathfrak{b} = \mathfrak{b}, \quad \tilde{Z}_-\mathfrak{h} = \mathfrak{h}. \quad (14.38)$$

As before the number of fermionic degrees of freedom stays the same. We started from 4+4 complex degrees of freedom from ψ_n and $\bar{\psi}_n$. At the end we are left with two Dirac fermions which amount to 8+8 complex degrees of freedom. Both are shortened since they satisfy the projection relations (14.38) and we end up with 4+4 degrees of freedom again.

Due to the redefinitions the multiplet splits into two separate sub-multiples. The first one contains the bosonic spinor and one of the fermions $(\mathfrak{q}, \mathfrak{c})$

$$\begin{aligned} \delta \mathfrak{q} &= \mathfrak{c}, \\ \delta \mathfrak{c} &= (i\mathcal{L}_v - \mathcal{G}_\Phi) \mathfrak{q}, \end{aligned} \quad (14.39)$$

where we omit the $Sp(k)$ indices for simplicity. The second sub-multiplet contains the remaining spinor and the auxiliary field (\mathbf{b}, \mathbf{h}) with variations

$$\begin{aligned}\delta \mathbf{b} &= i\mathbf{h} , \\ \delta \mathbf{h} &= (\mathcal{L}_v + i\mathcal{G}_\Phi)\mathbf{b} .\end{aligned}\tag{14.40}$$

Note that the gauge transformation parameter is defined in (14.22).

Finally as for the vector multiplet the square of the supersymmetric transformations takes the form

$$\delta^2 = i\mathcal{L}_v - \mathcal{G}_\Phi .\tag{14.41}$$

14.1.4 Localization of the gauged hypermultiplet

We review the localization computation for the gauged hypermultiplet following Paper III.

As it was explained in Section 6, one first needs to identify the localization locus. This is achieved by setting the fermions and their variations to zero. In this case this results to

$$(i\mathcal{L}_v - \mathcal{G}_\Phi)\mathbf{q} = 0 \quad \text{and} \quad \mathbf{h} = 0 .\tag{14.42}$$

However, by imposing reality conditions this leads to $\mathbf{q} = 0$.

The localizing action is of a similar form as (6.3)

$$V_{\text{loc}} = \frac{1}{4\tilde{\mathbf{z}}^i \tilde{\mathbf{z}}_i} (\delta \Psi_n)^* \Psi_n = \frac{8}{(s + \tilde{s})^3} [(\delta \mathbf{b}_n)^* \mathbf{b}_n + (\delta \mathbf{c}_n)^* \mathbf{c}_n] .\tag{14.43}$$

One may rewrite it for the purpose of computing the one loop determinant as

$$V_{\text{loc}} = \frac{8}{(s + \tilde{s})^3} \left(\overline{\delta \mathbf{q}}, \overline{\mathbf{b}} \right) \begin{pmatrix} D_{00} & D_{01} \\ D_{10} & D_{11} \end{pmatrix} \begin{pmatrix} \mathbf{q} \\ \delta \mathbf{b} \end{pmatrix} .\tag{14.44}$$

The matrix elements are defined by

$$\begin{aligned}D_{00} &= i\mathcal{L}_v - i\Phi - i[2\phi + i(s - \tilde{s})\varphi] , & D_{01} &= 0 , \\ D_{10} &= i(s + \tilde{s})\gamma^\mu (D_\mu + iT_\mu) - 2\iota_v G - g(s + \tilde{s})\varphi , & D_{11} &= -\mathbb{I} .\end{aligned}\tag{14.45}$$

The Gaussian integration is once more resulting in a ratio of the determinants of the $\delta^2 = i\mathcal{L}_v - \mathcal{G}_\Phi$ for bosons and fermions. Using the fact that D_{ij} commutes with δ^2 , the one loop determinant is

$$\text{sdet}(i\mathcal{L}_v - \mathcal{G}_\Phi)^{-1} = \frac{\det_{\mathbf{b}}(i\mathcal{L}_v - \mathcal{G}_\Phi)}{\det_{\mathbf{q}}(i\mathcal{L}_v - \mathcal{G}_\Phi)} = \frac{\det_{\text{Coker } D_{10}}(i\mathcal{L}_v - \mathcal{G}_\Phi)}{\det_{\text{Ker } D_{10}}(i\mathcal{L}_v - \mathcal{G}_\Phi)}\tag{14.46}$$

As explained in [18, 29], for the case of a transversally elliptic operator the 1-loop determinant can be computed by extracting the spectrum of

eigenvalues of δ^2 from the index of that operator. In the following we will show that D_{10} is transversally elliptic and we will give an explanation of the index derivation for plus and minus fixed points.

In order to prove that transversally ellipticity of D_{10} , we will consider its symbol

$$\sigma [D_{10}] = \frac{8 p_\mu}{(s + \bar{s})^2} \tilde{Z}_- \gamma^\mu Z_+ . \quad (14.47)$$

The symbol simplifies on the two fixed points as

$$\sigma [D_{10}]|_{\bar{s}=0} = \sigma \left[-\frac{8i}{s^2} \bar{\sigma}^\mu \partial_\mu \right] , \quad (14.48)$$

$$\sigma [D_{10}]|_{s=0} = \sigma \left[-\frac{8i}{\bar{s}^2} \sigma^\mu \partial_\mu \right] . \quad (14.49)$$

These are the symbols of the chiral Dirac operators. This behaviour hints to the operator being elliptic. By close examination of the case $s = \bar{s}$ away from the fixed points, we get

$$\sigma [D_{10}]|_{s=\bar{s}} = \frac{4}{s^2} \tilde{Z}_- \gamma_5 \gamma^{\mu\nu} p_\mu v_\nu , \quad (14.50)$$

The operator is not elliptic in this patch since for $p^\mu = v^\mu \neq 0$ we have $\sigma[D_{10}] = 0$. The symbol is invertible for any $p^\mu \neq 0$ orthogonal to v . Hence, the operator is transversally elliptic. As such we can compute the index of the operator which will have the information needed for the calculation of the 1-loop determinant.

As explained in [18] one can calculate the kernel and the co-kernel of such operators by summing over all the irreducible representations of the gauge group R_a with different multiplicities, m_a^0 for the kernel and m_a^1 for the co-kernel. We find that

$$\frac{\det_{\text{Coker} D_{10}}(i\mathcal{L}_v - \mathcal{G}_\Phi)}{\det_{\text{Ker} D_{10}}(i\mathcal{L}_v - \mathcal{G}_\Phi)} = \prod_a (\det R_a)^{m_a^0 - m_a^1} . \quad (14.51)$$

However, the index can be computed to be

$$\text{ind} D_{10} = \sum_a (m_a^0 - m_a^1) \epsilon^a , \quad (14.52)$$

where ϵ is the generator of one of the $U(1)$ actions. Therefore, if we calculate the index, we can use the coefficients of the $U(1)$ -generators, $(m_a^0 - m_a^1)$, in (14.51) to find the partition function.

In our case the index is

$$\text{ind}(D_{10})(t) = \sum_{x: \tilde{x}=x} \frac{\text{Tr}_{\mathfrak{q}} e^{-it\delta^2} - \text{Tr}_{\mathfrak{b}} e^{-it\delta^2}}{\det(1 - \partial\tilde{x}/\partial x)} , \quad (14.53)$$

where $t \in \mathbb{R}$, while x is a coordinate on \mathcal{M} and \tilde{x} its image under the torus action induced by δ^2 . The sum goes over the fixed points of the torus action, i.e. $\tilde{x} = x$.

Around the neighbourhood of a plus/minus fixed point the space looks locally flat. We can parametrize it by $(z_1, z_2) \in \mathbb{C}^2$ with a T^2 acting as (2.16). As seen in (2.11), the Killing vector takes the form

$$v = i\epsilon_1^{(+/-)}(z_1\partial_{z_1} - \bar{z}_1\partial_{\bar{z}_1}) + i\epsilon_2^{(+/-)}(z_2\partial_{z_2} - \bar{z}_2\partial_{\bar{z}_2}) , \quad (14.54)$$

where $\epsilon_{1,2}^{(+/-)}$ are the parameters of the torus action close to the plus or minus fixed points. The denominator of the index formula (14.53) becomes

$$\det\left(1 - \frac{\partial\tilde{z}_i}{\partial z_j}\right) = (1 - q_1)(1 - \bar{q}_1)(1 - q_2)(1 - \bar{q}_2) , \quad (14.55)$$

where $q_i = \exp(i\epsilon_i^{(+/-)}t)$ are the parameters of the group action.

In order to calculate the index we additionally need the action of δ^2 on the fields. As explained in Paper III or [18], we can embed T^2 into $\text{Spin}(4)$ using the spinor representation of the torus action $g \in \text{Spin}(4)$

$$g = \text{diag}\left(\sqrt{q_1\bar{q}_2}, \sqrt{q_1q_2}, \sqrt{\bar{q}_1q_2}, \sqrt{\bar{q}_1\bar{q}_2}\right) , \quad (14.56)$$

We then can define the coordinate \mathbf{z} matrix

$$\mathbf{z} = x_\mu \gamma^\mu = \begin{pmatrix} 0 & 0 & \bar{z}_2 & \bar{z}_1 \\ 0 & 0 & z_1 & -z_2 \\ -z_2 & -\bar{z}_1 & 0 & 0 \\ -z_1 & \bar{z}_2 & 0 & 0 \end{pmatrix} . \quad (14.57)$$

such that $\mathbf{z} \rightarrow g\mathbf{z}g^{-1}$. the spinor field $\Psi = \{\mathbf{q}, \mathbf{b}\}$ transforms as $\Psi \rightarrow g^{-1}\Psi$ under the \mathcal{L}_v action.

For the plus fixed points, the bosonic spinor \mathbf{q} is left-handed and the fermionic counterpart \mathbf{b} is right-handed as can be seen from their definitions (14.34) and (14.36) respectively. The \mathcal{L}_v action reads

$$\mathbf{q}_+ \rightarrow \sqrt{q_1q_2}\mathbf{q}_+ , \quad \mathbf{q}_- \rightarrow \sqrt{\bar{q}_1\bar{q}_2}\mathbf{q}_- , \quad (14.58)$$

$$\tilde{\mathbf{b}}^\dagger \rightarrow \sqrt{q_1\bar{q}_2}\tilde{\mathbf{b}}^\dagger , \quad \tilde{\mathbf{b}}^\dagger \rightarrow \sqrt{\bar{q}_1q_2}\tilde{\mathbf{b}}^\dagger . \quad (14.59)$$

Plugging this back into the index formula (14.53) we get

$$\text{ind}(D_{10})|_{\text{plus point}} = \frac{\sqrt{q_1q_2}}{(1 - q_1)(1 - q_2)} \sum_{\rho \in \mathcal{R}} e^{-t\rho(\Phi_0)} . \quad (14.60)$$

The last contribution is from the action of \mathcal{G}_Φ . It acts non-trivially. Thus, we sum over the weights ρ of the representation \mathcal{R} of the gauge group. Finally, Φ_0 is the BPS field that contains only the Coulomb branch modulus and fluxes. It takes the form

$$\Phi_0 = a_0 + k_+(\epsilon_1^+, \epsilon_2^+), \quad (14.61)$$

where a_0 is a Coulomb branch moduli and $k_+(\epsilon_1^+, \epsilon_2^+)$ is parametrizing the flux contribution at the plus fixed point.

For a minus fixed point, \mathfrak{q} is right- and \mathfrak{b} is left-handed i.e.

$$\mathfrak{b}_+ \rightarrow \sqrt{q'_1 q'_2} \mathfrak{b}_+, \quad \mathfrak{b}_- \rightarrow \sqrt{\bar{q}'_1 \bar{q}'_2} \mathfrak{b}_-, \quad (14.62)$$

$$\tilde{\mathfrak{q}}^{\dagger} \rightarrow \sqrt{q'_1 \bar{q}'_2} \tilde{\mathfrak{q}}^{\dagger}, \quad \tilde{\mathfrak{q}}^{\dot{-}} \rightarrow \sqrt{\bar{q}'_1 q'_2} \tilde{\mathfrak{q}}^{\dot{-}}. \quad (14.63)$$

The index becomes

$$\text{ind}(D_{10})|_{\text{minus point}} = -\frac{\sqrt{q'_1 \bar{q}'_2}}{(1 - q'_1)(1 - q'_2)} \sum_{\rho \in \mathcal{R}} e^{-t \rho(\Phi'_0)}. \quad (14.64)$$

Note that we used the primed version $q'_i = \exp(i \epsilon_i^{(-)} t)$ to visualize that the two indices have different results even if they look similar. Note that Φ'_0 is the BPS field with analogous expression as in (14.61) for the minus fixed point instead.

In order to find the partition function one needs to regularize the index. In Paper III, we were inspired by [31, 75, 93, 94] to use two different regularization schemas which we denoted by plus and minus

$$\left[\frac{1}{1 - q_i} \right]_+ = \sum_{n \geq 0} q_i^n, \quad \left[\frac{1}{1 - q_i} \right]_- = - \sum_{n \leq -1} q_i^n = - \sum_{n \geq 0} q_i^{-n-1}. \quad (14.65)$$

Their difference is

$$\left[\frac{1}{1 - q_i} \right]_+ - \left[\frac{1}{1 - q_i} \right]_- = \sum_{n \in \mathbb{Z}} q_i^n. \quad (14.66)$$

In Paper III we presented multiple regularization results depending on which regularization was used for q_1 and q_2 .

For the example of a plus fixed point with $(+, +)$ regularization, we use the plus regularization for both q_1 and q_2 . We find

$$\left[\text{ind}(D_{10})|_{\text{plus point}} \right]_{++} = + \sum_{\rho \in \mathcal{R}} \sum_{n_1, n_2 \in \mathbb{N}} q_1^{n_1 + \frac{1}{2}} q_2^{n_2 + \frac{1}{2}} e^{-t \rho(\Phi_0)}, \quad (14.67)$$

which translates to the partition function

$$\begin{aligned}
Z_{\epsilon_1^{(+)}, \epsilon_2^{(+)}}^{\text{HM}^{++}}(a_0, k_+) &= \prod_{\rho \in \mathcal{R}} \prod_{n_i \in \mathbb{N}} \left[\epsilon_1^{(+)} \left(n_1 + \frac{1}{2} \right) + \epsilon_2^{(+)} \left(n_2 + \frac{1}{2} \right) + i \rho(\Phi_0) \right]^{-1} \\
&= \prod_{\rho \in \mathcal{R}} \Gamma_2 \left(i \rho(\Phi_0) + ((\epsilon_1^{(+)} + \epsilon_2^{(+)})/2) | \epsilon_1^{(+)}, \epsilon_2^{(+)} \right) .
\end{aligned} \tag{14.68}$$

It is important to note that the regularization needs to be chosen case by case since it is far from an obvious task.

Combining this with the results of [38], we give a schematic answer for the full partition function for $\mathcal{N} = 2$

$$\begin{aligned}
Z_{\vec{\epsilon}_1, \vec{\epsilon}_2}(q, \bar{q}) &= \sum_{k_i} \int_{\mathbf{h}} da_0 e^{-S_{\text{cl}}} \prod_{i=1}^p Z_{\epsilon_1^{(i)}, \epsilon_2^{(i)}}^{\text{inst}}(a_0, k_i, q) Z_{\epsilon_1^{(i)}, \epsilon_2^{(i)}}^{\text{VM}}(a_0, k_i) Z_{\epsilon_1^{(i)}, \epsilon_2^{(i)}}^{\text{HM}}(a_0, k_i) \\
&\quad \times \prod_{i=p+1}^l Z_{\epsilon_1^{(i)}, \epsilon_2^{(i)}}^{\text{anti-inst}}(a_0, k_i, \bar{q}) \tilde{Z}_{\epsilon_1^{(i)}, \epsilon_2^{(i)}}^{\text{VM}}(a_0, k_i) \tilde{Z}_{\epsilon_1^{(i)}, \epsilon_2^{(i)}}^{\text{HM}}(a_0, k_i) .
\end{aligned} \tag{14.69}$$

The formula above holds for a manifold with p plus and $(l-p)$ minus fixed points. In (14.69) the integral is taken over the Cartan gauge subalgebra \mathbf{h} , whereas \bar{q}, q are counting parameters labelling (anti-)instantons. Z^{inst} , Z^{VM} and Z^{HM} are the instantons contribution, the vector multiplet 1-loop determinant, and the hypermultiplet 1-loop determinant at a plus fixed point respectively. Analogously, $Z^{\text{anti-inst}}$, \tilde{Z}^{VM} and \tilde{Z}^{HM} are the anti-instantons contribution, the vector multiplet 1-loop determinant, and the hypermultiplet 1-loop determinant at a minus fixed point. The 1-loop contributions Z^{HM} and \tilde{Z}^{HM} are in general Barnes double gamma functions [95].

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Svensk sammanfattning

Den största utmaningen inom teoretisk fysik idag är att komma fram till exakta resultat som beskriver naturen. Faktum är att sannolikhetsfördelningen av standardmodellpartiklar som sprids, något som kan observeras vid LHC, är mycket svårt att beräkna. Fysiker lockas då till att söka efter symmetrier för att förenkla denna uppgift. Ett exempel på en sådan symmetri är *supersymmetri* som spelar en avgörande roll för vår förståelse av naturen. Supersymmetri är intressant modell där varje partikel i universums början parades ihop med en superpartner. Mer specifikt så associerades de partiklar som vi kallar materia med partiklar som var av strålningsnatur och vice versa.

Dessa superpartners har dock ännu inte observerats experimentellt. En mekanism som skulle kunna förklara detta är att när universum blev svalare upphörde denna symmetri vid en viss energinivå, och tanken är då att superpartnerna bara kan observeras ovanför denna energinivå. Nedanför denna nivå kan experiment endast observera partiklarna i standardmodellen. Supersymmetri har löst många av de förbryllande problemen som standardmodellen har och fysiker är förväntansfulla över att se om LHC eller någon annan partikelaccelerator kan bekräfta att supersymmetri existerar genom att accelerera partiklar till högre energier i framtiden. Förutom att supersymmetri möjligen beskriver naturen, fungerar den även som en förenklad test-modell för att studera matematiska egenskaper, utveckla beräkningsverktyg och studera fysikaliska fenomen.

Denna avhandling undersöker sådana supersymmetriska teorier. En viktig teknik som används är något som kallas *localizing* ("lokalisering") som används för att beräkna observabler. Mer specifikt är lokalisering ett utmärkt verktyg som låter oss exakt beräkna vissa särskilt utmanande integraler. Metoden går ut på att man kan bevisa att endast några få speciella punkter bidrar till dessa integraler och att de kan reduceras till enklare integraler eller till och med summeringar. Mängden som dessa punkter utgör kallas *localization locus* ("lokaliseringens geometriska ort").

Även geometrin spelar en viktig roll i dessa beräkningar. I själva verket tyder observationer på att universum inte är platt, utan snarare aningen krökt vilket motsvarar en liten *kosmologisk konstant*, något som introducerades i Einsteins gravitationsmodell i hans berömda *allmänna relativitetsteori*. Medan Einsteins gravitationskraft klassiskt är en välfungerande teori, innebär däremot dess förenig med standardmodellen på kvantnivå

allvarliga problem. En mer omfattande modell som innefattar allmän relativitet som en klassisk gräns är *strängteori*. Supersymmetri och strängteori är sammankopplade på ett naturligt vis. Fungerande formuleringar av strängteori kräver ett universum som har tio rymdtdimensioner. Strängteorins grundidé är att dess fundamentala objekt är små strängar snarare än punktpartiklar. För att studera spridningen av sådana strängar måste man ta hänsyn till alla topologiskt olika vägar som en sträng kan ta. Därför är det viktigt att förstå spridning i olika geometrier.

I artiklar I och II utforskar vi supersymmetriska teorier i sju och tre dimensioner. En modells geometri har en mycket stor påverkan på beräkningen av observablerna. I den förstnämnda är studien i sju dimensioner särskilt intressant då det förekommer speciella geometrier som antyder intressanta mönster i resultaten. I den sistnämnda artikeln undersöker vi idén att beräkna sannolikheter för spridning för specifika operatorer i tre dimensioner som är naturligt förekommande i dessa teorier.

Artikel III fokuserar på fyrdimensionella teorier och en teknik som kallas *twisting* ("vridning"). Vridning undersöktes först av Witten då han försökte få teorier i platta rum att även fungera i krökta rum. Mer specifikt så börjar man med en supersymmetrisk teori i ett platt rum, varpå man vrider partikelinnehåll så att det resulterande ramverket är oberoende av vilket rum man faktiskt beskriver teorin i.

References

- [1] ATLAS collaboration, G. Aad et al., *Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC*, *Phys. Lett. B* **716** (2012) 1–29, [1207.7214].
- [2] P. W. Higgs, *Broken Symmetries and the Masses of Gauge Bosons*, *Phys. Rev. Lett.* **13** (1964) 508–509.
- [3] A. M. Jaffe and E. Witten, *Quantum Yang-Mills theory*, .
- [4] R. Rattazzi, V. S. Rychkov, E. Tonni and A. Vichi, *Bounding scalar operator dimensions in 4D CFT*, *JHEP* **12** (2008) 031, [0807.0004].
- [5] J.-L. Gervais and B. Sakita, *Field Theory Interpretation of Supergauges in Dual Models*, *Nucl. Phys. B* **34** (1971) 632–639.
- [6] Y. Golfand and E. Likhtman, *Extension of the Algebra of Poincare Group Generators and Violation of p Invariance*, *JETP Lett.* **13** (1971) 323–326.
- [7] G. 't Hooft, *Naturalness, chiral symmetry, and spontaneous chiral symmetry breaking*, *NATO Sci. Ser. B* **59** (1980) 135–157.
- [8] E. Witten, *Dynamical Breaking of Supersymmetry*, *Nucl. Phys. B* **188** (1981) 513.
- [9] M. Atiyah and R. Bott, *The Moment map and equivariant cohomology*, *Topology* **23** (1984) 1–28.
- [10] N. Berline and M. Vergne, *Classes caractéristiques équivariantes. formule de localisation en cohomologie équivariante*, *CR Acad. Sci. Paris* **295** (1982) 539–541.
- [11] E. Witten, *Supersymmetry and Morse theory*, *J. Diff. Geom.* **17** (1982) 661–692.
- [12] E. Witten, *Topological Quantum Field Theory*, *Commun. Math. Phys.* **117** (1988) 353.
- [13] N. A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, *Adv. Theor. Math. Phys.* **7** (2003) 831–864, [hep-th/0206161].
- [14] A. Losev, G. W. Moore and S. L. Shatashvili, *M & m 's*, *Nucl. Phys.* **B522** (1998) 105–124, [hep-th/9707250].
- [15] A. Losev, N. Nekrasov and S. L. Shatashvili, *Issues in topological gauge theory*, *Nucl. Phys. B* **534** (1998) 549–611, [hep-th/9711108].
- [16] G. W. Moore, N. Nekrasov and S. Shatashvili, *Integrating over Higgs branches*, *Commun. Math. Phys.* **209** (2000) 97–121, [hep-th/9712241].
- [17] A. Losev, N. Nekrasov and S. L. Shatashvili, *Testing Seiberg-Witten solution*, *NATO Sci. Ser. C* **520** (1999) 359–372, [hep-th/9801061].
- [18] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, *Commun. Math. Phys.* **313** (2012) 71–129, [0712.2824].
- [19] V. Pestun et al., *Localization techniques in quantum field theories*, *J. Phys.* **A50** (2017) 440301, [1608.02952].

- [20] N. Hama and K. Hosomichi, *Seiberg-Witten Theories on Ellipsoids*, *JHEP* **09** (2012) 033, [1206.6359].
- [21] V. Pestun, *Localization for $\mathcal{N} = 2$ Supersymmetric Gauge Theories in Four Dimensions*, in *New Dualities of Supersymmetric Gauge Theories* (J. Teschner, ed.), pp. 159–194. 2016. 1412.7134. DOI.
- [22] J. A. Minahan and M. Zabzine, *Gauge theories with 16 supersymmetries on spheres*, *JHEP* **03** (2015) 155, [1502.07154].
- [23] A. Kapustin, B. Willett and I. Yaakov, *Exact Results for Wilson Loops in Superconformal Chern-Simons Theories with Matter*, *JHEP* **03** (2010) 089, [0909.4559].
- [24] F. Benini and S. Cremonesi, *Partition Functions of $\mathcal{N} = (2, 2)$ Gauge Theories on S^2 and Vortices*, *Commun. Math. Phys.* **334** (2015) 1483–1527, [1206.2356].
- [25] N. Doroud, J. Gomis, B. Le Floch and S. Lee, *Exact Results in $D=2$ Supersymmetric Gauge Theories*, *JHEP* **05** (2013) 093, [1206.2606].
- [26] J. Kallen, J. Qiu and M. Zabzine, *The perturbative partition function of supersymmetric 5D Yang-Mills theory with matter on the five-sphere*, *JHEP* **1208** (2012) 157, [1206.6008].
- [27] J. Kallen and M. Zabzine, *Twisted supersymmetric 5D Yang-Mills theory and contact geometry*, *JHEP* **05** (2012) 125, [1202.1956].
- [28] H.-C. Kim and S. Kim, *M5-branes from gauge theories on the 5-sphere*, *JHEP* **05** (2013) 144, [1206.6339].
- [29] N. Hama, K. Hosomichi and S. Lee, *SUSY Gauge Theories on Squashed Three-Spheres*, *JHEP* **05** (2011) 014, [1102.4716].
- [30] J. A. Minahan, *Localizing gauge theories on S^d* , *JHEP* **04** (2016) 152, [1512.06924].
- [31] J. Qiu and M. Zabzine, *5D Super Yang-Mills on $Y^{p,q}$ Sasaki-Einstein manifolds*, *Commun. Math. Phys.* **333** (2015) 861–904, [1307.3149].
- [32] A. Rocén, *7D supersymmetric Yang-Mills on a 3-Sasakian manifold*, *JHEP* **11** (2018) 024, [1808.06917].
- [33] N. Iakovidis, J. Qiu, A. Rocén and M. Zabzine, *7D supersymmetric Yang-Mills on hypertoric 3-Sasakian manifolds*, *JHEP* **06** (2020) 026, [2003.12461].
- [34] M. Dedushenko, S. S. Pufu and R. Yacoby, *A one-dimensional theory for Higgs branch operators*, *JHEP* **03** (2018) 138, [1610.00740].
- [35] Y. Pan and W. Peelaers, *Chiral Algebras, Localization and Surface Defects*, *JHEP* **02** (2018) 138, [1710.04306].
- [36] C. Beem, M. Lemos, P. Liendo, W. Peelaers, L. Rastelli and B. C. van Rees, *Infinite Chiral Symmetry in Four Dimensions*, *Commun. Math. Phys.* **336** (2015) 1359–1433, [1312.5344].
- [37] S. Donaldson, *Polynomial invariants for smooth manifolds*, *Topology* **29** (1990) 257–315.
- [38] G. Festuccia, J. Qiu, J. Winding and M. Zabzine, *Twisting with a Flip (the Art of Pestunization)*, *Commun. Math. Phys.* **377** (2020) 341–385, [1812.06473].
- [39] M. Nakahara, *Geometry, topology, and physics*. Institute of Physics, Bristol, 2. ed., 2003.

- [40] A. C. da Silva, *Lectures on Symplectic Geometry*, vol. 1764 of *Lecture Notes in Mathematics*. Springer-Verlag Berlin Heidelberg, 2008, 10.1007/978-3-540-45330-7.
- [41] J. Schmude, *Localisation on Sasaki-Einstein manifolds from holomorphic functions on the cone*, *JHEP* **01** (2015) 119, [1401.3266].
- [42] J. Sparks, *Sasaki-Einstein Manifolds*, *Surveys Diff. Geom.* **16** (2011) 265–324, [1004.2461].
- [43] M. F. Atiyah, *Convexity and Commuting Hamiltonians*, *Bulletin of the London Mathematical Society* **14** (1982) 1–15.
- [44] D. Martelli and J. Sparks, *Notes on toric Sasaki-Einstein seven-manifolds and $AdS(4) / CFT(3)$* , *JHEP* **11** (2008) 016, [0808.0904].
- [45] W. Fulton, *Introduction to toric varieties*. No. 131 in *Annals of mathematics studies*. Princeton University Press, 1993.
- [46] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, vol. 203 of *Progress in Mathematics*. Birkhauser, 2nd ed., 2010.
- [47] H. Geiges, *An introduction to contact topology*, vol. 109. Cambridge University Press, 2008.
- [48] J.B.Etnyre, *Introductory Lectures on Contact Geometry*, 0111118.
- [49] J. Winding, *Multiple elliptic gamma functions associated to cones*, *Adv. Math.* **325** (2018) 56–86, [1609.02384].
- [50] J. Qiu, L. Tizzano, J. Winding and M. Zabzine, *Gluing Nekrasov partition functions*, *Commun. Math. Phys.* **337** (2015) 785–816, [1403.2945].
- [51] J. Milnor, *Construction of universal bundles. I*, *Ann. of Math.* **63** (1956) 272–284.
- [52] H. Cartan, *Notions d’algèbre différentielle; applications aux groupes de lie et aux variétés où opère un groupe de lie*, *Colloque de topologie (espaces fibres)*, *Bruxelles* (1950) 15–27.
- [53] V. Pestun, *Review of localization in geometry*, *J. Phys. A* **50** (2017) 443002, [1608.02954].
- [54] S. Cremonesi, *An Introduction to Localisation and Supersymmetry in Curved Space*, *PoS Modave2013* (2013) 002.
- [55] V. Pestun and M. Zabzine, *Introduction to localization in quantum field theory*, *J. Phys. A* **50** (2017) 443001, [1608.02953].
- [56] A. Gorantis, J. A. Minahan and U. Naseer, *Analytic continuation of dimensions in supersymmetric localization*, *JHEP* **02** (2018) 070, [1711.05669].
- [57] C.-N. Yang and R. L. Mills, *Conservation of Isotopic Spin and Isotopic Gauge Invariance*, *Phys. Rev.* **96** (1954) 191–195.
- [58] D. Baraglia and P. Hekmati, *Moduli spaces of contact instantons*, *Adv. Math.* **294** (2016) 562–595, [1401.5140v1].
- [59] M. Blau, *Killing spinors and SYM on curved spaces*, *Journal of High Energy Physics* **2000** (nov, 2000) 023–023.
- [60] C. P. Boyer and K. Galicki, *Sasakian Geometry*. Oxford University Press, USA, 2008.
- [61] C. Bär, *Real Killing spinors and holonomy*, *Communications in mathematical physics* **154** (1993) 509–521.
- [62] T. Friedrich, I. Kath, A. Moroianu and U. Semmelmann, *On nearly parallel*

- G2-structures*, *Journal of Geometry and Physics* **23** (1997) 259–286.
- [63] B. S. Acharya, J. Figueroa-O’Farrill, B. J. Spence and M. O’Loughlin, *Euclidean D-branes and higher dimensional gauge theory*, *Nucl. Phys. B* **514** (1998) 583–602, [[hep-th/9707118](#)].
 - [64] D. Prins, *Supersymmetric Gauge Theory on Curved 7-Branes*, *Fortsch. Phys.* **67** (2019) 1900009, [[1812.05349](#)].
 - [65] L. Brink, J. H. Schwarz and J. Scherk, *Supersymmetric Yang-Mills theories*, *Nuclear Physics B* **121** (1977) 77–92.
 - [66] N. Berkovits, *A Ten-dimensional superYang-Mills action with off-shell supersymmetry*, *Phys. Lett.* **B318** (1993) 104–106, [[hep-th/9308128](#)].
 - [67] M. Fujitsuka, M. Honda and Y. Yoshida, *Maximal super Yang-Mills theories on curved background with off-shell supercharges*, *JHEP* **01** (2013) 162, [[1209.4320](#)].
 - [68] J. A. Minahan and U. Naseer, *One-loop tests of supersymmetric gauge theories on spheres*, *JHEP* **07** (2017) 074, [[1703.07435](#)].
 - [69] J. Schmude, *Laplace operators on Sasaki-Einstein manifolds*, *JHEP* **04** (2014) 008, [[1308.1027](#)].
 - [70] R. Eager, J. Schmude and Y. Tachikawa, *Superconformal Indices, Sasaki-Einstein Manifolds, and Cyclic Homologies*, [1207.0573](#).
 - [71] J. J. Kohn and H. Rossi, *On the extension of holomorphic functions from the boundary of a complex manifold*, *Annals of Mathematics* (1965) 451–472.
 - [72] J. Qiu and M. Zabzine, *Review of localization for 5d supersymmetric gauge theories*, vol. A50, p. 443014, 2017. [1608.02966](#). DOI.
 - [73] A. Narukawa, *The modular properties and the integral representations of the multiple elliptic gamma functions*, *Adv. Math.* **189** (2004) 247–267.
 - [74] N. A. Nekrasov, *Instanton partition functions and M-theory*, in *Proceedings, 15th International Seminar on High Energy Physics (Quarks 2008)*, 2008.
 - [75] J. Qiu and M. Zabzine, *Factorization of 5D super Yang-Mills on $Y^{p,q}$ spaces*, *Phys.Rev.* **D89** (2014) 065040, [[1312.3475](#)].
 - [76] S. M. Chester, J. Lee, S. S. Pufu and R. Yacoby, *Exact Correlators of BPS Operators from the 3d Superconformal Bootstrap*, *JHEP* **03** (2015) 130, [[1412.0334](#)].
 - [77] F. Dolan and H. Osborn, *On short and semi-short representations for four-dimensional superconformal symmetry*, *Annals Phys.* **307** (2003) 41–89, [[hep-th/0209056](#)].
 - [78] J. Kinney, J. M. Maldacena, S. Minwalla and S. Raju, *An Index for 4 dimensional super conformal theories*, *Commun. Math. Phys.* **275** (2007) 209–254, [[hep-th/0510251](#)].
 - [79] A. Gadde, L. Rastelli, S. S. Razamat and W. Yan, *The 4d Superconformal Index from q-deformed 2d Yang-Mills*, *Phys. Rev. Lett.* **106** (2011) 241602, [[1104.3850](#)].
 - [80] A. Gadde, L. Rastelli, S. S. Razamat and W. Yan, *Gauge Theories and Macdonald Polynomials*, *Commun. Math. Phys.* **319** (2013) 147–193, [[1110.3740](#)].
 - [81] C. Beem, W. Peelaers and L. Rastelli, *Deformation quantization and superconformal symmetry in three dimensions*, *Commun. Math. Phys.* **354**

- (2017) 345–392, [1601.05378].
- [82] M. Mezei, S. S. Pufu and Y. Wang, *Chern-Simons theory from M5-branes and calibrated M2-branes*, *JHEP* **08** (2019) 165, [1812.07572].
 - [83] Y. Pan and W. Peelaers, *Schur correlation functions on $S^3 \times S^1$* , *JHEP* **07** (2019) 013, [1903.03623].
 - [84] F. Bonetti and L. Rastelli, *Supersymmetric localization in AdS_5 and the protected chiral algebra*, *JHEP* **08** (2018) 098, [1612.06514].
 - [85] K. Costello and J. Yagi, *Unification of integrability in supersymmetric gauge theories*, 1810.01970.
 - [86] E. Witten, *A New Look At The Path Integral Of Quantum Mechanics*, 1009.6032.
 - [87] K. Hosomichi, S. Lee and T. Okuda, *Supersymmetric vortex defects in two dimensions*, *JHEP* **01** (2018) 033, [1705.10623].
 - [88] R. Panerai, M. Poggi and D. Seminara, *Supersymmetric Wilson loops in two dimensions and duality*, *Phys. Rev. D* **100** (2019) 025011, [1812.01315].
 - [89] J. Labastida and M. Marino, *A Topological Lagrangian for monopoles on four manifolds*, *Phys. Lett. B* **351** (1995) 146–152, [hep-th/9503105].
 - [90] G. W. Moore, N. Nekrasov and S. Shatashvili, *D particle bound states and generalized instantons*, *Commun. Math. Phys.* **209** (2000) 77–95, [hep-th/9803265].
 - [91] N. A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, *Adv. Theor. Math. Phys.* **7** (2003) 831–864, [hep-th/0206161].
 - [92] G. Festuccia and N. Seiberg, *Rigid supersymmetric theories in curved superspace*, *JHEP* **06** (2011) 114, [1105.0689].
 - [93] G. Festuccia, J. Qiu, J. Winding and M. Zabzine, *$\mathcal{N} = 2$ supersymmetric gauge theory on connected sums of $S^2 \times S^2$* , *JHEP* **03** (2017) 026, [1611.04868].
 - [94] J. Qiu and M. Zabzine, *On twisted $N=2$ 5D super Yang-Mills theory*, *Lett. Math. Phys.* **106** (2016) 1–27, [1409.1058].
 - [95] E. Friedman and S. Ruijsenaars, *Shintani–barnes zeta and gamma functions*, *Advances in Mathematics* **187** (2004) 362 – 395.

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