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Essential orders on stratified algebras with duality and $\mathcal{S}$-subcategories in $\mathcal{O}$

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ABSTRACT

We prove uniqueness of the essential order for stratified algebras having simple preserving duality, generalizing a recent result of Coulembier for quasi-hereditary algebras. We apply this to classify, up to equivalence, regular integral blocks of $\mathcal{S}$-subcategories in the BGG category $\mathcal{O}$. We also describe various homological invariants of these blocks.

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1. Introduction and description of the results

The notion of a *highest weight category* was introduced in [7] being motivated, among others, by the properties of the BGG category $\mathcal{O}$ associated to a triangular decomposition of a semi-simple complex Lie algebra, see [5,22]. Highest weight categories satisfying reasonable finiteness conditions are described by *quasi-hereditary algebras*, see [14]. Many examples and applications of these structures appeared in the literature over the last 35 years (a naive MathSciNet search for the keyword "quasi-hereditary algebra" produces more than 600 results).

A quasi-hereditary algebra is a structure consisting of a finite dimensional algebra A and a choice of a partial order on the set of isomorphism classes of simple A-modules (this datum is assumed to satisfy some conditions). This datum is used to define the so-called *standard* A-modules which play the crucial role in the quasi-hereditary structure. Usually, there are many different choices for a partial order making a given algebra A quasi-hereditary (for example, A is hereditary if and only if it is quasi-hereditary for *any* choice

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of a linear order). The recent paper [9] established the very surprising fact that, under the assumption of existence of a simple preserving duality, the quasi-hereditary structure on A is, essentially, *unique*. More precisely, any flexibility in the order is unessential in the sense that it does not affect the standard modules.

There are various generalizations of quasi-hereditary algebras. One of them is that of *(standardly) stratified algebras*, introduced in [8]. Assuming the simple preserving duality, this leads to the class of *properly stratified* algebras, introduced in [13], see also [17]. Appearance of these kinds of structures in Lie theory was established in [18,19] with several application proposed in [30], see also [16,6].

The first main result of the present paper provides an analogue of the uniqueness result of [9] for properly stratified algebras, see Theorem 2 in Subsection 2.3.

In [9], the main application of its uniqueness result is the classification, up to equivalence, of (indecomposable) blocks of BGG category $\mathcal{O}$. Such blocks are always described by quasi-hereditary algebras admitting a simple preserving duality. The paper [18] introduces a generalization of $\mathcal{O}$ called $\mathcal{S}$*-subcategories in* $\mathcal{O}$ and shows that these are usually described by properly stratified algebras. The realization of $\mathcal{S}$-subcategories in $\mathcal{O}$ as subcategories of $\mathfrak{p}$-presentable modules in $\mathcal{O}$ was further studied in [30]. $\mathcal{S}$-subcategories in $\mathcal{O}$ appear naturally when studying the action of projective functors on blocks of $\mathcal{O}$, see [4,31]. In particular, the action of projective functors on the principal block of an $\mathcal{S}$-subcategory models the permutation module (i.e. the induced trivial module) for the Weyl group of $\mathfrak{g}$ with respect to the parabolic subgroup corresponding to $\mathfrak{p}$. Compared to $\mathcal{O}$, a natural parameterization of blocks for $\mathcal{S}$-subcategories in $\mathcal{O}$ has an additional independent parameter (with the original category $\mathcal{O}$ being the special case corresponding to a trivial value of that parameter). We address the problem of classification of blocks of $\mathcal{S}$-subcategories in $\mathcal{O}$, up to equivalence, but at the moment it seems quite difficult (the main result of [9] can be viewed as a special case of this problem). We discuss some partial results in this direction in Subsections 3.5 and 3.6. In particular, we use shuffling and twisting functors to establish two different types of derived equivalences between blocks of $\mathcal{S}$-subcategories in $\mathcal{O}$ in Propositions 10 and 11.

In Theorem 6, which can be found in Subsection 3.3, we solve another special case (the "opposite" one with respect to the one considered in [9]). In more detail, we classify, up to equivalence, regular blocks of $\mathcal{S}$-subcategories in $\mathcal{O}$. In our proof of Theorem 6 we use both Theorem 2 and the two main results of [9].

In Section 4, we also study homological properties of regular blocks of $\mathcal{S}$-subcategories in $\mathcal{O}$ in the spirit of [26], in particular, the finitistic dimension of these blocks (finiteness of which was established already in [29]). We provide a description of objects of finite projective dimension in Theorem 15 and use it to obtain an explicit formula for the finitistic dimension (in terms of Lusztig's $\mathbf{a}$-function) as well as explicit formulae for projective dimensions of indecomposable tilting and injective objects (and also a less explicit formula for projective dimensions of standard objects). For tilting and injective objects, we use Theorem 15 to reduce the question to known results for category $\mathcal{O}$ from [26,28]. For standard objects, we use Theorem 15 and the results of [10].

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2. Essential order on stratified algebras with duality

2.1. Stratified algebras

Let $\Bbbk$ be a field and A a finite dimensional (associative) $\Bbbk$-algebra. Let $\{L_\lambda : \lambda \in \Lambda\}$ be a complete and irredundant list of representatives of isomorphism classes of simple A-modules and define

$d_\lambda = \dim_{\Bbbk} \mathrm{End}_A(L_\lambda)$. Note that if $\Bbbk$ is algebraically closed, then $d_\lambda = 1$ for all $\lambda \in \Lambda$. Assume that Λ is equipped with a fixed partial order $\prec$. Denote by A-mod the category of all finite dimensional left A-modules.

Recall that, for two A-modules M and N, the *trace* of M in N is the submodule of N spanned by the images of all A-module homomorphisms from M to N.

For $\lambda \in \Lambda$, denote by P_λ the indecomposable projective cover of L_λ and by I_λ the indecomposable injective envelope of L_λ. Further, denote by Δ_λ the quotient of P_λ by the trace in P_λ of all P_μ, where $\mu \not\succeq \lambda$. Set $\overline{\Delta}_\lambda$ to be the maximal quotient of Δ_λ such that $[\overline{\Delta}_\lambda : L_\lambda] = 1$. Dually, denote by ∇_λ the intersection of the kernels of all morphisms from I_λ to I_μ, where $\mu \not\preceq \lambda$, and define $\overline{\nabla}_\lambda$ as the maximal submodule of ∇_λ such that $[\overline{\nabla}_\lambda : L_\lambda] = 1$. The modules Δ_λ, $\overline{\Delta}_\lambda$, ∇_λ and $\overline{\nabla}_\lambda$ are called *standard*, *proper standard*, *costandard* and *proper costandard*, respectively. If we need to emphasize the role of $\prec$ in the definition, we will use the notation $\Delta_\lambda^\prec$ and similarly for the others.

Recall from [8,17], that $(A, \prec)$ is called *standardly stratified* provided that, for each $\lambda \in \Lambda$, the kernel of the surjection $P_\lambda \twoheadrightarrow \Delta_\lambda$ has a filtration (called a *standard filtration*), each subquotient of which is isomorphic to Δ_μ, for some $\lambda \prec \mu$. Equivalently, A is standardly stratified provided that, for each $\lambda \in \Lambda$, the cokernel of the injection $\overline{\nabla}_\lambda \hookrightarrow I_\lambda$ has a filtration (called a *proper costandard filtration*), each subquotient of which is isomorphic to $\overline{\nabla}_\mu$, for some $\lambda \preceq \mu$, see [17].

A *simple preserving duality* is a contravariant anti-equivalence $\star$ of A-mod which preserves the isomorphism classes of simple A-modules. If a stratified algebra A has a simple preserving duality, then A^{op} is also stratified, in particular, A is *properly stratified* in the sense of [13].

2.2. Essential order

Let $(A, \prec)$ be a standardly stratified algebra. Define a new partial order $\prec_e$ on Λ, called the *essential order*, as the minimal partial order which contains all pairs $(\lambda, \mu) \in \Lambda \times \Lambda$ such that

- L_λ is a composition subquotient of Δ_μ or
- L_λ is a composition subquotient of $\overline{\nabla}_\mu$.

Clearly, $\lambda \prec_e \mu$ implies $\lambda \prec \mu$. Note that, by [17, Subsection 4.1], standard and proper costandard modules form homologically orthogonal families of modules. That is,

$$\mathcal{F}(\Delta) = \{M \in A\text{-mod} \mid \mathrm{Ext}^i(M, \overline{\nabla}_\lambda) = 0 \ \text{ for all } i > 0, \lambda \in \Lambda\}$$

and

$$\mathcal{F}(\overline{\nabla}) = \{M \in A\text{-mod} \mid \mathrm{Ext}^i(\Delta_\lambda, M) = 0 \ \text{ for all } i > 0, \lambda \in \Lambda\},$$

where $\mathcal{F}(\Delta)$ and $\mathcal{F}(\overline{\nabla})$ denote the full subcategories of modules with standard and proper costandard filtrations, respectively. Therefore the condition that L_λ is a composition subquotient of $\overline{\nabla}_\mu$ is equivalent to the condition that Δ_μ appears as a subquotient in a standard filtration of P_λ.

Lemma 1. *If $(A, \prec)$ is standardly stratified, then $(A, \prec_e)$ is standardly stratified. Moreover, the standard modules for both structures coincide and also the proper costandard modules for both structures coincide.*

Proof. It is enough to prove that the standard modules for $\prec_e$ coincide with the standard modules for $\prec$. Indeed, if we can do that, then the ordering conditions for filtrations are automatically satisfied by the definition of $\prec_e$. Furthermore, the fact that the proper costandard modules are the same follows from, e.g.

[17, Subsection 4.1], that is from the fact that standard and proper costandard modules form homologically orthogonal families and hence determine one another, as families of modules.

Since $\prec_e \subset \prec$, directly from the definition of standard modules, we get $\Delta_\lambda^{\prec} \twoheadrightarrow \Delta_\lambda^{\prec_e}$, for every $\lambda \in \Lambda$. We want to show that this surjection is, in fact, an isomorphism. We will prove the claim by induction on the cardinality of Λ. In case $|\Lambda| = 1$, both $\prec$ and $\prec_e$ coincide with the equality relation and hence we have nothing to prove.

Let λ be a maximal element in $(\Lambda, \prec)$. Then $P_\lambda = \Delta_\lambda^{\prec}$ and hence, for any $\mu \in \Lambda$ such that $[P_\lambda : L_\mu] \neq 0$, we have $\mu \prec_e \lambda$. Therefore $P_\lambda = \Delta_\lambda^{\prec_e}$.

Consider now the quotient B of A modulo the ideal $\mathcal{I}$ given by the trace of P_λ in A. The algebra B is standardly stratified with respect to the restriction of $\prec$ to $\Lambda \setminus \{\lambda\}$. Note that $|\Lambda \setminus \{\lambda\}| < |\Lambda|$. Therefore we may now apply the inductive assumption to the algebra B and conclude that standard B-modules with respect to $\prec$ and $\prec_e$ coincide. From the definitions, we see that all standard B-modules coincide with the corresponding standard A-modules. This completes the proof of the induction and of the lemma. □

Recall that a standardly stratified algebra $(A, \prec)$ is called *quasi-hereditary* provided that $\Delta_\lambda = \overline{\Delta}_\lambda$, for all $\lambda \in \Lambda$, see [7,14]. Equivalently, we may require the equality $\nabla_\lambda = \overline{\nabla}_\lambda$, for all $\lambda \in \Lambda$. For quasi-hereditary algebras, the essential order is discussed in [9, § 1.2.4]. Lemma 1 generalizes some of the claims in [9, § 1.2.4].

2.3. Uniqueness of the essential order for stratified algebras with simple preserving duality

The following result, which establishes uniqueness of the essential order for stratified algebras with simple preserving duality, generalizes [9, Theorem 2.1.1] where the case of quasi-hereditary algebras was considered.

Theorem 2. *Let $(A, \prec^1)$ and $(A, \prec^2)$ be standardly stratified. Assume that A-mod has a simple preserving duality. Then $\prec_e^1 = \prec_e^2$.*

Note that, under the assumption of existence of a simple preserving duality, each Δ_μ has a filtration with subquotients isomorphic to $\overline{\Delta}_\mu$, and each ∇_μ has a filtration with subquotients isomorphic to $\overline{\nabla}_\mu$. Therefore the condition that L_λ is a simple subquotient of Δ_μ is equivalent to the condition that L_λ is a simple subquotient of $\overline{\Delta}_\mu$.

In order to prove this, we will need some notation and lemmata. For $\lambda, \mu \in \Lambda$, we denote by $c_{\lambda,\mu}$ the composition multiplicity $[P_\lambda : L_\mu]$.

Lemma 3. *Let $(A, \prec)$ be standardly stratified with a simple preserving duality $\star$. Let $\lambda \in \Lambda$ be a maximal element with respect to $\prec$. Then $c_{\lambda,\lambda} \leq c_{\lambda,\mu}$, for any $\mu \in \Lambda$ such that $c_{\lambda,\mu} \neq 0$.*

Proof. Because of $\star$, the algebra A is properly stratified. Therefore the standard module $P_\lambda = \Delta_\lambda$ has a filtration with all subquotients isomorphic to $\overline{\Delta}_\lambda$. From the definition of $\overline{\Delta}_\lambda$ we see that this filtration has length $c_{\lambda,\lambda}$. Therefore any $c_{\lambda,\mu}$ is an integer multiple of $c_{\lambda,\lambda}$. The claim follows. □

Lemma 4. *Let $(A, \prec)$ be standardly stratified with a simple preserving duality $\star$. Let $\lambda \in \Lambda$ be such that $P_\lambda \neq \Delta_\lambda$. Then there exists $\mu \in \Lambda$ such that $0 < c_{\lambda,\mu} < c_{\lambda,\lambda}$.*

Proof. Choose $\mu \in \Lambda$ maximal among all those ν for which Δ_ν appears as a subquotient of a standard filtration of P_λ. Note that μ is also maximal among those ν such that $\overline{\Delta}_\nu$ appears as a subquotient of a proper standard filtration of P_λ. Then $c_{\lambda,\mu}$ coincides with the multiplicity of $\overline{\Delta}_\mu$ in a proper standard filtration of P_λ. In particular, $c_{\lambda,\mu} \neq 0$.

By the BGG reciprocity, see e.g. [2, Theorem 2.5], [17, Lemma 12], and the fact that A has a simple preserving duality, we have the following relation between the multiplicity of Δ_μ in a standard filtration of P_λ and the composition multiplicity of L_λ in $\overline{\Delta}_\mu$:

$$d_\mu[P_\lambda : \Delta_\mu] = d_\lambda[\overline{\Delta}_\mu : L_\lambda].$$

(We note that [17] only proves the result for algebras over an algebraically closed field, however, the general case, which takes d_λ and d_μ into account, is straightforward.) Consequently, the right hand side is non-zero and hence $[\overline{\Delta}_\mu : L_\lambda] \geq 1$. Therefore

$$\begin{aligned} c_{\lambda,\lambda} &\geq [P_\lambda : \overline{\Delta}_\lambda][\overline{\Delta}_\lambda : L_\lambda] + [P_\lambda : \overline{\Delta}_\mu][\overline{\Delta}_\mu : L_\lambda] \\ &\geq 1 + c_{\lambda,\mu}. \end{aligned}$$

This completes the proof. □

Proof of Theorem 2. What we really need to prove is that the corresponding standard modules with respect to $\prec_e^1$ and $\prec_e^2$ coincide. If we prove that, the fact that $\prec_e^1$ and $\prec_e^2$ coincide would follow directly from the definition of an essential order. We proceed by induction on $|\Lambda|$. If Λ is a singleton, then $\prec^1 = \prec^2 = \prec_e^1 = \prec_e^2$ and we have nothing to prove.

Lemmata 3 and 4 together imply that the elements $\lambda \in \Lambda$ having the property that $c_{\lambda,\lambda} \leq c_{\lambda,\mu}$, for all μ, are maximal with respect to all the orders $\prec^1$, $\prec^2$, $\prec_e^1$ and $\prec_e^2$. In particular, the corresponding standard module is projective and hence is the same for both $\prec_e^1$ and $\prec_e^2$.

Recall that, for any projective module P, the trace of P in the left regular A-module ${}_AA$ is a two-sided ideal of A. Let $\mathcal{I}$ be such an ideal given by the trace of P_λ in ${}_AA$. Factoring this ideal out, we can apply the inductive assumption to the stratified algebra $A/\mathcal{I}$ and conclude that all corresponding standard modules with respect to $\prec_e^1$ and $\prec_e^2$ coincide. This completes the proof. □

2.4. The weighted poset invariant

Let $(A, \prec)$ be a standardly stratified algebra with a simple preserving duality. Define the function $\nu : \Lambda \to \mathbb{Z}_{>0}$ as follows:

$$\nu(\lambda) := \dim_{\Bbbk} \operatorname{End}(\Delta_\lambda).$$

From Theorem 2 it follows that the weighted poset $(\Lambda, \prec_e, \nu)$ is an invariant of the Morita equivalence class of A.

Certainly, there are other natural choices for ν which give rise to similar invariants, for example, one could use $[\Delta_\lambda : L_\lambda]$ instead of $\dim_{\Bbbk} \operatorname{End}(\Delta_\lambda)$ (the two numbers coincide provided that $\Bbbk$ is algebraically closed). We have chosen $\dim_{\Bbbk} \operatorname{End}(\Delta_\lambda)$ as this number is more straightforward to compute in the situation where this invariant is used in Subsection 3.4.

3. $\mathcal{S}$-subcategories in $\mathcal{O}$

3.1. Category $\mathcal{O}$

Consider a simple finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{C}$. Fix a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+.$$

To this datum, we have the associated Bernstein-Gelfand-Gelfand (BGG) category $\mathcal{O}$, see [5,22]. We refer the reader to [5,22,12,23] for further details on category $\mathcal{O}$.

Let W be the Weyl group of $\mathfrak{g}$ and $\mathbf{R}$ be the corresponding root system in $\mathfrak{h}^*$, with the basis π corresponding to our choice of $\mathfrak{n}_+$. We denote by ρ the half of the sum of all positive roots. We have the corresponding *dot-action* of W on $\mathfrak{h}^*$ given by $w \cdot \lambda = w(\lambda + \rho) - \rho$.

Simple objects in $\mathcal{O}$ are the simple highest weight modules $L(\lambda)$, where $\lambda \in \mathfrak{h}^*$. In particular, we have infinitely many non-isomorphic simple objects. However, $\mathcal{O}$ decomposes into a direct sum of indecomposable *blocks*, each of which has finitely many simples and is equivalent to A-mod for some finite dimensional, associative and quasi-hereditary algebra A. In fact, if two simples $L(\lambda)$ and $L(\mu)$ belong to the same block, then $\lambda \in W \cdot \mu$.

Category $\mathcal{O}$ has a simple preserving duality, which we denote by $\star$. This duality is an involutive contravariant equivalence which fixes every isomorphism class of simple objects. We refer to [22, § 3.2] for details.

We denote by $\mathcal{O}_{\mathrm{int}}$ the full subcategory of $\mathcal{O}$ consisting of modules with integral weights. By [33], each block of $\mathcal{O}$ is equivalent to an *integral block*, that is, to a block of $\mathcal{O}_{\mathrm{int}}$ for some (possibly different) Lie algebra $\mathfrak{g}$. Using this we restrict our attention to $\mathcal{O}_{\mathrm{int}}$. For $\mathcal{O}_{\mathrm{int}}$, two simples $L(\lambda)$ and $L(\mu)$ belong to the same block if and only if $\lambda \in W \cdot \mu$. To simplify our notation, if λ is dominant and clear from the context, we write L_w instead of $L(w \cdot \lambda)$, for $w \in W$.

We denote by $P(\lambda)$ the indecomposable projective cover of $L(\lambda)$ in $\mathcal{O}$ and by $I(\lambda)$ the corresponding indecomposable injective envelope. We denote by $\Delta(\lambda)$ the standard module (also called Verma module) with top $L(\lambda)$ and by $\nabla(\lambda)$ the corresponding costandard module. Then $\nabla(\lambda) = \Delta(\lambda)^\star$, where $\star$ is the simple preserving duality on $\mathcal{O}$. We denote by $T(\lambda)$ the indecomposable tilting module with highest weight λ. Similarly to the above, we use the notation P_w etc. when appropriate.

For a dominant integral weight λ, we denote by $\mathcal{O}_\lambda$ the block of $\mathcal{O}$ containing $L(\lambda)$. Let W_λ denote the *dot-stabilizer* of λ. Then simple modules in $\mathcal{O}_\lambda$ are naturally parameterized by the set of longest coset representatives in W/W_λ. *Soergel's combinatorial description* for integral blocks of $\mathcal{O}$, see [33], implies that $\mathcal{O}_\lambda \cong \mathcal{O}_{\lambda'}$ if $W_\lambda = W_{\lambda'}$ (note that the latter is a genuine equality and not an isomorphism). This allows us to parameterize blocks of $\mathcal{O}$ by pairs (W, S), where W is a Weyl group and S a parabolic subgroup of W (i.e. the *singularity* of a block). One of the main results in [9] asserts that the blocks given by (W, S) and (W', S') are equivalent if and only if the partially ordered sets W/S and W'/S', with respect to the Bruhat order (more precisely, the restriction of the Bruhat order to the set of the longest coset representatives), are isomorphic.

3.2. $\mathcal{S}$-subcategories in $\mathcal{O}$

In this paper we will be interested in the extension of the results of [9] to the so-called $\mathcal{S}$-subcategories in $\mathcal{O}$, introduced in [18], see also [30, Section 2]. To define these we need to fix another parabolic subgroup G in W, completely unrelated to S.

So, we fix an integral dominant weight λ with dot-stabilizer S and split $W \cdot \lambda$ into two sets: the set X consisting of all $\mu \in W \cdot \lambda$ such that $w \cdot \mu \geq \mu$, for any $w \in G$, and the set $Y := (W \cdot \lambda) \setminus X$.

Let $\mathcal{C}$ denote the Serre subcategory of $\mathcal{O}_\lambda$ generated by all $L(\mu)$, where $\mu \in Y$. The *$\mathcal{S}$-subcategory* of $\mathcal{O}$ associated to λ and G is the abelian quotient

$$\mathcal{S}_{\lambda,G} := \mathcal{O}_\lambda / \mathcal{C}.$$

We denote by $\Phi : \mathcal{O}_\lambda \to \mathcal{S}_{\lambda,G}$ the corresponding exact quotient functor. An alternative description of $\mathcal{S}_{\lambda,G}$ is as follows: $\mathcal{S}_{\lambda,G}$ is equivalent to the category of eAe-modules, where A is the unique, up to isomorphism,

basic associative algebra such that $\mathcal{O}_\lambda$ is equivalent to A-mod, and e is the idempotent corresponding to all $\mu \in X$, that is:

$$eAe \cong \mathrm{End}_{\mathcal{O}}\big(\bigoplus_{\mu\in X} P(\mu)\big)^{\mathrm{op}}.$$

The algebra eAe is, in general, no longer quasi-hereditary, however, it is properly stratified, see [18]. Note that $\mathcal{S}_{\lambda,\{e\}} = \mathcal{O}_\lambda$.

Let $\mathfrak{p}$ be the parabolic subalgebra of $\mathfrak{g}$ corresponding to G. We have the Levi decomposition $\mathfrak{p} = \mathfrak{a}_\mathfrak{p} \oplus \mathfrak{h}_\mathfrak{p}^\perp \oplus \mathfrak{n}_\mathfrak{p}$, where $\mathfrak{n}_\mathfrak{p}$ is the nilpotent radical, $\mathfrak{l}_\mathfrak{p} := \mathfrak{a}_\mathfrak{p} \oplus \mathfrak{h}_\mathfrak{p}^\perp$ is the reductive Levi factor, $\mathfrak{a}_\mathfrak{p}$ is semi-simple and $\mathfrak{h}_\mathfrak{p}^\perp$ is the center of $\mathfrak{l}_\mathfrak{p}$. Sometimes we use the notation $\mathfrak{l}_G$, and similarly for the other factors, to emphasize the connection to G.

Putting the above things together, we can parameterize $\mathcal{S}$-subcategories in $\mathcal{O}$ using triples (W, S, G), where W is a Weyl group and S and G are two parabolic subgroups. We simply write $\mathcal{S}(W, S, G)$ for the corresponding $\mathcal{S}_{\lambda,G}$, which is well-defined, up to equivalence. Below we would like to address the following problem:

Problem 5. Classify the blocks of $\mathcal{S}$-subcategories in $\mathcal{O}$, up to equivalence.

This accounts to determining for which triples (W, S, G) and (W', S', G') we have $\mathcal{S}(W, S, G) \cong \mathcal{S}(W', S', G')$. Note that the main result of [9] solves a special case of this problem, namely the case $G = G' = \{e\}$. In this paper we are not able to give a full solution, but we do solve a special case as well, which can be seen as the "opposite" case to the one considered in [9].

We say that (W, S, G) and (W', S', G') are *isomorphic* if there is an isomorphism of Coxeter groups from W to W' which sends S to S' and G to G'. Clearly, such an isomorphism induces an equivalence of $\mathcal{S}(W, S, G)$ and $\mathcal{S}(W', S', G')$.

3.3. Regular blocks

Our classification result is for regular blocks, that is, the case $S = S' = \{e\}$. In this case we may take $\lambda = 0$, so that we work in the *principal block* $\mathcal{O}_0$. Then the simple objects in $\mathcal{S}(W, \{e\}, G)$ are naturally indexed by the set X_G of longest coset representatives in $G \setminus W$. We denote by B the unique, up to isomorphism, basic associative algebra such that B-mod is equivalent to $\mathcal{S}(W, \{e\}, G)$.

Theorem 6. *We have $\mathcal{S}(W, \{e\}, G) \cong \mathcal{S}(W', \{e\}, G')$ if and only if $(W, \{e\}, G)$ and $(W', \{e\}, G')$ are isomorphic.*

To prove this theorem, we will need more notation and some preliminary results. Denote by $\mathtt{C}_W$ the *coinvariant algebra* associated to W. It is defined as the quotient of $\mathbb{C}[\mathfrak{h}]$ modulo the ideal generated by the homogeneous W-invariant polynomials of non-zero degree. In $\mathcal{O}_0$, the algebra $\mathtt{C}_W$ can be found as the endomorphism algebra of the unique indecomposable projective injective module P_{w_0}, here w_0 is the longest element of W, see [33, Endomorphismensatz]. In particular, $\mathtt{C}_W$ is finite dimensional (of dimension $|W|$) and self-injective. We refer to [20] for more details on coinvariant algebras.

Lemma 7. *We have $\mathtt{C}_W \cong \mathtt{C}_{W'}$ if and only if W and W' are isomorphic as Coxeter groups.*

We note that the isomorphism of $\mathtt{C}_W \cong \mathtt{C}_{W'}$ in the formulation is just an isomorphism of algebras, not necessarily of graded algebras.

Proof. The "if" part is obvious, so we prove the "only if" part. To start with, we want to prove that $\mathtt{C}_W \cong \mathtt{C}_{W'}$ implies that the rank of W, i.e. the dimension of $\mathfrak{h}$, equals the rank of W'.

Directly from the definition we have that $\mathtt{C}_W$ is positively graded. For this proof, we choose the grading such that the generating elements of $\mathfrak{h}$ have degree 1 (note that this is different from the natural grading coming from $\mathcal{O}$ in which the generators have degree 2).

Since $\mathtt{C}_W$ is positively graded and generated by elements of degrees 0 and 1, by [3, Proposition 2.4.1], the graded filtration of $\mathtt{C}_W$ coincides with the radical filtration of the regular module ${}_{\mathtt{C}_W}\mathtt{C}_W$. Therefore the rank of W can be recovered from $\mathtt{C}_W$ as the dimension of $\mathrm{Rad}({}_{\mathtt{C}_W}\mathtt{C}_W)/\mathrm{Rad}(\mathrm{Rad}({}_{\mathtt{C}_W}\mathtt{C}_W))$.

Now, if we fix the rank of W, we can go through the finite list of finite Weyl groups of this fixed rank and check that non-isomorphic Coxeter groups of this fixed rank have different cardinalities. This is clear if we compare them with each other from well-known lists: the Weyl group cardinality for type A_n is $(n+1)!$, for type $B_n = C_n$ it is $2^n n!$ and for type D_n it is $2^{n-1}n!$. In the exceptional cases, we have the following cardinalities: for E_6 it is $72 \cdot 6!$; for E_7 it is $576 \cdot 7!$; for E_8 it is $17280 \cdot 8!$; for F_4 it is $48 \cdot 4!$ and in type G_2 it is $6 \cdot 2!$. The claim follows. □

Lemma 8. *The Bruhat order on X_G coincides with the essential order for B.*

Proof. This follows, for example, from the well-known fact that in category $\mathcal{O}$ $[\Delta_y : L_x] \neq 0$ if and only if $x \geq y$ with respect to the Bruhat order, see e.g. [12, Section 7]. □

Proof of Theorem 6. The "if" part of the claim is obvious, so we prove the "only if" part. Assume $\mathcal{S}(W, \{e\}, G) \cong \mathcal{S}(W', \{e\}, G')$. Both categories $\mathcal{S}(W, \{e\}, G)$ and $\mathcal{S}(W', \{e\}, G')$ contain a unique, up to isomorphism, indecomposable projective-injective module, namely the one whose index is the anti-dominant weight. Therefore, any equivalence between these categories must match these indecomposable projective-injective objects and induce an isomorphism between the endomorphism algebras of these objects. From [33, Endomorphismensatz] we thus get that there is an isomorphism $\mathtt{C}_W \cong \mathtt{C}_{W'}$. From Lemma 7 it follows that W and W' are isomorphic as Coxeter groups.

To complete the proof we just need to argue that there is an isomorphism of Coxeter groups from W to W' which maps G to G'. By Theorem 2, we know that the essential order on X_G is unique and, by Lemma 8, it coincides with the Bruhat order on X_G. Therefore any equivalence $\mathcal{S}(W, \{e\}, G) \cong \mathcal{S}(W', \{e\}, G')$ induces an isomorphism of the posets X_G and $X_{G'}$ with respect to the corresponding Bruhat orders.

Taking into account [9, Theorem 1], all such non-trivial isomorphisms are classified in [9, Theorem 2]. The corresponding lists (1)-(4) in [9, Theorem 2] are (written as a pair (type of W, type of G)):

- $(A_{2n+1}, A_{2n}) \cong (B_{n+1}, B_n)$, for $n \geq 2$;
- $(B_n, A_{n-1}) \cong (D_{n+1}, A_n)$, for $n \geq 3$;
- $(A_3, A_2) \cong (B_2, A_1)$;
- $(A_5, A_4) \cong (G_1, A_1) \cong (B_3, B_2)$.

We see that all non-trivial isomorphisms concern Weyl groups of different ranks. Therefore an isomorphism W to W' in our case can be chosen such that it maps G to G'. This completes the proof. □

3.4. The weighted poset invariant via the Bruhat order

By construction, simple objects in $\mathcal{S}(W, S, G)$ are parameterized by elements of the set $G\backslash W/S$ of double cosets (and they naturally correspond to the choice of the longest representatives is such cosets). From Lemma 8, it follows that the essential order on $G\backslash W/S$ is exactly the restriction of the Bruhat order to the set of the longest coset representatives for cosets in $G\backslash W/S$. We note that, by [21], this coincides with the restriction of the Bruhat order $\leq$ to the set of the shortest representatives in cosets from $G\backslash W/S$.

The next lemma describes the weight function ν for $(G\backslash W/S, \leq)$ as defined in Subsection 2.4.

Lemma 9. *The value of ν at a coset $\xi \in G\backslash W/S$ coincides with the number of right S-subcosets in ξ.*

Proof. The standard module Δ_ξ is parabolically induced from an indecomposable projective-injective module $P(\mu)$ in the category $\mathcal{O}$ for the Levi factor $\mathfrak{l}_G$ of the parabolic subalgebra $\mathfrak{p}$ corresponding to G, see [30, Proposition 2.9]. In other words, there exists an anti-dominant weight μ for $\mathfrak{l}_G$ such that $\Delta_\xi \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} P(\mu)$, where $P(\mu)$ is the projective cover of $L(\mu)$ in the category $\mathcal{O}$ for $\mathfrak{l}_G$.

Observe that the dimension of the endomorphism algebra of an indecomposable projective-injective module in category $\mathcal{O}$ equals the number of isomorphism classes of simple objects in its block. This follows by combining the following standard facts from the literature. First, indecomposable projective-injective modules in $\mathcal{O}$ are exactly the projective covers of simple Verma modules by [15]. Next, each Verma module has simple socle, this socle is a Verma module and none of the other simple subquotients of the original Verma module is a Verma module, see [12, Section 7]. Finally, each indecomposable block of category $\mathcal{O}$ contains a unique simple Verma module (since each Weyl group has a unique longest element) and hence, by the BGG reciprocity, the Verma flag of an indecomposable projective-injective module contains each Verma module from its block exactly once.

Since, in the situation of our lemma, the simple objects in the block of $\mathcal{O}$ which we consider correspond exactly to the right S-subcosets in ξ, we obtain that the dimension of the endomorphism algebra of $P(\mu)$ is given by the number of right S-subcosets in ξ.

The parabolic induction provides an embedding of the endomorphism algebra of $P(\mu)$ into the endomorphism algebra of Δ_ξ. That this embedding is an isomorphism, is obtained, using adjunction, from the following:

$$\mathrm{Hom}_{\mathfrak{g}}(\Delta_\xi, \Delta_\xi) \cong \mathrm{Hom}_{\mathfrak{p}}(P(\mu), \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}}\Delta_\xi) \cong \mathrm{Hom}_{\mathfrak{p}}(P(\mu), P(\mu)).$$

Here, to justify the second isomorphism, we note that the elements of $\mathfrak{h}^{\perp}_{\mathfrak{p}}$ act on $P(\mu)$ as scalars. Furthermore, the action of $\mathfrak{h}^{\perp}_{\mathfrak{p}}$ on $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}}\Delta_\xi$ is semisimple with $P(\mu)$ being an isotypic component. Since $\mathfrak{p}$ contains $\mathfrak{h}^{\perp}_{\mathfrak{p}}$, the image of any $\mathfrak{p}$-homomorphism from $P(\mu)$ to $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}}\Delta_\xi$ belongs to $P(\mu)$.

This completes the proof of the lemma. □

3.5. Some derived equivalences in type A

In type A, that is when $W = S_n$, a parabolic subgroup W' of W defines naturally a partition of n which bookkeeps the sizes of the connected blocks in $\{1, 2, \dots, n\}$, where i and $i+1$ are connected if $(i, i+1) \in W'$.

In the next two propositions, when we say that two module categories are derived equivalent we mean that the corresponding derived categories of right bounded complexes are equivalent. All derived equivalences which we construct are given by (generalized) tilting modules. Hence our derived equivalence induces an equivalence of the categories of perfect complexes.

Proposition 10. *Assume that W is the symmetric group S_n and that (W, S, G) and (W, S', G) are such that S and S' correspond to the same partition of n. Then the category $\mathcal{S}(W, S, G)$ is derived equivalent to the category $\mathcal{S}(W, S', G)$.*

Proof. We have the derived equivalence between $\mathcal{S}(W, S, \{e\})$ and $\mathcal{S}(W, S', \{e\})$ given by [11, Theorem B]. By construction in [11], this equivalence is defined in terms of projective and shuffling functors and such functors send $\mathcal{S}(W, S, G)$ to $\mathcal{S}(W, S', G)$. Since the inclusion of the homotopy category of projective objects

in $\mathcal{S}(W,S,G)$ into the homotopy category of projective objects in $\mathcal{S}(W,S,\{e\})$ is full and faithful, the above derived equivalence between $\mathcal{S}(W,S,\{e\})$ and $\mathcal{S}(W,S',\{e\})$ restricts to a derived equivalence between $\mathcal{S}(W,S,G)$ and $\mathcal{S}(W,S',G)$. □

Proposition 11. *Assume that W is the symmetric group S_n and that (W,S,G) and (W,S,G') are such that G and G' correspond to the same partition of n. Then the category $\mathcal{S}(W,S,G)$ is derived equivalent to the category $\mathcal{S}(W,S,G')$.*

Proof. This is very similar to the proof of [11, Theorem B], but much easier as we can use twisting functors from [1] which, in particular, commute with projective functors. It is enough to assume that G and G' differ by permuting two neighbor parts. We let w_0^1 denote the longest element of the parabolic subgroup corresponding to the first of these parts for G and we let w_0^2 denote the longest element of the parabolic subgroup corresponding to the second part. Note that w_0^1 and w_0^2 commute since they are permutations of sets of elements that do not intersect. Finally, we let $w_0^{1,2}$ denote the longest element of the parabolic subgroup corresponding to the merging of the two parts. Note that the parts of G' are obtained from the parts of G by conjugation with $w_0^{1,2}$. We will use various properties of $\mathcal{S}$-subcategories established in [18,30] and also various properties of twisting functor from [1].

First we claim that the twisting functor $\top_{w_0^{1,2}}$ induces a derived equivalence between $\mathcal{S}(W,\{e\},G)$ and $\mathcal{S}(W,\{e\},G')$. For this we can realize both subcategories inside $\mathcal{O}_0$. As usual, w_0^G denotes the longest element in G. Then w_0^G is the product of the longest element in each parabolic subgroup corresponding to a connected block in G. Set $w_0^r := w_0^G w_0^1 w_0^2$, then $w_0^G = w_0^r w_0^1 w_0^2$ is reduced. Note that w_0^r commutes with w_0^1, w_0^2 and $w_0^{1,2}$. Projectives in $\mathcal{S}(W,\{e\},G)$ are obtained by applying projective functors to the dominant projective (thus standard) module $P_{w_0^G} = \theta_{w_0^G}\Delta_e$, where θ_w is the indecomposable projective functor uniquely defined by $\theta_w\Delta_\lambda = P(w\cdot\lambda)$, see [4]. Applying $\top_{w_0^{1,2}}$ to this module, we have

$$\top_{w_0^{1,2}} P_{w_0^G} = \top_{w_0^{1,2}}\theta_{w_0^G}\Delta_e = \theta_{w_0^G}\top_{w_0^{1,2}}\Delta_e = \theta_{w_0^G}\Delta_{w_0^{1,2}}.$$

The module $P_{w_0^G}$ has a Verma flag with Verma subquotients of the form Δ_w, where $e \le w \le w_0^G$. As $\theta_{w_0^G}$ acts on the indices on the right, the module $\theta_{w_0^G}\Delta_{w_0^{1,2}}$ has a Verma flag with Verma subquotients of the form Δ_w, where w belongs to the coset $G'w_0^{1,2} = w_0^{1,2}G$. Note that

$$w_0^{1,2}w_0^G = w_0^{1,2}w_0^1w_0^2w_0^r = (w_0^{1,2}w_0^1w_0^{1,2})(w_0^{1,2}w_0^2w_0^{1,2})w_0^r w_0^{1,2} = w_0^{G'}w_0^{1,2}.$$

Let q be the longest element in the coset $G'w_0^{1,2}$. The module $\theta_{w_0^G}\Delta_{w_0^{1,2}}$ has simple top L_q and thus is a quotient of the indecomposable projective module P_q. From the Verma flag of $\theta_{w_0^G}\Delta_{w_0^{1,2}}$ described above, we see that the kernel of the projection $P_q \twoheadrightarrow \theta_{w_0^G}\Delta_{w_0^{1,2}}$ is generated by all Verma subquotients of P_q which do not appear in the Verma flag of $\theta_{w_0^G}\Delta_{w_0^{1,2}}$. This implies that $\top_{w_0^{1,2}}P_{w_0^G} = \theta_{w_0^G}\Delta_{w_0^{1,2}}$ is a standard object in $\mathcal{S}(W,\{e\},G')$.

Applying projective functors and using that they commute with twisting functors, we obtain that $\top_{w_0^{1,2}}$ sends projectives in $\mathcal{S}(W,\{e\},G)$ to modules with standard filtrations in $\mathcal{S}(W,\{e\},G')$. The inclusion of the homotopy category of projectives in $\mathcal{S}(W,\{e\},G)$ into the homotopy category of projectives in $\mathcal{O}_0$ is full and faithful. Since $\top_{w_0^{1,2}}$ is a derived equivalence for $\mathcal{O}$, it thus follows that the image of a projective generator from $\mathcal{S}(W,\{e\},G)$ under $\top_{w_0^{1,2}}$ is a generalized tilting module in $\mathcal{S}(W,\{e\},G')$. This implies that $\top_{w_0^{1,2}}$ induces a derived equivalence between $\mathcal{S}(W,\{e\},G)$ and $\mathcal{S}(W,\{e\},G')$.

Finally, as twisting functors commute with projective functors (in particular, with translations to and from the walls), the derived equivalence constructed above induces a derived equivalence between $\mathcal{S}(W,S,G)$ and $\mathcal{S}(W,S,G')$. □

3.6. Small rank cases

The weighted poset $G\backslash W/S$ with the weight function given by Lemma 9 gives a strong invariant for telling different blocks $\mathcal{S}(W,S,G)$ apart. This allows us to deal with many small rank cases. The poset itself can be computed using computer and suitable software. In Section 5 one can find the results of our computations using Sage. Below is a short analysis of these computations.

In type A_2, which can be found in Fig. 1, there is the obvious redundancy given by the symmetry of the root system. Also, all blocks in the last row are obviously equivalent (to the category of vector spaces). Taking this into account (i.e. identifying the blocks obtained from each other by the involution swapping s_1 and s_2 and collapsing the last row to a single element), we get an irredundant classification of blocks in type A_2, up to equivalence.

Similarly one can deal with the types B_2 and G_2, which can be found in Fig. 1, respectively.

Type A_3, which can be found in Fig. 2, is more interesting. There are two places for which the obvious redundancy as above is not enough. These are indicated by the magenta and the orange colors in the figure. The two orange cases, $(A_3,\langle s_1\rangle,A_3)$ and $(A_3,\langle s_2\rangle,A_3)$, correspond to module categories over the algebras of all $(1,2)$- and all $(2,3)$-invariants in the coinvariant algebra of S_4, respectively. As these algebras are naturally isomorphic (since $(1,2)$ and $(2,3)$ are conjugate in S_4 and hence the action on the coinvariant algebra of an element which conjugates $(1,2)$ to $(2,3)$ gives rise to an isomorphism between the corresponding algebras of $(1,2)$- and $(2,3)$-invariants), we obtain that the orange cases are equivalent. We do not know whether the magenta cases, $(A_3,\langle s_1\rangle,\langle s_2\rangle)$ and $(A_3,\langle s_2\rangle,\langle s_1\rangle)$, are equivalent or not, we suspect that they are. They are certainly derived equivalent since they both are derived equivalent to the teal case, $(A_3,\langle s_2\rangle,\langle s_2\rangle)$. The magenta case in the same column with the teal case is derived equivalent to the teal case by Proposition 10. The magenta case in the same row with the teal case is derived equivalent to the teal case by Proposition 11.

Type B_3, which can be found in Fig. 3, further supports the observation that, with a small number of exceptions, different blocks of $\mathcal{S}(W,S,G)$ will not be equivalent. The only cases of suspected, but not obvious by other kinds of symmetries, equivalences are indicated by the same color.

In general, it is not true that $\mathcal{S}(W,S,G)$ and $\mathcal{S}(W',S',G')$ are equivalent if and only if their weighted posets are isomorphic. A very degenerate counterexample would be the one element poset of weight 6. This can be realized, on the one hand, by the coinvariant algebra of type A_2 and, on the other hand, by the algebra of A_1-invariants in the coinvariant algebra of type G_2, see Fig. 1. These two algebras are not isomorphic (the first one has two generators while the second one has only one generator). Therefore, to tell the blocks $\mathcal{S}(W,S,G)$ apart, a more subtle invariant is necessary. A good starting point would be:

Problem 12. Classify, up to isomorphism, the algebras of parabolic invariants in all coinvariant algebras (alternatively, all $\mathcal{S}(W,S,G)$ having exactly one isomorphism class of simple objects).

Figs. 1–3 suggest the following conjecture which, for the moment, is completely open:

Conjecture 13. *The categories $\mathcal{S}(W,S,S)$ and $\mathcal{S}(W',S',S')$ are equivalent if and only if their weighted posets are isomorphic.*

4. Homological properties of $\mathcal{S}(W,\{e\},G)$

4.1. Objects of finite projective dimension

We take this opportunity to describe various homological invariants for $\mathcal{S}(W,\{e\},G)$, in particular, improving on the main result of [29]. The latter says that the finitistic dimension of $\mathcal{S}(W,\{e\},G)$ equals twice the projective dimension of the characteristic tilting module. This characteristic tilting module is an additive generator of the full subcategory of $\mathcal{S}(W,\{e\},G)$ consisting of all modules which have both standard

and proper costandard filtrations. We improve this result by giving an explicit formula for the projective dimension of this tilting module, reducing it to the known results in category $\mathcal{O}$ from [26,28].

Recall that X_G is the set of all longest coset representatives in $G \setminus W$ and it indexes simple objects in $\mathcal{S}(W, \{e\}, G)$ which we consider as the quotient category of $\mathcal{O}_0$. Recall that we denote by $\mathfrak{a}_G$ the semi-simple part of the parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ corresponding to G. Each M in the category $\mathcal{O}$ for $\mathfrak{g}$, when considered as an $\mathfrak{a}_G$-module, is a (possibly infinite) direct sum of modules from the category $\mathcal{O}$ for $\mathfrak{a}_G$. We say that M is *admissible* provided that all summands of this latter decomposition are projective-injective (in the category $\mathcal{O}$ for $\mathfrak{a}_G$). Our first observation is the following:

Lemma 14. *Let $M \in \mathcal{O}$ be admissible. Assume that $\mathrm{Ext}^i_{\mathcal{O}}(M, L_w) \neq 0$, for some $w \in W$ and some $i \geq 0$. Then $w \in X_G$.*

Proof. For a simple reflection s in G, consider the corresponding twisting functor $\top_s$, see [1]. The element s corresponds to a simple root in $\mathfrak{a}_G$, say α.

We claim that $\top_s M \cong M$, more precisely, that the natural map $\top_s M \to M$ given by [24, Theorem 4] is an isomorphism. Since M is admissible, when restricted to $\mathfrak{a}_G$, the module M is a direct sum of projective modules. Therefore the map $\top_s M \to M$ is injective by [24, Theorem 4].

On the other hand, since M is admissible, when restricted to $\mathfrak{a}_G$, the module M is a direct sum of injective modules. Therefore the action of $\mathfrak{g}_{-\alpha}$ is injective on any top constituent of M. This implies that the cokernel of the natural map $\top_s M \to M$ is zero and thus $\top_s M \cong M$.

Now let $w \notin X_G$. Then there is a simple reflection s as above such that $sw > w$. Let G_s denote the right adjoint of $\top_s$. Then $G_s \cong \star \top_s \star$ by [1, Theorem 4.1]. By [1, Theorem 6.1], we have $\mathcal{L}_1 \top_s M = 0$, since $\mathfrak{g}_{-\alpha}$ acts injectively on M, and thus $\top_s M \cong \mathcal{L} \top_s M$. Combining [1, Theorem 4.1] and [1, Theorem 6.1], we also get $\mathcal{R} G_s L_w \cong L_w[-1]$. Using the above, we compute:

$$\begin{aligned}
\mathrm{Ext}^i_{\mathcal{O}}(M, L_w) &= \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(M, L_w[i]) \\
&= \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\top_s M, L_w[i]) \\
&= \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(\mathcal{L} \top_s M, L_w[i]) \\
&= \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(M, \mathcal{R} G_s L_w[i]) \\
&= \mathrm{Hom}_{\mathcal{D}^b(\mathcal{O})}(M, L_w[i-1]).
\end{aligned}$$

Applying this procedure $i+1$ times we get a homomorphism in $\mathcal{D}^b(\mathcal{O})$ from a module to a module shifted to the right. This is certainly zero. The claim of the lemma follows. □

Theorem 15. *An object $N \in \mathcal{S}(W, \{e\}, G)$ has finite projective dimension if and only if there is an admissible $M \in \mathcal{O}_0$ such that $N \cong \Phi(M)$, where Φ is the quotient functor. Moreover, the minimal projective resolution of N in $\mathcal{S}(W, \{e\}, G)$ is obtained from a minimal projective resolution of M in $\mathcal{O}_0$ by applying Φ and the projective dimensions of N (in $\mathcal{S}(W, \{e\}, G)$) and M (in $\mathcal{O}_0$) coincide.*

Proof. Assume that $M \in \mathcal{O}_0$ is admissible. Let $\mathcal{P}_\bullet$ be a minimal projective resolution of M. By Lemma 14, the only indecomposable projectives appearing in $\mathcal{P}_\bullet$ are P_w, where $w \in X_G$. The images of these projective modules under the exact functor Φ are projective in $\mathcal{S}(W, \{e\}, G)$, moreover, Φ preserves the property of such projectives to be pairwise non-isomorphic. Therefore $\Phi(\mathcal{P}_\bullet)$ is a minimal projective resolution of $\Phi(M)$.

Conversely, assume that $N \in \mathcal{S}(W, \{e\}, G)$ has a finite projective resolution. Since any indecomposable projective in $\mathcal{S}(W, \{e\}, G)$ is, up to isomorphism, the image under Φ of some P_w, where $w \in X_G$, and since Φ is full and faithful on such projectives, we may assume that a minimal projective resolution of N has the form $\Phi(\mathcal{P}_\bullet)$, for some complex $\mathcal{P}_\bullet$ of projective modules in $\mathcal{O}_0$, in which each indecomposable projective has the form P_w, for some $w \in X_G$. We claim that $\mathcal{P}_\bullet$ has no homology in $\mathcal{O}_0$ outside of the homological

position 0, and that the latter homology is an admissible module M such that $\Phi(M) \cong N$ (this isomorphism is obvious by construction).

To this end, we first note that each P_w, for $w \in X_G$, is admissible. Indeed, if w_0^G is the longest element of G, then $P_{w_0^G}$ is parabolically induced from its $\mathfrak{l}_G$-direct summand on which the nilradical of the parabolic subalgebra acts as 0. This direct summand is the projective module in the category $\mathcal{O}$ for $\mathfrak{l}_G$ corresponding to the anti-dominant element w_0^G, hence injective. Since parabolic induction, when restricted to $\mathfrak{l}_G$, is an infinite direct sum of tensorings with finite dimensional $\mathfrak{l}_G$-modules and such tensorings preserve both injectivity and projectivity, it follows that $P_{w_0^G}$ is admissible.

All other P_w, for $w \in X_G$, are summands of images of $P_{w_0^G}$ under projective functors on $\mathcal{O}_0$, see [4]. These functors are summands of tensorings with finite dimensional $\mathfrak{g}$-modules. It follows that all P_w, for $w \in X_G$, are admissible.

Next we note that the cokernel of an injection between two admissible modules is admissible (since both modules are direct sums of projective-injective objects when restricted to $\mathfrak{l}_G$ and thus the same property must be inherited by the cokernel).

By our assumptions, $\Phi(\mathcal{P}_\bullet)$ is exact at all non-zero homological positions. Therefore the only possible simples occurring in the homology of $\mathcal{P}_\bullet$ in non-zero positions are L_w, for $w \notin X_G$. For each such w, there exists a simple reflection $s \in G$ such that, for the corresponding simple root α, the action of $\mathfrak{g}_{-\alpha}$ on L_w is locally nilpotent. Now let us take the leftmost non-zero homology of $\mathcal{P}_\bullet$. By the previous paragraph, applied inductively, this homology must be a submodule of an admissible module. However, as we have already mentioned before, all $\mathfrak{g}_{-\alpha}$ as above act injectively on admissible modules, a contradiction.

Consequently, all homologies in non-zero homological position of $\mathcal{P}_\bullet$ are zero and the argument in the previous paragraph also proves that in this case the zero homology of $\mathcal{P}_\bullet$ is admissible. This completes the proof. □

4.2. The finitistic dimension

Theorem 15 reduces computation of projective dimension in $\mathcal{S}(W, \{e\}, G)$ to the same question for $\mathcal{O}_0$. For the latter category, projective dimensions of structural modules are determined in [26,28], see also [10]. We first record the following result which determines the finitistic dimension of $\mathcal{S}(W, \{e\}, G)$ explicitly, thus strengthening the main result of [29]. Let $\mathbf{a} : W \to \mathbb{Z}_{\geq 0}$ denote Lusztig's $\mathbf{a}$-function, see [25].

Corollary 16. *The finitistic dimension of* $\mathcal{S}(W, \{e\}, G)$ *equals* $2\mathbf{a}(w_0^G w_0)$.

Proof. By the main result of [29], the finitistic dimension of $\mathcal{S}(W, \{e\}, G)$ equals twice the projective dimension of the characteristic tilting module. By [18], the indecomposable tilting modules in $\mathcal{S}(W, \{e\}, G)$ are exactly $\Phi(T_w)$, where w is a shortest coset representative in $G \setminus W$.

Observe that all such T_w are admissible by essentially the same arguments as for projective modules: the module $T_{w_0^G w_0}$ is parabolically induced from a projective-injective $\mathfrak{l}_G$-module and all other T_w are obtained from $T_{w_0^G w_0}$ by applying projective functors.

By the main result of [28], the maximal projective dimension for such T_w is achieved for $w = w_0^G w_0$ and has the value $\mathbf{a}(w_0^G w_0)$. The claim now follows from Theorem 15. □

4.3. Various explicit projective dimensions

For $w \in X_G$, we denote by $\Delta_w^{\mathcal{S}}$ the corresponding standard object in $\mathcal{S}(W, \{e\}, G)$, by $\nabla_w^{\mathcal{S}}$ the corresponding costandard object, by $T_w^{\mathcal{S}}$ the corresponding indecomposable tilting (and cotilting) object and by $I_w^{\mathcal{S}}$ the corresponding indecomposable injective object. For the homological invariants of indecomposable structural modules, we have the following result.

Corollary 17. *For $w \in X_G$, we have*

$$\mathrm{proj.dim}(T_w^{\mathcal{S}}) = \mathbf{a}(w_0^G w) \quad \text{and} \quad \mathrm{proj.dim}(I_w^{\mathcal{S}}) = 2\mathbf{a}(w_0 w).$$

Proof. As noted above, $T_w^{\mathcal{S}} = \Phi(T_{w_0^G w})$. Similarly to the proof of Corollary 16, the equality $\mathrm{proj.dim}(T_w^{\mathcal{S}}) = \mathbf{a}(w_0^G w)$ follows directly from Theorem 15 and the main results of [26,28].

Any costandard object has a finite resolution by cotilting objects. Since all cotilting objects are tilting and hence are images under Φ of admissible objects, the same argument as in the proof of Theorem 15 implies that all costandard objects are images under Φ of admissible objects. As any injective object has a filtration by costandard objects, it follows that any injective object is an image under Φ of an admissible object. Hence all injectives in $\mathcal{S}(W, \{e\}, G)$ have finite projective dimension by Theorem 15. Further, since $\Delta_{w_0^G}^{\mathcal{S}} = P_{w_0^G}^{\mathcal{S}}$, applying $\star$ (cf. [30, Section 2]) we obtain that $\nabla_{w_0^G}^{\mathcal{S}} = I_{w_0^G}^{\mathcal{S}} = \Phi(I_{w_0^G})$, where $I_{w_0^G}$ is admissible. Similarly, $I_w^{\mathcal{S}} = \Phi(I_w)$, for every $w \in X_G$, and each such I_w is admissible (because it is a summand of the image of the admissible module $I_{w_0^G}$ under a projective functor). Hence the desired equality $\mathrm{proj.dim}(I_w^{\mathcal{S}}) = 2\mathbf{a}(w_0 w)$ follows directly from Theorem 15 and the main results of [26,28]. □

Recall from [10, Page 1084] the function $\mathbf{d}_\lambda$, where λ is an integral dominant weight, which assigns to $w \in W$ the projective dimension of the (potentially singular) Verma module $\Delta(w \cdot \lambda)$. This is one of the two functions which were used in [10] to describe homological invariants in singular blocks of parabolic category $\mathcal{O}$.

Proposition 18. *For $w \in X_G$, we have* $\mathrm{proj.dim}(\Delta_w^{\mathcal{S}}) = \mathbf{d}_\lambda(w^{-1})$*, where λ is such that its dot-stabilizer coincides with G.*

Proof. This result follows by collecting various statements in the literature. In the natural $\mathbb{Z}$-grading of category $\mathcal{O}$ introduced in [33], the module $\Delta_w^{\mathcal{S}}$ has a linear projective resolution by [27, Corollary 8.4(i)]. Combining the Ringel self-duality of $\mathcal{S}(W, \{e\}, G)$, see [18, Theorem 3], with [27, Theorem 3.3 and Corollary 8.7], we can identify the linear projective resolution of $\Delta_w^{\mathcal{S}}$ with the maximal quotient N_w of $\Delta_{w_0 w^{-1}}$ having as composition subquotients only those L_x for which x is a minimal coset representative in W/G. Hence the projective dimension of $\Delta_w^{\mathcal{S}}$ coincides with the Loewy (or graded) length of N_w. Soergel's self-equivalence for the full subcategory of $\mathcal{O}_0$ consisting of modules on which the action of the center of $U(\mathfrak{g})$ is scalar, see [32], matches N_w with a parabolic Verma module (a maximal quotient of Δ_{ww_0} having as composition subquotients only those L_x for which x is a minimal coset representative in $G \setminus W$). Therefore the claim of the proposition follows from [10, Table 2]. □

An interesting problem seems to be:

Problem 19. Determine the projective dimension of $\nabla_w^{\mathcal{S}}$, where $w \in X_G$.

The following example suggests that Problem 19 is non-trivial.

Example 20. Let $\mathfrak{g} = \mathfrak{sl}_3$ with $W = \{e, s, t, st, ts, w_0\}$ and $G = \{e, s\}$. Then $X_G = \{s, st, w_0\}$. We have $\Delta_{w_0}^{\mathcal{S}} = \Phi(T_{ts})$. The module T_{ts} admits a short exact sequence $\Delta_{ts} \hookrightarrow T_{ts} \twoheadrightarrow \Delta_{w_0}$. Note that $\mathrm{proj.dim}(\Delta_{w_0}) = 3$, $\mathrm{proj.dim}(\Delta_{ts}) = 2$ while $\mathrm{proj.dim}(T_{ts}) = \mathrm{proj.dim}(\Delta_{w_0}^{\mathcal{S}}) = 1$ (in particular, it is strictly smaller than the projective dimension of the individual components of the standard filtration). For costandard filtrations it is even worse. One can note that the module T_{ts} admits a short exact sequence $\nabla_{w_0} \hookrightarrow T_{ts} \twoheadrightarrow \nabla_{ts}$ and that $\mathrm{proj.dim}(\nabla_{w_0}) = 3$ and $\mathrm{proj.dim}(\nabla_{ts}) = 4$. At the same time, we have $\nabla_{w_0}^{\mathcal{S}} = \Phi(T_{ts})$.

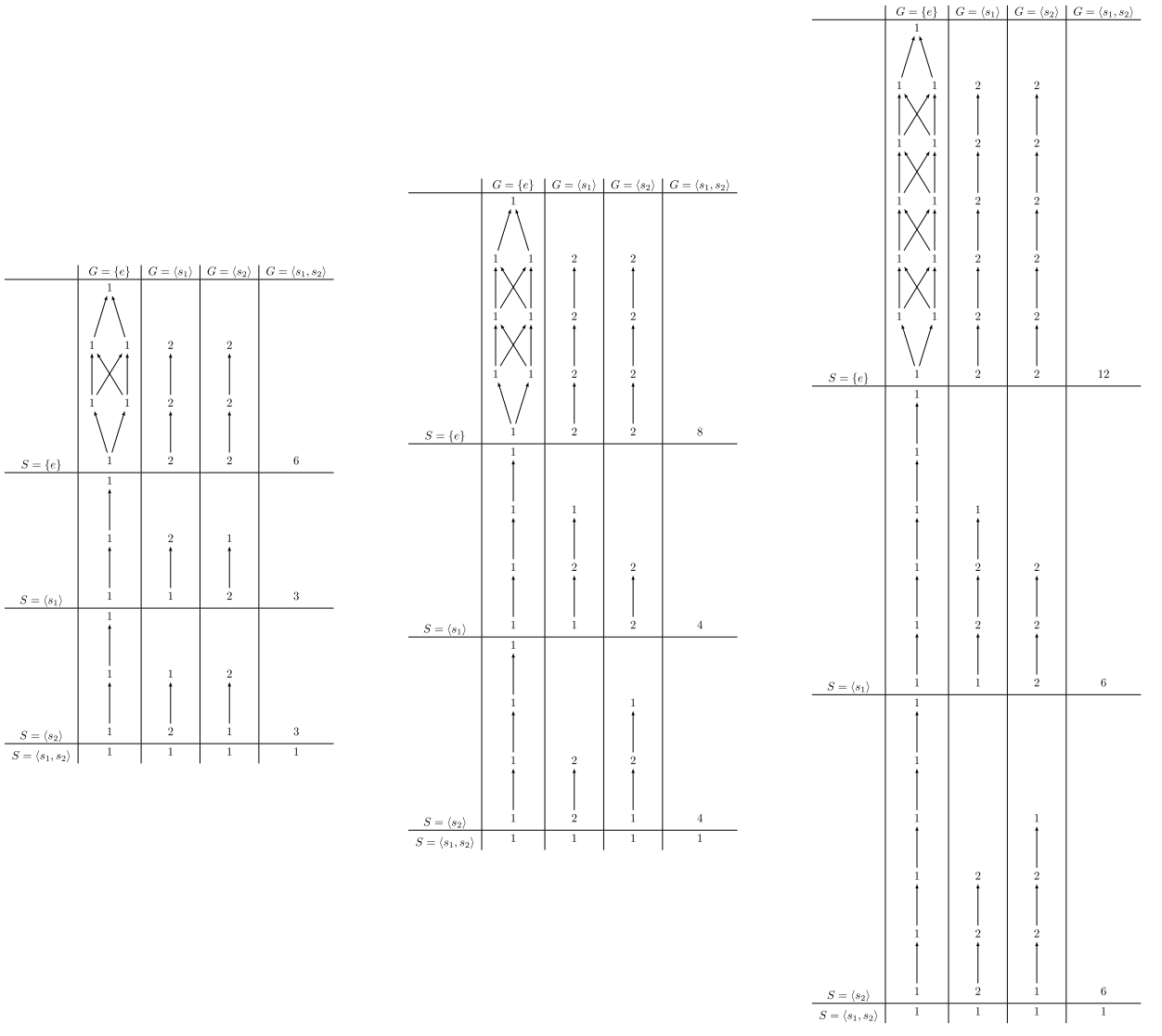

Fig. 1. Weighted posets $G\backslash W/S$ in types A_2, B_2 and G_2.

5. Weighted essential posets in types A_2, B_2, G_2, A_3 and B_3

In Figs. 1–3 one can find lists of the posets $G\backslash W/S$ (with respect to the Bruhat order on longest representatives) in types A_2, B_2, G_2, A_3 and B_3. The picture gives the Hasse diagram of the poset with each vertex represented by its weight as given by Lemma 9.

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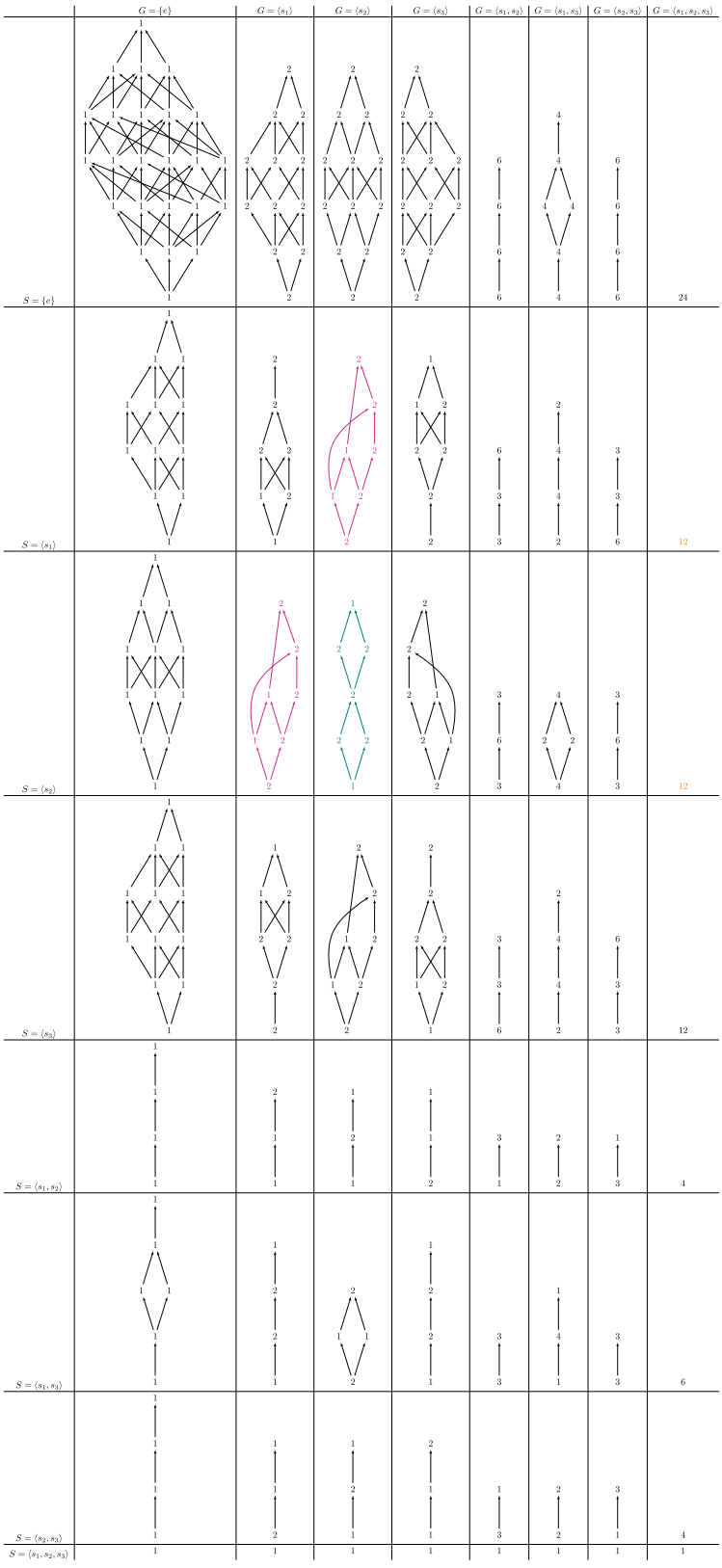

Fig. 2. Weighted posets $G\backslash W/S$ in type A_3. (For interpretation of the colors in the figures, the reader is referred to the web version of this article.)

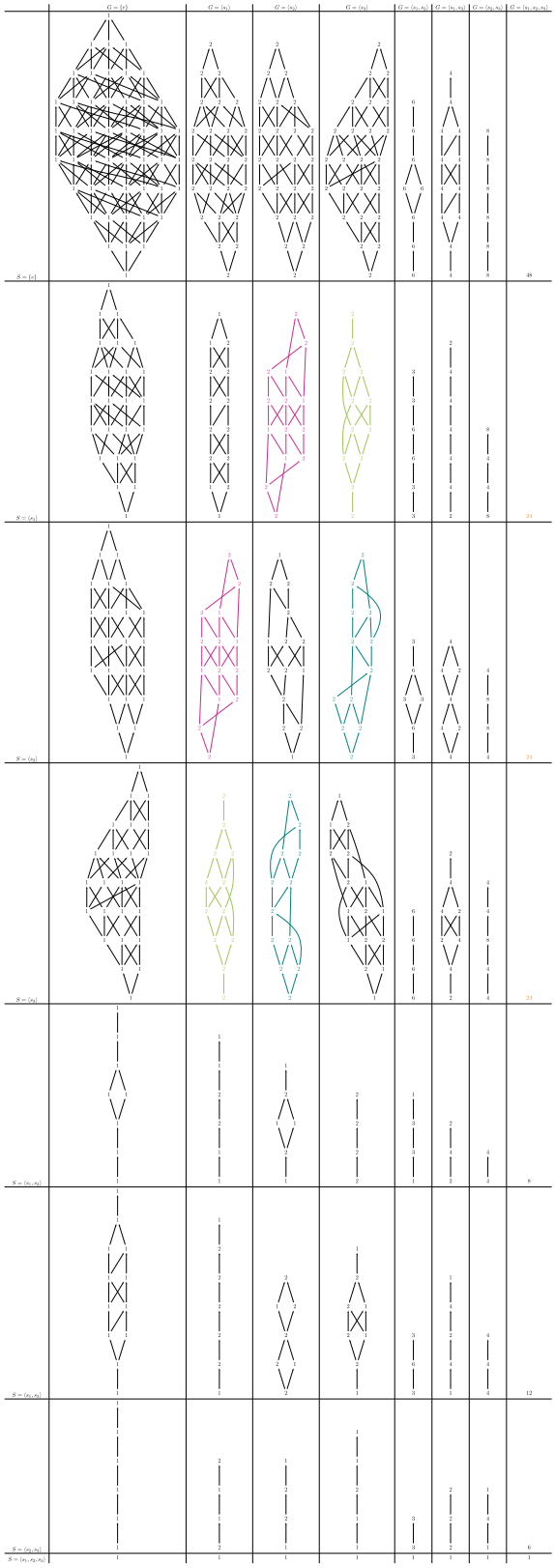

Fig. 3. Weighted posets $G\backslash W/S$ in type B_3. For interpretation of the colors in the figures, the reader is referred to the web version of this article.

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