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Twist-minimal trace formulas and the Selberg eigenvalue conjecture

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Abstract

We derive a fully explicit version of the Selberg trace formula for twist-minimal Maass forms of weight 0 and arbitrary conductor and nebentypus character, and apply it to prove two theorems. First, conditional on Artin's conjecture, we classify the even 2-dimensional Artin representations of small conductor; in particular, we show that the even icosahedral representation of smallest conductor is the one found by Doud and Moore (*J. Number Theory* 118 (2006) 62–70) of conductor 1951. Second, we verify the Selberg eigenvalue conjecture for groups of small level, improving on a result of Huxley (*Elementary and analytic theory of numbers*, Banach Center Publications 17 (PWN, Warsaw, 1985) 217–306) from 1985.

1. *Introduction*

In [6], the first and third authors derived a fully explicit version of the Selberg trace formula for cuspidal Maass newforms of squarefree conductor, and applied it to obtain partial results toward the Selberg eigenvalue conjecture and the classification of 2-dimensional Artin representations of small conductor. In this paper, we remove the restriction to squarefree conductor, with the following applications:

Theorem 1.1. *The Selberg eigenvalue conjecture is true for* $\Gamma_1(N)$ *for* $N \leqslant 880$, *and for* $\Gamma(N)$ *for* $N \leqslant 226$.

Theorem 1.2. *Assuming Artin's conjecture, Table* 1 *is the complete list, up to twist, of even, nondihedral, irreducible,* 2*-dimensional Artin representations of conductor* $\leqslant 2862$.

Remarks 1.3. • As we pointed out in [6], in the case of squarefree level the Selberg trace formula becomes substantially cleaner if one sieves down to *newforms*, and that also helps in numerical applications by thinning out the spectrum. For nonsquarefree level, this is no longer the case, as the newform sieve results in more complicated formulas in many cases.

Our main innovation in this paper is to introduce a further sieve down to *twist-minimal* forms, that is, those newforms whose conductor cannot be reduced by twisting. Although there are many technical complications to overcome in the intermediate stages, in the end we find that the twist-minimal trace formula is again significantly cleaner and helps to improve the numerics.

A natural question to explore in further investigations is whether one can skip the intermediate stages and derive the twist-minimal trace formula directly. A direct proof might

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shed light on why several complicated terms of the trace formula are annihilated by the twist-minimal sieve, and avoid messy calculations.

• Theorem 1.1 for $\Gamma(N)$ improves a 30-year-old result of Huxley [**16**], who proved the Selberg eigenvalue conjecture for groups of level $N \leqslant 18$. Treating nonsquarefree conductors is essential for this application, since a form of level N can have conductor[†] as large as N^2. Moreover, the reduction to twist-minimal spaces yields a substantial improvement in our numerical results by essentially halving the spectrum in the critical case of forms of prime level N and conductor N^2 (see (1.3)). This partially explains why our result for $\Gamma(N)$ is within a factor of 4 of that for $\Gamma_1(N)$, despite the conductors being much larger.

• By the Langlands–Tunnell theorem, the Artin conjecture is true for tetrahedral and octahedral representations, so the conclusion of Theorem 1.2 holds unconditionally for those types. In the icosahedral case, by [**3**] it is enough to assume the Artin conjecture for all representations in a given Galois conjugacy class; that is, if there is a twist-minimal, even icosahedral representation of conductor $\leqslant 2862$ that does not appear in Table 1, then Artin's conjecture is false for at least one of its Galois conjugates.

The entries of Table 1 were computed by Jones and Roberts [**20**] by a thorough search of number fields with prescribed ramification behavior, and we verified the completeness of the list via the trace formula. In principle, the number field search by Jones and Roberts [**19**] is exhaustive, so it should be possible to prove Theorem 1.2 unconditionally with a further computation, but that has not yet been carried out to our knowledge.

For comparison, we note that Buzzard and Lauder [**7**] have characterized the *odd* 2-dimensional representations of conductor $\leqslant 1500$ by computing bases of the associated spaces of weight 1 holomorphic modular forms.

• Theorem 1.1 for $\Gamma_1(N)$ improves on the result from [**6**] by extending to nonsquarefree N and increasing the upper bound from 854 to 880. To accomplish the latter, we computed a longer list of class numbers of the quadratic fields $\mathbb{Q}(\sqrt{t^2 \pm 4})$ using the algorithm from [**2**]. Nothing (other than limited patience of the user) prevents computing an even longer list and increasing the bounds in Theorem 1.1 a bit more. However, as explained in [**6**, § 6], our method suffers from an exponential barrier to increasing the conductor, so that by itself is likely to yield only marginal improvements. Some ideas for surmounting this barrier are described in [**6**, § 6]; in any case, as Theorem 1.2 shows, the first even icosahedral representation occurs at conductor $N = 1951$,[‡] so the bounds in Theorem 1.1 cannot be improved unconditionally beyond 1950.

TABLE 1. *Even, nondihedral Artin representations of conductor* $\leqslant 2862$ *up to twist. For each twist equivalence class we indicate the minimal Artin conductor and link to the* LMFDB *page of a representation in the class. It is twist minimal in all cases except those marked with* $*$.

Tetrahedral															
163	277	349	397	547	549	607	679	703	709	711	763	853	937	949	995
1009	1073	1143	1147	1197	1267	1267	1333	1343	1368	1399	1413	1699	1773	1777	1789
1879	1899	1899	1935	1951	1953	1957	1984	2051	2077	2097	2131	2135	2169	2169	2223
2311	2353	2439	2456	2587	2639	2689	2709	2743	2763	2797	2803	2817			
Octahedral															
785	1345	1940	2159*	2279	2313	2364	2424	2440	2713	2777	2777	2777	2857		
Icosahedral															
1951*	1951*	2141*	2141*	2804*	2804*										

[†]The level of a form f is the smallest N such that f is modular for $\Gamma(N)$. Its conductor is the conductor of the associated automorphic representation, or equivalently the smallest N such that f is modular for some conjugate of $\Gamma_1(N)$.

[‡]The existence of this representation was first shown by Doud and Moore [**10**], who also proved that 1951 is minimal among *prime* conductors of even icosahedral representations.

• All of our computations with real and complex numbers were carried out using the interval arithmetic package `Arb` [18]. Thus, modulo bugs in the software and computer hardware, our results are rigorous. The reader is invited to inspect our source code at [5].

We conclude the introduction with a brief outline of the paper. In §§ 1.1 and 1.2, we define the space of twist-minimal Maass forms and state our version of the Selberg trace formula for it. In § 2, we state and prove the full trace formula in general terms, and then specialize it to $\Gamma_0(N)$ with nebentypus character. In § 3, we apply the sieving process to pass from the full space to newforms, and then to twist-minimal forms. In § 4, we describe some details of the application of the trace formula to $\Gamma(N)$ and to Artin representations. Finally, in § 5, we make a few remarks on numerical aspects of the proofs of Theorems 1.1 and 1.2.

1.1. *Preliminaries on twist-minimal spaces of Maass forms*

Let $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$ denote the hyperbolic plane. Given a real number $\lambda > 0$ and an even Dirichlet character $\chi \pmod N$, let $\mathcal{A}_\lambda(\chi)$ denote the vector space of Maass cusp forms of eigenvalue λ, level N and nebentypus character χ, that is, the set of smooth functions $f : \mathbb{H} \to \mathbb{C}$ satisfying

(1) $f(\frac{az+b}{cz+d}) = \chi(d)f(z)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N)$;
(2) $\int_{\Gamma_0(N)\backslash\mathbb{H}} |f|^2 \frac{dx\,dy}{y^2} < \infty$;
(3) $-y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})f = \lambda f$.

We omit the level N from the notation $\mathcal{A}_\lambda(\chi)$ since it is determined implicitly as the modulus of χ (which might differ from its conductor, that is, χ need not be primitive). Any $f \in \mathcal{A}_\lambda(\chi)$ has a Fourier expansion of the form

$$f(x+iy) = \sqrt{y} \sum_{n \in \mathbb{Z}\setminus\{0\}} a_f(n) K_{\sqrt{\frac{1}{4}-\lambda}}(2\pi|n|y) e^{2\pi i n x},$$

for certain coefficients $a_f(n) \in \mathbb{C}$, where $K_s(y) = \frac{1}{2}\int_{\mathbb{R}} e^{st - y\cosh t}\,dt$ is the K-Bessel function.

For any $n \in \mathbb{Z} \setminus \{0\}$ coprime to N, let $T_n : \mathcal{A}_\lambda(\chi) \to \mathcal{A}_\lambda(\chi)$ denote the Hecke operator defined by

$$(T_n f)(z) = \frac{1}{\sqrt{|n|}} \sum_{\substack{a,d \in \mathbb{Z} \\ d>0, ad=n}} \chi(a) \sum_{b \pmod d} \begin{cases} f\left(\dfrac{az+b}{d}\right) & \text{if } n > 0, \\ f\left(\dfrac{a\bar{z}+b}{d}\right) & \text{if } n < 0. \end{cases}$$

We say that $f \in \mathcal{A}_\lambda(\chi)$ is a *normalized Hecke eigenform* if $a_f(1) = 1$ and f is a simultaneous eigenfunction of T_n for every n coprime to N. In this case, one has $T_n f = a_f(n) f$.

Let $\mathrm{cond}(\chi)$ denote the conductor of χ. For any $M \in \mathbb{Z}_{>0}$ with $\mathrm{cond}(\chi) \mid M$, let $\chi|_M$ denote the unique character mod M such that $\chi|_M(n) = \chi(n)$ for all n coprime to MN. Then for any M with $\mathrm{cond}(\chi) \mid M \mid N$ and any $d \mid \frac{N}{M}$, we have a linear map $\ell_{M,d} : \mathcal{A}_\lambda(\chi|_M) \to \mathcal{A}_\lambda(\chi)$ defined by $(\ell_{M,d} f)(z) = f(dz)$. Let

$$\mathcal{A}_\lambda^{\mathrm{old}}(\chi) = \sum_{\substack{M,d \in \mathbb{Z}_{>0} \\ \mathrm{cond}(\chi) \mid M \mid N,\ M < N \\ d \mid \frac{N}{M}}} \ell_{M,d} \mathcal{A}_\lambda(\chi|_M)$$

denote the span of the images of all lower level forms under these maps, and let $\mathcal{A}_\lambda^{\mathrm{new}}(\chi) \subseteq \mathcal{A}_\lambda(\chi)$ denote the orthogonal complement of $\mathcal{A}_\lambda^{\mathrm{old}}(\chi)$ with respect to the Petersson

inner product

$$\langle f, g\rangle = \int_{\Gamma_0(N)\backslash\mathbb{H}} f\bar{g}\frac{dx\,dy}{y^2}.$$

We call this the space of *newforms* of eigenvalue λ, conductor N and character χ.

By strong multiplicity one, we have $\mathcal{A}_\lambda^{\mathrm{new}}(\chi)\cap\mathcal{A}_\lambda^{\mathrm{new}}(\chi') = \{0\}$ unless χ and χ' have the same modulus and satisfy $\chi(n) = \chi'(n)$ for all n. In particular, any nonzero $f \in \mathcal{A}_\lambda^{\mathrm{new}}(\chi)$ uniquely determines its conductor, which we denote by $\mathrm{cond}(f)$. Moreover, the Hecke operators T_n map $\mathcal{A}_\lambda^{\mathrm{new}}(\chi)$ to itself, and $\mathcal{A}_\lambda^{\mathrm{new}}(\chi)$ has a unique basis consisting of normalized Hecke eigenforms.

Suppose that $f \in \mathcal{A}_\lambda^{\mathrm{new}}(\chi)$ is a normalized Hecke eigenform. Then for any Dirichlet character $\psi \pmod q$, there is a unique $M \in \mathbb{Z}_{>0}$ and a unique $g \in \mathcal{A}_\lambda^{\mathrm{new}}(\chi\psi^2|_M)$ such that $a_g(n) = a_f(n)\psi(n)$ for all $n \in \mathbb{Z}\setminus\{0\}$ coprime to q. We write $f\otimes\psi$ to denote the corresponding g. We say that f is *twist minimal* if $\mathrm{cond}(f\otimes\psi) \geqslant \mathrm{cond}(f)$ for every Dirichlet character ψ.

Let $\mathcal{A}_\lambda^{\mathrm{min}}(\chi) \subseteq \mathcal{A}_\lambda^{\mathrm{new}}(\chi)$ denote the subspace spanned by twist-minimal normalized Hecke eigenforms. Clearly any normalized Hecke eigenform can be expressed as $f\otimes\psi$ for some twist-minimal form f. In turn, for any normalized Hecke eigenform $f \in \mathcal{A}_\lambda^{\mathrm{min}}(\chi)$, we have

$$\mathrm{cond}(f\otimes\psi) = \mathrm{lcm}(\mathrm{cond}(f), \mathrm{cond}(\psi)\,\mathrm{cond}(\chi\psi)),$$

as implied by the following lemma, strengthening [**14**, Lemma 2.1] and [**9**, Lemma 2.7].

LEMMA 1.4. *Let F be a non-Archimedean local field, π an irreducible, admissible, generic representation of $\mathrm{GL}_2(F)$ with central character χ and ψ a character of $F^\times$. Let $a(\pi)$ and $a(\psi)$ denote the respective conductor exponents of π and ψ (written additively). Then we have*

$$a(\pi\otimes\psi) \leqslant \max\{a(\pi), a(\psi)+a(\chi\psi)\}, \tag{1.1}$$

with equality if π is twist minimal or $a(\pi) \neq a(\psi)+a(\chi\psi)$.

Proof. There are three cases to consider:

(1) $\pi \cong \chi_1 \boxplus \chi_2$ is a principal series representation, in which case

$$a(\pi\otimes\psi) = a(\chi_1\psi)+a(\chi_2\psi), \quad a(\pi) = a(\chi_1)+a(\chi_2), \quad \text{and} \quad \chi = \chi_1\chi_2.$$

(2) $\pi \cong \mathrm{St}\otimes\chi_1$ is a twist of the Steinberg representation, in which case

$$a(\pi\otimes\psi) = \max\{1, 2a(\chi_1\psi)\}, \quad a(\pi) = \max\{1, 2a(\chi_1)\}, \quad \text{and} \quad \chi = \chi_1^2.$$

(3) π is supercuspidal.

In the first two cases, we verify (1.1) by a laborious case-by-case analysis based on the observation that, for any characters ξ_1 and ξ_2, $a(\xi_1\xi_2) \leqslant \max\{a(\xi_1), a(\xi_2)\}$, with equality if $a(\xi_1) \neq a(\xi_2)$. This also shows that equality holds in (1.1) when $a(\pi) \neq a(\psi)+a(\chi\psi)$.

In the third case, Tunnell [**26**, Proposition 3.4] showed that $a(\pi\otimes\psi) \leqslant \max\{a(\pi), 2a(\psi)\}$, with equality if $a(\psi) \neq \frac{1}{2}a(\pi)$, and that $a(\chi) \leqslant \frac{1}{2}a(\pi)$. It follows that $\max\{a(\pi), 2a(\psi)\} = \max\{a(\pi), a(\psi)+a(\chi\psi)\}$, so Tunnell's proposition implies (1.1).

Let us prove that equality holds in (1.1) if $a(\pi) \neq a(\psi)+a(\chi\psi)$. By what we have already pointed out, we may assume $a(\psi) = \frac{1}{2}a(\pi)$, and then $a(\pi) \neq a(\psi)+a(\chi\psi)$ implies $a(\chi\psi) < \frac{1}{2}a(\pi)$. Applying Tunnell's proposition to the representation $\pi^\vee \cong \pi\otimes\chi^{-1}$ (which has $a(\pi^\vee) = a(\pi)$) and the character $\chi\psi$, we then obtain $a(\pi\otimes\psi) = a(\pi^\vee\otimes\chi\psi) = a(\pi)$, that is, equality holds in (1.1).

Finally, if π is twist minimal, then $a(\pi\otimes\psi) \geqslant a(\pi)$, so equality holds in (1.1) even if $a(\pi) = a(\psi)+a(\chi\psi)$. □

Thus, $f \mapsto f \otimes \psi$ extends to an injective linear map from $\mathcal{A}_\lambda^{\min}(\chi)$ to $\mathcal{A}_\lambda^{\mathrm{new}}(\chi\psi^2|_M)$, where $M = \operatorname{lcm}(N, \operatorname{cond}(\psi)\operatorname{cond}(\chi\psi))$.

In light of this, it is enough to consider the trace formula for twist-minimal spaces of forms. In fact, since some twist-minimal spaces are trivial, and for others a given form can have more than one representation as the twist of a twist-minimal form (that is, it is possible to have $\operatorname{cond}(f \otimes \psi) = \operatorname{cond}(f)$ and $f \otimes \psi \neq f$, cf. [14]), a further reduction of the nebentypus character is possible, as follows.

DEFINITION 1.5. Let $\chi = \prod_{p|N} \chi_p$ be a Dirichlet character modulo N, and put $e_p = \operatorname{ord}_p N$, $s_p = \operatorname{ord}_p \operatorname{cond}(\chi)$. We say that χ is *minimal* if the following statement holds for every prime $p \mid N$:

$$p > 2 \text{ and } \left[s_p \in \{0, e_p\} \text{ or } \chi_p \text{ has order } 2^{\operatorname{ord}_2(p-1)} \right]$$

$$\text{or } p = 2 \text{ and } s_p \in \{\lfloor e_p/2 \rfloor, e_p\} \cup \begin{cases} \{0\} & \text{if } e_p \leqslant 3, \\ \{0, 2\} & \text{if } e_p > 3 \text{ and } 2 \nmid e_p, \\ \emptyset & \text{if } e_p > 3 \text{ and } 2 \mid e_p. \end{cases}$$

Note that for an odd prime p, there are $2^{\operatorname{ord}_2(p-1)-1}$ choices of $\chi_p \pmod{p^{e_p}}$ of order $2^{\operatorname{ord}_2(p-1)}$, and for any such χ_p we have $s_p = 1$ and $\chi_p(-1) = -1$; in particular, when $p \equiv 3 \pmod 4$, the Legendre symbol $(\frac{\cdot}{p})$ is the unique such character. (For $p \equiv 1 \pmod 4$ and $e_p > 1$, the spaces resulting from different choices of χ_p of order $2^{\operatorname{ord}_2(p-1)}$ are twist equivalent, but there is no canonical choice. Similarly, for $p = 2$ the characters of conductor $2^{\lfloor e_2/2 \rfloor}$ and fixed parity yield twist-equivalent spaces.)

LEMMA 1.6. *Let $\chi \pmod N$ be a Dirichlet character, and suppose $\mathcal{A}_\lambda^{\min}(\chi) \neq \{0\}$. Then there exists $\psi \pmod N$ such that $\chi\psi^2$ is minimal and $\mathcal{A}_\lambda^{\min}(\chi) \xrightarrow{\otimes\psi} \mathcal{A}_\lambda^{\min}(\chi\psi^2)$ is an isomorphism.*

Proof. Since the map $f \mapsto f \otimes \psi$ is injective, by Lemma 1.4 it suffices to show that there is a $\psi \pmod N$ such that $\chi\psi^2$ is minimal and $\operatorname{cond}(\psi)\operatorname{cond}(\chi\psi) \mid N$. Writing $\psi = \prod_{p|N} \psi_p$, this is equivalent to finding $\psi_p \pmod{p^{\operatorname{ord}_p N}}$ such that $\chi_p\psi_p^2$ is minimal and

$$\operatorname{ord}_p \operatorname{cond}(\psi_p) + \operatorname{ord}_p \operatorname{cond}(\chi_p\psi_p) \leqslant \operatorname{ord}_p N \tag{1.2}$$

for each prime $p \mid N$.

Fix $p \mid N$ and set $e = \operatorname{ord}_p N$, $s = \operatorname{ord}_p \operatorname{cond}(\chi_p)$. If $e \in \{s, 1\}$, then χ_p is minimal, so we can take ψ_p equal to the trivial character mod p^e. Hence, we may assume that $e > \max\{s, 1\}$. In this case, since $\mathcal{A}_\lambda^{\min}(\chi)$ is nonzero, it follows from [1, Theorem 4.3'] that $s \leqslant \frac{1}{2}e$.

Suppose that p is odd, and let g be a primitive root mod p^e. Then $\chi_p(g) = \exp(2\pi i a/\varphi(p^e))$ for a unique $a \in \mathbb{Z} \cap [1, \varphi(p^e)]$. Set

$$b = \begin{cases} -a/2 & \text{if } 2 \mid a, \\ \left[\varphi(p^e)2^{-\operatorname{ord}_2(p-1)} - a\right]/2 & \text{if } 2 \nmid a, \end{cases}$$

and let ψ_p be the character defined by $\psi_p(g) = \exp(2\pi i b/\varphi(p^e))$. Then $\chi_p\psi_p^2$ is minimal, and since $p^{\operatorname{ord}_p a} \mid b$ and $e > 1$, it follows that $\operatorname{ord}_p \operatorname{cond}(\psi_p) \leqslant \frac{1}{2}e$, which implies the desired inequality (1.2).

Suppose now that $p = 2$. Since $s \leqslant \frac{1}{2}e$ and there is no character of conductor 2, χ_2 is already minimal if $e \leqslant 3$. Also, by the analogue of [1, Theorem 4.4(iii)] for Maass forms, if $e \geqslant 4$ is

even, then we must have $s = e/2$, so χ_2 is again minimal. Hence, we may assume that e is an odd number exceeding 3.

If $s \in \{0, 2, \lfloor e/2 \rfloor\}$, then χ_2 is minimal, so we may assume that $3 \leqslant s \leqslant (e-3)/2$. Recalling that $(\mathbb{Z}/2^e\mathbb{Z})^\times$ is generated by -1 and 5, we have $\chi_2(5) = \exp(2\pi i a/2^{s-2})$ for a unique odd number $a \in [1, 2^{s-2})$. Let ψ_2 be the character defined by $\psi_2(-1) = 1$ and $\psi_2(5) = \exp(-2\pi i a/2^{s-1})$. Then $\chi_2(5)\psi_2(5)^2 = 1$, so $\chi_2\psi_2^2$ has conductor at most 4, and is therefore minimal. Moreover, $\operatorname{cond}(\psi_2) = \operatorname{cond}(\chi_2\psi_2) = 2^{s+1}$, so that

$$\operatorname{ord}_2 \operatorname{cond}(\psi_2) + \operatorname{ord}_2 \operatorname{cond}(\chi_2\psi_2) = 2s + 2 < e,$$

as desired. □

Thus, we may restrict our attention to the spaces $\mathcal{A}_\lambda^{\min}(\chi)$ for minimal characters χ.

1.2. *The twist-minimal trace formula*

Let $g : \mathbb{R} \to \mathbb{C}$ be even, continuous and absolutely integrable, with Fourier transform $h(r) = \int_{\mathbb{R}} g(u)e^{iru}\,du$. We say that the pair (g, h) is of *trace class* if there exists $\delta > 0$ such that h is analytic on the strip $\Omega = \{r \in \mathbb{C} : |\Im(r)| < \frac{1}{2} + \delta\}$ and satisfies $h(r) \ll (1 + |r|)^{-2-\delta}$ for all $r \in \Omega$.

Let $\chi \pmod N$ be a Dirichlet character, and write $\chi = \prod_{p|N} \chi_p$, where each χ_p is a character modulo $p^{\operatorname{ord}_p N}$. The trace formula is an expression for $\sum_{\lambda>0} \operatorname{tr} T_n|_{\mathcal{A}_\lambda^{\min}(\chi)} h(\sqrt{\lambda - \frac{1}{4}})$ in terms of g and χ. Its terms are linear functionals of g with coefficients that are multiplicative functions of χ, that is, functions F satisfying $F(\chi) = \prod_{p|N} F(\chi_p)$.

In what follows we fix a prime p and a character $\chi \pmod{p^e}$ of conductor p^s, and define the local factor at p for various terms appearing in the trace formula. As a notational convenience, for any proposition P we write δ_P to denote the characteristic function of P, that is, $\delta_P = 1$ if P is true and $\delta_P = 0$ if P is false.

Identity term. Define

$$M(\chi) = \begin{cases} p^{e-1}(p+1) & \text{if } s = e, \\ \dfrac{p - \chi(-1)}{\gcd(2, p-1, e)}\varphi(p^{e-1}) & \text{if } s < e \leqslant 2, \\ \dfrac{p^2-1}{\gcd(2, p-1, e)}\varphi(p^{e-2}) & \text{if } e > \max\{s, 2\}. \end{cases} \tag{1.3}$$

Constant eigenfunction. Define

$$\mu(\chi) = \begin{cases} -1 & \text{if } s < e = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Parabolic terms. For $m \geqslant 1$ and $n \in \{\pm 1\}$, define

$$\Phi_{m,n}(\chi) = \begin{cases} \overline{\chi(m)} + \chi(n)\chi(m) & \text{if } s = e, \\ -1 & \text{if } s < e = 1 \text{ and } m = p^k \text{ for some } k > 0, \\ 0 & \text{otherwise} \end{cases} \tag{1.4}$$

and

$$\Psi_n(\chi) = \begin{cases} 1+\chi(n) & \text{if } s=e \text{ or } n=e=1, \\ \dfrac{p-\chi(-1)}{\gcd(2,p-1,e)} & \text{if } n=1, s<e=2, \\ \dfrac{p^{\lfloor \frac{e-3}{2} \rfloor}+p^{\lceil \frac{e-3}{2} \rceil}}{\gcd(2,p-1,e)}(p-1) & \text{if } n=1, e>\max\{s,2\}, \\ \dfrac{3}{2} & \text{if } n=-1, p=2, s<e=1, \\ \dfrac{1}{2} & \text{if } n=-1, p=2, s<e\in\{2,3\}, \\ 1 & \text{if } n=-1, p>2, s<e=1, \\ 0 & \text{otherwise.} \end{cases}$$

Elliptic and hyperbolic terms. Following the notation in [4, § 1.1], let $\mathcal{D}$ denote the set of discriminants, that is

$$\mathcal{D} = \{D \in \mathbb{Z} : D \equiv 0 \text{ or } 1 \pmod 4\}.$$

Any nonzero $D \in \mathcal{D}$ may be expressed uniquely in the form $d\ell^2$, where d is a fundamental discriminant and $\ell > 0$. We define $\psi_D(n) = (\frac{d}{n/\gcd(n,\ell)})$, where $(-)$ denotes the Kronecker symbol. Note that ψ_D is periodic modulo D, and if D is fundamental, then ψ_D is the usual quadratic character mod D. Set

$$L(z,\psi_D) = \sum_{n=1}^{\infty} \frac{\psi_D(n)}{n^z} \quad \text{for } \Re(z) > 1.$$

Then it is not hard to see that

$$L(z,\psi_D) = L(z,\psi_d) \prod_{p\mid\ell} \left[1 + (1-\psi_d(p)) \sum_{j=1}^{\operatorname{ord}_p \ell} p^{-jz}\right],$$

so that $L(z,\psi_D)$ has analytic continuation to $\mathbb{C}$, apart from a simple pole at $z=1$ when D is a square. In particular, if D is not a square, then we have

$$L(1,\psi_D) = L(1,\psi_d) \cdot \frac{1}{\ell} \prod_{p\mid\ell} \left[1 + (p-\psi_d(p)) \frac{p^{\operatorname{ord}_p \ell}-1}{p-1}\right].$$

Let $t \in \mathbb{Z}$ and $n \in \{\pm 1\}$ with $D = t^2 - 4n$ not a square. Then $D \in \mathcal{D}$, so we may write $D = d\ell^2$ as above. Define $r = \operatorname{ord}_p(D/2) + 1$,

$$\alpha = \frac{\psi_d(p)-1}{1+(p-\psi_d(p))\frac{p^{\operatorname{ord}_p \ell}-1}{p-1}} \quad \text{and} \quad \omega = \begin{cases} 1 & \text{if } 2 \nmid t \text{ or } p=2 \text{ and } r=2s, \\ 0 & \text{otherwise.} \end{cases}$$

For $s = e$, we set

$$H_{t,n}(\chi) = \begin{cases} \chi\left(\dfrac{t+p^s\omega}{2}\right)\left(2+\alpha\frac{p^e+p^{e-1}-2}{p-1}\right) & \text{if } r \geqslant 2e, \\ \chi\left(\dfrac{t+\sqrt{d}\ell}{2}\right) + \chi\left(\frac{t-\sqrt{d}\ell}{2}\right) & \text{if } r < 2e \text{ and } \psi_d(p) = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{1.5}$$

where $\sqrt{d}$ denotes a square root of d modulo $4p^e$. For $s < e$, we set

$$H_{t,n}(\chi) = \delta_{e=1 \text{ or } (\frac{n}{p})=1} \cdot \delta_{r \geqslant e-1} \cdot \frac{\alpha}{\gcd(2,e)} \chi\left(\frac{t+p^s\omega}{2}\right) p^{e-3} \cdot \left[p\big(p - \delta_{e=2}\chi(-1) - \delta_{r=e-1}(p+\delta_{2|e})\big) - \delta_{e>2}\right] \tag{1.6}$$

when $p > 2$, and

$$H_{t,n}(\chi) = \alpha\chi\left(\frac{t+2^{r/2}\omega}{2}\right) 2^{e-3} \begin{cases} 3 & \text{if } e \geqslant 3, r > e, \\ 1+2(-1)^e & \text{if } e \geqslant 3, r = e, \\ 1-2(-1)^d & \text{if } e \geqslant 3, r = e-1, 2 \nmid e, \\ 2 & \text{if } e = 2, r \geqslant e, \\ -1 & \text{if } e = 2, r < e, \\ 4 & \text{if } e = 1, \\ 0 & \text{otherwise} \end{cases} \tag{1.7}$$

when $p = 2$.

With the notation in place, we can now state the trace formula for T_n acting on $\mathcal{A}_\lambda^{\min}(\chi)$.

THEOREM 1.7. *Let $\chi \pmod N$ be a minimal character of conductor q, $n \in \{\pm 1\}$, and (g,h) a pair of test functions of trace class. Then*

$$\begin{aligned} &\sum_{\lambda>0} \operatorname{tr} T_n|_{\mathcal{A}_\lambda^{\min}(\chi)} h\left(\sqrt{\lambda - \tfrac{1}{4}}\right) \\ &= -\delta_{n=1}\left[\frac{M(\chi)}{12}\int_{\mathbb{R}} \frac{g'(u)}{\sinh(u/2)}\,du + \mu(\chi)\int_{\mathbb{R}} g(u)\cosh(u/2)\,du\right] \\ &\quad + \sum_{\substack{t\in\mathbb{Z} \\ D=t^2-4n \\ \sqrt{D}\notin\mathbb{Q}}} H_{t,n}(\chi) L(1,\psi_D) \begin{cases} g\left(2\log\dfrac{|t|+\sqrt{D}}{2}\right) & \text{if } D>0, \\ \dfrac{\sqrt{|D|}}{\pi}\displaystyle\int_{\mathbb{R}} \frac{g(u)\cosh(u/2)}{4\sinh^2(u/2)+|D|}\,du & \text{if } D<0 \end{cases} \\ &\quad + \Phi_{1,n}(\chi)\left\{\left(\gamma + \log\frac{2\pi}{N} + \frac{1}{2}\log\frac{2}{(2,N)}\right)g(0) - \int_0^\infty \log\left(\sinh\frac{u}{2}\right)g'(u)\,du\right. \\ &\quad \left. + \delta_{n=1}\left[\left(\gamma + \log\frac{2}{N} + \frac{1}{2}\log(2,N) + \frac{1}{2}\sum_{p|N}\log p\right)g(0) - \int_0^\infty \log\left(\tanh\frac{u}{4}\right)g'(u)\,du\right]\right\} \\ &\quad + 2\sum_{m=2}^\infty \frac{\Lambda(m)}{m}\Phi_{m,n}(\chi)g(2\log m) - \Lambda(N/q)\Psi_n(\chi)g(0) - \delta_{N=1}\frac{1}{4}\int_{\mathbb{R}} g(u)\,du. \end{aligned}$$

2. *Statement and proof of the full Selberg trace formula*

2.1. *Selberg trace formula in general form from* [6]

The group of all isometries (orientation preserving or not) of $\mathbb{H}$ can be identified with $\mathrm{G} = \mathrm{PGL}_2(\mathbb{R})$, where the action is defined by

$$T(z) = \begin{cases} \dfrac{az+b}{cz+d} & \text{if } ad-bc>0, \\ \dfrac{a\bar{z}+b}{c\bar{z}+d} & \text{if } ad-bc<0, \end{cases} \qquad \text{for } T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{G}\,.$$

The group of orientation preserving isometries, $\mathrm{G}^+ = \mathrm{PSL}_2(\mathbb{R})$, is a subgroup of index 2 in G. We write $\mathrm{G}^- = \mathrm{G} \setminus \mathrm{G}^+$ for the other coset in G.

Let Γ be a discrete subgroup of G such that the surface $\Gamma\backslash\mathbb{H}$ is noncompact but of finite area, and let χ be a (unitary) character on Γ. We set $\Gamma^+ := \Gamma \cap \mathrm{G}^+$ and assume $\Gamma^+ \neq \Gamma$. We let $L^2(\Gamma\backslash\mathbb{H}, \chi)$ be the Hilbert space of functions $f : \mathbb{H} \to \mathbb{C}$ satisfying the automorphy relation

$$f(\gamma z) = \chi(\gamma) f(z), \quad \forall \gamma \in \Gamma,$$

and

$$\int_{\Gamma\backslash\mathbb{H}} |f|^2 \, d\mu < \infty.$$

We let $\{\phi_j\}_{j\geqslant 1}$ be any orthonormal basis of the discrete spectrum of the Laplace operator $\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ on $L^2(\Gamma\backslash\mathbb{H}, \chi)$, that is, $\phi_j \in C^\infty(\mathbb{H}) \cap L^2(\Gamma\backslash\mathbb{H}, \chi)$ and $\Delta\phi_j = \lambda_j \phi_j$, say with increasing eigenvalues $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots$. We also let

$$r_j = \sqrt{\lambda_j - \tfrac{1}{4}} \in \mathbb{R}^+ \cup i\big[-\tfrac{1}{2}, 0\big].$$

The trace formula from [6, Theorem 2 and (2.37)] reads as follows. The even analytic function h and its Fourier transform g are given as in Section 1.2. The trace formula for (Γ, χ) is

$$\sum_{j\geqslant 1} h(r_j) = \mathrm{I}(\Gamma,\chi) + \mathrm{NEll}(\Gamma,\chi) + \mathrm{Ell}(\Gamma,\chi) + \mathrm{C}(\Gamma,\chi) + \mathrm{Eis}(\Gamma,\chi),$$

where

$$\mathrm{I}(\Gamma,\chi) = \frac{\mathrm{Area}(\Gamma\backslash\mathbb{H})}{4\pi} \int_{\mathbb{R}} h(r) r \tanh(\pi r)\, dr, \tag{2.1}$$

$$\mathrm{NEll}(\Gamma,\chi) = \sum_{\{T\}\subset M,\ \text{non-ell}} \frac{\log N(T_0)}{[Z_\Gamma(T) : [T_0]]} \frac{\chi(T)}{N(T)^{\frac{1}{2}} - \operatorname{sgn}(\det T) N(T)^{-\frac{1}{2}}} g(\log N(T)), \tag{2.2}$$

$$\mathrm{Ell}(\Gamma,\chi) = \sum_{\{T\}\subset M,\ \text{elliptic}} \frac{\chi(T)}{2|Z_\Gamma(T)|\sin(\theta(T))} \int_{\mathbb{R}} \frac{e^{-2\theta(T)r}}{1+e^{-2\pi r}} h(r)\, dr, \tag{2.3}$$

$$\begin{aligned} \mathrm{C}(\Gamma,\chi) = {} & \left(\sum_{j\in C_{\Gamma,\chi}, k(j)=j} \chi(T_{j,0}) \right) \left\{ \frac{1}{8} h(0) + \frac{1}{4\pi} \int_{\mathbb{R}} h(r) \left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right) - \frac{\Gamma'}{\Gamma}(1+ir) \right) dr \right\} \\ & + \frac{g(0)}{4} \left(\sum_{1\leqslant j\leqslant \kappa, k(j)=j} \sum_{v\in\{0,1\}} \chi(T_{j,v}) \log \mathbf{c}_{j,v} - 2 \sum_{1\leqslant j\leqslant\kappa, j\notin C_{\Gamma,\chi}} \log|1-\chi(T_j)| \right) \\ & + \frac{|C_{\Gamma,\chi}|}{2} \left\{ \frac{1}{4} h(0) - g(0)\log 2 - \frac{1}{2\pi} \int_{\mathbb{R}} h(r) \frac{\Gamma'}{\Gamma}(1+ir)\, dr \right\} \end{aligned} \tag{2.4}$$

and

$$\mathrm{Eis}(\Gamma,\chi) = \frac{1}{4\pi} \int_{\mathbb{R}} h(r) \frac{(\varphi^\Gamma)'}{\varphi^\Gamma}\left(\frac{1}{2}+ir\right) dr - \frac{1}{4} h(0) \operatorname{tr} \Phi^\Gamma\left(\frac{1}{2}\right). \tag{2.5}$$

Note that all the sums and integrals are absolutely convergent (see [**6**, Proposition 2.2]). In the remainder of this section, we explain the notation appearing in (2.1), (2.2), (2.3), (2.4) and (2.5).

The set M in $\mathrm{NEll}(\Gamma,\chi)$ and $\mathrm{Ell}(\Gamma,\chi)$ is given as

$$M=\{T\in\Gamma\ :\ T \text{ has no cusp of } \Gamma^+ \text{ as a fixpoint}\},$$

that is, M is the set of all $T\in\Gamma$ that do not fix any cusp of Γ^+ and '$\{T\}\subset M$' denotes that we add over a set of representatives for the Γ-conjugacy classes in M. We write $Z_\Gamma(T)$ for the centralizer of T in Γ. The sum in $\mathrm{NEll}(\Gamma,\chi)$ in (2.2) is over all non-elliptic conjugacy classes in M. Thus, T is hyperbolic, reflection or a glide reflection, and let $\alpha\in(-\infty,-1]\cup(1,\infty)$ be the unique number such that T is conjugate within G to $\left(\begin{smallmatrix}\alpha & 0\\ 0 & 1\end{smallmatrix}\right)$. Then we write

$$N(T)=|\alpha|.$$

We denote by T_0 some hyperbolic element or a glide reflection in $Z_\Gamma(T)$. Then the infinite cyclic group

$$[T_0]=\{T_0^n\ :\ n\in\mathbb{Z}\}$$

has finite index $[Z_\Gamma(T):[T_0]]$ in $Z_\Gamma(T)$ and the ratio

$$\frac{\log N(T_0)}{[Z_\Gamma(T):[T_0]]}$$

depends only on T and not on our choice of T_0. The sum in $\mathrm{Ell}(\Gamma,\chi)$ in (2.3) is over all elliptic conjugate classes in M, and we write $\theta(T)$ for the unique number $\theta\in(0,\frac{\pi}{2}]$ such that T is conjugate within G to $\left(\begin{smallmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{smallmatrix}\right)$.

Now we explain the notations appearing in $\mathrm{C}(\Gamma,\chi)$ in (2.4). Let

$$\eta_1,\dots,\eta_\kappa\in\partial\mathbb{H}=\mathbb{R}\cup\{\infty\}$$

be a set of representatives of the cusps of $\Gamma^+\backslash\mathbb{H}$, one from each Γ^+-equivalence class. For each $j\in\{1,\dots,\kappa\}$, we choose $\mathsf{N}_j\in\mathrm{G}^+$ such that $\mathsf{N}_j(\eta_j)=\infty$ and the stabilizer $\Gamma^+_{\eta_j}$ is $[T_j]$ where

$$T_j=\mathsf{N}_j^{-1}\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\mathsf{N}_j.$$

We write $C_{\Gamma,\chi}$ for the set of open cusp representatives, viz.

$$C_{\Gamma,\chi}=\{j\in\{1,\dots,\kappa\}\ :\ \chi(T_j)=1\}.$$

We fix, once and for all, an element $V\in\Gamma-\Gamma^+$. For each $j\in\{1,\dots,\kappa\}$, there exists $k(j)\in\{1,\dots,\kappa\}$ and $U_j\in\Gamma^+$ such that

$$V\eta_j=U_j\eta_{k(j)}.$$

Note that $k(j)$ is uniquely determined by the condition that $\eta_{k(j)}$ is the representative of the cusp $V\eta_j$ and then U_j is determined up to the right shifts with $T_{k(j)}$. By [**6**, p. 116 and (2.5)], U_j satisfies

$$U_j=V\mathsf{N}_j^{-1}\begin{pmatrix}-1 & x_j\\ 0 & 1\end{pmatrix}\mathsf{N}_{k(j)}\in\Gamma^+,\ \text{for some } x_j\in\mathbb{R}. \tag{2.6}$$

We note in particular that $k(k(j))=j$ and that $k(j)\in C_{\Gamma,\chi}$ if and only if $j\in C_{\Gamma,\chi}$. For each $j\in\{1,\dots,\kappa\}$ with $k(j)=j$, and each $v\in\mathbb{Z}$, we set

$$T_{j,v}=\mathsf{N}_j^{-1}\begin{pmatrix}-1 & x_j+v\\ 0 & 1\end{pmatrix}\mathsf{N}_j=U_j^{-1}VT_j^v\in\Gamma. \tag{2.7}$$

The last identity follows from (2.6). Then $T_{j,v}$ is a reflection fixing the point η_j. Also the other fixpoint of $T_{j,v}$ in $\partial\mathbb{H}$ must be a cusp; we write it as $V_2\eta_k$ for some $V_2 \in \Gamma^+$ and $k \in \{1,\dots,\kappa\}$, and then define the number $\mathbf{c}_{j,v} > 0$ by the relation (cf. [**6**, (2.26)])

$$\mathsf{N}_j V_2 \mathsf{N}_k^{-1} = \begin{pmatrix} * & * \\ \mathbf{c}_{j,v} & * \end{pmatrix}, \text{ where } \det(\mathsf{N}_j V_2 \mathsf{N}_k^{-1}) = 1. \tag{2.8}$$

It now remains to explain the notations φ^Γ and Φ^Γ appearing in $\mathrm{Eis}(\Gamma,\chi)$ in (2.5). For $j \in C_{\Gamma,\chi}$, let $E_j(z,s,\chi)$ be the Eisenstein series for $\langle\Gamma^+,\chi\rangle$ associated to the cusp η_j:

$$E_j(z,s,\chi) = \sum_{W\in[T_j]\backslash\Gamma^+} \chi(W^{-1})(\Im(\mathsf{N}_j W z))^s, \text{ for } z\in\mathbb{H} \text{ and } \Re(s) > 1, \tag{2.9}$$

continued meromorphically to all $s \in \mathbb{C}$. There is a natural Γ-analog of $E_j(z,s,\chi)$, which for every $j \in C_{\Gamma,\chi}$ is given by (cf. [**6**, (2.8)–(2.10)])

$$E_j^\Gamma(z,s,\chi) = E_j(z,s,\chi) + \chi(V^{-1})E_j(Vz,s,\chi). \tag{2.10}$$

We fix, once and for all, a subset $R_{\Gamma,\chi} \subset C_{\Gamma,\chi}$ such that

$$\forall j \in C_{\Gamma,\chi} : \begin{cases} \text{exactly one of } j, k(j) \in R_{\Gamma,\chi} & \text{if } k(j) \neq j, \\ j \in R_{\Gamma,\chi} \iff \chi(V^{-1}U_j) = 1 & \text{if } k(j) = j. \end{cases} \tag{2.11}$$

For each $j \in C_{\Gamma,\chi}$ with $k(j) = j$, it holds that $\chi(V^{-1}U_j) = \pm 1$. Hence, if $j \notin R_{\Gamma,\chi}$, then $E_j^\Gamma(z,s,\chi) \equiv 0$. In [**6**, (2.32)], meromorphic functions $\varphi_{j,\ell}^\Gamma(s)$ for $j,\ell \in R_{\Gamma,\chi}$ are constructed so that for each $j \in R_{\Gamma,\chi}$,

$$E_j^\Gamma(z,1-s,\chi) = \sum_{\ell\in R_{\Gamma,\chi}} \varphi_{j,\ell}^\Gamma(1-s)E_\ell^\Gamma(z,s,\chi). \tag{2.12}$$

The relations (2.12) together determine the functions $\varphi_{j,\ell}^\Gamma(s)$ uniquely (cf. [**6**, p. 125(bottom)]). We also note that all $\varphi_{j,\ell}^\Gamma(s)$ are holomorphic along the line $\Re(s) = \frac{1}{2}$. The $|R_{\Gamma,\chi}| \times |R_{\Gamma,\chi}|$ matrix

$$\Phi^\Gamma(s) = (\varphi_{j,\ell}^\Gamma(s))_{j,\ell\in R_{\Gamma,\chi}}$$

is called the scattering matrix for Γ and we write

$$\varphi^\Gamma(s) = \det \Phi^\Gamma(s).$$

We point out that this determinant $\varphi^\Gamma(s)$, as well as the trace of $\Phi^\Gamma(s)$ which also appears in (2.5) are independent of which ordering of $R_{\Gamma,\chi}$ we chose when defining the matrix $\Phi^\Gamma(s)$.

2.2. *Trace formula for* $\Gamma_0(N)$, χ

Throughout this section we will use the convention that all matrix representatives for elements in $\mathrm{G} = \mathrm{PGL}_2(\mathbb{R})$ are taken to have determinant 1 or -1. Let N be an arbitrary positive integer and set $\Gamma^+ = \Gamma_0(N) \subset \mathrm{G}^+$. Fix

$$V = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{G}$$

and note that $V^2 = 1_2$, the identity matrix, and $V\Gamma^+V^{-1} = \Gamma^+$. Hence, $\Gamma = \Gamma_0^\pm(N) = \langle\Gamma^+, V\rangle \subset \mathrm{G}$ is a supergroup of Γ^+ of index 2. We will use the standard notation $\omega(N)$ for the number of primes dividing N.

Let χ be a Dirichlet character modulo N. For any divisor $\alpha \mid N$ satisfying $\gcd(\alpha, N/\alpha) = 1$, we define

$$\chi_\alpha(x) = \chi(y)$$

for $y \equiv_\alpha x$ and $y \equiv_{N/\alpha} 1$. By the Chinese remainder theorem, such y is uniquely determined modulo N and χ_α is a Dirichlet character modulo α. It follows that $\chi = \chi_\alpha \chi_{N/\alpha}$ for all such divisors α. For any prime p, we set

$$\chi_p = \chi_{\gcd(N, p^\infty)}.$$

It follows that $\chi = \prod_{p|N} \chi_p$ (identity of functions on $\mathbb{Z}$); also $\chi_p \equiv 1$ for all $p \nmid N$.

Let us agree to call χ pure if $\chi_p(-1) = 1$ for every odd prime p. Note that if χ is even, then χ is pure if and only if $\chi_p(-1) = 1$ for all primes p. From now on we fix χ to be an even Dirichlet character modulo N, and we set $q = \mathrm{cond}(\chi)$. We introduce two indicator functions I_χ and $I_{q,4}$ via

$$I_\chi = \delta_{\chi \text{ is pure}}; \tag{2.13}$$

$$I_{q,4} = \delta_{p \equiv_4 1 \ \forall p|q}. \tag{2.14}$$

We view χ as a character on Γ^+ via

$$\chi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \chi(d) \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^+.$$

This character can be extended in two ways to a character of Γ: either by $\chi(V) = 1$ or $\chi(V) = -1$. These extensions are explicitly given by

$$\chi^{(\varepsilon)}(T) = \det(T)^\varepsilon \chi(d), \text{ for } T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \text{ where } \varepsilon \in \{0, 1\}. \tag{2.15}$$

Let $\mathrm{Tr}(\mathrm{G}, \chi^{(\varepsilon)})$ be the trace formula for $(\Gamma_0^\pm(N), \chi^{(\varepsilon)})$. For $n \in \{\pm 1\}$, we have

$$\sum_{\lambda \geqslant 0} \mathrm{tr}\, T_n|_{\mathcal{A}_\lambda(\chi)} h\left(\sqrt{\lambda - \tfrac{1}{4}}\right) = \mathrm{Tr}(\mathrm{G}, \chi^{(0)}) + n\, \mathrm{Tr}(\mathrm{G}, \chi^{(1)}). \tag{2.16}$$

For each prime $p \mid N$, define

$$\Psi_1(p^{e_p}, p^{s_p}) = \begin{cases} 2p^{e_p - s_p} & \text{if } e_p < 2s_p, \\ p^{\lfloor \frac{e_p}{2} \rfloor} + p^{\lfloor \frac{e_p - 1}{2} \rfloor} & \text{if } e_p \geqslant 2s_p, \end{cases} \tag{2.17}$$

$$\Psi_2(p^{e_p}, p^{s_p}) = \begin{cases} 1 & \text{if } e_p = 0, \\ \begin{cases} 2e_p & \text{if } e_p > 0 \text{ and } s_p = 0, \\ 2(e_p - 2s_p + 1) & \text{if } e_p \geqslant 2s_p \text{ and } s_p > 0, \end{cases} & \text{if } p \text{ is odd}, \\ \begin{cases} e_p + 1 & \text{if } e_p \in \{1, 2\} \text{ and } s_p = 0, \\ 2(e_p - 1) & \text{if } e_p \in \{4, 3\} \text{ and } s_p = 0, \\ 4(e_p - 3) & \text{if } e_p \geqslant 5 \text{ and } s_p = 0, \\ 4(e_p - 2s_p - 1) & \text{if } e_p \geqslant 2s_p + 1 \text{ and } s_p \geqslant 3, \end{cases} & \text{if } p = 2, \\ 0 & \text{otherwise}, \end{cases} \tag{2.18}$$

$$\tilde{\Psi}_2(p^{e_p},\chi_p)=\begin{cases}1 & \text{if } e_p=0,\\ \min\{e_2+1,4\} & \text{if } p=2 \text{ and } s_2=0,\\ 2 & \text{if } p\equiv_4 -1 \text{ and } s_p=0,\\ \chi_p(\sqrt{-1})\Psi_2(p^{e_p},p^{s_p}) & \text{if } p\equiv_4 1 \text{ and } \chi_p \text{ even},\\ 0 & \text{otherwise}\end{cases} \tag{2.19}$$

(where $\sqrt{-1}$ is a square root of -1 in $\mathbb{Z}/p\mathbb{Z}$), and

$$\Psi_3(p^{e_p},p^{s_p})=\begin{cases}0 & \text{if } e_p=0,\\ p^{e_p-s_p}(4s_p-1) & \text{if } e_p<2s_p,\\ \left(p^{\lfloor\frac{e_p}{2}\rfloor}+p^{\lfloor\frac{e_p-1}{2}\rfloor}\right)\left(e_p-\dfrac{1}{p-1}\right)+\dfrac{2}{p-1}+\delta_{s_p>0}\left(p^{s_p-1}+2\dfrac{p^{s_p-1}-1}{p-1}\right) & \text{if } e_p\geqslant 2s_p.\end{cases} \tag{2.20}$$

For $j\in\{1,2\}$, we set $\Psi_j(N,q)=\prod_{p|N}\Psi_j(p^{e_p},p^{s_p})$. Similarly, $\tilde{\Psi}_2(N,\chi)=\prod_{p|N}\tilde{\Psi}_2(p^{e_p},\chi_p)$.

Let

$$\Omega_1(N,q)=\begin{cases}1 & \text{if } 0\leqslant e_2\leqslant 1 \text{ or } s_2=e_2,\\ \dfrac{3}{2} & \text{if } e_2=2 \text{ and } s_2<e_2,\\ 2 & \text{if } e_2\geqslant 3 \text{ and } s_2<e_2,\end{cases} \tag{2.21}$$

$$\Omega_2(N,q)=\begin{cases}\dfrac{1}{2}-\dfrac{3}{4}e_2 & \text{if } 0\leqslant e_2\leqslant 2,\\ -2 & \text{if } e_2\geqslant 3, s_2=0,\\ -2s_2 & \text{if } e_2>s_2\geqslant 3,\\ -e_2 & \text{if } e_2=s_2\geqslant 3.\end{cases} \tag{2.22}$$

Finally we define

$$\Phi_\pm(p^{e_p},m)=\Psi_1(p^{e_p},p^{w_\pm(m)}) \qquad \text{with } w_\pm(m)=\max\{s_p,e_p-\operatorname{ord}_p(\pm m^2-1)\}. \tag{2.23}$$

THEOREM 2.1. *For $n\in\{\pm1\}$, we have*

$$\sum_{\lambda\geqslant 0}\operatorname{tr}T_n|_{\mathcal{A}_\lambda(\chi)}h\left(\sqrt{\lambda-\tfrac{1}{4}}\right)=\mathrm{I}(\Gamma,\chi;n)+(\mathrm{NEll}+\mathrm{Ell})(\Gamma,\chi;n)+(\mathrm{C}+\mathrm{Eis})(\Gamma,\chi;n), \tag{2.24}$$

where

$$\mathrm{I}(\Gamma,\chi;n)=\frac{1+n}{2}\frac{N\prod_{p|N}(1+p^{-1})}{12}\int_{\mathbb{R}}rh(r)\tanh(\pi r)\,dr,$$

$$(\mathrm{NEll}+\mathrm{Ell})(\Gamma,\chi;n)=\sum_{\substack{t\in\mathbb{Z},t^2-4n=d\ell^2,\\ \sqrt{t^2-4n}\notin\mathbb{Q}}}A_{t,n}[h]\mathbf{h}(d)\prod_p\mathbf{S}_p(p^{e_p},\chi_p;t,n),$$

where $A_{t,n}[h]$ *is defined in* (2.28), $\mathbf{h}(d)$ *is the narrow class number of* $\mathbb{Q}(\sqrt{d})$, $\mathbf{S}_p(p^{e_p}, \chi_p; t, n)$ *is given in* (2.34) *and Lemma* 2.8,

$$\begin{aligned}
&(\mathrm{C}+\mathrm{Eis})(\Gamma,\chi;1)\\
&\quad=\Psi_1(N,q)\Big\{\frac{1}{4}h(0)-\frac{1}{2\pi}\int_{\mathbb{R}}h(r)\Big(\frac{\Gamma'}{\Gamma}\Big(\frac{1}{2}+ir\Big)+\frac{\Gamma'}{\Gamma}(1+ir)\Big)\,dr\Big\}-\frac{1}{4}I_\chi\Psi_2(N,q)h(0)\\
&\qquad+\Big\{\Psi_1(N,q)\log\Big(\frac{\pi}{2}\Big)-\sum_{p\mid N}\Psi_3(p^{e_p},p^{s_p})\prod_{p'\mid N,p'\neq p}\Psi_1(p'^{e_{p'}},p'^{s_{p'}})\log p\Big\}g(0)\\
&\qquad+2\sum_{m\geqslant 1,\gcd(m,q)=1}\frac{\Lambda(m)}{m}\prod_{p\mid N,p\nmid m}[(\Re\chi_p(m))\Phi_+(p^{e_p},m)]\,g(2\log m)
\end{aligned}\tag{2.25}$$

and

$$\begin{aligned}
&(\mathrm{C}+\mathrm{Eis})(\Gamma,\chi;-1)\\
&\quad=\frac{1}{4}I_\chi\Big\{2^{\omega(N)}\Omega_1(N,q)-I_{q,4}\tilde{\Psi}_2(N,\chi)\Big\}h(0)-I_\chi 2^{\omega(N)}\Omega_1(N,q)\frac{1}{2\pi}\int_{\mathbb{R}}h(r)\frac{\Gamma'}{\Gamma}(1+ir)\,dr\\
&\qquad+I_\chi 2^{\omega(N)}\Big\{\Omega_1(N,q)\Big(\log\pi-\sum_{p\mid N,p>2}\max\{\tfrac{1}{2},s_p\}\log p\Big)+\Omega_2(N,q)\log 2\Big\}g(0)\\
&\qquad+2\sum_{m\geqslant 1,\gcd(m,q)=1}\frac{\Lambda(m)}{m}\prod_{p\mid N,p\nmid m}[(\{\Re/i\Im\}\chi_p(m))\Phi_-(p^{e_p},m)]\,g(2\log m),
\end{aligned}\tag{2.26}$$

where $\{\Re/i\Im\}\chi_p(m)=\frac{1}{2}(\overline{\chi_p(m)}+\chi_p(-m))$.

2.3. *Identity* $I(\Gamma,\chi)$

Lemma 2.2. *For* $n\in\{\pm1\}$, *we have*

$$I(\Gamma,\chi;n)=\frac{1+n}{2}\frac{N\prod_{p\mid N}(1+p^{-1})}{12}\int_{\mathbb{R}}rh(r)\tanh(\pi r)\,dr.$$

Proof. It is well known that $\mathrm{Area}(\Gamma^+\backslash\mathbb{H})=\frac{\pi}{3}N\prod_{p\mid N}(1+p^{-1})$ (cf. [**21**, Theorem 4.2.5(2)]), and the area of $\Gamma\backslash\mathbb{H}$ is half as large. By (2.1), we have

$$\mathrm{I}(\Gamma,\chi^{(\varepsilon)})=\frac{N\prod_{p\mid N}(1+p^{-1})}{24}\int_{\mathbb{R}}h(r)r\tanh(\pi r)\,dr\qquad\text{for }\ \varepsilon=\pm1.$$

□

2.4. *Non-cuspidal contributions* $\mathrm{NEll}+\mathrm{Ell}(\Gamma,\chi)$

The aim of this section is to prove the following proposition.

Proposition 2.3. *For* $n\in\{\pm1\}$, *we have*

$$\mathrm{NEll}(\Gamma,\chi;n)+\mathrm{Ell}(\Gamma,\chi;n)=\sum_{\substack{t\in\mathbb{Z},t^2-4n=d\ell^2,\\ \sqrt{t^2-4n}\notin\mathbb{Q}}}A_{t,n}[h]\,\mathbf{h}(d)\prod_p\mathbf{S}_p(p^{e_p},\chi_p;t,n),\tag{2.27}$$

where d *is a fundamental discriminant and* $\ell\in\mathbb{Z}_{>0}$.

The method we use to enumerate the conjugacy classes appearing in these sums is well known, cf., for example, [11, 21 27]. Here we will follow the setup in [21] fairly closely (as was done in [6] in the case of N squarefree).

We start by introducing some notations. Set

$$R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}) \ : \ c \equiv_N 0 \right\}$$

and

$$R^1 = \{T \in R \ : \ \det(T) = 1\}.$$

Let

$$P = \left\{ \langle n, t\rangle \in \{\pm 1\} \times \mathbb{Z} \ : \ \sqrt{t^2 - 4n} \notin \mathbb{Q}, \ \exists T_{n,t} \in R, \ \det(T_{n,t}) = n \text{ and } \operatorname{tr}(T_{n,t}) = t \right\}.$$

For each $\langle n, t\rangle \in P$, we fix one such element $T_{n,t} \in R$.

Given $\langle n, t\rangle \in P$, we let d and ℓ be the unique integers such that $t^2 - 4n = d\ell^2$, $\ell \in \mathbb{Z}_{\geqslant 1}$ and d is a fundamental discriminant (that is, $d \equiv_4 1$ is squarefree and $d \neq 1$ or else $d \equiv_4 0$, $d/4$ is squarefree and $d/4 \not\equiv_4 1$). Then the subalgebra $\mathbb{Q}[T_{n,t}] \subset \mathrm{M}_2(\mathbb{Q})$ is isomorphic to the quadratic field $\mathbb{Q}(\sqrt{d})$. For any positive integer f, we write $\mathfrak{r}[f]$ for the order $\mathbb{Z} + f\frac{d+\sqrt{d}}{2}\mathbb{Z}$ in $\mathbb{Q}(\sqrt{d})$. Using $\mathbb{Q}[T_{n,t}] \cong \mathbb{Q}(\sqrt{d})$ (either of the two possible isomorphisms), we let $\mathfrak{r}[f]$ denote also the corresponding order in $\mathbb{Q}[T_{n,t}]$. For any order $\mathfrak{r}$ in $\mathbb{Q}[T_{n,t}]$, we set

$$C(T_{n,t}, \mathfrak{r}) = \{\delta T_{n,t} \delta^{-1} \ : \ \delta \in \mathrm{GL}_2(\mathbb{Z}), \ \mathbb{Q}[T_{n,t}] \cap \delta^{-1} R \delta = \mathfrak{r}\}.$$

Note that $C(T_{n,t}, \mathfrak{r})$ is closed under conjugation by elements in R^1. We denote by $C(T_{n,t}, \mathfrak{r})//R^1$ a set of representatives for the inequivalent R^1-conjugacy classes in $C(T_{n,t}, \mathfrak{r})$. Also for $U = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in C(T_{n,t}, \mathfrak{r})$, we write

$$\chi^{(\varepsilon)}(U) = \det(U)^{\varepsilon} \chi(d).$$

For every $f \mid \ell$, it holds that $C(T_{n,t}, \mathfrak{r}[f]) \subset R$ and using this one checks that $\chi^{(\varepsilon)}(U)$ for $U \in C(T_{n,t}, \mathfrak{r}[f])$ only depends on the R^1-conjugacy class of U.

Let $\mathbf{h}(\mathfrak{r}[f])$ be the narrow class number for $\mathfrak{r}[f]$. We also define

$$A_{t,n}[h] = \begin{cases} \dfrac{\log \epsilon_1}{\sqrt{t^2 - 4n}} g\left(\log \dfrac{\left(|t| + \sqrt{t^2 - 4n}\right)^2}{4} \right), & \text{if } t^2 - 4n > 0, \\ \dfrac{2}{|\mathfrak{r}[1]^1| \sqrt{4n - t^2}} \displaystyle\int_{\mathbb{R}} \frac{e^{-2r \cdot \arccos(|t|/2)}}{1 + e^{-2\pi r}} h(r) \, dr, & \text{if } t^2 - 4n < 0, \end{cases} \tag{2.28}$$

where in the first case $\epsilon_1 > 1$ is the proper fundamental unit in $\mathbb{Q}(\sqrt{d})$.

Following the discussion in [6, pp. 136–137], we see that $\mathrm{NEll}(\Gamma, \chi^{(\varepsilon)}) + \mathrm{Ell}(\Gamma, \chi^{(\varepsilon)})$ can be collected as

$$\begin{aligned} &\mathrm{NEll}(\Gamma, \chi^{(\varepsilon)}) + \mathrm{Ell}(\Gamma, \chi^{(\varepsilon)}) \\ &= \frac{1}{2} \sum_{\langle n,t\rangle \in P} \left\{ \sum_{f \mid \ell} \left[\mathfrak{r}[1]^1 : \mathfrak{r}[f]^1\right] \left(\sum_{U \in C(T_{n,t}, \mathfrak{r}[f])//R^1} \chi^{(\varepsilon)}(U) \right) \right\} A_{t,n}[h] \begin{cases} 1 & \text{if } t^2 - 4n > 0, \\ \dfrac{1}{2} & \text{if } t^2 - 4n < 0. \end{cases} \end{aligned} \tag{2.29}$$

REMARK 2.4. The expression (2.29) would of course look slightly nicer if we did not include the factor '2' in the definition of $A_{t,n}[h]$ (2.28) in the case $t^2 - 4n < 0$. However, our definition makes the final expression slightly simpler.

The following lemma generalizes [24, Lemma 3.4].

LEMMA 2.5. *Let n, t be any integers satisfying $\sqrt{t^2-4n} \notin \mathbb{Q}$, and let d, ℓ be the unique integers such that $t^2 - 4n = d\ell^2$, $\ell > 0$ and d is a fundamental discriminant. Then there exists $T \in R$ satisfying $\det(T) = n$ and $\mathrm{tr}(T) = t$ if and only if, for each prime $p \mid N$, we have*

$$\mathrm{ord}_p(\ell) \geqslant \left\lceil \frac{e_p}{2} \right\rceil \text{ or } \left(\frac{d}{p}\right) = 1 \text{ or } \left[2 \nmid e_p,\ \mathrm{ord}_p(\ell) \geqslant \left\lfloor \frac{e_p}{2} \right\rfloor \text{ and } p \mid d\right]. \tag{2.30}$$

Here $e_p = \mathrm{ord}_p(N)$.

Proof. Such an element $T \in R$ exists if and only if there is some $a \in \mathbb{Z}$ satisfying $a(t-a) - n \equiv_N 0$, and this holds if and only if the same congruence equation is solvable modulo p^{e_p} for each prime $p \mid N$. The lemma now follows by completing the square in the expression $a(t-a) - n$, and splitting into the classes $2 \mid e_p$ and $2 \nmid e_p$, as well as $p = 2$ and $p > 2$. □

The R^1-conjugacy classes in each $C(T_{n,t}, \mathfrak{r})$ can be enumerated using a local-to-global principle. Let $T = T_{n,t}$ and $\mathfrak{r} = \mathfrak{r}[f]$ for some $f \mid \ell$. For each prime p, we set

$$C_p(T, \mathfrak{r}) = \left\{ xTx^{-1} \ : \ x \in \mathrm{GL}_2(\mathbb{Q}_p),\ \mathbb{Q}_p[T] \cap x^{-1} R_p x = \mathfrak{r}_p \right\},$$

where

$$R_p = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}_p) \ : \ \gamma \equiv_N 0 \right\}$$

is the closure of R in $\mathrm{M}_2(\mathbb{Q}_p)$ and $\mathfrak{r}_p$ is the closure of $\mathfrak{r}$ in $\mathbb{Q}_p(\sqrt{d}) \cong \mathbb{Q}_p[T]$. One checks that $C_p(T, \mathfrak{r})$ is closed under $R_p^\times$-conjugation. We also set

$$C_\infty(T, \mathfrak{r}) = \left\{ xTx^{-1} \ : \ x \in \mathrm{GL}_2(\mathbb{R}) \right\},$$

which is closed under $R_\infty^\times$-conjugation, where we have set $R_\infty^\times = \mathrm{GL}_2^+(\mathbb{R})$. Let us now mildly alter our previous definition of $C(T, \mathfrak{r})//R^1$ (a set of representatives), so as to instead let $C(T, \mathfrak{r})//R^1$ denote the set of R^1-conjugacy classes in $C(T, \mathfrak{r})$. Similarly for each place v of $\mathbb{Q}$, we let $C_v(T, \mathfrak{r})//R_v^\times$ be the set of $R_v^\times$-conjugacy classes in $C_v(T, \mathfrak{r})$. Clearly, we have a natural map

$$\theta : C(T, \mathfrak{r})//R^1 \to \prod_v C_v(T, \mathfrak{r})//R_v^\times. \tag{2.31}$$

By [21, Lemma 6.5.2] (trivially generalized as noted in [6]), the map θ is surjective, and in fact θ is exactly $\mathbf{h}(\mathfrak{r})$-to-1.

It remains to understand each factor in the right-hand side of (2.31). For $v = \infty$, we have, as in [21, (6.6.1)],

$$\left| C_\infty(T, \mathfrak{r}[f])//R_\infty^\times \right| = \begin{cases} 1 & \text{if } d > 0, \\ 2 & \text{if } d < 0. \end{cases}$$

Also for primes $p \nmid N$, we have $R_p \subset \mathrm{M}_2(\mathbb{Z}_p) \subset \mathrm{M}_2(\mathbb{Q}_p)$ and thus

$$\left| C_p(T, \mathfrak{r})//R_p^\times \right| = 1$$

(cf., for example, [21, Theorem 6.6.7]).

Finally, if $p \mid N$, let $e = \mathrm{ord}_p(N)$ and $\rho = \mathrm{ord}_p(\ell/f)$. For $\alpha \in \mathbb{Z}_{\geqslant 1}$, let

$$\Omega(p^\alpha; n, t) = \big\{\xi \in \mathbb{Z}_p \ : \ \xi^2 - t\xi + n \equiv_{p^\alpha} 0\big\}.$$

Then by [21, Theorem 6.6.6][†], a complete set of representatives for $C_p(T, \mathfrak{r})//R_p^\times$ (where $T = T_{n,t}$, $\mathfrak{r} = \mathfrak{r}[f]$ and $f \mid \ell$) is given by

$$\left\{ \begin{pmatrix} \xi & p^\rho \\ -\dfrac{\xi^2 - t\xi + n}{p^\rho} & t - \xi \end{pmatrix} \ : \ \xi \in \Omega(p^{e+2\rho}; n, t)/p^{e+\rho} \right\}$$

$$\bigcup \begin{cases} \left\{ \begin{pmatrix} t - \xi - \dfrac{t^2 - t\xi + n}{p^{e+\rho}} & \\ p^{e+\rho} & \xi \end{pmatrix} \ : \ \xi \in \Omega(p^{e+2\rho+1}; n, t)/p^{e+\rho} \right\} & \text{if } p^{2\rho+1} \mid \ell^2 d, \\ \emptyset & \text{otherwise.} \end{cases}$$

The sets $\Omega(p^{e+2\rho}; n, t)$ and $\Omega(p^{e+2\rho+1}; n, t)$ are both closed under addition with any element from $p^{e+\rho}\mathbb{Z}_p$, and $\Omega(p^{e+2\rho}; n, t)/p^{e+\rho}$ and $\Omega(p^{e+2\rho+1}; n, t)/p^{e+\rho}$ denote complete sets of representatives for $\Omega(p^{e+2\rho}; n, t) \mod p^{e+\rho}\mathbb{Z}_p$ and $\Omega(p^{e+2\rho+1}; n, t) \mod p^{e+\rho}\mathbb{Z}_p$, respectively.

Using the above facts together with $\chi = \prod_{p|N} \chi_p$ and the fact that $\xi^2 - t\xi + n = \xi(\xi - t) + n$ is invariant under $\xi \mapsto t - \xi$, it follows that, for any $f \mid \ell$,

$$\sum_{U \in C(T_{n,t}, \mathfrak{r}[f])//R^1} \chi^{(\varepsilon)}(U)$$

$$= \prod_{p|N} \left(\sum_{\xi \in \Omega(e^{e+2\rho}; n, t)/p^{e+\rho}} \chi_p(\xi) + \delta_{p^{2\rho+1} | \ell^2 d} \sum_{\xi \in \Omega(p^{e+2\rho+1}; n, t)/p^{e+\rho}} \chi_p(\xi) \right)$$

$$\times \mathbf{h}(\mathfrak{r}[f]) n^\varepsilon \begin{cases} 1 & \text{if } d > 0, \\ 2 & \text{if } d < 0. \end{cases} \tag{2.32}$$

For $p \mid N$ with $e = \mathrm{ord}_p(N)$ and $f = \mathrm{ord}_p(\ell)$, set

$$\mathbf{S}_p(p^e, \chi; t, n) = \sum_{\xi \in \Omega(p^{e+2f}; n, t)/p^{e+f}} \chi(\xi) + \delta_{p|d} \sum_{\xi \in \Omega(p^{e+2f+1}; n, t)/p^{e+f}} \chi(\xi)$$

$$+ \left(p - \left(\frac{d}{p}\right)\right) \sum_{\beta \in \{0,1\}} \sum_{k=1}^{f} p^{k-1} \sum_{\xi \in \Omega(p^{e+2f-2k+\beta}; n, t)/p^{e+f-k}} \chi(\xi). \tag{2.33}$$

Furthermore for $p \nmid N$, set

$$\mathbf{S}_p(1, 1; t, n) = 1 + \left(p - \left(\frac{d}{p}\right)\right) \frac{p^{\mathrm{ord}_p(\ell)} - 1}{p - 1}. \tag{2.34}$$

Since

$$\mathbf{h}(\mathfrak{r}[f]) \big[\mathfrak{r}[1]^1 : \mathfrak{r}[f]^1\big] = \mathbf{h}(d) f \prod_{p|f} p^{-1} \left(p - \left(\frac{d}{p}\right)\right),$$

[†]We correct for a misprint in [21, Theorem 6.6.6]: $\Omega'/p^{\nu+\rho+1}$ should be replaced by $\Omega'/p^{\nu+\rho}$.

for each $\langle n,t\rangle \in P$, we have

$$\sum_{f|\ell} [\mathfrak{r}[1]^1 : \mathfrak{r}[f]^1] \left(\sum_{U \in C(T_{n,t},\mathfrak{r}[f])//R^1} \chi^{(\varepsilon)}(U) \right) = \left(\prod_p \mathbf{S}_p(p^{e_p}, \chi_p; t, n) \right) n^{\varepsilon} \mathbf{h}(d) \begin{cases} 1 & \text{if } d > 0, \\ 2 & \text{if } d < 0. \end{cases} \tag{2.35}$$

Applying (2.35) to (2.29), then by (2.16),

$$\mathrm{NEll}(\Gamma, \chi; n) + \mathrm{Ell}(\Gamma, \chi; n) = \sum_{\substack{t \in \mathbb{Z} \\ (\langle n,t\rangle \in P)}} A_{t,n}[h] \mathbf{h}(d) \prod_p \mathbf{S}_p(p^{e_p}, \chi_p; t, n). \tag{2.36}$$

In the remainder of this section, we prove Lemma 2.8 and get an explicit formula for $\mathbf{S}_p(p^e, \chi; t, n)$. Let us first record the general solution to the congruence equation $\xi^2 - t\xi + n \equiv 0$ modulo a prime power in the following lemma.

LEMMA 2.6. *Let p be a prime and α a positive integer. Then*

$$\Omega(p^\alpha; n, t) = \begin{cases} \left\{ \dfrac{t + p^{\lceil \frac{\alpha}{2} \rceil} \omega}{2} + p^{\lceil \frac{\alpha}{2} \rceil} u \ : \ u \in \mathbb{Z}_p \right\} & \text{if } \mathrm{ord}_p(t^2 - 4n) \geqslant \alpha + \max\{\mathrm{ord}_p(d) - 1, 0\}, \\ \left\{ \dfrac{t}{2} \pm \dfrac{\sqrt{d}\ell}{2} + p^{\alpha - \mathrm{ord}_p(\ell)} u \ : \ u \in \mathbb{Z}_p \right\} & \text{if } \mathrm{ord}_p(t^2 - 4n) < \alpha \text{ and } \left(\frac{d}{p}\right) = 1, \\ \emptyset & \text{otherwise,} \end{cases} \tag{2.37}$$

where

$$\omega = \begin{cases} 1 & \text{if } p = 2, \alpha = \mathrm{ord}_p(t^2 - 4n) - 1 \text{ and } 8 \nmid d, \\ & \text{or } p = 2, \alpha = \mathrm{ord}_p(t^2 - 4n) \text{ and } 2 \nmid d, \\ & \text{or } p \text{ and } t \text{ odd}, \\ 0 & \text{otherwise,} \end{cases}$$

and $t^2 - 4n = d\ell^2$ with d a fundamental discriminant and $\ell \in \mathbb{Z}$.

Proof. Suppose first that p is odd. For $\alpha \geqslant 1$, since $\xi \in \Omega(p^\alpha; n, t)$, we have

$$\xi^2 - t\xi + n \equiv \left(\xi - \frac{t}{2}\right)^2 - \frac{t^2}{4} + n \equiv 0 \pmod{p^\alpha},$$

so that

$$\left(\xi - \frac{t}{2}\right)^2 \equiv \frac{d\ell^2}{4} \pmod{p^\alpha}. \tag{2.38}$$

When $\mathrm{ord}_p(d\ell^2) \geqslant \alpha$, this yields $(\xi - \frac{t}{2})^2 \equiv 0 \pmod{p^\alpha}$. For some $u \in \mathbb{Z}_p$, we have

$$\xi = \frac{t}{2} + p^{\lceil \frac{\alpha}{2} \rceil} u.$$

Next consider the case when $\mathrm{ord}_p(d\ell^2) < \alpha$. We must have $p \nmid d$ and $\mathrm{ord}_p(d\ell^2) = 2\,\mathrm{ord}_p(\ell) < \alpha$ for ξ to satisfy (2.38). In this case, (2.38) has a solution only if $(\frac{d}{p}) = 1$, and then

$\operatorname{ord}_p(\xi - \frac{t}{2}) = \operatorname{ord}_p(\ell) = \frac{\operatorname{ord}_p(d\ell^2)}{2}$. Therefore, for some $u \in \mathbb{Z}_p$,

$$\xi = \frac{t}{2} \pm \frac{\sqrt{d}\ell}{2} + p^{\alpha - \operatorname{ord}_p(\ell)} u.$$

Suppose now that $p = 2$. Since $n \in \{\pm 1\}$ and

$$\xi^2 - t\xi + n = \xi(\xi - t) + n \equiv 0 \pmod{2^\alpha},$$

both ξ and $\xi - t$ must be odd, so t is even. Note that when $\alpha = 1$ and t is even,

$$\Omega(2; n, t) = \{1 + 2u \ : \ u \in \mathbb{Z}_2\}.$$

From now on, we assume that $\alpha \geqslant 2$ and t is even. Then $d\ell^2 = t^2 - 4n = 4(\frac{t^2}{4} - n)$, so $4 \mid d$ or $2 \mid \ell$. Therefore,

$$\left(\xi - \frac{t}{2}\right)^2 \equiv \frac{t^2 - 4n}{4} \equiv \frac{d\ell^2}{4} \pmod{2^\alpha}. \tag{2.39}$$

If $\operatorname{ord}_2(d\ell^2) - 2 \geqslant \alpha$, we have $(\xi - \frac{t}{2})^2 \equiv 0 \pmod{2^\alpha}$, so

$$\Omega(2^\alpha; n, t) = \left\{ \frac{t}{2} + 2^{\lceil \frac{\alpha}{2} \rceil} u \ : \ u \in \mathbb{Z}_2 \right\}.$$

Next consider the case when $\operatorname{ord}_2(d\ell^2) - 2 < \alpha$. When $4 \mid d$, we have $(\operatorname{ord}_2(d) - 2) + 2\operatorname{ord}_2(\ell) < \alpha$. Dividing both sides of (2.39) by $2^{2\operatorname{ord}_2(\ell)}$, we get

$$\left(\frac{\xi - \frac{t}{2}}{2^{\operatorname{ord}_2(\ell)}}\right)^2 \equiv \frac{d}{4}\left(\frac{\ell}{2^{\operatorname{ord}_2(\ell)}}\right)^2 \pmod{2^{\alpha - 2\operatorname{ord}_2(\ell)}}.$$

Note that $d/4 \equiv_4 2$ or 3, that is, $d/4$ is not a square modulo 4, which implies $\Omega(2^\alpha; n, t) = \emptyset$ if $\alpha - 2\operatorname{ord}_2(\ell) \geqslant 2$. If $\alpha - 2\operatorname{ord}_2(\ell) = 1$, then

$$\frac{\xi - \frac{t}{2}}{2^{\operatorname{ord}_2(\ell)}} \equiv \frac{d}{4} \pmod{2}.$$

Thus, when $\alpha = 2\operatorname{ord}_2(\ell) + 1$ and $4 \mid d$, we get

$$\Omega(2^\alpha; n, t) = \left\{ \frac{t}{2} + 2^{\operatorname{ord}_2(\ell)}\left(\frac{d}{4} + 2u\right) \ : \ u \in \mathbb{Z}_2 \right\}.$$

When $d \equiv 1 \pmod 4$, we have $2\operatorname{ord}_2(\ell) - 2 < \alpha$. Dividing both sides of (2.39) by $2^{2\operatorname{ord}_2(\ell) - 2}$, we get

$$\left(\frac{\xi - \frac{t}{2}}{2^{\operatorname{ord}_2(\ell) - 1}}\right)^2 \equiv d\left(\frac{\ell}{2^{\operatorname{ord}_2(\ell)}}\right)^2 \pmod{2^{\alpha - 2\operatorname{ord}_2(\ell) + 2}}.$$

When $\alpha = 2\operatorname{ord}_2(\ell) - 1$ or $\alpha = 2\operatorname{ord}_2(\ell)$, we have

$$\Omega(2^\alpha; n, t) = \left\{ \frac{t}{2} + 2^{\operatorname{ord}_2(\ell) - 1}(1 + 2u) \ : \ u \in \mathbb{Z}_2 \right\}.$$

For $\alpha > 2\operatorname{ord}_2(\ell)$, following a similar procedure to the case when p is odd, we get

$$\Omega(2^\alpha; n, t) = \left\{ \frac{t}{2} \pm \frac{\sqrt{d}\ell}{2} + 2^{\alpha - \operatorname{ord}_2(\ell)} u \ : \ u \in \mathbb{Z}_2 \right\}. \qquad \square$$

Next, by Lemma 2.5, if $\langle n,t\rangle \in \{\pm 1\}\times\mathbb{Z}$ satisfies $\sqrt{t^2-4n}\notin\mathbb{Q}$ and $\langle n,t\rangle\notin P$, then there is a prime $p\mid N$ such that either

- $(\frac{d}{p})=-1$ and $e_p > 2\operatorname{ord}_p(\ell)$; or
- $p\mid d$ and $e_p > 2\operatorname{ord}_p(\ell)+1$.

By inspection in (2.37), we see that the summation condition '$\langle n,t\rangle\in P$' in (2.36) may be replaced by summation over all $\langle n,t\rangle\in\{\pm1\}\times\mathbb{Z}$ satisfying $\sqrt{t^2-4n}\notin\mathbb{Q}$, as in (2.27).

LEMMA 2.7. *Let p be a prime, $s\in\mathbb{Z}_{\geqslant 0}$ and ψ a primitive Dirichlet character of conductor p^s. For $a,b\in\mathbb{Z}_{\geqslant 0}$, $a+b\geqslant s$, we have*

$$\sum_{u \pmod{p^b}} \psi(x+p^a u) = \begin{cases} p^b\psi(x) & \text{if } s\leqslant a,\\ 0 & \text{otherwise.}\end{cases} \tag{2.40}$$

Proof. Since ψ is primitive, $\tau(\overline{\psi})\neq 0$ and we have

$$\psi(x+p^a u) = \frac{1}{\tau(\overline{\psi})}\sum_{\alpha \pmod{p^s}} \overline{\psi(\alpha)} e^{2\pi i(x+p^a u)\frac{\alpha}{p^s}}.$$

Then

$$\sum_{u \pmod{p^b}} \psi(x+p^a u) = \frac{1}{\tau(\overline{\psi})}\sum_{\alpha \pmod{p^s}} \overline{\psi(\alpha)} e^{2\pi i x\frac{\alpha}{p^s}} \begin{cases} \displaystyle\sum_{u \pmod{p^b}} e^{2\pi i\alpha\frac{u}{p^{s-a}}} & \text{if } s>a,\\ p^b & \text{if } s\leqslant a.\end{cases}$$

Since $a+b\geqslant s$, if $s>a$,

$$\sum_{u \pmod{p^b}} e^{2\pi i\alpha\frac{u}{p^{s-a}}} = \sum_{u \pmod{p^b}} e^{2\pi i\alpha p^{a+b-s}\frac{u}{p^b}} = \begin{cases} p^b & \text{if } p^{s-a}\mid\alpha,\\ 0 & \text{otherwise.}\end{cases}$$

Since $\psi(\alpha)=0$ when $s>0$ and $p\mid\alpha$, we get (2.40). □

LEMMA 2.8. *For $t\in\mathbb{Z}$ and $n\in\{\pm1\}$, let $t^2-4n=d\ell^2$ where d is a fundamental discriminant and $\ell\in\mathbb{Z}_{>0}$. For $e=\operatorname{ord}_p(N)\geqslant 1$ and $s=\operatorname{ord}_p(\operatorname{cond}(\chi))$, let $h=\max\{2s-1,e\}$, $g=\operatorname{ord}_p(t^2-4n)$ and $f=\frac{g-\operatorname{ord}_p(d)}{2}=\operatorname{ord}_p(\ell)$.*

When p is odd, we have

$$\begin{aligned}\mathbf{S}_p(p^e,\chi;t,n) = \delta_{g\geqslant h>0}\chi\Big(\frac{t+p^s\delta_{2\nmid t}}{2}\Big)\Big\{&p^{e-1}\Big(p^{f-\lfloor\frac{h-1}{2}\rfloor}+p^{f-\lfloor\frac{h}{2}\rfloor}\Big)\\ &+\Big(1-\Big(\frac{d}{p}\Big)\Big)\frac{p^{e-1}}{p-1}\Big(p^{f-\lfloor\frac{h-1}{2}\rfloor}+p^{f-\lfloor\frac{h}{2}\rfloor}-p-1\Big)\Big\}\\ &+\delta_{g\leqslant h-1,\ (\frac{d}{p})=1}\Big(\chi\Big(\frac{t+\sqrt{d}\ell}{2}\Big)+\chi\Big(\frac{t-\sqrt{d}\ell}{2}\Big)\Big)p^{f+\min\{e-s,f\}}.\end{aligned} \tag{2.41}$$

When $p = 2$, we have $\mathbf{S}_2(2^e, \chi; t, n) = 0$ if t is odd, while if t is even, then

$$\mathbf{S}_2(2^e,\chi;t,n)$$
$$= \chi\left(\frac{t}{2}\right)\begin{cases} \delta_{2\nmid d}2^{e-1}\left(2^{\frac{g}{2}-\left\lfloor\frac{h-1}{2}\right\rfloor}+2^{\frac{g}{2}-\left\lfloor\frac{h}{2}\right\rfloor}\right) & \\ +\left(1-\left(\frac{d}{2}\right)\right)2^{e-1}\left(2^{\left\lfloor\frac{g}{2}\right\rfloor-\left\lfloor\frac{h-1}{2}\right\rfloor}+2^{\left\lfloor\frac{g}{2}\right\rfloor-\left\lfloor\frac{h}{2}\right\rfloor}-3\right) & \text{if } g\geqslant h+1 \text{ and } 2\nmid g, \\ & \text{or } g\geqslant \max\{h,2s+2\} \text{ and } 2\mid g, \\ -\delta_{2\nmid d}2^{e+1}-\left(1-\left(\frac{d}{2}\right)\right)2^{e-1} & \text{if } g=2s\geqslant e+1, \\ -3\cdot 2^{e-1} & \text{if } g=2s=e \text{ and } 2\nmid d, \\ 0 & \text{otherwise} \end{cases}$$
$$+\delta_{\substack{g\leqslant h-1,\\ \left(\frac{d}{2}\right)=1}}\left(\chi\left(\frac{t+\sqrt{d}\ell}{2}\right)+\chi\left(\frac{t-\sqrt{d}\ell}{2}\right)\right)2^{f+\min\{e-s,f\}}. \tag{2.42}$$

Proof. By Lemma 2.6, setting $\alpha = e + 2(f-k) + \beta$ for $\beta \in \{0,1\}$ and $0 \leqslant k \leqslant f$ in (2.37), we get

$$\Omega(p^{e+2(f-k)+\beta};n,t)/p^{e+f-k}$$
$$= \begin{cases} \left\{\dfrac{t+p^{\lceil\frac{e+\beta}{2}\rceil+f-k}\omega}{2}+p^{\lceil\frac{e+\beta}{2}\rceil+f-k}u \ :\ u \pmod{p^{\lceil\frac{e+\beta}{2}\rceil-\beta}}\right\} & \text{if } 2k+\operatorname{ord}_p(d)\geqslant e+\beta+\delta, \\ \left\{\dfrac{t\pm\sqrt{d}\ell}{2}+p^{e+\beta+f-2k}u \ :\ u \pmod{p^{k-\beta}}\right\} & \text{if } 2k\leqslant e+\beta-1 \text{ and } \left(\frac{d}{p}\right)=1, \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\delta = \max\{\operatorname{ord}_p(d)-1, 0\}$ and

$$\omega = \begin{cases} 1 & \text{if } p=2, 2\nmid e+\beta, k=\dfrac{e+\beta+1}{2}-\dfrac{\operatorname{ord}_2(d)}{2}\leqslant f \text{ and } 8\nmid d, \\ & \text{or } p=2, 2\mid e+\beta, k=\dfrac{e+\beta}{2}\leqslant f \text{ and } 2\nmid d, \\ & \text{or } p\neq 2 \text{ and } t \text{ odd}, \\ 0 & \text{otherwise.} \end{cases}$$

When p is odd, applying Lemma 2.7,

$$\sum_{\xi\in\Omega(p^{e+2(f-k)+\beta};n,t)/p^{e+f-k}}\chi(\xi)$$
$$= \begin{cases} p^{\lfloor\frac{e+\beta}{2}\rfloor-\beta}\chi\left(\dfrac{t+\delta_{2\mid t}p^s}{2}\right) & \text{if } \left\lceil\dfrac{e+\beta-\operatorname{ord}_p(d)}{2}\right\rceil\leqslant k\leqslant f+\left\lceil\dfrac{e+\beta}{2}\right\rceil-\left\lceil\dfrac{h+\beta}{2}\right\rceil, \\ p^{k-\beta}\left(\chi\left(\dfrac{t+\sqrt{d}\ell}{2}\right)+\chi\left(\dfrac{t-\sqrt{d}\ell}{2}\right)\right) & \text{if } k\leqslant \min\left\{\left\lfloor\dfrac{e+\beta+f-s}{2}\right\rfloor,\left\lfloor\dfrac{e-1+\beta}{2}\right\rfloor,f\right\} \text{ and } \left(\dfrac{d}{p}\right)=1, \\ 0 & \text{otherwise.} \end{cases}$$

Recalling (2.33), we have

$$\begin{aligned}
&\mathbf{S}_p(p^e,\chi;t,n)\\
&= \delta_{\left(\frac{d}{p}\right)=1}\left(\chi\left(\frac{t+\sqrt{d}\ell}{2}\right)+\chi\left(\frac{t-\sqrt{d}\ell}{2}\right)\right)\left\{1+\sum_{\beta\in\{0,1\}}p^{-1-\beta}(p-1)\sum_{k-1}^{\left\lfloor\frac{\min\{e+\beta+f-s,e-1+\beta,2f\}}{2}\right\rfloor}p^{2k}\right\}\\
&\quad+\chi\left(\frac{t+\delta_{2\nmid t}p^s}{2}\right)\left(p-\left(\frac{d}{p}\right)\right)\sum_{\beta\in\{0,1\}}p^{\left\lfloor\frac{e+\beta}{2}\right\rfloor-\beta-1}\sum_{k=\left\lceil\frac{e+\beta-\mathrm{ord}_p(d)}{2}\right\rceil}^{f+\left\lceil\frac{e+\beta}{2}\right\rceil-\left\lceil\frac{h+\beta}{2}\right\rceil}p^k.
\end{aligned}$$

Since

$$\begin{aligned}
&1+\sum_{\beta\in\{0,1\}}p^{-1-\beta}(p-1)\sum_{k-1}^{\left\lfloor\frac{\min\{e+\beta+f-s,e-1+\beta,2f\}}{2}\right\rfloor}p^{2k}\\
&\qquad=p^{\min\{e+f-s,e-1,2f\}}=\begin{cases}p^{e-1} & \text{if } g\geqslant h,\\ p^{\min\{e+f-s,2f\}} & \text{if } g\leqslant h-1\end{cases}
\end{aligned}\tag{2.43}$$

and

$$\begin{aligned}
&\sum_{\beta\in\{0,1\}}p^{\left\lfloor\frac{e+\beta}{2}\right\rfloor-\beta-1}\sum_{k=\left\lceil\frac{e+\beta-\mathrm{ord}_p(d)}{2}\right\rceil}^{f+\left\lceil\frac{e+\beta}{2}\right\rceil-\left\lceil\frac{h+\beta}{2}\right\rceil}p^k\\
&\quad=p^{e-1}\frac{p^{\max\{0,\left\lfloor\frac{g}{2}\right\rfloor-\left\lceil\frac{h}{2}\right\rceil+1\}}-1}{p-1}+p^{e-1-\mathrm{ord}_p(d)}\frac{p^{\max\{0,\left\lceil\frac{g}{2}\right\rceil-\left\lfloor\frac{h}{2}\right\rfloor\}}-1}{p-1}\\
&\quad=\begin{cases}p^{e-1}\frac{p^{\left\lfloor\frac{g}{2}\right\rfloor-\left\lceil\frac{h}{2}\right\rceil+1}-1}{p-1}+p^{e-1-\mathrm{ord}_p(d)}\frac{p^{\left\lceil\frac{g}{2}\right\rceil-\left\lfloor\frac{h}{2}\right\rfloor}-1}{p-1} & \text{if } g\geqslant h,\\ 0 & \text{if } g\leqslant h-1,\end{cases}
\end{aligned}$$

we get (2.41).

Now assume $p=2$. If t is odd, then $d\equiv 5 \bmod 8$; thus $\left(\frac{d}{2}\right)=-1$, and $\Omega(p^\alpha;n,t)=\emptyset$ for all $\alpha\geqslant 1$ by Lemma 2.6; therefore all sums in (2.33) are empty, and so $\mathbf{S}_2(2^e,\chi;t,n)=0$. From now on we assume t is even. Applying Lemma 2.7, we get

$$\begin{aligned}
&\sum_{\xi\in\Omega(2^{e+2(f-k)+\beta};n,t)/2^{e+f-k}}\chi(\xi)\\
&=\begin{cases}
2^{\left\lfloor\frac{e+\beta}{2}\right\rfloor-\beta}\chi\left(\frac{t}{2}\right) & \text{if } \left\lceil\frac{e+\beta-\delta_{4\mid d}}{2}\right\rceil\leqslant k\leqslant f+\left\lceil\frac{e+\beta}{2}\right\rceil-\left\lceil\frac{h+\beta}{2}\right\rceil,\\
& \text{and } \left[2\nmid g \text{ or } 2\mid g \text{ and } \tfrac{g}{2}-1\right],\\
2^{\left\lfloor\frac{e+\beta}{2}\right\rfloor-\beta}\chi\left(\frac{t}{2}+2^{s-1}\right) & \text{if } g=2s, k=\left\lceil\frac{e+\beta}{2}\right\rceil\leqslant f \text{ and } 2\nmid d,\\
& \text{or } g=2s, 2\nmid e+\beta-1, k=\frac{e+\beta-1}{2}\leqslant f \text{ and } 4\mid d,\\
2^{k-\beta}\left(\chi\left(\frac{t+\sqrt{d}\ell}{2}\right)+\chi\left(\frac{t-\sqrt{d}\ell}{2}\right)\right) & \text{if } 2k\leqslant\min\{e+\beta+f-s,e+\beta-1,2f\} \text{ and } \left(\frac{d}{2}\right)=1,\\
0 & \text{otherwise.}
\end{cases}
\end{aligned}$$

Note that if $2 \nmid d$, then $d \equiv_4 1$. So for $s = 2$ and $(\frac{d}{2}) = 1$, $\sqrt{d} \pmod 4$ is well defined. Furthermore, for $2 \nmid d$, since $4 \mid d\ell^2$, so $f = \operatorname{ord}_p(\ell) \geqslant 1$. Also note that since $(x + 2^{s-1})^2 \equiv x^2 \pmod{2^s}$ for $s \geqslant 2$, we have $\chi(x + 2^{s-1}) = \pm\chi(x)$. Because χ is primitive, we get

$$\chi(x + 2^{s-1}) = -\chi(x).$$

By (2.33),

$$\begin{aligned}
&\mathbf{S}_2(2^e, \chi; t, n)\\
&= \delta_{(\frac{d}{2})=1}\left(\chi\left(\frac{t+\sqrt{d}\ell}{2}\right) + \chi\left(\frac{t-\sqrt{d}\ell}{2}\right)\right)\left\{1 + \sum_{\beta\in\{0,1\}} \sum_{k=1}^{\lfloor \frac{\min\{e+\beta+f-s,\, e+\beta-1,\, 2f\}}{2} \rfloor} 2^{2k-1-\beta}\right\}\\
&\quad + \delta_{\substack{2\nmid g \\ \text{or } 2\mid g \text{ and } s\leqslant \frac{g}{2}-1}}\left(2 - \left(\frac{d}{2}\right)\right)\chi\left(\frac{t}{2}\right) \sum_{\beta\in\{0,1\}} 2^{\lfloor \frac{e+\beta}{2}\rfloor - \beta} \sum_{k=\left\lceil \frac{e+\beta-\delta_{4|d}}{2}\right\rceil}^{f+\lceil \frac{e+\beta}{2}\rceil - \lceil \frac{h+\beta}{2}\rceil} 2^{k-1}\\
&\quad - \delta_{g=2s}\chi\left(\frac{t}{2}\right)\left(2 - \left(\frac{d}{2}\right)\right) \sum_{\beta\in\{0,1\}} 2^{\lfloor \frac{e+\beta}{2}\rfloor - \beta} \begin{cases} 2^{\lceil \frac{e+\beta}{2}\rceil - 1} & \text{if } \left\lceil \dfrac{e+\beta}{2}\right\rceil \leqslant f \text{ and } 2 \nmid d,\\ 2^{\frac{e+\beta-1}{2}-1} & \text{if } \dfrac{e+\beta-1}{2} \leqslant f \text{ and } 4 \mid d,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Applying (2.43) and

$$\begin{aligned}
&\sum_{\beta\in\{0,1\}} 2^{\lfloor \frac{e+\beta}{2}\rfloor - \beta - 1} \sum_{k=\left\lceil \frac{e+\beta-\delta_{4|d}}{2}\right\rceil}^{f+\lceil \frac{e+\beta}{2}\rceil - \lceil \frac{h+\beta}{2}\rceil} 2^k\\
&= \begin{cases} 2^{e-1-\delta_{4|d}}\left(2^{\lfloor \frac{g}{2}\rfloor - \lfloor \frac{h-1}{2}\rfloor} + 2^{\lfloor \frac{g}{2}\rfloor - \lfloor \frac{h}{2}\rfloor} - 2 - \delta_{4|d}\right) & \text{if } g \geqslant h + \delta_{4|d},\\ 0 & \text{otherwise,}\end{cases}
\end{aligned}$$

we get (2.42). □

2.5. *Cusps*

One easily verifies that every cusp of $\Gamma^+ = \Gamma_0(N)$ is equivalent to a cusp of the form

$$\frac{a}{c},\ a, c \in \mathbb{Z},\ c \geqslant 1,\ c \mid N \text{ and } \gcd(c, a) = 1. \tag{2.44}$$

Any two such cusps $\frac{a_1}{c_1}$ and $\frac{a_2}{c_2}$ are Γ^+-equivalent if and only if

$$c_1 = c_2 \text{ and } a_1 \equiv_{\gcd(c_1, N/c_1)} a_2$$

and we then write $\frac{a_1}{c_1} \sim_{\Gamma^+} \frac{a_2}{c_2}$. We fix, once and for all, a set C_Γ containing exactly one representative $\frac{a}{c}$ satisfying (2.44) for each cusp class, that is,

$$C_\Gamma = \left\{\tfrac{a}{c} \ :\ a, c \in \mathbb{Z},\ c \geqslant 1,\ c \mid N,\ \gcd(a, c) = 1\right\} / \sim_{\Gamma^+}.$$

We make our choice of C_Γ in such a way that

$$\begin{cases} \text{for every } \dfrac{a}{c} \in C_\Gamma \text{ with } \gcd(c, N/c) > 2, \text{ also } \dfrac{-a}{c} \in C_\Gamma; \\ \text{for every } c \mid N \text{ with } \gcd(c, N/c) \leqslant 2, \ \dfrac{1}{c} \in C_\Gamma. \end{cases} \tag{2.45}$$

This is possible since $\frac{a}{c} \not\sim_{\Gamma^+} \frac{-a}{c}$ whenever $\gcd(c, N/c) > 2$. Note also that it follows from our choice of C_Γ that for every $c \mid N$ with $\gcd(c, N/c) \leqslant 2$, $\frac{1}{c}$ is the unique cusp in C_Γ with denominator c.

For each $\frac{a}{c} \in C_\Gamma$, we fix a choice of scaling matrix

$$W_{a/c} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}).$$

We make these choices so that

$$W_{-a/c} = \begin{pmatrix} -a & b \\ c & -d \end{pmatrix} \text{ for } \gcd(c, N/c) > 2 \text{ and } W_{1/c} = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \text{ for } \gcd(c, N/c) \leqslant 2$$

holds for any $\frac{a}{c} \in C_\Gamma$ (cf. (2.45)). Note that $W_{a/c}(\infty) = \frac{a}{c}$, so $W_{a/c}^{-1}(\frac{a}{c}) = \infty$. We also set

$$\mathsf{N}_{a/c} = \begin{pmatrix} \sqrt{\dfrac{\gcd(c^2, N)}{N}} & 0 \\ 0 & \sqrt{\dfrac{N}{\gcd(c^2, N)}} \end{pmatrix} W_{a/c}^{-1} \tag{2.46}$$

and

$$T_{a/c} = \mathsf{N}_{a/c}^{-1} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \mathsf{N}_{a/c} \in \Gamma^+. \tag{2.47}$$

We then verify that the fixator subgroup of $\frac{a}{c}$ in Γ^+ is the cyclic subgroup generated by $T_{a/c}$. That is, for the cusp $\eta_j = \frac{a}{c}$, $\mathsf{N}_{a/c}$ and $T_{a/c}$ correspond to N_j and T_j in § 2.1.

LEMMA 2.9. *For any* $\frac{a}{c} \in C_\Gamma$,

$$T_{a/c} = \begin{pmatrix} 1 + ac\dfrac{N}{\gcd(c^2, N)} & -a^2 \frac{N}{\gcd(c^2, N)} \\ N\dfrac{c^2}{\gcd(c^2, N)} & 1 - ac\frac{N}{\gcd(c^2, N)} \end{pmatrix}$$

and

$$\chi(T_{a/c}) = \chi\biggl(1 - ac\frac{N}{\gcd(c^2, N)}\biggr). \tag{2.48}$$

Proof. This lemma follows from the definition of $T_{a/c}$ in (2.47). □

For later use, we prove the following lemma and then compute the set of open cusps. From now on, for each prime $p \mid N$, set $e_p = \mathrm{ord}_p(N)$ and $s_p = \mathrm{ord}_p(q)$, where $q = \mathrm{cond}(\chi)$.

LEMMA 2.10. *For every* $x \in \mathbb{R}$,

$$\sum_{c \mid N, q \mid \frac{N}{\gcd(c, N/c)}} \varphi(\gcd(c, N/c))(\gcd(c, N/c))^x$$

$$= \prod_{p \mid N} \left[2 + (p-1)p^x \frac{p^{\min\{\lfloor \frac{e_p}{2} \rfloor, e_p - s_p\}(x+1)} + p^{\min\{\lfloor \frac{e_p - 1}{2} \rfloor, e_p - s_p\}(x+1)} - 2}{p^{x+1} - 1} \right]. \tag{2.49}$$

Proof. By the multiplicativity of the Euler φ-function, the left-hand side of (2.49) becomes a product over primes dividing N:

$$\prod_{p|N}\left(\sum_{\substack{j\geqslant 0,\\ e_p-s_p\geqslant \min\{j,e_p-j\}}}\varphi(p^{\min\{j,e_p-j\}})p^{x\min\{j,e_p-j\}}\right).$$

For each prime $p \mid N$, the inner sum is

$$\begin{aligned}&\sum_{\substack{j\geqslant 0,\\ e_p-s_p\geqslant \min\{j,e_p-j\}}}\varphi(p^{\min\{j,e_p-j\}})p^{x\min\{j,e_p-j\}}\\ &= \sum_{j=0}^{\min\{\lfloor \frac{e_p}{2}\rfloor, e_p-s_p\}}\varphi(p^j)p^{jx} + \sum_{j=0}^{\min\{\lfloor \frac{e_p-1}{2}\rfloor, e_p-s_p\}}\varphi(p^j)p^{jx}\\ &= 2+(p-1)p^x\frac{p^{\min\{\lfloor \frac{e_p}{2}\rfloor, e_p-s_p\}(x+1)}-1}{p^{x+1}-1} + (p-1)p^x\frac{p^{\min\{\lfloor \frac{e_p-1}{2}\rfloor, e_p-s_p\}(x+1)}-1}{p^{x+1}-1}. \quad (2.50)\end{aligned}$$

□

LEMMA 2.11 (cf. [17, Lemma 13.5]). *The set of open cusps is*

$$C_{\Gamma,\chi} = \left\{\frac{a}{c}\in C_\Gamma \ : \ q \mid \frac{N}{\gcd(c,N/c)}\right\}. \quad (2.51)$$

Its cardinality is

$$|C_{\Gamma,\chi}| = \sum_{c|N, q|\frac{N}{\gcd(c,N/c)}}\varphi(\gcd(c,N/c)) = \Psi_1(N,q).$$

Here $q=\mathrm{cond}(\chi)$ *and* $\Psi_1(N,q)$ *is given in* (2.17).

Proof. Using (2.48) and $\mathrm{cond}(\chi)=q$, we have

$$\chi(T_{a/c}) = \chi\left(1-a\frac{N}{\gcd(c,N/c)}\right) = 1 \iff q \mid \frac{N}{\gcd(c,N/c)}.$$

By Lemma 2.10, taking $x=0$, we get

$$\begin{aligned}|C_{\Gamma,\chi}| &= \sum_{c|N, q|\frac{N}{\gcd(c,N/c)}}\varphi(\gcd(c,N/c))\\ &= \prod_{p|N}\left[p^{\min\{\lfloor \frac{e_p}{s}\rfloor, e_p-s_p\}} + p^{\min\{\lfloor \frac{e_p-1}{2}\rfloor, e_p-s_p\}}\right] = \Psi_1(N,q).\end{aligned}$$

□

Adapting the notation from § 2.1 to our present explicit setting, for each $\frac{a}{c}\in C_\Gamma$ we write $k(a/c)$ for the representative in C_Γ for the cusp $V\frac{a}{c} = \frac{-a}{c}$. Then by (2.45),

$$k(a/c) = \begin{cases}\dfrac{-a}{c} & \text{if } \gcd(c,N/c)>2,\\[2ex] \dfrac{1}{c} & \text{if } \gcd(c,N/c)\leqslant 2.\end{cases} \quad (2.52)$$

Furthermore, as in (2.6) we have fixed a choice of an element $U_{a/c} \in \Gamma^+$ satisfying

$$U_{a/c} = V\mathsf{N}_{a/c}^{-1} \begin{pmatrix} -1 & u \\ 0 & 1 \end{pmatrix} \mathsf{N}_{k(a/c)} \tag{2.53}$$

for some $u \in \mathbb{R}$ (which may depend on $\frac{a}{c}$).

LEMMA 2.12. *For $\frac{a}{c} \in C_\Gamma$, when $\gcd(c, N/c) > 2$, we have*

$$U_{a/c} = V\mathsf{N}_{a/c}^{-1} \begin{pmatrix} -1 & u \\ 0 & 1 \end{pmatrix} \mathsf{N}_{-a/c} = \begin{pmatrix} -1 + au\dfrac{N}{\gcd(c, N/c)} & a^2u\dfrac{N/c}{\gcd(c, N/c)} \\ -Nu\dfrac{c}{\gcd(c, N/c)} & -1 - au\dfrac{N}{\gcd(c, N/c)} \end{pmatrix},$$

for some $u \in \mathbb{Z}$. When $\gcd(c, N/c) \leqslant 2$,

$$U_{1/c} = V\mathsf{N}_{1/c}^{-1} \begin{pmatrix} -1 & u_1\dfrac{\gcd(c, N/c)}{N/c} \\ 0 & 1 \end{pmatrix} \mathsf{N}_{1/c} \tag{2.54}$$

for some $u_1 \in \mathbb{Z}$ satisfying $cu_1 \equiv_{N/c} -2$. Furthermore, for every $\frac{a}{c} \in C_{\Gamma,\chi}$,

$$\chi(U_{a/c}) = \chi_{\alpha(N,c)}(-1), \tag{2.55}$$

where

$$\alpha(N, c) = \begin{cases} N & \text{if } \gcd(c, N/c) > 2, \\ \dfrac{N}{c} & \text{if } \gcd(c, N/c) = 1, \\ \dfrac{N}{2c} & \text{if } \gcd(c, N/c) = 2 \text{ and } \dfrac{N}{2c} \text{ is odd}, \\ \dfrac{2N}{c} & \text{if } \gcd(c, N/c) = 2 \text{ and } \dfrac{N}{2c} \text{ is even}. \end{cases} \tag{2.56}$$

Proof. Given $\frac{a}{c} \in C_\Gamma$, recalling (2.53),

$$U_{a/c} = V\mathsf{N}_{a/c}^{-1} \begin{pmatrix} -1 & u \\ 0 & 1 \end{pmatrix} \mathsf{N}_{k(a/c)} \in \Gamma^+,$$

for some $u \in \mathbb{R}$, which is uniquely determined modulo one. If $\gcd(c, N/c) > 2$, then $k(a/c) = -a/c$ and we get

$$U_{a/c} = \begin{pmatrix} -1 + acu\dfrac{N}{\gcd(c^2, N)} & a^2u\dfrac{N}{\gcd(c^2, N)} \\ -Nu\dfrac{c^2}{\gcd(c^2, N)} & -1 - acu\dfrac{N}{\gcd(c^2, N)} \end{pmatrix}.$$

One easily checks that $U_{a/c} \in \Gamma^+ = \Gamma_0(N)$ if and only if $u \in \mathbb{Z}$. If $\frac{a}{c} \in C_{\Gamma,\chi}$, then $q \mid \frac{N}{\gcd(c,N/c)}$. Because

$$-1 - acu\frac{N}{\gcd(c^2, N)} = -1 - au\frac{N}{\gcd(c, N/c)} \equiv_q -1$$

and χ is even, we get $\chi(U_{a/c}) = \chi(-1) = 1$.

If $\gcd(c, N/c) \leqslant 2$, then

$$U_{1/c} = \begin{pmatrix} 1 + cu\dfrac{N}{\gcd(c^2,N)} & -u\dfrac{N}{\gcd(c^2,N)} \\ -2c - c^2u\dfrac{N}{\gcd(c^2,N)} & 1 + cu\dfrac{N}{\gcd(c^2,N)} \end{pmatrix}.$$

Let $u_1 = u\frac{N}{\gcd(c^2,N)}$. Then $U_{1/c} \in \Gamma_0(N)$ if and only if $u_1 \in \mathbb{Z}$ and satisfying $cu_1 \equiv_{N/c} -2$. Then $1 + cu_1 \equiv_{N/c} -1$. Moreover,

$$\chi(U_{1/c}) = \chi(1 + cu_1).$$

If $\gcd(c, N/c) = 1$, then $\chi = \chi_c\chi_{N/c}$, so

$$\chi(U_{1/c}) = \chi_{N/c}(1 + cu_1) = \chi_{N/c}(-1).$$

Assume that $\gcd(c, N/c) = 2$ and $\frac{1}{c} \in C_{\Gamma,\chi}$, so that $q \mid \frac{N}{2}$. If $\frac{N}{2c}$ is odd, then $\gcd(2c, \frac{N}{2c}) = 1$ and $\chi = \chi_{2c}\chi_{\frac{N}{2c}}$. Since $q = \operatorname{cond}(\chi_{2c})\operatorname{cond}(\chi_{\frac{N}{2c}})$ divides $\frac{N}{2}$ and $\operatorname{cond}(\chi_{\frac{N}{2c}}) \mid \frac{N}{2c}$, we have $\operatorname{cond}(\chi_{2c}) \mid c$. Thus

$$\chi(U_{1/c}) = \chi_{2c}(1 + cu_1)\chi_{\frac{N}{2c}}(1 + cu_1) = \chi_{\frac{N}{2c}}(-1).$$

If $\frac{N}{2c}$ is even, then $\frac{c}{2}$ is odd, so $\gcd(\frac{2N}{c}, \frac{c}{2}) = 1$ and $\chi = \chi_{\frac{2N}{c}}\chi_{\frac{c}{2}}$. Since $q = \operatorname{cond}(\chi_{\frac{2N}{c}})\operatorname{cond}(\chi_{\frac{c}{2}})$ divides $\frac{N}{2}$, $\operatorname{cond}(\chi_{\frac{2N}{c}})$ must divide $\frac{N}{c}$, so we get

$$\chi(U_{1/c}) = \chi_{\frac{c}{2}}(1 + cu_1)\chi_{\frac{2N}{c}}(1 + cu_1) = \chi_{\frac{2N}{c}}(-1). \qquad \square$$

Note that in our present setting, (2.7) says that for each $c \mid N$ with $\gcd(c, N/c) \leqslant 2$, and each $v \in \mathbb{Z}$, we set

$$T_{1/c,v} = U_{1/c}^{-1} V T_{1/c}^{v}. \tag{2.57}$$

Also, for any such c and v, we have a number $\mathbf{c}_{1/c,v} > 0$ defined by (2.8). The following two lemmas evaluate $\chi^{(\varepsilon)}(T_{1/c,v})$ and $\mathbf{c}_{1/c,v}$. Recall that $\chi^{(\varepsilon)}(V) = (-1)^{\varepsilon}$.

LEMMA 2.13. *For $c \mid N$ and $\gcd(c, N/c) \leqslant 2$, we have*

$$\chi^{(\varepsilon)}(T_{1/c,v}) = \begin{cases} (-1)^{\varepsilon}\chi_{\alpha(N,c)}(-1) & \text{if } \dfrac{1}{c} \in C_{\Gamma,\chi}, \\ (-1)^{\varepsilon+v+w_c} & \text{if } \dfrac{1}{c} \notin C_{\Gamma,\chi}, \end{cases} \tag{2.58}$$

where $w_c \in \{0, 1\}$ is a constant determined by $(-1)^{w_c} = \chi(U_{1/c})$. Here $\alpha(N, c)$ as in (2.56).

Proof. It follows from (2.57) that $\chi^{(\varepsilon)}(T_{1/c,v}) = (-1)^{\varepsilon}\overline{\chi(U_{1/c})}\chi(T_{1/c})^v$.

If $\frac{1}{c} \in C_{\Gamma,\chi}$, then $\chi(T_{1/c}) = 1$, and using (2.55) we have

$$\chi^{(\varepsilon)}(T_{1/c,v}) = (-1)^{\varepsilon}\chi_{\alpha(N,c)}(-1),$$

for all $v \in \mathbb{Z}$.

Now assume $\frac{1}{c} \notin C_{\Gamma,\chi}$. Then $\gcd(c, N/c) = 2$ and $q \nmid \frac{N}{2}$, and by Lemma 2.9, we have

$$\chi(T_{1/c}) = \chi\big(1 - \tfrac{N}{2}\big) = -1,$$

since $(1 - \frac{N}{2})^2 \equiv_N 1$ and $q \nmid \frac{N}{2}$. By Lemma 2.12, $\chi(U_{1/c}) = \chi(1 + cu_1)$ for some $u_1 \in \mathbb{Z}$ satisfying $cu_1 \equiv_{N/c} -2$. Then

$$(1 + cu_1)^2 = 1 + 2cu_1\big(1 + \tfrac{c}{2}u_1\big) \equiv_N 1,$$

so $\chi(U_{1/c})^2 = 1$ and $\chi(U_{1/c}) \in \{\pm 1\}$. Choosing $w_c \in \{0,1\}$ so that $\chi(U_{1/c}) = (-1)^{w_c}$, we get

$$\chi^{(\varepsilon)}(T_{1/c,v}) = (-1)^{\varepsilon+v+w_c}. \qquad \square$$

LEMMA 2.14. *Assume that $c \mid N$ and $\gcd(c, N/c) \leqslant 2$. If $4 \mid N$, then $\mathbf{c}_{1/c,v} = \sqrt{N}$ for all v. On the other hand, if $4 \nmid N$, then there is some $s \in \{0,1\}$ (which depends on c and $U_{1/c}$) such that for all v,*

$$\mathbf{c}_{1/c,v} = \begin{cases} \sqrt{N} & \text{if } v \equiv_2 s, \\ \sqrt{2N} & \text{if } v \not\equiv_2 s \text{ and } 2 \mid N \\ 2\sqrt{N} & \text{if } v \not\equiv_2 s \text{ and } 2 \nmid N. \end{cases} \tag{2.59}$$

Proof. For given c and v, following the definition of $\mathbf{c}_{1/c,v}$ in (2.8), we recall that $T_{1/c,v}$ is a reflection fixing the point $\frac{1}{c}$, and that also the other fixpoint of $T_{1/c,v}$ in $\partial\mathbb{H}$ must be a Γ^+-cusp, which we call η. We choose $V_2 \in \Gamma^+$ and $\frac{a'}{c'} \in C_\Gamma$ so that $\eta = V_2(\frac{a'}{c'})$. Then $\frac{a'}{c'}$ is a fixpoint of $V_2^{-1} T_{\frac{1}{c},v} V_2 \in \Gamma^+ V$, and this forces $k(\frac{a'}{c'}) = \frac{a'}{c'}$, that is, $\gcd(c', N/c') \leqslant 2$ and $a' = 1$.

Set $M = \gcd(c, N/c) \leqslant 2$. From (2.54) and (2.57), we get

$$\mathsf{N}_{1/c} T_{1/c,v} \mathsf{N}_{1/c}^{-1} = \begin{pmatrix} -1 & u_1 \dfrac{M}{N/c} + v \\ 0 & 1 \end{pmatrix}, \tag{2.60}$$

where u_1 is an integer satisfying $cu_1 \equiv_{N/c} -2$; and the two fixpoints in $\partial\mathbb{H}$ of the reflection in (2.60) are ∞ and $\frac{1}{2}(u_1 \frac{M}{N/c} + v)$. Hence,

$$\eta = \mathsf{N}_{1/c}^{-1}\left(\frac{1}{2}\left(u_1 \frac{M}{N/c} + v\right)\right) = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \frac{u_2}{2} = \frac{u_2}{cu_2 + 2}, \qquad \text{with } u_2 := u_1 + v\frac{N/c}{M}.$$

But we know from above that the C_Γ-representative for the cusp η is $\frac{1}{c'}$, and one easily verifies that the C_Γ-representative of an arbitrary Γ^+-cusp $\frac{\alpha}{\gamma}$ with $\alpha, \gamma \in \mathbb{Z}$ has denominator $\gcd\big(N, \frac{\gamma}{\gcd(\alpha,\gamma)}\big)$. Hence, letting $u_3 := \gcd(u_2, cu_2 + 2) = \gcd(u_2, 2) \in \{1, 2\}$, we have

$$c' = \gcd\left(N, \frac{cu_2 + 2}{u_3}\right) = \frac{1}{u_3} \gcd\left(Nu_3, cu_2 + 2\right) = \frac{N}{cu_3} \gcd\left(cu_3, \frac{cu_2 + 2}{N/c}\right).$$

In the last equality, we used the fact that $\frac{N}{c} \mid cu_2 + 2$, since $cu_1 \equiv_{N/c} -2$. To make the last expression more explicit, note that $\gcd(cu_3, \frac{cu_2+2}{N/c}) \in \{1, 2\}$, since $\gcd(cu_3, cu_2 + 2) = \gcd(cu_3, 2) \in \{1, 2\}$. Therefore,

$$c' = \begin{cases} N/c & \text{if } u_2 \equiv \dfrac{cu_2 + 2}{N/c} \pmod 2, \\ N/(2c) & \text{if } 2 \mid u_2 \text{ and } 2 \nmid \dfrac{cu_2 + 2}{N/c}, \\ 2N/c & \text{if } 2 \nmid u_2 \text{ and } 2 \mid \dfrac{cu_2 + 2}{N/c}. \end{cases} \tag{2.61}$$

Let us write $W_{1/c} V_2 W_{1/c'} = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $\gamma \geqslant 0$. Since $\frac{\alpha}{\gamma} = W_{1/c} V_2 W_{1/c'}(\infty) = \frac{u_2}{2}$, we get $\gamma = \frac{2}{u_3}$. Using this in the definition of $\mathbf{c}_{1/c,v}$, (2.8), we get

$$\mathbf{c}_{1/c,v} = \frac{2}{u_3} \sqrt{\frac{N/c}{\gcd(c, N/c)}} \sqrt{\frac{N/c'}{\gcd(c', N/c')}}. \tag{2.62}$$

Let us set $f = \mathrm{ord}_2(c)$ and $g = \mathrm{ord}_2(N/c)$; then $\gcd(c, N/c) \leqslant 2$ implies $\min(f,g) \leqslant 1$. Using (2.61), the formula (2.62) can be re-expressed as $\mathbf{c}_{1/c,v} = \sqrt{N} \cdot 2^{\rho/2}$, where

$$\rho = \begin{cases} -2\min(f,g) & \text{if } 2 \mid u_2 \text{ and } 2 \mid \dfrac{cu_2+2}{N/c}, \\ 1 - \min(f,g) - \min(f+1, g-1) & \text{if } 2 \mid u_2 \text{ and } 2 \nmid \dfrac{cu_2+2}{N/c}, \\ 1 - \min(f,g) - \min(f-1, g+1) & \text{if } 2 \nmid u_2 \text{ and } 2 \mid \dfrac{cu_2+2}{N/c}, \\ 2 - 2\min(f,g) & \text{if } 2 \nmid u_2 \text{ and } 2 \nmid \dfrac{cu_2+2}{N/c}. \end{cases} \tag{2.63}$$

Let us first assume $\min(f,g) = 1$, that is, both c and $\frac{N}{c}$ are even. In this case, if either $4 \mid c$ or $2 \mid u_2$, then $cu_2 + 2 \equiv_4 2$, which implies $2 \nmid \frac{cu_2+2}{N/c}$ and $4 \nmid \frac{N}{c}$. Hence, the first case in (2.63) cannot occur, and if $f \geqslant 2$ ($\Rightarrow g = 1$), then also the third case in (2.63) is excluded, while if $g \geqslant 2$ ($\Rightarrow f = 1$), then the second case in (2.63) is excluded. By inspection, it then follows that $\rho = 0$. Next assume $\min(f,g) = 0$. Note that if $f \geqslant 1$ ($\Rightarrow g = 0$), then $2 \mid \frac{cu_2+2}{N/c}$, and on the other hand if if $g \geqslant 1$ ($\Rightarrow f = 0$), then the condition $cu_1 \equiv_{N/c} -2$ forces u_1 to be even, and then also $u_2 = u_1 + v\frac{N/c}{M}$ is even. If $f \geqslant 2$ or $g \geqslant 2$, then these observations imply $\rho = 0$, by (2.63). We have thus proved the lemma in the case $4 \mid N$, and it only remains to consider the three cases with $f, g \leqslant 1$, not both 1. If $f = 1$ ($\Rightarrow g = 0$), then (2.63) implies $\rho = \delta_{2\nmid u_2}$, and so, recalling $u_2 = u_1 + v\frac{N/c}{M}$, it follows that (2.59) holds with $s \equiv_2 u_1$. If $g = 1$ ($\Rightarrow f = 0$), then (2.63) implies $\rho = \delta_{2\nmid \frac{cu_2+2}{N/c}} = \delta_{4\mid u_2}$, and so (2.59) holds with $s \equiv_2 1 + \frac{u_1}{2}$. Finally if $f = g = 0$, that is, $2 \nmid c$ and $2 \nmid \frac{N}{c}$, then $u_2 \equiv_2 \frac{cu_2+2}{N/c}$, and thus (2.63) implies $\rho = 2\delta_{2\nmid u_2}$; therefore (2.59) again holds with $s \equiv_2 u_1$. □

2.6. *Cuspidal contributions* $\mathrm{C}(\Gamma, \chi)$

Recall that we write $\mathrm{C}(N, \chi; n)$ for the cuspidal contribution in the trace formula in Theorem 2.1; thus $\mathrm{C}(N, \chi; n) = \mathrm{C}(\Gamma, \chi^{(0)}) + n\,\mathrm{C}(\Gamma, \chi^{(1)})$ where $C(\Gamma, \chi^{(\varepsilon)})$ is the cuspidal contribution (cf. (2.4)) in the trace formula for $(\Gamma_0^{\pm}(N), \chi^{(\varepsilon)})$. Our aim in this section is to prove the following proposition, giving an explicit formula for $\mathrm{C}(N, \chi; n)$. Let

$$\varepsilon_N = \begin{cases} 1 & \text{if } N \text{ is odd}, \\ \dfrac{1}{2} & \text{if } 2 \parallel N, \\ 0 & \text{otherwise}, \end{cases} \tag{2.64}$$

and for each prime $p \mid N$, define

$$\Psi_4(p^{e_p}, p^{s_p}) = p^{e_p - s_p} \max\{2s_p - e_p - 1, 0\}. \tag{2.65}$$

Proposition 2.15. *We have*

$$\begin{aligned} \mathrm{C}(N, \chi; 1) = \Psi_1(N,q)\Big\{ \frac{1}{4}h(0) - \frac{1}{2\pi}\int_{\mathbb{R}} h(r)\frac{\Gamma'}{\Gamma}(1+ir)\,dr \Big\} \\ - \Big\{ \Psi_1(N,q) \sum_{p \mid N} \frac{\Psi_4(p^{e_p}, p^{s_p})}{\Psi_1(p^{e_p}, p^{s_p})} \log p + \Psi_1(N,q)\log 2 \Big\} g(0) \end{aligned} \tag{2.66}$$

and

$$\mathrm{C}(N,\chi;-1) = I_\chi 2^{\omega(N)}\Omega_1(N,q)\left\{\frac{1}{4}h(0)+\frac{1}{2\pi}\int_{\mathbb{R}} h(r)\left(\frac{\Gamma'}{\Gamma}\left(\frac{1}{2}+ir\right)-\frac{\Gamma'}{\Gamma}(1+ir)\right)dr\right\}$$
$$+\left\{I_\chi 2^{\omega(N)-1}(\Omega_1(N,q)\log N+\varepsilon_N\log 2)\right\}g(0). \tag{2.67}$$

Here Ψ_1, Ψ_4, Ω_1 and ε_N are given in (2.17), (2.65), (2.21) *and* (2.64), *respectively.*

The proposition will be proved by evaluating the various sums appearing in (2.4) in our explicit setting. By (2.51), we can write

$$\sum_{j\in C_{\Gamma,\chi},k(j)=j}\chi(T_{j,0}) = \sum_{c|N,\gcd(c,N/c)\leqslant 2,q|\frac{N}{\gcd(c,N/c)}}\chi^{(\varepsilon)}(T_{1/c,0}); \tag{2.68}$$

$$\sum_{1\leqslant j\leqslant\kappa,k(j)=j}\sum_{v\in\{0,1\}}\chi(T_{j,v})\log\mathbf{c}_{j,v} = \sum_{c|N,\gcd(c,N/c)\leqslant 2}\sum_{v\in\{0,1\}}\chi^{(\varepsilon)}(T_{1/c,v})\log\mathbf{c}_{1/c,v}; \tag{2.69}$$

$$\sum_{\substack{1\leqslant j\leqslant\kappa,\\ j\notin C_{\Gamma,\chi}}}\log|1-\chi(T_j)| = \sum_{\substack{\frac{a}{c}\in C_\Gamma,\\ q\nmid\frac{N}{\gcd(c,N/c)}}}\log\left|1-\chi(T_{a/c})\right|. \tag{2.70}$$

We get formulas for each of these sums in the following lemmas.

Lemma 2.16.

$$\sum_{c|N,\gcd(c,N/c)\leqslant 2,q|\frac{N}{\gcd(c,N/c)}}\chi^{(\varepsilon)}(T_{1/c,0}) = (-1)^\varepsilon I_\chi 2^{\omega(N)}\Omega_1(N,q),$$

where $\Omega_1(N,q)$ is given in (2.21).

Proof. By Lemma 2.13 and the definition of $\alpha(N,c)$ in (2.56), we have

$$\sum_{c|N,\gcd(c,N/c)\leqslant 2,q|\frac{N}{\gcd(c,N/c)}}\chi^{(\varepsilon)}(T_{1/c,0}) = (-1)^\varepsilon\sum_{c|N,\gcd(c,N/c)\leqslant 2,q|\frac{N}{\gcd(c,N/c)}}\chi_{\alpha(N,c)}(-1)$$
$$= (-1)^\varepsilon\left\{\sum_{c|N,\gcd(c,N/c)=1}\chi_{N/c}(-1)+\delta_{4|N,q|\frac{N}{2}}\sum_{c|N,\gcd(c,N/c)=2}\left\{\delta_{2\nmid\frac{N}{2c}}\chi_{\frac{N}{2c}}(-1)+\delta_{2|\frac{N}{2c}}\chi_{\frac{2N}{c}}(-1)\right\}\right\}.$$

For $\gcd(c,N/c)=1$, we have

$$\sum_{c|N,\gcd(c,N/c)=1}\chi_{N/c}(-1) = \prod_{p|N}(1+\chi_p(-1)) = I_\chi 2^{\omega(N)}. \tag{2.71}$$

When $4\mid N$, $q\mid\frac{N}{2}$ and $\gcd(c,N/c)=2$, then either $\frac{N}{2c}$ is odd so $2^{e_2-1}\parallel c$ or $\frac{N}{2c}$ is even, $8\mid N$ and $2\parallel c$. For $\frac{N}{2c}$ odd, we have

$$\sum_{c|N,\gcd(c,N/c)=2,2\nmid\frac{N}{2c}}\chi_{\frac{N}{2c}}(-1) = \prod_{p|N,p>2}(1+\chi_p(-1)) = I_\chi 2^{\omega(N)-1}.$$

For $8\mid N$, $q\mid\frac{N}{2}$ and $\frac{N}{2c}$ even, we have

$$\sum_{c|N,\gcd(c,N/c)=2,2|\frac{N}{2c}}\chi_{\frac{2N}{c}}(-1) = \chi_2(-1)\prod_{p|N,p>2}(1+\chi_p(-1)) = I_\chi 2^{\omega(N)-1}.$$

Combining these, we obtain the formula stated in the lemma. □

LEMMA 2.17.

$$\sum_{c|N,\gcd(c,N/c)\leqslant 2}\sum_{v\in\{0,1\}}\chi^{(\varepsilon)}(T_{1/c,v})\log \mathbf{c}_{1/c,v}=(-1)^{\varepsilon}I_{\chi}2^{\omega(N)}(\Omega_1(N,q)\log N+\varepsilon_N\log 2).$$

Here $\Omega_1(N,q)$ and ε_N are given in (2.21) and (2.64), respectively.

Proof. Recalling (2.58) and (2.59), when $4\mid N$, we get

$$\begin{aligned}&\sum_{c|N,\gcd(c,N/c)\leqslant 2}\sum_{v\in\{0,1\}}\chi^{(\varepsilon)}(T_{1/c,v})\log \mathbf{c}_{1/c,v}\\&\quad=(-1)^{\varepsilon}\log N\left\{\sum_{c|N,\gcd(c,N/c)=1}\chi_{\alpha(N,c)}(-1)+\frac{1}{2}\sum_{c|N,\gcd(c,N/c)=2}\left\{\begin{array}{ll}2\chi_{\alpha(N,c)}(-1) & \text{if } q\mid \frac{N}{2},\\ \sum_{v\in\{0,1\}}(-1)^{v+w_c} & \text{if } q\nmid \frac{N}{2}\end{array}\right\}\right\}\\&\quad=(-1)^{\varepsilon}\log N\sum_{c|N,\gcd(c,N/c)\leqslant 2,q|\frac{N}{\gcd(c,N/c)}}\chi_{\alpha(N,c)}(-1)=(-1)^{\varepsilon}I_{\chi}2^{\omega(N)}\Omega_1(N,q)\log N,\end{aligned}$$

where the last equality holds by Lemma 2.16 (and its proof).

For $4\nmid N$, by (2.58) and (2.59), we have

$$\begin{aligned}&\sum_{c|N,\gcd(c,N/c)\leqslant 2}\sum_{v\in\{0,1\}}\chi^{(\varepsilon)}(T_{1/c,v})\log \mathbf{c}_{1/c,v}\\&\quad=(-1)^{\varepsilon}\sum_{c|N,\gcd(c,N/c)=1}\chi_{\alpha(N,c)}(-1)\left(\log N+\frac{1}{\gcd(2,N)}\log 2\right)\\&\quad=(-1)^{\varepsilon}I_{\chi}2^{\omega(N)}\Omega_1(N,q)\left(\log N+\frac{1}{\gcd(2,N)}\log 2\right),\end{aligned}$$

where we again used Lemma 2.16. The desired formula follows if we also note that $4\nmid N$ implies $\Omega_1(N,q)=1$. □

LEMMA 2.18.

$$\sum_{\substack{\frac{a}{c}\in C_\Gamma,\\ q\nmid\frac{N}{\gcd(c,N/c)}}}\log\left|1-\chi(T_{a/c})\right|=\Psi_1(N,q)\sum_{p|N}\frac{\Psi_4(p^{e_p},p^{s_p})}{\Psi_1(p^{e_p},p^{s_p})}\log p.$$

Here Ψ_1 and Ψ_4 are given in (2.17) and (2.65).

Proof. Recalling (2.48), and writing $M=\gcd(c,N/c)$, we have

$$\sum_{\substack{\frac{a}{c}\in C_\Gamma,\\ q\nmid\frac{N}{M}}}\log\left|1-\chi(T_{a/c})\right|=\sum_{c|N,q\nmid\frac{N}{M}}\sum_{a\in(\mathbb{Z}/M\mathbb{Z})^\times}\log\left|1-\chi\big(1-a\tfrac{N}{M}\big)\right|.\tag{2.72}$$

Now let c be fixed subject to the conditions $c\mid N$ and $q\nmid\frac{N}{M}$. Note that $N\mid(N/M)^2$, and therefore the map $a\mapsto 1-aN/M$ is a homomorphism from the additive group $\mathbb{Z}/M\mathbb{Z}$ to the multiplicative group $(\mathbb{Z}/N\mathbb{Z})^\times$. In particular, there is some $A\in\mathbb{Z}$ (uniquely determined modulo M) such that $\chi(1-aN/M)=e(aA/M)$ for all $a\in\mathbb{Z}$, and we have $M\nmid A$ since $q\nmid\frac{N}{M}$. It follows

that, with $B = M/\gcd(A, M) > 1$,

$$\sum_{a\in(\mathbb{Z}/M\mathbb{Z})^\times} \log\left|1-\chi\left(1-a\tfrac{N}{M}\right)\right| = \frac{\varphi(M)}{\varphi(B)} \log\left(\prod_{y\in(\mathbb{Z}/B\mathbb{Z})^\times} \left|1-e\left(\tfrac{y}{B}\right)\right|\right)$$
$$= \frac{\varphi(M)}{\varphi(B)} \begin{cases} \log p & \text{if } B = p^r \text{ for some prime } p \text{ and } r \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

(cf., for example, [**23**, § 46(7)]). Note that for $a \in \mathbb{Z}$ we have $\chi(1 - aN/M) = e(aA/M) = 1$ if and only if $B \mid a$; hence in fact

$$B = \frac{q}{\gcd(N/M, q)}. \tag{2.73}$$

For any prime $p \mid N$, let $f_p = \mathrm{ord}_p(c)$. It follows from (2.73) that $B = p^r$ holds for some prime p and some $r \geqslant 1$ if and only if $\min\{f_p, e_p - f_p\} > e_p - s_p$ while $\min\{f_{p'}, e_{p'} - f_{p'}\} \leqslant e_{p'} - s_{p'}$ for every prime $p' \neq p$. Furthermore, when this holds, we have

$$\frac{\varphi(M)}{\varphi(B)} = p^{e_p - s_p} \prod_{p'\mid N, p'\neq p} \varphi\Big(p'^{\min\{e_{p'}-f_{p'}, f_{p'}\}}\Big).$$

Hence the expression in (2.72) equals

$$\sum_{p\mid N} \left[\prod_{p'\mid N, p'\neq p} \left(\sum_{\substack{0\leqslant f_{p'}\leqslant e_{p'},\\ \min\{f_{p'}, e_{p'}-f_{p'}\}\leqslant e_{p'}-s_{p'}}} \varphi\Big(p'^{\min\{e_{p'}-f_{p'}, f_{p'}\}}\Big)\right)\right] p^{e_p-s_p}\log p \sum_{\substack{0\leqslant f_p\leqslant e_p\\ \min\{f_p, e_p-f_p\}>e_p-s_p}} 1.$$

Here the sum over f_p equals $\max\{2s_p - e_p - 1, 0\}$, and each sum over $f_{p'}$ can be evaluated using (2.50) (with $x = 0$). This leads to the statement of the lemma. □

Proof of Proposition 2.15. The proposition follows from

$$\mathrm{C}(N, \chi; n) = \mathrm{C}(\Gamma, \chi^{(0)}) + n\,\mathrm{C}(\Gamma, \chi^{(1)})$$

by evaluating the various sums appearing in (2.4) using the above three lemmas and also the formula $|C_{\Gamma,\chi}| = \Psi_1(N, q)$ (cf. Lemma 2.11). □

2.7. *Eisenstein series*

Huxley [**15**] gave explicit expressions (involving Dirichlet L-functions) for the scattering matrix and its determinant for the congruence subgroups $\Gamma^0(N)$, $\Gamma^1(N)$ and $\Gamma(N)$ of the modular group, for arbitrary level N. Note that $\Gamma^0(N)$ is conjugate to the group $\Gamma_0(N)$, which we consider here. The case of squarefree N was previously considered in [**12**, Chapter 11]. The calculations in this section could likely be shortened by appealing to work of Young [**28**] that was written subsequently to this paper; we have included our original treatment for completeness.

In our setting, the Eisenstein series for $\langle\Gamma^+, \chi\rangle$ associated to an open cusp $\frac{a}{c} \in C_{\Gamma,\chi}$ is given by (cf. (2.9))

$$E_{a/c}(z, s, \chi) = \sum_{U\in[T_{a/c}]\backslash\Gamma^+} \overline{\chi(U)}(\Im(\mathsf{N}_{a/c}Uz))^s, \text{ for } z \in \mathbb{H} \text{ and } \Re(s) > 1. \tag{2.74}$$

Substituting $U = W_{a/c}\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)$, using (2.46) and (2.47), and writing $z = x + iy$ and $W_{a/c} = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, one sees that (2.74) can be rewritten as

$$E_{a/c}(z, s, \chi) = \frac{\gcd(c^2, N)^s}{2N^s} \sum_{\langle\gamma,\delta\rangle} \chi(a\alpha + b\gamma) \frac{y^s}{|\gamma z + \delta|^{2s}}, \tag{2.75}$$

where now the sum runs over all $\langle\gamma, \delta\rangle \in \mathbb{Z}^2$ satisfying $\gcd(\gamma, \delta) = 1$, $\gcd(\gamma, N) = c$ and $\delta\frac{\gamma}{c} \equiv_{\gcd(c,N/c)} -a$. Also in (2.75), $\alpha = \alpha(\gamma, \delta) \in \mathbb{Z}$ denotes any integer satisfying both $\alpha\delta \equiv_\gamma 1$ and $\alpha \equiv_{N/c} -d\frac{\gamma}{c}$. Such α exists for every relevant pair $\langle\gamma, \delta\rangle$.

We wish to make the factor $\chi(a\alpha + b\gamma)$ somewhat more explicit. For a prime p, from now on we are using the following notations:

$$e_p = \mathrm{ord}_p(N), \quad s_p = \mathrm{ord}_p(q) \quad \text{and} \quad f_p = \mathrm{ord}_p(c).$$

Since $\frac{a}{c} \in C_{\Gamma,\chi}$, by Lemma 2.11, we have $s_p \leqslant \max\{e_p - f_p, f_p\}$. Also $\gcd(\gamma, N) = c$ implies that $\mathrm{ord}_p(\gamma) \geqslant f_p$ with equality unless $f_p = e_p$. Now the conditions on $\alpha = \alpha(\gamma, \delta)$ imply

$$a\alpha + b\gamma \equiv \begin{cases} a\delta^{-1} & \mod p^{f_p},\\ a\left(-d\dfrac{\gamma}{c}\right) + b\gamma \equiv -\frac{\gamma}{c} & \mod p^{e_p - f_p}. \end{cases}$$

Hence

$$\chi_p(a\alpha + b\gamma) = \begin{cases} \chi_p(a)\overline{\chi_p(\delta)} & \text{if } e_p \leqslant 2f_p,\\ \chi_p(-\gamma/c) & \text{if } e_p > 2f_p. \end{cases} \tag{2.76}$$

Note that both relations are valid in the special case $e_p = 2f_p$.

As in [**15**] we now introduce a family of sums similar to but simpler than (2.75). It will turn out that these sums can be expressed as linear combinations of the Eisenstein series $E_{a/c}(z, s, \chi)$ (cf. Lemma 2.19), and a key step in computing the scattering matrix will be to invert these linear relations.

Throughout this and the next section, we let χ_1 and χ_2 denote primitive Dirichlet characters satisfying $\chi_1(-1) = \chi_2(-1)$. For $j \in \{1, 2\}$, we write $q_j = \mathrm{cond}(\chi_j)$ and $\chi_{j,p} = (\chi_j)_p$ for any prime p. As in [**15**, p. 143], for $\Re(s) > 1$ and $z = x + iy$, we define

$$B_{\chi_1}^{\chi_2}(z, s) = \sum_{(c,d)\in\mathbb{Z}^2-\{(0,0)\}} \chi_1(c)\chi_2(d) \frac{y^s}{|cz + d|^{2s}}$$

and

$$E_{\chi_1}^{\chi_2}(z, s) = \sum_{(c,d)\in\mathbb{Z}^2,\gcd(c,d)=1} \chi_1(c)\chi_2(d) \frac{y^s}{|cz + d|^{2s}} = L(2s, \chi_1\chi_2)^{-1} B_{\chi_1}^{\chi_2}(z, s).$$

As in [**15**], we have

$$B_{\chi_1}^{\chi_2}\left(\begin{pmatrix} A & B\\ C & D\end{pmatrix} z, s\right) = \chi_1(D)\chi_2(A) B_{\chi_1}^{\chi_2}(z, s), \tag{2.77}$$

for all $\left(\begin{smallmatrix}A & B\\ C & D\end{smallmatrix}\right) \in \Gamma_0(1)$ with $q_1 \mid C$ and $q_2 \mid B$. Similarly,

$$B_{\chi_1}^{\chi_2}(Vz, s) = \chi_1(-1) B_{\chi_1}^{\chi_2}(z, s). \tag{2.78}$$

The same transformation formulas also hold for $E_{\chi_1}^{\chi_2}(z, s)$.

We let

$$F = \{(m, \chi_1, \chi_2) \ : \ m \in \mathbb{Z}_{\geqslant 1},\ mq_1 \mid N,\ q_2 \mid m,\ \mathrm{cond}(\chi\chi_2\overline{\chi_1}) = 1\}. \tag{2.79}$$

Note that $\mathrm{cond}(\chi\chi_2\overline{\chi_1}) = 1$ if and only if $\chi\chi_2(x) = \chi_1(x)$ for all $x \in (\mathbb{Z}/N\mathbb{Z})^\times$; however, this does not imply $\chi\chi_2 = \chi_1$ since the product $\chi\chi_2$ is not necessarily primitive. It follows from (2.77) that for every $(m, \chi_1, \chi_2) \in F$, the function $E_{\chi_1}^{\chi_2}(mz, s)$ is $\langle\Gamma^+, \chi\rangle$-invariant, that is $E_{\chi_1}^{\chi_2}(mTz, s) = \chi(T)E_{\chi_1}^{\chi_2}(mz, s)$ for all $T \in \Gamma^+$. In fact, each such function $E_{\chi_1}^{\chi_2}(mz, s)$ may be expressed as a linear combination of the Eisenstein series for $\langle\Gamma^+, \chi\rangle$ as follows.

LEMMA 2.19. *For any $(m, \chi_1, \chi_2) \in F$, we have*

$$E_{\chi_1}^{\chi_2}(mz, s) = \frac{2N^s}{m^s} \sum_{\frac{a}{c} \in C_{\Gamma,\chi}} \frac{\gcd(m, c)^{2s}}{\gcd(c^2, N)^s}\, \chi_1\Big(-\frac{c}{\gcd(m, c)}\Big)\, \chi_2\Big(a\frac{m}{\gcd(m, c)}\Big)\, E_{a/c}(z, s, \chi).$$

REMARK 2.20. With some work, one can see that Lemma 2.19 is equivalent to [**28**, Theorem 7.1]. However, our proof is significantly different; indeed note that the proof in [**28**] is based on inversion of a formula expressing $E_{a/c}(z, s, \chi)$ as a linear combination of functions of the form $E_{\chi_1}^{\chi_2}(mz, s)$ [**28**, Theorem 6.1].

Proof. For $(m, \chi_1, \chi_2) \in F$, by [**15**, p. 146(top)], $E_{\chi_1}^{\chi_2}(mz, s)$ can be expressed as

$$\sum_{m=gh} \sum_{\substack{e \pmod{N/h},\\ \gcd(e,g)=1}} \sum_{\substack{f \pmod{N},\\ \gcd(f,he,N)=1}} \chi_2(g)\chi_1(e)\chi_2(f)\Big(\frac{h}{g}\Big)^s \sum_{\gamma \in he+N\mathbb{Z}} \sum_{\substack{\delta \in f+N\mathbb{Z},\\ \gcd(\gamma,\delta)=1}} \frac{y^s}{|\gamma z + \delta|^{2s}}. \tag{2.80}$$

Given any g, h, e, f as in this sum, let a and c be the uniquely determined integers such that

$$c = \gcd(he, N), \quad a \equiv_{\gcd(c,N/c)} -f\frac{he}{c} \quad \text{and} \quad \tfrac{a}{c} \in C_\Gamma.$$

One easily checks that this transformation gives a bijection between the set of tuples

$$(g, h, e, f, \gamma, \delta)$$

appearing in the sum, and the set

$$\Big\{(a, c, \gamma, \delta) \ : \ \tfrac{a}{c} \in C_\Gamma, \gamma, \delta \in \mathbb{Z}, \gcd(\gamma, \delta) = 1, \gcd(\gamma, N) = c, \delta \equiv_{\gcd(c,N/c)} -a\Big(\frac{\gamma}{c}\Big)^{-1}\Big\}.$$

The inverse map is given by

$$h = \gcd(m, c), \quad g = \frac{m}{h}, \quad e \equiv_{N/h} \frac{\gamma}{h} \text{ and } f \equiv_N \delta.$$

Now let (a, c, γ, δ) be any tuple in the above set, with corresponding $(g, h, e, f, \gamma, \delta)$. We then have, for each prime $p \mid N$,

$$\begin{aligned} &\chi_{2,p}(g)\chi_{1,p}(e)\chi_{2,p}(f) \\ &\quad = \chi_{2,p}(g)\chi_{2,p}(a)\chi_{1,p}\Big(-\frac{c}{h}\Big)\begin{cases} \chi_p(a)\overline{\chi_p(\delta)} & \text{if } \mathrm{ord}_p(N/c) \leqslant \mathrm{ord}_p(c), \\ \chi_p(-\gamma/c) & \text{if } \mathrm{ord}_p(N/c) > \mathrm{ord}_p(c). \end{cases} \end{aligned} \tag{2.81}$$

To prove this, first note that if $\mathrm{ord}_p(q_2) > \mathrm{ord}_p(c)$, then $g = \frac{m}{\gcd(c,m)} \equiv_p 0$, and thus $\chi_{2,p}(g) = 0$ so (2.81) holds. Hence, from now on we may assume $\mathrm{ord}_p(q_2) \leqslant \mathrm{ord}_p(c)$.

Let us first assume that $\mathrm{ord}_p(c) > \mathrm{ord}_p(m)$. Then $p \mid e$ but $p \nmid a$, $p \nmid \delta$ (since $\gcd(a, c) = 1$ and $\gcd(\gamma, \delta) = 1$). If $p \mid q_1$, then both sides of (2.81) are 0. On the other hand, if $p \nmid q_1$, viz. $\chi_{1,p} \equiv 1$, then $\mathrm{cond}(\chi_p\chi_{2,p}) = 1$ and so $\chi_{2,p}(f) = \overline{\chi_p(f)} = \overline{\chi_p(\delta)}$ and $\chi_{2,p}(a) = \overline{\chi_p(a)}$; and if $\mathrm{ord}_p(N/c) > \mathrm{ord}_p(c)$, then also $\delta \equiv_{\gcd(c,p^\infty)} -a(\frac{\gamma}{c})^{-1}$; hence (2.81) holds.

It remains to consider the case $\mathrm{ord}_p(c) \leqslant \mathrm{ord}_p(m)$. Then since $mq_1 \mid N$, we have $\mathrm{ord}_p(q_1) \leqslant \mathrm{ord}_p(N/c)$ and hence $\chi_{1,p}(e) = \chi_{1,p}(-\gamma/c)\chi_{1,p}(-c/\gcd(m,c))$. Also, using

$$f \equiv \delta \equiv -a\Big(\frac{\gamma}{c}\Big)^{-1} \pmod{\gcd(c, N/c)},$$

we get

$$\begin{cases} \chi_{1,p}(-\gamma/c) = \chi_{1,p}(a)\overline{\chi_{1,p}(\delta)} & \text{if } \mathrm{ord}_p(N/c) \leqslant \mathrm{ord}_p(c); \\ \chi_{2,p}(f) = \chi_{2,p}(a)\overline{\chi_{2,p}(-\gamma/c)} & \text{if } \mathrm{ord}_p(N/c) > \mathrm{ord}_p(c). \end{cases}$$

The second identity holds since we are assuming $\mathrm{ord}_p(q_2) \leqslant \mathrm{ord}_p(c)$ from start. From this (2.81) follows and so we have proved that (2.81) holds in all cases.

Next let us note that

$$\chi_2(g)\chi_1(e) = 0 \text{ whenever } \tfrac{a}{c} \notin C_{\Gamma,\chi}. \tag{2.82}$$

To prove this, assume $\frac{a}{c} \notin C_{\Gamma,\chi}$; then by Lemma 2.11, there is a prime $p \mid N$ such that

$$\mathrm{ord}_p(q) > \mathrm{ord}_p\Big(\frac{N}{\gcd(c, N/c)}\Big) = \max\{\mathrm{ord}_p(N/c), \mathrm{ord}_p(c)\}.$$

As noted above, if $\mathrm{ord}_p(q_2) > \mathrm{ord}_p(c)$, then $\chi_{2,p}(g) = 0$ and (2.82) holds. On the other hand, if $\mathrm{ord}_p(q_2) \leqslant \mathrm{ord}_p(c)$, then using $\mathrm{cond}(\chi\chi_2\overline{\chi_1}) = 1$ we get $\mathrm{ord}_p(q_1) = \mathrm{ord}_p(q)$ and since $mq_1 \mid N$ we have

$$\mathrm{ord}_p(m) < \mathrm{ord}_p(\gcd(c, N/c)) \leqslant \mathrm{ord}_p(c),$$

hence $p \mid e$ so $\chi_{1,p}(e) = 0$ and (2.82) holds.

This lemma is now a direct consequence of (2.80), the bijection between $\{(g,h,e,f,\gamma,\delta)\}$ and $\{(a,c,\gamma,\delta)\}$ and (2.81), (2.82), (2.75) and (2.76). □

Next, recalling the definition of F in (2.79), for any $m \mid N$ we now set

$$F_m = \{(\chi_1, \chi_2) \ : \ (m, \chi_1, \chi_2) \in F\}. \tag{2.83}$$

The following lemma gives an explicit enumeration of the set F_m. For any $c \in \mathbb{Z}_{\geqslant 1}$, we write

$$X_c = \{\psi \text{ primitive Dirichlet character } : \ \mathrm{cond}(\psi) \mid c\}.$$

This set is in one-to-one correspondence with the dual of the group $(\mathbb{Z}/c\mathbb{Z})^\times$. In particular, $|X_c| = \varphi(c)$. Recall that $q = \mathrm{cond}(\chi)$. Let us write

$$G_{N,q} = \Big\{ m \mid N \ : \ q \mid \frac{N}{\gcd(m, N/m)} \Big\}.$$

LEMMA 2.21. *Given $m \mid N$, the set F_m is nonempty if and only if $m \in G_{N,q}$. When this holds, the map*

$$(\chi_1, \chi_2) \mapsto \prod_{p \mid N} \begin{cases} \chi_{1,p} & \text{if } \mathrm{ord}_p(q) \leqslant \mathrm{ord}_p(m), \\ \chi_{2,p} & \text{if } \mathrm{ord}_p(q) > \mathrm{ord}_p(m) \end{cases} \tag{2.84}$$

is a bijection from F_m onto $X_{\gcd(m,N/m)}$. In particular $|F_m| = \varphi(\gcd(m, N/m))$.

Proof. By the definition of F, two primitive Dirichlet characters χ_1 and χ_2 satisfy $(\chi_1, \chi_2) \in F_m$ if and only if, for every prime $p \mid N$,

$$s_2 \leqslant \mathrm{ord}_p(m), \quad s_1 \leqslant \mathrm{ord}_p(N/m) \quad \text{and} \quad \mathrm{cond}(\chi_p\overline{\chi_{1,p}}\chi_{2,p}) = 1, \tag{2.85}$$

Here $s_j = \mathrm{ord}_p(q_j)$ for $j \in \{1,2\}$. Let $s = \mathrm{ord}_p(q)$. Note that $\mathrm{cond}(\chi_p\overline{\chi_{1,p}}\chi_{2,p}) = 1$ implies that $s \leqslant \max\{s_1, s_2\}$. Hence by (2.85),

$$s \leqslant \max\{\mathrm{ord}_p(m), \mathrm{ord}_p(N/m)\} = \mathrm{ord}_p\Big(\tfrac{N}{\gcd(m,N/m)}\Big).$$

Since this must hold for every $p \mid N$, it follows that F_m can be nonempty only if $m \in G_{N,q}$.

From now on we assume $m \in G_{N,q}$. We will prove that the map in (2.84) is a bijection from F_m onto $X_{\gcd(m,N/m)}$. In particular, this will imply that F_m is nonempty, with $|F_m| = \varphi(\gcd(m, N/m))$.

First consider any prime $p \mid N$ satisfying $s \leqslant \mathrm{ord}_p(m)$ (with $s = \mathrm{ord}_p(q)$). For such p, the first and third relations in (2.85) imply $s_1 \leqslant \mathrm{ord}_p(m)$, which together with the second relation in (2.85) implies $s_1 \leqslant \mathrm{ord}_p(\gcd(m, N/m))$. Conversely, if $\chi_{1,p}$ is *any* primitive character with conductor p^{s_1} subject to $s_1 \leqslant \mathrm{ord}_p(\gcd(m, N/m))$, and if $\chi_{2,p}$ is the unique primitive character satisfying $\mathrm{cond}(\chi_p\overline{\chi}_{1,p}\chi_{2,p}) = 1$, then all the conditions in (2.85) are fulfilled.

Next consider any $p \mid N$ for which $s > \mathrm{ord}_p(m)$. Recall that we are assuming $m \in G_{N,q}$; hence $s \leqslant \max\{\mathrm{ord}_p(m), \mathrm{ord}_p(N/m)\}$ and so $s \leqslant \mathrm{ord}_p(N/m)$. Then the two last relations in (2.85) together imply $s_2 \leqslant \mathrm{ord}_p(N/m)$, and in combination with the first relation this gives $s_2 \leqslant \mathrm{ord}_p(\gcd(m, N/m))$. Conversely if $\chi_{2,p}$ is *any* primitive character with conductor p^{s_2} subject to $s_2 \leqslant \mathrm{ord}_p(\gcd(m, N/m))$, and if $\chi_{1,p}$ is the unique primitive character satisfying $\mathrm{cond}(\chi_p\overline{\chi}_{1,p}\chi_{2,p}) = 1$, then all the conditions in (2.85) are fulfilled.

The above observations imply that the map in (2.84) is indeed a bijection as stated. □

We now turn to the full group $\Gamma = \Gamma_0^{\pm}(N)$. Let $\frac{a}{c} \in C_{\Gamma,\chi}$ be given. By Lemma 2.12 and [**6**, (2.8)], we have

$$E_{a/c}(Vz, s, \chi) = \chi_{\alpha(N,c)}(-1)E_{k(a/c)}(z, s, \chi), \tag{2.86}$$

where $\alpha(N,c)$ is as in (2.56) and $k(a/c)$ as in (2.52). Hence (2.10) gives

$$E^{\Gamma}_{a/c}(z, s, \chi^{(\varepsilon)}) = \begin{cases} E_{a/c}(z,s,\chi) + (-1)^{\varepsilon}E_{-a/c}(z,s,\chi) & \text{if } \gcd(c, N/c) > 2, \\ \big(\chi_{\alpha(N,c)}(-1)(-1)^{\varepsilon} + 1\big)E_{1/c}(z,s,\chi) & \text{if } \gcd(c,N/c) \leqslant 2. \end{cases} \tag{2.87}$$

It is also natural to introduce, for any primitive Dirichlet characters χ_1 and χ_2 satisfying $\chi_1(-1) = \chi_2(-1)$ as before,

$${}^{\varepsilon}B^{\chi_2}_{\chi_1}(z,s) = B^{\chi_2}_{\chi_1}(z,s) + \chi^{(\varepsilon)}(V^{-1})B^{\chi_2}_{\chi_1}(Vz,s) = \begin{cases} 2B^{\chi_2}_{\chi_1}(z,s) & \text{if } \chi_1(-1) = (-1)^{\varepsilon}, \\ 0 & \text{otherwise} \end{cases}$$

(cf. (2.78)). This function is $\langle\Gamma,\chi\rangle$-invariant, since one easily checks that it is $\langle\Gamma^+,\chi\rangle$-invariant and also transforms correctly under V. We also set

$${}^{\varepsilon}E^{\chi_2}_{\chi_1}(z,s) = L(2s,\chi_1\chi_2)^{-1}\,{}^{\varepsilon}B^{\chi_2}_{\chi_1}(z,s) = \begin{cases} 2E^{\chi_2}_{\chi_1}(z,s) & \text{if } \chi_1(-1) = (-1)^{\varepsilon}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.88}$$

Recall from (2.11) that we have fixed a subset $R_{\Gamma,\chi} \subset C_{\Gamma,\chi}$, which in our present notation (cf. also (2.52) and Lemma 2.12) has the property that for every $\frac{a}{c} \in C_{\Gamma,\chi}$ with $\gcd(c, N/c) \leqslant 2$, $\frac{a}{c}$ lies in $R_{\Gamma,\chi}$ if and only if $\chi_{\alpha(N,c)}(-1) = (-1)^{\varepsilon}$. Set

$$\begin{aligned} F^{\varepsilon} &= \{(m,\chi_1,\chi_2) \in F \ :\ \chi_1(-1) = (-1)^{\varepsilon}\} \\ &= \{(m,\chi_1,\chi_2) \ :\ m \in \mathbb{Z}_{\geqslant 1},\ q_2 \mid m,\ mq_1 \mid N,\ \mathrm{cond}(\chi\chi_2\overline{\chi_1}) = 1,\ \chi_1(-1) = (-1)^{\varepsilon}\}. \end{aligned}$$

We now have the following analogue of Lemma 2.19.

LEMMA 2.22. *For any* $(m,\chi_1,\chi_2)\in F^\varepsilon$,

$$ {}^\varepsilon E^{\chi_2}_{\chi_1}(mz,s)=\frac{2N^s}{m^s}\sum_{\frac{a}{c}\in R_{\Gamma,\chi}}\frac{\gcd(m,c)^{2s}}{\gcd(c^2,N)^s}\chi_1\Big(-\frac{c}{\gcd(m,c)}\Big)\chi_2\Big(\frac{am}{\gcd(m,c)}\Big)I_c\,E^\Gamma_{a/c}(z,s,\chi^{(\varepsilon)}),$$

where $I_c=1+\delta_{\gcd(c,N/c)>2}$.

Proof. Fix $(m,\chi_1,\chi_2)\in F^\varepsilon$. Then ${}^\varepsilon E^{\chi_2}_{\chi_1}(mz,s)=2E^{\chi_2}_{\chi_1}(mz,s)\not\equiv 0$ and this is expressed in Lemma 2.19 as a linear combination of $E_{a/c}(z,s,\chi)$ over all $\frac{a}{c}\in C_{\Gamma,\chi}$. For $\frac{a}{c}\in C_{\Gamma,\chi}$ with $\gcd(c,N/c)>2$, we see immediately that the combined contribution from $\frac{a}{c}\in C_{\Gamma,\chi}$ and $\frac{-a}{c}\in C_{\Gamma,\chi}$ to the sum in Lemma 2.19 (doubled) is

$$\frac{4N^s}{m^s}\frac{\gcd(m,c)^{2s}}{\gcd(c^2,N)^s}\chi_1\Big(-\frac{c}{\gcd(m,c)}\Big)\chi_2\Big(a\frac{m}{\gcd(m,c)}\Big)E^\Gamma_{a/c}(z,s,\chi^{(\varepsilon)}),$$

since $\chi_2(-1)=\chi_1(-1)=(-1)^\varepsilon$.

Next take $\frac{a}{c}\in C_{\Gamma,\chi}$ with $\gcd(c,N/c)\leqslant 2$ (thus $a=1$). If $\chi_{\alpha(N,c)}(-1)=-(-1)^\varepsilon$, then as we will prove below

$$\chi_1\Big(-\frac{c}{\gcd(m,c)}\Big)\chi_2\Big(\frac{m}{\gcd(m,c)}\Big)=0, \tag{2.89}$$

so that the contribution from $E_{1/c}(z,s,\chi)$ in Lemma 2.19 vanishes. If $\chi_{\alpha(N,c)}(-1)=(-1)^\varepsilon$, then the contribution to the doubled sum is

$$\frac{4N^s}{m^s}\frac{\gcd(c,m)^{2s}}{\gcd(c^2,N)^s}\chi_1\Big(-\frac{c}{\gcd(m,c)}\Big)\chi_2\Big(\frac{m}{\gcd(m,c)}\Big)\frac{1}{2}E^\Gamma_{1/c}(z,s,\chi^{(\varepsilon)}),$$

since $E^\Gamma_{1/c}(z,s,\chi^{(\varepsilon)})=2E_{1/c}(z,s,\chi)$. Hence, adding up the contributions, we obtain (2.88).

It remains to prove the claim (2.89) for every $\frac{1}{c}\in C_{\Gamma,\chi}$ satisfying $\gcd(c,N/c)\leqslant 2$ and $\chi_{\alpha(N,c)}(-1)=-(-1)^\varepsilon$. To do so, let us fix $s>1$. Using (2.78) and $\chi_1(-1)=(-1)^\varepsilon$, we see that $E^{\chi_2}_{\chi_1}(mVz,s)\equiv(-1)^\varepsilon E^{\chi_2}_{\chi_1}(mz,s)$ and applying Lemma 2.19 and (2.86), this leads to a linear relation between the functions $z\mapsto E_{a/c}(z,s,\chi)$ for $\frac{a}{c}$ running through $C_{\Gamma,\chi}$. However, these functions are linearly independent, since $E_{a/c}(z,s,\chi)$ grows like $(\Im(\mathsf{N}_{a/c}(z)))^s$ when z approaches the cusp $\frac{a}{c}$ but is bounded by $O((\Im(\mathsf{N}_{a'/c'}(z)))^{1-s})$ as z approaches any other cusp $\frac{a'}{c'}\in C_\Gamma$; cf. [**12**, (Proposition 3.7), p. 280]. It follows that all coefficients in the linear relation which we have obtained mush vanish. In particular for $\frac{1}{c}\in C_{\Gamma,\chi}$ with $\gcd(c,N/c)\leqslant 2$, this implies

$$(\chi_{\alpha(N,c)}(-1)-(-1)^\varepsilon)\chi_1\Big(-\frac{c}{\gcd(m,c)}\Big)\chi_2\Big(\frac{m}{\gcd(m,c)}\Big)=0,$$

and (2.89) follows. □

REMARK 2.23. The claim (2.89) for every $\frac{1}{c}\in C_{\Gamma,\chi}$ satisfying $\gcd(c,N/c)\leqslant 2$ and $\chi_{\alpha(N,c)}(-1)=-(-1)^\varepsilon$ may alternatively be proved directly from the definition of $\alpha(N,c)$, using no facts about Eisenstein series. However, we have not found any way of doing this without going through a somewhat lengthy case-by-case analysis.

Mimicking now [**15**, p. 147], we let $\mathbf{B}(s)$ be the vector of the functions

$$\mathbf{B}(s)=({}^\varepsilon B^{\chi_2}_{\chi_1}(mz,s))_{(m,\chi_1,\chi_2)\in F^\varepsilon}.$$

So far our discussion has been for $\Re(s) > 1$. However, the functions $B_{\chi_1}^{\chi_2}(z,s)$ can be meromorphically continued to all $s \in \mathbb{C}$; hence so can $E_{\chi_1}^{\chi_2}(z,s)$, ${}^{\varepsilon}B_{\chi_1}^{\chi_2}(z,s)$ and ${}^{\varepsilon}E_{\chi_1}^{\chi_2}(z,s)$; and we have the functional equation (cf. [15, p. 145])

$$\mathbf{B}(s) = \mathbf{D}(s) P \mathbf{B}(1-s),$$

where P is a permutation which in each row $(m,\chi_1,\chi_2) \in F^{\varepsilon}$ has an entry one in column $(q_1 m/q_2, \overline{\chi_2}, \overline{\chi_1})) \in F^{\varepsilon}$ and zeros elsewhere, and $\mathbf{D}(s)$ is a diagonal matrix with entries

$$\frac{q_1^{1-s}}{q_2^s} \pi^{2s-1} \frac{\tau(\chi_2)}{\tau(\overline{\chi_1})} \frac{\Gamma(1-s)}{\Gamma(s)}.$$

Using the fact that the map $(m,\chi_1,\chi_2) \mapsto (q_1 m/q_2, \overline{\chi_2}, \overline{\chi_1})$, which is the permutation given by P, is an involution of the set F^{ε}, we obtain

$$\det(P) = (-1)^{\frac{|F^{\varepsilon}| - |F_0^{\varepsilon}|}{2}}, \tag{2.90}$$

where

$$F_0^{\varepsilon} = \{(m,\chi_1,\chi_2) \in F^{\varepsilon} \ : \ \chi_1 = \overline{\chi_2}\} \tag{2.91}$$

is the set of fixpoints of the involution. Making use of the same permutation we also see that

$$\det(\mathbf{D}(s)) = \left(\frac{\pi^{2s-1}\Gamma(1-s)}{\Gamma(s)}\right)^{|F^{\varepsilon}|} \prod_{(m,\chi_1,\chi_2)\in F^{\varepsilon}} q_1^{1-2s}. \tag{2.92}$$

Let $\mathbf{E}^{\Gamma}(s)$ be the vector of the functions $E_{a/c}^{\Gamma}(z,s,\chi^{(\varepsilon)})$ for all $\frac{a}{c} \in R_{\Gamma,\chi}$. By (2.88) and Lemma 2.22, we have

$$\mathbf{B}(s) = L(s) M(s) \mathbf{E}^{\Gamma}(s), \tag{2.93}$$

where $L(s)$ is the $|F^{\varepsilon}| \times |F^{\varepsilon}|$ diagonal matrix with entries $L(2s, \chi_1\chi_2)$ and $M(s)$ is an $|F^{\varepsilon}| \times |R_{\Gamma,\chi}|$ matrix, whose entry in row $(m,\chi_1,\chi_2) \in F^{\varepsilon}$, column $\frac{a}{c} \in R_{\Gamma,\chi}$ is

$$\frac{2N^s}{m^s} \frac{\gcd(m,c)^{2s}}{\gcd(c^2,N)^s} \chi_1\Big(-\frac{c}{\gcd(m,c)}\Big)\chi_2\Big(a\frac{m}{\gcd(m,c)}\Big) I_c. \tag{2.94}$$

At first (2.93) holds for $\Re(s) > 1$; however, $\mathbf{E}^{\Gamma}(s)$ can be meromorphically continued to all $s \in \mathbb{C}$ (cf. (2.87) and [12, Chapters 6.11 and 8.3]) and (2.93) remains valid for $s \in \mathbb{C}$.

Let $Q(s)$ be the $|F^{\varepsilon}| \times |F^{\varepsilon}|$ matrix whose entry in row $(m,\chi_1,\chi_2) \in F^{\varepsilon}$ and column $(m',\chi_1',\chi_2') \in F^{\varepsilon}$ is

$$\begin{cases} \mu\big(\frac{m}{m'}\big)\chi_2\big(\frac{m}{m'}\big)\big(\frac{m}{m'}\big)^{-s} & \text{if } m' \mid m, \chi_1' = \chi_1 \text{ and } \chi_2' = \chi_2, \\ 0 & \text{otherwise.} \end{cases} \tag{2.95}$$

Then the product matrix $Q(s)M(s)$ has the following entry in row $(m,\chi_1,\chi_2) \in F^{\varepsilon}$ and column $\frac{a}{c} \in R_{\Gamma,\chi}$:

$$\begin{aligned} & I_c \sum_{m'|m} \mu\Big(\frac{m}{m'}\Big)\chi_2\Big(\frac{m}{m'}\Big)\Big(\frac{m}{m'}\Big)^{-s} \frac{2N^s}{m'^s} \frac{\gcd(c,m')^{2s}}{\gcd(c^2,N)^s} \chi_1\Big(-\frac{c}{\gcd(c,m')}\Big)\chi_2\Big(a\frac{m'}{\gcd(c,m')}\Big) \\ &= \frac{2N^s}{m^s \gcd(c^2,N)^s} I_c \sum_{d|m} \mu(d) \gcd(m/d,c)^{2s} \chi_1\Big(-\frac{c}{\gcd(m/d,c)}\Big)\chi_2\Big(a\frac{m/d}{\gcd(m/d,c)}\Big) \\ &= \frac{2N^s}{m^s} \frac{\gcd(c,m)^{2s}}{\gcd(c^2,N)^s} I_c \chi_1\Big(-\frac{c}{\gcd(c,m)}\Big)\chi_2\Big(a\frac{m}{\gcd(c,m)}\Big) \end{aligned}$$

$$\times \prod_{p\mid m}\left(1-\begin{cases}1 & \text{if } \operatorname{ord}_p(c)<\operatorname{ord}_p(m),\\ \chi_1(p)\chi_2(p)p^{-2s} & \text{if } \operatorname{ord}_p(c)\geqslant \operatorname{ord}_p(m)\end{cases}\right)$$

$$= \begin{cases} 0 & \text{if } m\nmid c,\\ \dfrac{2N^s m^s}{\gcd(c^2,N)^s} I_c \chi_1\Big(-\dfrac{c}{m}\Big)\chi_2(a)\displaystyle\prod_{p\mid m}(1-\chi_1(p)\chi_2(p)p^{-2s}) & \text{if } m\mid c.\end{cases} \tag{2.96}$$

The following lemma is immediate from Lemma 2.11 and the definition of $R_{\Gamma,\chi}$ in (2.11).

LEMMA 2.24. *For every $c\in G_{N,q}$, we have*

$$\big|\{a \bmod \tfrac{a}{c}\in R_{\Gamma,\chi}\}\big| = \frac{1}{2}\begin{cases}\varphi(\gcd(c,N/c)) & \text{if } \gcd(c,N/c)>2,\\ (-1)^{\varepsilon}\chi_{\alpha(N,c)}(-1)+1 & \text{if } \gcd(c,N/c)\leqslant 2.\end{cases}$$

Here $\chi^{(\varepsilon)}(V)=(-1)^{\varepsilon}$ and $\alpha(N,c)$ is given in (2.56).

For any $m\mid N$, we set

$$F_m^{\varepsilon} = \{(\chi_1,\chi_2) \,:\, (m,\chi_1,\chi_2)\in F^{\varepsilon}\} = \{(\chi_1,\chi_2)\in F_m \,:\, \chi_1(-1)=(-1)^{\varepsilon}\}.$$

LEMMA 2.25. *Assume $m\in G_{N,q}$. Then for any $(\chi_1,\chi_2)\in F_m$, $(\chi_1,\chi_2)\in F_m^{\varepsilon}$ holds if and only if*

$$\varpi(-1)\prod_{p\mid N,\operatorname{ord}_p(q)>\operatorname{ord}_p(m)}\chi_p(-1)=(-1)^{\varepsilon}, \tag{2.97}$$

where $\varpi\in X_{\gcd(m,N/m)}$ is the image of (χ_1,χ_2) under the map (2.84). It follows that

$$|F_m^{\varepsilon}| = \frac{1}{2}\begin{cases}\varphi(\gcd(m,N/m)) & \text{if } \gcd(m,N/m)>2,\\ (-1)^{\varepsilon}\chi_{\alpha(N,m)}(-1)+1 & \text{if } \gcd(m,N/m)\leqslant 2,\end{cases} \tag{2.98}$$

Proof. Using (2.84) and $\chi_{1,p}(-1)=\chi_{2,p}(-1)\chi_p(-1)$, which holds for every prime p since $q(\chi\chi_2\overline{\chi_1})=1$, we see that for the left-hand side of (2.97) equals to $\chi_1(-1)$. This proves the first statement of the lemma. It follows that $|F_m^{\varepsilon}|=\frac{1}{2}\varphi(\gcd(m,N/m))$ if $\gcd(m,N/m)>2$ since then exactly half of the characters of the group $(\mathbb{Z}/\gcd(m,N/m)\mathbb{Z})^{\times}$ are even and half are odd.

It remains to prove the statement (2.98) in the case $\gcd(m,N/m)\leqslant 2$. In this case, $X_{\gcd(m,N/m)}$ consists of the trivial character only, and we see that our task is to prove that

$$\prod_{p\mid N,\operatorname{ord}_p(q)>\operatorname{ord}_p(m)}\chi_p(-1)=\chi_{\alpha(N,m)}(-1).$$

But $\chi_p(-1)=1$ whenever $p\nmid q$. Hence we are done if we can show that for all primes p

$$\operatorname{ord}_p(\alpha(N,m)) = \begin{cases}\operatorname{ord}_p(N) & \text{if } \operatorname{ord}_p(q)>\operatorname{ord}_p(m),\\ 0 & \text{if } 1\leqslant \operatorname{ord}_p(q)\leqslant \operatorname{ord}_p(m).\end{cases} \tag{2.99}$$

When $p>2$ this claim is immediate from the definition of $\alpha(N,m)$ in (2.56), since it follows from $\gcd(m,N/m)\leqslant 2$ that $\operatorname{ord}_p(m)\in\{0,\operatorname{ord}_p(N)\}$ for every such prime p. Similarly the claim is immediate also for $p=2$ when $\gcd(m,N/m)=1$.

Finally assume $p=2$ and $\gcd(m,N/m)=2$. Then $\operatorname{ord}_p(N)\geqslant 2$, $\operatorname{ord}_p(m)\in\{1,\operatorname{ord}_p(N)-1\}$ and (2.56) implies that $\operatorname{ord}_p(\alpha)=0$ if $\operatorname{ord}_p(m)=\operatorname{ord}_p(N)-1$, otherwise $\operatorname{ord}_p(\alpha)=\operatorname{ord}_p(N)$.

Also $\mathrm{ord}_p(q) < \mathrm{ord}_p(N)$ since $m \in G_{N,q}$; and recall that $\mathrm{ord}_p(q)$ cannot equal to 1 since there is no primitive character modulo 2. From these observations, we see that (2.99) holds also when $p = 2$ and $\gcd(m, N/m) = 2$. □

It follows from Lemmas 2.24 and 2.25 that

$$|F^\varepsilon| = \sum_{m \in G_{N,q}} |F^\varepsilon_m| = |R_{\Gamma,\chi}|. \tag{2.100}$$

In particular, $M(s)$ is a square matrix.

LEMMA 2.26. *We have* $\det(Q(s)) = 1$ *and*

$$\det(M(s)) = c(N,\chi) \prod_{m \in G_{N,q}} \left(\frac{N}{\gcd(m, N/m)}\right)^{|F^\varepsilon_v| s} \prod_{(m,\chi_1,\chi_2) \in F^\varepsilon} \prod_{p \mid m} (1 - \chi_1\chi_2(p) p^{-2s}), \tag{2.101}$$

for some nonzero constant $c(N,\chi) \in \mathbb{C}$. *Here* $Q(s)$ *and* $M(s)$ *are given as in* (2.95) *and* (2.94).

Proof. Let us fix an ordering F^ε such that (m, χ_1, χ_2) comes before (m', χ_1', χ_2') whenever $m < m'$. Giving both the rows and columns of $Q(s)$ this ordering, then $Q(s)$ is lower triangular by definition, and has all diagonal entries equal to 1. Therefore, $\det(Q(s)) = 1$.

It also follows that $\det(M(s)) = \det(Q(s)M(s))$. Now $Q(s)M(s)$ is a square matrix whose rows (respectively columns) are naturally indexed by $(m, \chi_1, \chi_2) \in F^\varepsilon$ (respectively, $\frac{a}{c} \in R_{\Gamma,\chi}$). Let us view $Q(s)M(s)$ as a block matrix, with the blocks indexed by m (row) and c (column). Note that Lemmas 2.24 and 2.25 then show that each diagonal block is a square matrix; furthermore, (2.96) implies that $Q(s)M(s)$ is upper block triangular. Hence, again using (2.96), we see that the determinant is given by the expression in the right-hand side of (2.101), for some constant $C = c(N,\chi) \in \mathbb{C}$.

It also follows that in order to prove $C \neq 0$, it suffices to check that for any $c \in G_{N,q}$ for which

$$R_c = \left\{ a \ : \ \frac{a}{c} \in R_{\Gamma,\chi} \right\} \tag{2.102}$$

is nonempty, the $|R_c| \times |R_c|$-matrix $(\chi_2(a))$, with rows indexed by $(\chi_1, \chi_2) \in F_c$ and columns indexed by $a \in R_c$ (recall $|F_c| = |R_c|$) has nonvanishing determinant. This is trivial if $\gcd(c, N/c) \leqslant 2$ since then $R_c = \{1\}$ (if $R_c \neq \emptyset$). Now assume that $\gcd(c, N/c) > 2$, and set $t = |F_c| = |R_c| = \frac{1}{2}\varphi(\gcd(c, N/c))$. Using Lemmas 2.21 and 2.25, we find that by multiplying the columns with appropriate constants of absolute value one, the determinant can be transformed into $\det(\psi_i(g_j))$, where $g_1, \dots, g_t$ are the elements and $\psi_1, \dots, \psi_t$ are the characters of the Abelian group $(\mathbb{Z}/\gcd(c, N/c)\mathbb{Z})^\times/\{\pm 1\}$. Such determinant is nonvanishing, since multiplying the matrix with its conjugate transpose gives t times the $t \times t$ identity matrix (cf. [**15**, p. 149]). □

Now by (2.88) and (2.93), we have

$$\mathbf{E}^\Gamma(s) = M(s)^{-1} L(s)^{-1} D(s) P L(1-s) M(1-s) \mathbf{E}^\Gamma(1-s)$$

and hence (by the same type of uniqueness argument as in [**6**, below (2.32)]) the 'Γ-scattering matrix' introduced in [**6**, (2.33)] is

$$\Phi^\Gamma(s) = M(s)^{-1} L(s)^{-1} D(s) P L(1-s) M(1-s).$$

Hence by (2.90), (2.92) and Lemma 2.26,

$$\det(\Phi^{\Gamma}(s)) = (-1)^{\frac{|F^{\varepsilon}|-|F_0^{\varepsilon}|}{2}} \left(\prod_{m \in G_{N,q}} \left(\frac{N}{\gcd(m, N/m)} \right)^{|F_m^{\varepsilon}|} \right)^{1-2s} \left(\frac{\pi^{2s-1}\Gamma(1-s)}{\Gamma(s)} \right)^{|F^{\varepsilon}|} \times \prod_{(m,\chi_1,\chi_2) \in F^{\varepsilon}} \left(q_1^{1-2s} \frac{L(2-2s, \chi_1\chi_2\omega_m)}{L(2s, \chi_1\chi_2\omega_m)} \right), \tag{2.103}$$

where ω_m denotes the trivial Dirichlet character modulo m. We note that for each $(m, \chi_1, \chi_2) \in F^{\varepsilon}$, $\chi_1\chi_2\omega_m$ is a Dirichlet character modulo mq_1.

Remark 2.27. By an entirely similar discussion one computes the determinant of the scattering matrix $\Phi(s)$ for $\langle \Gamma^+, \chi \rangle$. We will not need this formula, but we state it here for possible future reference:

$$\det(\Phi(s)) = (-1)^{\frac{|F|-|F_0|}{2}} \left(\prod_{m \in G_{N,q}} \left(\frac{N}{\gcd(m, N/m)} \right)^{\varphi(\gcd(m,N/m))} \right)^{1-2s} \left(\frac{\pi^{2s-1}\Gamma(1-s)}{\Gamma(s)} \right)^{|F|} \times \prod_{(m,\chi_1,\chi_2) \in F} \left(q_1^{1-2s} \frac{L(2-2s, \chi_1\chi_2\omega_m)}{L(2s, \chi_1\chi_2\omega_m)} \right). \tag{2.104}$$

From (2.103), we obtain:

$$\frac{(\varphi^{\Gamma})'}{\varphi^{\Gamma}}(s) = -2 \sum_{m \in G_{N,q}} |F_m^{\varepsilon}| \log\left(\frac{N}{\gcd(m, N/m)} \right) + |F^{\varepsilon}| \left(2\log\pi - \frac{\Gamma'}{\Gamma}(1-s) - \frac{\Gamma'}{\Gamma}(s) \right) - 2 \sum_{(m,\chi_1,\chi_2) \in F^{\varepsilon}} \left(\log q_1 + \frac{L'}{L}(2-2s, \chi_1\chi_2\omega_m) + \frac{L'}{L}(2s, \chi_1\chi_2\omega_m) \right). \tag{2.105}$$

We can now evaluate the integral appearing in (2.5):

Lemma 2.28.

$$\frac{1}{4\pi} \int_{\mathbb{R}} h(r) \frac{(\varphi^{\Gamma})'}{\varphi^{\Gamma}}\left(\frac{1}{2} + ir \right) dr = \left(|F^{\varepsilon}| \log\pi - \sum_{m \in G_{N,q}} |F_m^{\varepsilon}| \log\left(\frac{N}{\gcd(m, N/m)} \right) - \sum_{(m,\chi_1,\chi_2) \in F^{\varepsilon}} \log q_1 \right) g(0) - \frac{|F^{\varepsilon}|}{2\pi} \int_{\mathbb{R}} h(r) \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + ir \right) dr - \frac{|F_0^{\varepsilon}|}{2} h(0) + 2 \sum_{r=1}^{\infty} \frac{\Lambda(r)\{\chi\}_{\varepsilon}(r)}{r} g(2\log r), \tag{2.106}$$

where

$$\{\chi\}_{\varepsilon}(r) = \sum_{(m,\chi_1,\chi_2) \in F^{\varepsilon}} \chi_1\chi_2\omega_m(r). \tag{2.107}$$

Proof. This follows from (2.105) using

$$\frac{1}{4\pi}\int_{\mathbb{R}} h(r)\left(\frac{L'}{L}(1-2ir,\chi_1\chi_2\omega_m)+\frac{L'}{L}(1+2ir,\chi_1\chi_2\omega_m)\right)dr$$
$$=-\sum_{r=1}^{\infty}\frac{\Lambda(r)\chi_1\chi_2\omega_m(r)}{r}g(2\log r)+\delta_{\chi_1=\overline{\chi_2}}\frac{1}{4}h(0),$$

which follows by imitating [**12**, p. 509] and (2.91). □

REMARK 2.29. We stress that our $\{\chi\}_\varepsilon(r)$ (cf. (2.107)) is not exactly a generalization of the number which was denoted by that symbol in [**6**, (2.41)]. Rather, in the special case when N is squarefree, '$\{\chi\}_\varepsilon(r)$' in [**6**] equals $2^{1-\omega(N/\gcd(r,N))}\{\chi\}_\varepsilon(r)$ in our present notation.

In order to evaluate (2.5), we also need to compute $\operatorname{tr}\Phi^\Gamma(\frac{1}{2})$.

LEMMA 2.30. *We have*

$$\operatorname{tr}\Phi^\Gamma\left(\tfrac{1}{2}\right)=-|F_0^\varepsilon|. \tag{2.108}$$

Proof. We have

$$\operatorname{tr}\Phi^\Gamma\left(\frac{1}{2}\right)=\lim_{s\to\frac{1}{2}}\operatorname{tr}(M(s)^{-1}L(s)^{-1}D(s)PL(1-s)M(1-s))$$
$$=\lim_{s\to\frac{1}{2}}\operatorname{tr}(L(s)^{-1}D(s)PL(1-s)M(1-s)M(s)^{-1}),$$

and using the definition of $L(s)$, $D(s)$ and P, we find that $L(s)^{-1}D(s)PL(1-s)$ is the $|F^\varepsilon|\times|F^\varepsilon|$ matrix that in row $(m,\chi_1,\chi_2)\in F^\varepsilon$ has all entries zero except in column $(q_1m/q_2,\overline{\chi_2},\overline{\chi_1})$, where the entry is

$$\frac{q_1^{1-s}}{q_2}\pi^{2s-1}\frac{\tau(\chi_2)}{\tau(\overline{\chi_1})}\frac{\Gamma(1-s)}{\Gamma(s)}\frac{L(2-2s,\overline{\chi_1\chi_2})}{L(2s,\chi_1\chi_2)}. \tag{2.109}$$

If $\chi_1\neq\overline{\chi_2}$, then $\chi_1\chi_2$ is a nontrivial Dirichlet character and thus $L(s,\chi_1\chi_2)$ is holomorphic at $s=1$ with $L(1,\chi_1\chi_2)\neq0$ (cf., for example, [**8**, § X.11]), and similarly $L(1,\overline{\chi_1\chi_2})\neq0$. However, if $\chi_1=\overline{\chi_2}$, then

$$L(2s,\chi_1\chi_2)=L(2s,\overline{\chi_1\chi_2})=\zeta(2s)\prod_{p\mid q_1}(1-p^{-2s})=\left(\prod_{p\mid q_1}(1-p^{-1})\right)(2s-1)^{-1}+O(1)$$

as $s\to\frac{1}{2}$. Hence, when $s\to\frac{1}{2}$, the number in (2.109) tends to

$$\begin{cases}-1 & \text{if } \chi_1=\overline{\chi_2},\\ \sqrt{\dfrac{q_1}{q_2}}\dfrac{\tau(\chi_2)}{\tau(\overline{\chi_1})}\dfrac{L(1,\overline{\chi_1\chi_2})}{L(1,\chi_1\chi_2)} & \text{if } \chi_1\neq\overline{\chi_2}.\end{cases}$$

Also note that $M(1-s)M(s)^{-1}$ tends to the $|F^\varepsilon|\times|F^\varepsilon|$ identity matrix as $s\to\frac{1}{2}$. Indeed, $M(s)$ is by definition a holomorphic function of s in the entire complex plane, and Lemma 2.26 shows that $M(s)^{-1}$ is also holomorphic when $\Re(s)\neq0$. Hence, we obtain the stated formula. □

2.8. *Continuous spectrum* $\operatorname{Eis}(\Gamma,\chi)$

Recall that we write $\operatorname{Eis}(N,\chi;n)$ for the contribution from the continuous spectrum in the trace formula in Theorem 2.1; thus $\operatorname{Eis}(N,\chi;n)=\operatorname{Eis}(\Gamma,\chi^{(0)})+n\operatorname{Eis}(\Gamma,\chi^{(1)})$, where $\operatorname{Eis}(\Gamma,\chi^{(\varepsilon)})$ is

the corresponding contribution (cf. (2.5)) in the trace formula for $(\Gamma_0^{\pm}(N), \chi^{(\varepsilon)})$. Our aim in this section is to prove the following proposition, giving an explicit formula for $\mathrm{Eis}(N, \chi; n)$. Let

$$\Psi_5(p^{e_p}, p^{s_p}) = \frac{\Psi_1(p^{e_p}, p^{s_p}) - 2}{p-1} + \max\left\{\left\lceil \tfrac{e_p}{2}\right\rceil, s_p\right\} p^{\min\left\{\left\lfloor \frac{e_p}{2}\right\rfloor, e_p - s_p\right\}} + \max\left\{\left\lceil \tfrac{e_p+1}{2}\right\rceil, s_p\right\} p^{\min\left\{\left\lfloor \frac{e_p-1}{2}\right\rfloor, e_p - s_p\right\}}. \tag{2.110}$$

Also let

$$B_p(x) = \begin{cases} xp^x - 2\dfrac{p^x - 1}{p-1} & \text{if } x \in \mathbb{Z}_{\geqslant 0}, \\ 0 & \text{if } x < 0. \end{cases}$$

and

$$\Psi_6(p^{e_p}, p^{s_p}) = \begin{cases} B_p(e_p - s_p) + s_p p^{e_p - s_p} & \text{if } e_p < 2s_p, \\ B_p\Big(\Big\lfloor \dfrac{e_p}{2}\Big\rfloor\Big) + B_p\Big(\Big\lfloor \dfrac{e_p - 1}{2}\Big\rfloor\Big) - B_p(s_p - 1) + s_p p^{s_p - 1} & \text{if } e_p \geqslant 2s_p. \end{cases} \tag{2.111}$$

Finally, let

$$\Psi_7(p^{e_p}, p^{s_p}) = \begin{cases} \begin{cases} -e_p + 1 & \text{if } s_p = 0, \\ -e_p + 2s_p & \text{if } s_p \geqslant 1, \end{cases} & \text{if } p > 2, \\ \begin{cases} -e_p + 3 & \text{if } s_p = 0 \text{ and } e_p \geqslant 3, \\ -e_p + 2s_p + 1 & \text{if } e_p > s_p \geqslant 2, \\ e_p & \text{if } e_p = s_p \geqslant 2, \\ 0 & \text{otherwise,} \end{cases} & \text{if } p = 2. \end{cases} \tag{2.112}$$

Proposition 2.31.

$$\begin{aligned} \mathrm{Eis}(\Gamma, \chi; 1) = {} & \left\{ \Psi_1(N, q) \log \pi - \Psi_1(N, q) \sum_{p \mid N} \frac{\Psi_5(p^{e_p}, p^{s_p}) + \Psi_6(p^{e_p}, p^{s_p})}{\Psi_1(p^{e_p}, p^{s_p})} \log p \right\} g(0) \\ & - \Psi_1(N, q) \frac{1}{2\pi} \int_{\mathbb{R}} h(r) \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + ir\right) dr - \frac{1}{4} I_\chi \Psi_2(N, q) h(0) \\ & + 2 \sum_{m=1}^{\infty} \frac{\Lambda(m)}{m} \prod_{p \mid N} \left[(\Re \chi_p(m)) \Phi_+(p^{e_p}; m)\right] g(2 \log m) \end{aligned} \tag{2.113}$$

and

$$\begin{aligned} & \mathrm{Eis}(\Gamma, \chi; -1) \\ & = -I_\chi \left\{ 2^{\omega(N)} (\Omega_1(N, q)(\log N - \log(2\pi)) + \log 2) + \Omega_1(N, q) 2^{\omega(N) - 1} \sum_{p \mid N} \Psi_7(p^{e_p}, p^{s_p}) \log p \right\} g(0) \\ & \quad - I_\chi 2^{\omega(N)} \Omega_1(N, q) \frac{1}{2\pi} \int_{\mathbb{R}} h(r) \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + ir\right) dr \end{aligned}$$

$$-\frac{1}{4}I_\chi\left\{I_{q,4}\,\chi(\sqrt{-1})\,\min\{e_2+1,4\}\prod_{p|N,p\equiv_4 1}\Psi_2(p^{e_p},p^{s_p})\prod_{p|N,p\equiv_4 -1}2\right\}h(0)$$

$$+2\sum_{m=1}^{\infty}\frac{\Lambda(m)}{m}\prod_{p|N}[(\{\Re/i\Im\}\chi_p(m))\Phi_-(p^{e_p};m)]\,g(2\log m). \tag{2.114}$$

Here Ψ_1, Ψ_5, Ψ_6, Ψ_7, Ψ_2 and $\Phi_\pm$ are given in (2.17), (2.110), (2.111), (2.112), (2.18) *and* (2.23), *respectively. Also $\sqrt{-1}$ denotes a square root of -1 in $\mathbb{Z}/q\mathbb{Z}$ (as exists when $I_{q,4}=1$; and $\chi(\sqrt{-1})$ is independent of the choice of the square root when $I_\chi=1$).*

The proposition will be proved by making the right-hand sides in the formulas in Lemmas 2.28 and 2.30 more explicit. We will carry this out in a sequence of lemmas.

LEMMA 2.32. *Recall that $\varepsilon\in\{0,1\}$ is given by $\chi^{(\varepsilon)}(V)=(-1)^\varepsilon$. We have*

$$\sum_{m\in G_{N,q}}|F_m^\varepsilon|\log\left(\frac{N}{\gcd(m,N/m)}\right)$$

$$=\frac{1}{2}\Psi_1(N,q)\sum_{p|N}\frac{\Psi_5(p^{e_p},p^{s_p})}{\Psi_1(p^{e_p},p^{s_p})}\log p$$

$$+\frac{1}{2}(-1)^\varepsilon I_\chi 2^{\omega(N)}\{\Omega_1(N,q)\log N-(\Omega_1(N,q)-1)\log 2\}. \tag{2.115}$$

Here Ψ_1, Ψ_5 and Ω_1 are given in (2.17), (2.110) *and* (2.21), *respectively. Furthermore,*

$$|F^\varepsilon|=\frac{1}{2}\Psi_1(N,q)+\frac{1}{2}(-1)^\varepsilon I_\chi 2^{\omega(N)}\Omega_1(N,q). \tag{2.116}$$

Proof. By (2.98), we have

$$\sum_{m\in G_{N,q}}|F_m^\varepsilon|\log\left(\frac{N}{\gcd(m,N/m)}\right)$$

$$=\frac{1}{2}\sum_{m|N,q|\frac{N}{\gcd(m,N/m)}}\varphi(\gcd(m,N/m))\log\left(\frac{N}{\gcd(m,N/m)}\right)$$

$$+\frac{1}{2}(-1)^\varepsilon\sum_{\substack{m|N,\gcd(m,N/m)\leqslant 2,\\ q|\frac{N}{\gcd(m,N/m)}}}\chi_{\alpha(N,m)}(-1)\log\left(\frac{N}{\gcd(m,N/m)}\right).$$

For the first term, recalling Lemma 2.10, let

$$f(x)=\sum_{m|N,q|\frac{N}{\gcd(m,N/m)}}\varphi(\gcd(m,N/m))(\gcd(m,N/m))^x.$$

Then we have

$$f(0)=\sum_{m|N,q|\frac{N}{\gcd(m,N/m)}}\varphi(\gcd(m,N/m))=\Psi_1(N,q) \tag{2.117}$$

and

$$f'(0) = \sum_{m|N,q|\frac{N}{\gcd(m,N/m)}} \varphi(\gcd(m,N/m))\log(\gcd(m,N/m)).$$

By (2.49),

$$f'(0) = \sum_{p|N}\left(\prod_{p'|N,p'\neq p}\Psi_1(p'^{e_{p'}},p'^{s_{p'}})\right)\log p\Big[\frac{-\Psi_1(p^{e_p},p^{s_p})+2}{p-1}$$

$$+\min\{\lfloor\tfrac{e_p}{2}\rfloor,e_p-s_p\}p^{\min\{\lfloor\frac{e_p}{2}\rfloor,e_p-s_p\}}+\min\{\Big\lfloor\frac{e_p-1}{2}\Big\rfloor,e_p-s_p\}p^{\min\{\lfloor\frac{e_p-1}{2}\rfloor,e_p-s_p\}}\Big].$$

Note that by definition in (2.17), $\Psi_1(p^{e_p},p^{s_p})\neq 0$. We have

$$\sum_{m|N,q|\frac{N}{\gcd(m,N/m)}}\varphi(\gcd(m,N/m))\log\left(\frac{N}{\gcd(m,N/m)}\right)$$

$$=\log N f(0)-f'(0)=\Psi_1(N,q)\sum_{p|N}\frac{\log p}{\Psi_1(p^{e_p},p^{s_p})}\Psi_5(p^{e_p},p^{s_p}).$$

For the second term, we get

$$\sum_{\substack{m|N,\gcd(m,N/m)\leqslant 2,\\ q|\frac{N}{\gcd(m,N/m)}}}\chi_{\alpha(N,m)}(-1)\log\left(\frac{N}{\gcd(m,N/m)}\right)$$

$$=\log N\sum_{\substack{m|N,\gcd(m,N/m)\leqslant 2,\\ q|\frac{N}{\gcd(m,N/m)}}}\chi_{\alpha(N,m)}(-1)-\log 2\sum_{\substack{m|N,\gcd(m,N/m)=2,\\ q|\frac{N}{\gcd(m,N/m)}}}\chi_{\alpha(N,m)}(-1).$$

As in Lemma 2.16 and its proof, we have

$$\sum_{\substack{m|N,\gcd(m,N/m)\leqslant 2,\\ q|\frac{N}{\gcd(m,N/m)}}}\chi_{\alpha(N,m)}(-1)=I_\chi 2^{\omega(N)}\Omega_1(N,q), \tag{2.118}$$

and

$$\sum_{\substack{m|N,\gcd(m,N/m)=2,\\ q|\frac{N}{\gcd(m,N/m)}}}\chi_{\alpha(N,m)}(-1)=I_\chi 2^{\omega(N)}(\Omega_1(N,q)-1).$$

Similarly, recalling (2.100) and (2.98),

$$|F^\varepsilon|=\sum_{m\in G_{N,q}}|F_m^\varepsilon|=\frac{1}{2}\sum_{m|N,q|\frac{N}{\gcd(m,N/m)}}\varphi(\gcd(m,N/m))$$

$$+\frac{1}{2}(-1)^\varepsilon\sum_{\substack{m|N,\gcd(m,N/m)\leqslant 2,\\ q|\frac{N}{\gcd(m,N/m)}}}\chi_{\alpha(N,m)}(-1),$$

so by (2.117) and (2.118), we get (2.116). □

For $p\mid N$, let

$$S_p=\{0\leqslant f\leqslant e_p\ :\ \min\{f,e_p-f\}\leqslant e_p-s_p\}. \tag{2.119}$$

Then note that

$$S_p = \{0 \leqslant f \leqslant e_p \ : \ f \leqslant e_p - s_p \text{ or } f \geqslant s_p\}.$$

LEMMA 2.33. *Fix a prime $p \mid N$. For $f \in S_p$, if $f \geqslant s_p$,*

$$F_{p^f} = \Big\{(\chi_1, \chi_2) \ : \ \chi_1 \in X_{p^{\min\{f, e_p - f\}}}, \chi_2 \text{ primitive, determined by } \operatorname{cond}(\chi_p \chi_2 \overline{\chi_1}) = 1\Big\}$$

and if $f < s_p$,

$$F_{p^f} = \Big\{(\chi_1, \chi_2) \ : \ \chi_2 \in X_{p^{\min\{f, e_p - f\}}}, \chi_1 \text{ primitive, determined by } \operatorname{cond}(\chi_p \chi_2 \overline{\chi_1}) = 1\Big\}.$$

Here F_{p^f} is given in (2.83).

Proof. Similar to the proof of Lemma 2.21. □

LEMMA 2.34.

$$|F_0^\varepsilon| = \frac{1}{2} I_\chi \left\{ \Psi_2(N, q) + (-1)^\varepsilon I_{q,4}\, \chi(\sqrt{-1}) \min\{e_2 + 1, 4\} \prod_{p \mid N, p \equiv_4 1} \Psi_2(p^{e_p}, p^{s_p}) \prod_{p \mid N, p \equiv_4 -1} 2 \right\}$$

Here Ψ_2 is given in (2.18), and $\sqrt{-1}$ denotes a square root of -1 in $\mathbb{Z}/q\mathbb{Z}$ (as exists when $I_{q,4} = 1$; and $\chi(\sqrt{-1})$ is independent of the choice of the square root when $I_\chi = 1$).

Proof. Recalling (2.91), we can write

$$F_0^\varepsilon = \big\{(m, \chi_1) \ : \ m \in \mathbb{Z}_{\geqslant 1},\ m \mid N,\ q_1 \mid \gcd(m, N/m),\ \operatorname{cond}(\chi \overline{\chi_1}^2) = 1,\ \chi_1(-1) = (-1)^\varepsilon\big\}.$$

Note that $F_0^\varepsilon = \emptyset$ unless χ is square in the sense that there exists a primitive character χ_1 such that $\chi \overline{\chi_1}^2(n) = 1$ for any n with $\gcd(n, q) = 1$. There exists such a primitive character if and only if for every prime $p \mid N$, $\chi_p(-1) = 1$, that is, χ is pure. From now on, we assume that χ is pure.

Let

$$F_0(N) = \big\{(m, \chi_1) \ : \ m \in \mathbb{Z}_{\geqslant 1},\ m \mid N,\ q_1 \mid \gcd(m, N/m),\ \operatorname{cond}(\chi \overline{\chi_1}^2) = 1\big\}$$

and

$$S_0^\pm = \sum_{(m, \chi_1) \in F_0(N)} \chi_1(\pm 1).$$

Then

$$|F_0^\varepsilon| = \frac{1}{2}\big(S_0^+ + (-1)^\varepsilon S_0^-\big).$$

Note that $S_0^\pm$ is multiplicative. One can write

$$S_0^\pm = \prod_{p \mid N} \left(\sum_{f \in S_p} \sum_{(p^f, \chi_1) \in F_0(p^{e_p})} \chi_1(\pm 1) \right).$$

So for $p \mid N$, let

$$S_{0,p}^\pm = \sum_{f \in S_p} \sum_{(p^f, \chi_1) \in F_0(p^{e_p})} \chi_1(\pm 1).$$

For every prime $p \mid N$, fix a primitive Dirichlet character ψ_p of minimal conductor subject to $\mathrm{cond}(\chi_p\overline{\psi_p}^2) = 1$. We further assume that

$$\mathrm{ord}_p(\mathrm{cond}(\psi_p)) = \begin{cases} 0 & \text{if } p = 2 \text{ and } s_p = 0, \\ s_p + 1 & \text{if } p = 2 \text{ and } s_p \geqslant 3, \\ s_p & \text{if } p > 2. \end{cases}$$

Note that s_2 cannot be 1 or 2.

When p is odd, let ξ_p be the quadratic character modulo p. When $p = 2$, let ξ_4 be the quadratic character modulo 4 and ξ_8 be the even quadratic character modulo 8. Then $\xi_4\xi_8$ is another quadratic character, and ξ_8, $\xi_4\xi_8$ are the only primitive characters modulo 8.

When χ_1 is the primitive character such that $\mathrm{cond}(\chi_p\overline{\chi_1}^2) = 1$, then

$$\chi_1 \in \{\psi_p\xi_p^{u_p} \ : \ u_p \in \{0,1\}\},$$

when p is odd, and

$$\chi_1 \in \{\psi_2\xi_4^{u_2}\xi_8^{v_2} \ : \ u_2, v_2 \in \{0,1\}\},$$

when $p = 2$. Here ψ_p is as fixed above, so $q(\chi\overline{\chi_1}^2) = 1$. Note that

$$\mathrm{ord}_p(q_1) = \begin{cases} \max\{s_p, u_p\} & \text{if } p \text{ odd }, \\ s_p + 1 & \text{if } p = 2 \text{ and } s_p \geqslant 3, \\ \max\{2u_2, 3v_2\} & \text{if } p = 2 \text{ and } s_p = 0. \end{cases}$$

When p is odd, we have

$$\begin{aligned} S_{0,p}^{\pm} &= \sum_{u_p \in \{0,1\}} \sum_{\max\{s_p,u_p\} \leqslant \min\{f, e_p - f\}} \psi_p\xi_p^{u_p}(\pm 1) \\ &= \psi_p(\pm 1) \begin{cases} (e_p - 1)(1 + \xi_p(\pm 1)) + 2 & \text{if } s_p = 0, \\ (e_p - 2s_p + 1)(1 + \xi_p(\pm 1)) & \text{if } s_p \geqslant 1 \text{ and } 2s_p \leqslant e_p, \\ 0 & \text{if } 2s_p > e_p. \end{cases} \end{aligned}$$

Note that $1 + \xi_p(-1) = 0$ if $s_p > 0$ and $p \equiv_4 -1$. So

$$S_{0,p}^{+} = \Psi_2(p^{e_p}, p^{s_p})$$

and

$$S_{0,p}^{-} = \psi_p(-1) \begin{cases} 2 & \text{if } s_p = 0 \text{ and } p \equiv_4 -1, \\ \Psi_2(p^{e_p}, p^{s_p}) & \text{if } p \equiv_4 1, \\ 0 & \text{if } s_p > 0 \text{ and } p \equiv_4 -1. \end{cases}$$

When $p = 2$, we have

$$S_{0,p}^{\pm} = \sum_{u_2, v_2 \in \{0,1\}} \begin{cases} \displaystyle\sum_{\max\{2u_2, 3v_2\} \leqslant \min\{f, e_p - f\}} \xi_4^{u_2}\xi_8^{v_2}(\pm 1) & \text{if } s_p = 0, \\ \displaystyle\sum_{s_p + 1 \leqslant \min\{f, e_p - f\}} \psi_p\xi_4^{u_2}\xi_8^{v_2}(\pm 1) & \text{if } s_p \geqslant 3. \end{cases}$$

So we get

$$S_{0,p}^{\pm} = \psi_p(\pm 1) \begin{cases} e_p + 1 + \max\{e_p - 5, 0\}(1 + \xi_4(\pm 1)) + \max\{e_p - 3, 0\}\xi_4(\pm 1) & \text{if } s_p = 0, \\ 2\max\{e_p - 2s_p - 1, 0\}\psi_p(\pm 1)(1 + \psi_4(\pm 1)) & \text{if } s_p \geqslant 3. \end{cases}$$

Hence

$$S^{+}_{0,p} = \Psi_2(p^{e_p}, p^{s_p})$$

and

$$S^{-}_{0,p} = \begin{cases} 4 & \text{if } e_p \geqslant 3 \text{ and } s_p = 0, \\ e_p + 1 & \text{if } e_p \in \{1,2\} \text{ and } s_p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

□

LEMMA 2.35.

$$\sum_{(m,\chi_1,\chi_2)\in F^{\varepsilon}} \log q_1 = \frac{1}{2}\Psi_1(N,q)\sum_{p\mid N}\frac{\Psi_6(p^{e_p},p^{s_p})}{\Psi_1(p^{e_p},p^{s_p})}\log p + (-1)^{\varepsilon} I_{\chi} 2^{\omega(N)-2}\Omega_1(N,q)\sum_{p\mid N}\Psi_7(p^{e_p},p^{s_p})\log p.$$

Here Ψ_1, Ω_1, Ψ_6 *and* Ψ_7 *are given in* (2.17), (2.21), (2.111) *and* (2.112).

Proof. Let

$$S^{\pm} = \sum_{(m,\chi_1,\chi_2)\in F} \chi_1(\pm 1)\log q_1.$$

Then

$$\sum_{(m,\chi_1,\chi_2)\in F^{\varepsilon}} \log q_1 = \frac{1}{2}\big(S^{+} + (-1)^{\varepsilon} S^{-}\big).$$

For $n \in \{\pm 1\}$, let

$$f_n(x) = \sum_{(m,\chi_1,\chi_2)\in F} \chi_1(n) q_1^{x}.$$

Then

$$f_n'(0) = \begin{cases} S^{+} & \text{if } n = 1, \\ S^{-} & \text{if } n = -1. \end{cases}$$

Recalling (2.119),

$$f_n(x) = \prod_{p\mid N}\left(\sum_{f\in S_p}\sum_{(\chi_1,\chi_2)\in F_{pf}} \chi_1(n) q_1^{x}\right).$$

For each prime $p \mid N$,

$$f_{n,p}(x) = \sum_{f\in S_p}\sum_{(\chi_1,\chi_2)\in F_{pf}} \chi_1(n) q_1^{x}.$$

Then

$$f_n'(0) = \sum_{p\mid N} f_{n,p}'(0) \prod_{p'\mid N, p'\neq p} f_{n,p'}(0).$$

For each prime $p \mid N$ and $f \in S_p$, by Lemma 2.33, we have

$$\sum_{(\chi_1,\chi_2)\in F_{pf}} \chi_1(n) q_1^x = \begin{cases} \sum_{\ell=0}^{\min\{f,e_p-f\}} \left(\sum_{\substack{\psi \text{ primitive} \\ \mathrm{cond}(\psi)=p^\ell}} \psi(n) \right) p^{\ell x} & \text{if } f \geqslant s_p, \\ \sum_{\ell=0}^{\min\{f,e_p-f\}} \left(\sum_{\substack{\psi \text{ primitive} \\ \mathrm{cond}(\psi)=p^\ell}} \chi_p\psi(n) \right) p^{s_p x} & \text{if } f < s_p. \end{cases}$$

Note that

$$\sum_{\substack{\psi \text{ primitive} \\ \mathrm{cond}(\psi)=p^\ell}} \psi(n) = \begin{cases} 1 & \text{if } \ell = 0, \\ p^{\ell-2}(p-1)^2 & \text{if } n = 1 \text{ and } \ell \geqslant 2, \\ p-2 & \text{if } n = 1 \text{ and } \ell = 1, \\ -1 & \text{if } p > 2, n = -1 \text{ and } \ell = 1, \\ & \text{or } p = 2, n = -1 \text{ and } \ell = 2, \\ 0 & \text{otherwise.} \end{cases}$$

So when $f \geqslant s_p$,

$$\sum_{(\chi_1,\chi_2)\in F_{pf}} \chi_1(n) q_1^x$$

$$= \begin{cases} 1 & \text{if } \min\{f, e_p - f\} = 0, \\ & \text{or } p = 2, n = -1 \text{ and } \min\{f, e_p - f\} = 1, \\ \frac{(p-1)^2 p^{\min\{f,e_p-f\}(x+1)+x-1} - (p^x-1)^2}{p^{x+1}-1} & \text{if } n = 1 \text{ and } \min\{f, e_p - f\} \geqslant 1, \\ 1 - p^{x(1+\delta_{p=2})} & \text{if } n = -1 \text{ and } \min\{f, e_p - f\} \geqslant 1 + \delta_{p=2}. \end{cases}$$

When $f < s_p$, we have

$$\sum_{(\chi_1,\chi_2)\in F_{pf}} \chi_1(n) q_1^x$$

$$= \chi_p(n) p^{s_p x} \begin{cases} 1 & \text{if } \min\{f, e_p - f\} = 0, \\ & \text{or } p = 2, n = -1 \text{ and } \min\{f, e_p - f\} = 1, \\ (p-1)p^{\min\{f,e_p-f\}-1} & \text{if } n = 1 \text{ and } \min\{f, e_p - f\} \geqslant 1, \\ 0 & \text{if } n = -1 \text{ and } \min\{f, e_p - f\} \geqslant 1 + \delta_{p=2}. \end{cases}$$

For $n = 1$ and $s_p > \lfloor \frac{e_p}{2} \rfloor$, we have

$$f_{1,p}(x) = \sum_{f\in S_p} \sum_{(\chi_1,\chi_2)\in F_{pf}} \chi_1(n) q_1^x$$

$$= p^{s_p x + e_p - s_p} + 1 + \frac{1}{p^{x+1}-1} \left((p-1)^2 p^{2x} \frac{p^{(e_p-s_p)(x+1)} - 1}{p^{x+1}-1} - (p^x-1)^2 (e_p - s_p + 1) \right).$$

For $s_p \leqslant \lfloor \frac{e_p}{2} \rfloor$, we have

$$f_{1,p}(x) = p^{s_p(x+1)-1} + 1 + \frac{(p-1)^2 p^{x-1}}{(p^{x+1}-1)^2} \left(p^{(\lfloor \frac{e_p}{2} \rfloor + 1)(x+1)} + p^{\lceil \frac{e_p}{2} \rceil (x+1)} - p^{s_p(x+1)} - p^{x+1} \right)$$

$$-(e-s)\frac{(p^x-1)^2}{p^{x+1}-1},$$

So we have

$$f_{1,p}(0) = \begin{cases} 2p^{e_p - s_p} & \text{if } e_p < 2s_p, \\ p^{\lfloor \frac{e_p}{2} \rfloor} + p^{\lfloor \frac{e_p - 1}{2} \rfloor} & \text{if } e_p \geqslant 2s_p \end{cases} = \Psi_1(p^{e_p}, p^{s_p})$$

and

$$f'_{1,p}(0) = \log p \begin{cases} B_p(e_p - s_p) + s_p p^{e_p - s_p} & \text{if } e_p < 2s_p, \\ B_p\big(\lfloor \frac{e_p}{2} \rfloor\big) + B_p\Big(\frac{e_p - 1}{2}\Big) - B_p(s_p - 1) + s_p p^{s_p - 1} & \text{if } e_p \geqslant 2s_p. \end{cases}$$

Hence

$$S^+ = \Psi_1(N, q) \sum_{p \mid N} \frac{\Psi_6(p^{e_p}, p^{s_p})}{\Psi_1(p^{e_p}, p^{s_p})} \log p.$$

For $n = -1$ and $p > 2$, we have

$$f_{-1,p}(x) = \begin{cases} 2 + (e_p - 1)(1 - p^x) & \text{if } s_p = 0, \\ 1 + \chi_p(-1) p^{s_p x} + (e_p - s_p)(1 - p^x) & \text{if } s_p > 0. \end{cases}$$

So

$$f_{-1,p}(0) = 1 + \chi_p(-1)$$

and

$$f'_{-1,p}(0) = \log p \begin{cases} -e_p + 1 & \text{if } s_p = 0, \\ s_p(1 + \chi_p(-1)) - e_p & \text{if } s_p > 0. \end{cases}$$

For $n = -1$ and $p = 2$, we have

$$f_{-1,p}(x) = \begin{cases} 1 + \chi_p(-1) p^{s_p x} & \text{if } e_p = s_p, \\ 2(1 + \chi_p(-1) p^{s_p x}) + (e_p - s_p - 1)(1 - p^{2x}) & \text{if } e_p > s_p \geqslant 2, \\ 2 & \text{if } s_p = 0 \text{ and } e_p = 1, \\ 3 & \text{if } s_p = 0 \text{ and } e_p = 2, \\ 4 + (e_p - 3)(1 - p^{2x}) & \text{if } s_p = 0 \text{ and } e_p \geqslant 3. \end{cases}$$

So we get

$$f_{-1,p}(0) = (1 + \chi_p(-1)) \begin{cases} 1 & \text{if } e_p = s_p \geqslant 2 \text{ or } e_p = 1, \\ 2 & \text{if } e_p > s_p \text{ and } e_p \geqslant 3, \\ \frac{3}{2} & \text{if } s_p = 0 \text{ and } e_p = 2, \end{cases}$$

and

$$f'_{-1,p}(0) = 2 \log p \begin{cases} \frac{1}{2} s_p \chi_p(-1) & \text{if } e_p = s_p \geqslant 2, \\ s_p \chi_p(-1) - e_p + s_p + 1 & \text{if } e_p > s_p \geqslant 2, \\ -e_p + 3 & \text{if } s_p = 0 \text{ and } e_p \geqslant 3, \\ 0 & \text{otherwise.} \end{cases}$$

Since $1 + \chi_p(-1) = 0$ unless χ_p is even, we can conclude that $S^- = 0$ unless χ is pure. Hence we get

$$S^- = \sum_{p \mid N} f'_{-1,p}(0) \prod_{p' \mid N, p' \neq p} f_{-1,p'}(0) = I_\chi 2^{\omega(N) - 1} \Omega_1(N, q) \sum_{p \mid N} \Psi_7(p^{e_p}, p^{s_p}) \log p. \qquad \square$$

LEMMA 2.36. *Let $r \in \mathbb{Z}$. If $\gcd(r,q) > 1$, then $\{\chi\}_\varepsilon(r) = 0$, while if $\gcd(r,q) = 1$, then*

$$\{\chi\}_\varepsilon(r) = \frac{1}{2}\left\{ \prod_{p\mid N, p\nmid r} (\Re\chi_p(r))\Phi_+(p^{e_p}, r) + (-1)^\varepsilon \prod_{p\mid N, p\nmid r} (\{\Re/i\Im\}\chi_p(r))\Phi_-(p^{e_p}, r) \right\},$$

where $\{\Re/i\Im\}\chi_p(r) = \frac{1}{2}(\overline{\chi_p(r)} + \chi_p(-r))$, and Φ_+ and Φ_- are given in (2.23).

Proof. Let

$$f_\chi^\pm(r) = \sum_{(m,\chi_1,\chi_2)\in F} \chi_1(\pm 1)\chi_1\chi_2\omega_m(r).$$

Then, recalling (2.107),

$$\{\chi\}_\varepsilon(r) = \frac{1}{2}\big(f_\chi^+(r) + (-1)^\varepsilon f_\chi^-(r)\big).$$

Note that $f_\chi^\pm(r)$ is multiplicative. For a prime $p \mid N$, let

$$f_{\chi,p}^\pm(r) = \sum_{f\in S_p} \omega_{p^f}(r) \sum_{(\chi_1,\chi_2)\in F_p} \chi_1(\pm 1)\chi_1\chi_2(r).$$

Then $f_\chi^\pm = \prod_{p\mid N} f_{\chi,p}^\pm$.

When $p \mid r$, since $\omega_{p^f}(r) = 0$ unless $f = 0$, so $f_{\chi,p}^\pm(r) = 0$ unless $s_p = 0$. From now on, assume $p \nmid r$.

Recalling (2.119) and Lemma 2.33,

$$f_{\chi,p}^\pm(r) = \sum_{f\in S_p} \begin{cases} \overline{\chi_p(r)} \sum_{\psi\in X_{p^{\min\{f,e_p-f\}}}} \psi(\pm r^2) & \text{if } f \geqslant s_p, \\ \chi_p(\pm r) \sum_{\psi\in X_{p^{\min\{f,e_p-f\}}}} \psi(\pm r^2) & \text{if } f < s_p. \end{cases}$$

Set $\operatorname{ord}_p(0) = \infty$. Then

$$\sum_{\psi\in X_{p^{\min\{f,e_p-f\}}}} \psi(\pm r^2) = \begin{cases} \varphi(p^{\min\{f,e_p-f\}}) & \text{if } \min\{f, e_p - f\} \leqslant \operatorname{ord}_p(\pm r^2 - 1), \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\sum_{f=0}^k \varphi(p^f) = p^k$.

Let $u_\pm = \operatorname{ord}_p(\pm r^2 - 1)$. When $e_p < 2s_p$, we have

$$f_{\chi,p}^\pm(r) = (\chi_p(\pm r) + \overline{\chi_p(r)}) \sum_{f=0}^{\min\{e_p-s_p,u_\pm\}} \varphi(p^f) = (\chi_p(\pm r) + \overline{\chi_p(r)})p^{e_p - w_\pm(r)}.$$

When $e_p \geqslant 2s_p$, we have

$$f_{\chi,p}^\pm(r) = \chi_p(\pm r) \sum_{f=0}^{\min\{s_p-1,u_\pm\}} \varphi(p^f) + \overline{\chi_p(r)} \left(\sum_{f=s_p}^{\min\{\lfloor \frac{e_p}{2}\rfloor, u_\pm\}} \varphi(p^f) + \sum_{f=0}^{\min\{u_\pm, \lfloor \frac{e_p-1}{2}\rfloor\}} \varphi(p^f) \right).$$

For the second term, we have two cases: $s_p > \min\{\lfloor \frac{e_p}{2}\rfloor, u_\pm\}$ or $s_p \leqslant \min\{\lfloor \frac{e_p}{2}\rfloor, u_\pm\}$.

When $s_p > \min\{\lfloor \frac{e_p}{2}\rfloor, u_\pm\}$, since $e_p \geqslant 2s_p$, we get $u_\pm \leqslant s_p - 1$. So we have

$$f_{\chi,p}^\pm(r) = (\chi_p(\pm r) + \overline{\chi_p(r)}) \sum_{f=0}^{u_\pm} \varphi(p^f) = (\chi_p(\pm r) + \overline{\chi_p(r)})p^{u_\pm}.$$

When $s_p \leqslant \min\{\lfloor \frac{e_p}{2}\rfloor, u_\pm\}$, that is, $s_p \leqslant u_\pm$, we get

$$\chi_p(\pm r^2) = \chi_p(\pm 1)\chi_p(r)^2 = 1.$$

We get

$$f^{\pm}_{\chi,p}(r) = (\chi_p(\pm r) + \overline{\chi_p(r)})\left(p^{\min\{\lfloor \frac{e_p-1}{2} \rfloor, u_\pm\}} + \frac{1}{2}\delta_{2|e_p, \frac{e_p}{2} \leqslant u_\pm} \varphi(p^{\frac{e_p}{2}})\right).$$

So we have

$$f^{\pm}_{\chi,p}(r) = (\chi_p(\pm r) + \overline{\chi_p(r)}) \begin{cases} \frac{1}{2}\left(p^{\lfloor \frac{e_p-1}{2} \rfloor} + p^{\lfloor \frac{e_p}{2} \rfloor}\right) & \text{if } w_\pm(r) \leqslant \lfloor \frac{e_p}{2} \rfloor, \\ p^{e_p - w_\pm(r)} & \text{if } w_\pm(r) > \lfloor \frac{e_p}{2} \rfloor. \end{cases}$$

□

Proof of Proposition 2.31. The proposition follows from

$$\mathrm{Eis}(N, \chi; n) = \mathrm{Eis}(\Gamma, \chi^{(0)}) + n\,\mathrm{Eis}(\Gamma, \chi^{(1)}),$$

by evaluating $\mathrm{Eis}(\Gamma, \chi^{(0)})$ and $\mathrm{Eis}(\Gamma, \chi^{(1)})$ using Lemmas 2.28 and 2.30 from the previous section together with Lemmas 2.32–2.36 in the present section. □

2.9. *Cuspidal and Continuous contributions* $(\mathrm{C} + \mathrm{Eis})(\Gamma, \chi)$

Finally we conclude the proof of Theorem 2.1 by computing the sum of $\mathrm{C}(N, \chi; n)$ from Proposition 2.15 and $\mathrm{Eis}(\Gamma, \chi; n)$ from Proposition 2.31.

For $n = 1$, we combine (2.66) and (2.113), and note that

$$(\Psi_4 + \Psi_5 + \Psi_6)(p^{e_p}, p^{s_p})$$

$$= e_p\left(p^{\lfloor \frac{e_p}{2} \rfloor} + p^{\lfloor \frac{e_p-1}{2} \rfloor}\right) - \frac{p^{\lfloor \frac{e_p}{2} \rfloor} + p^{\lfloor \frac{e_p-1}{2} \rfloor} - 2}{p-1} + p^{s_p-1} + 2\frac{p^{s_p-1}-1}{p-1} = \Psi_3(p^{e_p}, p^{s_p}).$$

Here Ψ_3 is given in (2.20). Hence we obtain (2.25).

For $n = -1$, we combine (2.67) and (2.114). Here one notes that, if $I_\chi = I_{q,4} = 1$,

$$\chi(\sqrt{-1}) \min\{e_2 + 1, 4\} \prod_{p|N, p \equiv_4 1} \Psi_2(p^{e_p}, p^{s_p}) \prod_{p|N, p \equiv_4 -1} 2 = \tilde{\Psi}_2(N, \chi),$$

with $\tilde{\Psi}_2$ as in (2.19). Recall also that if $I_\chi = 1$, that is, if χ is pure, then $s_2 = 0$ or $s_2 \geqslant 3$. Therefore,

$$I_\chi 2^{\omega(N)} \tfrac{1}{2}(\Omega_1(N, q)(-e_2 + 2) + \varepsilon_N - 2 - \Omega_1(N, q)\Psi_7(2^{e_2}, 2^{s_2})) = I_\chi 2^{\omega(N)} \Omega_2(N, q),$$

where Ω_2 is given in (2.22). Finally for $p \mid N$ odd, recalling (2.112),

$$\frac{1}{2}(\Psi_7(p^{e_p}, p^{s_p}) + e_p) = \max\left\{\frac{1}{2}, s_p\right\}.$$

Hence we obtain (2.26).

Recall also that the formulas for $\mathrm{I}(\Gamma, \chi; n)$ and $(\mathrm{NEll} + \mathrm{Ell})(\Gamma, \chi; n)$ stated in Theorem 2.1 were proved in Lemma 2.2 and Proposition 2.3, respectively. Hence the proof of Theorem 2.1 is now complete.

3. *Sieving*

Our goal in this section is to prove Theorem 1.7, by sieving out the contribution from the twist-minimal Hecke eigenforms in the trace formula in Theorem 2.1.

The sieving is carried out in two steps: we first sieve for newforms, and then sieve for twist-minimal forms among the newforms. The first step is as in [**6**, § 2.3]: for any Dirichlet character

χ modulo N of conductor $\mathrm{cond}(\chi)$, any $\lambda > 0$ and $n \in \{\pm 1\}$, we have

$$\operatorname{Tr} T_n|_{\mathcal{A}_\lambda^{\mathrm{new}}(\chi)} = \sum_{M \mid \frac{N}{\mathrm{cond}(\chi)}} \beta\Big(\frac{N/\mathrm{cond}(\chi)}{M}\Big) \operatorname{Tr} T_n|_{\mathcal{A}_\lambda(\chi|_{M\,\mathrm{cond}(\chi)})}, \tag{3.1}$$

where $\beta(m)$ is the multiplicative function given by $\zeta(s)^{-2} = \sum_{m=1}^{\infty} \beta(m) m^{-s}$.

For the second step, sieving down to $\mathcal{A}_\lambda^{\min}(\chi)$, we assume that χ is minimal as in Definition 1.5. Let S_χ be the set of pairs $\langle M, \psi\rangle$ where M is a (positive) divisor of N and ψ is a primitive Dirichlet character such that $\mathrm{lcm}(M, \mathrm{cond}(\psi)\,\mathrm{cond}(\chi\psi)) = N$, $\mathrm{cond}(\chi\psi^2) \mid M$ and $\chi\psi^2|_M$ is minimal. Also let $\sim$ be the equivalence relation on S_χ defined by $\langle M, \psi\rangle \sim \langle M', \psi'\rangle$ if and only if $M' = M$ and $\mathcal{A}_\lambda^{\min}(\chi\psi^2|_M) \otimes \overline{\psi} = \mathcal{A}_\lambda^{\min}(\chi\psi'^2|'_M) \otimes \overline{\psi}'$. It then follows from Lemmas 1.4 and 1.6 that for any $\lambda > 0$, we have the direct sum decomposition

$$\mathcal{A}_\lambda^{\mathrm{new}}(\chi) = \bigoplus_{\langle M,\psi\rangle \in S_\chi/\sim} \big(\mathcal{A}_\lambda^{\min}(\chi\psi^2|_M) \otimes \overline{\psi}\big), \tag{3.2}$$

where $S_\chi/\sim$ denotes any set of representatives for S_χ modulo $\sim$. Hence, also using the fact that $\overline{\psi(n)} = \psi(n)$ for $n \in \{\pm 1\}$,

$$\operatorname{Tr} T_n|_{\mathcal{A}_\lambda^{\mathrm{new}}(\chi)} = \sum_{\langle M,\psi\rangle \in S_\chi/\sim} \psi(n) \operatorname{Tr} T_n|_{\mathcal{A}_\lambda^{\min}(\chi\psi^2|_M)}. \tag{3.3}$$

Note that the pairs $\langle M, \psi\rangle$ in S_χ with $M = N$ form a single equivalence class, and this class contributes via the term $\operatorname{Tr} T_n|_{\mathcal{A}_\lambda^{\min}(\chi)}$ to the sum in (3.3).

In order to invert the formula (3.3), we first note that the set S_χ and the relation $\sim$ can be fully described by local conditions: writing $\chi = \prod_{p|N} \chi_p$ as usual, and setting $N_p = p^{e_p}$ ($e_p = \mathrm{ord}_p(N)$) and $M_p = p^{\mathrm{ord}_p(M)}$, we have that a pair $\langle M, \psi\rangle$ lies in S_χ if and only if $\langle M_p, \psi_p\rangle$ lies in S_{χ_p} for each prime p, and furthermore $\langle M, \psi\rangle \sim \langle M', \psi'\rangle$ holds if and only if $\langle M_p, \psi_p\rangle \sim \langle M'_p, \psi'_p\rangle$ for all primes $p \mid N$. Also the equivalence classes in S_{χ_p} are easily classified: if $p = 2$ or $e_p \leqslant 1$ or $2 \nmid e_p$ or $s_p > 1$, then *all* elements in S_{χ_p} are equivalent with $\langle N_p, 1\rangle$. In the remaining case when $p > 2$, $e_p \geqslant 2$, $2 \mid e_p$ and $s_p \in \{0, 1\}$, the set of elements in S_{χ_p} outside the equivalence class of $\langle N_p, 1\rangle$ equals

$$\{\langle p^{e_p/2}, \psi_p\rangle \,:\, \mathrm{cond}(\psi_p) = p^{e_p/2},\ \psi_p \neq \overline{\chi}_p\} \qquad \text{if } e_p \geqslant 4 \text{ or } s_p = 1, \tag{3.4}$$

and

$$\{\langle p, \psi_p\rangle \,:\, \mathrm{cond}(\psi_p) = p\} \cup \{\langle 1, (\tfrac{\cdot}{p})\rangle\} \qquad \text{if } e_p = 2 \text{ and } s_p = 0. \tag{3.5}$$

In the case of (3.4), each $\sim$ equivalence class in that set has exactly two elements. In the case of (3.5), the elements $\langle p, (\frac{\cdot}{p})\rangle$ and $\langle 1, (\frac{\cdot}{p})\rangle$ form singleton equivalence classes, while the remaining $p - 3$ elements group together into equivalence classes with exactly two elements each.

In particular, it follows from the above description that (3.3) can be rewritten as

$$\operatorname{Tr} T_n|_{\mathcal{A}_\lambda^{\mathrm{new}}(\chi)} = \sum_{\langle M,\psi\rangle \in S'_\chi} 2^{-k(N,M,\psi)}\, \psi(n) \operatorname{Tr} T_n|_{\mathcal{A}_\lambda^{\min}(\chi\psi^2|_M)},$$

where S'_χ is the subset of all $\langle M, \psi\rangle \in S_\chi$ satisfying $[M_p < N_p$ or $\psi_p = 1]$ for each prime $p \mid N$, and $k(N, M, \psi)$ is the number of primes $p \mid N$ for which $M_p < N_p$ and $[s_p = 1$ or $\psi_p \neq (\frac{\cdot}{p})]$. This formula can now be inverted as follows:

$$\operatorname{Tr} T_n|_{\mathcal{A}_\lambda^{\min}(\chi)} = \sum_{\langle M,\psi\rangle \in S'_\chi} (-1)^{k'(N,M)} 2^{-k(N,M,\psi)}\, \psi(n) \operatorname{Tr} T_n|_{\mathcal{A}_\lambda^{\mathrm{new}}(\chi\psi^2|_M)}, \tag{3.6}$$

where $k'(N, M)$ is the number of primes $p \mid N$ for which $M_p < N_p$.

We will apply the sieving in (3.1) and then (3.6) to the trace formula in Theorem 2.1. Note that the right-hand side of that formula is a sum of terms $f(N,\chi)$ each of which is multiplicative with respect to χ, in the sense that $f(N,\chi)=\prod_{p|N} f(p^{e_p},\chi_p)$ for any Dirichlet character χ modulo N. One verifies that this multiplicativity property is preserved by the sieving, that is, for any multiplicative function $f(N,\chi)$, if we define f^{new} and f^{min} via

$$f^{\mathrm{new}}(N,\chi)=\sum_{M|\frac{N}{\mathrm{cond}(\chi)}}\beta\Big(\frac{N/\mathrm{cond}(\chi)}{M}\Big)f(M\,\mathrm{cond}(\chi),\chi) \tag{3.7}$$

and

$$f^{\mathrm{min}}(N,\chi)=\sum_{\langle M,\psi\rangle\in S'_\chi}(-1)^{k'(N,M)}2^{-k(N,M,\psi)}\,\psi(n)f^{\mathrm{new}}(M,\chi\psi^2), \tag{3.8}$$

then also f^{new} and f^{min} are multiplicative. Hence our task is reduced to computing $f^{\mathrm{min}}(p^{e_p},\chi_p)$ for each term $f(N,\chi)$ appearing in the trace formula in Theorem 2.1. Writing $e=e_p$ and $s=s_p$, we note that (3.7) implies

$$f^{\mathrm{new}}(p^e,\chi_p)=\sum_{j=0}^{e-s}\beta(p^j)f(p^{e-j},\chi_p), \tag{3.9}$$

and we have, for each $j\geqslant 0$,

$$\beta(p^j)=\begin{cases}1 & \text{if } j\in\{0,2\},\\ -2 & \text{if } j=1,\\ 0 & \text{otherwise.}\end{cases}$$

Next assume again that $\chi=\prod_p\chi_p$ is minimal as in Definition 1.5. It now follows from (3.8) and the description of the equivalence classes of S_{χ_p} given around (3.4) and (3.5) that, for each odd prime $p\mid N$:

$$\begin{aligned}f^{\mathrm{min}}(p^e,\chi_p)=f^{\mathrm{new}}(p^e,\chi_p)-\delta_{e=2,s=0}\Big(\tfrac{n}{p}\Big)\big(f^{\mathrm{new}}(1,1)+\tfrac12 f^{\mathrm{new}}(p,1)\big)\\ -\delta_{2|e,s\leqslant 1}\cdot\frac12\sum_{\substack{\mathrm{cond}(\psi)=p^{\frac{e}{2}}\\ \psi\neq\overline{\chi}_p}}\psi(n)f^{\mathrm{new}}(p^{\frac{e}{2}},\chi_p\psi^2).\end{aligned} \tag{3.10}$$

On the other hand for $p=2$, we have simply

$$f^{\mathrm{min}}(2^e,\chi_2)=f^{\mathrm{new}}(2^e,\chi_2). \tag{3.11}$$

3.1. $\mathrm{I}(\Gamma,\chi)$

Recalling Lemma 2.2, by (3.10) and (3.11), we have

$$\mathrm{I}^{\mathrm{min}}(\Gamma,\chi;n)=\frac{1+n}{2}\frac{M(\chi)}{12}\int_{\mathbb{R}}rh(r)\tanh(\pi r)\,dr.$$

Here $M(\chi)$ is given in (1.3). By [**6**, p. 141],

$$\int_{\mathbb{R}}rh(r)\tanh(\pi r)\,dr=-\int_{-\infty}^{\infty}\frac{g'(u)}{\sinh(u/2)}\,du,$$

so we get

$$\mathrm{I}^{\mathrm{min}}(\Gamma,\chi;n)=-\frac{1+n}{2}\frac{M(\chi)}{12}\int_{-\infty}^{\infty}\frac{g'(u)}{\sinh(u/2)}\,du.$$

3.2. $\mathbf{S}_p(p^e, \chi; t, n)$

Using the Dirichlet class number formula and the formula for $\mathbf{S}_p(1,1;t,n)$ in (2.34), the elliptic and hyperbolic terms in Theorem 2.1 can be rewritten as

$$(\mathrm{NEll}+\mathrm{Ell})(\Gamma,\chi;n) = \sum_{\substack{t\in\mathbb{Z}\\ D=t^2-4n\\ \sqrt{D}\notin\mathbb{Q}}} \left(\prod_{p\mid N} \frac{\mathbf{S}_p(p^e,\chi_p;t,n)}{1+(p-(\frac{d}{p}))\frac{p^{\mathrm{ord}_p\,\ell}-1}{p-1}}\right) L(1,\psi_D)$$
$$\cdot \begin{cases} g\Big(2\log\frac{|t|+\sqrt{D}}{2}\Big) & \text{if } D>0,\\ \frac{\sqrt{|D|}}{\pi}\int_{\mathbb{R}} \frac{g(u)\cosh(u/2)}{4\sinh^2(u/2)+|D|}\,du & \text{if } D<0, \end{cases}$$

where the character ψ_D is as in Section 1.2 (cf. also [4, § 2]). Hence to show that we obtain the corresponding sum in Theorem 1.7, it remains to prove that for each prime p with $e=e_p>0$,

$$\frac{\mathbf{S}_p^{\min}(p^e,\chi_p;t,n)}{1+(p-(\frac{d}{p}))\frac{p^{\mathrm{ord}_p\,\ell}-1}{p-1}} = H_{t,n}(\chi_p). \tag{3.12}$$

From now on until further notice, let us assume that p *is odd*. If $e=s>0$, then $\mathbf{S}_p^{\min}=\mathbf{S}_p^{\mathrm{new}}=\mathbf{S}_p$ by (3.9) and (3.10), and one then verifies directly from (1.5) and (2.41) (with $h=\max\{2e-1,e\}=2e-1$) that (3.12) holds.

Next assume $s\in\{0,1\}$ and $e>2$. Then by (2.41) (where now $h=\max\{2s-1,e\}=e$) and (3.9), we get

$$\mathbf{S}_p^{\mathrm{new}}(p^e,\chi;t,n) = \mathbf{S}_p(p^e,\chi;t,n) - 2\mathbf{S}_p(p^{e-1},\chi;t,n) + \mathbf{S}_p(p^{e-2},\chi;t,n)$$
$$= \chi\Big(\frac{t+\delta_{2\nmid t}p^s}{2}\Big)p^{e-3}\begin{cases} (p-1)\Big(\delta_{2\mid e}\Big(p-\Big(\frac{d}{p}\Big)\Big)p^{f-\frac{e}{2}+1}+\Big(\Big(\frac{d}{p}\Big)-1\Big)(p+1)\Big) & \text{if } g\geqslant e,\\ \Big(\Big(\frac{d}{p}\Big)-1\Big)(p-1)(1+\delta_{2\nmid e}p) & \text{if } g=e-1,\\ 1-\Big(\frac{d}{p}\Big)p & \text{if } g=e-2,\\ 0 & \text{otherwise.}\end{cases} \tag{3.13}$$

Note that for $\alpha\geqslant 2$,

$$\sum_{\substack{\psi \text{ primitive}\\ \mathrm{cond}(\psi)=p^\alpha}} \psi(x) = \varphi(p^{\alpha-1})\begin{cases} p-1 & \text{if } \alpha\leqslant \mathrm{ord}_p(x-1),\\ -1 & \text{if } \alpha=\mathrm{ord}_p(x-1)+1,\\ 0 & \text{otherwise.}\end{cases}$$

To evaluate the last sum in (3.10), we need the following lemma.

LEMMA 3.1. *When* $(\frac{d}{p})=1$, $n(\frac{t}{2}\pm\frac{\sqrt{d}\ell}{2})^2\equiv_{p^\alpha}1$ *if and only if* $\ell\equiv_{p^\alpha}0$.

Proof. Set $x\equiv_{p^\alpha}\frac{t}{2}\pm\frac{\sqrt{d}\ell}{2}$ and $y\equiv_{p^\alpha}\frac{t}{2}\mp\frac{\sqrt{d}\ell}{2}$. Then since $t^2-4n=d\ell^2$, we get $xy\equiv_{p^\alpha}n$ and $x^2\equiv_{p^\alpha}tx-n$.

Assume $n(tx-n)=nx^2\equiv_{p^\alpha}1$. Then $ntx\equiv_{p^\alpha}2$. Multiplying by y on both sides, we get

$$t\equiv 2y\equiv t-\sqrt{d}\ell \pmod{p^\alpha}.$$

So $p^\alpha\mid\ell$.

Conversely, assume $\ell \equiv_{p^\alpha} 0$. Then

$$nx^2 \equiv ntx - 1 \equiv n\frac{t^2}{2} - 1 = n\frac{t^2-4n}{2} + 1 \equiv 1 \pmod{p^\alpha}.$$

□

By the above lemma, for $e > 2$, $2 \mid e$, we have

$$\sum_{\mathrm{cond}(\psi)=p^{\frac{e}{2}}} \psi(n)\, \mathbf{S}_p^{\mathrm{new}}(p^{\frac{e}{2}}, \chi\psi^2; t, n) = \chi\Big(\tfrac{t+\delta_{2\nmid t}p^s}{2}\Big)(p-1)p^{e-3}$$

$$\times \begin{cases} 2(p-1)p^{f-\frac{e}{2}+1} + \Big(\Big(\frac{d}{p}\Big) - 1\Big)\big(-2p^{f-\frac{e}{2}+1} + p + 1\big) & \text{if } g \geqslant e-1, \\ -2 & \text{if } g = e-2 \text{ and } \Big(\frac{d}{p}\Big) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

By (3.10) and (3.13), we now obtain, when $s \in \{0, 1\}$ and $e > 2$,

$$\mathbf{S}_p^{\min}(p^e, \chi; t, n) = \Big(\Big(\frac{d}{p}\Big) - 1\Big)\frac{1+\delta_{2\nmid e}}{2}\chi\Big(\tfrac{t+\delta_{2\nmid t}p^s}{2}\Big)p^{e-3}\begin{cases} (p-1)(p+1) & \text{if } g \geqslant e-1, \\ -\delta_{2|e}p - 1 & \text{if } g = e-2, \\ 0 & \text{otherwise.} \end{cases}$$

Hence (3.12) again holds; cf. (1.6).

We next turn to the case $e \in \{1, 2\}$, $s < e$. By (3.10), when $e = 1$ and $s = 0$, we have

$$\mathbf{S}_p^{\min}(p, 1; t, n) = \mathbf{S}_p^{\mathrm{new}}(p, 1; t, n) = \mathbf{S}_p(p, 1; t, n) - 2\mathbf{S}_p(1, 1; t, n) = \Big(\frac{d}{p}\Big) - 1.$$

For $e = 2$,

$$\mathbf{S}_p^{\mathrm{new}}(p^2, \chi; t, n)$$

$$= \chi\Big(\tfrac{t+\delta_{2\nmid t}p^s}{2}\Big)\begin{cases} p^f(p-2-s) + \Big(\Big(\frac{d}{p}\Big) - 1\Big)\frac{-p^f(p-2-s)+p^2-p-1-s}{p-1} & \text{if } g \geqslant 2, \\ -1-s & \text{if } g = 1, \\ 0 & \text{if } g = 0 \end{cases}$$

$$+\delta_{g=0}\left\{(1-s)\frac{1-\big(\frac{d}{p}\big)}{2} - \frac{1}{2-s}\frac{1+\big(\frac{d}{p}\big)}{2}\Big(\chi\Big(\tfrac{t+\sqrt{d}\ell}{2}\Big) + \chi\Big(\tfrac{t-\sqrt{d}\ell}{2}\Big)\Big)\right\}.$$

Assuming first $s = 1$, $e = 2$, we compute:

$$\sum_{\substack{\psi \pmod p \\ \psi \notin \{\overline{\chi}_p, 1\}}} \mathbf{S}_p^{\mathrm{new}}(p, \chi\psi^2; t, n)\psi(n)$$

$$= \begin{cases} \chi\Big(\dfrac{t+\delta_{2\nmid t}p}{2}\Big)(p-3)\Big(2p^f + \Big(\Big(\dfrac{d}{p}\Big) - 1\Big)\dfrac{-2p^f+p+1}{p-1}\Big) & \text{if } g \geqslant 1, \\ -2\Big(\chi\Big(\dfrac{t+\sqrt{d}\ell}{2}\Big) + \chi\Big(\dfrac{t-\sqrt{d}\ell}{2}\Big)\Big) & \text{if } g = 0 \text{ and } \Big(\dfrac{d}{p}\Big) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Hence by (3.10),

$$\mathbf{S}_p^{\min}(p^2,\chi;t,n)=\mathbf{S}_p^{\mathrm{new}}(p^2,\chi;t,n)-\frac{1}{2}\sum_{\substack{\psi \pmod p\\ \psi\notin\{\overline{\chi}_p,1\}}}\mathbf{S}_p^{\mathrm{new}}(p^2,\chi\psi^2;t,n)\psi(n)$$

$$=\begin{cases}\chi\Big(\dfrac{t+\delta_{2\nmid t}p^s}{2}\Big)\Big(\Big(\dfrac{d}{p}\Big)-1\Big)\dfrac{p+1}{2} & \text{if } g\geqslant 1,\\ 0 & \text{if } g=0.\end{cases}$$

Finally for $s=0$, $e=2$, we have

$$\Big(\tfrac{n}{p}\Big)\big(\mathbf{S}_p(1,1;t,n)+\tfrac{1}{2}\mathbf{S}_p^{\mathrm{new}}(p,1;t,n)\big)+\frac{1}{2}\sum_{\substack{\psi \pmod p\\ \psi\neq 1}}\mathbf{S}_p^{\mathrm{new}}(p,\psi^2;t,n)\psi(n)$$

$$=\begin{cases}(p-2)p^f+\Big(\Big(\dfrac{d}{p}\Big)-1\Big)\dfrac{\frac{1}{2}(p^2-3)-p^f(p-2)}{p-1} & \text{if } g\geqslant 1,\\ -1+\Big(\Big(\dfrac{d}{p}\Big)-1\Big)\dfrac{\Big(\frac{n}{p}\Big)-1}{2} & \text{if } g=0,\\ 0 & \text{otherwise,}\end{cases}$$

and so, by (3.10),

$$\mathbf{S}_p^{\min}(p^2,1;t,n)=\frac{\Big(\frac{d}{p}\Big)-1}{2}\begin{cases}p-1 & \text{if } g\geqslant 1,\\ -\Big(\frac{n}{p}\Big)-1 & \text{if } g=0,\\ 0 & \text{otherwise.}\end{cases}$$

In all these cases, we again see that (3.12) holds; cf. (1.6).

Finally we turn to the case $p=2$. Since χ is minimal, we have to consider the following subcases (cf. Definition 1.5 and recall that $s=1$ is impossible when $p=2$):

$$s=\begin{cases}e & \text{if } e\geqslant 2,\\ \Big\lfloor\dfrac{e}{2}\Big\rfloor & \text{if } e\geqslant 4,\\ 2 & \text{if } e\geqslant 5 \text{ and } 2\nmid e,\\ 0 & \text{if } 2\nmid e \text{ or } e=2.\end{cases}\tag{3.14}$$

Recalling (2.42), for $s\geqslant 2$ and $s\leqslant e\leqslant 2s$, we have

$$\mathbf{S}_2(2^e,\chi;t,n)$$

$$=\chi\big(\tfrac{t}{2}\big)2^{e-1}\begin{cases}\delta_{2\nmid d}2^{\frac{g}{2}-s}(4-\delta_{e=2s})+\big(\big(\frac{d}{2}\big)-1\big)(3-2^{\lfloor\frac{g}{2}\rfloor-s}(4-\delta_{e=2s})) & \text{if } g\geqslant 2s+1,\\ -\delta_{2\nmid d}(4-\delta_{e=2s})+\delta_{e\leqslant 2s-1}\big(\big(\frac{d}{2}\big)-1\big) & \text{if } g=2s,\\ 0 & \text{otherwise}\end{cases}$$

$$+\delta_{\left(\frac{d}{2}\right)=1,g\leqslant 2s-2}\left(\chi\left(\frac{t+\sqrt{d}\ell}{2}\right)+\chi\left(\frac{t-\sqrt{d}\ell}{2}\right)\right)2^{f+\min\{e-s,f\}},$$

For $e = 2s+1$, we have

$$\mathbf{S}_2(2^e,\chi;t,n) = \chi\big(\tfrac{t}{2}\big)2^{e-1}\begin{cases}\delta_{2\nmid d}2^{\frac{g}{2}-s+1} + \big(\big(\frac{d}{2}\big)-1\big)\big(3-2^{\lfloor\frac{g}{2}\rfloor-s+1}\big) & \text{if } g \geqslant 2s+2,\\ 0 & \text{otherwise}\end{cases}$$
$$+\delta_{\left(\frac{d}{2}\right)=1,g\leqslant 2s}\left(\chi\left(\frac{t+\sqrt{d}\ell}{2}\right)+\chi\left(\frac{t-\sqrt{d}\ell}{2}\right)\right)2^{2f}.$$

Here $f = \operatorname{ord}_2(\ell)$. Then by (3.11), for $s = \lfloor\frac{e}{2}\rfloor$, $e \geqslant 4$, we have

$$\mathbf{S}_2^{\min}(2^e,\chi;t,n) = \mathbf{S}_2^{\mathrm{new}}(2^e,\chi;t,n) = \chi\big(\tfrac{t}{2}\big)\big(\big(\tfrac{d}{2}\big)-1\big)2^{e-3}\begin{cases}3 & \text{if } g \geqslant e+1,\\ -1-\delta_{2|e}2 & \text{if } g = e,\\ 1-\delta_{2\nmid d}4 & \text{if } g = e-1, 2\nmid e,\\ 0 & \text{otherwise}\end{cases}$$
$$-\delta_{\left(\frac{d}{2}\right)=1,g=e-2}\left(\chi\left(\frac{t+\sqrt{d}\ell}{2}\right)+\chi\left(\frac{t-\sqrt{d}\ell}{2}\right)\right)2^{g-1}.$$

For $e = 2s$ and $g = 2s-2$, we claim that $\chi(\frac{t+\sqrt{d}\ell}{2}) + \chi(\frac{t-\sqrt{d}\ell}{2}) = 0$. Note that $g = 4$ cannot occur, so $s \geqslant 4$. Moreover, if a is odd, then

$$\chi\left(\frac{t}{2}+a2^{s-2}\right)^2 = \chi\left(\frac{t^2}{4}+a2^{s-1}\right) = -\chi\left(\frac{t^2}{4}\right) = -\chi(1) = -1,$$

since $t^2 - 4n = d\ell^2$ and $g = \operatorname{ord}_2(d\ell^2) = 2s-2 > 4$, which forces $n = 1$. This implies

$$\chi\left(\frac{t}{2}+a2^{s-2}\right) = \pm i.$$

Since $(\frac{t+\sqrt{d}\ell}{2})(\frac{t-\sqrt{d}\ell}{2}) = 1$, we get $\chi(\frac{t+\sqrt{d}\ell}{2})\chi(\frac{t-\sqrt{d}\ell}{2}) = 1$, so $\chi(\frac{t+\sqrt{d}\ell}{2}) = \overline{\chi(\frac{t-\sqrt{d}\ell}{2})}$. Then $\chi(\frac{t+\sqrt{d}\ell}{2}) + \chi(\frac{t-\sqrt{d}\ell}{2}) = 0$ as claimed.

Now consider the cases for $s \in \{0,2\}$. For $e \geqslant \max\{1, 2s-1\}$, we have

$$\mathbf{S}_2(2^e,\chi;t,n)$$
$$= \chi\big(\tfrac{t}{2}\big)\begin{cases}\delta_{2\nmid d}2^{\frac{g}{2}+\lfloor\frac{e}{2}\rfloor}(1+2^{\lceil\frac{e}{2}\rceil-\lfloor\frac{e}{2}\rfloor-1}) & \\ +\big(\big(\frac{d}{2}\big)-1\big)(3\cdot 2^{e-1}-2^{\lfloor\frac{g}{2}\rfloor+\lfloor\frac{e}{2}\rfloor}(1+2^{\lceil\frac{e}{2}\rceil-\lfloor\frac{e}{2}\rfloor-1})) & \text{if } g \geqslant e+1 \text{ and } 2\nmid g,\\ & \text{or } g \geqslant \max\{e, 2s+2\} \text{ and } 2 \mid g,\\ 0 & \text{otherwise}\end{cases}$$
$$+\delta_{\left(\frac{d}{2}\right)=1,g\leqslant e-1}\Big(\chi\Big(\tfrac{t+\sqrt{d}\ell}{2}\Big)+\chi\Big(\tfrac{t-\sqrt{d}\ell}{2}\Big)\Big)2^{g}.$$

For $e = s = 0$, we have

$$\mathbf{S}_2(1,1;t,n) = \delta_{2\nmid d}2^{\frac{g}{2}} + \big(\big(\tfrac{d}{2}\big)-1\big)(1-2^{\lfloor\frac{g}{2}\rfloor}).$$

For $e = 4$ and $s = 2$, we have

$$\mathbf{S}_2^{\min}(2^4, \chi_{-4}; t, n) = \chi_{-4}\left(\tfrac{t}{2}\right)\left(\left(\tfrac{d}{2}\right) - 1\right)\begin{cases} 6 & \text{if } g \geqslant 5, \\ 0 & \text{otherwise.} \end{cases}$$

For $s = 0$ and $e \in \{1, 2\}$, we have

$$\mathbf{S}_2^{\min}(2^e, 1; t, n) = \begin{cases} \left(\tfrac{d}{2}\right) - 1 & \text{if } g \geqslant 2, \\ -2 & \text{if } e = 1 \text{ and } g = 0, \\ 1 & \text{if } e = 2 \text{ and } g = 0. \end{cases}$$

For $s \in \{0, 2\}$ and $e \geqslant \max\{3, 2s + 1\}$ and $2 \nmid e$, we have

$$\mathbf{S}_2^{\min}(2^e, \chi; t, n) = \chi\left(\tfrac{t}{2}\right)\left(\left(\tfrac{d}{2}\right) - 1\right)2^{e-3}\begin{cases} 3 & \text{if } g \geqslant e + 1, \\ \delta_{2 \nmid d} 4 - 1 & \text{if } g \in \{e, e - 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

In all these cases we again see that (3.12) holds; cf. (1.7).

3.3. Ψ_1, Ψ_2, $\tilde{\Psi}_2$ *and* Ψ_3

Note that Ψ_1 appears only when $n = 1$.

LEMMA 3.2. *We have*

$$\Psi_1^{\min}(p^e, p^s) = \delta_{e=s} 2. \tag{3.15}$$

So

$$\Psi_1^{\min}(N, q) = \delta_{N=q} 2^{\omega(N)}. \tag{3.16}$$

Proof. Recalling (2.17), when p is odd and $e = s \geqslant 1$ or $e \geqslant \max\{1, 2s\}$ for $s \in \{0, 1\}$, we have

$$\Psi_1^{\text{new}}(p^e, p^s) = \begin{cases} p^{\frac{e}{2}-2}(p-1)^2 & \text{if } e \geqslant \max\{3, 2s + 2\} \text{ and } 2 \mid e, \\ p - 2 - s & \text{if } e = 2 \text{ and } s \in \{0, 1\}, \\ 2 & \text{if } e = s \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then by (3.10), we get (3.15). When $p = 2$ and s is as in (3.14), we have by (3.11)

$$\Psi_1^{\min}(2^e, 2^s) = \Psi_1^{\text{new}}(2^e, 2^s) = \begin{cases} 2 & \text{if } e = s \geqslant 2, \\ 0 & \text{otherwise.} \end{cases}$$

□

Recalling (2.18), let

$$\Psi_2(p^e, \chi) = \frac{\chi(-1) + 1}{2}\Psi_2(p^e, p^s).$$

Note that Ψ_2 appears only when $n = 1$. But it also occurs in the definition of $\tilde{\Psi}_2$ when $n = -1$.

LEMMA 3.3. *For $p \equiv_4 1$, we have*

$$\Psi_2^{\min}(p^e, \chi) = 0, \tag{3.17}$$

so that $\Psi_2^{\min}(N, \chi) = \delta_{N=1}$.

Proof. Recall (2.18). When p is odd, $e \geqslant \max\{1, 2s\}$, $s \in \{0, 1\}$ or $e = s \geqslant 1$, we have

$$\Psi_2^{\mathrm{new}}(p^e, \chi) = \delta_{e=2, s=0}.$$

Then by (3.10) and (3.11), we get (3.17). □

Note that $\tilde{\Psi}_2$ appears only when $n = -1$.

LEMMA 3.4. *We have*

$$\tilde{\Psi}_2^{\min}(p^e, \chi) = 0,$$

so that $\tilde{\Psi}_2^{\min}(N, \chi) = \delta_{N=1}$.

Proof. Recalling (2.19), for $p \not\equiv_4 1$ and $s = 0$, we have

$$\tilde{\Psi}_2^{\mathrm{new}}(p^e, 1) = \begin{cases} -1 & \text{if } p \equiv_4 -1 \text{ and } e = 2, \\ 0 & \text{otherwise.} \end{cases}$$

For $p \equiv_4 1$, by (3.10), (3.11) and (3.17), we have $\tilde{\Psi}_2^{\min}(p^e, \chi) = 0$. □

Note that Ψ_3 appears only for $n = 1$.

LEMMA 3.5. *When p is odd, we have*

$$\Psi_3^{\min}(p^e, p^s) = \begin{cases} 4e - 1 & \text{if } e = s > 0, \\ 2 & \text{if } e = 1 \text{ and } s = 0, \\ \frac{1}{2}(p - 1 + 2s) & \text{if } e = 2 \text{ and } s \in \{0, 1\}, \\ p^{\lfloor \frac{e-3}{2} \rfloor}(p-1)\frac{\delta_{2|e}p + \delta_{2\nmid e}3 + 1}{2} & \text{if } e \geqslant 3 \text{ and } s \in \{0, 1\}. \end{cases}$$

For $p = 2$, we have

$$\Psi_3^{\min}(2^e, 2^s) = \begin{cases} 4e - 1 & \text{if } e = s \geqslant 2, \\ 2 & \text{if } e \in \{1, 3\} \text{ and } s = 0, \\ 1 & \text{if } e = 2 \text{ and } s = 0, \\ 3 \cdot 2^{\frac{e}{2} - 2} & \text{if } e = 2s \geqslant 4, \\ 2^{\frac{e-1}{2}} & \text{if } e \geqslant 5, 2 \nmid e \text{ and } s \in \{0, 2, \frac{e-1}{2}\}. \end{cases}$$

Thus, combining with (3.15),

$$\sum_{p|N} \Psi_3^{\min}(p^{e_p}, p^{s_p}) \prod_{p'|N, p' \neq p} \Psi_1^{\min}(p'^{e_{p'}}, p'^{s_{p'}}) \log p$$

$$= \begin{cases} 2^{\omega(N)}\left(\log N^2 - \frac{1}{2}\sum_{p|N} \log p\right) & \text{if } N = q, \\ 2^{\omega(N)-1}\Psi_3^{\min}(p^{e_p}, p^{s_p}) \log p & \text{if } \frac{N}{q} = p^{e_p - s_p} > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Recalling (2.20), for $e = s > 0$ or $e \geqslant \max\{1, 2s\}$ for $s \in \{0, 1, 2\}$, we have

$$\Psi_3^{\mathrm{new}}(p^e, p^s) = \begin{cases} 4e-1 & \text{if } e = s > 0, \\ p^{\frac{e}{2}-2}(p-1)(e(p-1)+1) & \text{if } e \geqslant \max\{3, 2s+2\} \text{ and } 2 \mid e, \\ 2p^{\frac{e-3}{2}}(p-1) & \text{if } e \geqslant \max\{3, 2s+2\} \text{ and } 2 \nmid e, \\ 2p-3 & \text{if } e = 2 \text{ and } s = 0, \\ 2 & \text{if } e = 1 \text{ and } s = 0, \\ 2(p-1) & \text{if } e = 3 \text{ and } s = 1, \\ 2(p-2) & \text{if } e = 2 \text{ and } s = 1, \\ e-1 & \text{if } p = 2, e \in \{4, 5\} \text{ and } s = 2, \end{cases}$$

For $n = 1$, by applying (3.10) and (3.11), we prove the lemma. □

3.4. $\Phi_\pm$

Let us define

$$\Phi_\pm(p^e, \chi; m) = \begin{cases} 1 & \text{if } s = 0,\ p \mid m \\ \frac{1}{2}\left(\overline{\chi(m)} + \chi(\pm m)\right)\Phi_\pm(p^e, m) & \text{otherwise.} \end{cases}$$

Then in the trace formula in Theorem 2.1, the sums over m appearing in the last lines of (2.25) (for $n = 1$) and (2.26) (for $n = -1$) can be expressed as

$$2\sum_{m=2}^{\infty} \frac{\Lambda(m)}{m} \left(\prod_{p \mid N} \Phi_n(p^e, \chi_p; m) \right) g(2 \log m).$$

Hence to show that we obtain the corresponding sum in Theorem 1.7, we have to prove that for each prime $p \mid N$, writing now χ in place of χ_p,

$$\Phi_n^{\min}(p^e, \chi; m) = \Phi_{m,n}(\chi) = \begin{cases} \overline{\chi(m)} + \chi(nm) & \text{if } s = e, \\ -1 & \text{if } p \mid m,\ e = 1 \text{ and } s = 0, \\ 0 & \text{otherwise} \end{cases} \tag{3.18}$$

(cf. (1.4)).

By (2.23) we have, for $e > 0$,

$$\Phi_\pm(p^e, \chi; m) = \begin{cases} \delta_{s=0} & \text{if } p \mid m, \\ \left(\overline{\chi(m)} + \chi(\pm m)\right) p^{e - w_\pm} & \text{if } p \nmid m,\ e < 2w_\pm, \\ \left(\overline{\chi(m)} + \chi(\pm m)\right) \frac{1}{2}(p^{\lfloor e/2 \rfloor} + p^{\lfloor (e-1)/2 \rfloor}) & \text{if } p \nmid m,\ e \geqslant 2w_\pm, \end{cases}$$

where $w_\pm = w_\pm(m) = \max\{s, e - \mathrm{ord}_p(\pm m^2 - 1)\}$. Let $f_\pm = \mathrm{ord}_p(\pm m^2 - 1)$. For $s = 0$, we compute

$$\Phi_\pm^{\mathrm{new}}(p^e, \chi; m) = \begin{cases} -1 & \text{if } p \mid m \text{ and } e = 1, \\ \begin{cases} -1 & \text{if } e = 2 \text{ and } f_\pm = 0, \\ -p^{f_\pm} + p^{f_\pm - 1} & \text{if } e = 2f_\pm + 2 \text{ and } f_\pm \geqslant 1, \\ p^{\frac{e}{2}-2}(p-1)^2 & \text{if } 3 \leqslant e \leqslant 2f_\pm \text{ and } 2 \mid e, \\ p-2 & \text{if } e = 2 \text{ and } f_\pm \geqslant 1, \end{cases} & \text{if } p \nmid m, \\ 0 & \text{otherwise.} \end{cases}$$

For $s = 1$, p odd and $e \geqslant 2$, and assuming $p \nmid m$, we compute

$$\Phi_{\pm}^{\mathrm{new}}(p^e, \chi; m)$$
$$= \frac{1}{2}\Big(\overline{\chi(m)} + \chi(\pm m)\Big) \begin{cases} -2 & \text{if } e = 2 \text{ and } f_{\pm} = 0, \\ \begin{cases} p - 3 & \text{if } e = 2, \\ p^{\frac{e}{2}-2}(p-1)^2 & \text{if } 4 \leqslant e \leqslant 2f_{\pm} \text{ and } 2 \mid e, \\ -p^{f_{\pm}} + p^{f_{\pm}-1} & \text{if } e = 2f_{\pm} + 2, \end{cases} & \text{if } f_{\pm} \geqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

For p odd, using the above formulas together with (3.10) and the fact that $\Phi_{\pm}^{\mathrm{new}} = \Phi_{\pm}$ when $s = e > 0$, the desired result (3.18) now follows by a direct computation in each case.

Next assume $p = 2$. Recalling (3.11) and (3.14), we consider the following cases. For $e \geqslant 5$, $2 \nmid e$ and $s \in \{0, 2\}$, we get

$$\Phi_{\pm}^{\min}(2^e, \chi; m) = 0,$$

and for $s = e \geqslant 2$, we get

$$\Phi_{\pm}^{\min}(2^e, \chi; m) = \overline{\chi(m)} + \chi(nm).$$

For $s \geqslant 2$ and $e \in \{2s, 2s + 1\}$, we obtain

$$\Phi_{\pm}^{\min}(2^e, \chi; m) = \frac{1}{2}\Big(\overline{\chi(m)} + \chi(\pm m)\Big) \begin{cases} -2^{f_{\pm}} & \text{if } 2 \nmid m, e = 2s \text{ and } f_{\pm} = s - 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $f_{\pm} = s - 1$, we have $m^2 = \pm 1 + a \cdot 2^{s-1}$ for an odd integer a. Then

$$\chi(m)^2 = \chi(\pm 1 + a \cdot 2^{s-1}) = -\chi(\pm 1).$$

For Φ_+, we get $\chi(m) = \pm i$ and $\overline{\chi(m)} + \chi(m) = 0$. For Φ_-, if χ is even, then $\chi(m) = \pm i$. If χ is odd, then $\chi(m) = \pm 1$. For either case, we get $\overline{\chi(m)} + \chi(-m) = 0$. So finally we conclude that $\Phi_{\pm}^{\min}(2^e, \chi; m) = 0$ for $e \in \{2s, 2s + 1\}$.

For $s = 0$ and $e \in \{1, 2, 3\}$, we get

$$\Phi_{\pm}^{\min}(2^e, \chi; m) = \begin{cases} -1 & \text{if } 2 \mid m \text{ and } e = 1, \\ & \text{or } 2 \nmid m, e = 2 \text{ and } f_{\pm} = 0, \\ 0 & \text{otherwise.} \end{cases}$$

This formula agrees with (3.18), since in fact $f_{\pm} = \mathrm{ord}_p(\pm m^2 - 1) \geqslant 1$ must hold when $p = 2$ and $2 \nmid m$.

Now (3.18) has been proved in all cases.

3.5. Ω_1 *and* Ω_2

For $j \in \{1, 2\}$, define

$$\tilde{\Omega}_j(p^e, \chi) = \begin{cases} 2\Omega_j(N, q) & \text{if } p = 2, e \geqslant 1 \text{ and } \chi \text{ even}, \\ \Omega_j(N, q) & \text{if } p = 2 \text{ and } e = 0, \\ 2 & \text{if } p \text{ odd}, e \geqslant 1 \text{ and } \chi \text{ even}, \\ 1 & \text{if } p \text{ odd and } e = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here Ω_j is given in (2.21) and (2.22). Then

$$\tilde{\Omega}(N,\chi)=\prod_{p|N}\tilde{\Omega}_j(p^{e_p},\chi_p)=I_\chi 2^{\omega(N)}\Omega_j(N,q).$$

LEMMA 3.6. *When p is odd and $n=-1$, we have*

$$\tilde{\Omega}_j^{\min}(p^e,\chi)=\delta_{e=s\geqslant 1}(1+\chi(-1)). \tag{3.19}$$

When $p=2$ and $n=-1$, we have

$$\tilde{\Omega}_1^{\min}(2^e,\chi)=\delta_{e=s\geqslant 2}(1+\chi(-1)) \tag{3.20}$$

and

$$\tilde{\Omega}_2^{\min}(2^e,\chi)=\begin{cases}\frac{1}{2} & \text{if } e=0,\\ -2e & \text{if } e=s\geqslant 3 \text{ and } \chi \text{ even},\\ -\frac{3}{2} & \text{if } e=1 \text{ and } s=0,\\ -\frac{1}{2} & \text{if } e\in\{2,3\} \text{ and } s=0,\\ 0 & \text{otherwise.}\end{cases} \tag{3.21}$$

Proof. When p is odd, for $e=s\geqslant 1$ or $e\geqslant\max\{1,2s\}$, we have

$$\tilde{\Omega}_j^{\text{new}}(p^e,\chi)=\begin{cases}2 & \text{if } e=s>0 \text{ and } \chi \text{ even},\\ -2 & \text{if } e=2, s=1 \text{ and } \chi \text{ even},\\ -1 & \text{if } e=2 \text{ and } s=0,\\ 0 & \text{otherwise.}\end{cases}$$

By (3.10), for $j\in\{1,2\}$ and $n=-1$, we get (3.19).

When $p=2$, for $j\in\{1,2\}$, by (3.11), since $\tilde{\Omega}_j^{\min}(2^e,\chi)=\tilde{\Omega}_j^{\text{new}}(2^e,\chi)$, we get (3.20) and (3.21). □

3.6. *Concluding the proof of Theorem 1.7*

It follows from (3.1) and (3.6) that for any minimal character $\chi \pmod N$, any $n\in\{\pm 1\}$ and any pair of test functions of trace class (g,h), we have

$$\sum_{\lambda>0}\operatorname{tr}T_n|_{\mathcal{A}_\lambda^{\min}(\chi)}h\left(\sqrt{\lambda-\tfrac{1}{4}}\right)=\sum_{\langle M,\psi\rangle\in S'_\chi}(-1)^{k'(N,M)}2^{-k(N,M,\psi)}\,\psi(n)\operatorname{Tr}^{\text{new}}\left(M,\chi\psi^2\right), \tag{3.22}$$

where

$$\operatorname{Tr}^{\text{new}}(N',\chi')=\sum_{M|\frac{N'}{\operatorname{cond}(\chi')}}\beta\left(\frac{N'/\operatorname{cond}(\chi')}{M}\right)\sum_{\lambda>0}\operatorname{Tr}T_n|_{\mathcal{A}_\lambda(\chi'|_{M\operatorname{cond}(\chi')})}h\left(\sqrt{\lambda-\tfrac{1}{4}}\right). \tag{3.23}$$

Now the inner sum in (3.23) can be evaluated by using Theorem 2.1 and compensating for the possible contribution from the Laplace eigenvalue $\lambda=0$. Specifically, we have $\mathcal{A}_0(\chi)=\mathbb{C}$ if χ is the trivial character, otherwise $\mathcal{A}_0(\chi)=\{0\}$; therefore $\lambda=0$ contributes with a term $h(\frac{i}{2})$ in the left-hand side of (2.24) if and only if $n=1$ and χ is trivial. This means that for $n=1$ and χ' trivial, when using the right-hand side of (2.24) to evaluate (3.23), we need to compensate by subtracting a term $(\sum_{M|N'}\beta(M))h(\frac{i}{2})=\mu(N')h(\frac{i}{2})$ from the resulting formula. This compensation has an effect in the right-hand side of (3.22) only if $n=1$, χ is trivial and $\operatorname{ord}_p(N)\leqslant 2$ for all p; and if $\operatorname{ord}_p(N)=2$ for at least one p, then the compensations cancel each other out; hence it is only for N squarefree that the net effect is nonzero. Using

$h(\frac{i}{2}) = \int_{\mathbb{R}} g(u)\cosh(u/2)\,du$, we now see that the total contribution from these compensations is exactly the term $-\delta_{n=1}\mu(\chi)\int_{\mathbb{R}} g(u)\cosh(u/2)\,du$ appearing in the formula of Theorem 1.7.

Taking the results of Sections 3.1–3.5 into account, we now see that the formula in Theorem 1.7 follows by using (3.23) and (2.24) to evaluate (3.22). Hence the proof of Theorem 1.7 is now complete.

We are grateful to the referee for suggesting a remark along the following lines.

REMARK 3.7. As is discussed in [**28**, Section 8], the newform part of the continuous part of the spectrum of the Laplace operator on $L^2(\Gamma_0(N)\backslash\mathbb{H},\chi)$ is spanned by the Eisenstein series $E^{\chi_2}_{\chi_1}(q_2 z, \frac{1}{2}+it)$ (cf. Section 2.7) with $\langle\chi_1,\chi_2\rangle$ running through all pairs of primitive Dirichlet characters satisfying $q_1q_2 = N$ (with $q_j = \mathrm{cond}(\chi_j)$) and $\mathrm{cond}(\chi\chi_2\overline{\chi}_1) = 1$.

It is natural to define the twist of the Eisenstein series $E^{\chi_2}_{\chi_1}$ with an arbitrary primitive Dirichlet character ψ to be $E^{\chi_2'}_{\chi_1'}$, where χ_1' and χ_2' are the primitive Dirichlet characters corresponding to the products $\chi_1\psi$ and $\chi_2\psi$, respectively. (This is most naturally seen in the language of automorphic representations of the adele group of GL_2, where twisting by ψ simply corresponds to multiplying any automorphic representation by the 1-dimensional representation $\psi\circ\det$.) It then follows that the *twist-minimal part* of the continuous spectrum is spanned by $E^{\chi_2}_{\chi_1}(q_2 z, \frac{1}{2}+it)$ with $\langle\chi_1,\chi_2\rangle$ running through the set described above but further restricted by the condition $\gcd(q_1,q_2)=1$. This nice and simple description of the twist-minimal part of the continuous spectrum is likely to play an important role in a potential future more direct proof of the twist-minimal trace formula avoiding the intermediate steps of Theorem 2.1 and sieving.

4. *Artin representations and* $\Gamma(N)$

In this section we prove two lemmas that restrict the computations of twist-minimal spaces needed to prove Theorem 1.1 for $\Gamma(N)$ and Theorem 1.2.

LEMMA 4.1. *For any $N\in\mathbb{Z}_{>0}$, the Selberg eigenvalue conjecture holds for $\Gamma(N)$ if and only if $\mathcal{A}^{\min}_\lambda(\chi) = \{0\}$ for all $\lambda\in(0,\frac{1}{4})$ and $\chi \pmod M$ with $\mathrm{lcm}(M,\mathrm{cond}(\chi)^2)\mid N^2$.*

Proof. Given $\lambda>0$, let $\mathcal{A}_\lambda(N)$ denote the space of Maass cusp forms of eigenvalue λ that are invariant under the action of $\Gamma(N)$. Then, as shown in [**13**, § 3.5], we have the isomorphism

$$\mathcal{A}_\lambda(N) \cong \bigoplus_{\chi \pmod N} \mathcal{A}_\lambda(\chi|_{N^2}),$$

where to a given $f\in\mathcal{A}_\lambda(\chi|_{N^2})$ we associate the function $z\mapsto f(N^{-1}z)$, which is an element of $\mathcal{A}_\lambda(N)$. In turn, for a given pair $\lambda>0$ and $\chi \pmod N$, we have

$$\mathcal{A}_\lambda(\chi|_{N^2}) = \sum_{\substack{M,d\in\mathbb{Z}_{>0}\\ \mathrm{cond}(\chi)\mid M\mid N^2\\ d\mid \frac{N^2}{M}}} \ell_{M,d}\,\mathcal{A}^{\mathrm{new}}_\lambda(\chi|_M),$$

in the notation of Section 1.1.

By Lemma 1.4, the space $\mathcal{A}^{\mathrm{new}}_\lambda(\chi|_M)$ is spanned by the Hecke eigenforms $f\otimes\psi$, where f is twist minimal and $M = \mathrm{lcm}(\mathrm{cond}(f),\mathrm{cond}(\psi)\,\mathrm{cond}(\chi\overline{\psi}))$. Put $M' = \mathrm{cond}(f)$ and $\chi' = \chi\overline{\psi}^2|_{M'}$, so that $f\in\mathcal{A}^{\min}_\lambda(\chi')$. Since $M\mid N^2$ and $\mathrm{cond}(\chi)\mid N$, we have $M'\mid N^2$ and $\mathrm{cond}(\chi')\mid N$.

Thus, if $\mathcal{A}^{\min}_\lambda(\chi') = \{0\}$ whenever $\lambda<\frac{1}{4}$ and $\chi' \pmod{M'}$ satisfies $M'\mid N^2$ and $\mathrm{cond}(\chi')\mid N$, then the Selberg eigenvalue conjecture holds for $\Gamma(N)$. Conversely, we have $\mathcal{A}^{\min}_\lambda(\chi')\subseteq\mathcal{A}^{\mathrm{new}}_\lambda(\chi')$ so the Selberg conjecture for $\Gamma(N)$ implies $\mathcal{A}^{\min}_\lambda(\chi') = \{0\}$ for $\lambda<\frac{1}{4}$. □

LEMMA 4.2. *Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{C})$ be a nondihedral, irreducible Artin representation of conductor N, and let $\chi = \prod_{p|N} \chi_p$ be the Dirichlet character associated to $\det \rho$ via class field theory. If $p \mid N$ is a prime such that $\mathrm{ord}_p N \in \{1, \mathrm{ord}_p \mathrm{cond}(\chi)\}$, then χ_p has order 2, 3, 4 or 5. Further, if p and q are two such primes, then $\chi_p \chi_q$ cannot have order 20.*

Proof. Let ρ_p denote the restriction of ρ to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$. Then ρ_p factors through $G = \mathrm{Gal}(L/\mathbb{Q}_p)$ for some finite extension $L/\mathbb{Q}_p$. Let G_i, $i = 0, 1, 2, \ldots$, denote the ramification subgroups of G, with G_0 the inertia group. Then ρ_p and $\det \rho_p$ have conductor exponents

$$e = \frac{1}{\#G_0} \sum_{i \geqslant 0} \sum_{g \in G_i} (2 - \mathrm{tr}\, \rho_p(g)) \quad \text{and} \quad s = \frac{1}{\#G_0} \sum_{i \geqslant 0} \sum_{g \in G_i} (1 - \det \rho_p(g)),$$

respectively. Note that the average of $\mathrm{tr}\, \rho_p$ over G_i is the number of copies of the trivial representation in $\rho_p|G_i$. If this is nonzero, then $\rho_p|G_i \cong \det \rho_p|G_i \oplus 1$, from which it follows that the ith terms of the two sums above are the same. If $\rho_p|G_i$ does not contain the trivial representation, then

$$\frac{1}{\#G_i} \sum_{g \in G_i} (2 - \mathrm{tr}\, \rho_p(g)) = 2 > 1 \geqslant \frac{1}{\#G_i} \sum_{g \in G_i} (1 - \det \rho_p(g)).$$

Thus, the ith term of the formula for e is always $\geqslant$ the ith term of the formula for s, with equality if and only if $\rho_p|G_i$ contains the trivial representation. If $e \in \{1, s\}$, then equality must hold for every term; in particular, $\rho_p|G_0$ contains the trivial representation, so that $\rho_p|G_0 \cong \det \rho_p|G_0 \oplus 1$. When $e = 1$, this in turn implies that $s = 1$, so $\det \rho_p|G_0$ is nontrivial.

Let $\bar{\rho}_p$ denote the composition of ρ_p with the canonical projection $\mathrm{GL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$. Then when $e \in \{1, s\}$, the natural maps $\rho_p(G_0) \to \det \rho_p(G_0)$ and $\rho_p(G_0) \to \bar{\rho}_p(G_0)$ are isomorphisms. Since $\det \rho_p(G_0)$ is a nontrivial cyclic subgroup of $\mathbb{C}^\times$ and $\bar{\rho}_p(G_0)$ is a subgroup of A_4, S_4 or A_5, it follows that $\det \rho_p(G_0) \cong \bar{\rho}_p(G_0)$ is cyclic of order 2, 3, 4 or 5. Since the Dirichlet character χ_p associated to $\det \rho_p$ is determined by $\det \rho_p|G_0$, they have the same order, which implies the first claim.

Finally, order 4 (respectively, 5) can only occur when ρ is octahedral (respectively, icosahedral). These possibilities are mutually exclusive, whence the second claim. □

In the A_5 case, we may also take advantage of the fact that icosahedral representations occur in Galois-conjugate pairs that are not twist equivalent. Thus, assuming Artin's conjecture, we can still rule out the existence of an icosahedral representation of a given conductor when our computation accommodates one representation (in total over all characters, modulo twist equivalence) but not two. We used this trick to rule out icosahedral representations with conductor $N \in \{2221, 2341, 2381, 2529, 2799\}$.

5. *Numerical remarks*

To prove Theorems 1.1 and 1.2, we applied the numerical method described in [**6**, § 4]. Briefly, we consider test functions of the form

$$h(r) = \left(\mathrm{sinc}^2\left(\frac{\delta r}{2}\right) \sum_{j=0}^{M-1} x_j \cos(j\delta r) \right)^2,$$

where

$$\mathrm{sinc}\, r := \begin{cases} \sin(r)/r & \text{if } r \neq 0, \\ 1 & \text{if } r = 0, \end{cases}$$

$\delta = X/2M$ for $X, M \in \mathbb{Z}_{>0}$ and $x_0, \dots, x_{M-1} \in \mathbb{R}$ are arbitrary. For each $\chi \pmod N$ and $\epsilon \in \{0,1\}$, set

$$m_\chi = \#\{\psi \pmod N : \psi(-1) = 1,\ \psi^2 = 1,\ \mathrm{cond}(\psi)\,\mathrm{cond}(\chi\psi) \mid N\}$$

and let $n_{\chi,\epsilon}$ be a lower bound for the number of twist-minimal forms of character χ and parity ϵ arising from Artin representations; we use `PARI/GP` [25] to compute the contribution from dihedral representations, as described in [6, § 3.2] (see also the source code at [5]), and the data from Table 1 for the rest. Then the quantity

$$Q_{\chi,\epsilon}(x_0, \dots, x_{M-1}) := \frac{1}{m_\chi}\left[\sum_{\lambda>0} \mathrm{tr}\, \tfrac{1}{2}(T_1 + (-1)^\epsilon T_{-1})|_{\mathcal{A}_\lambda^{\min}(\chi)} h\left(\sqrt{\lambda - \tfrac{1}{4}}\right) - n_{\chi,\epsilon} h(0)\right],$$

is a positive-definite quadratic form in the x_j. By standard trigonometric identities, the matrix of $Q_{\chi,\epsilon}$ is determined from the traces of $\mathrm{sinc}^4(\delta r/2)\cos(j\delta r)$ for $0 \leqslant j \leqslant 2M-2$. We apply the trace formula to compute these, and then minimize $Q_{\chi,\epsilon}$ with respect to the constraint $\sum_{j=0}^{M-1} x_j = 1$.

In every case, it turned out that the criterion from [6, § 4.3] applied, so that the optimal test function h satisfied $h(r) \geqslant 1$ for $r \in i\mathbb{R}$. As explained in [6, § 3.4], every non-CM form occurs with multiplicity m_χ. Thus, since the CM forms satisfy Selberg's conjecture,[†] whenever the resulting minimal value of $Q_{\chi,\epsilon}$ is less than 1, we deduce both the Selberg conjecture and the completeness of the list of nondihedral Artin representations for twist-minimal forms of character χ.

To ensure the accuracy of our numerical computations, we used the interval arithmetic library `Arb` [18] throughout. To handle the integral terms of the trace formula, for each basis function we first computed $\int_0^\infty g'(u)\log u\, du$ symbolically, which allowed us to replace $\log(\sinh(u/2))$ and $\log(\tanh(u/4))$ by the real-analytic functions $\log(\sinh(u/2)/u)$ and $\log(\tanh(u/4)/u)$, respectively. Thus, in every integral term, the integrand agrees with an analytic function on each interval $[j\delta, (j+1)\delta]$. After applying a suitable affine transformation to replace the interval by $[-1,1]$, we use the following rigorous numerical quadrature estimate of Molin [22]:

LEMMA 5.1 (Molin). *Let f be an analytic function on an open neighborhood of $D = \{z \in \mathbb{C} : |z| \leqslant 2\}$. Then for any $n \geqslant 1$, we have*

$$\left|\int_{-1}^{1} f(x)\,dx - \sum_{k=-n}^{n} a_k f(x_k)\right| \leqslant \exp(4 - 5/h) \sup_{z\in\partial D} |f(z)|,$$

where $h = \log(5n)/n$, $a_k = \frac{h\cosh(kh)}{\cosh^2(\sinh(kh))}$ and $x_k = \tanh(\sinh(kh))$.

Note that the error term decays exponentially in $n/\log n$. To obtain a bound for $|f|$ on ∂D, we write $\partial D = \bigcup_{j=0}^{n-1}\{2e(\theta) : \theta \in [j/n, (j+1)/n)\}$ and use interval arithmetic to bound $|f(2e(\theta))|$ on each segment.

Using the algorithm from [2], we computed the class numbers of $\mathbb{Q}(\sqrt{t^2 \pm 4})$ for all $t \leqslant e^{20}$, which enables us to take X as large as 40 in the above. Taking $M = 200$, various $X \leqslant 40$ and χ as indicated by Lemmas 4.1 and 4.2 sufficed to prove Theorems 1.1 and 1.2.

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[†]For squarefree N, we actually remove the contribution from CM forms, following [6, § 3.3]. With more work one could generalize that approach to arbitrary N, but it turns out not to be necessary for our applications.

sharing their data in the LMFDB. Finally, we thank Andrew Knightly for helpful comments and corrections.

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