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One-loop matrix elements of effective superstring interactions: α'-expanding loop integrands

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Abstract: In the low-energy effective action of string theories, non-abelian gauge interactions and supergravity are augmented by infinite towers of higher-mass-dimension operators. We propose a new method to construct one-loop matrix elements with insertions of operators $D^{2k}F^n$ and $D^{2k}R^n$ in the tree-level effective action of type-I and type-II superstrings. Inspired by ambitwistor string theories, our method is based on forward limits of moduli-space integrals using string tree-level amplitudes with two extra points, expanded in powers of the inverse string tension α'. Similar to one-loop ambitwistor computations, intermediate steps feature non-standard linearized Feynman propagators which eventually recombine to conventional quadratic propagators. With linearized propagators the loop integrand of the matrix elements obey one-loop versions of the monodromy and KLT relations. We express a variety of four- and five-point examples in terms of quadratic propagators and formulate a criterion on the underlying genus-one correlation functions that should make this recombination possible at all orders in α'. The ultraviolet divergences of the one-loop matrix elements are crosschecked against the non-separating degeneration of genus-one integrals in string amplitudes. Conversely, our results can be used as a constructive method to determine degenerations of elliptic multiple zeta values and modular graph forms at arbitrary weight.

Keywords: Scattering Amplitudes, Superstrings and Heterotic Strings

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Contents

1 Introduction

Recent years have witnessed remarkable synergies between properties and building blocks of massless string amplitudes and new structures of field-theory amplitudes. A central example is the double-copy structure of perturbative gravity [1–3], which is manifest in the Kawai-Lewellen-Tye open/closed string relations [4], the chiral-splitting description of string amplitudes [5, 6], and kinematic numerators in multiloop field-theory amplitudes [2, 7, 8]. Ambitwistor string theories [9–12] added important facets to the elegant realization of double copy via string worldsheets and extracted key insights on (possibly non-supersymmetric) loop integrands from forward limits of tree-level diagrams [13–23].

The double-copy structures of string amplitudes have profound implications on the effective field theories that govern the low-energy dynamics of the gauge and gravity multiplets from the massless string excitations. For open (type-I) superstrings and closed (type-II) superstrings, the leading orders in the low-energy effective actions at tree level are schematically given by (see e.g. [24–35] and references therein)

$$\begin{aligned} S_{\rm eff}^{\rm open} &= \int {\rm d}^D x \, {\rm Tr}\Big\{ F^2 + \alpha'^2 \zeta_2 F^4 + \alpha'^3 \zeta_3 (D^2F^4 {+} F^5) + \alpha'^4 \zeta_4 (D^4 F^4 {+} D^2 F^5 {+} F^6) + \ldots \Big\} \\ S_{\rm eff}^{\rm closed} &= \int {\rm d}^D x \, \Big\{ R + \alpha'^3 \zeta_3 R^4 + \alpha'^5 \zeta_5 (D^4 R^4 {+} D^2 R^5 {+} R^6) + \ldots \Big\} \end{aligned} \tag{1.1}$$

with Riemann zeta values $\zeta_n = \sum_{k=1}^\infty k^{-n}$, non-abelian gauge-field strengths F, Riemann or Ricci curvature R and their respective covariant derivatives D. The ellipses refer to infinite power series in the inverse string tension α' that organize the low-energy expansion[1] as well as the supersymmetric completions, see e.g. [36–45]. From an effective-field-theory perspective, the infinitely many coefficients in (1.1) are exactly determined by string perturbation theory at tree level. Furthermore, loop corrections to (1.1) and non-perturbative effects are related to string dualities and intricate properties of string vacua, which are beyond the scope of this work.

The elegant structures of superstring tree-level amplitudes involving gauge and gravity supermultiplets expose a rich web of relations among the matrix elements of the higher-mass-dimension operators in (1.1). Moreover, properties of bosonic- and heterotic-string amplitudes extend the harmonious interplay of tree-level matrix elements to non-supersymmetrizable effective operators including $\alpha'{\rm Tr}(F^3), \alpha' R^2$ and $\alpha'^2 R^3$ [46–48].

In this work, we initiate a systematic construction and structural understanding of one-loop matrix elements involving the effective tree-level interactions of open and closed superstrings as in (1.1). More specifically, we use forward limits of string tree amplitudes to propose expressions for, and relations among, loop integrands involving insertions of the $D^{2k}F^n$ and $D^{2k}R^n$ operators in tree-level effective actions. Typical diagrams in these loop integrands are drawn in figure 1, and we will give the explicit form of the kinematic factors for four and five gauge or gravity states at various orders in α'. In fact, the vertices and internal

[1] More precisely, low-energy- or α'-expansions refer to series expansion of string amplitudes in the dimensionless Mandelstam invariants $\alpha' k_i \cdot k_j$ with external momenta k_i. The latter signal gauge- or diffeomorphism covariant derivatives upon translation to the effective interactions in (1.1).

Figure 1. Contributions to the one-loop matrix elements with single- and double-insertion of the operator $\alpha'^2\mathrm{Tr}(F^4)$ in the open-string effective action (1.1).

lines in figure 1 represent the full supersymmetry multiplets of the gluon and graviton, and our results incorporate cancellations between bosonic and fermionic states in the loop.

Our construction of one-loop matrix elements applies to arbitrary multiplicities, spacetime dimensions and orders in α', and it bypasses a variety of difficulties that would arise in other lines of attack. First, a Feynman-diagrammatic computation in supersymmetry components involving different n-gon diagrams and the fermionic superpartners of the operators in (1.1) appears hopeless by its combinatorial complexity. Second, superspace descriptions of $D^{2k}F^n$ and $D^{2k}R^n$ operators are only available at low mass dimensions, but not for generic orders in α' that are aimed for in this work. Third, the α'-expansion of one-loop string amplitudes in the literature usually proceeds by integrating out the loop momentum in early stages of the computation. Hence, the desired momentum-space representations of one-loop matrix elements cannot be extracted from string-theory references.

The one-loop matrix elements computed from our method determine the non-analytic thresholds in the low-energy expansion of one-loop string amplitudes, i.e. their branch cuts due to massless states in the string loop. However, the matrix elements of this work do not fix the analytic contributions to one-loop string amplitudes, i.e. the terms with power-series behaviour in the Mandelstam variables. Such analytic terms and the associated loop corrections to the superstring effective actions (1.1) can be thought of as partially originating from purely massive loop integrals, which have no branch cuts below the ultraviolet (UV) cutoff of the effective field theory. They are beyond the reach of the computations in this work since all the propagators in the α'-expansion of our one-loop matrix elements (see e.g. figure 1) refer to massless states.

1.1 Inspiration from ambitwistor strings

Our construction of one-loop matrix elements is inspired by the ambitwistor-string formulae for loop integrands in gauge theories and (super)gravity. Although ambitwistor theories only involve *closed* strings, selected features of their one-loop amplitudes to be detailed below will be exported to propose matrix elements of type-I *open* superstrings. In ambitwistor string theories, scattering equations reduce the genus-one worldsheets for one-loop integrands to nodal Riemann spheres [13, 16] and introduce forward limits of various tree-level building blocks [15, 17] akin to the Q-cuts of [49].[2] All the Feynman propagators in these loop integrands can be derived from forward limits of (doubly-ordered) tree amplitudes of bi-

[2] In fact, the role of forward limits in one-loop amplitudes was already understood to a great detail through the Feynman tree theorem [50], see for instance [51] and references therein.

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adjoint scalars. The polarization degrees of freedom in turn are organized by a conformal field theory on the ambitwistor-string worldsheet whose correlators can be freely interchanged with those of conventional strings (see [21, 52–54] for a standard ambitwistor derivation, and [55–58] for a more stringy interpretation in the context of the so-called chiral strings).

The tree amplitudes of bi-adjoint scalars also appear from the $\alpha' \to 0$ limit of moduli-space integrals over disk and sphere worldsheets in tree-level amplitudes of conventional strings [59, 60]. The key idea in our proposal is to also export higher orders in the α'-expansion of disk and sphere integrals into forward-limit formulae for one-loop quantities. In other words, we convert the low-energy expansion of string tree integrals into α'-uplifts of the loop integrands from ambitwistor strings. The resulting power series only involve propagators and momentum invariants, i.e. the entire α'-dependence in our one-loop matrix elements of gauge and gravity multiplets is carried by simple *scalar* ingredients. Their dressing by polarization degrees of freedom follows the thoroughly studied conformal-field-theory mechanisms that are common to ambitwistor strings and conventional strings.

1.2 Comparison with conventional strings

In later sections, we will also motivate our proposal for one-loop matrix elements from first principles of conventional string theories in the chiral-splitting formalism. Although we are not providing a mathematically rigorous derivation, our proposal is supported by two kinds of consistency checks.

- The UV divergences in the α'-expansions of the one-loop matrix elements in various spacetime dimensions are checked to match expectations from string theory [61–63]. Even though the one-loop amplitudes of open superstrings with gauge group SO(32) and closed superstrings are UV finite, the effective-field-theory viewpoint introduces spurious UV divergences order by order in α' which can be matched with degenerations of the string worldsheet.

- The discontinuities of gravitational four-point examples at the orders of $\alpha'^{\leq 8}$ are shown to be consistent with the all-order results on the non-analytic sector of the four-point closed-string amplitude [64].[3] The method of this work should be a key steppingstone to evaluate the non-analytic sector of open-string amplitudes and ($n \geq 5$)-point closed-string amplitudes at one loop, and similar techniques are hoped to be applicable to higher loops.

Furthermore, the study of matrix elements with interactions between gauge and gravity multiplets (such as $F^m R^n$) and the application of our method to heterotic strings are relegated to the future. On the one hand, the loop diagrams with $D^{2k} F^n$ insertions are closely related to the cylinder diagram in one-loop open-string amplitudes which also propagates gravitational states in the non-planar case. On the other hand, the closed-string exchange in non-planar cylinders leads to analytic dependence on the kinematic variables, whereas the results of this work are only sensitive to the non-analytic parts of one-loop string amplitudes. This is consistent with the absence of gravitational states in the loop

[3] Also see for instance [65–68] for earlier work on the discontinuity structure at four points.

of the ambitwistor-string prescription for gauge-theory amplitudes [13] that serves as a starting point for our proposal. Indeed, the massless spectra of heterotic ambitwistor strings and conventional strings are different [69, 70], also see [16] for mismatches between ambitwistor-string amplitudes and field-theory limits of conventional heterotic strings.

1.3 Linearized versus quadratic propagators

The forward limit we consider identifies the loop momentum ℓ with a lightlike external momentum of a tree diagram, and consequently most of the inverse Feynman propagators in ambitwistor-string amplitudes are linear in ℓ. The traditional form of inverse Feynman propagators quadratic in ℓ is often better suited for integration, for instance since the Feynman $i\epsilon$ prescription is the familiar one, and such integrals have been studied for many decades. The two types of propagators are related via partial-fraction manipulations combined with shifts of loop momenta [13, 16]. The conversions from linearized to quadratic propagators has been actively discussed in the recent literature [71–76] and will also be performed for the one-loop matrix elements in this work. While the α'-expansions of the sphere and disk integrals in our proposal do not involve ℓ^2, all of our examples of one-loop matrix elements will be presented after conversion to quadratic propagators.

In spite of its drawbacks in extracting UV properties and relating to traditional Feynman vertices, the linearized-propagator form of loop integrands offers additional flexibility in manifesting the color-kinematics duality and double-copy structures. At the time of writing, the linearized propagators derived from ambitwistor strings are the only framework where a one-loop KLT formula is known in field theory [20, 21]. Our gravitational one-loop matrix elements obey a stringy (i.e. α'-dressed) version of the field-theory KLT relations for loop integrands in the references which follow from forward limits of their tree-level counterparts [4]. Similarly, our one-loop matrix elements of gauge multiplets enjoy an echo of the monodromy relations among open-string tree-level amplitudes [77–79].[4] These relations among loop integrands are exact in α' and mix the building blocks for the planar and non-planar sectors of the matrix elements.

The recombination of linearized propagators to quadratic ones is well-known to be possible for the complete one-loop integrands in ambitwistor-string formulae. As a refinement of this, we will propose a criterion to identify smaller subsectors of the correlation functions which allow for quadratic-propagator representations. Our criterion concerns the double-periodicity of correlators on a torus worldsheet *prior to* its degeneration to the nodal Riemann sphere [13, 16]. Translations of the torus punctures around the homology B-cycle require a simultaneous shift of loop momentum [6]. Functions of the punctures and ℓ with an invariance of this combined homology action have been studied under the name of generalized elliptic integrands [87, 88] or *homology invariants* [89].

The Feynman integrals descending from the moduli-space integration of homology invariants in ambitwistor string theories are conjectured to always have a quadratic-propagator representation. Moreover, the same recombination to quadratic propagators is claimed to

[4]It would be interesting to connect the monodromy relations of one-loop matrix elements in this work with those of the full-fledged one-loop open-string amplitudes [80–86].

be possible for our α'-uplifts of the ambitwistor-string integrals. We will illustrate these statements at the five-point level, where the chiral worldsheet correlators for the gauge multiplet are built from seven homology invariants to be denoted by $E_{\dots}$ that *individually* recombine to quadratic propagators.

1.4 Scope of our method and results

The loop integrands of the matrix elements in this work are universal to any number of spacetime dimensions. The examples discussed in this work are tailored to type-II superstrings with 32 supercharges and the gauge sector of type-I superstrings with 16 supercharges. Maximal supersymmetry in spacetime dimensions $D < 10$ can be attained by compactification of the initially ten-dimensional theories on a $(10-D)$-torus. The one-loop matrix elements in this work refer to the effective field theory of massless states in D dimensions and therefore exclude Kaluza-Klein and winding modes from their propagators. One-loop string amplitudes, by contrast, depend on the number of dimensions and radii of the compactification torus through their analytic part which is currently inaccessible to our method.

Our method can also be applied to type-I and type-II string amplitudes with reduced supersymmetry due to K3 or Calabi-Yau compactifications [90–93]. This follows from the fact that the associated ambitwistor-string correlators of [21–23] have a natural uplift to forward limits of disk and sphere integrals. Like this, one can for instance obtain one-loop matrix elements of the operator R^2 and its completion under 16 supercharges which plays a prominent role for the UV structure of four-dimensional $\mathcal{N} = 4$ supergravity [94–97].

1.5 Further lines of motivation

As another line of motivation, the one-loop matrix elements in this work serve as testing grounds whether large numbers of loop momenta in the numerator can be reconciled with the color-kinematics duality [1, 2]. While BCJ numerators are known up to four loops in maximal super-Yang-Mills (SYM) and supergravity [2, 7, 8], the corresponding four-point five-loop amplitude has so far resisted a manifest realization of color-kinematics duality; that is, with all kinematic Jacobi identities among cubic diagrams imposed and with local numerators. Instead, the five-loop integrand was obtained through a relaxed approach as a generalized double copy [98, 99], and subsequently used to determine the UV behavior [100]. The exact cause of a putative five-loop obstruction is difficult to pin down, due to the complexity of a five-loop calculation. If lower-loop calculations encounter similar problems, then they would provide much simpler testing ground for overcoming the obstacles. For example, in [101] such an obstacle was overcome at two loops in non-supersymmetric Yang-Mills and gravity through an increase of the powers of loop momenta. One may expect that such effects can also be observed at one loop by varying the numbers of loop-momentum factors in the numerators.

The infinite number of operators in the open-string effective action which are know to be compatible with tree-level color-kinematics duality [46, 47, 60], generate helpful illustrative examples for probing this question. Already their one-loop matrix elements involve high powers of loop momenta and might serve as toy examples to settle their compatibility with kinematic Jacobi relations. For instance, the moderately sized one-loop matrix elements in section 4 with $\alpha'^3\zeta_3 D^2F^4$ and $\alpha'^5\zeta_5 D^6F^4$ insertions furnish a manageable laboratory for manifestly color-kinematic dual representations that bypasses the combinatorial complexity of the five-loop problem in field theory.

Finally, the results of this work offer a new line of attack for a challenging problem at the interface of string amplitudes, number theory and algebraic geometry: intermediate steps in computing the low-energy expansion of one-loop closed-string amplitudes introduce a fascinating class of non-holomorphic modular forms dubbed *modular graph forms* [102, 103]. Many of their properties and their relations have been well-understood on the basis of their differential equations in the modular parameter τ [68, 103–107] which still necessitate further input on their behaviour at the cusp $\tau \to i\infty$. In spite of steady progress in the mathematics and physics literature [108–112], computing the cusp degeneration of generic modular graph forms is still a daunting task.

As will be detailed in section 6, the UV properties of our one-loop matrix elements are in one-to-one correspondence with the expansion of modular graph forms around the cusp. The Feynman integrals at fixed order in the α'-expansion of one-loop matrix elements still carry information on any higher α'-order through their higher-dimensional UV divergences. In this way, infinitely many contributions to degenerate modular graph forms can be inferred from straightforward UV properties of triangle and bubble integrals. In simple cases where the behaviour of modular graph forms around the cusp is known, this connection has been used as a check of the loop integrands in the matrix elements. At higher orders in α', the logic can be turned around such as to use the matrix element as a new method to infer mathematical information.

1.6 Outline

The main body of this work begins with a review of various ideas from ambitwistor-string amplitudes, α'-expansions of genus-zero integrals and superstring effective actions in section 2. Then, the key definitions of one-loop matrix elements as well as their KLT- and monodromy relations can be found in section 3, followed by detailed four- and five-point examples in section 4. Their connection with superstring amplitudes and the conjectural origin of quadratic propagators from homology invariance is discussed in section 5. In section 6, we relate the UV structure of one-loop matrix elements to the non-separating degeneration of genus-one string integrals. Section 7 is dedicated to the discontinuities of one-loop matrix elements and superstring amplitudes. Finally, we conclude with a brief summary and outlook in section 8. Several appendices complement the material in the main text.

2 Review

2.1 One-loop amplitudes from ambitwistor strings

Our starting point is the prescription for one-loop SYM and supergravity amplitudes from ambitwistor string theories. Both the RNS [9, 11] and the pure-spinor formulation [10, 12] of ambitwistor strings express one-loop amplitudes via moduli-space integrals over punctured

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tori that are localized on the solutions of the scattering equations. These integrals were reduced to forward limits of tree-level building blocks [13, 16] from the boundary of moduli space where the torus degenerates to a nodal Riemann sphere.

More specifically, the ambitwistor prescription for n-point one-loop amplitudes in D-dimensional theories with double-copy structure $L \otimes R$ is given by[5]

$$M^{\text{1-loop}}_{n,L\otimes R} = \int \frac{\mathrm{d}^D \ell}{\ell^2} \lim_{k_\pm \to \pm \ell} \int_{\mathbb{C}^{n-1}} \mathrm{d}\mu^{\text{tree}}_{n+2}\, \mathcal{I}_L(\ell)\, \mathcal{I}_R(\ell)\,. \tag{2.1}$$

We define the volume element in loop-momentum space as $\mathrm{d}^D \ell = -i \mathrm{d}\ell^0\, \mathrm{d}\ell^1 \ldots \mathrm{d}\ell^{D-1}$ with a factor of $-i$ to simplify later formulae, and use a spacetime metric of signature $(-,+,+,\ldots,+)$. The forward limit $k_\pm \to \pm \ell$ amounts to integrating over $(n{+}2)$-punctured Riemann spheres with the well-known measure from the CHY formulae for tree-level amplitudes in [59, 113, 114]

$$\mathrm{d}\mu^{\text{tree}}_{n+2} = \mathrm{d}\sigma_2\, \mathrm{d}\sigma_3 \, \ldots \, \mathrm{d}\sigma_n \prod_{i=2}^{n} \delta\left(2 \sum_{j\neq i}^{n+2} \frac{k_i \cdot k_j}{\sigma_{ij}}\right)\,, \qquad \sigma_{ij} = \sigma_i - \sigma_j \tag{2.2}$$

and identifying the momenta $k_+ = k_{n+1},\; k_- = k_{n+2}$ of the last two legs with (plus or minus) the D-dimensional loop momentum ℓ. The half integrands $\mathcal{I}_{L,R}(\ell)$ in (2.1) depend on the punctures $\sigma_1, \sigma_2, \ldots, \sigma_n$, the internal and external momenta ℓ, k_j and the polarization- or color degrees of freedom specific to the theory. In this work, we need the following half integrands for color $\mathcal{I}_{\text{col}}$ and kinematics $\mathcal{I}_{\text{kin}}(\ell)$:

$$\begin{aligned} \text{SYM}: \quad & \mathcal{I}_L(\ell)\mathcal{I}_R(\ell) \to \mathcal{I}_{\text{col}}\mathcal{I}_{\text{kin}}(\ell) \\ \text{supergravity}: \quad & \mathcal{I}_L(\ell)\mathcal{I}_R(\ell) \to \mathcal{I}_{\text{kin}}(\ell)\bar{\mathcal{I}}_{\text{kin}}(\ell) \end{aligned} \tag{2.3}$$

As indicated by the bar notation, the kinematic half integrands $\mathcal{I}_{\text{kin}}(\ell)$ and $\bar{\mathcal{I}}_{\text{kin}}(\ell)$ of supergravity may carry different amounts of supersymmetry.

It will be convenient to express the half integrands in terms of Parke-Taylor factors

$$\mathrm{PT}(+,1,2,\ldots,n,-) = \frac{1}{\sigma_{+1}\sigma_{12}\sigma_{23}\ldots\sigma_{n-1,n}}\,, \tag{2.4}$$

which are already adapted to our ubiquitous choice of $\mathrm{SL}_2(\mathbb{C})$-frame on the Riemann sphere (implicitly used in (2.2) by the absence of $\mathrm{d}\sigma_1$)

$$\sigma_1 = 1\,, \qquad \sigma_+ = 0\,, \qquad \sigma_- \to \infty\,. \tag{2.5}$$

The half integrand for the color dependence of SYM is given in terms of structure constants f^{abc} of some gauge group $\mathcal{G}$ (with generators t^a, commutation relations $[t^a, t^b] = f^{abc}t^c$ and adjoint indices $a = 1, 2, \ldots, \dim(\mathcal{G})$) [13, 16],

$$\begin{aligned} \mathcal{I}_{n,\text{col}} &= \sum_{\rho\in S_n} c_{\rho(12\ldots n)} \mathrm{PT}(+, \rho(1,2,\ldots,n), -) \\ c_{12\ldots n} &= f^{za_1 b} f^{ba_2 c} f^{ca_3 d} \ldots f^{ya_n z}\,. \end{aligned} \tag{2.6}$$

[5] Note that the D-dimensional loop integrand in (2.1) can also be obtained from CHY representations of tree-level amplitudes in $D{+}1$ dimensions [17].

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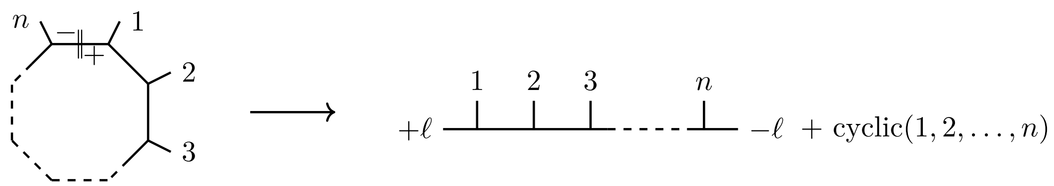

Figure 2. Left panel: n-gon diagram associated with the color factor $c_{12\ldots n}$ in (2.6). Right panel: the partial-fraction decomposition to be reviewed in (2.17) relates each n-gon with a cyclic orbit of the depicted $(n{+}2)$-point tree-level diagram.

The color tensors $c_{12\ldots n}$ arise from the n-gon diagrams in figure 2. By the Jacobi identities among contracted structure constants $f^{e[ab}f^{c]de} = 0$, they form a basis for those of all other cubic diagrams in planar or non-planar one-loop SYM amplitudes. By the cyclic invariance $c_{12\ldots n} = c_{23\ldots n1}$, color ordered amplitudes following from (2.6) involve a cyclic sum of Parke-Taylor factors (2.4),

$$A^{\text{1-loop}}_{\text{SYM}}(1,2,\ldots,n) = \int \frac{\mathrm{d}^D\ell}{\ell^2} \lim_{k_\pm \to \pm\ell} \int_{\mathbb{C}^{n-1}} \mathrm{d}\mu^{\text{tree}}_{n+2}\, \mathcal{I}_n(\ell) \sum_{\gamma \in \text{cyc}(1,2,\ldots,n)} \text{PT}(+,\gamma(1,2,\ldots,n),-)\,, \tag{2.7}$$

where we drop the subscript $_{\text{kin}}$ of $\mathcal{I}_n(\ell) = \mathcal{I}_{n,\text{kin}}(\ell)$ here and below to avoid cluttering. The Parke-Taylor decomposition (2.6) of the color-half integrand can also be attained for its kinematic counterpart [21]

$$\mathcal{I}_n(\ell) = \sum_{\rho \in S_n} N_{+|\rho(12\ldots n)|-}(\ell)\text{PT}(+,\rho(1,2,\ldots,n),-)\,, \tag{2.8}$$

with $n!$ independent kinematic numerators $N_{+|\rho(12\ldots n)|-}(\ell)$ associated with n-gon diagrams. More specifically, each $N_{+|12\ldots n|-}(\ell)$ corresponds to an $(n{+}2)$-point tree-level diagram in figure 2 obtained from the partial-fraction decomposition to be reviewed in (2.17) below.

While the color factor (2.6) is a thoroughly tested prescription, the kinematic half integrands (2.8) are derived from vertex operators for SYM and supergravity multiplets. Their worldsheet correlation functions on the torus can be imported from those of conventional strings [115–117] in the chiral-splitting formalism [5, 6]. The degeneration of these torus correlators to the Riemann sphere yields linear combinations of Parke-Taylor factors as in (2.8) after repeated use of integration by parts or scattering equations [21]. Alternatively, (2.8) can be obtained from $(n{+}2)$-point correlators on the Riemann sphere by taking the forward limit in pairs of bosons and fermions [23], also see [16, 22] for an earlier discussion of the bosonic forward limits. In both approaches, worldsheet techniques yield explicit expressions for the numerators $N_{+|\rho(12\ldots n)|-}(\ell)$ in theories with variable amount of supersymmetry.

For non-maximal supersymmetry the forward limit can be subtle since the collinear momenta ℓ and $-\ell$ may in principle introduce singularities corresponding to tadpole and external-bubble diagrams. However, they can often be removed with proper treatment such as the Minahaning procedure [118] in the organization of string amplitudes with half-maximal supersymmetry in [92, 93]. Alternatively, see the approach of [119] for identifying external bubbles that arise from 0/0 cancelations in half-maximally supersymmetric field theories. In non-supersymmetric cases, forward-limit divergences are regularized by the dropout of singular solutions of the scattering equations [15, 17] which has been used to obtain explicit numerators in [22].

For maximally supersymmetric SYM with 16 supercharges, the numerators $N_{+|12\ldots n|-}(\ell)$ in (2.8) are well-known to be degree-$(n-4)$ polynomials in the loop momentum. Manifestly supersymmetric expressions in pure-spinor superspace [120, 121] can be found in [122, 123]. For external gluons, the four- and five-point examples to be discussed in section 4 are expressible in terms of the standard t_8-tensor contracting polarization vectors.

2.2 Linearized versus quadratic propagators

By the organization of the color and kinematic half integrands in (2.6) and (2.8), the σ_j-integrals in the one-loop amplitudes (2.1) boil down to those over pairs of Parke-Taylor factors. From the early days of CHY formulae, such Parke-Taylor integrals are given by doubly-partial amplitudes $m(\alpha|\beta)$ of bi-adjoint scalars [59], and the scattering equations at one loop pick up their forward limits

$$\lim_{k_\pm \to \pm\ell} \int_{\mathbb{C}^{n-1}} \mathrm{d}\mu^{\text{tree}}_{n+2} \, \text{PT}(\alpha(+,1,2,\ldots,n,-))\text{PT}(\beta(+,1,2,\ldots,n,-)) = \lim_{k_\pm \to \pm\ell} m(\alpha|\beta) \,, \tag{2.9}$$

where $\alpha, \beta \in S_{n+2}$. Doubly partial amplitudes are defined via tree-level amplitudes $M^{\text{tree}}_{n,\phi^3}$ of bi-adjoint scalars $\Phi = \phi_{a,\tilde{b}} t^a \otimes \tilde{t}^{\tilde{b}}$ with cubic interaction $\mathcal{L}_{\text{int}} \sim \phi_{a,\tilde{a}}\phi_{b,\tilde{b}}\phi_{c,\tilde{c}} f^{abc} \tilde{f}^{\tilde{a}\tilde{b}\tilde{c}}$ by color-decomposing w.r.t. traces of the two types of Lie-algebra generators t^a and $\tilde{t}^{\tilde{b}}$,

$$M^{\text{tree}}_{n,\phi^3} = \sum_{\alpha,\beta \in S_n/\mathbb{Z}_n} \text{Tr}(t^{a_{\alpha(1)}} t^{a_{\alpha(2)}} \ldots t^{a_{\alpha(n)}}) \text{Tr}\Big(\tilde{t}^{\tilde{b}_{\beta(1)}} \tilde{t}^{\tilde{b}_{\beta(2)}} \ldots \tilde{t}^{\tilde{b}_{\beta(n)}}\Big) \, m(\alpha|\beta) \,. \tag{2.10}$$

Accordingly, doubly-partial amplitudes depend on the external momenta $k_{j=1,2,\ldots,n}$ and $k_\pm$ through the inverse propagators

$$s_{ab} = 2k_a \cdot k_b \,, \qquad s_{a_1 a_2 \ldots a_p} = 2 \sum_{1 \leq i < j}^{p} k_{a_i} \cdot k_{a_j} \,, \tag{2.11}$$

of cubic diagrams that are compatible with the cyclic orderings of both α and β. Any $m(\alpha|\beta)$ can be straightforwardly computed via graphical methods [59] or through the Berends-Giele recursion given in [124]. These methods yield the $\mathbb{C}^{n-1}$ integrals in the one-loop amplitudes (2.6) and (2.8) as a rational function of $s_{a_1\ldots a_p}$ with $a_i \in \{1, 2, \ldots, n, +, -\}$.

By the vanishing of $s_{12\ldots n-1}$ and $s_{12\ldots n}$ in the momentum phase space of n massless particles subject to momentum conservation $\sum_{j=1}^n k_j = 0$, individual $m(\alpha|\beta)$ in one-loop amplitudes may introduce forward-limit divergences. In supersymmetric amplitudes, these divergences can be efficiently cancelled by first performing the permutation sums over $m(\alpha|\beta)$ prescribed by $\mathcal{I}_L(\ell)\mathcal{I}_R(\ell)$ before taking the forward limit and imposing phase-space constraints.

The inverse Feynman propagators in the loop integrands of scattering amplitudes are of the form $(\ell + K)^2$ quadratic in ℓ, with K a sum of external momenta. However, the

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SYM and supergravity amplitudes resulting from (2.1), (2.7), (2.8) and (2.9),

$$A^{\text{1-loop}}_{\text{SYM}}(1,2,\ldots,n) = \int \frac{\mathrm{d}^D\ell}{\ell^2} \lim_{k_\pm\to\pm\ell} \sum_{\substack{\gamma\in\text{cyc}(1,2,\ldots,n)\\ \rho\in S_n}} m(+,\gamma,-|+,\rho,-)N_{+|\rho|-}(\ell)\,, \tag{2.12}$$

$$M^{\text{1-loop}}_{n,\text{SUGRA}} = \int \frac{\mathrm{d}^D\ell}{\ell^2} \lim_{k_\pm\to\pm\ell} \sum_{\gamma,\rho\in S_n} m(+,\gamma,-|+,\rho,-)N_{+|\gamma|-}(\ell)\bar{N}_{+|\rho|-}(\ell)\,, \tag{2.13}$$

mostly feature linearized inverse propagators $2\ell\cdot K + K^2$ instead of $(\ell+K)^2$ from the variables $s_{ij\ldots p,\pm}$ in doubly partial amplitudes. Using the notation

$$s_{a_1a_2\ldots a_p,\pm\ell} = s_{a_1a_2\ldots a_p} \pm 2\ell\cdot k_{a_1a_2\ldots a_p}\,, \qquad k_{a_1a_2\ldots a_p} = \sum_{i=1}^{p} k_{a_i}\,, \tag{2.14}$$

a representative four-point example of linearized propagators is

$$\sum_{\rho\in S_4} m(+,1,2,3,4,-|+,\rho(1,2,3,4),-) = \frac{1}{s_{1,\ell}s_{12,\ell}s_{123,\ell}}\,. \tag{2.15}$$

The denominator is obtained by manually dropping the ℓ^2 in three of the Feynman propagators on the left-hand side of the four-point box integral

$$\int \frac{\mathrm{d}^D\ell}{\ell^2(\ell{+}k_1)^2(\ell{+}k_{12})^2(\ell{+}k_{123})^2} = \int\frac{\mathrm{d}^D\ell}{\ell^2}\left(\frac{1}{s_{1,\ell}s_{12,\ell}s_{123,\ell}} + \text{cyc}(1,2,3,4)\right). \tag{2.16}$$

In passing to the right-hand side, however, partial-fraction manipulations and shifts of the loop momenta $\ell\to\ell\pm k_i$ have been used to rewrite the box integral in terms of linearized-propagator expressions as in (2.16). The same manipulations apply to n-gons

$$\begin{aligned}\int \frac{\mathrm{d}^D\ell}{\ell^2(\ell{+}k_1)^2(\ell{+}k_{12})^2\ldots(\ell{+}k_{12\ldots n-1})^2} &= \sum_{i=0}^{n-1}\int\frac{\mathrm{d}^D\ell}{(\ell{+}k_{12\ldots i})^2}\prod_{j\neq i}^{n}\frac{1}{(\ell{+}k_{12\ldots j})^2-(\ell{+}k_{12\ldots i})^2}\\ &= \sum_{i=0}^{n-1}\int\frac{\mathrm{d}^D\ell}{\ell^2}\prod_{j=0}^{i-1}\frac{1}{s_{j+1,j+2,\ldots,i,-\ell}}\prod_{j=i+1}^{n-1}\frac{1}{s_{i+1,i+2,\ldots,j,\ell}}\,,\end{aligned} \tag{2.17}$$

and their generalizations to massive corners with momenta $k_A = k_{a_1}+k_{a_2}+\ldots+k_{a_p}$ for $A=a_1a_2\ldots a_p$, see (2.14). The example of the box integral in (2.15) and (2.16) illustrates that the recombination of linearized propagators to quadratic ones generically relies on the sum over cyclic permutations γ in the color-ordered SYM amplitudes (2.12). Similarly, the quadratic-propagator representation of supergravity amplitudes is far from manifest in (2.13) and relies on the interplay between different terms in the permutation sum over $\gamma,\rho\in S_n$.

Consider a generic numerator $\mathfrak{N}(\ell)$ belonging to a $(p\leq n)$-gon diagram with quadratic propagators, then after inserting it into the cyclic sum of (2.17) each term will become a shifted version $\mathfrak{N}(\ell{-}k_{\ldots})$. For the box integral, we get

$$\begin{aligned}\int\frac{\mathfrak{N}(\ell)\,\mathrm{d}^D\ell}{\ell^2(\ell{+}k_1)^2(\ell{+}k_{12})^2(\ell{+}k_{123})^2} = \int\frac{\mathrm{d}^D\ell}{\ell^2}\Bigg(&\frac{\mathfrak{N}_{+|1234|-}(\ell)}{s_{1,\ell}s_{12,\ell}s_{123,\ell}}+\frac{\mathfrak{N}_{+|2341|-}(\ell)}{s_{2,\ell}s_{23,\ell}s_{234,\ell}}\\ &+\frac{\mathfrak{N}_{+|3412|-}(\ell)}{s_{3,\ell}s_{34,\ell}s_{341,\ell}}+\frac{\mathfrak{N}_{+|4123|-}(\ell)}{s_{4,\ell}s_{41,\ell}s_{412,\ell}}\Bigg)\,,\end{aligned} \tag{2.18}$$

where

$$\mathfrak{N}_{+|1234|-}(\ell) = \mathfrak{N}(\ell)\,, \qquad \mathfrak{N}_{+|2341|-}(\ell) = \mathfrak{N}(\ell - k_1)\,, \qquad (2.19)$$
$$\mathfrak{N}_{+|3412|-}(\ell) = \mathfrak{N}(\ell - k_{12})\,, \qquad \mathfrak{N}_{+|4123|-}(\ell) = \mathfrak{N}(\ell - k_{123})\,.$$

Therefore, we can trivially transform a p-gon integrand in the linearized-propagator representation back to standard quadratic propagators if

$$\mathfrak{N}_{+|i+1,i+2,\ldots,p,1,\ldots,i|-}(\ell) = \mathfrak{N}_{+|123\ldots p|-}(\ell - k_{12\ldots i})\,, \qquad 1 \leq i \leq p-1\,, \qquad (2.20)$$

for example,

$$\mathfrak{N}_{+|23\ldots p1|-}(\ell) = \mathfrak{N}_{+|123\ldots p|-}(\ell - k_1)\,, \quad \mathfrak{N}_{+|34\ldots p12|-}(\ell) = \mathfrak{N}_{+|123\ldots p|-}(\ell - k_{12})\,, \text{ etc.} \qquad (2.21)$$

However, there is in principle no need for the numerators with $p \leq n$ to obey (2.20) — they may depart from this relation through contact terms that cancel out in the full amplitude.

The appearance of linearized propagators from the ambitwistor-string prescription was firstly noticed in [13] and their equivalence with quadratic propagators has been actively discussed in the recent literature including [71–76]. In this work, we will find the first instance of this phenomenon in matrix elements of higher-mass-dimension operators of the schematic form $\mathrm{Tr}\{D^{2k} F^n\}$ or $D^{2k} R^n$ in the low-energy effective action of open and closed superstrings to be reviewed in section 2.4. Moreover, motivated by the five-point examples in SYM and supergravity, the possibility to recombine linearized to quadratic propagators is conjectured to descend from homology invariance in the chiral-splitting formalism, see section 5.

2.3 Disk and sphere integrals

The doubly-partial amplitudes $m(\alpha|\beta)$ furnish an important building block in the explicit evaluation of moduli-space integrals (2.9) of ambitwistor string theories. Also in conventional string theories, the $m(\alpha|\beta)$ feature prominently in the field-theory limit of integrals Z over punctured disks and J over punctured spheres in open- and closed-string tree-level amplitudes,

$$Z(\gamma, n|\rho, n) = (-1)^n (\alpha')^{n-3} \int_{\mathfrak{D}(\gamma)} \mathrm{d}\sigma_2\, \mathrm{d}\sigma_3 \ldots \mathrm{d}\sigma_{n-2} \prod_{1\leq i<j}^{n-1} |\sigma_{ij}|^{\alpha' s_{ij}}\, \mathrm{PT}(\rho(1,2,\ldots,n-1),n) \qquad (2.22)$$

$$J(\gamma, n|\rho, n) = \left(\frac{\alpha'}{\pi}\right)^{n-3} \int_{\mathbb{C}^{n-3}} \mathrm{d}^2\sigma_2\, \mathrm{d}^2\sigma_3 \ldots \mathrm{d}^2\sigma_{n-2} \prod_{1\leq i<j}^{n-1} |\sigma_{ij}|^{2\alpha' s_{ij}}$$
$$\times \overline{\mathrm{PT}(\gamma(1,2,\ldots,n-1),n)}\, \mathrm{PT}(\rho(1,2,\ldots,n-1),n) \qquad (2.23)$$

indexed by two permutations $\gamma, \rho \in S_{n-1}$ of legs $1, 2, \ldots, n-1$. We have already picked SL_2-frames with $(\sigma_1, \sigma_{n-1}, \sigma_n) \to (0, 1, \infty)$ such that the Parke-Taylor factors have the chain structure of (2.4), and the integration domain $\mathfrak{D}(\gamma)$ for the punctures on the disk boundary is given by

$$\mathfrak{D}(\gamma) = \Big\{\sigma_{j=1,2,\ldots,n-1} \in \mathbb{R},\ -\infty < \sigma_{\gamma(1)} < \sigma_{\gamma(2)} < \ldots < \sigma_{\gamma(n-1)} < \infty\Big\}\,. \qquad (2.24)$$

Throughout this work, the normalization convention for α' is tailored to open strings, and the accurate numerical factors in closed-string quantities can be restored by rescaling $\alpha' \to \alpha'/4$.

The low-energy expansion of the disk and sphere integrals yields a Laurent series in the dimensionless Mandelstam invariants $\alpha' s_{ij}$ whose coefficients are $\mathbb{Q}$-linear combinations of multiple zeta values (MZVs) [125, 126]

$$\zeta_{n_1,n_2,\ldots,n_r} = \sum_{0<k_1<k_2<\ldots<k_r}^{\infty} k_1^{-n_1} k_2^{-n_2} \ldots k_r^{-n_r} \,, \qquad n_1,\ldots,n_r \in \mathbb{N} \,, \qquad n_r \geq 2 \,. \tag{2.25}$$

By the normalization of (2.22) and (2.23), their $\alpha' \to 0$ limit is free of MZVs and yields the doubly partial amplitudes $m(\alpha|\beta)$ in case of both disk integrals [59] and sphere integrals [60],

$$\lim_{\alpha'\to 0} Z(\gamma,n|\rho,n) = \lim_{\alpha'\to 0} J(\gamma,n|\rho,n) = m(\gamma,n|\rho,n) \,, \tag{2.26}$$

so Z and J are natural uplifts of the $m(\alpha|\beta)$ in SYM and supergravity amplitudes (2.12) and (2.13), respectively. The subleading orders in α' beyond the field-theory limit (2.26) comprise Laurent polynomials $p_d(s_{ij})$ of homogeneity degree $d = 3-n+w$ in $s_{ij\ldots p}$ along with α'^w. The weight $n_1+\ldots+n_r = w$ of the accompanying MZVs $\zeta_{n_1,\ldots,n_r}$ matches the order of α'^w, i.e. both Z and J are said to be uniformly transcendental.

On top of their identical field-theory limits (2.26), the α'-expansions of sphere integrals are completely determined by those of the disk integrals through the single-valued map "sv" of MZVs [127, 128],[6] e.g. ($k \in \mathbb{N}$)

$$\text{sv}\,\zeta_{2k} = 0 \,, \qquad \text{sv}\,\zeta_{2k+1} = 2\zeta_{2k+1} \,, \qquad \text{sv}\,\zeta_{3,5} = -10\zeta_3\zeta_5 \tag{2.27}$$

which acts order by order in the α'-expansion of [32, 60, 129–132]

$$J(\gamma,n|\rho,n) = \text{sv}\, Z(\gamma,n|\rho,n) \,. \tag{2.28}$$

At the lowest orders in the Laurent-expansion beyond the field-theory limit (2.26), the single-valued map (2.28) produces

$$\begin{aligned} Z &= m + \alpha'^2\zeta_2 p_{5-n}(s_{ij}) + \alpha'^3\zeta_3 p_{6-n}(s_{ij}) + \alpha'^4\zeta_4 p_{7-n}(s_{ij}) \\ &\quad + \alpha'^5[\zeta_5 p_{8-n}(s_{ij}) + \zeta_2\zeta_3 \hat{p}_{8-n}(s_{ij})] + \mathcal{O}(\alpha'^6) \,, \end{aligned} \tag{2.29}$$

$$J = m + 2\alpha'^3\zeta_3 p_{6-n}(s_{ij}) + 2\alpha'^5\zeta_5 p_{8-n}(s_{ij}) + \mathcal{O}(\alpha'^6) \,. \tag{2.30}$$

We have suppressed the common labels $(\gamma,n|\rho,n)$ of the doubly partial amplitudes $m = p_{3-n}(s_{ij})$ and the degree-d Laurent-polynomials $p_d(s_{ij})$. The single-valued map removes the appearance of ζ_2, ζ_4, $\zeta_2\zeta_3$ and doubles the coefficients of ζ_3, ζ_5 in passing from disk to sphere integrals at the orders $\alpha'^{\leq 5}$, i.e. the polynomials $p_{6-n}(s_{ij})$ and $p_{8-n}(s_{ij})$ are universal to (2.29) and (2.30). The coefficients of $\zeta_w \alpha'^w$ determine those of MZVs at higher depth $r \geq 2$ and arbitrary products such as $\zeta_2\zeta_3$ in (2.29) [32] which can be attributed to the group-like behaviour of the Z-integrals under the motivic coaction [133].

[6] Strictly speaking, the single-valued map is only well defined for the motivic versions $\zeta^{\mathfrak{m}}_{n_1,n_2,\ldots,n_r}$ of MZVs.

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The pattern of Laurent polynomials at the leading orders of (2.29) can be illustrated through the six-point examples

$$\begin{aligned} Z(1,2,3,4,5,6|1,4,2,3,6,5) &= \frac{1}{s_{23}s_{234}s_{56}} \\ &+ \alpha'^2\zeta_2\Big(-\frac{s_{16}}{s_{23}s_{234}} - \frac{s_{123}}{s_{23}s_{56}} - \frac{s_{34}}{s_{234}s_{56}} + \frac{1}{s_{23}} + \frac{1}{s_{56}}\Big) \\ &+ \alpha'^3\zeta_3\Big(\frac{s_{16}(s_{16}+s_{56})}{s_{23}s_{234}} + \frac{s_{123}(s_{123}+s_{234})}{s_{23}s_{56}} + \frac{(s_{23}+s_{34})s_{34}}{s_{234}s_{56}} \\ &\quad - \frac{s_{234}}{s_{23}} - \frac{2s_{45}}{s_{23}} - \frac{s_{56}}{s_{23}} - \frac{2s_{12}}{s_{56}} - \frac{s_{23}}{s_{56}} - \frac{s_{234}}{s_{56}}\Big) + \mathcal{O}(\alpha'^4) \end{aligned} \tag{2.31}$$

as well as

$$\begin{aligned} \sum_{\rho\in S_4} Z(1,2,\ldots,6|1,\rho(2,3,4,5),6) &= \frac{1}{s_{12}s_{123}s_{56}} - \alpha'^2\zeta_2\Big(\frac{s_{45}}{s_{12}s_{123}} + \frac{s_{23}}{s_{123}s_{56}} + \frac{s_{34}}{s_{12}s_{56}}\Big) \\ &+ \alpha'^3\zeta_3\left(\frac{s_{45}(s_{45}+s_{56})}{s_{12}s_{123}} + \frac{(s_{12}+s_{23})s_{23}}{s_{123}s_{56}} + \frac{(s_{123}+s_{34})s_{34}}{s_{12}s_{56}} - \frac{s_{35}}{s_{12}} - \frac{s_{24}}{s_{56}}\right) + \mathcal{O}(\alpha'^4)\,. \end{aligned} \tag{2.32}$$

The results in section 4 for one-loop matrix elements of higher-mass-dimension operators crucially rely on the availability of six- and seven-point α'-expansions which we have generated by means of the Berends-Giele recursion in [134, 135]. Alternative methods to determine the polynomial structure of the α'-expansions at all multiplicities are based on the Drinfeld associator [136] (also see [133, 137]) or polylogarithm manipulations [138], with machine-readable expressions available for download on [139]. Another approach to generate α'-expansions of disk integrals (2.22) at $n \leq 7$ points is to exploit their connection with hypergeometric functions [30, 31, 140–142].

2.4 Tree-level effective action of type I and II superstrings

The schematic form of the tree-level effective action of type I and II superstrings at leading orders in α' has been previewed in (1.1). In spite of all-multiplicity formulae including the exact tensor structure of tree amplitudes of gauge and gravity multiplets [32, 143], the detailed tensor structure of the effective operators with more than four powers of the non-abelian field strength F is only known up to the α'^4 order. One can anticipate from the bosonic components at the subleading order of

$$\begin{aligned} S^{\rm open}_{\rm eff} = \int \mathrm{d}^D x\, \mathrm{Tr}\bigg\{ \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\alpha'^2\zeta_2}{4}\Big[&-8F^{\mu}{}_{\lambda}F^{\lambda}{}_{\nu}F^{\rho}{}_{\mu}F^{\nu}{}_{\rho} - 4F^{\mu}{}_{\lambda}F^{\lambda}{}_{\nu}F^{\nu}{}_{\rho}F^{\rho}{}_{\mu} \\ &+ 2F^{\mu\nu}F_{\mu\nu}F^{\lambda\rho}F_{\lambda\rho} + F^{\mu\nu}F^{\lambda\rho}F_{\mu\nu}F_{\lambda\rho}\Big] + \ldots \bigg\} \end{aligned} \tag{2.33}$$

that the number of terms grows drastically with the order in α'. A state-of-the-art method to determine 8-term and 96-term-expressions for the Lagrangians $\alpha'^3\zeta_3 \mathrm{Tr}\{D^2F^4{+}F^5\}$ and

$\alpha'^4\zeta_2^2\text{Tr}\{D^4F^4{+}D^2F^5{+}F^6\}$ is described in [33], see for instance [26, 29] and [28, 30] for earlier results at the respective orders. The supersymmetric completion of (2.33) and its gravitational counterpart is discussed from a variety of perspectives in [36–45] and references therein.

The bosonic terms in (2.33) and their gravitational counterparts in (1.1) are universal to any torus compactification of the initially ten-dimensional type-I and type-II theories to D spacetime dimensions. Such compactifications preserve all supersymmetries and lead to a straightforward dimensional reduction of the massless tree-level interactions, since Kaluza-Klein and winding modes are only produced in pairs. Accordingly, the one-loop matrix elements to be obtained from forward limits of tree amplitudes in the massless multiplets have a dimension-agnostic loop integrand.

For abelian external open-string states, the leading order in the low-energy effective action is described by the Born-Infeld theory [144]. In four dimensions, one-loop n-point amplitudes of Born-Infeld have been discussed in [145–147], and the abelian limit of our results contain the D-dimensional uplift of the loop integrands in the references. A detailed comparison is left for the future.

Low-energy effective actions are usually determined by reverse-engineering the Feynman rules that generate the α'-expansions of massless string amplitudes. When expressing the tree-level amplitudes of the open superstring in terms of Z-integrals [138, 143] in (2.22), the contributions of ζ_2 and ζ_3 to their expansion (2.29) yield matrix elements of the effective operators $\text{Tr}\{F^4\}$ and $\text{Tr}\{D^2F^4{+}F^5\}$, respectively. Similarly, the ζ_3- and ζ_5-terms in the expansion (2.30) of J-integrals (2.23) correspond to R^4 and $D^4R^4{+}D^2R^5{+}R^6$ couplings of type-II superstrings, respectively. In both cases, each $\mathbb{Q}$-independent combination of MZVs in the expansion of the scalar genus-zero integrals is associated with a separate effective operator.

The gauge-multiplet operators due to weight-w MZVs in the disk integral (2.29) have bosonic components of the schematic form $\text{Tr}\{D^{2(w-2)}F^4{+}D^{2(w-3)}F^5{+}\ldots{+}F^{w+2}\}$. Similarly the single-valued MZVs of weight w in the sphere integral (2.30) lead to gravitational components $D^{2(w-3)}R^4{+}D^{2(w-4)}R^5{+}\ldots{+}R^{w+1}$ in the type-II effective action. The maximal supersymmetry of the type-I and type-II theories usually interlocks operators with different numbers of F and R while balancing their mass dimension through the number of covariant derivatives $D^2 \leftrightarrow F$ and $D^2 \leftrightarrow R$. In fact, relations of the schematic form $D^2F \cong F^2$ and $D^2R \cong R^2$ together with field redefinitions introduce immense ambiguities into the form of the effective action. Up to and including weight $w = 6$, these composite gauge and gravity operators can be represented as seen in table 1.

Starting from weight $w = 8$, indecomposable MZVs $\zeta_{n_1,\ldots,n_r}$ of depth $r \geq 2$ occur in the conjectural $\mathbb{Q}$-bases (see e.g. [148]), for instance $\zeta_{3,5}, \zeta_{3,7}$ and $\zeta_{3,3,5}$ at weight $8, 10$ and 11, respectively. Since the four-point Z- and J-integrals are expressible in terms of depth-one ζ_w, one can avoid four-field interactions $D^{2k}F^4$ along with irreducible MZVs beyond depth one, i.e. eliminate $\alpha'^8\zeta_{3,5}D^{12}F^4, \alpha'^{10}\zeta_{3,7}D^{16}F^4$ and $\alpha'^{11}\zeta_{3,3,5}D^{18}F^4$. The ubiquity of indecomposable higher-depth MZVs in ($n \geq 5$)-point string amplitudes [32] leads to combinations of $F^{\geq 5}$-operators such as $\alpha'^8\zeta_{3,5}(D^{10}F^5{+}D^8F^6{+}\ldots)$. For closed strings, $\zeta_{3,5}$ and $\zeta_{3,7}$ are removed from the α'-expansion through the single-valued map, see (2.27), so the first appearance of irreducible MZVs is via $\alpha'^{11}\zeta_{3,3,5}(D^{14}R^5{+}D^{12}R^6{+}\ldots)$ without any four-point admixture $\sim D^{16}R^4$.

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gauge interactions	gauge interactions	gravitational interactions
$1 \leftrightarrow F^2$	$\alpha'^5\zeta_5 \leftrightarrow D^6F^4+\ldots+F^7$	$1 \leftrightarrow R$
$\alpha'^2\zeta_2 \leftrightarrow F^4$	$\alpha'^5\zeta_2\zeta_3 \leftrightarrow D^6F^4+\ldots+F^7$	$\alpha'^3\zeta_3 \leftrightarrow R^4$
$\alpha'^3\zeta_3 \leftrightarrow D^2F^4+F^5$	$\alpha'^6\zeta_2^3 \leftrightarrow D^8F^4+\ldots+F^8$	$\alpha'^5\zeta_5 \leftrightarrow D^4R^4+D^2R^5+R^6$
$\alpha'^4\zeta_2^2 \leftrightarrow D^4F^4+D^2F^5+F^6$	$\alpha'^6\zeta_3^2 \leftrightarrow D^8F^4+\ldots+F^8$	$\alpha'^6\zeta_3^2 \leftrightarrow D^6R^4+\ldots+R^7$

Table 1. Schematic form of gauge and gravity operators at the orders of $\alpha'^{\leq 6}$ in the tree-level effective action of type-I and type-II superstrings (suppressing the trace of $\text{Tr}\{D^{2k}F^n\}$ operators to avoid cluttering).

The appearance of $\zeta_{3,5}$ in $F^{\geq 5}$ operators and its dropout from R^5 and R^6 interactions was firstly reported in [31], and the all-order systematics of MZVs in massless n-point amplitudes of open and closed superstrings is described in [32]. In the same way as irreducible MZVs of depth ≥ 2 decouple at the four-point level, five-point amplitudes still feature dropouts of MZVs at higher weight ≥ 18 [32].[7] At sufficiently large multiplicity, any conjecturally $\mathbb{Q}$-independent combination of MZVs is expected to occur in the α'-expansion of disk integrals and therefore the open-superstring effective action. Accordingly, closed-string tree-level interactions should feature all $\mathbb{Q}$-independent single-valued MZVs.

3 One-loop matrix elements for the superstring effective action

We shall now combine the ingredients reviewed in the previous section to propose expressions for one-loop matrix elements with single and multiple insertions of any operator $\text{Tr}\{D^{2k}F^n\}$ and $D^{2k}R^n$ in the above tree-level effective actions.

3.1 α' uplifts for one-loop amplitudes

The key idea of this work can be phrased as systematically introducing α'-corrections to one-loop ambitwistor-string amplitudes. For this purpose, the doubly-partial amplitudes m in the expressions (2.12) and (2.13) for SYM and supergravity amplitudes are promoted to disk and sphere integrals (2.22) and (2.23), respectively,[8]

$$A^{\text{1-loop}}_{\text{eff}}(1,2,\ldots,n) = \int \frac{\mathrm{d}^D\ell}{\ell^2} \lim_{k_\pm \to \pm\ell} \sum_{\substack{\gamma\in \text{cyc}(1,2,\ldots,n)\\ \rho\in S_n}} Z(+,\gamma,-|+,\rho,-) N_{+|\rho|-}(\ell)\,, \tag{3.1}$$

$$M^{\text{1-loop}}_{n,\text{eff}} = \int \frac{\mathrm{d}^D\ell}{\ell^2} \lim_{k_\pm \to \pm\ell} \sum_{\gamma,\rho\in S_n} J(+,\gamma,-|+,\rho,-) N_{+|\gamma|-}(\ell) \bar{N}_{+|\rho|-}(\ell)\,. \tag{3.2}$$

[7] At weight $w = 18$ for instance, the dropout of one MZV specified in [32] can be used to bring one effective operator into the form $\alpha'^{18}(D^{28}F^6 + \ldots)$ without any admixtures of $\alpha'^{18}D^{30}F^5$ and $\alpha'^{18}D^{32}F^4$.

[8] Throughout this work, we suppress the dependence on the gauge coupling and gravitational constant, given by global prefactors $(g_{\text{YM}})^n$ in case of $A^{\text{1-loop}}_{\text{eff}}(1,\ldots,n)$ and κ^n in case of $M^{\text{1-loop}}_{n,\text{eff}}$, respectively. Normalization factors including powers of i and $\sqrt{2}$ that apply to the entire α'-expansion are not tracked.

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By the common field-theory limit (2.26) of Z and J, the leading orders in α' are engineered to comprise SYM and supergravity amplitudes,

$$A^{\text{1-loop}}_{\text{eff}}(1,2,\ldots,n) = A^{\text{1-loop}}_{\text{SYM}}(1,2,\ldots,n)+\mathcal{O}(\alpha'^2)\,, \qquad M^{\text{1-loop}}_{n,\text{eff}} = M^{\text{1-loop}}_{n,\text{SUGRA}}+\mathcal{O}(\alpha'^3)\,. \tag{3.3}$$

The α'-expansion of (3.1) and (3.2) is expected to reproduce the matrix elements for single and multiple insertions of the higher-mass-dimension operators in the superstring effective actions of (1.1) and section 2.4. However, the expressions for $A^{\text{1-loop}}_{\text{eff}}(1,2,\ldots,n)$ and $M^{\text{1-loop}}_{n,\text{eff}}$ are *not* claimed to be the full massless one-loop amplitudes of open and closed superstrings: as will be explained in section 5.1, they are lacking important contributions from the bulk of the moduli space of punctured genus-one surfaces. Indeed, localizing to the boundary of the moduli space naturally induces a truncation of the superstring one-loop amplitudes where massive string excitations in the loop are dropped at any fixed order in the α'-expansion. By resumming the α'-expansion of these boundary contributions, in turn, one may in principle recover the subset of the massive propagators which stem from genus-zero integrals Z and J.

In the open-string case, (3.1) corresponds to the planar or single-trace amplitudes in the color decomposition

$$\begin{aligned} A^{\text{1-loop}}_{n,\text{eff}} &= \text{Tr}\{1\} \sum_{\gamma\in S_{n-1}} \text{Tr}\{t^{a_1}t^{a_{\gamma(2)}}\ldots t^{a_{\gamma(n)}}\} A^{\text{1-loop}}_{\text{eff}}(1,\gamma(2,\ldots,n)) \\ &\quad + \sum_{k=2}^{\lfloor n/2\rfloor} \Big[\text{Tr}\{t^{a_1}t^{a_2}\ldots t^{a_k}\}\text{Tr}\{t^{a_{k+1}}t^{a_{k+2}}\ldots t^{a_n}\} \\ &\qquad\qquad \times A^{\text{1-loop}}_{\text{eff}}(1,2,\ldots,k|k{+}1,k{+}2,\ldots,n) + \text{permutations}\Big]\,, \end{aligned} \tag{3.4}$$

and a proposal for their non-planar counterparts will be motivated below. The permutation sum is understood to involve all cyclically inequivalent double-trace configurations without double-counting of $A^{\text{1-loop}}_{\text{eff}}(1,\ldots,k|k{+}1,\ldots,n) = A^{\text{1-loop}}_{\text{eff}}(k{+}1,\ldots,n|1,\ldots,k)$.

In order to further explore the structure and relations of the one-loop matrix elements (3.1) and (3.2), we introduce an important substructure of the single-trace $A^{\text{1-loop}}_{\text{eff}}$: the kinematic half integrand $\mathcal{I}_n(\ell)$ in the ambitwistor-string formula (2.7) for one-loop SYM amplitudes is gauge invariant on the support of the scattering equations. Hence, $\mathcal{I}_n(\ell)$ dressed with any of the individual Parke-Taylor factors in the cyclic sum of (2.7) integrates to gauge invariant objects dubbed *partial integrands* [20, 21],

$$\begin{aligned} a_{\text{SYM}}(\gamma(+,1,2,\ldots,n,-)) &= \lim_{k_\pm\to\pm\ell} \int_{\mathbb{C}^{n-1}} \text{d}\mu^{\text{tree}}_{n+2}\, \text{PT}(\gamma(+,1,2,\ldots,n,-))\, \mathcal{I}_n(\ell) \\ &= \lim_{k_\pm\to\pm\ell} \sum_{\rho\in S_n} m(\gamma(+,1,2,\ldots,n,-)|+,\rho,-)N_{+|\rho|-}(\ell)\,, \end{aligned} \tag{3.5}$$

with $\gamma \in S_{n+2}$ which compose single- and double-trace one-loop amplitudes via

$$A^{\text{1-loop}}_{\text{SYM}}(1,2,\ldots,n) = \int \frac{\text{d}^D\ell}{\ell^2}\, a_{\text{SYM}}(+,1,2,\ldots,n,-) + \text{cyc}(1,2,\ldots,n)\,, \tag{3.6}$$

$$A^{\text{1-loop}}_{\text{SYM}}(1,2,\ldots,k|k{+}1,\ldots,n) = \int \frac{\text{d}^D\ell}{\ell^2} \sum_{\substack{\beta\in\text{cyc}(1,2,\ldots,k)\\ \gamma\in\text{cyc}(k+1,\ldots,n)}} a_{\text{SYM}}(+,\beta(1,2,\ldots,k),-,\gamma(k{+}1,\ldots,n))\,.$$

In the same way as the promotion $m \to Z$ in $A^{\text{1-loop}}_{\text{SYM}}$ led to (3.1), we introduce α'-corrections to the partial integrand (3.5) by inserting disk integrals into

$$a_{\text{eff}}(\gamma(+,1,2,\ldots,n,-)) = \lim_{k_\pm \to \pm \ell} \sum_{\rho \in S_n} Z(\gamma(+,1,2,\ldots,n,-)|+,\rho,-) N_{+|\rho|-}(\ell) \,. \tag{3.7}$$

The ambitwistor-string definition of partial SYM integrands in the first line of (3.5) yields forward limits of color-ordered tree-level amplitudes where tentative divergences are regularized by the dropout of singular solutions. The transition to the second line of (3.5) requires supersymmetry in order to reliably evaluate the $\mathrm{d}\mu^{\text{tree}}_{n+2}$ integrals in terms of doubly-partial amplitudes which do not diverge in the forward limit. Accordingly, we only define α'-dressed partial integrands (3.7) in a supersymmetric setup for now such as the open superstring in toroidal, K3 or Calabi-Yau compactifications to D-dimensional Minkowski spacetime.[9] They are forward limits of color-ordered tree-level amplitudes with massless external states and therefore inherit their relations. In particular, (3.7) is the forward limit of the $n!$-term representation of $(n+2)$-point open-superstring tree amplitudes in terms of BCJ master numerators given in appendix B.3 of [138].

With the α'-dressed partial integrands (3.7), the assembly (3.6) of SYM single- and double-trace amplitudes generalizes to

$$\begin{aligned} A^{\text{1-loop}}_{\text{eff}}(1,2,\ldots,n) &= \int \frac{\mathrm{d}^D \ell}{\ell^2}\, a_{\text{eff}}(+,1,2,\ldots,n,-) + \text{cyc}(1,2,\ldots,n)\,, \\ A^{\text{1-loop}}_{\text{eff}}(1,2,\ldots,k|k{+}1,\ldots,n) &= \int \frac{\mathrm{d}^D \ell}{\ell^2} \sum_{\substack{\beta \in \text{cyc}(1,2,\ldots,k) \\ \gamma \in \text{cyc}(k+1,\ldots,n)}} a_{\text{eff}}(+,\beta(1,2,\ldots,k),-,\gamma(k{+}1,\ldots,n))\,. \end{aligned} \tag{3.8}$$

The single-trace relation is equivalent to (3.1), and the second line defines our proposal for the double-trace sector of the one-loop matrix elements in the color decomposition (3.4).

3.2 Relations among one-loop matrix elements

The forward limit in the definition (3.7) of partial integrands for one-loop matrix elements will preserve the monodromy relations among disk integrals with different orderings $\gamma \in S_{n+2}$ of the punctures on a disk boundary [77–79]

$$\begin{aligned} &\sum_{j=0}^{n} e^{2\pi i \alpha' \ell \cdot k_{12\ldots j}} a_{\text{eff}}(1,2,\ldots,j,+,j{+}1,\ldots,n,-) = 0 \\ &\sum_{j=1}^{n} e^{2\pi i \alpha' k_1 \cdot k_{23\ldots j}} a_{\text{eff}}(2,3,\ldots,j,1,j{+}1,\ldots,n,-,+) = e^{2\pi i \alpha' \ell \cdot k_1} a_{\text{eff}}(2,3,\ldots,n,-,1,+)\,. \end{aligned} \tag{3.9}$$

In the field-theory limit, the exponentials can be replaced by their Taylor expansion to the first order in α' and imply the generalization of Kleiss-Kuijf [149] and BCJ relations [1]

[9] Still, the treatment of forward-limit divergences in deriving non-supersymmetric gauge-theory amplitudes from ambitwistor strings [22] is expected to have smooth α'-uplifts via $m \to Z$ or $m \to J$.

that apply to partial integrands [20, 21], e.g.

$$\sum_{j=1}^{n-1} \ell \cdot k_{12\ldots j} a_{\rm SYM}(1,2,\ldots,j,+,j{+}1,\ldots,n,-) = 0\,, \qquad (3.10)$$

$$\sum_{j=1}^{n} a_{\rm SYM}(2,3,\ldots,j,1,j{+}1,\ldots,n,-,+) = a_{\rm SYM}(2,3,\ldots,n,-,1,+)\,.$$

By combining suitable permutations of (3.9), one can express the constituents $a_{\rm eff}(+,\beta,-,\gamma)$ in the double-trace sector (3.8) in terms of the single-trace quantities $a_{\rm eff}(+,\ldots,-)$. In the field-theory limit, the resulting relations among $a_{\rm SYM}(+,\beta,-,\gamma)$ and $a_{\rm SYM}(+,\ldots,-)$ imply those between single- and double-trace amplitudes (3.6) in general gauge theories [150]. The monodromy relations (3.9) leave at most $(n{-}1)!$ independent orderings of $a_{\rm eff}(\ldots)$, but spacetime supersymmetry in general implies additional relations: by the 16 supercharges of the type-I superstring, $(n \leq 3)$-point instances of $a_{\rm eff}$ vanish, and all permutations of $a_{\rm eff}(\gamma(+,1,2,3,4,-))$ at four points have the same kinematic factor and only differ in their series-expansion in k_j and ℓ. It would be interesting to relate the monodromy relations (3.9) among the one-loop matrix elements to those among the full-fledged one-loop open-string amplitudes [80–86].

In preparation for relations between one-loop matrix elements of ${\rm Tr}\{D^{2k}F^n\}$ and $D^{2k}R^n$ operators, we note that the BCJ relations (3.10) of the partial SYM integrands qualify them to enter a KLT construction. The field-theory KLT formula [4] among tree amplitudes of SYM and supergravity generalizes as follows to partial integrands (3.5) and one-loop supergravity integrands [20, 21]

$$M^{\text{1-loop}}_{n,{\rm SUGRA}} = \int \frac{{\rm d}^D \ell}{\ell^2} \sum_{\gamma,\rho \in S_{n-1}} a_{\rm SYM}(+,\gamma(1,2,\ldots,n{-}1),n,-) \\ \times S_0(\gamma|\rho)_\ell \, \bar a_{\rm SYM}(+,\rho(1,2,\ldots,n{-}1),-,n)\,. \qquad (3.11)$$

The field-theory KLT matrix S_0 encoding the all-multiplicity formula at tree level [151] is used in its representation within the momentum-kernel formalism [152] and obeys the recursion

$$S_0(A,j|B,j,C)_\ell = k_j \cdot (\ell + k_B) S_0(A|B,C)_\ell\,, \qquad S_0(\emptyset|\emptyset)_\ell = 1\,, \qquad (3.12)$$

where $A = a_1, a_2, \ldots$ as well as $B = b_1, b_2, \ldots$ and $C = c_1, c_2, \ldots$ are ordered sequences of external legs. In the α'-dressed supergravity integrand (3.2), the forward limit preserves the relations between the sphere integrals J and the disk integrals Z that drive the string-theory KLT relations at tree level [4]. As a consequence, we obtain the following uplift of (3.11) to one-loop matrix elements,

$$M^{\text{1-loop}}_{n,{\rm eff}} = \int \frac{{\rm d}^D \ell}{\ell^2} \sum_{\gamma,\rho \in S_{n-1}} a_{\rm eff}(+,\gamma(1,2,\ldots,n{-}1),n,-) \\ \times S_{\alpha'}(\gamma|\rho)_\ell \, \bar a_{\rm eff}(+,\rho(1,2,\ldots,n{-}1),-,n)\,, \qquad (3.13)$$

with a string-theory KLT kernel subject to

$$S_{\alpha'}(A,j|B,j,C)_\ell = \frac{\sin(2\pi\alpha' k_j \cdot (\ell + k_B))}{2\pi\alpha'} S_{\alpha'}(A|B,C)_\ell\,, \qquad S_{\alpha'}(\emptyset|\emptyset)_\ell = 1\,. \qquad (3.14)$$

At the same time, the string-theory KLT formula is equivalent to a field-theory double-copy formula involving single-valued open-string amplitudes [32, 60, 129–132]. In the forward limit, this reasoning leads to the following rewriting of (3.13)

$$\begin{aligned} M_{n,\text{eff}}^{\text{1-loop}} = \int \frac{\mathrm{d}^D \ell}{\ell^2} \sum_{\gamma,\rho \in S_{n-1}} & a_{\text{SYM}}(+, \gamma(1,2,\ldots,n{-}1), n, -) \\ & \times S_0(\gamma|\rho)_\ell \,\text{sv}\, \bar{a}_{\text{eff}}(+, \rho(1,2,\ldots,n{-}1), -, n)\,. \end{aligned} \tag{3.15}$$

As in the relation (2.28) between sphere and disk integrals, the single-valued map acts on the (motivic) MZVs in the α'-expansion according to (2.27), also see [127, 128] for $\text{sv}(\zeta_{n_1,\ldots,n_r})$ at general depth r. One may wonder about an echo of the KLT relations (3.13) among one-loop matrix elements for the full-fledged closed-string one-loop amplitudes. As we will see in section 6, the appearance of single-valued disk integrals in the expressions (3.2) and (3.15) for $M_{n,\text{eff}}^{\text{1-loop}}$ resonates with recent progress in relating the degeneration limits of genus-one integrals of open and closed strings [105, 111, 112, 153, 154].

4 Examples of numerator extractions

This section is dedicated to four- and five-point examples of the one-loop matrix elements (3.1) and (3.2) of massless SYM and supergravity states and the associated effective tree-level interactions. The six- and seven-point tree-level disk integrals used in (3.1) and (3.2) are expanded by the `BGap` package [135], and the kinematic one-loop numerators for SYM are well-known from [21, 23, 122, 155]. As detailed in section 4.1.2, the conversion from linearized to quadratic propagators is facilitated by `BGap`: at both four and five points, the kinematic numerators at the orders of $\alpha'^{\leq 7}$ turned out to readily satisfy the criteria (2.20) under the forward limit.

4.1 Four points, open strings

At four points, all the polarization dependence of the kinematic half integrand (2.8) is captured by the permutation invariant kinematic factor [155]

$$t_8(f_1, f_2, f_3, f_4) = \text{tr}(f_1 f_2 f_3 f_4) - \frac{1}{4}\text{tr}(f_1 f_2)\text{tr}(f_3 f_4) + \text{cyc}(2,3,4) \tag{4.1}$$

and its supersymmetrization [156]. The traces are over the Lorentz indices of the linearized field strength $f_j^{\mu\nu} = k_j^\mu \epsilon_j^\nu - k_j^\nu \epsilon_j^\mu$ involving transverse polarization vectors ϵ_j^μ of the external states $j = 1,2,3,4$ (with $\mu,\nu = 0,1,\ldots,D{-}1$). The accompanying Parke-Taylor factors also form a permutation invariant,

$$\mathcal{I}_4 = t_8(f_1, f_2, f_3, f_4) \sum_{\rho \in S_4} \text{PT}(+, \rho(1,2,3,4), -)\,, \tag{4.2}$$

which is the ambitwistor-string explanation for the absence of triangle and bubble diagrams in the loop integrand (2.7) of SYM. One can read off the master numerators

$$N_{+|\rho(1234)|-} = t_8(f_1, f_2, f_3, f_4) \tag{4.3}$$

from (4.2) which do not depend on the permutation ρ of legs 1,2,3,4 in the half-ladder diagram of figure 2. When assembling the four-point instances of the partial integrands (3.7) for one-loop matrix elements, it is convenient to peel off the polarization dependent t_8-factor,

$$a_{\rm eff}(\gamma(+,1,2,3,4,-)) = t_8(f_1,f_2,f_3,f_4)\hat{a}_{\rm eff}(\gamma(+,1,2,3,4,-))\,, \\ \hat{a}_{\rm eff}(\gamma(+,1,2,3,4,-)) = \lim_{k_\pm\to\pm\ell}\sum_{\rho\in S_4} Z(\gamma(+,1,2,3,4,-)|+,\rho(1,2,3,4),-)\,. \tag{4.4}$$

With the leading orders in the α'-expansion (2.32) of the six-point disk integrals on the right-hand sides, the forward limit yields an α'-dressed partial integrand of the form

$$\begin{aligned}\hat{a}_{\rm eff}(+,1,2,3,4,-) = {} & \frac{1}{s_{1,\ell}s_{12,\ell}s_{123,\ell}} - \alpha'^2\zeta_2\left(\frac{s_{34}}{s_{1,\ell}s_{12,\ell}}+\frac{s_{12}}{s_{12,\ell}s_{123,\ell}}+\frac{s_{23}}{s_{1,\ell}s_{123,\ell}}\right) \\ & +\alpha'^3\zeta_3\left(\frac{s_{34}(s_{34}+s_{123,\ell})}{s_{1,\ell}s_{12,\ell}}+\frac{(s_{1,\ell}+s_{12})s_{12}}{s_{12,\ell}s_{123,\ell}}+\frac{(s_{12,\ell}+s_{23})s_{23}}{s_{1,\ell}s_{123,\ell}}-\frac{s_{24}}{s_{1,\ell}}-\frac{s_{13}}{s_{123,\ell}}\right)+\mathcal{O}(\alpha'^4)\,.\end{aligned} \tag{4.5}$$

By the conventions (2.14) for ℓ-dependent Mandelstam invariants, we have for instance $s_{1,\ell} = 2k_1\cdot\ell$ and $s_{12,\ell} = s_{12} + 2k_{12}\cdot\ell$. In the one-loop matrix elements assembled via (3.8),

$$A^{\text{1-loop}}_{\rm eff}(1,2,3,4) = t_8(f_1,f_2,f_3,f_4)\hat{A}^{\text{1-loop}}_{\rm eff}(1,2,3,4)\,, \\ \hat{A}^{\text{1-loop}}_{\rm eff}(1,2,3,4) = \int\frac{{\rm d}^D\ell}{\ell^2}\lim_{k_\pm\to\pm\ell}\sum_{\gamma\in{\rm cyc}(1,2,3,4)}\sum_{\rho\in S_4}Z(+,\gamma(1,2,3,4),-|+,\rho(1,2,3,4),-)\,, \tag{4.6}$$

the cyclic permutations of $a_{\rm eff}(+,1,2,3,4,-)$ in the loop integrand conspire to the quadratic propagators upon shifts of ℓ as in (2.17). The resulting box-, triangle- and bubble diagrams tie in with the Feynman vertices of the operators F^2, $\alpha'^2\zeta_2F^4$, $\alpha'^3\zeta_3(D^2F^4+F^5),\ldots$, in the low-energy effective action $S^{\rm open}_{\rm eff}$ in (1.1) and their supersymmetric completions. In particular, the field-theory limit $(s_{1,\ell}s_{12,\ell}s_{123,\ell})^{-1}$ of (4.5) recombines to the box integral (2.16) with the well-known t_8-numerator in maximally supersymmetric gauge theories [155].

4.1.1 α'^2 and α'^3 corrections

The α'-corrections to the Z-integrals in (4.5) augment the box integrals of SYM by triangles, bubbles and tadpoles. At the subleading orders, their explicit form resulting from (4.6) is

$$\begin{aligned}\hat{A}^{\text{1-loop}}_{\rm eff}(1,2,3,4) = {} & \int{\rm d}^D\ell\,\frac{1}{\ell^2(\ell+k_1)^2(\ell+k_{12})^2(\ell+k_{123})^2} \\ & -\alpha'^2\zeta_2\int{\rm d}^D\ell\left[\frac{s_{12}}{\ell^2(\ell+k_1)^2(\ell+k_{12})^2}+{\rm cyc}_\ell(1,2,3,4)\right] \\ & +\alpha'^3\zeta_3\int{\rm d}^D\ell\left[\frac{s_{34}(s_{3,\ell}-s_{4,\ell}+s_{34})}{2\ell^2(\ell+k_1)^2(\ell+k_{12})^2}-\frac{3s_{13}}{2\ell^2(\ell+k_1)^2}+{\rm cyc}_\ell(1,2,3,4)\right] \\ & +\mathcal{O}(\alpha'^4)\,,\end{aligned} \tag{4.7}$$

where the numerators are properly symmetrized with respect to the graph automorphism groups of the underlying propagator topologies. The "$+\,{\rm cyc}_\ell(1,2,\ldots,n)$" prescription involves *a sum over the cyclic permutations of the external legs together with a proper shift of*

(a) $N^{(\zeta_2)}_{\text{triangle}}(1,2,34)$ **(b)** $N^{(\zeta_3)}_{\text{bubble}}(1,234)$

Figure 3. Diagrammatic representation of the terms $\sim \zeta_2$ in the left panel and the contribution $\sim \frac{s_{13}}{\ell^2(\ell+k_1)^2}$ to the third line in the α'-expansion (4.7) in the right panel.

loop momentum such that it always points from leg n to leg 1*.* For example, the scalar triangle at the $\alpha'^2\zeta_2$-order and the vector integral along with $\alpha'^3\zeta_3$ are symmetrized as follows:

$$\frac{s_{12}}{\ell^2(\ell{+}k_1)^2(\ell{+}k_{12})^2} + \text{cyc}_\ell(1,2,3,4) = \frac{s_{12}}{\ell^2(\ell{+}k_1)^2(\ell{+}k_{12})^2} + \frac{s_{23}}{(\ell{+}k_1)^2(\ell{+}k_{12})^2(\ell{+}k_{123})^2} + \frac{s_{34}}{\ell^2(\ell{+}k_{12})^2(\ell{+}k_{123})^2} + \frac{s_{14}}{\ell^2(\ell{+}k_1)^2(\ell{+}k_{123})^2}\,, \tag{4.8}$$

$$\frac{s_{34}s_{3,\ell}}{\ell^2(\ell{+}k_1)^2(\ell{+}k_{12})^2} + \text{cyc}_\ell(1,2,3,4) = \frac{s_{34}s_{3,\ell}}{\ell^2(\ell{+}k_1)^2(\ell{+}k_{12})^2} + \frac{s_{14}(s_{4,\ell}+s_{14})}{(\ell{+}k_1)^2(\ell{+}k_{12})^2(\ell{+}k_{123})^2} + \frac{s_{12}(s_{1,\ell}+s_{12})}{\ell^2(\ell{+}k_{12})^2(\ell{+}k_{123})^2} + \frac{s_{23}(s_{2,\ell}+s_{12}+s_{23})}{\ell^2(\ell{+}k_1)^2(\ell{+}k_{123})^2}\,. \tag{4.9}$$

Our convention of fixing the position of the loop momentum corresponds to fixing one of the punctures in one-loop string amplitudes by translation invariance on a genus-one surface.

The term proportional to ζ_2 in the second line of (4.7) corresponds to the triangle diagram in figure 3a with a single-insertion of an F^4 operator. The propagators $\frac{1}{\ell^2(\ell+k_1)^2}$ in the third line of (4.7) signal a bubble in external legs as visualized in figure 3b, where the quintic vertex can be attributed to the non-linear extensions of D^2F^4 beyond $\partial^2(\partial A)^4$. Although such external bubbles integrate to zero in dimensional regularization, we still keep them in view of their contributions to UV divergences in four and fewer dimensions, see section 6. Further triangle diagrams as in figure 3 also arise with F^4 replaced by any higher-derivative operator $D^{2k}F^4$ in the low-energy effective action. While the four derivatives of the F^4 vertex in the second line of (4.7) exclusively contribute external momenta such as the triangle numerator s_{12}, the D^2F^4 vertex in the first term of the third line yields up to one loop momentum in the numerator $s_{34}(s_{3,\ell} - s_{4,\ell} + s_{34})$.

4.1.2 Linearized versus quadratic propagators and bubbles on external legs

Given a linearized-propagator representation of generic one-loop integrands, the numerators usually do not have the properties (2.20) under cyclic shifts such that converting back to quadratic propagators may become challenging. In our setup where the numerators arise from α'-expansions of the Z-integrals in (3.1), the conversion can be greatly facilitated: if we take forward limits on the output of the `BGap` package [135], the numerators of one-loop four- and five-point integrands are observed to automatically satisfy the criteria (2.20).

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First of all, BGap uses different bases for the Mandelstam variables appearing in the numerators and denominators. For the disk ordering of $Z(+,i,i{+}1,\ldots,n,1,2,\ldots,i{-}1,-|\ldots)$, the denominators are expressed via region variable $s_{\ell,i,i+1,\ldots}$ respecting the ordering in the first slot, while the numerators are expressed in terms of two-particle s_{ab} in which $k_-{=}{-}\ell$ and $s_{\ell,i-1}$ do not appear. One can check that both bases consist of $\frac{1}{2}(n{+}2)(n{-}1)$ elements, which are indeed minimal for $(n{+}2)$-point kinematics prior to the forward limit. The cyclic properties (2.20) of generic numerators usually depend on the choice whether identities like

$$\frac{s_{2,\ell}}{s_{1,\ell}s_{12,\ell}} = \frac{1}{s_{1,\ell}} - \frac{1}{s_{12,\ell}} - \frac{s_{12}}{s_{1,\ell}s_{12,\ell}} \tag{4.10}$$

are used to mix p-gon topologies with $(p{-}1)$-gons, i.e. triangles and bubbles in this case. At least at four and five points, the output of BGap singles out a scheme of applying (4.10): we have checked up to α'^7, the highest order implemented in BGap, that the criteria (2.20) are satisfied by the numerators along with each combination of MZVs. Therefore, we can obtain quadratic-propagator representations without much effort up to and including α'^7.

In contrary, the expressions for the six-point Z-integrals at α'^8 and α'^9 order given in [139] are not of this desired form. Consequently, the above shortcuts to get quadratic propagators from the forward limit do not apply here in an obvious way. We instead relied on the interpretation of forward limits as one-particle unitarity cuts to find a convenient management of the kinematic variables in the α'-expansions from [139].

It is well known that scaleless integrals like scalar bubbles on an external leg are not visible to the linearized-propagator representation. In order to still specify their numerators in the α'-expansion of $\widehat{A}^{\text{1-loop}}_{\text{eff}}$ in a systematic way, we first rewrite (2.17) as

$$\int \frac{\mathrm{d}^D\ell \; \mathfrak{N}(\ell)}{\ell^2(\ell{+}k_1)^2} = \int \frac{\mathrm{d}^D\ell}{\ell^2}\left(\frac{\mathfrak{N}(\ell)}{s_{1,\ell}} - \frac{\mathfrak{N}(\ell{-}k_1)}{s_{1,\ell}}\right) = \int \frac{\mathrm{d}^D\ell}{\ell^2}\left(\frac{\mathfrak{N}(\ell)}{s_{1,\ell}} + \frac{\mathfrak{N}(\ell{-}k_1)}{s_{234,\ell}}\right). \tag{4.11}$$

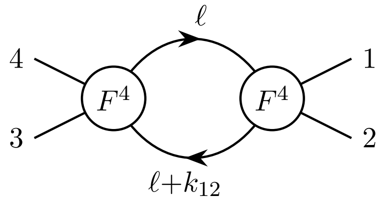
The forward limit (4.6) taken on six-point disk integrals will naturally lead to the form in the rightmost step before we impose the consequence $s_{1,\ell}=-s_{234,\ell}$ of four-point kinematics. This is due to the above choice of Mandelstam basis in the six-point disk integrals: one can systematically distinguish $s_{1,\ell}$ from $-s_{234,\ell}$ since the denominators in the α'-expansion of $Z(+,i,i{+}1,\ldots|\ldots)$ from [134] are expressed via region variables $s_{\ell,i,i+1\ldots}$. Then, if the numerators of $s_{1,\ell}$ and $s_{234,\ell}$ are related by the shift $\ell\to\ell{-}k_1$, we can recover the bubbles on external legs by reversing (4.11) as done at the ζ_3-order of (4.7). As a consistency check, our prescription indeed produces the expected UV divergence of external bubbles in $D\leq 4$ dimensions, see section 6.4.

4.1.3 Higher orders

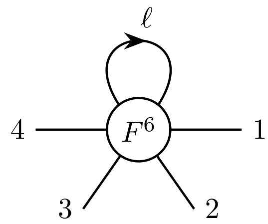
In order to compactly denote higher-order α'-corrections to (4.7), we write[10]

$$\begin{aligned}\widehat{A}^{\text{1-loop}}_{\text{eff}}(1,2,3,4) = \int \mathrm{d}^D\ell \Bigg\{ &\frac{1}{\ell^2(\ell{+}k_1)^2(\ell{+}k_{12})^2(\ell{+}k_{123})^2} + \Bigg[\frac{N_{\text{triangle}}(1,2,34)}{\ell^2(\ell{+}k_1)^2(\ell{+}k_{12})^2} \\ &+ \frac{1}{2}\frac{N_{\text{bubble}}(12,34)}{\ell^2(\ell{+}k_{12})^2} + \frac{N_{\text{bubble}}(1,234)}{\ell^2(\ell{+}k_1)^2} + \frac{1}{4}\frac{N_{\text{tadpole}}(1234)}{\ell^2} + \mathrm{cyc}_\ell(1,2,3,4)\Bigg]\Bigg\} + \mathcal{O}(\alpha'^9).\end{aligned} \tag{4.12}$$

[10]Note that the 1/2 and 1/4 multiplying the two-mass bubble and tadpole diagrams are the symmetry factors that compensate the overcounting in the sum over cyclic permutations.



(a) $N^{(\zeta_2^2)}_{\rm bubble}(12,34)$



(b) $N^{(\zeta_2^2)}_{\rm tadpole}(1234)$

Figure 4. Diagrammatic representation of the two-mass bubble in the left panel and the tadpole diagrams in the right panel for the numerators in (4.15).

In the supplementary material attached to this paper, we provide complete results for the numerators up to α'^8 order, which are organized for each diagram $\Gamma\in\{\text{triangle, bubble, tadpole}\}$ via

$$\begin{aligned} N_\Gamma = {}& \alpha'^2\zeta_2 N_\Gamma^{(\zeta_2)} + \alpha'^3\zeta_3 N_\Gamma^{(\zeta_3)} + \alpha'^4\zeta_2^2 N_\Gamma^{(\zeta_2^2)} + \alpha'^5\Big[\zeta_5 N_\Gamma^{(\zeta_5)} + \zeta_2\zeta_3 N_\Gamma^{(\zeta_2\zeta_3)}\Big] \\ &+ \alpha'^6\Big[\zeta_3^2 N_\Gamma^{(\zeta_3^2)} + \zeta_2^3 N_\Gamma^{(\zeta_2^3)}\Big] + \alpha'^7\Big[\zeta_7 N_\Gamma^{(\zeta_7)} + \zeta_2\zeta_5 N_\Gamma^{(\zeta_2\zeta_5)} + \zeta_2^2\zeta_3 N_\Gamma^{(\zeta_2^2\zeta_3)}\Big] \\ &+ \alpha'^8\Big[\zeta_{3,5} N_\Gamma^{(\zeta_{3,5})} + \zeta_3\zeta_5 N_\Gamma^{(\zeta_3\zeta_5)} + \zeta_2\zeta_3^2 N_\Gamma^{(\zeta_2\zeta_3^2)}\Big]\,, \end{aligned} \tag{4.13}$$

with a separate numerator for each element in the tentative $\mathbb{Q}$-bases of MZVs [148] at given weight. Note that up to α'^8 order only the topologies given in (4.12) show up. The expressions in (4.7) are equivalent to

$$\begin{aligned} &N^{(\zeta_2)}_{\rm triangle}(1,2,34) = -s_{12}\,, \qquad N^{(\zeta_2)}_{\rm bubble}(12,34) = N^{(\zeta_2)}_{\rm bubble}(123,4) = N^{(\zeta_2)}_{\rm tadpole}(1234) = 0\,, \\ &N^{(\zeta_3)}_{\rm triangle}(1,2,34) = \frac{1}{2}s_{34}(s_{3,\ell}-s_{4,\ell}+s_{34})\,, \qquad N^{(\zeta_3)}_{\rm bubble}(12,34) = 0\,, \\ &N^{(\zeta_3)}_{\rm bubble}(1,234) = -\frac{3}{2}s_{13}\,, \qquad N^{(\zeta_3)}_{\rm tadpole}(1234) = 0\,, \end{aligned} \tag{4.14}$$

and the first higher-order correction beyond (4.7) which follows from the α'-expansion of the six-point disk integrals in (4.6) is given by

$$\begin{aligned} N^{(\zeta_2^2)}_{\rm triangle}(1,2,34) &= -\frac{1}{20}s_{34}(4s_{3,\ell}^2 + 4s_{4,\ell}^2 - 7s_{3,\ell}s_{34} - s_{4,\ell}s_{34} + 11s_{34}^2)\,, \\ N^{(\zeta_2^2)}_{\rm bubble}(12,34) &= s_{12}^2\,, \qquad N^{(\zeta_2^2)}_{\rm tadpole}(1234) = \frac{24}{5}s_{13}\,, \\ N^{(\zeta_2^2)}_{\rm bubble}(1,234) &= \frac{20s_{2,\ell}s_{24}-4s_{3,\ell}(4s_{34}+s_{23})-13s_{34}^2-4s_{1,\ell}(2s_{34}+3s_{23})-12s_{23}s_{34}-9s_{23}^2}{20}\,. \end{aligned} \tag{4.15}$$

Apart from the triangle diagram (with a $\zeta_2^2 D^4F^4$ vertex in the place of $\zeta_2 F^4$ in figure 3a), the order of $\zeta_2^2 = \frac{5}{2}\zeta_4$ is the first instance of a two-mass bubble diagram with a double insertion of $\zeta_2 F^4$ depicted in figure 4a. Moreover, the tadpole numerator receives contributions from the $\zeta_2^2 F^6$ vertex in figure 4b.

Starting from the order of α'^5, the open-string effective action exhibits two combinations of zeta values ζ_5 and $\zeta_2\zeta_3$ associated with numerators (see table 1),

$$\begin{aligned}
N^{(\zeta_5)}_{\text{triangle}}(1,2,34) &= \frac{1}{2}s_{34}\Big[s_{3,\ell}^3 - s_{4,\ell}^3 - (s_{3,\ell}^2 - 2s_{4,\ell}^2)s_{34} + (s_{3,\ell} - 2s_{4,\ell})s_{34}^2 + s_{34}^3\Big]\,,\\
N^{(\zeta_5)}_{\text{bubble}}(12,34) &= 0\,, \qquad N^{(\zeta_5)}_{\text{tadpole}}(1234) = 2(s_{12}^2 + 4s_{12}s_{23} + s_{23}^2)\,,\\
N^{(\zeta_5)}_{\text{bubble}}(1,234) &= \frac{1}{4}\Big[s_{34}(8s_{2,\ell}^2 + 10s_{3,\ell}^2 + 6s_{4,\ell}^2 + 12s_{2,\ell}s_{3,\ell} + 6s_{3,\ell}s_{4,\ell})\\
&\quad + s_{23}(6s_{2,\ell}^2 + 8s_{4,\ell}^2 + 10s_{3,\ell}^2 + 12s_{3,\ell}s_{4,\ell} + 6s_{2,\ell}s_{3,\ell})\\
&\quad + s_{34}^2(3s_{2,\ell} - 9s_{3,\ell} - s_{4,\ell}) - s_{23}^2(5s_{2,\ell} - s_{3,\ell} - s_{4,\ell})\\
&\quad - s_{23}s_{34}(4s_{2,\ell} + 14s_{3,\ell} + 2s_{4,\ell}) - 2s_{24}(4s_{24}^2 + 3s_{34}^2)\Big]\,,
\end{aligned} \tag{4.16}$$

as well as

$$\begin{aligned}
N^{(\zeta_2\zeta_3)}_{\text{triangle}}(1,2,34) &= -\frac{1}{2}s_{34}^2\Big[s_{3,\ell}^2 + s_{4,\ell}^2 - (s_{3,\ell} + s_{4,\ell})s_{34}\Big]\,,\\
N^{(\zeta_2\zeta_3)}_{\text{bubble}}(12,34) &= -s_{12}^2(s_{1,\ell} + s_{3,\ell} + s_{12})\,, \qquad N^{(\zeta_2\zeta_3)}_{\text{tadpole}}(1234) = 0\,,\\
N^{(\zeta_2\zeta_3)}_{\text{bubble}}(1,234) &= \frac{s_{34}^2(s_{2,\ell}-2s_{4,\ell}) - s_{23}^2(s_{4,\ell}-2s_{2,\ell}) + 2s_{23}s_{34}(s_{2,\ell}-s_{4,\ell}) - s_{23}(s_{23}^2 - 6s_{34}s_{24})}{2}\,,
\end{aligned} \tag{4.17}$$

and similar expressions for the numerators at order α'^6 can be found in appendix A.1. Note that the two-mass bubble numerator $N^{(\zeta_2\zeta_3)}_{\text{bubble}}(12,34)$ in (4.17) captures one insertion of both $\zeta_2 F^4$ and $\zeta_3 D^2 F^4$ similar to figure 4a.

The simplest interaction $\alpha'^8\zeta_{3,5}D^{10}F^5$ with an irreducible MZV beyond depth one gives rise to the external-bubble and tadpole numerators in appendix A.1. By unitarity, the absence of triangles and two-mass bubbles $\sim \zeta_{3,5}$ is a consequence of only having ζ_w of depth one in the four-point tree-level amplitude. The contributions at α'^7 and the remaining terms at α'^8 are sufficiently complicated that we do not reproduce them in text, but instead include them only in the supplementary material.

4.2 Four points, closed strings

The double copy of the four-point half integrand (4.2) leads to the following simple form of the gravitational one-loop matrix element (3.2),

$$\begin{aligned}
M^{\text{1-loop}}_{4,\text{eff}} &= |t_8(f_1,f_2,f_3,f_4)|^2\,\widehat{M}^{\text{1-loop}}_{4,\text{eff}}\,,\\
\widehat{M}^{\text{1-loop}}_{4,\text{eff}} &= \int \frac{\mathrm{d}^D\ell}{\ell^2}\lim_{k_\pm\to\pm\ell}\sum_{\gamma,\rho\in S_4} J(+,\gamma(1,2,3,4),-|+,\rho(1,2,3,4),-)\,,
\end{aligned} \tag{4.18}$$

where $|t_8(f_1,f_2,f_3,f_4)|^2$ instructs to replace $\epsilon_j^\mu \to \bar\epsilon_j^\mu$ in one of the chiral halves (with analogous replacements for the fermionic completion). By the single-valued map in (2.28), the leading α'-orders of the six-point sphere integrals in (4.18) can be obtained by setting $\zeta_2 \to 0$ and $\zeta_3 \to 2\zeta_3$ in (4.5). More specifically, by comparing the permutation sum over γ

in (4.18) with the cyclic sums over γ in (4.4) and (4.6), we are led to

$$\widehat{M}^{\text{1-loop}}_{4,\text{eff}} = \sum_{\rho\in S_4} \text{sv}\, \widehat{A}^{\text{1-loop}}_{\text{eff}}(1,\rho(2,3,4))\,, \qquad (4.19)$$
$$M^{\text{1-loop}}_{4,\text{eff}} = t_8(\bar f_1,\bar f_2,\bar f_3,\bar f_4) \sum_{\rho\in S_4} \text{sv}\, A^{\text{1-loop}}_{\text{eff}}(1,\rho(2,3,4))\,.$$

These relations holds to all orders in α', and one can readily assemble the orders of $\alpha'^{\leq 6}$ from (4.7), (4.16) and (A.3). However, the permutation sum in (4.19) leads to additional cancellations and improved power counting as one can for instance see from the explicit form of the first orders of the α'-expansion

$$\begin{aligned}\widehat{M}^{\text{1-loop}}_{4,\text{eff}} = \int \mathrm{d}^D\ell \Bigg[&\frac{1}{4}\frac{1}{\ell^2(\ell+k_1)^2(\ell+k_{12})^2(\ell+k_{123})^2} + \frac{\alpha'^3\zeta_3 s_{12}^2}{\ell^2(\ell+k_1)^2(\ell+k_{12})^2} \\ &+ \frac{\alpha'^5\zeta_5 s_{12}^2[s_{3,\ell}^2+s_{4,\ell}^2-(s_{3,\ell}+s_{4,\ell})s_{12}+2s_{12}^2]}{2\ell^2(\ell+k_1)^2(\ell+k_{12})^2} \\ &- \frac{5\alpha'^5\zeta_5 s_{12}s_{13}s_{14}}{2\ell^2(\ell+k_1)^2} + \mathcal{O}(\alpha'^6)\Bigg] + \text{perm}(1,2,3,4)\,.\end{aligned} \qquad (4.20)$$

The coefficients of ζ_3 and ζ_5 are the one-loop matrix elements with a single insertion of the operators R^4 and D^4R^4 in the low-energy effective action (1.1), respectively. Similar to the power counting of the triangle numerators from F^4 in (4.7), the eight derivatives of R^4 conspire to the external momenta in the numerator $\alpha'^3\zeta_3 s_{12}^2$ of the first line of (4.20). As a closed-string analogue of the D^2F^4 numerator in (4.7) which is linear in loop momentum, the triangle numerator from D^4R^4 in the second line of (4.20) is quadratic in ℓ akin to the double-copy property of supergravity numerators. The diagrams in (4.20) are identical to those in figure 3 with R^4 and D^4R^4 in the place of F^4 and D^2F^4, respectively.

By analogy with (4.12) and (4.13) we organize higher orders in α' according to[11]

$$\begin{aligned}\widehat{M}^{\text{1-loop}}_{4,\text{eff}} = 2\int \mathrm{d}^D\ell \Bigg[&\frac{1}{8}\frac{1}{\ell^2(\ell+k_1)^2(\ell+k_{12})^2(\ell+k_{123})^2} + \frac{1}{4}\frac{\mathcal{N}_{\text{triangle}}(1,2,34)}{\ell^2(\ell+k_1)^2(\ell+k_{12})^2} \\ &+\frac{1}{16}\frac{\mathcal{N}_{\text{bubble}}(12,34)}{\ell^2(\ell+k_{12})^2} + \frac{1}{12}\frac{\mathcal{N}_{\text{bubble}}(1,234)}{\ell^2(\ell+k_1)^2} + \mathcal{O}(\alpha'^9)\Bigg] + \text{perm}(1,2,3,4)\,,\end{aligned} \qquad (4.21)$$

and we provide complete results up to α'^8 order,

$$\mathcal{N}_\Gamma = \alpha'^3\zeta_3\mathcal{N}_\Gamma^{(\zeta_3)} + \alpha'^5\zeta_5\mathcal{N}_\Gamma^{(\zeta_5)} + \alpha'^6\zeta_3^2\mathcal{N}_\Gamma^{(\zeta_3^2)} + \alpha'^7\zeta_7\mathcal{N}_\Gamma^{(\zeta_7)} + \alpha'^8\zeta_3\zeta_5\mathcal{N}_\Gamma^{(\zeta_3\zeta_5)}\,. \qquad (4.22)$$

The subleading orders of (4.20) translate into

$$\mathcal{N}^{(\zeta_3)}_{\text{triangle}}(1,2,34) = 2s_{12}^2\,, \qquad \mathcal{N}^{(\zeta_3)}_{\text{bubble}}(12,34) = \mathcal{N}^{(\zeta_3)}_{\text{bubble}}(1,234) = 0\,, \qquad (4.23)$$

[11] The overall factor of 2 reconciles the symmetry factors of the diagrams in the square brackets (compensating for an overcounting by the permutation sum) with the normalization of the one-loop matrix element (3.2).

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as well as

$$\mathcal{N}^{(\zeta_5)}_{\text{triangle}}(1,2,34) = s_{12}^2\left[s_{3,\ell}^2 + s_{4,\ell}^2 - (s_{3,\ell}+s_{4,\ell})s_{12} + 2s_{12}^2\right] ,$$
$$\mathcal{N}^{(\zeta_5)}_{\text{bubble}}(1,234) = -15 s_{12}s_{13}s_{14} , \qquad \mathcal{N}^{(\zeta_5)}_{\text{bubble}}(12,34) = 0 . \tag{4.24}$$

At higher orders, similar calculations yield

$$\begin{aligned}
\mathcal{N}^{(\zeta_3^2)}_{\text{triangle}}(1,2,34) &= s_{12}^3\left[s_{3,\ell}^2 + s_{4,\ell}^2 - (s_{3,\ell}+s_{4,\ell})s_{12}\right] ,\\
\mathcal{N}^{(\zeta_3^2)}_{\text{bubble}}(12,34) &= -4s_{12}(s_{1,\ell}+s_{2,\ell})(s_{1,\ell}+s_{2,\ell}+2s_{12}) ,\\
\mathcal{N}^{(\zeta_3^2)}_{\text{bubble}}(1,234) &= s_{12}^4 + s_{13}^4 + s_{14}^4 + 3s_{1,\ell}s_{12}s_{13}s_{14} + 2s_{\ell,1}\left[s_{2,\ell}s_{12}^2 + s_{3,\ell}s_{13}^2 + s_{4,\ell}s_{14}^2\right] ,
\end{aligned} \tag{4.25}$$

where the two-mass bubble $\mathcal{N}^{(\zeta_3^2)}_{\text{bubble}}(12,34)$ in the second line is due to double insertion of $\zeta_3 R^4$ and can be viewed as a closed-string analogue of the double insertion of F^4 in figure 4a. Two-mass bubbles of this type do not occur along with single zeta values such as

$$\begin{aligned}
\mathcal{N}^{(\zeta_7)}_{\text{triangle}}(1,2,34) &= s_{12}^2\left[s_{3,\ell}^4+s_{4,\ell}^4-2(s_{3,\ell}^3+s_{4,\ell}^3)s_{12}+3(s_{3,\ell}^2+s_{4,\ell}^2)s_{12}^2-2(s_{3,\ell}+s_{4,\ell})s_{12}^3+2s_{12}^4\right],\\
\mathcal{N}^{(\zeta_7)}_{\text{bubble}}(12,34) &= 0,\\
\mathcal{N}^{(\zeta_7)}_{\text{bubble}}(1,234) &= -\frac{13}{8}s_{1,\ell}^2 s_{12}s_{13}s_{14}+4s_{1,\ell}(s_{12}^4+s_{13}^4+s_{14}^4)-14s_{12}s_{13}s_{14}(s_{12}^2+s_{13}^2+s_{14}^2)\\
&\quad -\left[s_{2,\ell}^2(6s_{12}s_{13}s_{14}+5s_{12}^3)-8s_{2,\ell}s_{3,\ell}s_{14}^3+3s_{2,\ell}s_{12}^2(2s_{12}^2-3s_{13}^2-3s_{14}^2)+\text{cyc}(2,3,4)\right],
\end{aligned} \tag{4.26}$$

and it is tedious but straightforward to obtain the next orders in α' along the same lines.

4.3 The five-point kinematic half integrand

The five-point worldsheet correlator in one-loop superstring amplitudes has been studied from a variety of perspectives [115, 117, 157–159], and we shall now review the representation of [21] for its analogue in the ambitwistor setup [11, 13]. Its kinematic dependence can be compactly encoded in the two-particle polarization ϵ_{12}^μ and field strength $f_{12}^{\mu\nu}$ [160, 161] given by

$$\begin{aligned}
\epsilon_{12}^\mu &= (k_2\cdot\epsilon_1)\epsilon_2^\mu - (k_1\cdot\epsilon_2)\epsilon_1^\mu + \frac{1}{2}(\epsilon_1\cdot\epsilon_2)(k_1^\mu - k_2^\mu)\\
&= \frac{1}{2}\left[(k_2\cdot\epsilon_1)\epsilon_2^\mu + (\epsilon_1)_\nu f_2^{\nu\mu} - (1\leftrightarrow 2)\right] ,\\
f_{12}^{\mu\nu} &= (k_2\cdot\epsilon_1)f_2^{\mu\nu} - (k_1\cdot\epsilon_2)f_1^{\mu\nu} + f_1^{\mu}{}_\lambda f_2^{\lambda\nu} - f_2^{\mu}{}_\lambda f_1^{\lambda\nu}\\
&= k_{12}^\mu\epsilon_{12}^\nu - k_{12}^\nu\epsilon_{12}^\mu - (k_1\cdot k_2)(\epsilon_1^\mu\epsilon_2^\nu - \epsilon_1^\nu\epsilon_2^\mu) ,
\end{aligned} \tag{4.27}$$

and subject to

$$\epsilon_{12}^\mu = -\epsilon_{21}^\mu , \qquad f_{12}^{\mu\nu} = -f_{21}^{\mu\nu} = -f_{12}^{\nu\mu} . \tag{4.28}$$

As a natural generalization of the four-point kinematic factor in (4.2), the t_8-tensor (4.1) in the five-point correlator either contracts the two-particle field strength or is dressed by additional polarization vectors

$$t_8(12,3,4,5) = t_8(f_{12},f_3,f_4,f_5) = -t_8(21,3,4,5) \tag{4.29}$$
$$t_\pm^\mu(1,2,3,4,5) = \left[\epsilon_1^\mu t_8(f_2,f_3,f_4,f_5) + (1\leftrightarrow 2,3,4,5)\right] \pm \frac{i}{2}\varepsilon_{10}^\mu(\epsilon_1,f_2,f_3,f_4,f_5) ,$$

where $t_8(12,3,4,5)$ and $t^\mu_\pm(1,2,3,4,5)$ are permutation-invariant in $\{3,4,5\}$ and $\{1,2,3,4,5\}$, respectively.[12] The $t_\pm$ notation for the vector accommodates two sign choices for the parity-odd term $\varepsilon^\mu_{10}(\epsilon_1,f_2,f_3,f_4,f_5) = (\varepsilon_{10})^\mu{}_{\nu\lambda_2\rho_2\lambda_3\rho_3\lambda_4\rho_4\lambda_5\rho_5}\epsilon^\nu_1 f_2^{\lambda_2\rho_2} f_3^{\lambda_3\rho_3} f_4^{\lambda_4\rho_4} f_5^{\lambda_5\rho_5}$ in order to later on unify expressions for type-IIA and type-IIB theories.

With the above building blocks, the five-point kinematic half integrand can be compactly written as

$$\mathcal{I}_5(\ell) = \frac{\ell_\mu t^\mu_+(1,2,3,4,5) + \big[G_{12} t_8(12,3,4,5) + (1,2|1,2,3,4,5)\big]}{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5} \tag{4.30}$$

with the following shorthand for the Green function on the nodal sphere

$$G_{ij} = \frac{\sigma_i + \sigma_j}{2(\sigma_i - \sigma_j)} \,. \tag{4.31}$$

The notation $+(1,2|1,2,3,4,5)$ in (4.30) instructs to add the nine permutations of the term $G_{12}t_8(f_{12},f_3,f_4,f_5)$ in the square bracket with 12 replaced by any other pair ij with $1 \leq i < j \leq 5$. By arranging the σ_j dependence of (4.30) and (4.31) into Parke-Taylor factors [21], we arrive at the following pentagon numerators in the expansion (2.8),

$$\begin{aligned}
\mathcal{I}_5(\ell) &= \sum_{\rho\in S_5} N_{+|\rho(12345)|-}(\ell)\mathrm{PT}(+,\rho(1,2,3,4,5),-)\,, \\
N_{+|12345|-}(\ell) &= \ell_\mu t^\mu_+(1,2,3,4,5) - \frac{1}{2}\Big\{t_8(12,3,4,5) + t_8(13,2,4,5) \\
&\quad + t_8(14,2,3,5) + t_8(15,2,3,4) + t_8(23,1,4,5) + t_8(24,1,3,5) \\
&\quad + t_8(25,1,3,4) + t_8(34,1,2,5) + t_8(35,1,2,4) + t_8(45,1,2,3)\Big\}\,.
\end{aligned} \tag{4.32}$$

In contrast to the permutation-invariant four-point numerators $N_{+|\rho(1234)|-}{=}t_8(f_1,f_2,f_3,f_4)$, the five-point numerators depend on the ordering ρ of the pentagon. Its antisymmetric part

$$N_{+|12345|-} - N_{+|21345|-} = -t_8(12,3,4,5) \tag{4.33}$$

no longer depends on ℓ, inherits the symmetry in $3,4,5$ of the t_8-tensor[13] and corresponds to a one-mass box numerator. The structure of (4.32) and its compatibility with the color-kinematics duality was firstly discussed in [122]. The ancestors of the kinematic building blocks (4.29) in pure-spinor superspace described in the reference can be readily imported to supersymmetrize the t_8-tensors in the α'-expansions below. Color-kinematics dual five-point numerators in terms of four-dimensional spinor-helicity variables can be found in [7, 162].

For later reference, we note the linearized gauge variation of the above building blocks which we shall implement via

$$\delta = \sum_{j=1}^5 \omega_j k^\mu_j \frac{\partial}{\partial \epsilon^\mu_j}\,. \tag{4.34}$$

[12] Momentum conservation also implies the symmetry $\varepsilon^\mu_{10}(\epsilon_1,f_2,f_3,f_4,f_5) = \varepsilon^\mu_{10}(\epsilon_2,f_1,f_3,f_4,f_5)$ of the parity-odd contributions.

[13] In other words, we also obtain $-t_8(12,3,4,5)$ from $N_{+|31245|-} - N_{+|32145|-}$, $N_{+|34125|-} - N_{+|34215|-}$ and $N_{+|34512|-} - N_{+|34521|-}$ as well as their permutations in $3,4,5$.

The formal bookkeeping variables ω_j track the transformation $\epsilon_j \to k_j$ in the j^{th} leg,

$$\begin{aligned}
\delta t_8(12,3,4,5) &= \frac{1}{2} s_{12}\left[\omega_1 t_8(f_2,f_3,f_4,f_5) - \omega_2 t_8(f_1,f_3,f_4,f_5)\right] \\
\delta t_+^\mu(1,2,3,4,5) &= k_1^\mu \omega_1 t_8(f_2,f_3,f_4,f_5) + (1 \leftrightarrow 2,3,4,5) \\
\delta N_{+|12345|-} &= \frac{\omega_1}{2}(\ell_1^2 - \ell^2) t_8(f_2,f_3,f_4,f_5) + \frac{\omega_2}{2}(\ell_{12}^2 - \ell_1^2) t_8(f_1,f_3,f_4,f_5) \\
&\quad + \frac{\omega_3}{2}(\ell_{123}^2 - \ell_{12}^2) t_8(f_1,f_2,f_4,f_5) + \frac{\omega_4}{2}(\ell_{1234}^2 - \ell_{123}^2) t_8(f_1,f_2,f_3,f_5) \\
&\quad + \frac{\omega_5}{2}(\ell^2 - \ell_{1234}^2) t_8(f_1,f_2,f_3,f_4)\,,
\end{aligned} \tag{4.35}$$

where we introduced the shorthand notations

$$\ell_1 = \ell + k_1\,, \qquad \ell_{12} = \ell + k_{12}\,, \qquad \ell_{123} = \ell + k_{123}\,, \qquad \ell_{1234} = \ell + k_{1234}\,, \tag{4.36}$$

and we have used five-point momentum conservation to simplify the variation of $N_{+|12345|-}$. The operator (4.34) implementing linearized gauge variations is engineered to mimic the BRST transformations of the supersymmetric numerators in [122].

4.4 Five points, open string

In contrast to the four-point case, already the field-theory limit of the one-loop matrix elements (3.1) at five points is a combination of a pentagon and five massive boxes,

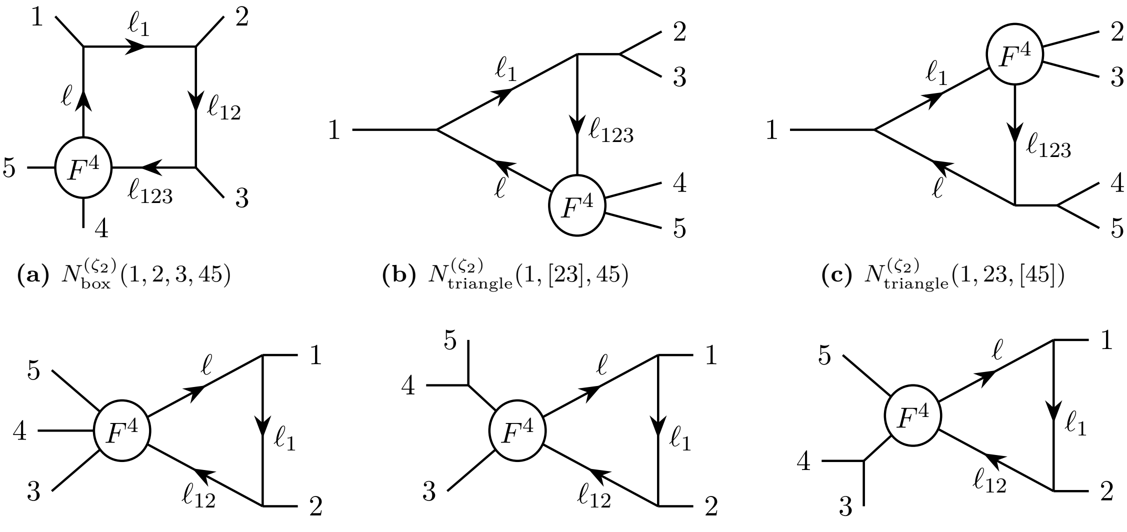
$$\begin{aligned}
A^{\text{1-loop}}_{\text{SYM}}(1,2,3,4,5) = \int \mathrm{d}^D\ell \Bigg[&\frac{N_{+|12345|-}}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2\ell_{1234}^2} - \frac{t_8(12,3,4,5)}{s_{12}\ell^2\ell_{12}^2\ell_{123}^2\ell_{1234}^2} - \frac{t_8(23,1,4,5)}{s_{23}\ell^2\ell_1^2\ell_{123}^2\ell_{1234}^2} \\
&- \frac{t_8(34,1,2,5)}{s_{34}\ell^2\ell_1^2\ell_{12}^2\ell_{1234}^2} - \frac{t_8(45,1,2,3)}{s_{12}\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2} - \frac{t_8(51,2,3,4)}{s_{12}\ell_1^2\ell_{12}^2\ell_{123}^2\ell_{1234}^2}\Bigg]\,.
\end{aligned} \tag{4.37}$$

In (4.37) and its α'-corrections, the recombination of the linearized propagators to quadratic ones relies on shifts of loop momenta in $\ell_\mu t_+^\mu(1,2,3,4,5)$ and the relation

$$(k_1)_\mu t_\pm^\mu(1,2,3,4,5) = -t_8(12,3,4,5) - t_8(13,2,4,5) - t_8(14,2,3,5) - t_8(15,2,3,4)\,. \tag{4.38}$$

Based on the α'-expansion of the seven-point disk integrals [134] in the one-loop matrix element (3.1), we organize the corrections to (4.37) via

$$\begin{aligned}
A^{\text{1-loop}}_{\text{eff}}(1,2,3,4,5) = A^{\text{1-loop}}_{\text{SYM}}(1,2,3,4,5) + \int \mathrm{d}^D\ell \Bigg[&\frac{N_{\text{box}}(1,2,3,45)}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2} + \frac{N_{\text{triangle}}(1,2,345)}{\ell^2\ell_1^2\ell_{12}^2} \\
&+ \frac{N_{\text{triangle}}(1,2,[34]5)}{\ell^2\ell_1^2\ell_{12}^2 s_{34}} + \frac{N_{\text{triangle}}(1,2,3[45])}{\ell^2\ell_1^2\ell_{12}^2 s_{45}} + \frac{N_{\text{triangle}}(1,23,45)}{\ell^2\ell_1^2\ell_{12}^2} \\
&+ \frac{N_{\text{triangle}}(1,[23],45)}{\ell^2\ell_1^2\ell_{12}^2 s_{23}} + \frac{N_{\text{triangle}}(1,23,[45])}{\ell^2\ell_1^2\ell_{12}^2 s_{45}} + \frac{N_{\text{bubble}}(12,345)}{\ell^2\ell_{12}^2} \\
&+ \frac{N_{\text{bubble}}([12],345)}{\ell^2\ell_{12}^2 s_{12}} + \frac{N_{\text{bubble}}(12,[34]5)}{\ell^2\ell_{12}^2 s_{34}} + \frac{N_{\text{bubble}}(12,3[45])}{\ell^2\ell_{12}^2 s_{45}} \\
&+ \frac{N_{\text{bubble}}(1,2345)}{\ell^2\ell_1^2} + \frac{N_{\text{bubble}}(1,[23]45)}{\ell^2\ell_1^2 s_{23}} + \frac{N_{\text{bubble}}(1,2[34]5)}{\ell^2\ell_1^2 s_{34}} \\
&+ \frac{N_{\text{bubble}}(1,23[45])}{\ell^2\ell_1^2 s_{45}} + \frac{1}{5}\frac{N_{\text{tadpole}}(12345)}{\ell^2} + \text{cyc}_\ell(1,2,3,4,5)\Bigg] + \mathcal{O}(\alpha'^6)\,,
\end{aligned} \tag{4.39}$$



(a) $N^{(\zeta_2)}_{\rm box}(1,2,3,45)$ **(b)** $N^{(\zeta_2)}_{\rm triangle}(1,[23],45)$ **(c)** $N^{(\zeta_2)}_{\rm triangle}(1,23,[45])$

(d) $N^{(\zeta_2)}_{\rm triangle}(1,2,345)$ **(e)** $N^{(\zeta_2)}_{\rm triangle}(1,2,3[45])$ **(f)** $N^{(\zeta_2)}_{\rm triangle}(1,2,[34]5)$

Figure 5. Diagrammatic representation of the contribution to the ζ_2-order of (4.39) with numerators in (4.41). Other ζ orders will have the same topologies with different operator insertions in place of F^4.

where square brackets as in $N_{\rm triangle}(1,[23],45)$ or $N_{\rm bubble}([12],345)$ indicate the presence of propagators s_{ij}^{-1} as opposed to the contact diagrams $N_{\rm triangle}(1,23,45)$ and $N_{\rm bubble}(12,345)$ at different mass dimensions. The notation ${\rm cyc}_\ell(1,2,3,4,5)$ is again understood to include a shift of loop momentum such that ℓ points from leg 5 to leg 1 which may affect the numerators as exemplified in (4.8). We give complete results up to α'^5 order,

$$N_\Gamma = \alpha'^2\zeta_2 N_\Gamma^{(\zeta_2)} + \alpha'^3\zeta_3 N_\Gamma^{(\zeta_3)} + \alpha'^4\zeta_2^2 N_\Gamma^{(\zeta_2^2)} + \alpha'^5\Big[\zeta_5 N_\Gamma^{(\zeta_5)} + \zeta_2\zeta_3 N_\Gamma^{(\zeta_2\zeta_3)}\Big]\,. \tag{4.40}$$

The α'-expansion of the numerators in (4.39) is organized according to MZVs as in (4.13): the coefficients of $\alpha'^2\zeta_2$,

$$N^{(\zeta_2)}_{\rm box}(1,2,3,45) = -s_{45}N_{+|12345|-} + \frac{1}{2}(s_{4,\ell}-s_{5,\ell}-s_{45})t_8(45,1,2,3)\,, \tag{4.41a}$$

$$N^{(\zeta_2)}_{\rm triangle}(1,[23],45) = s_{45}t_8(23,1,4,5)\,, \tag{4.41b}$$

$$N^{(\zeta_2)}_{\rm triangle}(1,23,[45]) = s_{23}t_8(45,1,2,3)\,, \tag{4.41c}$$

$$N^{(\zeta_2)}_{\rm triangle}(1,2,345) = \frac{1}{2}t_8(34,1,2,5) - t_8(35,1,2,4) + \frac{1}{2}t_8(45,1,2,3)\,, \tag{4.41d}$$

$$N^{(\zeta_2)}_{\rm triangle}(1,2,[34]5) = (s_{35}+s_{45})t_8(34,1,2,5)\,, \tag{4.41e}$$

$$N^{(\zeta_2)}_{\rm triangle}(1,2,3[45]) = (s_{34}+s_{35})t_8(45,1,2,3)\,, \tag{4.41f}$$

comprise triangle and box diagrams with F^4-insertions depicted in figure 5. At the order of $\alpha'^3\zeta_3(D^2F^4+F^5)$, we have the same types of diagrams as in (4.41) with D^2F^4 in the place of F^4,

$$N^{(\zeta_3)}_{\text{box}}(1,2,3,45) = \frac{1}{2}s_{45}(s_{4,\ell} - s_{5,\ell} + s_{45})N_{+|12345|-} - \frac{1}{2}\Big[s^2_{4,\ell} + s^2_{5,\ell} - s_{45}(s_{4,\ell} + s_{5,\ell})\Big]t_8(45,1,2,3)\,, \tag{4.42a}$$

$$N^{(\zeta_3)}_{\text{triangle}}(1,[23],45) = -\frac{1}{2}s_{45}(s_{4,\ell} - s_{5,\ell} + s_{45})t_8(23,1,4,5)\,, \tag{4.42b}$$

$$N^{(\zeta_3)}_{\text{triangle}}(1,23,[45]) = -\frac{1}{2}s_{23}(s_{2,\ell} - s_{3,\ell} + s_{12} + s_{23} - s_{13})t_8(45,1,2,3)\,, \tag{4.42c}$$

$$\begin{aligned} N^{(\zeta_3)}_{\text{triangle}}(1,2,345) = &\ \frac{1}{2}(s_{34} + s_{45} - 2s_{35})N_{+|12345|-} \\ &+ \frac{1}{2}(2s_{4,\ell} + s_{5,\ell} - s_{34} - 5s_{35} - 3s_{45})t_8(34,1,2,5) \\ &- \frac{1}{2}(s_{3,\ell} + 2s_{4,\ell} - s_{45} + 4s_{35})t_8(45,1,2,3) \\ &+ (s_{3,\ell} - s_{5,\ell} + s_{34} + 2s_{45})t_8(35,1,2,4)\,, \end{aligned} \tag{4.42d}$$

$$N^{(\zeta_3)}_{\text{triangle}}(1,2,[34]5) = -\frac{1}{2}(s_{35} + s_{45})(s_{3,\ell} + s_{4,\ell} - s_{5,\ell} + s_{12} - 3s_{34})t_8(34,1,2,5)\,, \tag{4.42e}$$

$$N^{(\zeta_3)}_{\text{triangle}}(1,2,3[45]) = -\frac{1}{2}(s_{34} + s_{35})(s_{\ell,3} - s_{4,\ell} - s_{5,\ell} + s_{34} + s_{35})t_8(45,1,2,3)\,, \tag{4.42f}$$

and additionally the following types of bubbles depicted in figure 6,

$$\begin{aligned} N^{(\zeta_3)}_{\text{bubble}}(1,2345) = &-\frac{1}{2}t_8(23,1,4,5) + t_8(24,1,3,5) - t_8(25,1,3,4) \\ &+ t_8(35,1,2,4) - \frac{1}{2}t_8(45,1,2,3)\,, \end{aligned} \tag{4.43a}$$

$$N^{(\zeta_3)}_{\text{bubble}}([12],345) = -\frac{1}{2}(s_{34} + s_{45} - 2s_{35})t_8(12,3,4,5)\,, \tag{4.43b}$$

$$N^{(\zeta_3)}_{\text{bubble}}(1,[23]45) = \frac{1}{2}(s_{14} + 2s_{25} + 2s_{35})t_8(23,1,4,5)\,, \tag{4.43c}$$

$$N^{(\zeta_3)}_{\text{bubble}}(1,2[34]5) = \frac{1}{2}(s_{34} + 3s_{25})t_8(34,1,2,5)\,, \tag{4.43d}$$

$$N^{(\zeta_3)}_{\text{bubble}}(1,23[45]) = \frac{1}{2}(s_{13} + 2s_{24} + 2s_{25})t_8(45,1,2,3)\,. \tag{4.43e}$$

The quintic vertex in figure 5 is due to the non-linear extensions of the field strengths in $\alpha'^2\zeta_2 F^4$, and the vertices in figure 6 of valence > 4 can be attributed to both $\alpha'^3\zeta_3 D^2F^4$ and $\alpha'^3\zeta_3 F^5$.

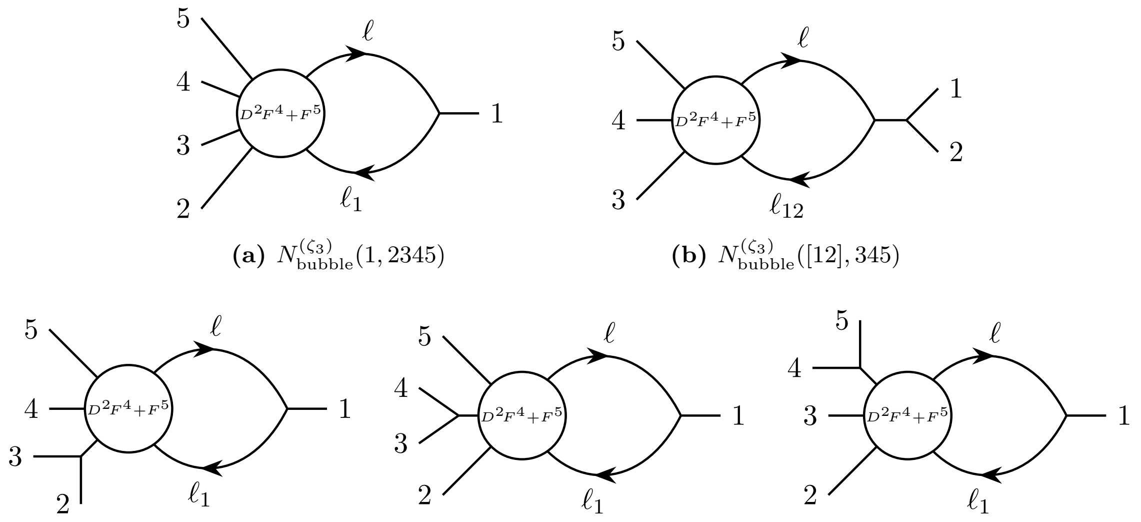
The order of $\alpha'^4\zeta_2^2$ is again the onset of tadpoles and diagrams with double-insertions of higher-mass-dimension operators, see (4.15) and figure 4 for their four-point instances. At five points, particularly characteristic numerators in (4.42) are

$$N^{(\zeta_2^2)}_{\text{bubble}}(12,345) = -\frac{1}{2}s_{12}\Big[t_8(34,1,2,5) - 2t_8(35,1,2,4) + t_8(45,1,2,3)\Big]\,, \tag{4.44a}$$

$$N^{(\zeta_2^2)}_{\text{bubble}}(12,[34]5) = -s_{12}(s_{12} - s_{34})t_8(34,1,2,5)\,, \tag{4.44b}$$

$$N^{(\zeta_2^2)}_{\text{bubble}}(12,3[45]) = s_{12}(s_{13} + s_{23})t_8(45,1,2,3)\,, \tag{4.44c}$$

$$N^{(\zeta_2^2)}_{\text{tadpole}}(12345) = \frac{12}{5}\left[\frac{s_{34} + s_{45} - s_{35}}{s_{12}}t_8(12,3,4,5) - t_8(13,2,4,5) + \text{cyc}(1,2,3,4,5)\right]\,, \tag{4.44d}$$



(a) $N^{(\zeta_3)}_{\text{bubble}}(1,2345)$ **(b)** $N^{(\zeta_3)}_{\text{bubble}}([12],345)$

(c) $N^{(\zeta_3)}_{\text{bubble}}(1,[23]45)$ **(d)** $N^{(\zeta_3)}_{\text{bubble}}(1,2[34]5)$ **(e)** $N^{(\zeta_3)}_{\text{bubble}}(1,23[45])$

Figure 6. Bubble contributions to the $\alpha'^3\zeta_3$-order of (4.39) with numerators given by (4.43). The blobs represent insertions of $D^2F^4 + F^5$. Other ζ orders contain the same topologies but with different operator insertions.

see figure 7 for the associated diagrams. The remaining numerators contributing to this order can be found in appendix A.2.

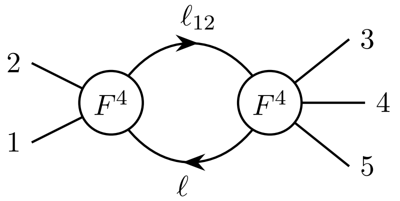
As a drawback of the tadpole numerator (4.44), gauge invariance of the α'^4-order of (4.39) can only be seen after shifts of loop momentum. One can attain gauge invariance pointwise in ℓ by employing the more lengthy alternative expression,

$$\begin{aligned} N^{(\zeta_2^2)}_{\text{tadpole}}(12345) &\to 2t_8(12,3,4,5)-3t_8(13,2,4,5)+3t_8(14,2,3,5)-2t_8(15,2,3,4) \\ &\quad+3t_8(23,1,4,5)+3t_8(25,1,3,4)+3t_8(34,1,2,5)-3t_8(35,1,2,4)+2t_8(45,1,2,3) \\ &\quad-6\left[\frac{s_{35}}{s_{12}}t_8(12,3,4,5)+\frac{s_{14}}{s_{23}}t_8(23,1,4,5)+\frac{s_{25}}{s_{34}}t_8(34,1,2,5)+\frac{s_{13}}{s_{45}}t_8(45,1,2,3)\right]. \end{aligned} \tag{4.45}$$

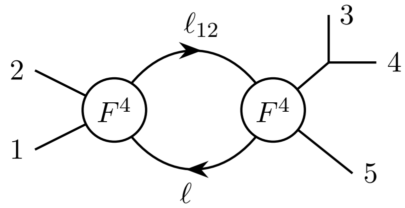
4.5 Five points, closed string

For the one-loop matrix elements (3.2) of closed strings, the ordering-dependence of the five-point numerators in (4.32) does not allow for a straightforward uplift of the four-point relation (4.19) to single-valued open-string matrix elements (though n-point relations of this type are available in (3.15) at the level of linearized propagators). Instead, we insert the five-point numerators into (3.2) and assemble the α'-expansion of the sphere integrals from single-valued seven-point disk integrals. At the leading orders in α', we have

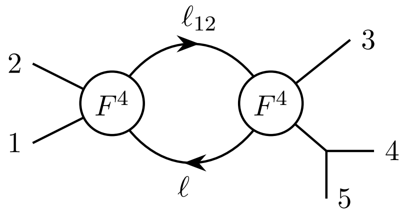
$$\begin{aligned} M^{\text{1-loop}}_{5,\text{eff}} &= M^{\text{1-loop}}_{5,\text{SUGRA}} + 2\int \mathrm{d}^D\ell \left[\frac{1}{4}\frac{\mathcal{N}_{\text{box}}(1,2,3,45)}{\ell^2(\ell+k_1)^2(\ell+k_{12})^2(\ell+k_{123})^2}+\frac{1}{12}\frac{\mathcal{N}_{\text{triangle}}(1,2,345)}{\ell^2(\ell+k_1)^2(\ell+k_{12})^2}\right. \\ &\quad+\frac{1}{4}\frac{\mathcal{N}_{\text{triangle}}(1,2,[34]5)}{\ell^2(\ell+k_1)^2(\ell+k_{12})^2 s_{34}}+\frac{1}{4}\frac{\mathcal{N}_{\text{triangle}}(1,23,[45])}{\ell^2(\ell+k_1)^2(\ell+k_{123})^2 s_{45}}+\frac{1}{48}\frac{\mathcal{N}_{\text{bubble}}(1,2345)}{\ell^2(\ell+k_1)^2} \\ &\quad\left.+\frac{1}{8}\frac{\mathcal{N}_{\text{bubble}}(1,[23]45)}{\ell^2(\ell+k_1)^2 s_{23}}+\frac{1}{24}\frac{\mathcal{N}_{\text{bubble}}([12],345)}{\ell^2(\ell+k_1)^2 s_{12}}+\text{perm}(1,2,3,4,5)\right]+\mathcal{O}(\alpha'^6). \end{aligned} \tag{4.46}$$



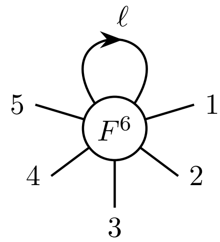
(a) $N^{(\zeta_2^2)}_{\text{bubble}}(12,345)$



(b) $N^{(\zeta_2^2)}_{\text{bubble}}(12,[34]5)$



(c) $N^{(\zeta_2^2)}_{\text{bubble}}(12,3[45])$



(d) $N^{(\zeta_2^2)}_{\text{tadpole}}(12345)$

Figure 7. Diagrammatic representation of the two-mass bubbles and the tadpole diagrams at the α'^4 order of the five-point one-loop matrix element.

In this work, we have computed the numerators up to $\alpha'^5\zeta_5$ order,

$$\mathcal{N}_\Gamma = \alpha'^3\zeta_3\mathcal{N}^{(\zeta_3)} + \alpha'^5\zeta_5\mathcal{N}^{(\zeta_5)}\,. \tag{4.47}$$

The loop integrand $M^{\text{1-loop}}_{5,\text{SUGRA}}$ of supergravity[14] is followed by box- and triangle diagrams with a single insertion of $\alpha'^3\zeta_3 R^4$ (in the place of $\alpha'^2\zeta_2 F^4$ in figure 5) and numerators

$$\begin{aligned}
\mathcal{N}^{(\zeta_3)}_{\text{box}}(1,2,3,45) &= s_{45}^2\left(\left|N_{+|12345|-}\right|^2 + \left|N_{+|12354|-}\right|^2\right)\\
&\quad + (s_{\ell,4}-s_{\ell,5})s_{45}\left(\left|N_{+|12345|-}\right|^2 - \left|N_{+|12354|-}\right|^2\right)\\
&\quad + \left[s_{\ell,4}^2 + s_{\ell,5}^2 - (s_{\ell,4}+s_{\ell,5})s_{45}\right]|t_8(45,1,2,3)|^2\,, && (4.48a)\\
\mathcal{N}^{(\zeta_3)}_{\text{triangle}}(1,2,345) &= \Big[(s_{12}+3s_{35})\big(t_8(34,1,2,5)\bar{t}_8(45,1,2,3) + (t_8\leftrightarrow\bar{t}_8)\big)\\
&\quad - 3(s_{15}+s_{25})\,|t_8(34,1,2,5)|^2 + \text{cyc}(3,4,5)\Big]\,, && (4.48b)\\
\mathcal{N}^{(\zeta_3)}_{\text{triangle}}(1,2,[34]5) &= 2(s_{35}+s_{45})(2s_{35}+2s_{45}-s_{12})\,|t_8(34,1,2,5)|^2\,, && (4.48c)\\
\mathcal{N}^{(\zeta_3)}_{\text{triangle}}(1,23,[45]) &= 2s_{23}^2\,|t_8(45,1,2,3)|^2\,, && (4.48d)
\end{aligned}$$

and $\mathcal{N}^{(\zeta_3)}_{\text{bubble}} = 0$ at this order. All the six topologies in (4.46) show up at $\alpha'^5\zeta_5$ order. The expressions for these numerators are lengthy and they are provided in the supplementary material.

[14] The explicit form of $M^{\text{1-loop}}_{5,\text{SUGRA}}$ can for instance be obtained from a cubic-graph expansion with the double-copy of the SYM numerators in (4.37) [2, 7, 122] which follows from the ambitwistor-string formula (2.1) with half integrands $\mathcal{I}_L, \mathcal{I}_R \to \mathcal{I}_5$.

4.5.1 Type IIA versus type IIB

Note that $M^{\text{1-loop}}_{5,\text{SUGRA}}$ and (4.48a) feature bilinears in the vector building block $t^\mu_\pm$ in (4.29) that differ between the type-IIA and IIB theories: the relative sign of the parity-odd terms need to be chosen as $t^\mu_+ \bar{t}^\nu_+$ for the chiral type-IIB theory and as $t^\mu_+ \bar{t}^\nu_-$ in the non-chiral type-IIA theory. Accordingly, the one-loop matrix elements in the two theories differ by

$$\begin{aligned} M^{\text{1-loop}}_{5,\text{eff}}\Big|_{\text{IIA}} = M^{\text{1-loop}}_{5,\text{eff}}\Big|_{\text{IIB}} &- i \int \frac{\mathrm{d}^D \ell}{\ell^2}\, \ell_\mu \varepsilon^\mu_{10}(\bar{\epsilon}_1, \bar{f}_2, \bar{f}_3, \bar{f}_4, \bar{f}_5) \\ &\times \lim_{k_\pm \to \pm \ell} \sum_{\gamma,\rho \in S_5} J(+, \gamma(1,2,3,4,5), -|+, \rho(1,2,3,4,5), -) N_{+|\rho(12345)|-}(\ell)\,, \end{aligned} \tag{4.49}$$

see (4.32) for the numerators $N_{+|\rho(12345)|-}(\ell)$. When the external states are chosen to be five gravitons h^5 with $\bar{\epsilon}_j \to \epsilon_j$ (rather than B-fields or dilatons), the general formula (4.49) for the difference specializes as follows

$$\begin{aligned} M^{\text{1-loop}}_{5,\text{eff}}\Big|^{h^5}_{\text{IIA}} = M^{\text{1-loop}}_{5,\text{eff}}\Big|^{h^5}_{\text{IIB}} &+ \frac{1}{2} \int \frac{\mathrm{d}^D \ell}{\ell^2}\, \ell_\mu \varepsilon^\mu_{10}(\epsilon_1, f_2, f_3, f_4, f_5) \ell_\nu \varepsilon^\nu_{10}(\epsilon_1, f_2, f_3, f_4, f_5) \\ &\times \lim_{k_\pm \to \pm \ell} \sum_{\gamma,\rho \in S_5} J(+, \gamma(1,2,3,4,5), -|+, \rho(1,2,3,4,5), -)\,, \end{aligned} \tag{4.50}$$

where only the tensor structure $\ell_\mu \ell_\nu \sim \eta_{\mu\nu}$ contributes to the loop integral.

4.5.2 Higher orders in α'

At the orders of $\alpha'^5 \zeta_5 (D^4 R^4 + D^2 R^5 + R^6)$ and beyond, expressions similar to (4.48) can be obtained from (3.2). However, the computational effort grows drastically with the number of terms in the α'-expansion of sphere integrals which need to be combined to quadratic propagators as in (4.46). As will be detailed below, the reorganization (5.34) of the five-point one-loop matrix element in terms of certain polarization-independent integrals "$\langle E|E \rangle$" with quadratic-propagator representations furnishes an improved starting point to obtain explicit results at higher orders in α'. By analogy with the four-point results in (4.24) and (4.25), the onset of external and two-mass bubble diagrams is expected at the orders of $\alpha'^5 \zeta_5$ and $\alpha'^6 \zeta_3^2$, respectively.

5 One-loop matrix elements and quadratic propagators from superstrings

In this section, we relate the one-loop matrix elements to superstring amplitudes and reorganize the transition from linearized to quadratic propagators according to the homology invariance of worldsheet correlators on the torus.

5.1 Motivation from chiral splitting

In earlier sections, the proposal (3.1) and (3.2) for one-loop matrix elements of higher-mass-dimension operators has been motivated from the ambitwistor-string perspective. We shall now provide a complementary motivation from certain limits of one-loop amplitudes of conventional strings based on the chiral-splitting formalism [5, 6].

The key idea of chiral splitting is to identify loop momenta as the joint zero modes of the worldsheet fields $\partial_z X^\mu$ and $\partial_{\bar z} X^\mu$ in the left- and right-moving sector. By separating the loop-momentum integral from the rest of the path integral that defines the string integrand, the contributions of the left- and right movers can be kept in a factorized form. In particular, the evaluation and simplification of the correlation function of vertex operators that determines the basis of kinematic factors can be performed separately for the two chiral halves of the string amplitude. Accordingly, chiral splitting is the driving force in the recent construction of the correlators in multiparticle one-loop [117], five-point two-loop [89] and three-loop four-point [163] string amplitudes.

In the chiral-splitting approach to one-loop closed-string amplitudes, the integrand of[15]

$$M^{\text{1-loop}}_{n,\text{closed}} = \int \mathrm{d}^D\ell \int_{\mathfrak{F}} \mathrm{d}^2\tau \int_{\mathfrak{T}_\tau^{n-1}} \mathrm{d}^2 z_2 \ldots \mathrm{d}^2 z_n \left|\mathcal{K}_n(\ell,\tau)\, \mathcal{J}_n(\ell,\tau)\right|^2 \tag{5.1}$$

is the absolute-value square of the chiral amplitude $\mathcal{K}_n \mathcal{J}_n$ which is a meromorphic function of the moduli z_j and τ with $z_1 = 0$ by translation invariance. We have organized the chiral amplitude into the Koba-Nielsen factor

$$\mathcal{J}_n(\ell,\tau) = \exp\left(2\pi i\alpha'\tau\ell^2 + 4\pi i\alpha' \sum_{j=1}^n (\ell\cdot k_j) z_j + 2\alpha' \sum_{1\leq i<j}^n k_i\cdot k_j \log\theta_1(z_{ij},\tau)\right) \tag{5.2}$$

and a kinematic factor $\mathcal{K}_n$ that carries all the dependence on the external polarization. While (5.2) is the correlator of the universal plane-wave parts $e^{ik_j\cdot X}$ of vertex operators, the Wick contractions of the remaining worldsheet fields determine $\mathcal{K}_n$ in terms of derivatives of the odd Jacobi theta function[16]

$$\theta_1(z,\tau) = 2iq^{1/8}\sin(\pi z)\prod_{j=1}^{\infty}(1-q^j)(1-e^{2\pi i z}q^j)(1-e^{-2\pi i z}q^j) = -\theta_1(-z,\tau)\,, \qquad q = e^{2\pi i\tau}\,. \tag{5.3}$$

In fact, the maximally supersymmetric $\mathcal{K}_n$ are degree-$(n-4)$ polynomials in loop momenta, see (5.9) below for a relation to the half integrand $\mathcal{I}_n(\ell)$ in (2.7), (4.2) and (4.32).

Finally, the integration domains for the moduli in (5.1) are the fundamental domain $\mathfrak{F}$ of the modular group $\mathrm{SL}_2(\mathbb{Z})$ and the torus worldsheet $\mathfrak{T}_\tau = \frac{\mathbb{C}}{\mathbb{Z}+\tau\mathbb{Z}}$ in its parametrization as a parallelogram.

The open-string counterpart of (5.1) in the chiral-splitting formalism is

$$A^{\text{1-loop}}_{\text{open}}(\gamma(1,2,\ldots,n)) = \int \mathrm{d}^D\ell \int_0^{i\infty} \mathrm{d}\tau \int_{\mathfrak{C}_\tau(\gamma)} \mathrm{d}z_2 \ldots \mathrm{d}z_n\, \mathcal{K}_n(\ell,\tau)\,|\mathcal{J}_n(\ell,\tau)|\,, \tag{5.4}$$

[15]For ease of notation, the chiral-splitting representation (5.1) is adapted to the case of real invariants ℓ^2, $\ell\cdot k_i$ and $k_i\cdot k_j$. One can still accommodate complex kinematics by employing $-k_j^*$, $-\ell^*$ in the place of k_j, ℓ in one of the chiral halves before complex conjugation of $\mathcal{K}_n(\ell,\tau)\,\mathcal{J}_n(\ell,\tau)$. In this way, the real parts of τ and z_j drop out from the first and second term in the exponent of (5.2) when combining the left- and right-movers.

[16]More precisely, the chiral correlators $\mathcal{K}_n$ for massless states in type-II, heterotic and bosonic string theories are polynomials in loop momenta, holomorphic Eisenstein series and expansion coefficients of a meromorphic Kronecker-Eisenstein series [116, 153, 164].

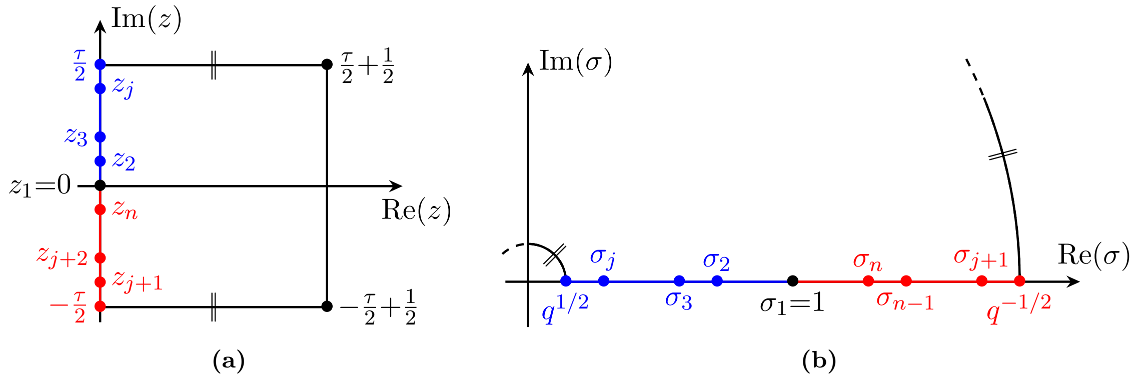


Figure 8. Parametrization of the integration domain on the cylinder boundary in planar one-loop open-string amplitudes.

where $z_1 = 0$ by translation invariance, the modular parameter τ is integrated over the positive imaginary axis instead of $\mathfrak{F}$, and the punctures $z_i = \tau u_i$ (with real u_i) are inserted on the cylinder boundary

$$\mathfrak{C}_\tau(\gamma) = \bigoplus_{j=1}^{n} \left\{ -\tfrac{1}{2} < u_{\gamma(j+1)} < u_{\gamma(j+2)} < \ldots < u_{\gamma(n)} < 0 < u_{\gamma(2)} < u_{\gamma(3)} < \ldots < u_{\gamma(j)} < \tfrac{1}{2} \right\}. \tag{5.5}$$

The integration domains in (5.4) are tailored to the amplitude contributions from planar cylinder diagrams and can be adapted to the non-planar cylinder or the Möbius strip by shifting $z_j \to z_j + \frac{1}{2}$ or $\tau \to \tau + \frac{1}{2}$, respectively [165]. As visualized in figure 8a, the cylinder worldsheet in (5.4) and (5.5) arises from a rectangular torus whose homology cycle $\mathfrak{B}$ sets the cylinder boundary where all the state insertions occur. The second cylinder boundary relevant for non-planar one-loop open-string amplitudes is displaced by half the homology cycle $\mathfrak{A}$, i.e. by $z_j \to z_j + \frac{1}{2}$. As reflected by the appearance of negative $\mathrm{Im}(z_i)$ in (5.5), we have chosen the fundamental cell of the parental torus to be the rectangle with corners at $\pm\frac{\tau}{2}$ and $1 \pm \frac{\tau}{2}$, and the cylinder covers its half bounded by $\pm\frac{\tau}{2}$ and $\frac{1}{2} \pm \frac{\tau}{2}$ seen in figure 8a.

5.1.1 The $\tau \to i\infty$ degeneration

One-loop amplitudes of SYM and supergravity arise in the limit $\tau \to i\infty$ where the $\mathfrak{A}$-cycle pinches, and the torus degenerates to a nodal sphere. In other words, the field-theory limit $\alpha' \to 0$ is traditionally performed while keeping the dimensionful quantity $\alpha' \,\mathrm{Im}\,\tau$ finite and identifying it with one of the Schwinger parameters of Feynman integrals [155, 166–173].

The $\tau \to i\infty$ limits of the constituents of the one-loop open-string amplitude (5.4) are most conveniently analyzed in terms of the coordinate σ_j on the nodal sphere

$$\sigma_j = e^{2\pi i z_j}\,, \qquad \mathrm{d}z_j = \frac{\mathrm{d}\sigma_j}{2\pi i \sigma_j}\,, \qquad \mathrm{d}^2 z_j = \frac{\mathrm{d}^2\sigma_j}{4\pi^2 \sigma_j \bar{\sigma}_j}\,. \tag{5.6}$$

First, the cylinder ordering in the parametrization of (5.5) tends to a cyclic combination of

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disk orderings (2.24) in the σ_j-variable, see figure 8b,

$$\lim_{\tau\to i\infty} \mathfrak{C}_\tau(1,2,\ldots,n) = \bigoplus_{j=1}^{n} \mathfrak{D}_j\,, \qquad \mathfrak{D}_j = \mathfrak{D}(+,j,j{-}1,\ldots,2,1,n,n{-}1,\ldots,j{+}1,-)\,, \tag{5.7}$$

where $\sigma_+ = 0$ and $\sigma_- \to \infty$ as in (2.5). Second, the chiral Koba-Nielsen factor (5.2) turns into the forward limit of the tree-level one in (2.22) after removing the scaling by $e^{2\pi i\alpha'\tau\ell^2}$,

$$\lim_{\tau\to i\infty} e^{-2\pi i\alpha'\tau\ell^2}\mathcal{J}_n(\ell,\tau) = \prod_{j=1}^{n}\sigma_{j+}^{2\alpha'\ell\cdot k_j}\prod_{1\le i<j}^{n}|\sigma_{ij}|^{2\alpha' k_i\cdot k_j}\,, \tag{5.8}$$

where we used $\sigma_1 = 1$ and momentum conservation.[17] Third, the polarization dependent part $\mathcal{K}_n$ of the chiral correlator yields the kinematic half integrand of the ambitwistor string once its z_{ij}-dependence has been brought into logarithmic form via integration by parts [21],

$$\mathcal{I}_n(\ell) = \lim_{\tau\to i\infty}\frac{\mathcal{K}_n(\ell,\tau)}{(2\pi i)^{n-4}\sigma_1\sigma_2\ldots\sigma_n}\,. \tag{5.9}$$

The inverse factors of σ_j in (5.9) stem from the conversion of $\mathrm{d}z_j$ into $\mathrm{d}\sigma_j$ in (5.6). The four- and five-point instances of $\mathcal{I}_n(\ell)$ in (4.2) and (4.32) arise from

$$\mathcal{K}_4(\ell,\tau) = t_8(f_1,f_2,f_3,f_4)\,, \tag{5.10}$$
$$\mathcal{K}_5(\ell,\tau) = 2\pi i\ell_\mu t_+^\mu(1,2,3,4,5) + \big[\partial_{z_1}\log\theta_1(z_{12},\tau)t_8(12,3,4,5) + (1,2|1,2,3,4,5)\big]\,,$$

which can be checked by $\lim_{\tau\to i\infty}\partial_{z_1}\log\theta_1(z_{12},\tau) = 2\pi i G_{12}$, see (4.29) for the five-point kinematic factors and (4.31) for the nodal Green function G_{12}. The $(n\le 5)$-point examples in (5.10) are inevitably in logarithmic form, while the analogous six- and seven-point correlators [117] admit both logarithmic and non-logarithmic representations that are related by integration-by-parts manipulations detailed in the reference. The vector building blocks in the bilinear $|\mathcal{K}_5(\ell,\tau)|^2$ of the closed-string five-point amplitude (5.1) are $t_+^\mu\bar t_+^\nu$ for type IIB and $t_+^\mu\bar t_-^\nu$ for type IIA.

5.1.2 α'-expansions within the $\tau\to i\infty$ degeneration

We shall now argue that our proposal (3.1) and (3.2) for one-loop matrix elements arises naturally when expanding the $\tau\to i\infty$ degeneration of one-loop string amplitudes in α'. In comparison to the full loop integrand of string amplitudes obtained from performing the z_j- and τ-integrals in (5.1) and (5.4), the $\tau\to i\infty$ degeneration is insensitive to terms without any massless propagators ℓ^{-2} or $(\ell+K)^{-2}$ which result in rational loop integrals. For open strings, the dominant contributions from the degeneration are most conveniently found by studying the τ-integrand of (5.4) in terms of the quantity

$$F(\ell,\tau) = \int_{\mathfrak{C}_\tau(\gamma)}\mathrm{d}z_2\ldots\mathrm{d}z_n\, e^{-2\pi i\alpha'\tau\ell^2}\,\mathcal{J}_n(\ell,\tau)\,\mathcal{K}_n(\ell,\tau)\,, \tag{5.11}$$

[17] More precisely, we start from the degeneration $\log\theta_1(z_{ij},\tau) = \frac{i\pi\tau}{4} + \log(\frac{\sigma_{ij}}{\sqrt{\sigma_i\sigma_j}}) + \mathcal{O}(e^{2\pi i\tau})$ and apply momentum conservation in the form of $\sum_{j\neq i}^{n} s_{ij} = 0$ to cancel the contributions from $\frac{i\pi\tau}{4}$ and $-\frac{1}{2}(\log\sigma_i + \log\sigma_j)$ to $\sum_{1\le i<j}^{n} s_{ij}\log\theta_1(z_{ij},\tau) = \sum_{1\le i<j}^{n} s_{ij}\log\sigma_{ij} + \mathcal{O}(e^{2\pi i\tau})$.

and by inserting $0 = -F(\ell, i\infty) + F(\ell, i\infty)$ into

$$\begin{aligned} A^{\text{1-loop}}_{\text{open}}(1,2,\ldots,n) &= \int \mathrm{d}^D\ell \int_0^{i\infty} \mathrm{d}\tau\, e^{2\pi i\alpha'\tau\ell^2} F(\ell,\tau) \\ &= \int \mathrm{d}^D\ell \int_0^{i\infty} \mathrm{d}\tau\, e^{2\pi i\alpha'\tau\ell^2} \Big\{ \underbrace{F(\ell,\tau) - F(\ell,i\infty)}_{(i)} + \underbrace{F(\ell,i\infty)}_{(ii)} \Big\}. \end{aligned} \tag{5.12}$$

The factor of $e^{-2\pi i\alpha'\tau\ell^2}$ in (5.11) is engineered to cancel the ℓ^2-dependence of $\mathcal{J}_n(\ell,\tau)$ in (5.8) and to ensure that $F(\ell,\tau)$ is independent on ℓ^2. The first part (i) in (5.12) is suppressed at the cusp since $F(\ell,\tau) - F(\ell,i\infty)$ vanishes as $\tau \to i\infty$. In the second part (ii), one can directly perform the τ integral

$$\int_0^{i\infty} \mathrm{d}\tau\, e^{2\pi i\alpha'\tau\ell^2} = \frac{i}{2\pi\alpha'\ell^2} \tag{5.13}$$

under the assumption that ℓ^2 has a positive real part to drop the contribution from the integration limit $\tau \to i\infty$. The accompanying limit $F(\ell, i\infty)$ can be simplified by means of the results in section 5.1.1

$$\begin{aligned} \lim_{\tau\to i\infty} F(\ell,\tau) &= \frac{1}{(2\pi i)^{n-1}} \sum_{j=1}^n \int_{\mathfrak{D}_j} \frac{\mathrm{d}\sigma_2\,\mathrm{d}\sigma_3\ldots\mathrm{d}\sigma_n}{\sigma_2\sigma_3\ldots\sigma_n} \lim_{\tau\to i\infty} e^{-2\pi i\alpha'\tau\ell^2} \mathcal{K}_n(\ell,\tau)\mathcal{J}_n(\ell,\tau) \\ &= \frac{1}{(2\pi i)^3} \sum_{j=1}^n \int_{\mathfrak{D}_j} \mathrm{d}\sigma_2\,\ldots\,\mathrm{d}\sigma_n\, \mathcal{I}_n(\ell) \prod_{j=1}^n \sigma_{j+}^{2\alpha'\ell\cdot k_j} \prod_{1\le i<j}^n |\sigma_{ij}|^{2\alpha' k_i\cdot k_j} \\ &= \frac{-1}{(2\pi i)^3(\alpha')^{n-1}} \sum_{\rho\in S_n} N_{+|\rho(12\ldots n)|-}(\ell) \\ &\quad \times \lim_{k_\pm\to\pm\ell} \sum_{j=1}^n Z(+,j,j{-}1,\ldots,2,1,n,\ldots,j{+}1,-|+,\rho(1,2,\ldots,n),-) \\ &= \frac{-1}{(2\pi i)^3(\alpha')^{n-1}} \sum_{j=1}^n a_{\text{eff}}(+,j,j{-}1,\ldots,2,1,n,\ldots,j{+}1,-)\,, \end{aligned} \tag{5.14}$$

based on the limits (5.7), (5.8) and (5.9). The integration domains $\mathfrak{D}_j$ defined in (5.7) are the disk orderings of the Z-integrals identified in the third step, and we have expanded the kinematic half integrand in terms of Parke-Taylor factors and n-gon numerators $N_{+|\rho|-}$ according to (2.8).

In summary, by combining (5.12), (5.13) and (5.14), the part (ii) of one-loop open-string amplitudes that carries the dominant contribution at the cusp coincides with the proposal (3.1) for one-loop matrix elements,

$$A^{\text{1-loop}}_{\text{open}}(1,2,\ldots,n)\big|_{(ii)} = \frac{A^{\text{1-loop}}_{\text{eff}}(n,n{-}1,\ldots,2,1)}{16\pi^4(\alpha')^n} = \frac{A^{\text{1-loop}}_{\text{eff}}(1,2,\ldots,n)}{16\pi^4(-\alpha')^n}\,. \tag{5.15}$$

In particular, the $\alpha' \to 0$ limit of (5.14) reduces $Z \to m$ and extracts the representation (2.12) of SYM amplitudes from conventional strings, complementing its earlier derivation from ambitwistor strings.

The other part (i) of the one-loop open-string amplitude in (5.12) does not have the pole in ℓ^2 which enters (ii) via (5.13). Since the quantity $F(\ell,\tau)$ only depends on $\ell\cdot k_j$ but not on ℓ^2, there is no chance for $A^{\text{1-loop}}_{\text{open}}(1,2,\ldots,n)\big|_{(i)}$ to have a massless quadratic propagator $(\ell+K)^{-2}$. Upon loop integration, part (i) is therefore free of branch cuts in the Mandelstam invariants corresponding to massless particles and belongs to the analytic part of the one-loop open-string amplitude.

A similar decomposition can be performed for the one-loop closed-string amplitude (5.1). As detailed in appendix B, the proposal (3.2) for gravitational one-loop matrix elements can be motivated from the dominant contributions at $\tau\to i\infty$. Note that the absolute normalization of the string amplitudes (5.1) and (5.4) has not been tracked; it can be determined from unitarity or S-duality of type-IIB superstrings.

5.1.3 Universal one-loop matrix elements from torus compactifications

In preparation for one-loop string amplitudes in supersymmetry-preserving toroidal compactifications, we have already adapted the loop integration measure $\mathrm{d}^D\ell$ in (5.1) and (5.4) to a generic number D of spacetime dimensions. Upon compactification on a $10{-}D$ torus, open-string integrands additionally feature a τ-dependent partition function to account for Kaluza-Klein modes, and closed-string integrands involve a similar partition function encoding both Kaluza-Klein and winding modes [155, 174]. Both types of partition functions reduce to a constant in the limit $\tau\to i\infty$ relevant to one-loop matrix elements in (5.15) and (B.6). By absorbing the $\tau\to i\infty$ degeneration of the partition functions into the normalization, the loop integrands of the matrix elements (3.1) and (3.2) turn out to be dimension agnostic under torus compactification of the underlying string theory. This is consistent with the construction of the one-loop matrix elements from forward limits in the massless states of tree amplitudes, without any Kaluza-Klein or winding modes in the propagators.

5.2 Quadratic propagators from homology invariance

The ambitwistor approach to SYM and supergravity amplitudes does not manifest that the linearized propagators from the Parke-Taylor integrals (2.9) eventually recombine to quadratic ones. In our proposal (3.1) and (3.2), the translation between linearized and quadratic propagators [71–76] has to be performed order by order in the α'-expansion of the Z- and J integrals reviewed in section 2.3. This procedure led to the four- and five-point results in section 4, and there is no fundamental obstruction to higher-point extensions. Still, higher and higher orders in α' introduce a growing number of factors $\ell\cdot k_j$ into the numerators, so the efforts to find quadratic-propagator representations will grow accordingly by the shifts of loop momenta in the second step of (2.17) or in (2.18).

We shall now discuss a condition on amplitude building blocks to admit a quadratic-propagator representation. In particular, we pinpoint the smallest contributions to the half integrands $\mathcal{I}_n(\ell)$ which can integrate to quadratic propagators and thereby identify a particularly economic organization scheme to obtain them from linearized propagators. On the one hand, having an integration measure $\mathrm{d}\mu^{\text{tree}}_{n+2}$ of forward-limit form in (2.2) is tied to the degeneration $\tau\to i\infty$. On the other hand, the salient point for compatibility with quadratic propagators will be the behaviour of the chiral integrands under translations $z_j \to z_j + \tau$ around the $\mathfrak{B}$-cycle which can only be meaningfully studied at finite τ.

The torus integrals in one-loop closed-string amplitudes (5.1) are only well-defined for doubly-periodic integrands under $z_j \to z_j + 1$ and $z_j \to z_j + \tau$ for all of $j = 1, 2, \ldots, n$. By the integral over $\ell \in \mathbb{R}^D$, it will be sufficient to attain double-periodicity up to shifts of the loop momentum. By momentum conservation and the quasi-periodicity $\theta_1(z+\tau, \tau) = -e^{-i\pi\tau - 2\pi i z}\theta_1(z, \tau)$, one can easily check the chiral Koba-Nielsen factor (5.2) to exhibit the following $\mathfrak{A}$- and $\mathfrak{B}$-cycle monodromies [6]

$$
\begin{aligned}
\mathcal{J}_n(\ell, \tau)\big|_{z_j \to z_j+1} &= e^{4\pi i \alpha' \ell \cdot k_j} \mathcal{J}_n(\ell, \tau)\,, \\
\mathcal{J}_n(\ell, \tau)\big|_{z_j \to z_j+\tau} &= \mathcal{J}_n(\ell + k_j, \tau)\,.
\end{aligned}
\tag{5.16}
$$

The phases under the $\mathfrak{A}$-cycle shift $z_j \to z_j + 1$ drop out after combining the Koba-Nielsen factors (5.2) of both chiral halves, so their contributions to the closed-string amplitude (5.1) have the required doubly-periodicity.[18] In order to have the same doubly-periodicity properties for type-II and heterotic one-loop amplitudes, the left- and right-moving chiral correlators $\mathcal{K}_n(\ell)$ in (5.1) need to separately satisfy [6]

$$
\begin{aligned}
\mathcal{K}_n(\ell, \tau)\big|_{z_j \to z_j+1} &= \mathcal{K}_n(\ell, \tau)\,, \\
\mathcal{K}_n(\ell, \tau)\big|_{z_j \to z_j+\tau} &= \mathcal{K}_n(\ell + k_j, \tau)\,.
\end{aligned}
\tag{5.17}
$$

Functions of z_j, ℓ, k_j, τ that obey (5.17) were dubbed *generalized elliptic integrands* and systematically constructed in [87, 88, 117]. We will attribute the shifts on both sides of (5.17) to the action of the homology group on z_j, ℓ and follow the terminology of [89]:

$$
E(z_j, \ell, \tau) = E(z_j + \tau, \ell - k_j, \tau) \quad \Longrightarrow \quad E \text{ is referred to as } \textit{homology invariant}\,. \tag{5.18}
$$

Clearly, elliptic (i.e. meromorphic and doubly-periodic) functions of $z_1, z_2, \ldots, z_n$ are automatically homology invariant. Moreover, as becomes pressing for ($n \geq 6$)-point examples, the notion of homology invariance extends to the situation where (5.18) holds up to total Koba-Nielsen derivatives in the z_j,[19]

$$
\begin{aligned}
\int \bigg(\prod_{i=1}^n \mathrm{d}z_i \bigg) \mathcal{J}_n(\ell, \tau) E(z_j, \ell, \tau) &= \int \bigg(\prod_{i=1}^n \mathrm{d}z_i \bigg) \mathcal{J}_n(\ell, \tau) E(z_j + \tau, \ell - k_j, \tau) \\
&\Longrightarrow \quad E \text{ is referred to as } \textit{homology invariant}\,.
\end{aligned}
\tag{5.19}
$$

[18] While the prescription (5.1) with $|\mathcal{J}_n(\ell, \tau)|^2$ in the integrand is tailored to real kinematic invariants $\ell^2, \ell \cdot k_j$ and $k_i \cdot k_j$, one can reconcile homology invariance with complex kinematics by employing $-k_j^*, -\ell^*$ instead of k_j, ℓ in one of the chiral halves. Still, the loop integral in (5.1) only converges for real kinematics after Wick rotating the zeroth component of both ℓ and k_j.

[19] For instance, the following quantity relevant to six-point correlators [87, 88, 117] is homology invariant in the sense of (5.19) but does not obey the stronger condition (5.18),

$$
E_{1|2|3,4,5,6} = -2s_{12} g^{(2)}(z_{12}, \tau) + \bigg(2\pi i \ell \cdot k_2 + \sum_{j=3}^{6} s_{2j} \partial_{z_2} \log \theta_1(z_{2j}, \tau) \bigg) \partial_{z_1} \log \theta_1(z_{12}, \tau)\,.
$$

The first term features $g^{(2)}(z, \tau) = \frac{1}{2}(\partial_z \log \theta_1(z, \tau))^2 - \frac{1}{2}\wp(z, \tau)$, a meromorphic Kronecker-Eisenstein coefficient ubiquitous to the ($n \geq 6$)-point correlators in the references involving the Weierstraß function $\wp(z, \tau)$.

The same types of shifts $\ell \to \ell - k_j$ arise in the sufficient condition (2.20) for numerators of generic ($p \leq n$)-gon diagrams in the α'-expansions to admit a quadratic-propagator representation. Indeed, the associated cyclic relabellings $\mathfrak{N}_{+|\ldots j|-} \leftrightarrow \mathfrak{N}_{+|j\ldots|-}$ of the numerators in (2.20) descend from the shift $z_j \to z_j + \tau$ around the $\mathfrak{B}$-cycle. This is a first piece of evidence that homology invariance has important implications for the appearance of quadratic propagators.

5.2.1 Five-point examples

The four-point correlator $\mathcal{K}_4$ in (5.10) is independent on z_j and ℓ and thereby trivially a homology invariant by (5.18). The simplest non-trivial examples of homology invariants occur in the one-loop five-point correlator [87, 117]

$$E_{1|23,4,5} = \partial_z \log \theta_1(z_{12}, \tau) + \partial_z \log \theta_1(z_{23}, \tau) + \partial_z \log \theta_1(z_{31}, \tau) \tag{5.20}$$

$$E^{\mu}_{1|2,3,4,5} = 2\pi i \ell^{\mu} + \left[k_2^{\mu} \partial_z \log \theta_1(z_{12}, \tau) + (2 \leftrightarrow 3, 4, 5)\right] . \tag{5.21}$$

Given the transformations $\partial_z \log \theta_1(z \pm 1, \tau) = \partial_z \log \theta_1(z, \tau)$ as well as $\partial_z \log \theta_1(z \pm \tau, \tau) = \partial_z \log \theta_1(z, \tau) \mp 2\pi i$, the first example (5.20) is elliptic and therefore homology invariant by virtue of the cyclic sum of theta functions. For the vectorial function in (5.21), homology invariance w.r.t. z_j is tied to the interplay of ℓ with the monodromies of the theta functions and in case of z_1 requires five-point momentum conservation $k_{12345} = 0$. Based on the kinematic identity (4.38) for $(k_j)_{\mu} t^{\mu}_{+}(1, 2, 3, 4, 5)$, the five-point correlator in (5.10) can be written in manifestly homology-invariant form as [87, 117]

$$\mathcal{K}_5(\ell) = (E_{1|2,3,4,5})_{\mu} t^{\mu}_{+}(1, 2, 3, 4, 5) + \left[E_{1|23,4,5} t_8(23, 1, 4, 5) + (2, 3|2, 3, 4, 5)\right] , \tag{5.22}$$

where $+(2, 3|2, 3, 4, 5)$ instructs to add five inequivalent permutations of $E_{1|23,4,5} t_8(23, 1, 4, 5)$ obtained from $23 \leftrightarrow 24, 25, 34, 35, 45$.[20]

In extracting the kinematic half integrand via (5.9), we obtain the following contributions to $\mathcal{I}_5$ and $N_{+|\rho(12345)|-}$ from the $\tau \to i\infty$ degeneration of (5.20) and (5.21)

$$e_{1|23,4,5} = \lim_{\tau \to i\infty} \frac{E_{1|23,4,5}}{2\pi i} = G_{12} + G_{23} + G_{31} \tag{5.23}$$

$$e^{\mu}_{1|2,3,4,5} = \lim_{\tau \to i\infty} \frac{E^{\mu}_{1|2,3,4,5}}{2\pi i} = \ell^{\mu} + \left[k_2^{\mu} G_{12} + (2 \leftrightarrow 3, 4, 5)\right] , \tag{5.24}$$

see (4.31) for the nodal Green function G_{ij}. As will be detailed in subsection 5.3 below, each of the 7 terms in the alternative form

$$\mathcal{I}_5(\ell) = \frac{(e_{1|2,3,4,5})_{\mu} t^{\mu}_{+}(1, 2, 3, 4, 5) + \left[e_{1|23,4,5} t_8(23, 1, 4, 5) + (2, 3|2, 3, 4, 5)\right]}{\sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5} \tag{5.25}$$

of (4.30) separately integrates to quadratic propagators order by order in the open-string α'-expansion of (3.1). Similarly, the double copy of (5.25) organizes the gravitational one-loop matrix element (3.2) into 7×7 terms with integrands $e_{\ldots} \bar{e}_{\ldots}$ that separately integrate

[20] This selection of permutations can be understood from the properties $E_{1|23,4,5} = E_{1|23,5,4} = -E_{1|32,4,5}$. We also note that $E_{1|23,4,5} = E_{2|31,4,5}$ and $E_{2|34,1,5} = E_{1|34,2,5} + E_{1|23,4,5} - E_{1|24,3,5}$. Moreover, $E^{\mu}_{1|2,3,4,5}$ is permutation symmetric in $2, 3, 4, 5$, but $1 \leftrightarrow 2$ acts via $E^{\mu}_{2|1,3,4,5} = E^{\mu}_{1|2,3,4,5} + [k_3^{\mu} E_{1|23,4,5} + (3 \leftrightarrow 4, 5)]$.

to quadratic propagators at each order in α'. This refines the observations of sections 4.4 and 4.5 that the overall expressions for $A^{\text{1-loop}}_{\text{eff}}(1,2,3,4,5)$ and $M^{\text{1-loop}}_{5,\text{eff}}$ can be expanded in terms of conventional Feynman integrals with quadratic propagators.

At the level of practicalities, the number of terms conspiring in the recombination of linearized propagators to quadratic ones is reduced by a factor of 49 on average for each bilinear of the $e_{\ldots}$ in (5.25). At six points, by the 51 homology invariants contributing to $\mathcal{K}_6$ [87, 117], the size of typical expressions that recombine to quadratic propagators is reduced by a factor of 2601 on average. This applies to both the loop integrand in supergravity and the α'-expansion of $M^{\text{1-loop}}_{6,\text{eff}}$.

5.2.2 General conjectures

The above five-point observations and a variety of checks at six points[21] motivate us to formulate the following general conjecture: for any homology invariant in the sense of (5.19), the contribution to $A^{\text{1-loop}}_{\text{eff}}(1,2,\ldots,n)$ in (3.1) can be rewritten in terms of quadratic propagators. The same is claimed to hold for any pair of meromorphic and antimeromorphic homology invariants contributing to $M^{\text{1-loop}}_{n,\text{eff}}$ in (3.2). In the field-theory limit $\alpha'\to 0$, this conjecture can be formulated within the framework of ambitwistor strings:

Conjecture 1 (on quadratic propagators in field-theory amplitudes)**.**

$$
\begin{aligned}
&E_P(z_j,\ell,\tau),\ E_Q(z_j,\ell,\tau) \text{ homology invariant}\\
&\Longrightarrow \int \frac{\mathrm{d}^D\ell}{\ell^2}\int_{\mathbb{C}^{n-1}} \mathrm{d}\mu^{\text{tree}}_{n+2}\, \frac{E_P(\sigma_j,\ell,\tau\to i\infty)}{\sigma_1\sigma_2\ldots\sigma_n} \sum_{\gamma\in\text{cyc}(1,2,\ldots,n)} \text{PT}(+,\gamma(1,2,\ldots,n),-)\\
&\qquad \text{can be recombined to quadratic propagators} \qquad (5.26)\\
&\Longrightarrow \int \frac{\mathrm{d}^D\ell}{\ell^2}\int_{\mathbb{C}^{n-1}} \mathrm{d}\mu^{\text{tree}}_{n+2}\, \frac{E_P(\sigma_j,\ell,\tau\to i\infty)\,E_Q(\sigma_j,\ell,\tau\to i\infty)}{|\sigma_1\sigma_2\ldots\sigma_n|^2}\\
&\qquad \text{can be recombined to quadratic propagators}\,, \qquad (5.27)
\end{aligned}
$$

where the labels P and Q of the homology invariants may represent any combination of subscripts and vector indices such as $P\to 1|23,4,5$ or $P\to {}^{\mu}_{1|2,3,4,5}$ in case of (5.20) or (5.21).

The α'-corrections to the one-loop matrix elements in (3.1) and (3.2) rely on the open- and closed-string motivated measures

$$
\begin{aligned}
\int_{12\ldots n}\mathrm{d}\mu^{\text{open}}_{n+2} &= \int_{0=\sigma_1<\sigma_2<\ldots<\sigma_n<\infty} \mathrm{d}\sigma_2\ldots\mathrm{d}\sigma_n \prod_{j=1}^n|\sigma_{j+}|^{2\alpha'\ell\cdot k_j}\prod_{1\leq i<j}^n|\sigma_{ij}|^{2\alpha' k_i\cdot k_j} + \text{cyc}(1,2,\ldots,n)\\
\int \mathrm{d}\mu^{\text{closed}}_{n+2} &= \int_{\mathbb{C}^{n-1}} \mathrm{d}^2\sigma_2\ldots\mathrm{d}^2\sigma_n \prod_{j=1}^n|\sigma_{j+}|^{4\alpha'\ell\cdot k_j}\prod_{1\leq i<j}^n|\sigma_{ij}|^{4\alpha' k_i\cdot k_j}\,. \qquad (5.28)
\end{aligned}
$$

In this setting, we additionally claim that, order by order in the α'-expansion,

[21] We are grateful to Sebastian Mizera for testing (5.26) for all the six-point instances of E_P seen in [87, 117].

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Conjecture 2 (on quadratic propagators in one-loop matrix elements)**.**

$$\begin{aligned} & E_P(z_j,\ell,\tau),\ E_Q(z_j,\ell,\tau) \text{ homology invariant} \\ & \Longrightarrow \int \frac{\mathrm{d}^D\ell}{\ell^2} \int_{12\ldots n} \mathrm{d}\mu^{\text{open}}_{n+2} \frac{E_P(\sigma_j,\ell,\tau\to i\infty)}{\sigma_1\sigma_2\ldots\sigma_n} \\ & \qquad \text{can be recombined to quadratic propagators}, \end{aligned} \tag{5.29}$$

$$\begin{aligned} & \Longrightarrow \int \frac{\mathrm{d}^D\ell}{\ell^2} \int\limits_{\mathbb{C}^{n-1}} \mathrm{d}\mu^{\text{closed}}_{n+2} \frac{E_P(\sigma_j,\ell,\tau\to i\infty)\,\overline{E_Q(\sigma_j,\ell,\tau\to i\infty)}}{|\sigma_1\sigma_2\ldots\sigma_n|^2} \\ & \qquad \text{can be recombined to quadratic propagators}. \end{aligned} \tag{5.30}$$

These conditions are sufficient (though not necessary) for the one-loop matrix elements at four and five points to admit quadratic-propagator representations at all orders in α' as exemplified in section 4. At the same time, (5.26) to (5.30) do not follow from the quadratic-propagator representations of the overall n-point matrix elements with $n>4$ since kinematic relations (4.38) may in principle accommodate deviations at the level of the individual homology invariants in (5.22). For both the five-point examples in (5.20), (5.21) and the higher-point homology invariants E_P in [87, 88, 117], the integrands of (5.29) and (5.30) can be brought into Parke-Taylor form by the techniques in [21]. Hence, (5.29) and (5.30) boil down to combinations of disk and sphere integrals and line up with terms in the one-loop matrix elements (3.1) and (3.2), respectively.

The BCJ and KLT relations of section 3.2 and their α'-uplifts are formulated in terms of the linearized propagators obtained from m or the α'-expansion of Z and J. It would be interesting to reformulate these relations in terms of quadratic propagators, for instance on the basis of the conjectures (5.26) to (5.30). BCJ relations involving quadratic propagators have been discussed in [175–178], whereas quadratic-propagator formulations of one-loop KLT relations are uncharted territory at the time of writing.

5.3 α'-expansions due to individual homology invariants

We shall now illustrate the conjectures of the previous subsection by spelling out various examples of α'-expansions in terms of quadratic propagators. At four points, the open-string expressions (4.7), (4.12) and closed-string expressions (4.20), (4.21) of this type can be attributed to a single trivial homology invariant $E_{1|2,3,4}=1$. That is why in most of our nontrivial examples of (5.29) and (5.30), the quantities E_P, E_Q are specialized to the five-point homology invariants in (5.20) and (5.21).

5.3.1 Shorthands

It will be convenient to introduce shorthand notations for the contributions of homology invariants and their bilinears to open- and closed-string one-loop matrix elements (3.1) and (3.2), respectively. For open strings, the single-trace objects $A^{\text{1-loop}}_{\text{eff}}(1,2,\ldots,n)$ are built from dual pairings of cycles $a_1a_2\ldots a_n \cong a_2a_3\ldots a_na_1$ and a single homology invariant

$$[a_1a_2\ldots a_n\,|\,E_P\rangle = \alpha'^{n-1}\int\frac{\mathrm{d}^D\ell}{\ell^2}\int_{a_1a_2\ldots a_n}\mathrm{d}\mu^{\text{open}}_{n+2}\,\frac{E_P(\sigma_j,\ell,\tau\to i\infty)}{(2\pi i)^{n-4}\sigma_1\sigma_2\ldots\sigma_n}\,, \tag{5.31}$$

see (5.28) for the definition of the α'-dependent measures $\mathrm{d}\mu_{n+2}^{\text{open}}, \mathrm{d}\mu_{n+2}^{\text{closed}}$. One-loop matrix elements of closed strings in turn are built from sphere integrals describing the pairing of meromorphic and antimeromorphic homology invariants

$$\langle E_Q \,|\, E_P\rangle = \left(\frac{\alpha'}{\pi}\right)^{n-1} \int \frac{\mathrm{d}^D\ell}{\ell^2} \int \mathrm{d}\mu_{n+2}^{\text{closed}} \, \frac{E_P(\sigma_j,\ell,\tau\to i\infty)\,\overline{E_Q(\sigma_j,\ell,\tau\to i\infty)}}{(2\pi)^{2n-8}|\sigma_1\sigma_2\ldots\sigma_n|^2}\,. \tag{5.32}$$

By comparison with (5.29) and (5.30), the conjectures in the previous section state that the α'-expansions of any such $[a_1a_2\ldots a_n|E_P\rangle$ and $\langle E_Q|E_P\rangle$ are expressible in terms of quadratic propagators. By (5.9) and (5.22), the five-point examples below can be assembled to yield the one-loop matrix elements in section 4 via

$$\begin{aligned}A_{\text{eff}}^{\text{1-loop}}(1,2,3,4,5) &= [12345\,|\,E^{\mu}_{1|2,3,4,5}\rangle t_{+\mu}(1,2,3,4,5) \\ &\quad + [12345\,|\,E_{1|23,4,5}\rangle t_8(23,1,4,5) + [12345\,|\,E_{1|24,3,5}\rangle t_8(24,1,3,5) \\ &\quad + [12345\,|\,E_{1|25,3,4}\rangle t_8(25,1,3,4) + [12345\,|\,E_{1|34,2,5}\rangle t_8(34,1,2,5) \\ &\quad + [12345\,|\,E_{1|35,2,4}\rangle t_8(35,1,2,4) + [12345\,|\,E_{1|45,2,3}\rangle t_8(45,1,2,3)\,,\end{aligned} \tag{5.33}$$

as well as (with $\bar{t}_{\pm\nu} \to \bar{t}_{+\nu}$ for type-IIB amplitudes and $\bar{t}_{\pm\nu} \to \bar{t}_{-\nu}$ for those of type IIA)

$$\begin{aligned}M_{5,\text{eff}}^{\text{1-loop}} &= \bar{t}_{\pm\nu}(1,2,3,4,5)\langle E^{\nu}_{1|2,3,4,5}\,|\,E^{\mu}_{1|2,3,4,5}\rangle t_{+\mu}(1,2,3,4,5) \\ &+ \Big[\bar{t}_8(23,1,4,5)\Big(\langle E_{1|23,4,5}\,|\,E^{\mu}_{1|2,3,4,5}\rangle t_{+\mu}(1,2,3,4,5) \\ &\quad + \langle E_{1|23,4,5}\,|\,E_{1|23,4,5}\rangle t_8(23,1,4,5) + \langle E_{1|23,4,5}\,|\,E_{1|24,3,5}\rangle t_8(24,1,3,5) \\ &\quad + \langle E_{1|23,4,5}\,|\,E_{1|25,3,4}\rangle t_8(25,1,3,4) + \langle E_{1|23,4,5}\,|\,E_{1|34,2,5}\rangle t_8(34,1,2,5) \\ &\quad + \langle E_{1|23,4,5}\,|\,E_{1|35,2,4}\rangle t_8(35,1,2,4) + \langle E_{1|23,4,5}\,|\,E_{1|45,2,3}\rangle t_8(45,1,2,3)\Big) \\ &\quad + \bar{t}_{\pm\nu}(1,2,3,4,5)\langle E^{\nu}_{1|2,3,4,5}\,|\,E_{1|23,4,5}\rangle t_8(23,1,4,5) + (2,3|2,3,4,5)\Big]\,.\end{aligned} \tag{5.34}$$

At four points, in turn, the one-loop matrix elements (4.6) and (4.18) both result from a single pairing of the form (5.31) and (5.32) with trivial homology invariants $E_P, E_Q \to 1$,

$$\begin{aligned}A_{\text{eff}}^{\text{1-loop}}(1,2,3,4) &= t_8(f_1,f_2,f_3,f_4)\,[1234\,|\,E{=}1\rangle\,, \\ M_{4,\text{eff}}^{\text{1-loop}} &= |t_8(f_1,f_2,f_3,f_4)|^2\,\langle E{=}1\,|\,E{=}1\rangle\,.\end{aligned} \tag{5.35}$$

The results in sections 4.1 and 4.2 are then equivalent to the following leading orders

$$\begin{aligned}[1234\,|\,E{=}1\rangle &= \int \frac{\mathrm{d}^D\ell}{\ell^2} \lim_{k_\pm\to\pm\ell} \sum_{\gamma\in\text{cyc}(1,2,3,4)} \sum_{\rho\in S_4} Z(+,\gamma(1,2,3,4),-|+,\rho(1,2,3,4),-) \\ &= \int \mathrm{d}^D\ell\,\bigg\{\frac{1}{\ell^2(\ell{+}k_1)^2(\ell{+}k_{12})^2(\ell{+}k_{123})^2} \\ &\qquad - \alpha'^2\zeta_2\bigg[\frac{s_{12}}{\ell^2(\ell{+}k_1)^2(\ell{+}k_{12})^2} + \text{cyc}_\ell(1,2,3,4)\bigg] + \mathcal{O}(\alpha'^3)\bigg\}\,,\end{aligned} \tag{5.36}$$

$$
\begin{aligned}
\langle E{=}1\,|\,E{=}1\rangle &= \int \frac{\mathrm{d}^D\ell}{\ell^2} \lim_{k_\pm\to\pm\ell} \sum_{\gamma,\rho\in S_4} J(+,\gamma(1,2,3,4),-|+,\rho(1,2,3,4),-) \\
&= \int \mathrm{d}^D\ell \,\bigg\{ \bigg[\frac{1}{\ell^2(\ell{+}k_1)^2(\ell{+}k_{12})^2(\ell{+}k_{123})^2} + \mathrm{perm}(2,3,4)\bigg] \\
&\qquad + \alpha'^3\zeta_3 \bigg[\frac{s_{12}^2}{\ell^2(\ell{+}k_1)^2(\ell{+}k_{12})^2} + \mathrm{perm}(1,2,3,4)\bigg] + \mathcal{O}(\alpha'^5)\bigg\}\,,
\end{aligned}
\tag{5.37}
$$

see (4.7) and (4.20) for a further subleading order of both.

5.3.2 Five-point open-string examples

With the shorthands $\ell_{12\ldots j} = \ell{+}k_{12\ldots j}$ of (4.36) for the momenta in the propagators of the planar $(1,2,3,4,5)$-ordering, the cyclically inequivalent α'-expansions relevant to (5.33) are

$$
\begin{aligned}
[12345\,|\,E_{1|23,4,5}\rangle &= \int \mathrm{d}^D\ell \,\bigg\{ \frac{-1}{2\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2\ell_{1234}^2} - \frac{1}{s_{12}\ell^2\ell_{12}^2\ell_{123}^2\ell_{1234}^2} - \frac{1}{s_{23}\ell^2\ell_1^2\ell_{123}^2\ell_{1234}^2} \\
&\quad + \alpha'^2\zeta_2 \bigg[\frac{s_{15}}{2\ell_1^2\ell_{12}^2\ell_{123}^2\ell_{1234}^2} + \frac{s_{34}}{2\ell^2\ell_1^2\ell_{12}^2\ell_{1234}^2} + \frac{s_{45}}{2\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2} - \frac{s_{12}{+}2s_{\ell,2}}{2\ell^2\ell_{12}^2\ell_{123}^2\ell_{1234}^2} \\
&\quad - \frac{s_{23}{+}2s_{13}{+}2s_{\ell,3}}{2\ell^2\ell_1^2\ell_{123}^2\ell_{1234}^2} + \frac{s_{45}}{s_{23}\ell^2\ell_1^2\ell_{123}^2} + \frac{s_{45}}{s_{12}\ell^2\ell_{12}^2\ell_{123}^2} + \frac{s_{15}}{s_{23}\ell^2\ell_1^2\ell_{1234}^2} + \frac{s_{15}}{s_{23}\ell_1^2\ell_{123}^2\ell_{1234}^2} \\
&\quad + \frac{s_{34}}{s_{12}\ell^2\ell_{12}^2\ell_{1234}^2} + \bigg(\frac{1}{s_{12}} + \frac{1}{s_{23}}\bigg)\frac{s_{45}}{\ell^2\ell_{123}^2\ell_{1234}^2} + \bigg(\frac{s_{34}}{s_{12}} - 1\bigg)\frac{1}{\ell_{12}^2\ell_{123}^2\ell_{1234}^2}\bigg] + \mathcal{O}(\alpha'^3)\bigg\}
\end{aligned}
\tag{5.38}
$$

$$
\begin{aligned}
[12345\,|\,E_{1|24,3,5}\rangle &= \int \mathrm{d}^D\ell \,\bigg\{ \frac{-1}{2\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2\ell_{1234}^2} - \frac{1}{s_{12}\ell^2\ell_{12}^2\ell_{123}^2\ell_{1234}^2} \\
&\quad + \alpha'^2\zeta_2 \bigg[\frac{s_{15}}{2\ell_1^2\ell_{12}^2\ell_{123}^2\ell_{1234}^2} + \frac{s_{23}}{2\ell^2\ell_1^2\ell_{123}^2\ell_{1234}^2} + \frac{s_{34}}{2\ell^2\ell_1^2\ell_{12}^2\ell_{1234}^2} + \frac{s_{45}}{2\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2} \\
&\quad - \frac{s_{12}{+}2s_{\ell,2}}{2\ell^2\ell_{12}^2\ell_{123}^2\ell_{1234}^2} - \frac{1}{\ell^2\ell_1^2\ell_{1234}^2} - \frac{1}{\ell_1^2\ell_{12}^2\ell_{123}^2} + \frac{s_{45}}{s_{12}\ell^2\ell_{12}^2\ell_{123}^2} + \frac{s_{45}}{s_{12}\ell^2\ell_{123}^2\ell_{1234}^2} \\
&\quad + \frac{s_{34}}{s_{12}\ell^2\ell_{12}^2\ell_{1234}^2} + \bigg(\frac{s_{34}}{s_{12}} - 1\bigg)\frac{1}{\ell_{12}^2\ell_{123}^2\ell_{1234}^2}\bigg] + \mathcal{O}(\alpha'^3)\bigg\}
\end{aligned}
\tag{5.39}
$$

$$
\begin{aligned}
{*}[12345\,|\,E^\mu_{1|2,3,4,5}\rangle &= \int \mathrm{d}^D\ell \,\bigg\{ \frac{\ell^\mu + \frac{1}{2}k_1^\mu}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2\ell_{1234}^2} + \frac{k_5^\mu}{s_{15}\ell_1^2\ell_{12}^2\ell_{123}^2\ell_{1234}^2} - \frac{k_2^\mu}{s_{12}\ell^2\ell_{12}^2\ell_{123}^2\ell_{1234}^2} \\
&\quad + \alpha'^2\zeta_2 \bigg[-\bigg(\ell^\mu + \frac{1}{2}k_1^\mu\bigg)\bigg(\frac{s_{15}}{\ell_1^2\ell_{12}^2\ell_{123}^2\ell_{1234}^2} + \frac{s_{23}}{\ell^2\ell_1^2\ell_{123}^2\ell_{1234}^2} + \frac{s_{34}}{\ell^2\ell_1^2\ell_{12}^2\ell_{1234}^2} + \frac{s_{45}}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2}\bigg) \\
&\quad - \frac{(\ell^\mu + \frac{1}{2}k_1^\mu + k_2^\mu)s_{12}}{\ell^2\ell_{12}^2\ell_{123}^2\ell_{1234}^2} - \frac{k_2^\mu s_{\ell,2}}{\ell^2\ell_{12}^2\ell_{123}^2\ell_{1234}^2} + \frac{k_5^\mu s_{\ell,1}}{\ell_1^2\ell_{12}^2\ell_{123}^2\ell_{1234}^2} - \frac{k_3^\mu}{\ell^2\ell_{123}^2\ell_{1234}^2} + \frac{k_4^\mu}{\ell_1^2\ell_{12}^2\ell_{123}^2} \\
&\quad + \frac{k_2^\mu s_{45}}{s_{12}\ell^2\ell_{12}^2\ell_{123}^2} + \frac{k_2^\mu s_{45}}{s_{12}\ell^2\ell_{123}^2\ell_{1234}^2} + \frac{k_2^\mu s_{34}}{s_{12}\ell^2\ell_{12}^2\ell_{1234}^2} - \frac{k_5^\mu s_{34}}{s_{15}\ell_1^2\ell_{12}^2\ell_{1234}^2} - \frac{k_5^\mu s_{34}}{s_{15}\ell_{12}^2\ell_{123}^2\ell_{1234}^2} \\
&\quad - \frac{k_5^\mu s_{23}}{s_{15}\ell_1^2\ell_{123}^2\ell_{1234}^2} + \bigg(1 - \frac{s_{23}}{s_{15}}\bigg)\frac{k_5^\mu}{\ell_1^2\ell_{12}^2\ell_{123}^2} + \bigg(\frac{s_{34}}{s_{12}} - 1\bigg)\frac{k_2^\mu}{\ell_{12}^2\ell_{123}^2\ell_{1234}^2}\bigg] + \mathcal{O}(\alpha'^3)\bigg\}.
\end{aligned}
\tag{5.40}
$$

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By inserting into (5.33) and using the kinematic identity (4.38), we reproduce the SYM integrand (4.37) in the $\alpha' \to 0$ limit and the $\alpha'^2\zeta_2 F^4$- numerators (4.41) from the subleading order. Note that the expressions in (5.38) to (5.40) are compatible with the Koba-Nielsen dervatives

$$\frac{1}{\alpha'}\partial_{z_2}\log\mathcal{J}_5(\ell,\tau)=(k_2)_\mu E^\mu_{1|2,3,4,5}+k_2\cdot k_3 E_{1|23,4,5}+k_2\cdot k_4 E_{1|24,3,5}+k_2\cdot k_5 E_{1|25,3,4} \tag{5.41}$$

following from (5.2) which drop out from one-loop string amplitudes. At the level of one-loop matrix elements and their building blocks (5.31) and (5.32), the total derivatives (5.41) clearly lead to

$$\begin{aligned}0 = (k_2)_\mu [a_1a_2\ldots a_5\,|\,E^\mu_{1|2,3,4,5}\rangle + k_2\cdot k_3\,[a_1a_2\ldots a_5\,|\,E_{1|23,4,5}\rangle \\ + k_2\cdot k_4\,[a_1a_2\ldots a_5\,|\,E_{1|24,3,5}\rangle + k_2\cdot k_5\,[a_1a_2\ldots a_5\,|\,E_{1|25,3,4}\rangle\end{aligned} \tag{5.42}$$

and analogous integration-by-parts identities with $\langle E_Q|$ in the place of $[a_1a_2\ldots a_5|$.

5.3.3 Five-point closed-string examples

The gravitational five-point one-loop matrix element (5.34) involves five pairings (5.32) of homology invariants that are inequivalent under permutations of $2,3,4,5$:

- $\langle E_{1|23,4,5}\,|\,E_{1|23,4,5}\rangle$
- $\langle E_{1|23,4,5}\,|\,E_{1|24,3,5}\rangle$
- $\langle E_{1|23,4,5}\,|\,E_{1|45,2,3}\rangle$
- $\langle E^\mu_{1|2,3,4,5}\,|\,E_{1|23,4,5}\rangle$
- $\langle E^\mu_{1|2,3,4,5}\,|\,E^\nu_{1|2,3,4,5}\rangle$

Already the leading orders of their α'-expansion are considerably longer than their open-string counterparts and are therefore relegated to the supplementary material. For instance, the field-theory limit

$$\begin{aligned}\langle E_{1|23,4,5}\,|\,E_{1|24,3,5}\rangle = \int \mathrm{d}^D\ell\,\Bigg\{\sum_{\rho\in S_3}\frac{1}{s_{12}\ell^2\ell_{12}^2\ell^2_{12\rho(3)}\ell^2_{12\rho(34)}} \\ + \sum_{\rho\in S_4}\frac{\mathrm{sgn}^\rho_{23}\mathrm{sgn}^\rho_{24}}{4\ell^2\ell_1^2\ell^2_{1\rho(2)}\ell^2_{1\rho(23)}\ell_{1\rho(234)}} + \mathcal{O}(\alpha'^3)\Bigg\}\end{aligned} \tag{5.43}$$

involves 6 boxes and 24 pentagons. The signs of the pentagons are governed by

$$\mathrm{sgn}^\rho_{ij} = \begin{cases} +1: & i \text{ is on the right of } j \text{ in } \rho(2,3,4,5) \\ -1: & i \text{ is on the left of } j \text{ in } \rho(2,3,4,5)\end{cases} \tag{5.44}$$

introduced by the interplay of the nodal Green function (4.31) with Parke-Taylor factors, see section 3 of [21] for details. Note that the integration-by-parts identity (5.42) of the open string straightforwardly carries over to

$$\begin{aligned}0 = (k_2)_\mu\,\langle E_P\,|\,E^\mu_{1|2,3,4,5}\rangle + k_2\cdot k_3\,\langle E_P\,|\,E_{1|23,4,5}\rangle \\ + k_2\cdot k_4\,\langle E_P\,|\,E_{1|24,3,5}\rangle + k_2\cdot k_5\,\langle E_P\,|\,E_{1|25,3,4}\rangle\,.\end{aligned} \tag{5.45}$$

5.4 All-multiplicity formulae

While a detailed study of one-loop matrix elements at $n \geq 6$ points is relegated to the future, we shall now present all-multiplicity conjectures for scalar integrals over symmetrized Parke-Taylor factors. As an α'-uplift of the n-gon formulae in the ambitwistor setup [13, 15, 16] (with $\ell_{12\ldots j} = \ell + k_{12\ldots j}$),

$$\int \frac{\mathrm{d}^D \ell}{\ell^2} \int \frac{\mathrm{d}\mu_{n+2}^{\rm tree}}{\sigma_2\sigma_3\ldots\sigma_n} \sum_{\gamma\in {\rm cyc}(1,2,\ldots,n)} \mathrm{PT}(+,\gamma(1,2,\ldots,n),-) = \int \frac{(-1)^n \, \mathrm{d}^D\ell}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2\ldots\ell_{12\ldots n-1}^2}\,,$$
$$\int \frac{\mathrm{d}^D \ell}{\ell^2} \int \frac{\mathrm{d}\mu_{n+2}^{\rm tree}}{|\sigma_2\sigma_3\ldots\sigma_n|^2} = \int \frac{\mathrm{d}^D\ell}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2\ldots\ell_{12\ldots n-1}^2} + \mathrm{perm}(2,3,\ldots,n)\,, \tag{5.46}$$

we shall investigate the first string corrections to the symmetrized integrals

$$Z^{\text{1-loop}}_{\rm symm}(1,2,\ldots,n) = [12\ldots n\,|\,(2\pi i)^{n-4}\rangle = \alpha'^{n-1}\int\frac{\mathrm{d}^D\ell}{\ell^2}\int_{12\ldots n}\frac{\mathrm{d}\mu^{\rm open}_{n+2}}{\sigma_2\sigma_3\ldots\sigma_n} \tag{5.47}$$
$$= \int\frac{\mathrm{d}^D\ell}{\ell^2}\lim_{k_\pm\to\pm\ell}\sum_{\substack{\gamma\in{\rm cyc}(1,2,\ldots,n)\\ \rho\in S_n}} Z(+,\gamma(1,2,\ldots,n),-|+,\rho(1,2,\ldots,n),-)\,,$$

$$J^{\text{1-loop}}_{n,{\rm symm}} = \langle(2\pi i)^{n-4}\,|\,(2\pi i)^{n-4}\rangle = \left(\frac{\alpha'}{\pi}\right)^{n-1}\int\frac{\mathrm{d}^D\ell}{\ell^2}\int\frac{\mathrm{d}\mu^{\rm closed}_{n+2}}{|\sigma_2\sigma_3\ldots\sigma_n|^2} \tag{5.48}$$
$$= \int\frac{\mathrm{d}^D\ell}{\ell^2}\lim_{k_\pm\to\pm\ell}\sum_{\gamma,\rho\in S_n} J(+,\gamma(1,2,\ldots,n),-|+,\rho(1,2,\ldots,n),-)\,,$$

see (5.28) for the definition of the measures $\mathrm{d}\mu^{\rm open}_{n+2}$ and $\mathrm{d}\mu^{\rm closed}_{n+2}$. These all-multiplicity families of integrals are engineered to reproduce the n-gon integrals (5.46) in their field-theory limits. The four- and five-point instances of the symmetrized open-string integrals,

$$Z^{\text{1-loop}}_{\rm symm}(1,2,3,4) = \int \mathrm{d}^D\ell\,\bigg\{\frac{1}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2} - \alpha'^2\zeta_2\bigg[\frac{s_{12}}{\ell^2\ell_{12}^2\ell_{123}^2} + {\rm cyc}_\ell(1,2,3,4)\bigg] \tag{5.49}$$
$$+\frac{\alpha'^3\zeta_3}{2}\bigg[\frac{s_{12}^2+s_{12}(s_{1,\ell}-s_{2,\ell})}{\ell^2\ell_{12}^2\ell_{123}^2} + \frac{s_{12}-2s_{13}+s_{23}}{\ell^2\ell_{123}^2} + {\rm cyc}_\ell(1,2,3,4)\bigg] + \mathcal{O}(\alpha'^4)\bigg\}\,,$$
$$Z^{\text{1-loop}}_{\rm symm}(1,2,3,4,5) = \int \mathrm{d}^D\ell\,\bigg\{\frac{1}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2\ell_{1234}^2} - \alpha'^2\zeta_2\bigg[\frac{s_{12}}{\ell^2\ell_{12}^2\ell_{123}^2\ell_{1234}^2} + {\rm cyc}(1,2,3,4,5)\bigg]$$
$$+\frac{\alpha'^3\zeta_3}{2}\bigg[\frac{s_{12}^2+s_{12}(s_{1,\ell}-s_{2,\ell})}{\ell^2\ell_{12}^2\ell_{123}^2\ell_{1234}^2} + \frac{s_{12}-2s_{13}+s_{23}}{\ell^2\ell_{123}^2\ell_{1234}^2} + {\rm cyc}_\ell(1,2,3,4,5)\bigg] + \mathcal{O}(\alpha'^4)\bigg\}\,,$$

augment the n-gon integral in the prescribed color ordering by

- a cyclic orbit of $(n{-}1)$ gons at the order of the $\alpha'^2\zeta_2F^4$ and $\alpha'^3\zeta_3(D^2F^4+F^5)$ interactions
- by additional one-mass $(n{-}2)$ gons at the $\alpha'^3\zeta_3(D^2F^4+F^5)$-order.

The notation $+{\rm cyc}_\ell(1,2,\ldots,n)$ instructs to shift loop momenta in the cyclic images of the numerators $\sim s_{1,\ell}{-}s_{2,\ell}$ as explained below (4.7).

The same patterns of $(n-1)$ and $(n-2)$ gons are verified to arise at six points which motivates the n-point conjecture,

$$Z^{\text{1-loop}}_{\text{symm}}(1,2,\ldots,n) = \int \mathrm{d}^D\ell \left\{ \frac{1}{\ell^2\ell_1^2\ell_{12}^2\ldots\ell_{12\ldots n-1}^2} - \left[\frac{\alpha'^2\zeta_2 s_{12}}{\ell^2\ell_{12}^2\ldots\ell_{12\ldots n-1}^2} + \text{cyc}_\ell(1,2,\ldots,n)\right] \right.$$
$$\left. + \frac{\alpha'^3\zeta_3}{2}\left[\frac{s_{12}^2+s_{12}(s_{1,\ell}-s_{2,\ell})}{\ell^2\ell_{12}^2\ell_{123}^2\ldots\ell_{12\ldots n-1}^2} + \frac{s_{12}-2s_{13}+s_{23}}{\ell^2\ell_{123}^2\ldots\ell_{12\ldots n-1}^2} + \text{cyc}_\ell(1,2,\ldots,n)\right] + \mathcal{O}(\alpha'^4)\right\}. \tag{5.50}$$

Similarly, the closed-string results at four and five points,

$$J^{\text{1-loop}}_{4,\text{symm}} = \int \mathrm{d}^D\ell \left\{ \left[\frac{1}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2} + \text{perm}(2,3,4)\right] \right.$$
$$\left. + \alpha'^3\zeta_3\left[\frac{s_{12}^2}{\ell^2\ell_{12}^2\ell_{123}^2} + \text{perm}(1,2,3,4)\right] + \mathcal{O}(\alpha'^5)\right\}, \tag{5.51}$$
$$J^{\text{1-loop}}_{5,\text{symm}} = \int \mathrm{d}^D\ell \left\{ \left[\frac{1}{\ell^2\ell_1^2\ell_{12}^2\ell_{123}^2\ell_{1234}^2} + \text{perm}(2,3,4,5)\right] \right.$$
$$\left. + \alpha'^3\zeta_3\left[\frac{s_{12}^2}{\ell^2\ell_{12}^2\ell_{123}^2\ell_{1234}^2} + \text{perm}(1,2,3,4,5)\right] + \mathcal{O}(\alpha'^5)\right\},$$

involve a characteristic permutation sum over scalar $(n-1)$-gons at the order of the $\alpha'^3\zeta_3 R^4$ interaction. We propose the n-point generalization,

$$J^{\text{1-loop}}_{n,\text{symm}} = \int \mathrm{d}^D\ell \left\{ \left[\frac{1}{\ell^2\ell_1^2\ell_{12}^2\ldots\ell_{12\ldots n-1}^2} + \text{perm}(2,3,\ldots,n)\right] \right.$$
$$\left. + \alpha'^3\zeta_3\left[\frac{s_{12}^2}{\ell^2\ell_{12}^2\ldots\ell_{12\ldots n-1}^2} + \text{perm}(1,2,\ldots,n)\right] + \mathcal{O}(\alpha'^5)\right\}, \tag{5.52}$$

which has also been tested at six points. In fact, these expressions for $J^{\text{1-loop}}_{n,\text{symm}}$ follow from our conjecture (5.50) for $Z^{\text{1-loop}}_{\text{symm}}(1,2,\ldots,n)$ since the definitions (5.47) and (5.48) together with the tree-level relation $J = \text{sv}\, Z$ imply that

$$J^{\text{1-loop}}_{n,\text{symm}} = \text{sv}\, Z^{\text{1-loop}}_{\text{symm}}(1,2,\ldots,n) + \text{perm}(2,3,\ldots,n)\,. \tag{5.53}$$

The permutation sum removes the antisymmetric part $s_{1,\ell}-s_{2,\ell}$ from the $(n-1)$-gon numerators of sv Z and furthermore cancels the $(n-2)$-gons since $s_{12}-2s_{13}+s_{23}+\text{perm}(1,2,3) = 0$.

6 UV divergences versus degenerations of one-loop string amplitudes

In this section, we support our proposal for one-loop matrix elements with insertions of $D^{2k}F^n$ and $D^{2k}R^n$ by comparing their UV divergences with expectations from string theory. More specifically, the behaviour of genus-one string integrands at the boundary $\tau \to i\infty$ of moduli space encodes the UV divergences of one-loop matrix elements computed from low-energy effective actions [61–63]. We will determine the UV divergences of the one-loop matrix elements presented in earlier sections and verify their compatibility with degenerate genus-one integrals of the corresponding string amplitude.

6.1 Basics of degenerating one-loop closed-string amplitudes

In relating one-loop matrix elements to string amplitudes in section 5.1 and appendix B, we have first performed the integration over τ and then z_j within a certain approximation while leaving the loop momentum ℓ unintegrated. The discussion of the present section will follow the more conventional procedure [165, 174] to first integrate over ℓ which manifests modular invariance of closed-string genus-one amplitudes.

Based on properties of modular graph forms [102, 103], we will investigate the integrals over ℓ followed by z_j in the expansion around the cusp $\tau \to i\infty$. The dominant behaviour of these integrals at the cusp is governed by Laurent polynomials in $\pi \operatorname{Im} \tau$ with $\mathbb{Q}$-linear combinations of MZVs (conjecturally single-valued MZVs [102, 108]) in their coefficients. Each term is in one-to-one correspondence with the UV divergences of effective field theories in various spacetime dimensions [62, 63].

6.1.1 Modular graph forms and their degeneration

After performing the Gaussian loop integral, the closed-string four-point amplitude (5.1) is proportional to[22]

$$M^{\text{1-loop}}_{4,\text{closed}} \sim |t_8(f_1,f_2,f_3,f_4)|^2 \int_{\mathfrak{F}} \frac{\mathrm{d}^2\tau}{(\operatorname{Im}\tau)^2} \int_{\mathfrak{T}^3_\tau} \left(\prod_{j=2}^4 \frac{\mathrm{d}^2 z_j}{\operatorname{Im}\tau} \right) \exp\left(-\alpha' \sum_{1\leq i<j}^4 s_{ij} \mathcal{G}(z_{ij},\tau) \right) \tag{6.1}$$

with modular invariant measures $\frac{\mathrm{d}^2\tau}{(\operatorname{Im}\tau)^2}$ and $\frac{\mathrm{d}^2 z_j}{\operatorname{Im}\tau}$. The modular invariant closed-string Green function is given in terms of the θ_1 function (5.3), the Dedekind eta function $\eta(\tau) = q^{1/24} \prod_{n=1}^\infty (1-q^n)$ and the comoving coordinates $u, v \in \mathbb{R}$ for $z = u\tau + v$:

$$\mathcal{G}(z,\tau) = -\log\left|\frac{\theta_1(z,\tau)}{\eta(\tau)}\right|^2 + \frac{2\pi(\operatorname{Im} z)^2}{\operatorname{Im}\tau} = \frac{\operatorname{Im}\tau}{\pi} \sum_{\substack{m,n\in\mathbb{Z} \\ (m,n)\neq(0,0)}} \frac{e^{2\pi i(nu-mv)}}{|m\tau+n|^2} \,. \tag{6.2}$$

The key information on the low-energy effective action can be obtained by Taylor-expanding the integrand of (6.1) in α': upon integration over z_j, this leads to a power series in $\alpha' s_{ij}$ whose coefficients are the modular invariant functions of τ that result from integrating monomials in Green functions $\mathcal{G}(z_{ij},\tau)$ over their first arguments. These z_j-integrations can be systematically performed using the lattice-sum representation in (6.2) and the resulting nested lattice sums are known as modular graph forms [102, 103]. The study of modular graph forms became a vibrant research topic at the interface of string theory, algebraic geometry and number theory, see e.g. [179, 180] for a conjectural mathematical framework and the PhD thesis [181] for a recent overview from a physics perspective.

In an expansion of modular graph forms around the cusp $\tau \to i\infty$, the dominant contributions are Laurent polynomials in

$$y = \pi \operatorname{Im}\tau \tag{6.3}$$

[22] By different conventions for the definition (2.11) of s_{ij} in this work, the expressions in (6.1) and (6.4) can be obtained from results of [68, 102] by setting $s_{ij} \to -\alpha' s_{ij}$ in the references.

up to terms that are exponentially suppressed with q or $\bar{q}$, i.e. e^{-2y} in both cases. For the leading orders of the τ-integrand in (6.1), we have the degeneration

$$\begin{aligned} m^{\text{1-loop}}_{4,\text{closed}}(\tau) &= (4\alpha')^5 (\operatorname{Im}\tau)^2 \int_{\mathfrak{T}^3_\tau} \prod_{j=2}^4 \mathrm{d}^2 z_j \int \mathrm{d}^{10}\ell \, |\mathcal{J}_4(z_j,\ell,\tau)|^2 \Big|_{q^0\bar{q}^0} \\ &= \int_{\mathfrak{T}^3_\tau} \left(\prod_{j=2}^4 \frac{\mathrm{d}^2 z_j}{\operatorname{Im}\tau} \right) \exp\left(-\alpha' \sum_{1\le i<j}^4 s_{ij} \mathcal{G}(z_{ij},\tau) \right) \Big|_{q^0\bar{q}^0} \\ &= 1 + 2\alpha'^2 (s_{12}^2 - s_{13}s_{23}) \left(\frac{y^2}{45} + \frac{\zeta_3}{y} \right) - \alpha'^3 s_{12}s_{13}s_{23} \left(\frac{2y^3}{189} + \zeta_3 + \frac{15\zeta_5}{4y^2} \right) \\ &\quad + \alpha'^4 (s_{12}^2 - s_{13}s_{23})^2 \left(\frac{16y^4}{14175} + \frac{8y\zeta_3}{45} + \frac{5\zeta_5}{3y} + \frac{\zeta_3^2}{y^2} + \frac{\zeta_7}{y^3} \right) \\ &\quad + \alpha'^5 s_{12}s_{13}s_{23}(s_{13}s_{23} - s_{12}^2) \left(\frac{8y^5}{22275} + \frac{2y^2\zeta_3}{21} + \frac{58\zeta_5}{45} + \frac{2\zeta_3^2}{y} + \frac{49\zeta_7}{8y^2} + \frac{\zeta_3\zeta_5}{2y^3} + \frac{63\zeta_9}{32y^4} \right) \\ &\quad + \mathcal{O}(\alpha'^6) \,, \end{aligned} \tag{6.4}$$

see for instance [107–109, 111, 112] for the systematics at higher order. The notation $|_{q^0\bar{q}^0}$ in the first two lines instructs to only keep the $m=n=0$ terms in the expansion $\sim (\operatorname{Im}\tau)^a q^m \bar{q}^n$ around the cusp. As will be detailed in section 6.5 and appendix C below, each term in (6.4) accurately matches the UV divergences of the closed-string one-loop matrix elements in section 4.2, where the number of spacetime dimensions correlates with the power of y.

6.1.2 Analytic versus non-analytic sector of one-loop amplitudes

The step of performing the α'-expansion of (6.1) at the level of the τ-integrand is oblivious to the intricate branch cuts of the closed-string one-loop amplitudes [65]. Among other things, the power series in (6.4) does not take the discontinuities of the form $\log(\alpha' s_{ij})$ into account which can for instance be determined from the Feynman integrals in section 4 (see section 7 for four-point examples). Instead, (6.4) captures the rational functions in s_{ij} obtained as the residues of the UV divergences, and the $\mathcal{O}(e^{-2y})$ terms which are accurately known to the order of $(\alpha' s_{ij})^6$ can be used to infer the one-loop effective action. The interplay between the power-series behaved terms (the "analytic sector") of closed-string one-loop amplitudes and the branch cuts (the "non-analytic sector") has for instance been discussed in [64, 67, 68]. In section 7, we will spell out the discontinuities of one-loop four-point matrix elements which determine the non-analytic sector of closed- and open-string amplitudes.

6.1.3 Degenerate modular graph forms at five points

The low-energy expansion of the closed-string five-point function can again be organized into modular graph forms [182] after integrating $|\mathcal{K}_5\mathcal{J}_5|^2$ over the loop momenta. Since the chiral five-point correlator $\mathcal{K}_5$ in (5.10) is a sum of seven homology invariants E_P, it will be convenient to introduce a shorthand for the five-point analogues of the four-point building block in (6.4),

$$\langle\!\langle E_Q|E_P\rangle\!\rangle_\tau = 2^{10}(\alpha')^6(\operatorname{Im}\tau)^2\int_{\mathfrak{T}^4_\tau}\prod_{j=2}^{5}\mathrm{d}^2z_j\int \mathrm{d}^{10}\ell\, E_P(z_j,\ell,\tau)\,\overline{E}_Q(z_j,\ell,\tau)\,|\mathcal{J}_5(z_j,\ell,\tau)|^2\Big|_{q^0\bar q^0}\,. \tag{6.5}$$

Similar to section 5.3.3, the 7×7 bilinears $\overline{E}_Q E_P$ arising from $|\mathcal{K}_5|^2$ can be organized into five permutation inequivalent cases

$$\begin{aligned}
\langle\!\langle E_{1|23,4,5}|E_{1|23,4,5}\rangle\!\rangle_\tau &= -\frac{1}{s_{12}}-\frac{1}{s_{13}}-\frac{1}{s_{23}}+2\alpha'^2\Big(\frac{y^2}{45}+\frac{\zeta_3}{y}\Big)\Big(\frac{s_{34}s_{35}}{s_{12}}-\frac{s_{45}^2}{s_{12}}+\frac{s_{24}s_{25}}{s_{13}}-\frac{s_{45}^2}{s_{13}}\\
&\qquad +\frac{s_{14}s_{15}}{s_{23}}-\frac{s_{45}^2}{s_{23}}+s_{45}\Big)+\mathcal{O}(\alpha'^3)\,,\\
\langle\!\langle E_{1|23,4,5}|E_{1|24,3,5}\rangle\!\rangle_\tau &= -\frac{1}{s_{12}}+\alpha'^2\Big(\frac{y^2}{45}+\frac{\zeta_3}{y}\Big)\Big(\frac{2(s_{35}s_{45}-s_{34}^2)}{s_{12}}-s_{12}+5s_{34}\Big)+\mathcal{O}(\alpha'^3)\,,\\
\langle\!\langle E_{1|23,4,5}|E_{1|45,2,3}\rangle\!\rangle_\tau &= \alpha'^2\Big(\frac{y^2}{45}+\frac{\zeta_3}{y}\Big)(s_{24}+s_{35}-s_{25}-s_{34})+\mathcal{O}(\alpha'^3)\,,\\
\langle\!\langle E_{1|23,4,5}|E^\mu_{1|2,3,4,5}\rangle\!\rangle_\tau &= -\frac{k_2^\mu}{s_{12}}+\frac{k_3^\mu}{s_{13}}+\alpha'^2\Big(\frac{y^2}{45}+\frac{\zeta_3}{y}\Big)\Big(k_3^\mu\frac{(s_{24}^2+s_{25}^2+s_{45}^2)}{s_{13}}-k_2^\mu\frac{(s_{34}^2+s_{35}^2+s_{45}^2)}{s_{12}}\\
&\qquad +(k_3^\mu-k_2^\mu)s_{45}+k_4^\mu(s_{24}-s_{34})+k_5^\mu(s_{25}-s_{35})\Big)+\mathcal{O}(\alpha'^3)\,,\\
\langle\!\langle E^\nu_{1|2,3,4,5}|E^\mu_{1|2,3,4,5}\rangle\!\rangle_\tau &= -\frac{k_2^\mu k_2^\nu}{s_{12}}-\frac{k_3^\mu k_3^\nu}{s_{13}}-\frac{k_4^\mu k_4^\nu}{s_{14}}-\frac{k_5^\mu k_5^\nu}{s_{15}}\\
&\quad +2\alpha'^2\Big(\frac{y^2}{45}+\frac{\zeta_3}{y}\Big)\Big(k_2^\mu k_2^\nu\Big[\frac{(s_{34}s_{35}+s_{34}s_{45}+s_{35}s_{45})}{s_{12}}-s_{12}\Big]+(2\leftrightarrow 3,4,5)\Big)\\
&\quad +\alpha'^2\Big(\frac{y^2}{45}+\frac{\zeta_3}{y}\Big)\Big[(k_2^\mu k_3^\nu+k_2^\nu k_3^\mu)s_{23}+(2,3|2,3,4,5)\Big]+\mathcal{O}(\alpha'^3)\,.
\end{aligned} \tag{6.6}$$

While the loop integrals in (6.4) and (6.5) have been performed in $D=10$ dimensions, we will later on be interested in UV divergences in arbitrary dimensions. The D-dimensional versions of the closed-string loop integrals follow from inserting a factor of $(2\sqrt{\alpha'\operatorname{Im}\tau})^{10-D}$ due to the modified Gaussian integral and a series in non-integer powers of $q,\bar q$ to accommodate Kaluza-Klein and winding modes of toroidal compactifications [155]. These $q,\bar q$ series do not contribute to the $\tau\to i\infty$ limit relevant for our one-loop matrix elements, see section 5.1.3.

6.2 Basics of degenerating one-loop open-string amplitudes

After integrating out the loop momentum in one-loop open-string amplitudes (5.4), the integrand w.r.t. the modular parameter τ admits similar expansions around the cusp as in the closed-string case: when focusing on the analytic sector in $\alpha' s_{ij}$, the integration over the punctures can be systematically performed in terms of elliptic multiple zeta values [116, 183]. The dominant contributions as $\tau\to i\infty$ are Laurent polynomials in

$$T=\pi\tau \tag{6.7}$$

instead of the $y = \pi\,\mathrm{Im}\,\tau$ in the closed-string degenerations, with $\mathbb{Q}$-linear combinations of MZVs in its coefficients. In the four-point amplitude (5.4), we have [105, 153][23]

$$\begin{aligned}
a^{\text{1-loop}}_{\text{open}}(1,2,3,4;\tau) &= (-2i\alpha')^5\tau^2 \int_{\mathfrak{C}_\tau(1,2,3,4)} \prod_{j=2}^{4} \mathrm{d}z_j \int \mathrm{d}^{10}\ell\, |\mathcal{J}_4(z_j,\ell,\tau)| \Big|_{q^0} \\
&= \int_{0<u_2<u_3<u_4<1} \Big(\prod_{j=2}^{4} \mathrm{d}u_j \Big) \exp\Big(\alpha' \sum_{1\leq i<j}^{4} s_{ij} \big[\log\theta_1(|u_{ij}|\tau,\tau) + i\pi u_{ij}^2 \tau \big] \Big) \Big|_{q^0} \\
&= \frac{1}{6} + \alpha' s_{13}\Big(-\frac{iT}{60} + \frac{i\zeta_2}{2T} + \frac{3\zeta_3}{2T^2} - \frac{3i\zeta_4}{2T^3} \Big) + \alpha'^2 (s_{12}^2 - s_{13}s_{23}) \Big(-\frac{T^2}{540} + \frac{\zeta_2}{9} + \frac{i\zeta_3}{3T} + \frac{\zeta_4}{2T^2} \Big) \\
&\quad + \alpha'^2 (s_{12}^2 + 4 s_{12}s_{23} + s_{23}^2) \Big(\frac{T^2}{3780} - \frac{\zeta_2}{36} + \frac{\zeta_4}{4T^2} - \frac{i\zeta_5}{T^3} - \frac{5\zeta_6}{4T^4} \Big) \\
&\quad + \alpha'^3 s_{12}s_{23}s_{13} \Big(-\frac{iT^3}{4536} + \frac{iT\zeta_2}{45} - \frac{\zeta_3}{12} - \frac{35 i\zeta_4}{24T} - \frac{2\zeta_2\zeta_3}{T^2} + \frac{5\zeta_5}{4T^2} + \frac{17 i\zeta_6}{12T^3} \Big) \\
&\quad + \alpha'^3 s_{13} (s_{12}^2 - s_{13}s_{23}) \Big(\frac{iT^3}{7560} - \frac{iT\zeta_2}{90} + \frac{\zeta_3}{20} + \frac{3i\zeta_4}{4T} + \frac{5\zeta_5}{2T^2} - \frac{\zeta_2\zeta_3}{2T^2} \\
&\qquad\qquad - \frac{29 i\zeta_6}{12T^3} - \frac{3\zeta_7}{T^4} - \frac{3\zeta_3\zeta_4}{2T^4} + \frac{21 i\zeta_8}{4T^5} \Big) + \mathcal{O}(\alpha'^4)\,,
\end{aligned} \tag{6.8}$$

where $z_j = u_j\tau$ and $|_{q^0}$ instructs to only keep the $n=0$ terms in the expansion $\sim \tau^a q^n$ around the cusp. The methods of [105, 116, 154] can also be used to infer the exponentially suppressed terms $\mathcal{O}(e^{2iT})$ at any order in α' in terms of B-cycle elliptic multiple zeta values [183–185]. However, we have only displayed the dominant terms in the expansion of the τ-integrand of string amplitudes around the cusp in (6.8) since this is the regime relevant for one-loop matrix elements and their UV divergences.

6.2.1 Open- versus closed-string Laurent polynomials

A direct comparison between (6.8) and the closed-string counterpart (6.4) can be made upon symmetrization over the four-point color orderings. Mandelstam identities imply that for instance the entire first α'-order as well as the Laurent coefficients $\alpha'^2\zeta_5$ and $\alpha'^3\zeta_7$ drop out from (6.8) upon symmetrization,

$$\begin{aligned}
\sum_{\gamma\in S_3} a^{\text{1-loop}}_{4,\text{open}}(1,\gamma(2,3,4);\tau) &= 1 + \alpha'^2 (s_{12}^2 - s_{13}s_{23}) \Big(-\frac{T^2}{90} + \frac{2\zeta_2}{3} + \frac{2i\zeta_3}{T} + \frac{3\zeta_4}{T^2} \Big) \\
&\quad + \alpha'^3 s_{12}s_{23}s_{13} \Big(-\frac{iT^3}{756} + \frac{2iT\zeta_2}{15} - \frac{\zeta_3}{2} - \frac{35 i\zeta_4}{4T} - \frac{12\zeta_2\zeta_3}{T^2} + \frac{15\zeta_5}{2T^2} + \frac{17 i\zeta_6}{2T^3} \Big) + \mathcal{O}(\alpha'^4)\,.
\end{aligned} \tag{6.9}$$

As firstly pointed out in [105], these symmetrized open-string building blocks can be related to the closed-string degerations (6.4) via

$$m^{\text{1-loop}}_{4,\text{closed}}(\tau) = \mathrm{sv} \sum_{\gamma\in S_3} a^{\text{1-loop}}_{4,\text{open}}(1,\gamma(2,3,4);\tau) \tag{6.10}$$

[23] By the different conventions for the definition of s_{ij}, the following expressions can be obtained from the results of [105, 153] by setting $s_{ij} \to -\alpha' s_{ij}$ in the references.

if the action of the single-valued map (2.27) is extended to

$$\mathrm{sv}\, T = 2iy\,. \tag{6.11}$$

The intimate relation between degenerate elliptic multiple zeta values and modular graph forms has been proven for the two-point analogue of (6.10) [111]. Moreover, a conjectural generalization of (6.10) to individual n-point open-string integration cycles can be found in [154]. As we will see in section 6.5, the relation (6.10) between closed- and open-string Laurent polynomials can be understood from the UV divergences in the relation (4.19) between one-loop matrix elements.

6.2.2 Degenerate elliptic multiple zeta values at five points

At five points, we introduce a shorthand similar to (6.5) for the contributions of individual homology invariants to the τ integrand of (5.4)

$$[\![a_1 \dots a_5 \,|\, E_P \rangle\!\rangle_\tau = -32i(\alpha')^6 \tau^2 \int_{\mathfrak{C}_\tau(a_1,\dots,a_5)} \prod_{j=2}^{5} \mathrm{d}z_j \int \mathrm{d}^{10}\ell\, E_P(z_j,\ell,\tau)\, |\mathcal{J}_5(z_j,\ell,\tau)| \Big|_{q^0}\,. \tag{6.12}$$

For a given integration ordering $a_1 a_2 \dots a_5 \to 12345$ (see (5.5) for the definition of $\mathfrak{C}_\tau$), there are three cyclically inequivalent cases to consider,

$$\begin{aligned}
[\![12345 \,|\, E_{1|23,4,5} \rangle\!\rangle_\tau &= \frac{1}{6s_{12}} + \frac{1}{6s_{23}} + \alpha'\Big(3 - \frac{2s_{14}}{s_{23}} - \frac{2s_{35}}{s_{12}}\Big) \\
&\qquad \times \Big(-\frac{iT}{120} + \frac{i\zeta_2}{4T} + \frac{3\zeta_3}{4T^2} - \frac{3i\zeta_4}{4T^3}\Big) + \mathcal{O}(\alpha'^2)\,,\\
[\![12345 \,|\, E_{1|24,3,5} \rangle\!\rangle_\tau &= \frac{1}{6s_{12}} + \alpha'\Big(1 + \frac{2s_{35}}{s_{12}}\Big)\Big(-\frac{iT}{120} + \frac{i\zeta_2}{4T} + \frac{3\zeta_3}{4T^2} - \frac{3i\zeta_4}{4T^3}\Big) + \mathcal{O}(\alpha'^2)\,,\\
[\![12345 \,|\, E^\mu_{1|2,3,4,5} \rangle\!\rangle_\tau &= \frac{k_2^\mu}{6s_{12}} - \frac{k_5^\mu}{6s_{15}} + \alpha'\Big(k_{24}^\mu - k_{35}^\mu - \frac{2s_{35}}{s_{12}} k_2^\mu + \frac{2s_{24}}{s_{15}} k_5^\mu\Big) \\
&\qquad \times \Big(-\frac{iT}{120} + \frac{i\zeta_2}{4T} + \frac{3\zeta_3}{4T^2} - \frac{3i\zeta_4}{4T^3}\Big) + \mathcal{O}(\alpha'^2)\,.
\end{aligned} \tag{6.13}$$

We have again displayed the ten-dimensional loop integrals in (6.8) and (6.12), and the expressions in different dimensions follow by inserting an extra factor of $\sqrt{-2i\alpha'\tau}^{\,10-D}$ and a series in q to take the Kaluza-Klein modes from a toroidal compactification into account [155]. Again, the q-series of the Kaluza-Klein modes do not contribute to the $\tau \to i\infty$ limit relevant for the one-loop matrix elements under investigation, see section 5.1.3.

6.3 String Laurent polynomials, Schwinger parameters and UV divergences

In the field-theory limit of one-loop string amplitudes, the limit $\alpha' \to 0$ is performed while degenerating the torus or cylinder worldsheet to a worldline via $\mathrm{Im}\,\tau \to \infty$ [122, 155, 166–173]. In this process, Feynman integrals depending on momenta K_j with possibly non-

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vanishing K_j^2 are retained in their Schwinger parametrization

$$\begin{aligned}\int \frac{\mathrm{d}^D\ell}{\pi^{D/2}} \frac{(T_0 + T_1^\mu \ell_\mu + T_2^{\mu\nu}\ell_\mu\ell_\nu + \ldots)}{\ell^2(\ell{+}K_1)^2(\ell{+}K_{12})^2\ldots(\ell{+}K_{12\ldots n-1})^2} &= \int_0^\infty \frac{\mathrm{d}t}{t}\, t^{n-D/2} \\ \times \int\limits_{0<u_2<u_3<\ldots<u_n<1} \mathrm{d}u_2\,\mathrm{d}u_3\ldots\mathrm{d}u_n\, \exp&\left[-t\sum_{1\leq i<j}^n K_i\cdot K_j(u_{ij}^2 - |u_{ij}|)\right] \\ \times \Big(T_0 + T_1^\mu L_\mu + T_2^{\mu\nu}&\Big[L_\mu L_\nu + \frac{\eta_{\mu\nu}}{2t}\Big] + \ldots\Big)\,,\end{aligned} \tag{6.14}$$

where the worldline proper times u_j are the first comoving coordinate of the worldsheet punctures $z_j = u_j\tau{+}v_j$ with $u_j, v_j \in \mathbb{R}$. As will be detailed below, the worldline length t is identified with $2\pi\alpha' \operatorname{Im}\tau$ for open strings and $4\pi\alpha' \operatorname{Im}\tau$ for closed strings and remains a dynamical integration variable in the simultaneous limit $\alpha' \to 0$ and $\tau \to i\infty$. Moreover, (6.14) exemplifies the worldline treatment of loop momenta in the numerators, where $T_0, T_1^\mu, T_2^{\mu\nu}$ are arbitrary ℓ-independent tensors. The quantity

$$L^\mu = \sum_{j=1}^n K_j^\mu u_j \tag{6.15}$$

stems from the shift of loop momentum to complete the square in the Gaussian integral over ℓ, and higher powers $\ell^{N\geq 2}$ involve additional Wick contractions $\ell_\mu\ell_\nu \to \frac{\eta_{\mu\nu}}{2t}$. Finally, the exponential in the second line of (6.14) is the worldline counterpart of the string-theory Koba-Nielsen factors in (6.1) and (6.8) with worldline Green function $u_{ij}^2 - |u_{ij}|$. We emphasize that (6.14) also applies to any number of massive corners, i.e. momenta in $K_{12\ldots p} = K_1{+}K_2{+}\ldots{+}K_p$ subject to $K_j^2 \neq 0$.

In the Schwinger parametrization (6.14), UV divergences stem from the integration region $t \to 0$ of small worldline lengths, reflecting their origin from short-distance effects (see, for instance, [186]). Thus to study these UV divergences we first Taylor-expand the right-hand side of (6.14) as

$$\begin{aligned}\int \frac{\mathrm{d}^D\ell}{\pi^{D/2}} \frac{(T_0{+}T_1^\mu\ell_\mu{+}T_2^{\mu\nu}\ell_\mu\ell_\nu{+}\ldots)}{\ell^2(\ell{+}K_1)^2(\ell{+}K_{12})^2\ldots(\ell{+}K_{12\ldots n-1})^2}\Bigg|_{\mathrm{UV}} &= \int_0^{t_{\max}} \mathrm{d}t \sum_{M=0}^\infty \frac{t^{n+M-1-D/2}}{M!} \\ \times \int\limits_{0<u_2<u_3<\ldots<u_n<1} \mathrm{d}u_2\ldots\mathrm{d}u_n \left[\sum_{1\leq i<j}^n K_i{\cdot}K_j(u_{ji}{-}u_{ji}^2)\right]^M &\Big(T_0{+}T_1^\mu L_\mu{+}T_2^{\mu\nu}\Big[L_\mu L_\nu{+}\frac{\eta_{\mu\nu}}{2t}\Big]{+}\ldots\Big),\end{aligned} \tag{6.16}$$

where the integration over the simplex $0{=}u_1{<}u_2{<}u_3{<}\ldots{<}u_n{<}1$ is straightforward for any M and n, leading to a Laurent series in t as the integrand. The series expansion of the exponential in (6.14) generically does not commute with the integration over t, and we employ the notation $\int\ldots|_{\mathrm{UV}}$ on the left-hand side of (6.16) to indicate that the right-hand side does not provide an exact expression for the original Feynman integral. Still, the right-hand side of (6.16) accurately reproduces the UV divergences in various spacetime dimensions while the series expansion in $K_i \cdot K_j$ does not capture the discontinuities such as the $\log(K_i \cdot K_j)$ in the ϵ-expansions of appendix C.4. Moreover, we have introduced an

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infrared cutoff $t_{\max}$ into the integration domain for t to decouple infrared divergences from the poles in the dimensional-regularization parameter ϵ: if we view the Feynman integrals as analytic functions in $D = D_{\rm crit} - 2\epsilon$, then the UV divergences of (6.16) in $D_{\rm crit} = 2N$ are the residues at the $\frac{1}{\epsilon}$ pole of

$$\int_0^{t_{\max}} \mathrm{d}t\, t^{N-1-D/2} = \frac{1}{\epsilon} + \mathcal{O}(\epsilon^0)\,, \tag{6.17}$$

where the dependence on the cutoff $0 < t_{\max} < \infty$ is relegated to the regular terms $\mathcal{O}(\epsilon^0)$. Upon comparison with (6.16), its $M^{\rm th}$ order in $K_i \cdot K_j$ captures the UV divergence of a scalar n-gon in the critical dimension $2n{+}2M$. For tensor integrals, the loop momenta in the numerator lower the critical dimension through the Wick contractions $\ell_\mu \ell_\nu \to \frac{\eta_{\mu\nu}}{2t}$. On these grounds, the power series (6.16) in $tK_i \cdot K_j$ obtained from integrating out the u_j order by order in t serves as a generating function of higher-dimensional UV divergences. In practice, we truncate the power series and only track terms up to a maximum order of t corresponding to the UV divergences up to some maximum spacetime dimension.

In the cases of a massless scalar box with $(K_1, K_2, K_3, K_4) \to (k_1, k_2, k_3, k_4)$ and a one-mass scalar triangle with $(K_1, K_2, K_3) \to (k_{12}, k_3, k_4)$, the series expansion in the integrand of (6.16) yields

$$\begin{aligned}\int \frac{\mathrm{d}^D \ell}{\pi^{D/2}} \frac{1}{\ell^2 (\ell{+}k_1)^2 (\ell{+}k_{12})^2 (\ell{+}k_{123})^2}\bigg|_{\rm UV} &= \int_0^{t_{\max}} \mathrm{d}t\, t^{3-D/2} \bigg\{ \frac{1}{6} + \frac{t s_{13}}{120} \\ &\quad + \frac{t^2}{5040}(2s_{12}^2{+}s_{12}s_{23}{+}2s_{23}^2) + \frac{t^3 s_{13}}{181440}(3s_{12}^2{-}2s_{12}s_{23}{+}3s_{23}^2) + \mathcal{O}(t^4) \bigg\}, \\ \int \frac{\mathrm{d}^D \ell}{\pi^{D/2}} \frac{1}{\ell^2 (\ell{+}k_{12})^2 (\ell{+}k_{123})^2}\bigg|_{\rm UV} &= \int_0^{t_{\max}} \mathrm{d}t\, t^{2-D/2} \bigg\{ \frac{1}{2} - \frac{t s_{12}}{24} + \frac{t^2 s_{12}^2}{360} - \frac{t^3 s_{12}^3}{6720} + \mathcal{O}(t^4) \bigg\}.\end{aligned} \tag{6.18}$$

The scalar box in particular already provides an opportunity to verify the identifications between t and the string moduli introduced above. The details of the mapping will be discussed as part of the amplitude matchings below.

An alternative method of constructing the Laurent polynomials directly from logarithmic divergences in dimensional regularization is provided in appendix C.

6.4 UV divergences from open-string effective actions

We start by matching the UV divergences of one-loop matrix elements of gauge multiplets with the contributions to one-loop string amplitudes from the cusp $\tau \to i\infty$. For type-I superstrings with gauge group SO(32), UV divergences at one loop cancel between cylinder and Moebius-strip worldsheets in the single-trace sector and are regulated by closed-string exchange in the double-trace sector [174]. Hence, the Laurent polynomials (6.8) encoding the dominant contributions at the cusp may be viewed as the "would-be"-UV-divergences of the effective field theory that drop out from the complete one-loop amplitude of the SO(32) open superstring.

6.4.1 Four points

As in section 4.1, we omit the overall factor of $t_8(f_1,f_2,f_3,f_4)$ and study four-point UV divergences of the scalar integral $\widehat{A}^{\text{1-loop}}_{\text{eff}}(1,2,3,4)$ in (4.6). Our starting points are the Feynman integrals in (4.7) and (4.12) as well as their higher-order counterparts in the α'-expansion which will now be matched with Laurent polynomials in (6.8). It will be natural to collect the results according to the zeta values, rather than the α' striation of (6.8).

For the box integral and the scalar triangles in (4.7), the UV divergences are given by (6.18). At the order of $\alpha'^3\zeta_3$, we additionally encounter antisymmetrized vector triangles and bubbles on external legs with the following UV contributions according to (6.16):

$$\begin{aligned}\int\frac{\mathrm{d}^D\ell}{\pi^{D/2}}\,\frac{(s_{1,\ell}-s_{2,\ell})}{\ell^2(\ell+k_{12})^2(\ell+k_{123})^2}\bigg|_{\text{UV}}&=\int_0^{t_{\max}}\mathrm{d}t\,t^{2-D/2}\,(s_{23}-s_{13})\left[\frac{1}{6}-\frac{ts_{12}}{120}+\frac{t^2s_{12}^2}{2520}-\frac{t^3s_{12}^3}{60480}+\mathcal{O}(t^4)\right]\\ \int\frac{\mathrm{d}^D\ell}{\pi^{D/2}}\,\frac{1}{\ell^2(\ell+k_1)^2}\bigg|_{\text{UV}}&=\int_0^{t_{\max}}\mathrm{d}t\,t^{1-D/2}\,.\end{aligned}\tag{6.19}$$

On these grounds, the Feynman integrals in (4.7) have the combined UV behaviour

$$\begin{aligned}\pi^{-D/2}\widehat{A}^{\text{1-loop}}_{\text{eff}}(1,2,3,4)\big|_{\text{UV}}=\int_0^{t_{\max}}\mathrm{d}t\,t^{3-D/2}\bigg\{&\bigg[\frac{1}{6}+\frac{ts_{13}}{120}+\frac{t^2(2s_{12}^2+s_{12}s_{23}+2s_{23}^2)}{5040}\\ &+\frac{t^3s_{13}(3s_{12}^2-2s_{12}s_{23}+3s_{23}^2)}{181440}+\ldots\bigg]+\alpha'^2\zeta_2\bigg[\frac{s_{13}}{t}+\frac{s_{12}^2+s_{23}^2}{12}-\frac{t(s_{12}^3+s_{23}^3)}{180}+\ldots\bigg]\\ &+\alpha'^3\zeta_3\bigg[-\frac{6s_{13}}{t^2}+\frac{2(s_{12}^2+s_{12}s_{23}+s_{23}^2)}{3t}+\frac{s_{13}(3s_{12}^2-2s_{12}s_{23}+3s_{23}^2)}{60}+\ldots\bigg]+\mathcal{O}(\alpha'^4)\bigg\}.\end{aligned}\tag{6.20}$$

Higher orders in α' additionally involve two-mass bubbles, tadpoles and multiple powers of loop momenta in the numerators which can also be addressed by the techniques of section 6.3. At the order of $\alpha'^4\zeta_4$ for instance, the numerators in (4.15) give rise to UV divergences

$$\begin{aligned}\pi^{-D/2}\widehat{A}^{\text{1-loop}}_{\text{eff}}(1,2,3,4)\big|^{\alpha'^4\zeta_4}_{\text{UV}}=\int_0^{t_{\max}}\mathrm{d}t\,t^{3-D/2}\bigg\{&\frac{12s_{13}}{t^3}-\frac{3s_{13}^2}{t^2}\\ &+\frac{s_{13}(18s_{12}^2-17s_{12}s_{23}+18s_{23}^2)}{12t}+\ldots\bigg\}.\end{aligned}\tag{6.21}$$

All of these UV contributions are in perfect agreement in the Laurent polynomials in (6.8): with the dictionary

$$\text{open strings:}\quad t=-2\pi i\alpha'\tau=-2i\alpha'T\,,\tag{6.22}$$

between the purely imaginary modular parameter τ of the cylinder and the real Schwinger parameter t on the worldline,[24] any power of T at the $\alpha'^{\leq 3}$-orders of (6.8) is accurately

[24] The dictionary (6.22) can be further validated by comparing the exponent $-tk_i\cdot k_ju_{ij}^2$ in the Schwinger parametrization (6.14) of n-gons with the contributions $\sim e^{i\pi\alpha'\tau s_{ij}u_{ij}^2}$ to the open-string Koba-Nielsen factor in the second line of (6.8). Moreover, by rewriting (6.23) as $\widehat{A}^{\text{1-loop}}_{\text{eff}}(1,2,3,4)\big|_{\text{UV}}=\int_0^{t_{\max}}\mathrm{d}t\,t^3(\frac{\pi}{t})^{D/2}\ldots$, the dimension-dependence $\sim(\frac{\pi}{t})^{D/2}$ on the right-hand side is consistent with the factor of $\sqrt{-2i\alpha'\tau}^{10-D}$ in $(D\neq 10)$-dimensional open-string amplitudes after loop integration.

α'-order / eff. op's	0	1	2		3	
F^2	8	10	12	12	14	14
$\alpha'^2\zeta_2 F^4$		6	8	8	10	10
$\alpha'^3\zeta_3 D^2F^4$		4	6		8	8
$\alpha'^4\zeta_2^2 D^4F^4$		2	4	4	6	6
$\alpha'^5\zeta_5 D^6F^4$				2	4	4
$\alpha'^5\zeta_2\zeta_3 D^6F^4$					4	4
$\alpha'^6\zeta_2^3 D^8F^4$				0	2	2
$\alpha'^6\zeta_3^2 D^8F^4$						
$\alpha'^7\zeta_7 D^{10}F^4$						0
$\alpha'^8\zeta_2^4 D^{12}F^4$						-2

Table 2. The critical dimensions where the UV divergences of the operators in the first column appear in the α'-expansion of the degeneration $a^{\text{1-loop}}_{\text{open}}$ of the one-loop open-string amplitude, see (6.8). Dashed vertical lines separate the contributions that survive upon symmetrizing open-string amplitudes over the color orderings (left) from those that cancel upon symmetrization (right).

mapped to a UV power divergence in the effective field theory, with precise matching of the coefficients. In other words, we have confirmed up to and including α'^3 that

$$\pi^{-D/2}\hat{A}^{\text{1-loop}}_{\text{eff}}(1,2,3,4)\Big|_{\text{UV}} = \int_0^{t_{\max}} \mathrm{d}t\, t^{3-D/2} a^{\text{1-loop}}_{\text{open}}\Big(1,2,3,4;\tau{=}\frac{it}{2\pi\alpha'}\Big) \tag{6.23}$$

reproduces the UV regime (6.16) of the Schwinger parametrized one-loop matrix elements. Note that the leading orders $\alpha'^{\leq 3}$ in the Laurent polynomials are already sensitive to the UV divergences due to higher α'-orders of the one-loop matrix elements. For instance, the coefficients of $\alpha'^2\zeta_5$, $\alpha'^3\zeta_2\zeta_3$ and $\alpha'^2\zeta_2^3$ in $a^{\text{1-loop}}_{\text{open}}(1,2,3,4;\tau)$ already crosscheck the numerators of $\hat{A}^{\text{1-loop}}_{\text{eff}}$ at α'^5 (see (4.16), (4.17)) and at α'^6 (see appendix A.1), respectively. Our checks of (6.23) therefore include the UV divergences

$$\begin{aligned}
\pi^{-D/2}\hat{A}^{\text{1-loop}}_{\text{eff}}(1,2,3,4)\Big|^{\alpha'^6\zeta_6}_{\text{UV}} &= \int_0^{t_{\max}} \mathrm{d}t\, t^{3-D/2}\Big[-\frac{20}{t^4}(s_{12}^2{+}4s_{12}s_{23}{+}s_{23}^2){+}\ldots\Big]\,,\\
\pi^{-D/2}\hat{A}^{\text{1-loop}}_{\text{eff}}(1,2,3,4)\Big|^{\alpha'^7\zeta_7}_{\text{UV}} &= \int_0^{t_{\max}} \mathrm{d}t\, t^{3-D/2}\Big[-\frac{48}{t^4}\,s_{13}(s_{12}^2{+}s_{12}s_{23}{+}s_{23}^2){+}\ldots\Big]\,, \qquad (6.24)\\
\pi^{-D/2}\hat{A}^{\text{1-loop}}_{\text{eff}}(1,2,3,4)\Big|^{\alpha'^8\zeta_8}_{\text{UV}} &= \int_0^{t_{\max}} \mathrm{d}t\, t^{3-D/2}\Big[\frac{168}{t^5}\,s_{13}(s_{12}^2{+}s_{12}s_{23}{+}s_{23}^2){+}\ldots\Big]\,,
\end{aligned}$$

which occur in formally vanishing or negative numbers of spacetime dimensions by the power of the Schwinger parameter t in these examples, see the discussion around (6.16). Table 2 summarizes all the UV divergences that we have matched with the $\alpha'^{\leq 3}$-orders of $a^{\text{1-loop}}_{\text{open}}$.

By imposing the well-tested correspondence (6.23) to hold for higher orders of $a^{\text{1-loop}}_{\text{open}}$ beyond α'^3, one can efficiently predict the Laurent polynomials of elliptic multiple zeta values up to the transcendentality where $\widehat{A}^{\text{1-loop}}_{\text{eff}}$ is available. We have determined the loop integrand of $\widehat{A}^{\text{1-loop}}_{\text{eff}}(1,2,3,4)$ up to and including the order of α'^8, as well as $\alpha'^9\zeta_9$, giving the following new predictions for the coefficients of the leading orders in the Laurent polynomials

$$\begin{aligned}
\widehat{A}^{\text{1-loop}}_{\text{eff}}(1,2,3,4)\Big|^{\alpha'^8\zeta_3\zeta_5}_{\text{UV}} &\to \frac{-4}{t^4}\left(4s_{12}^4+11s_{12}^3s_{23}+6s_{12}^2s_{23}^2+11s_{12}s_{23}^3+4s_{23}^4\right)+\dots\\
\widehat{A}^{\text{1-loop}}_{\text{eff}}(1,2,3,4)\Big|^{\alpha'^8\zeta_{3,5}}_{\text{UV}} &\to \frac{4}{5t^4}\left(10s_{12}^4+17s_{12}^3s_{23}-6s_{12}^2s_{23}^2+17s_{12}s_{23}^3+10s_{23}^4\right)\\
&\quad+\frac{4s_{13}}{5t^3}(2s_{12}^4+5s_{12}^3s_{23}+10s_{12}^2s_{23}^2+5s_{12}s_{23}^3+2s_{23}^4)+\dots \qquad (6.25)\\
\widehat{A}^{\text{1-loop}}_{\text{eff}}(1,2,3,4)\Big|^{\alpha'^8\zeta_2\zeta_3^2}_{\text{UV}} &\to \frac{2}{t^3}s_{12}s_{13}s_{23}\left(s_{12}^2+s_{23}^2\right)+\dots\\
\widehat{A}^{\text{1-loop}}_{\text{eff}}(1,2,3,4)\Big|^{\alpha'^9\zeta_9}_{\text{UV}} &\to \frac{4}{t^5}\left(94s_{12}^4+227s_{12}^3s_{23}+78s_{12}^2s_{23}^2+227s_{12}s_{23}^3+94s_{23}^4\right)+\dots.
\end{aligned}$$

The arrows indicate that the integration measure $\pi^{D/2}\int_0^{t_{\max}}\mathrm{d}t\,t^{3-D/2}$ has been suppressed on the right-hand sides. The methods in both section 6.3 and appendix C are straightforward to extend such that combinatorics become the limiting factor in extracting any desired order in t and s_{ij}, which we plan to address in a future work. In this way, integrating polynomials in proper times u_j can in principle give access to the MZVs of any weight for which the expansion of six-point disk integrals is tractable, at arbitrary α'-orders of $a^{\text{1-loop}}_{\text{open}}$, bypassing the degeneration limits of large numbers of elliptic multiple zeta values.

6.4.2 Five points

At five points, the UV expansion of the three series of Feynman integrals in (5.38) to (5.40) determines all other cases via permutations of the external legs. The Schwinger parametrization of these momentum-space expressions in the UV regime perfectly lines up with the Laurent polynomials in (6.13),

$$\pi^{-D/2}\left[a_1a_2a_3a_4a_5\,|\,E_P\rangle\right|_{\text{UV}} = -\int_0^{t_{\max}}\mathrm{d}t\,t^{3-D/2}\,[\![a_1a_2a_3a_4a_5\,|\,E_P\rangle\!\rangle_\tau\,\Big|_{\tau=\frac{it}{2\pi\alpha'}}\,, \qquad (6.26)$$

for instance

$$\begin{aligned}
\pi^{-D/2}[12345|E_{1|23,4,5}\rangle\Big|_{\text{UV}} &= \int_0^{t_{\max}}\mathrm{d}t\,t^{3-D/2}\bigg\{-\frac{1}{6s_{12}}-\frac{1}{6s_{23}}\\
&\quad+\left(\frac{2s_{14}}{s_{23}}+\frac{2s_{35}}{s_{12}}-3\right)\left(\frac{t}{240}+\frac{\alpha'^2\zeta_2}{2t}-\frac{3\alpha'^3\zeta_3}{t^2}+\frac{6\alpha'^4\zeta_4}{t^3}\right)\bigg\}\\
\pi^{-D/2}[12345|E_{1|24,3,5}\rangle\Big|_{\text{UV}} &= \int_0^{t_{\max}}\mathrm{d}t\,t^{3-D/2}\bigg\{-\frac{1}{6s_{12}} \qquad (6.27)\\
&\quad-\left(1+\frac{2s_{35}}{s_{12}}\right)\left(\frac{t}{240}+\frac{\alpha'^2\zeta_2}{2t}-\frac{3\alpha'^3\zeta_3}{t^2}+\frac{6\alpha'^4\zeta_4}{t^3}\right)+\mathcal{O}(s_{ij})\bigg\}\\
\pi^{-D/2}[12345|E^{\mu}_{1|2,3,4,5}\rangle\Big|_{\text{UV}} &= \int_0^{t_{\max}}\mathrm{d}t\,t^{3-D/2}\bigg\{\frac{k_5^\mu}{6s_{15}}-\frac{k_2^\mu}{6s_{12}}+\left(k_{35}^\mu-k_{24}^\mu+\frac{2s_{35}}{s_{12}}k_2^\mu-\frac{2s_{24}}{s_{15}}k_5^\mu\right)\\
&\quad\times\left(\frac{t}{240}+\frac{\alpha'^2\zeta_2}{2t}-\frac{3\alpha'^3\zeta_3}{t^2}+\frac{6\alpha'^4\zeta_4}{t^3}\right)+\mathcal{O}(s_{ij})\bigg\}.
\end{aligned}$$

The full one-loop matrix element can be assembled from the loop integrals in $[12345|E_P\rangle$ together with the t_8-tensors as prescribed in (5.33). For the scalars $t_8(12,3,4,5)$ and vectors $t_+^\mu(1,2,3,4,5)$ defined in (4.29), the linearized gauge transformations are detailed in (4.35). At each order in t and α', the Schwinger integrand resulting from (6.27) is a gauge invariant combination of t_8-tensors which can be expressed in terms of $A^{\rm tree}_{\rm SYM}$ by repeated use of (4.38) and

$$\frac{t_8(12,3,4,5)}{s_{12}} + \text{cyc}(1,2,3) = s_{45}\Big[s_{34}A^{\rm tree}_{\rm SYM}(1,2,3,4,5) - s_{24}A^{\rm tree}_{\rm SYM}(1,3,2,4,5)\Big]\,. \tag{6.28}$$

By furthermore reducing the SYM trees to the BCJ basis $\{A^{\rm tree}_{\rm SYM}(1{,}2{,}3{,}4{,}5), A^{\rm tree}_{\rm SYM}(1{,}3{,}2{,}4{,}5)\}$, each order in the Schwinger integrands of $A^{\text{1-loop}}_{\rm eff}(1,2,3,4,5)$ can be characterized by a 2×2 matrix with s_{ij}-dependent entries acting on the two-component vector of $A^{\rm tree}_{\rm SYM}$,

$$\begin{aligned}\begin{pmatrix} A^{\text{1-loop}}_{\rm eff}(1,2,3,4,5) \\ A^{\text{1-loop}}_{\rm eff}(1,3,2,4,5)\end{pmatrix}\Bigg|_{\rm UV} &= \int_0^{t_{\max}} {\rm d}t\, t^{3-D/2}\bigg\{\frac{1}{6}P_2 \\ &+ \bigg(\frac{t}{120} + \frac{\alpha'^2\zeta_2}{t} - \frac{6\alpha'^3\zeta_3}{t^2} + \frac{12\alpha'^4\zeta_4}{t^3}\bigg)M_3 + \mathcal{O}(s_{ij}^4)\bigg\}\begin{pmatrix} A^{\rm tree}_{\rm SYM}(1,2,3,4,5) \\ A^{\rm tree}_{\rm SYM}(1,3,2,4,5)\end{pmatrix}\,.\end{aligned} \tag{6.29}$$

The entries of the 2×2 matrices P_w and M_w are degree-w polynomials in s_{ij} with rational coefficients [32], for instance

$$P_2 = \begin{pmatrix} s_{12}s_{34} - s_{34}s_{45} - s_{15}s_{12} & s_{13}s_{24} \\ s_{12}s_{34} & s_{13}s_{24} - s_{24}s_{45} - s_{15}s_{13}\end{pmatrix}\,. \tag{6.30}$$

The cases in (6.29) are known from the α'-expansion of five-point disk integrals, and their explicit form at $w \leq 22$ can be downloaded from [139]. The leading term in (6.29) arises from

$$\frac{t_8(12,3,4,5)}{s_{12}} + \text{cyc}(1,2,3,4,5) = -\Big[(P_2)_{11}A^{\rm tree}_{\rm SYM}(1,2,3,4,5) + (P_2)_{12}A^{\rm tree}_{\rm SYM}(1,3,2,4,5)\Big]\,, \tag{6.31}$$

which is the representation of [143] for the color-ordered tree-level matrix element with a single-insertion of the effective operator $\alpha'^2\zeta_2{\rm Tr}(F^4)$. Higher orders of (6.30) will involve additional 2×2 matrices beyond the P_w and M_w from genus-zero integrals, starting with L_4 in section 5.1 of [116].

6.5 UV divergences from closed-string effective actions

Closed strings feature a similar correspondence between UV divergences in the effective field theory and Laurent polynomials in the expansion of torus integrals around the cusp. The closed-string amplitude is free of UV divergences as can be understood from the bound ${\rm Im}(\tau) \geq \sqrt{3}/2$ for the modular parameters $\tau \in \mathfrak{F}$ in (5.1): by the shape of the fundamental domain of ${\rm SL}_2(\mathbb{Z})$, modular invariance imposes an UV cutoff on the Schwinger parameter t which is absent for the Feynman integrals in the α'-expansion of the matrix elements $M^{\text{1-loop}}_{n,{\rm eff}}$.

6.5.1 Four points

We again peel off a global factor of $|t_8(f_1,f_2,f_3,f_4)|^2$ and discuss four-point gravitational one-loop matrix elements at the level of the scalar integrals $\widehat{M}^{\text{1-loop}}_{4,\text{eff}}$ in (4.18). Based on standard Schwinger parametrizations of the Feynman integrals in (4.20), we find

$$\begin{aligned}\pi^{-D/2}\widehat{M}^{\text{1-loop}}_{4,\text{eff}}\Big|_{\text{UV}} = \int_0^{t_{\max}} \mathrm{d}t\, t^{3-D/2}\bigg\{\bigg[1+\frac{t^2}{360}(s_{12}^2-s_{13}s_{23})-\frac{t^3 s_{12}s_{13}s_{23}}{6048}+\ldots\bigg] \\ +\alpha'^3\zeta_3\bigg[\frac{8}{t}(s_{12}^2-s_{13}s_{23})-s_{12}s_{13}s_{23}+\ldots\bigg]+\alpha'^5\zeta_5\bigg[-\frac{60 s_{12}s_{13}s_{23}}{t^2}+\ldots\bigg]+\mathcal{O}(\alpha'^6)\bigg\}.\end{aligned} \tag{6.32}$$

At higher order $\alpha'^6\zeta_3^2$ and $\alpha'^7\zeta_7$, the numerators in (4.25) and (4.26) yield UV divergences

$$\begin{aligned}\pi^{-D/2}\widehat{M}^{\text{1-loop}}_{4,\text{eff}}\Big|^{\alpha'^6\zeta_3^2}_{\text{UV}} &= \int_0^{t_{\max}}\mathrm{d}t\, t^{3-D/2}\bigg\{\frac{32\sigma_4}{t^2}-\frac{8\sigma_5}{t}+\frac{12\sigma_6+\sigma_3^2}{15}-\frac{43\sigma_7}{630}t+\ldots\bigg\}, \\ \pi^{-D/2}\widehat{M}^{\text{1-loop}}_{4,\text{eff}}\Big|^{\alpha'^7\zeta_7}_{\text{UV}} &= \int_0^{t_{\max}}\mathrm{d}t\, t^{3-D/2}\bigg\{\frac{128\sigma_4}{t^3}-\frac{98\sigma_5}{t^2}-\frac{36\sigma_6-46\sigma_3^2}{5t} \\ &\qquad -\frac{89\sigma_7}{210}+\frac{1959\sigma_8+121\sigma_4^2}{18900}t+\ldots\bigg\},\end{aligned} \tag{6.33}$$

where the Mandelstam variables are organized in terms of the symmetric polynomials

$$\sigma_k = \frac{1}{k}\,(s_{12}^k+s_{23}^k+s_{13}^k)\,. \tag{6.34}$$

Similar to the open-string discussion in the previous section, we compare the t-integrand of (6.32) and (6.33) with the Laurent polynomials of the modular graph forms in (6.4). For closed strings, the dictionary between Schwinger parameter of the worldline and modular parameter of the worldsheet is[25]

$$\text{closed strings:}\quad t = 4\pi\alpha'\,\mathrm{Im}(\tau) = 4\alpha' y\,. \tag{6.35}$$

Indeed, (6.32) and higher-order terms in t or α' are consistent with

$$\pi^{-D/2}\widehat{M}^{\text{1-loop}}_{4,\text{eff}}\Big|_{\text{UV}} = \int_0^{t_{\max}}\mathrm{d}t\, t^{3-D/2}\, m^{\text{1-loop}}_{4,\text{closed}}\Big(\mathrm{Im}\,\tau=\frac{t}{4\pi\alpha'}\Big)\,. \tag{6.36}$$

The $\zeta_3\zeta_5$ term is a further verification of this correspondence, given by

$$\pi^{-D/2}\widehat{M}^{\text{1-loop}}_{4,\text{eff}}\Big|^{\alpha'^8\zeta_3\zeta_5}_{\text{UV}} = \int_0^{t_{\max}}\mathrm{d}t\, t^{3-D/2}\bigg\{\frac{32}{t^3}s_{12}s_{23}s_{13}(s_{13}s_{23}-s_{12}^2)+\ldots\bigg\} \tag{6.37}$$

which indeed matches the corresponding coefficient in (6.4). Moreover, the absence of t^{-4} terms in (6.37) is consistent with the single-valued map (4.19) and the vanishing of

[25] Similar to (6.22), the dictionary (6.35) can be validated by matching the contribution $e^{-ts_{ij}u_{ij}^2/t}$ to the worldline Koba-Nielsen factor with the contribution $e^{-2\pi\alpha'\,\mathrm{Im}\,\tau s_{ij}u_{ij}^2}$ to the closed-string Koba-Nielsen factor in (6.1). Alternatively, one can compare the D-dependence on the right-hand side of $\widehat{M}^{\text{1-loop}}_{4,\text{eff}}\big|_{\text{UV}} = \int_0^{t_{\max}}\mathrm{d}t\, t^3(\frac{\pi}{t})^{D/2}$ with the factor of $(2\sqrt{\alpha'\,\mathrm{Im}\,\tau})^{10-D}$ in $D\neq 10$-dimensional closed-string amplitudes from the D-dependent loop integral.

eff. op's \ α'-order	0	1	2	3	4	5
R	8		12	14	16	18
$\alpha'^3\zeta_3 R^4$			6	8	10	12
$\alpha'^5\zeta_5 D^4 R^4$				4	6	8
$\alpha'^6\zeta_3^2 D^6 R^4$					4	6
$\alpha'^7\zeta_7 D^8 R^4$					2	4
$\alpha'^8\zeta_3\zeta_5 D^{10} R^4$						2
$\alpha'^9\zeta_9 D^{12} R^4$						0
$\alpha'^9\zeta_3^3 D^{12} R^4$						

Table 3. The critical dimensions where the UV divergences of the operators in the first column appear in the α'-expansion of genus-one amplitudes in their Laurent polynomials (6.4) at the cusp.

the $\zeta_3\zeta_5 t^{-4}, \zeta_{3,5} t^{-4}$ terms in (6.25) upon symmetrization in s_{ij}. Again, a fixed α'-order of $m^{\text{1-loop}}_{4,\text{closed}}$ is already sensitive to UV divergences from effective operators at higher order in α'.

As a closed-string analogue of (6.24) and (6.25), we also obtain UV divergences from Schwinger integrands $\mathrm{d}t\, t^{-D/2-1}$ and lower powers of t, which in the language of appendix C correspond to a divergence in formal spacetime dimensions ≤ 0. The simplest such example is a zero-dimensional UV divergence from

$$\widehat{M}^{\text{1-loop}}_{4,\text{eff}}\Big|^{\alpha'^9\zeta_9}_{\text{UV}} = \int_0^{t_{\max}} \mathrm{d}t\, t^{3-D/2} \left\{ \frac{504}{t^4} s_{12}s_{13}s_{23}(s_{13}s_{23}-s_{12}^2) + \dots \right\} \tag{6.38}$$

which we have assembled from (6.25) and the single-valued map, see (4.19). The UV divergences of gravitational effective interactions that we have matched with Laurent polynomials in (6.4) are summarized in table 3.

One can again reverse the logic and rely on the well-tested loop integrands in (4.20) and (4.21) to determine Laurent polynomials at higher order in α'. The numerators in (4.23) to (4.26) together with the straightforward integrals over Schwinger proper times in (6.16) can be used to determine the appearance of $\zeta_3, \zeta_5, \zeta_3^2$ and ζ_7 in the Laurent polynomials at arbitrary α'-orders of $m^{\text{1-loop}}_{4,\text{closed}}$. This settles a large number of terms in the degeneration of infinitely many four-point modular graph forms. Since most of the closed-form results on Laurent polynomials concern the considerably simpler *two-point* modular graph forms [107, 109, 111], the correspondence (6.36) with UV divergences is a valuable source of mathematical information.

Finally, the comparison of UV divergences of gauge and gravitational one-loop matrix elements offers a new perspective on the single-valued map (6.10) between Laurent polynomials of elliptic multiple zeta values and modular graph forms. Based on tree-level results, $\widehat{M}^{\text{1-loop}}_{4,\text{eff}}$ and the symmetrized version of $\widehat{A}^{\text{1-loop}}_{\text{eff}}(1,2,3,4)$ are related by the single-valued map of MZVs as in (4.19). Moreover, the respective dictionaries (6.22) and (6.35) between UV divergences and Laurent polynomials preserve the single-valued map $\pi\tau \to 2\pi i \,\mathrm{Im}\,\tau$

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in (6.11). On these grounds, the relation (4.19) between $\widehat{M}^{\text{1-loop}}_{4,\text{eff}}$ and $\widehat{A}^{\text{1-loop}}_{\text{eff}}$ implies the connection (6.10) between Laurent polynomials of elliptic multiple zeta values and modular graph forms.

6.5.2 Five points

We continue the matching at five points, analyzing the pairings $\langle E_Q|E_P\rangle$ in (5.32). We have verified that the Schwinger parametrizations of the Feynman integrals match the Laurent polynomials of the torus integrals $\langle\!\langle E_Q|E_P\rangle\!\rangle_\tau$ in (6.5). For instance, the leading orders in the UV expansions of the results in section 5.3.3 such as

$$\begin{aligned}
\pi^{-D/2}\,\langle E_{1|23,4,5}|E_{1|24,3,5}\rangle\Big|_{\text{UV}} &= \int_0^{t_{\max}} \mathrm{d}t\, t^{3-D/2}\bigg\{-\frac{1}{s_{12}} \\
&\quad + \bigg(\frac{t^2}{720}+\frac{4\alpha'^3\zeta_3}{t}\bigg)\bigg(\frac{2(s_{35}s_{45}-s_{34}^2)}{s_{12}}-s_{12}+5s_{34}\bigg)+\ldots\bigg\} \\
\pi^{-D/2}\langle E_{1|23,4,5}|E_{1|45,2,3}\rangle\Big|_{\text{UV}} &= \int_0^{t_{\max}}\mathrm{d}t\, t^{3-D/2}\bigg\{\bigg(\frac{t^2}{720}+\frac{4\alpha'^3\zeta_3}{t}\bigg)(s_{24}+s_{35}-s_{25}-s_{34})+\ldots\bigg\} \\
\pi^{-D/2}\,\langle E_{1|23,4,5}|E^{\mu}_{1|2,3,4,5}\rangle\Big|_{\text{UV}} &= \int_0^{t_{\max}} \mathrm{d}t\, t^{3-D/2}\bigg\{\frac{k_3^\mu}{s_{13}}-\frac{k_2^\mu}{s_{12}}+\bigg(\frac{t^2}{720}+\frac{4\alpha'^3\zeta_3}{t}\bigg) \\
&\quad\times\bigg(k_3^\mu\frac{(s_{24}^2+s_{25}^2+s_{45}^2)}{s_{13}}-k_2^\mu\frac{(s_{34}^2+s_{35}^2+s_{45}^2)}{s_{12}} \\
&\quad + (k_3^\mu-k_2^\mu)s_{45}+k_4^\mu(s_{24}-s_{34})+k_5^\mu(s_{25}-s_{35})\bigg)+\ldots\bigg\}
\end{aligned}\tag{6.39}$$

are related to the degenerate torus integrals in (6.6) according to the general dictionary

$$\pi^{-D/2}\,\langle E_Q|E_P\rangle\Big|_{\text{UV}} = -\int_0^{t_{\max}}\mathrm{d}t\, t^{3-D/2}\,\langle\!\langle E_Q|E_P\rangle\!\rangle_\tau\,\big|_{\operatorname{Im}\tau=\frac{t}{4\pi\alpha'}}\,, \tag{6.40}$$

see (6.26) for its open-string analogue.

The one-loop matrix element $M^{\text{1-loop}}_{5,\text{eff}}$ is given in terms of 7×7 pairings $\langle E_Q|E_P\rangle$ and bilinears of t_8-tensors specified in (5.34). In the UV regime (6.16) of the Schwinger parametrization, gauge invariance again holds order by order in t and α'. Similar to the open-string five-point matrix element (6.29), the tensor structures of type-IIB components can be expressed in terms of $A^{\text{tree}}_{\text{SYM}}$. The type-IIA components are related via (4.49) by a sign-flip in one of the parity-odd terms in (4.29). The coefficients of $A^{\text{tree}}_{\text{SYM}}$ bilinears in $M^{\text{1-loop}}_{5,\text{eff}}\,|_{\text{IIB}}$ now depend on the total R-symmetry charges of the external legs in the type-IIB supergravity multiplet. For instance, the leading order in the UV expansion leads to the following R^4-type kinematic factor [182, 187]

$$\begin{aligned}
&\bigg[\frac{t_8(12,3,4,5)\bar t_8(12,3,4,5)}{s_{12}}+(1{,}2|1{,}2{,}3{,}4{,}5)\bigg]+\frac{1}{2}t_+^\mu(1,2,3,4,5)\eta_{\mu\nu}\bar t_+^\nu(1,2,3,4,5)\,|_{\text{IIB}} \\
&\quad = \begin{pmatrix} A^{\text{tree}}_{\text{SYM}}(1{,}2{,}3{,}5{,}4) \\ A^{\text{tree}}_{\text{SYM}}(1{,}3{,}2{,}5{,}4)\end{pmatrix}^{\!\text{T}} S_0\cdot M_3 \begin{pmatrix} A^{\text{tree}}_{\text{SYM}}(1{,}2{,}3{,}4{,}5) \\ A^{\text{tree}}_{\text{SYM}}(1{,}3{,}2{,}4{,}5)\end{pmatrix}\begin{cases} -1 \;:\; h^5 \\ \frac{D-8}{6} \;:\; \phi_D h^4 \end{cases},
\end{aligned}\tag{6.41}$$

where the 2×2 matrix $S_0 = \big(\begin{smallmatrix} s_{12}(s_{13}+s_{23}) & s_{12}s_{13} \\ s_{12}s_{13} & s_{13}(s_{12}+s_{23})\end{smallmatrix}\big)$ is the five-point instance of the field-theory KLT kernel (3.12). The extra factor of $(D{-}8)/6$ arises for four gravitons h and one

D-dimensional dilaton ϕ_D as well as any other external-state configuration with the same overall R-symmetry charges as $\phi_D h^4$. The type-IIA counterpart of the eight-dimensional UV divergence (6.41) has $\bar{t}^{\nu}_{-}$ in the place of $\bar{t}^{\nu}_{+}$. The type-IIA components with four gravitons and one B-field are parity odd and proportional to $\varepsilon_{10}(B_1, f_2, f_3, f_4, f_5)t_8(f_2, f_3, f_4, f_5)$ [182] whereas Bh^4 interactions in the type-IIB theory vanish to all orders in α' [188].

In the complete five-point type-IIB matrix element, we find bilinears in $A^{\text{tree}}_{\text{SYM}}$ at all orders of the UV expansion. For five external gravitons (or other R-symmetry-conserving state configurations), the coefficients at the leading orders are

$$M^{\text{1-loop}}_{5,\text{eff}}\Big|^{\text{IIB},h^5}_{\text{UV}} = \begin{pmatrix} A^{\text{tree}}_{\text{SYM}}(1,2,3,5,4) \\ A^{\text{tree}}_{\text{SYM}}(1,3,2,5,4) \end{pmatrix}^{\text{T}} \cdot S_0 \int_0^{t_{\max}} \mathrm{d}t\, t^{3-D/2} \Big[-M_3 - 2\Big(\frac{t^2}{720} + \frac{4\alpha'^3\zeta_3}{t}\Big) M_5 \\ - \Big(\frac{t^3}{6048} + \alpha'^3\zeta_3 + \frac{60\alpha'^5\zeta_5}{t^2}\Big) M_3^2 + \mathcal{O}(s_{ij}^7) \Big] \begin{pmatrix} A^{\text{tree}}_{\text{SYM}}(1,2,3,4,5) \\ A^{\text{tree}}_{\text{SYM}}(1,3,2,4,5) \end{pmatrix}, \quad (6.42)$$

where higher orders in s_{ij} involve 2×2 matrices $M'_{w\geq 7}$ beyond the tree-level ones M_w [182]. For R-symmetry violating type-IIB states such as $\phi_D h^4$, the numerical coefficients change to

$$M^{\text{1-loop}}_{5,\text{eff}}\Big|^{\text{IIB},\phi_D h^4}_{\text{UV}} = \begin{pmatrix} A^{\text{tree}}_{\text{SYM}}(1,2,3,5,4) \\ A^{\text{tree}}_{\text{SYM}}(1,3,2,5,4) \end{pmatrix}^{\text{T}} \cdot S_0 \int_0^{t_{\max}} \mathrm{d}t\, t^{3-D/2} \\ \times \Big[\frac{D-8}{6} M_3 + \frac{D-12}{5}\Big(\frac{t^2}{720} + \frac{4\alpha'^3\zeta_3}{t}\Big) M_5 \quad (6.43) \\ + \frac{D-14}{12}\Big(\frac{t^3}{6048} + \alpha'^3\zeta_3 + \frac{60\alpha'^5\zeta_5}{t^2}\Big) M_3^2 + \mathcal{O}(s_{ij}^7) \Big] \begin{pmatrix} A^{\text{tree}}_{\text{SYM}}(1,2,3,4,5) \\ A^{\text{tree}}_{\text{SYM}}(1,3,2,4,5) \end{pmatrix}.$$

The D-dimensional relative factors in comparison to (6.42) play an important role for the S-duality properties of type-IIB string amplitudes [182, 187, 189, 190]. From a supergravity point of view, the factors of $D-8$, $D-12$ and $D-14$ in (6.43) prevent R-symmetry violation through the respective UV divergence in $8, 12$ and 14 spacetime dimensions [63, 187, 190].

7 Comparison with non-analytic terms in string amplitudes

On top of the analysis of UV divergences in the previous section, we can also relate the discontinuities of one-loop matrix elements to string-theory computations. Unitarity at the level of the α'-expansion implies that the discontinuities of one-loop string amplitudes in s_{ij} agree with those of $A^{\text{1-loop}}_{\text{eff}}(1, 2, \ldots, n)$ and $M^{\text{1-loop}}_{n,\text{eff}}$. We shall first confirm this for the four-point closed-string results in the literature and then spell out predictions for the non-analytic sector of one-loop four-point open-string amplitudes.

7.1 Reproducing non-analytic terms of closed superstrings

The non-analytic part of the four-point one-loop amplitude of type-II superstrings is known to all orders in α' [64], also see [66–68] for earlier results at the leading α'-orders. Based on the Feynman integrals in section 4.2, we find the following logarithmic behaviour in the low-energy expansion

$$M^{\text{1-loop}}_{4,\text{eff}}\big|_{\text{disc}} = M^{\text{1-loop}}_{4,\text{SUGRA}}\big|_{\text{disc}} + |t_8(f_1,f_2,f_3,f_4)|^2\bigg\{-\frac{\alpha'^4\zeta_3}{45}s_{12}^4 + \frac{\alpha'^6\zeta_5}{1260}s_{12}^4(s_{13}s_{23}-22s_{12}^2) + \frac{\alpha'^7\zeta_3^2}{1260}s_{12}^5(12s_{12}^2+s_{13}s_{23}) - \frac{\alpha'^8\zeta_7}{18900}s_{12}^4(260s_{12}^4-25s_{12}^2s_{13}s_{23}+2s_{13}^2s_{23}^2) + \frac{\alpha'^9\zeta_3\zeta_5}{4725}s_{12}^5(70s_{12}^4+5s_{12}^2s_{13}s_{23}-s_{13}^2s_{23}^2) + \mathcal{O}(\alpha'^{10})\bigg\}\log\left(\frac{s_{12}}{\mu^2}\right) + \text{cyc}(2,3,4)\,, \tag{7.1}$$

where $|_{\text{disc}}$ instructs to disregard any analytic dependence on α'. In this way, the polynomials in s_{ij} introduced by a change of the scale μ do not affect the terms under consideration, see appendix C.4 for the origin of the terms $\log(\frac{s_{ij}}{\mu^2})$. To the orders shown, (7.1) exactly matches the non-analytic terms of [64] obtained from the α'-expansions in Theorem 4.1 in version 2 of the reference.[26]

A naive power-series expansion of the integrals (6.4) over closed-string punctures cannot detect the logarithmic dependence on s_{ij}. Still, the divergent τ-integrals over certain modular graph forms in the α'-expansion of (6.4) have crucial information on the non-analytic sector of one-loop closed-string amplitudes, see [64, 66–68] for further details.

7.2 Predicting non-analytic terms of open superstrings

As an open-string counterpart of (7.1), we can assemble the discontinuities of the Feynman integrals in the four-point one-loop matrix element of section 4.1,

$$A^{\text{1-loop}}_{\text{eff}}(1,2,3,4)\big|_{\text{disc}} = A^{\text{1-loop}}_{\text{SYM}}(1,2,3,4)\big|_{\text{disc}} + t_8(f_1,f_2,f_3,f_4)\bigg\{f(s_{12},s_{23})\log\left(\frac{s_{12}}{\mu^2}\right) + f(s_{23},s_{12})\log\left(\frac{s_{23}}{\mu^2}\right)\bigg\}\,, \tag{7.2}$$

where

$$f(s_{12},s_{23}) = \frac{\alpha'^3\zeta_2}{180}s_{12}^3 - \frac{\alpha'^4\zeta_3}{1260}s_{12}^3(4s_{12}+s_{23}) + \frac{\alpha'^5\zeta_2^2}{12600}s_{12}^3(46s_{12}^2-2s_{12}s_{23}+s_{23}^2) - \frac{\alpha'^6\zeta_2\zeta_3}{5040}s_{12}^4(12s_{12}^2+s_{13}s_{23}) - \frac{\alpha'^6\zeta_5}{15120}s_{12}^3(38s_{12}^2+12s_{12}^2s_{23}+3s_{12}s_{23}^2+s_{23}^2) + \frac{\alpha'^7\zeta_3^2}{30240}s_{12}^4(20s_{12}^3+2s_{12}^2s_{23}-3s_{12}s_{23}^2-s_{23}^3) + \frac{\alpha'^7\zeta_2^3}{5292000}s_{12}^3(12320s_{12}^4-275s_{12}^3s_{23}+357s_{12}^2s_{23}^2-36s_{12}s_{23}^3+32s_{23}^4) - \frac{\alpha'^8\zeta_7}{831600}s_{12}^3(1660s_{12}^5+645s_{12}^4s_{23}+296s_{12}^3s_{23}^2+137s_{12}^2s_{23}^3+36s_{12}s_{23}^4+10s_{23}^5) - \frac{\alpha'^8\zeta_2^2\zeta_3}{756000}s_{12}^4(1200s_{12}^4-115s_{12}^3s_{23}-48s_{12}^2s_{23}^2-s_{12}s_{23}^3-8s_{23}^4) - \frac{\alpha'^8\zeta_2\zeta_5}{37800}s_{12}^4(70s_{12}^4-5s_{12}^3s_{23}-6s_{12}^2s_{23}^2-2s_{12}s_{23}^3-s_{23}^4) + \mathcal{O}(\alpha'^9)\,. \tag{7.3}$$

[26] We are grateful to Eric D'Hoker and Michael Green for correspondence on a factor of 4 that was changed since the first version of [64].

This sets our prediction for the non-analytic terms in the planar-cylinder contribution to the four-point one-loop open-string amplitude. In addition to the above terms, we note that the $\zeta_{3,5}$ integrand only contains external bubbles (i.e. figure 3b) and tadpoles, which integrate to zero in dimensional reduction. As such, there are no terms of the form $\alpha'^9 \zeta_{3,5} \log(\frac{s_{ij}}{\mu^2})$ as required by unitarity (there is no $\zeta_{3,5}$ in the four-point tree-level amplitude). Again, the logarithms in s_{ij} can be anticipated from the divergent τ integrals in a naive power-series expansion of the integrals over the open-string punctures (with elliptic multiple zeta values in the τ-integrand). With the notable exception at the subleading α'-order discussed in section 4.2 of [81], the interplay between the analytic and non-analytic sector of one-loop open-string amplitudes has not yet been studied at the level of explicit higher-order results.

8 Conclusion and outlook

In this work, we have used a confluence of ideas from ambitwistor, conventional string and (effective) field theories to obtain one-loop matrix elements of higher-mass-dimension operators $D^{2k}F^n$ and $D^{2k}R^n$. Our method applies to the supersymmetric operators in the tree-level low-energy effective actions of type-I and type-II superstrings and is driven by the forward limits of the disk and sphere integrals in massless tree amplitudes. Both the one-loop matrix elements and their tree-level building blocks are organized in a series expansion in the inverse string tension α'. Moreover, one can further distinguish different operators at the same mass dimension from the conjecturally $\mathbb{Q}$-linearly independent multiple zeta values in their coefficients.

Our construction passes a variety of consistency checks including a matching of the matrix elements' UV divergences and discontinuities with expectations based on one-loop superstring amplitudes.

- The UV divergences of the loop integrals associated with various α'-orders and spacetime dimensions line up with the degeneration limits of genus-one integrals over torus or cylinder punctures. This was confirmed in all cases we considered, and by imposing consistency at higher orders, our proposal offers a new window into the expansion of modular graph forms and elliptic multiple zeta values around the cusp.

- The discontinuities of the one-loop matrix elements in this work via logarithmic dependence on Mandelstam invariants reproduce those of the one-loop four-graviton amplitude of type II superstrings (which is verified up to and including the order of α'^9 [64]). For open strings at ≥ 4 points and closed strings at ≥ 5 points, our one-loop matrix elements encode new results on the non-analytic sectors of one-loop string amplitudes.

By their construction from genus-zero integrals, the one-loop matrix elements evidently enjoy an echo of the monodromy relations [77–79] and KLT relations [4] of open- and closed-string tree amplitudes, respectively. These properties are spelt out for the loop integrand of n-point matrix elements and furnish an α'-uplift of one-loop BCJ and KLT relations among field-theory amplitudes in the formulation of [20, 21]. The monodromy and KLT relations of one-loop matrix elements apply to the representation of their loop integrand where the inverse Feynman propagators are linearized in loop momenta as familiar from ambitwistor strings [13, 16]. It would be interesting to further explore the practical implications of the one-loop monodromy relations of open-string amplitudes as given in [80–86], as well as tentative KLT relations between one-loop open- and closed-string amplitudes that do not rely on linearized propagators, or that hold after integration.

Monodromy and KLT relations of one-loop matrix elements reflect the underlying color-kinematics duality and double-copy properties of effective-field-theory operators, order by order in α'. This suggests an important line of follow-up work to manifest a one-loop cubic-diagram formulation of these properties. In particular, it would be very interesting to construct kinematic numerators at higher orders in α' that obey Jacobi identities on quadratic Feynman propagators. The numerators due to higher-derivative operators will involve large numbers of loop momenta and thereby serve as a useful laboratory for the rich loop-momentum dependence of field-theory numerators at higher loop orders. It is plausible that one-loop matrix elements may even provide new input on the difficulties in finding local numerators with kinematic Jacobi relations for the five-loop integrand of maximal supergravity [98, 99].

One-loop matrix elements at four and five points are spelt out to various orders in α', with all-multiplicity conjectures for the $\alpha'^{\leq 3}$ orders of certain scalar toy integrals. An obvious open problem for the future concerns explicit representations of higher-point matrix elements. Similarly, our proposal for non-planar or double-trace matrix elements of gauge multiplets calls for explicit examples and consistency checks. Particularly interesting features can be expected at six points: first, our method qualifies for explicit investigations of α'-corrections to the one-loop hexagon anomaly of ten-dimensional super-Yang-Mills [191, 192] and its cancellation for type-I superstrings [193, 194]. Second, one-loop six-point amplitudes in field theory pose additional challenges [122, 123] in manifesting the color-kinematics duality and double copy as compared to ≤ 5 points, at least in $D = 10$ dimensions and through quadratic Feynman propagators [21]. More specifically, while a four-dimensional construction have been understood for some time [195], it remains to iron out details of the double copy for the SYM loop integrand recently obtained in [123].

The one-loop matrix elements in this work incorporate diagrams with single and multiple insertions of higher-mass-dimension operators, see e.g. figure 1. In the open-string case, diagrams with single insertions encode the zero-momentum limit of maximally supersymmetric one-loop form factors involving the respective $\mathrm{Tr}\{D^{2k}F^n\}$ operator. Supersymmetric form factors have prominently featured in the recent amplitudes literature, for instance through their construction via unitarity-based methods [196–202] or their color-kinematics duality [203–205].

We expect fruitful connections between the one-loop matrix elements in this work and stringy canonical forms [206–208]. For instance, the $\alpha' \to 0$ limits of stringy canonical forms for finite-type cluster algebras produce the integrands for one-loop tadpole diagrams in case of type B, C and the one-loop planar ϕ^3 diagrams in case of type D. It would be interesting to compare the α'-expansions of such stringy canonical forms with the forward limits of disk and sphere integrals that carry the α'-dependence in our results. Moreover,

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our results together with follow-up studies of analytic contributions to one-loop string amplitudes may lead to interesting connections with the EFT-hedron [209]. Relatedly, it will be rewarding to study a loop-level formulation of the recent advances in finding new double-copy constructions at higher mass dimension [210–212] and to compare with the one-loop matrix elements in this work.

Finally, it would also be rewarding to investigate similar proposals for matrix elements at two loops and beyond. A promising starting point is offered by the ambitwistor-string construction of two-loop field-theory amplitudes via bi-nodal Riemann spheres [71, 213–215], also see [163] for an impressive three-loop result in this framework. It is conceivable that suitable α'-uplifts via double-forward limits of disk and sphere integrals yield two-loop matrix elements of $D^{2k}F^n$ and $D^{2k}R^n$ that resonate with string-theory expectations. The main crosschecks would concern the discontinuity structure of two-loop string amplitudes [62, 63] and the tropical degeneration of genus-two modular graph forms [63, 173, 190, 216–218]. Conversely, matrix elements of effective operators beyond one loop have enormous potential to infer physical and mathematical information on multiloop string amplitudes.

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A Numerators at higher order in α'

This appendix gathers higher-order examples in the α'-expansion of the four- and five-point one-loop matrix elements in section 4.

A.1 Four-point open strings at higher orders in α'

In this appendix, we display the $\zeta_2^3\alpha'^6, \zeta_3^2\alpha'^6$ and $\zeta_{3,5}\alpha'^8$ contributions to the numerators in the four-point one-loop matrix elements (4.12). The square of ζ_3 is accompanied by

$$
\begin{aligned}
N^{(\zeta_3^2)}_{\text{triangle}}(1,2,34) &= \frac{1}{4}s_{34}^2\Big[s_{\ell,3}^3 - s_{\ell,4}^3 - s_{34}(s_{\ell,3}^2 - 2s_{\ell,4}^2) - s_{34}^2 s_{\ell,4}\Big]\,, &\text{(A.1)}\\
N^{(\zeta_3^2)}_{\text{bubble}}(12,34) &= -\frac{1}{2}s_{12}^2\Big[s_{\ell,2}(s_{\ell,1}+s_{\ell,2}) - s_{\ell,3}(s_{\ell,1}-s_{\ell,2}) + s_{12}(s_{\ell,2}-s_{\ell,3})\Big]\,, &\text{(A.2)}\\
N^{(\zeta_3^2)}_{\text{bubble}}(1,234) &= \frac{1}{4}\Big[s_{34}^2(-2s_{\ell,2}^2 + 2s_{\ell,3}^2 - 3s_{\ell,4}^2 - s_{\ell,2}s_{\ell,3} - 2s_{\ell,3}s_{\ell,4}) + (2\leftrightarrow 4)\\
&\quad - 2s_{23}s_{34}(2s_{\ell,2}^2 + s_{\ell,3}^2 + 2s_{\ell,4}^2 + 2s_{\ell,2}s_{\ell,3} + 2s_{\ell,3}s_{\ell,4})\\
&\quad - s_{34}^3(2s_{\ell,2} + 2s_{\ell,3} - 3s_{\ell,4}) - s_{23}^3(s_{\ell,2} + 3s_{\ell,3} + s_{\ell,4})\\
&\quad + 3s_{23}s_{34}s_{24}(s_{\ell,3}+s_{\ell,3}) + 3s_{34}^2 s_{23}s_{\ell,4}\\
&\quad - s_{23}s_{24}(8s_{34}^2 + 7s_{34}s_{23} + s_{23}^2)\Big]\,, &\text{(A.3)}\\
N^{(\zeta_3^2)}_{\text{tadpole}}(1234) &= -3s_{13}(s_{13}^2 - s_{12}s_{23})\,, &\text{(A.4)}
\end{aligned}
$$

while even zeta values $\zeta_2^3 = \frac{35}{8}\zeta_6$ introduce the more lengthy expressions

$$
\begin{aligned}
N^{(\zeta_2^3)}_{\text{triangle}}(1,2,34) &= -\frac{1}{140}s_{34}\Big[16(s_{\ell,3}^4+s_{\ell,4}^4) - 4s_{34}(13s_{\ell,3}^3+3s_{\ell,4}^3) + s_{34}^3(83s_{\ell,3}^2+23s_{\ell,4}^2)\\
&\qquad - 2s_{34}^3(31s_{\ell,3}+6s_{\ell,4}) + 47s_{34}^4\Big]\,, &\text{(A.5)}\\
N^{(\zeta_2^3)}_{\text{bubble}}(12,34) &= \frac{1}{10}s_{12}^2\Big[4(s_{\ell,1}^2+s_{\ell,2}^2+s_{\ell,3}^2+s_{\ell,1}s_{\ell,2}+s_{\ell,1}s_{\ell,3}+s_{\ell,2}s_{\ell,3})\\
&\qquad + s_{12}(s_{\ell,1}+4s_{\ell,2}-3s_{\ell,3}) + 11s_{12}^2\Big]\,, &\text{(A.6)}\\
N^{(\zeta_2^3)}_{\text{bubble}}(1,234) &= -\frac{124}{105}s_{12}s_{23}s_{\ell,1}s_{\ell,2} + \frac{58}{105}s_{12}s_{23}s_{\ell,1}s_{\ell,3} - \frac{271}{420}s_{12}^2 s_{23}s_{\ell,1} - \frac{62}{105}s_{12}s_{23}s_{\ell,1}^2\\
&\quad - \frac{23}{420}s_{12}s_{23}^2 s_{\ell,1} + \frac{68}{105}s_{12}s_{23}s_{\ell,2}s_{\ell,3} - \frac{127}{210}s_{12}^2 s_{23}s_{\ell,2} - \frac{76}{105}s_{12}s_{23}s_{\ell,2}^2\\
&\quad + \frac{31}{210}s_{12}s_{23}^2 s_{\ell,2} + \frac{79}{70}s_{12}^2 s_{23}s_{\ell,3} - \frac{8}{21}s_{12}s_{23}s_{\ell,3}^2 + \frac{23}{105}s_{12}s_{23}^2 s_{\ell,3}\\
&\quad - \frac{8}{7}s_{12}^3 s_{23} - \frac{87}{70}s_{12}^2 s_{23}^2 - \frac{47}{70}s_{12}s_{23}^3 - \frac{48}{35}s_{12}s_{\ell,1}s_{\ell,2}s_{\ell,3} - \frac{128}{105}s_{12}^2 s_{\ell,1}s_{\ell,2}\\
&\quad - \frac{48}{35}s_{12}s_{\ell,1}s_{\ell,2}^2 - \frac{48}{35}s_{12}s_{\ell,1}^2 s_{\ell,2} - \frac{1}{5}s_{12}^2 s_{\ell,1}s_{\ell,3} - \frac{16}{35}s_{12}s_{\ell,1}s_{\ell,3}^2\\
&\quad - \frac{24}{35}s_{12}s_{\ell,1}^2 s_{\ell,3} - \frac{64}{105}s_{12}^3 s_{\ell,1} - \frac{64}{105}s_{12}^2 s_{\ell,1}^2 - \frac{16}{35}s_{12}s_{\ell,1}^3 - \frac{1}{5}s_{12}^2 s_{\ell,2}s_{\ell,3}\\
&\quad - \frac{8}{5}s_{12}s_{\ell,2}s_{\ell,3}^2 - \frac{64}{35}s_{12}s_{\ell,2}^2 s_{\ell,3} - \frac{593}{420}s_{12}^3 s_{\ell,2} - \frac{25}{21}s_{12}^2 s_{\ell,2}^2 - \frac{36}{35}s_{12}s_{\ell,2}^3\\
&\quad - \frac{79}{210}s_{12}^3 s_{\ell,3} + \frac{1}{3}s_{12}^2 s_{\ell,3}^2 - \frac{24}{35}s_{12}s_{\ell,3}^3 - \frac{113}{140}s_{12}^4 - \frac{8}{7}s_{23}s_{\ell,1}s_{\ell,2}s_{\ell,3}
\end{aligned}
$$

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$$
\begin{aligned}
&-\frac{12}{7}s_{23}s_{\ell,1}s_{\ell,2}^2-\frac{12}{7}s_{23}s_{\ell,1}^2s_{\ell,2}-\frac{2}{105}s_{23}^2s_{\ell,1}s_{\ell,2}-\frac{4}{7}s_{23}s_{\ell,1}s_{\ell,3}^2-\frac{4}{7}s_{23}s_{\ell,1}^2s_{\ell,3}\\
&-\frac{2}{105}s_{23}^2s_{\ell,1}s_{\ell,3}-\frac{4}{7}s_{23}s_{\ell,1}^3-\frac{1}{105}s_{23}^2s_{\ell,1}{}^2-\frac{89}{420}s_{23}^3s_{\ell,1}-\frac{36}{35}s_{23}s_{\ell,2}s_{\ell,3}^2\\
&-\frac{44}{35}s_{23}s_{\ell,2}^2s_{\ell,3}-\frac{13}{105}s_{23}^2s_{\ell,2}s_{\ell,3}-\frac{36}{35}s_{23}s_{\ell,2}^3+\frac{1}{15}s_{23}^2s_{\ell,2}^2-\frac{109}{420}s_{23}^3s_{\ell,2}\\
&-\frac{4}{35}s_{23}s_{\ell,3}^3-\frac{11}{35}s_{23}^2s_{\ell,3}{}^2-\frac{9}{140}s_{23}^3s_{\ell,3}-\frac{11}{35}s_{23}^4\,, \qquad \text{(A.7)}
\end{aligned}
$$

$$
\begin{aligned}
N^{(\zeta_2^3)}_{\text{tadpole}}(1234)=&-\frac{104}{35}s_{12}^2s_{23}-\frac{104}{35}s_{12}s_{23}^2-\frac{48}{7}s_{12}s_{\ell,1}s_{\ell,2}-\frac{16}{7}s_{12}s_{\ell,1}s_{\ell,3}+\frac{152}{35}s_{12}^2s_{\ell,1}\\
&-\frac{32}{7}s_{12}s_{\ell,1}^2-\frac{80}{7}s_{12}s_{\ell,2}s_{\ell,3}-\frac{64}{7}s_{12}s_{\ell,2}^2+\frac{152}{35}s_{12}^2s_{\ell,3}-\frac{48}{7}s_{12}s_{\ell,3}^2-\frac{76}{35}s_{12}^3\\
&-\frac{80}{7}s_{23}s_{\ell,1}s_{\ell,2}-\frac{16}{7}s_{23}s_{\ell,1}s_{\ell,3}-\frac{48}{7}s_{23}s_{\ell,1}^2-\frac{152}{35}s_{23}^2s_{\ell,1}-\frac{48}{7}s_{23}s_{\ell,2}s_{\ell,3}\\
&-\frac{64}{7}s_{23}s_{\ell,2}^2-\frac{32}{7}s_{23}s_{\ell,3}^2-\frac{152}{35}s_{23}^2s_{\ell,3}-\frac{76}{35}s_{23}^3\,. \qquad \text{(A.8)}
\end{aligned}
$$

Finally, as an example of the weight-8 contributions, we present the $\zeta_{3,5}$ numerators. Of note is that the triangle and two-mass bubble do not contribute, namely

$$
N^{(\zeta_{3,5})}_{\text{triangle}}(1,2,34)=0\,,\qquad N^{(\zeta_{3,5})}_{\text{bubble}}(12,34)=0\,, \qquad \text{(A.9)}
$$

while there is an external-bubble contribution

$$
\begin{aligned}
N^{(\zeta_{3,5})}_{\text{bubble}}(1,234)=\frac{s_{13}}{2560}\bigg(&s_{\ell,1}^3\left[16s_{\ell,2}(s_{12}+s_{23})+2s_{\ell,3}(5s_{12}+3s_{23})+3\left(s_{12}^2-s_{23}^2\right)\right]\\
&+3s_{\ell,1}^2\bigg[s_{\ell,2}\left(2s_{\ell,3}[5s_{12}+3s_{23}]+3\left[s_{12}^2-s_{23}^2\right]\right)+8s_{\ell,2}^2(s_{12}+s_{23})\\
&+s_{\ell,3}\left(5s_{12}s_{23}+5s_{12}^2+2s_{23}^2\right)+5s_{12}^2s_{23}+5s_{12}s_{23}^2+s_{12}^3-2s_{23}s_{\ell,3}^2+s_{23}^3\bigg]\\
&+s_{\ell,1}\bigg[3s_{\ell,2}^2\left(2s_{\ell,3}[5s_{12}+3s_{23}]+3\left[s_{12}^2-s_{23}^2\right]\right)+6s_{\ell,2}\Big(s_{\ell,3}\left[5s_{12}s_{23}+5s_{12}^2+2s_{23}^2\right]\\
&+5s_{12}^2s_{23}+5s_{12}s_{23}^2+s_{12}^3-2s_{23}s_{\ell,3}^2+s_{23}^3\Big)+16s_{\ell,2}^3(s_{12}+s_{23})-2s_{\ell,3}^3(5s_{12}+2s_{23})\\
&+3s_{\ell,3}^2\left(5s_{12}s_{23}+5s_{12}^2+2s_{23}^2\right)+3s_{23}s_{\ell,3}\left(5s_{12}s_{23}+5s_{12}^2+2s_{23}^2\right)\\
&+2\left(5s_{12}^3s_{23}-5s_{12}s_{23}^3+2s_{12}^4-2s_{23}^4\right)\bigg]+4s_{\ell,1}^4(s_{12}+s_{23})+2\bigg[s_{\ell,2}^3\Big(2s_{\ell,3}[5s_{12}+3s_{23}]\\
&+3\left[s_{12}^2-s_{23}^2\right]\Big)+3s_{\ell,2}^2\Big(s_{\ell,3}\left[5s_{12}s_{23}+5s_{12}^2+2s_{23}^2\right]+5s_{12}^2s_{23}+5s_{12}s_{23}^2+s_{12}^3\\
&-2s_{23}s_{\ell,3}^2+s_{23}^3\Big)+s_{\ell,2}\Big(-2s_{\ell,3}^3[5s_{12}+2s_{23}]+3s_{\ell,3}^2\left[5s_{12}s_{23}+5s_{12}^2+2s_{23}^2\right]\\
&+3s_{23}s_{\ell,3}\left[5s_{12}s_{23}+5s_{12}^2+2s_{23}^2\right]+2\left[5s_{12}^3s_{23}-5s_{12}s_{23}^3+2s_{12}^4-2s_{23}^4\right]\Big)\\
&+4s_{\ell,2}^4[s_{12}+s_{23}]+s_{12}s_{\ell,3}\Big(3s_{\ell,3}^2[s_{12}+2s_{23}]+s_{\ell,3}\left[6s_{12}s_{23}-3s_{12}^2+6s_{23}^2\right]\\
&+6s_{12}^2s_{23}-6s_{12}s_{23}^2+4s_{12}^3-4s_{23}^3-4s_{\ell,3}^3\Big)\bigg]\bigg) \qquad \text{(A.10)}
\end{aligned}
$$

and a tadpole

$$N^{(\zeta_{3,5})}_{\text{tadpole}}(1234) = \frac{1}{640}\Bigg(s_{\ell,1}\Big[s_{\ell,2}\left(81s_{12}^2s_{23} - 9s_{12}s_{23}^2 + 43s_{12}^3 - 7s_{23}^3\right)$$
$$+ s_{13}\Big(10\left[3s_{12}^2s_{23} - 3s_{12}s_{23}^2 + 2s_{12}^3 - 2s_{23}^3\right] - 3s_{\ell,3}\left[-4s_{12}s_{23} + 5s_{12}^2 + 5s_{23}^2\right]\Big)\Big]$$
$$+ s_{\ell,1}^2\left[42s_{12}^2s_{23} - 3s_{12}s_{23}^2 + 29s_{12}^3 + 4s_{23}^3\right] + s_{\ell,2}s_{\ell,3}\left[81s_{12}s_{23}^2 - 9s_{12}^2s_{23} - 7s_{12}^3 + 43s_{23}^3\right]$$
$$+ 18s_{\ell,2}^2\left[2s_{12}^2s_{23} + 2s_{12}s_{23}^2 + s_{12}^3 + s_{23}^3\right] + s_{\ell,3}\Big[s_{\ell,3}\left(42s_{12}s_{23}^2 - 3s_{12}^2s_{23} + 4s_{12}^3 + 29s_{23}^3\right)$$
$$+ 10\left(-5s_{12}^3s_{23} + 5s_{12}s_{23}^3 - 2s_{12}^4 + 2s_{23}^4\right)\Big]\Bigg). \tag{A.11}$$

A.2 The five-point one-loop matrix element at the order of α'^4

In the parametrization (4.39) of one-loop matrix elements at five points, the $\alpha'^4\zeta_2^2$-order of the numerators has been partially given in (4.44). The remaining contributions to this order are given as follows (see (4.32) for the pentagon numerator $N_{+|12345|-}$ of SYM):

$$N^{(\zeta_2^2)}_{\text{box}}(1,2,3,45) = -\frac{1}{20}s_{45}\left[4(s_{\ell,4}^2+s_{\ell,5}^2) - s_{45}(7s_{\ell,4}+s_{\ell,5}) + 11s_{45}^2\right]N_{+|12345|-}$$
$$+\frac{1}{20}\Big[4(s_{\ell,4}^3 - s_{\ell,5}^3) - s_{45}(11s_{\ell,4}^2 - s_{\ell,5}^2)$$
$$+2s_{45}^2(7s_{\ell,4} - 2s_{\ell,5}) - 7s_{45}^3\Big]t_8(45,1,2,3)\,, \tag{A.12}$$

$$N^{(\zeta_2^2)}_{\text{triangle}}(1,2,[34]5) = \frac{1}{20}(s_{35}+s_{45})\Big[4(s_{\ell,3}+s_{\ell,4})^2 + 4s_{\ell,5}^2 - s_{\ell,5}(s_{35}+s_{45})$$
$$-(s_{\ell,3}+s_{\ell,4})(16s_{34}+7s_{35}+7s_{45}) + 16s_{34}^2$$
$$+14s_{34}(s_{35}+s_{45}) + 11(s_{35}+s_{45})^2\Big]t_8(34,1,2,5)\,, \tag{A.13}$$

$$N^{(\zeta_2^2)}_{\text{triangle}}(1,2,3[45]) = \frac{1}{20}(s_{34}+s_{35})\Big[4s_{\ell,3}^2 + 4(s_{\ell,4}+s_{\ell,5})^2 - 7s_{\ell,3}(s_{34}+s_{35})$$
$$-(s_{\ell,4}+s_{\ell,5})(s_{34}+s_{35}) + 11(s_{34}+s_{35})^2\Big]t_8(45,1,2,3)\,, \tag{A.14}$$

$$N^{(\zeta_2^2)}_{\text{triangle}}(1,2,345) = \frac{1}{20}\Big[4s_{34}(2s_{\ell,4}+s_{\ell,5}) - 4s_{45}(s_{\ell,3}+2s_{\ell,4}) + 8s_{35}(s_{\ell,3}-s_{\ell,5}) - s_{34}^2$$
$$-6s_{35}^2 + 7s_{45}^2 + 2s_{34}s_{35} + 16s_{34}s_{45} + 14s_{35}s_{45}\Big]N_{+|12345|-}$$
$$+\frac{1}{20}\Big[12s_{\ell,4}(s_{\ell,4}+s_{\ell,5}) + 4s_{\ell,5}^2 + 2s_{\ell,3}(5s_{35}-s_{45})$$
$$+2s_{\ell,4}(8s_{35}-s_{34}-3s_{45}) - s_{\ell,5}(s_{34}+2s_{35}+2s_{45}) + 4s_{34}^2 + 15s_{35}^2$$
$$-5s_{45}^2 + 10s_{34}s_{35} + 22s_{35}s_{45}\Big]t_8(34,1,2,5)$$
$$+\frac{1}{20}\Big[12s_{\ell,4}(s_{\ell,3}+s_{\ell,4}) + 4s_{\ell,3}^2 - s_{\ell,3}(18s_{34}+6s_{35}+11s_{45})$$
$$-2s_{\ell,4}(15s_{34}+14s_{35}+11s_{45}) + 2s_{\ell,5}(s_{34}-5s_{35}) + 15s_{34}^2 + 27s_{35}^2$$
$$+14s_{45}^2 + 25s_{34}s_{45} + 52s_{34}s_{35} + 47s_{35}s_{45}\Big]t_8(45,1,2,3)$$

$$
\begin{aligned}
&-\frac{1}{20}\Big[12s_{\ell,3}^2+8s_{\ell,4}^2+12s_{\ell,5}^2+12s_{\ell,4}(s_{\ell,3}+s_{\ell,5})\\
&-s_{\ell,3}(17s_{34}+22s_{35}+5s_{45})-s_{\ell,4}(9s_{34}+12s_{35}+19s_{45})\\
&-s_{\ell,5}(7s_{34}+2s_{35}+19s_{45})+29s_{34}^2+18s_{35}^2+35s_{45}^2+36s_{34}s_{45}\\
&+39s_{34}s_{35}+29s_{35}s_{45}\Big]t_8(35,1,2,4)\,, \qquad \text{(A.15)}
\end{aligned}
$$

$$
\begin{aligned}
N^{(\zeta_2^2)}_{\text{triangle}}(1,23,45) &= s_{23}s_{45}N_{+|12345|-}\\
&-\frac{1}{2}s_{45}(s_{\ell,2}-s_{\ell,3}+s_{12}-s_{13}-s_{23})t_8(23,1,4,5)\\
&+\frac{1}{2}s_{23}(s_{\ell,5}-s_{\ell,4}+s_{45})t_8(45,1,2,3)\,, \qquad \text{(A.16)}\\
N^{(\zeta_2^2)}_{\text{triangle}}(1,[23],45) &= \frac{1}{20}s_{45}\Big[4(s_{\ell,4}^2+s_{\ell,5}^2)-(7s_{\ell,4}+s_{\ell,5})s_{45}+11s_{45}^2\Big]t_8(23,1,4,5)\,,\\
N^{(\zeta_2^2)}_{\text{triangle}}(1,23,[45]) &= \frac{1}{20}s_{23}\Big[4(s_{\ell,2}^2+s_{\ell,3}^2)+s_{\ell,2}(8s_{12}+s_{23})+s_{\ell,3}(8s_{13}+7s_{23})\\
&+4s_{12}^2+4s_{13}^2+11s_{23}^2+s_{12}s_{23}+7s_{13}s_{23}\Big]t_8(45,1,2,3)\,, \qquad \text{(A.17)}\\
N^{(\zeta_2^2)}_{\text{bubble}}([12],345) &= \frac{1}{20}\Big[4s_{\ell,3}(s_{45}-2s_{35})+8s_{\ell,4}(s_{45}-s_{34})+4s_{\ell,5}(2s_{35}-s_{34})+(s_{34}-s_{35})^2\\
&-7(s_{35}+s_{45})^2+4(3s_{35}^2-4s_{34}s_{45})\Big]t_8(12,3,4,5)\,, \qquad \text{(A.18)}\\
N^{(\zeta_2^2)}_{\text{bubble}}(1,2345) &= \frac{1}{5}(3s_{24}-s_{25}+s_{34}+3s_{35})N_{+|12345|-}\\
&-\frac{1}{20}(12s_{\ell,3}+8s_{\ell,4}+4s_{\ell,5}+s_{23}+20s_{25}-2s_{35})t_8(23,1,4,5)\\
&+\frac{1}{20}(12s_{\ell,3}+24s_{\ell,4}+12s_{\ell,5}-5s_{23}+8s_{25}-17s_{34}-2s_{45})t_8(24,1,3,5)\\
&+\frac{1}{10}\Big[6(s_{\ell,2}-s_{\ell,5})+2(s_{\ell,3}-s_{\ell,4})-4(s_{23}-s_{45})-7s_{24}-8s_{25} \qquad \text{(A.19)}\\
&\quad -3s_{35}\Big]t_8(25,1,3,4)+\frac{1}{10}(6s_{24}-8s_{25}+s_{34}+6s_{35})t_8(34,1,2,5)\\
&-\frac{1}{20}(12s_{\ell,2}+24s_{\ell,3}+12s_{\ell,4}-10s_{23}+4s_{25}+5s_{34}+17s_{45})t_8(35,1,2,4)\\
&+\frac{1}{20}(4s_{\ell,2}+8s_{\ell,3}+12s_{\ell,4}-2s_{24}-12s_{25}-8s_{34}+4s_{35}-s_{45})t_8(45,1,2,3)\,,\\
N^{(\zeta_2^2)}_{\text{bubble}}(1,[23]45) &= \frac{1}{20}\Big[4(s_{\ell,2}+s_{\ell,3})(s_{45}-2s_{14}+2s_{23})+8s_{\ell,4}(2s_{45}+s_{14})\\
&+4s_{\ell,5}(s_{45}+3s_{14}-2s_{23})+10(s_{12}+s_{13})^2+6s_{14}(s_{12}+s_{13})\\
&+9s_{14}^2+24s_{23}(s_{12}+s_{13})+8s_{14}s_{23}+4s_{23}^2\Big]t_8(23,1,4,5)\,, \qquad \text{(A.20)}\\
N^{(\zeta_2^2)}_{\text{bubble}}(1,2[34]5) &= \frac{1}{20}\Big[8(s_{\ell,3}+s_{\ell,4})(s_{12}-s_{15})-4s_{\ell,2}(s_{15}+3s_{25})+4s_{\ell,5}(s_{12}+3s_{25})\\
&+46s_{12}^2+3(11s_{12}-3s_{15})(s_{13}+s_{14})\\
&+12(3s_{12}-2s_{34})(s_{23}+s_{24})\Big]t_8(34,1,2,5)\,, \qquad \text{(A.21)}\\
N^{(\zeta_2^2)}_{\text{bubble}}(1,23[45]) &= \frac{1}{20}\Big[4s_{\ell,2}(2s_{12}-s_{13}+s_{23})-8s_{\ell,3}(s_{13}+2s_{23})-4(s_{\ell,4}+s_{\ell,5})(2s_{12}+3s_{23})\\
&+6s_{12}^2-7s_{13}^2+4s_{23}^2-14s_{12}s_{13}-12s_{13}s_{23}\Big]t_8(45,1,2,3)\,. \qquad \text{(A.22)}
\end{aligned}
$$

B Gravitational one-loop matrix element from closed strings

In this appendix, we sketch the closed-string analogue of the discussion in section 5.1, where one-loop matrix elements of gauge multiplets were recovered from the $\tau \to i\infty$ contributions to open-string amplitudes. The key idea for the gravitational counterparts is to decompose the expression (5.1) for closed-string one-loop amplitudes in direct analogy with (5.12),

$$M^{\text{1-loop}}_{n,\text{closed}} = \int \mathrm{d}^D\ell \int_{\mathfrak{F}} \mathrm{d}^2\tau \, e^{-4\pi\alpha' \operatorname{Im}\tau\, \ell^2} \Big\{ \underbrace{\mathcal{F}(\ell,\tau) - \mathcal{F}(\ell,i\infty)}_{(i)} + \underbrace{\mathcal{F}(\ell,i\infty)}_{(ii)} \Big\}\,, \tag{B.1}$$

where the following quantity no longer depends on ℓ^2

$$\mathcal{F}(\ell,\tau) = \int_{\mathfrak{T}_\tau^{n-1}} \mathrm{d}^2 z_2 \dots \mathrm{d}^2 z_n \, e^{4\pi\alpha' \operatorname{Im}\tau\, \ell^2} \left|\mathcal{K}_n(\ell,\tau)\, \mathcal{J}_n(\ell,\tau)\right|^2 . \tag{B.2}$$

We focus on the dominant contribution (ii), where the worldsheet correlator and the integrals over the punctures are evaluated on a degenerate torus. From the leftover integral over τ in the expression (B.1) for $M^{\text{1-loop}}_{n,\text{closed}}\big|_{(ii)}$, one can extract

$$\int_{\mathfrak{F}} \mathrm{d}^2\tau \, e^{-4\pi\alpha' \operatorname{Im}\tau \ell^2} = \frac{1}{4\pi\alpha'\ell^2} + \mathcal{O}((\ell^2)^0) \tag{B.3}$$

as the dominant contribution from the cusp, see (5.13) for the open-string counterpart. By largely following the steps in (5.14), the quantity $\mathcal{F}(\ell,i\infty)$ will now be related to the loop integrand in the proposal (3.2) for gravitational one-loop matrix elements.

Based on the change of variables (5.6) and the relation (5.9) between chiral integrands and half integrands of the ambitwistor string, we can rewrite

$$\begin{aligned} \lim_{\tau\to i\infty} \mathcal{F}(\ell,\tau) &\to \frac{1}{(4\pi^2)^{n-1}} \int_{\mathbb{C}^{n-1}} \frac{\mathrm{d}^2\sigma_2 \dots \mathrm{d}^2\sigma_n}{|\sigma_2 \dots \sigma_n|^2} \lim_{\tau\to i\infty} e^{4\pi\alpha' \operatorname{Im}\tau \ell^2} \left|\mathcal{K}_n(\ell,\tau)\, \mathcal{J}_n(\ell,\tau)\right|^2 \\ &= \frac{1}{(4\pi^2)^3} \int_{\mathbb{C}^{n-1}} \mathrm{d}^2\sigma_2 \dots \mathrm{d}^2\sigma_n \, |\mathcal{I}_n(\ell)|^2 \prod_{j=1}^n |\sigma_{j+}|^{4\alpha'\ell\cdot k_j} \prod_{1\le i<j}^n |\sigma_{ij}|^{4\alpha' k_i\cdot k_j} . \end{aligned} \tag{B.4}$$

In the parametrization of the torus worldsheet through the parallelogram with corners $-\frac{\tau}{2}$, $-\frac{\tau}{2}{+}1$, $\frac{\tau}{2}{+}1$ and $\frac{\tau}{2}$, the limit $\tau \to i\infty$ recovers $\mathbb{C}^{n-1}$ from the integration domain in (B.2).

As a next step, we insert the Parke-Taylor decomposition (2.8) of the ambitwistor-string correlators into (B.4) and identify the sphere integrals defined in (2.23),

$$\begin{aligned} \lim_{\tau\to i\infty} \mathcal{F}(\ell,\tau) &= \frac{1}{(4\pi^2)^3} \left(\frac{\pi}{\alpha'}\right)^{n-1} \sum_{\rho,\gamma\in S_{n-1}} N_{+|\rho|-} \bar{N}_{+|\gamma|-} \lim_{k_\pm\to\pm\ell} J(+,\gamma,-|+,\rho,-) \\ &= \frac{1}{(4\pi^2)^3} \left(\frac{\pi}{\alpha'}\right)^{n-1} M^{\text{1-loop}}_{n,\text{eff}} . \end{aligned} \tag{B.5}$$

In summary, the above assumptions and the restriction to the singular term ℓ^{-2} in (B.3) lead to a tentative string origin

$$M^{\text{1-loop}}_{n,\text{closed}}\big|_{(ii)} = \frac{1}{(2\pi)^8} \left(\frac{\pi}{\alpha'}\right)^n M^{\text{1-loop}}_{n,\text{eff}} + \text{analytic} \tag{B.6}$$

of gravitational one-loop matrix elements (3.2). Regular terms $(\ell^2)^{\geq 0}$ in (B.3) may additionally introduce analytic contributions to (B.6), i.e. without any branch cuts corresponding to massless particles similar to the ones in part (i) of (B.1).

Similar to the partitioning of open-string amplitudes in (5.12), gravitational one-loop matrix elements have been formally recovered from the terms (ii) in the closed-string amplitude (B.1) which gather the dominant contributions from $\tau \to i\infty$. This is the boundary of moduli space where the torus degenerates to a nodal sphere and the origin of all the poles in ℓ^2. Conversely, the one-loop matrix elements do not account for the terms (i) of closed-string amplitudes and the regular terms in (B.3) which are free of massless quadratic propagators $(\ell+K)^{-2}$ (with external momenta K) after integration over the punctures. Hence, $M^{\text{1-loop}}_{n,\text{eff}}$ is insensitive to the analytic dependence of closed-string one-loop amplitudes that arises from (i) and the regular terms in (B.3) and plays a crucial role for the one-loop effective action of type-II theories.

C Laurent series from logarithmic divergences

In this appendix, we discuss an alternative approach to constructing the Laurent polynomials in the Schwinger parameter t from section 6.3 via direct momentum-space calculations of logarithmic UV divergences in dimensional reduction. The following integrals have been used for the one-loop four and five-point matrix elements,

$$\text{bubble:} \qquad I_2(k) = \mu^{2\epsilon} e^{\epsilon\gamma_\text{E}} \int \frac{\mathrm{d}^D\ell}{\pi^{D/2}} \frac{1}{\ell^2(\ell+k)^2}\,, \tag{C.1a}$$

$$\text{one-mass triangle:} \qquad I_3(k_1,k_2) = \mu^{2\epsilon} e^{\epsilon\gamma_\text{E}} \int \frac{\mathrm{d}^D\ell}{\pi^{D/2}} \frac{1}{\ell^2(\ell+k_1)^2(\ell+k_{12})^2}\,, \tag{C.1b}$$

$$\text{two-mass triangle:} \qquad I_3(k_1,k_2+k_3) = \mu^{2\epsilon} e^{\epsilon\gamma_\text{E}} \int \frac{\mathrm{d}^D\ell}{\pi^{D/2}} \frac{1}{\ell^2(\ell+k_1)^2(\ell+k_{123})^2}\,, \tag{C.1c}$$

$$\text{one-mass box:} \qquad I_4(k_1,k_2,k_3) = \mu^{2\epsilon} e^{\epsilon\gamma_\text{E}} \int \frac{\mathrm{d}^D\ell}{\pi^{D/2}} \frac{1}{\ell^2(\ell+k_1)^2(\ell+k_{12})^2(\ell+k_{123})^2}\,, \tag{C.1d}$$

where μ^2 is an arbitrary mass scale that compensates the dimension shift $D = \mathcal{D} - 2\epsilon$ such that the mass dimension of the integral is not changed, and $e^{\epsilon\gamma_\text{E}}$ factor absorbs all the Euler constant γ_E in the ϵ-expansion. All the k_i in (C.1b) to (C.1d) are lightlike, while the momentum in I_2 is in general off-shell, i.e. $k^2 \neq 0$. We recall that the signature of the spacetime metric is $(-,+,+,\ldots,+)$ and the integration measure $\mathrm{d}^D\ell = -i\mathrm{d}\ell^0\,\mathrm{d}\ell^1\ldots\mathrm{d}\ell^{D-1}$ is taken to absorb the factor of i which appears in the more common conventions for Feynman integrals.

C.1 Logarithmic in one dimension is power in another

The first point to make is what the section title says: UV power divergences in a specific dimension, $\mathcal{D}$, can be probed using logarithmic UV divergences in other dimensions. This idea is well known, if not often discussed, in the amplitudes and QFT communities. However, since the reader may not be familiar with this idea, we present a brief argument for it here.

The idea is most obvious when considering the Schwinger representation of (6.14) and (6.16) where, in $D_{\rm crit} = \mathcal{D}$, an integrand that carries t^{-1} will produce the logarithm divergence in $\mathcal{D}$ while all other powers of t will contribute to power divergences. However, from the point of view of the t integration, there is nothing special about the choice of D in integrand prefactor of $t^{n-D/2}$. If we look at the coefficient of t^{-1+m} within the t integration, in addition to interpreting the term as sourcing a power divergence we could instead think of m as shifting the dimension away from $\mathcal{D}$. Running this logic in reverse, we can find the coefficient of t^{-1+m} in $\mathcal{D}$ dimensions by evaluating the logarithmic divergence of the same momentum-space integrand in $D_{\rm crit} = \mathcal{D} + 2m$ dimensions instead. Importantly, this should be thought of as an integration trick and not an actual change to the underlying theory: we are not manipulating the dimension the theory and its states live in, merely exploring the analytic structure of the loop integral.

C.2 Momentum-space perspective on UV divergences

We now present an alternate method for directly extracting the logarithmic UV divergence in a given dimension directly from the momentum-space integral, which is the one-loop analogue of the methods employed in many studies of multi-loop super-Yang-Mills and supergravity (see [8, 100, 219, 220] for a small sample of these studies). We begin by scaling the loop momenta via $\ell \to \beta\ell$ and series expanding the integrand near $\beta \to \infty$. The resulting expressions will contain only powers of ℓ^2 in the denominator, with all of the $\ell \cdot k_i$ relegated to the numerator. Such terms require only no-scale tensor reduction, e.g. $\ell^\mu \ell^\nu \to \frac{\ell^2}{D}\eta^{\mu\nu}$. After the reduction, the numerators n_λ will be free of ℓ and the denominators will be powers $\lambda \in \mathbb{Z}$ of ℓ^2,

$$\int \frac{{\rm d}^D\ell}{\pi^{D/2}} I(\ell)\bigg|_{\rm UV} \to \int \frac{{\rm d}^D\ell}{\pi^{D/2}} \sum_\lambda \frac{\bar{n}_\lambda}{\beta^{2\lambda}(\ell^2)^\lambda}\bigg|_{\rm UV}, \tag{C.2}$$

where $I(\ell)$ refers to a generic loop integrand, and the $\bar{n}_\lambda$ are polynomials in the external momenta. In dimensional regularization, integrals of this form are set to zero via a cancellation between UV and IR divergences. Since we are interested in the UV divergence specifically, we can introduce a mass to regulate the IR divergence separately from the UV divergence. The logarithmic UV divergence in $D = D_{\rm crit} - 2\epsilon$ spacetime dimensions (given the mostly-plus metric convention and $s_{ij} > 0$ region) is given by

$$\int \frac{{\rm d}^D\ell}{\pi^{D/2}} \frac{1}{(\ell^2)^{D_{\rm crit}}}\bigg|_{\rm UV} \to \int \frac{{\rm d}^D\ell}{\pi^{D/2}} \frac{1}{(\ell^2 - m^2)^{D_{\rm crit}/2}} = \frac{\Gamma(\epsilon)}{\Gamma(\frac{D_{\rm crit}}{2})(m^2)^\epsilon} = \frac{1}{\Gamma(\frac{D_{\rm crit}}{2})\epsilon} + \mathcal{O}(m, \epsilon^0)\,. \tag{C.3}$$

The UV divergence is the residue of the simple pole $\frac{1}{\epsilon}$, with the IR regulator decoupled into the non-singular terms of order $\epsilon^{\geq 0}$. Thus, the various orders of β in the large-ℓ expansion of (C.2) correspond to logarithmic UV divergences via the mapping

$$\int \frac{{\rm d}^D\ell}{\pi^{D/2}} I(\ell)\bigg|_{\rm UV} \to \bar{n}_{D_{\rm crit}/2} \int \frac{{\rm d}^D\ell}{\pi^{D/2}} \frac{1}{\beta^{D_{\rm crit}}(\ell^2)^{D_{\rm crit}/2}}\bigg|_{\rm UV} \to \frac{\bar{n}_{D_{\rm crit}/2}}{\Gamma(\frac{D_{\rm crit}}{2})\epsilon}\,. \tag{C.4}$$

As discussed above, these UV divergences directly correspond to terms in the expansion of the Schwinger exponent. For example, we can apply (C.4) to the scalar one-mass triangle (C.1b),

$$I_3(k_1,k_2) \xrightarrow{\text{UV}} \begin{array}{|c|c|c|c|c|}\hline D_{\text{crit}}=6 & D_{\text{crit}}=8 & D_{\text{crit}}=10 & D_{\text{crit}}=12 & D_{\text{crit}}=14 \\ \hline \frac{1}{\epsilon} & -\frac{s_{12}}{24\epsilon} & \frac{s_{12}^2}{360\epsilon} & -\frac{s_{12}^3}{6720\epsilon} & \frac{s_{12}^4}{151200\epsilon} \\ \hline \end{array}\,, \tag{C.5}$$

which exactly matches the expansion in (6.18). The constructions in section 6.4 have all been reproduced using this method as well. Then, as expected from the arguments in appendix C.1, the D-dimensional UV divergences at the residues of the simple poles in $\epsilon = \frac{1}{2}(D_{\text{crit}} - D)$ are related to the Laurent polynomials via

$$\text{coef}_{1/\epsilon}\, \pi^{-D/2} \widehat{A}^{\text{1-loop}}_{\text{eff}}(1,2,3,4)\,|_{\text{UV}} = \text{coef}_{(-2i\alpha' T)^{D_{\text{crit}}/2-4}}\, a^{\text{1-loop}}_{\text{open}}\left(1,2,3,4;\tau=\frac{T}{\pi}\right), \tag{C.6}$$

which is equivalent to (6.23). The procedure can be just as easily applied to the closed-string integrands which leads to the expected pattern of UV divergences

$$\text{coef}_{1/\epsilon}\, \pi^{-D/2} \widehat{M}^{\text{1-loop}}_{4,\text{eff}}\,|_{\text{UV}} = \text{coef}_{(4\alpha' y)^{D_{\text{crit}}/2-4}}\, m^{\text{1-loop}}_{4,\text{closed}}\left(\text{Im}\,\tau=\frac{y}{\pi}\right), \tag{C.7}$$

equivalent to (6.36).

C.3 Momentum-space and $D_{\text{crit}} \leq 0$

For the UV divergences along with higher ζ values, like those in (6.24), (6.25) and (6.38), we encounter powers like $T^{-4}, T^{-5}, \ldots$ in the associated Laurent polynomial (6.8) for open string, and $y^{-4}, \ldots$ in (6.4) for closed string. By (C.6) and (C.7), these UV divergences occur in formally vanishing or negative spacetime dimensions $D \leq 0$. There are two points of interest in these cases. The first is that the momentum-space tensor reduction generates a prefactor $\frac{1}{D(D+2)\ldots}$ that will contribute a $\frac{1}{\epsilon}$ pole when $D_{\text{crit}} = -2N$ is even and non-positive. Second, the Γ-function representation that (C.4) comes from,

$$\frac{\Gamma(\epsilon)}{\Gamma\left(\frac{D_{\text{crit}}}{2}\right)}, \tag{C.8}$$

has an infinity-over-infinity ambiguity and thus needs to be properly regulated. Since the tensor reduction already provides the $\frac{1}{\epsilon}$ pole, the simplest regulation scheme should lead to divergence-free contributions. To this end, we propose the simple scheme,

$$\frac{\Gamma(\epsilon)}{\Gamma\left(\frac{D_{\text{crit}}}{2} - \epsilon\right)}. \tag{C.9}$$

For $D_{\text{crit}} > 0$, the additional ϵ regulator is irrelevant, reducing the expression back to (C.4) in the leading-ϵ expansion. However, for $D_{\text{crit}} \leq 0$, we instead get

$$\frac{\Gamma(\epsilon)}{\Gamma\left(\frac{D_{\text{crit}}}{2} - \epsilon\right)} \xrightarrow[D_{\text{crit}} \text{ even and } D_{\text{crit}} \leq 0]{\epsilon\to 0} (-1)^{\frac{D_{\text{crit}}}{2}+1}\Gamma\left(\frac{|D_{\text{crit}}|}{2}+1\right). \tag{C.10}$$

Applying this regularization scheme to the leading terms of ζ_6, ζ_7, and ζ_8 in (6.24) exactly lines up with the established correspondence of (C.6) as desired.

C.4 Integrals with closed formulas

As yet another check, we can extract UV divergence and non-analytic terms from those integrals with closed formulas. The bubble integral (C.1a) and the one-mass triangle (C.1b) are given by

$$\begin{aligned} I_2(k) &= \mu^{2\epsilon} e^{\epsilon\gamma_E} \frac{\Gamma(2-\frac{D}{2})\Gamma(\frac{D}{2}-1)^2}{\Gamma(D-2)} (k^2)^{\frac{D}{2}-2} \\ &\xrightarrow{D=10-2\epsilon} -\frac{k^6}{840}\left[\frac{1}{\epsilon} - \log\frac{k^2}{\mu^2}\right] - \frac{44k^6}{11025} + \mathcal{O}(\epsilon)\,, \end{aligned} \tag{C.11a}$$

$$\begin{aligned} I_3(k_1,k_2) &= \mu^{2\epsilon} e^{\epsilon\gamma_E} \frac{\Gamma(3-\frac{D}{2})\Gamma(\frac{D}{2}-2)^2}{\Gamma(D-3)} (s_{12})^{\frac{D}{2}-3} \\ &\xrightarrow{D=10-2\epsilon} \frac{s_{12}^2}{360}\left[\frac{1}{\epsilon} - \log\frac{s_{12}}{\mu^2}\right] + \frac{17 s_{12}^2}{1800} + \mathcal{O}(\epsilon)\,. \end{aligned} \tag{C.11b}$$

Both of the formulas can be found in [186]. On the other hand, the two-mass triangle (C.1c) is not a master integral. It can be reduced to a combination of two bubbles

$$I_3(k_1, k_2{+}k_3) = \frac{2(D-3)}{D-4}\,\frac{I_2(k_2{+}k_3) - I_2(k_1{+}k_2{+}k_3)}{s_{12}+s_{13}}\,. \tag{C.12}$$

The one-mass box (C.1d) also has a closed formula [221, 222],

$$I_4(k_1,k_2,k_3) = \frac{\mu^{2\epsilon}e^{\epsilon\gamma_E}C_D}{s_{12}s_{23}}\left[(s_{12})^{\frac{D}{2}-2}\mathcal{F}\Big(\frac{s_{13}}{s_{23}}\Big) + (s_{23})^{\frac{D}{2}-2}\mathcal{F}\Big(\frac{s_{13}}{s_{12}}\Big) - (s_{123})^{\frac{D}{2}-2}\mathcal{F}\Big(\frac{s_{13}s_{123}}{s_{12}s_{23}}\Big)\right], \tag{C.13}$$

where

$$C_D = \frac{8\Gamma(3-\frac{D}{2})\Gamma(\frac{D}{2}-1)^2}{(D-4)^2\Gamma(D-3)}\,, \qquad \mathcal{F}(x) = {}_2F_1\left[\begin{matrix}1, \frac{D}{2}-2 \\ \frac{D}{2}-1\end{matrix}; -x\right]. \tag{C.14}$$

The UV divergence of the one-mass box at $D = 10{-}2\epsilon$ is

$$I_4(k_1,k_2,k_3) \xrightarrow{D=10-2\epsilon} -\frac{s_{12}+s_{23}+s_{123}}{120\epsilon} + \mathcal{O}(\epsilon^0)\,. \tag{C.15}$$

The non-analytic terms at ϵ^0 order contain logarithms and first-order derivatives of hypergeometric functions.

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