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Properties of invariant sets in certain two-dimensional dynamical systems

Renormalization and beyond

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Abstract
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This thesis consists of an introduction and four research papers concerning dynamical systems, focusing on renormalization in two dimensions.

Paper I (joint with Jordi-Lluís Figueras) studies a generalization of the anti-integrable limit of skew-product systems based on a Frenkel-Kontorova model. It is shown that under certain regularity conditions the orbits of such a system has a Cantor set structure and the existence of a non-smooth folding bifurcation is deduced.

Paper II studies the invariant Cantor sets of period doubling type of infinitely renormalizable area-preserving maps in the universality class of the Eckmann-Koch-Wittwer renormalization fixed point. It is shown that for such maps the invariant Cantor set is always contained in a Lipschitz curve but never in a smooth (meaning C^1) curve.

Paper III (joint with Denis Gaidashev) continues the study of the invariant Cantor sets of infinitely renormalizable area-preserving maps, focusing on their geometry. It is shown that there is always a positive measure of unbounded geometry. Additionally, some results giving control over the average expansion of vectors by the renormalization microscope are also provided.

Finally paper IV (joint with Denis Gaidashev) considers renormalization of two-dimensional perturbations of almost commuting pairs of holomorphic maps. A renormalization operator is developed and statements about the hyperbolicity, universality and non-rigidity of this operator are proven.

Keywords: Dynamical systems, renormalization, area-preserving maps, unbounded geometry, almost commuting pairs, universality, non-rigidity

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To Malin

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I J.-L. Figueras and D. Lilja, Existence of non-smooth bifurcations of uniformly hyperbolic invariant manifolds in skew product systems, *Nonlinearity* (2018), 31, pp. 5573-5588

II D. Lilja, On the Invariant Cantor sets of Period Doubling Type of Infinitely Renormalizable Area-Preserving Maps, *Commun. Math. Phys.* (2018) 358, pp. 1027-1039

III D. Gaidashev and D. Lilja, On the geometry of period doubling invariant sets for area-preserving maps, *Manuscript*, 2022

IV D. Gaidashev and D. Lilja, Universality but no rigidity for two-dimensional perturbations of almost commuting pairs, arXiv: 2204.02775, 2022, submitted for publication

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1. Introduction

This thesis consists of the following introduction, in which we give a historical context for the work in the thesis and relevant preliminaries, followed by a summary of the papers included in the thesis, a shorter summary in Swedish and finally the papers themselves.

1.1 Background

Certain dynamical systems have the quite remarkable property that iterate of it contains a small copy of the original system. An important and illustrative example is given by the *logistic family* $F_r(x) = rx(1-x)$ where $x \in [0,1]$ and $r \in [2,4]$. We can clearly see this small copy of $F_{3.4}$ by looking at a graph of $F_{3.4}^2$.

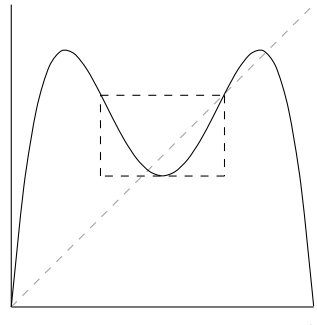

Figure 1.1. The graph of $F_{3.4}^2$.

When encountering such a situation it is sometimes possible to rescale this smaller copy back onto the domain of the original dynamical system. This gives us a new dynamical system sharing many properties with the original system and we call this process *renormalization*.

The idea of renormalization in dynamical systems is to act as a microscope for the dynamics: zooming in on an invariant set we can create a new dynamical system which captures the dynamics of the original system at a smaller scale. By iterating this process we have created a *renormalization operator* acting on our chosen class of dynamical systems which can be analyzed as a dynamical system in its own right. Questions regarding fixed points and hyperbolicity can then be posed of the renormalization operator, the answers to which can tell us something about the dynamics of every dynamical system in

our class or some subset of it. For example if such a hyperbolic fixed point of renormalization exists it is *infinitely renormalizable* and has a great deal of self-similarity. Likewise any map contained in the stable manifold of the fixed point must also be infinitely renormalizable and one can often use the convergence towards the fixed point to derive further similarities with the fixed point itself, a typical example being conjugacy of the dynamics on an invariant Cantor set.

Renormalization techniques in dynamical systems were introduced independently by Feigenbaum ([14], [15]) and Coullet and Tresser ([48]) in order to explain the remarkable universality of the period doubling cascade in families of unimodal maps of the interval, characterized by, e.g., the Feigenbaum constants. In those papers the authors define a renormalization operator and provide numerical evidence that the operator has a hyperbolic fixed point with a one-dimensional unstable direction which, if proven, would explain the observed universality.

Such a proof was first provided by Lanford using validated numerics ([34]). A completely analytical proof took longer and required the efforts of many but was eventually achieved by Lyubich ([38])[1].

Following on the success of renormalization for interval maps it was naturally applied to other types of dynamical systems as well. The application of period doubling renormalization to higher dimensional dynamical systems was started by Collet, Eckmann and Koch in [5] in which they define a renormalization operator for analytic maps $\mathbb{C}^n \to \mathbb{C}^n$ which are real on $\mathbb{R}^n \subset \mathbb{C}^n$, perturbations of the one dimensional renormalization fixed point and highly dissipative. Assuming certain, at the time unproven but substantiated, properties they then proved the existence of a hyperbolic fixed point for these renormalizations.

A closer study of those infinitely renormalizable two-dimensional maps was initiated by Gambaudo, van Strien and Tresser in [29]. The purpose of that paper was in fact to answer a question by Smale whether there are any Kupka-Smale diffeomorphisms of S^2 with neither sinks nor sources. In order to answer this question they studied the small scale gemoetry of the infinitely renormalizable maps which are close enough to the renormalization fixed point, showing that they all have an invariant Cantor set on which the dynamics is conjugate to the dyadic adding machine.

A more extensive study of the topology and geometry of those invariant Cantor sets was initiated by de Carvalho, Lyubich and Martens in [3]. There they redefine renormalization of two-dimensional maps in a more geometric way compared to the analytical definition in [5] and centered around the critical value rather than the critical point. They show that the previous results carry over to this new setting and then begin a more in-depth study of the invariant Cantor sets, showing, for example, that they have unbounded geometry and

[1]For other major contributions, see e.g. [46], [40].

are not rigid yet still have universality. These results, together with others in their paper, highlight the distinctly two-dimensional phenomena not present in one-dimensional renormalization.

The above two-dimensional renormalizaion was done for highly dissipative Hénon-like maps which are perturbations of one-dimensional unimodal maps. At the other extreme are the area-preserving maps. The detailed study of period-doubling renormalization of area-preserving maps was also initiated, arguably, by Collet, Eckmann and Koch in the paper [4] motivated by previous results indicating universal phenomena in this setting as well. At this point it was not clear which renormalization transformation to use and in this paper they use numerics to argue for the appropriate choice of renormalization operator in this setting.

Following up on this Eckmann, Koch and Wittwer were able to prove existence and hyperbolicity of a renormalization fixed point for this setup using computer assistance in their paper [12], thus explaining the observed universality in area-preserving maps.

Further study was performed by Gaidashev and Johnson in two articles ([20], [21]) where they prove existence of an invariant stable Cantor set for infinitely renormalizable area-preserving maps as well as a horseshoe and give bounds on their respective Hausdorff dimension. A subsequent paper by Gaidashev, Johnson and Martens ([23]), similarly to [3], defines a renormalization operator which differs slightly from the one defined by [12] and with the help of the companion paper [22] proves, in addition to the previously known properties, rigidity of the dynamics on the invariant Cantor sets.

Renormalization has also been applied in other types of spaces. Its use for critical circle maps was pioneered by Feigenbaum, Kadanoff and Shenker in [16] and, in parallel, Ostlund, Rand, Sethna and Siggia in [43]. Because the space of critical circle maps is ill-suited for defining a renormalization operator[2] it is instead done for a larger space of pairs of maps on an interval called *critical commuting pairs*. In both papers the authors define a renormalization operator and use numerics to indicate the existence of a fixed point. In [16] this renormalization was only defined for maps with golden mean rotation number whereas [43] defined it for arbitrary rotation numbers. The existence and hyperbolicity of a fixed point for the golden mean rotation number was proven by Mestel in [41] though the techniques used are only local, hence they cannot tell whether this fixed point is unique or whether hyperbolicity extends beyond a small neighborhood of it.

Major breakthroughs were achieved by de Faria and de Melo who were able to prove rigidity of analytic critical circle maps with rotation number of bounded type in the papers [7], [8] and [9]. Following this Yampolsky worked

[2] see, e.g., the discussion leading up to Definition 2.1 of [52]

on rotation numbers of unbounded type, proving the existence of the full renormalization horseshoe in [51]. He also developed a new type of renormalization of critical circle maps called *cylinder renormalization* to first prove hyperbolicity of all periodic orbits of the renormalization operator in [52] and later hyperbolicity for the full renormalization horseshoe in [53].

Like unimodal renormalization, renormalization of critical circle maps has also been generalized to other settings. Yampolsky's cylinder renormalization can be seen as an example of this, being defined for maps of subsets of cylinders $\mathbb{C}/\mathbb{Z}$ and being used to study Siegel disks in [28] and [26]. More recently the technique of using commuting pairs has been adapted to *almost commuting pairs* by Gaidashev and Yampolsky in [24] and further generalized to almost commuting pairs of maps of $\mathbb{C}^2$ by Gaidashev and Yampolsky in [27] and by Yang in [54]. Both of these papers use ideas in the same spirit as that of [5] and [3] in that they are dissipative perturbations of one-dimensional systems, however they differ in the type of perturbation. While [27] uses a perturbation in which the one-dimensional systems appear diagonally embedded in the space of two-dimensional pairs [54] instead uses a Hénon-like embedding. Furthermore, while [27] considers arbitrary rotation numbers [54] only considers the golden mean rotation number but manages to prove a higher degree of universality.

1.2 Preliminaries

In this section we will give a concise introduction to some of the fundamental and relevant topics within dynamical systems theory. In the first subsection we will discuss the general theory of dynamical systems, presenting basic definitions and fundamental theorems. This will be followed by subsections dealing specifically with skew products, symplectic twist maps and critical circle maps to prepare for the topics of the papers included in the thesis.

1.2.1 Dynamical systems

The modern study of dynamical systems was initiated by Poincaré who used them to, among other things, study the n-body problem. This development came out of his submission for a prize, organized by Mittag-Leffler in honor of King Oscar II of Sweden and Norway and announced in *Acta Mathematica*. Four questions had been announced for this prize. The first and most important question as well as the one that Poincaré was trying to answer was the following, as found in [11]:

> Given a system of arbitrarily many mass points that attract each other according to Newton's laws, under the assumption that no two points ever collide, try to find a representation of the coordinates of each point as a series in a variable that

is some known function of time and for all of whose values the series converges uniformly.

Though he was ultimately unable to solve this problem his contributions were of such importance that he was awarded the prize. Up until that point solutions to differential equations had been studied by mostly analytical means, analyzing the local existence and quantitative behavior of solutions. Poincaré instead used methods of geometry and topology to study more qualitative aspects of solutions.

Today dynamical systems have found use in all areas of science, going far beyond the initial application to Hamiltonian mechanics by Poincaré. This relationship has also gone the other way with discoveries in the sciences leading to significant advances in dynamical systems. An excellent example of this is the history of chaotic dynamical systems. Although aspects of chaos in dynamical systems have been observed mathematically since the work of Poincaré it only started receiving serious attention following the work of Lorenz in [36]. In this work he developed a system of ordinary differential equations now called the *Lorenz system* as a simplified model of atmospheric convection and noticed the phenomenon now called *sensitive dependence on initial conditions*, meaning that small differences in initial conditions can lead to large deviations over time.

In this thesis we will consider only what are called *discrete dynamical systems*. We begin by giving a definition.

Definition 1. A *discrete dynamical system* is a continuous map $F\colon\ M \to M$ of a topological manifold M. The manifold M is called the *phase space* of the dynamical system.

Note that the dynamical systems we will consider in this thesis will mostly have more structure than this. For example we will be considering area-preserving diffeomorphisms of $\mathbb{R}^2$.

One of the most fundamental objects of study in dynamical systems are *attractors*. As their name suggests these are subsets of phase space that attract nearby points. A formal definition goes as follows.

Definition 2. A compact set $A \subset M$ is called an *attractor* if there is an open set U containing A such that $F(\bar{U}) \subset U$ and $A = \bigcap_{n\geq 0} F^n(U)$.

The simplest example of an attractor is an attracting fixed point, i.e. a point $p \in M$ such that $F(p) = p$ for which there is an open set U containing p such that for any $q \in U$ the sequence of points $f^n(q)$ converges to p as $n \to \infty$.

A more complicated type of attractor are the *strange attractors*. These are attractors with some kind of fractal structure. As was suspected for a long time,

and proven by Tucker (see e.g. [49]), the Lorenz system contains just such a strange attractor. This strange attractor is responsible for the chaotic dynamics observed in the Lorenz system though it should be noted that not all strange attractors are chaotic.

Another important object of study are *hyperbolic sets*.

Definition 3. Let F be a diffeomorphism. A compact F-invariant subset $\Lambda \subset M$ is called a *hyperbolic set* if there are constants $\lambda \in (0,1)$ and $C > 0$ and the tangent bundle of M over Λ admits a splitting $T_\Lambda M = E^s \oplus E^u$ such that for every $p \in \Lambda$:

1. the splitting is invariant under F:
$$D_p F(E^s(p)) \subset E^s(F(p)) \text{ and } D_p F(E^u(p)) \subset E^u(F(p)),$$
2. $\|D_p F^n(v)\| \leq C\lambda^n \|v\|$ for all $v \in E^s(p)$ and $n \geq 0$,
3. $\|D_p F^{-n}(v)\| \leq C\lambda^n \|v\|$ for all $v \in E^u(p)$ and $n \geq 0$.

The simplest examples of hyperbolic sets are again given by fixed points. A fixed point x_0 is called *hyperbolic* if $D_{x_0}F$ has no eigenvalue λ on the unit circle. The splitting of T_pM is then given by the generalized eigenvectors of $D_{x_0}F$ in the sense that E^s is spanned by the generalized eigenvectors corresponding to eigenvalues $|\lambda| < 1$ and E^u is spanned by the generalized eigenvectors corresponding to eigenvalues $|\lambda| > 1$. Similarly we call a periodic point $F^p(x_0) = x_0$ *hyperbolic* if $D_{x_0}F^p$ has no eigenvalues on the unit circle. If the entire phase space M of F is a hyperbolic set then the diffeomorphism F is called an *Anosov diffeomorphism*.

More generally we can allow the constants λ and C to depend on $x \in \Lambda$ in such a way that the rates of contraction and expansion cannot be uniformly bounded away from 1. Such sets are called *nonuniformly hyperbolic sets*. Though such sets and the study of nonuniformly hyperbolic dynamics is an important area of study we will not discuss it further here.

The *stable manifold theorem* is one of the most important results about hyperbolic sets, finding applications not only in dynamical systems. For simplicity we formulate it here only for hyperbolic fixed points though the theorem is true for any hyperbolic set and even extends to infinite dimensions, see [33].

Theorem 4 (Stable manifold theorem)**.** *Let $F\colon M \to M$ be a diffeomorphism with a hyperbolic fixed point p. Then*

1. *The set $W^s(p) = \{q \in M\colon \lim_{n\to\infty} F^n(q) = p\}$ is a smooth embedded disk and $T_pW^s(p) = E^s(p)$.*
2. *The set $W^u(p) = \{q \in M\colon \lim_{n\to\infty} F^{-n}(q) = p\}$ is a smooth embedded disk and $T_pW^u(p) = E^u(p)$.*

1.2.2 Skew products

Skew products are one of the simplest ways of coupling two dynamical systems. This is commonly used to model the effect of an external force acting on a dynamical system. Formally, we will consider the following situation.

Definition 5. A *skew product system* is a pair of maps $h\colon M \to M$ and $f\colon M \times \mathbb{R}^n \to \mathbb{R}^n$, where M is a compact manifold, that form a dynamical system $F\colon (\theta, x) \mapsto (h(\theta), f(\theta, x))$.

Remark 6. More generally skew products can be defined for $f\colon M \times N$ where N is any phase space though we will restrict attention to the case $N = \mathbb{R}^n$.

In this formalism h models the dynamics of the external force while f models the dynamics of the system under influence of the external force. One important special case is given by $M = \mathbb{T}^n$ with $h(\theta) = \theta + \omega$ where $\omega \in \mathbb{R}^n$ is totally irrational, i.e. $m \cdot \omega \neq 0$ for all nonzero $m \in \mathbb{Z}^n$. These systems are called *quasiperiodically forced*.

A central object of study in skew product systems, especially quasiperiodically forced systems, are the so called *fiberwise hyperbolic invariant manifolds* (FHIM). These manifolds are the graph of a function $K\colon M \to \mathbb{R}^n$. Invariance means that $f(\theta, K(\theta)) = K(h(\theta))$. Lastly being fiberwise hyperbolic means that $M \times \mathbb{R}^n$, when considered as a trivial fiber bundle over M, decomposes into a Whitney sum $E^s \oplus E^u$ such that there are constants $C > 0$ and $0 < \lambda < 1$ satisfying

$$
\begin{aligned}
(\theta, v) \in E^s &\;\Rightarrow\; \|\pi_{\mathbb{R}^n}(h, D_2 f)(\theta, v)\| \leq C\lambda^k \|v\| \quad \forall k \geq 0, \\
(\theta, v) \in E^u &\;\Rightarrow\; \|\pi_{\mathbb{R}^n}(h, D_2 f)(\theta, v)\| \leq C\lambda^k \|v\| \quad \forall k \leq 0.
\end{aligned}
$$

For quasiperiodically forced systems FHIM are among the simplest persistent objects to study since fixed points cannot exist. Adding further importance to the study of FHIM it was shown in [31] that bifurcations of attracting FHIM, i.e. with $E^u = M \times \{0\}$, can create *strange nonchaotic attractors*[3]. Thus bifurcations of FHIM can be used to study the boundaries of chaos.

The study of bifurcations of non-attracting FHIM has proven more difficult. Recently, in [17], the authors studied bifurcations of non-attracting FHIM in the quasiperiodically driven Chirikov standard map $(R_\omega, f) : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{T} \times \mathbb{R}^2$ given by the system

$$
\begin{cases}
\bar{\theta} &= \theta + \omega \mod 1 \\
\bar{x} &= x + \bar{y} \\
\bar{y} &= y - \frac{\partial W}{\partial x}(\theta, x)
\end{cases}
$$

[3] In fact, in this paper the authors also proved the existence of strange nonchaotic attractors.

With ω irrational and $W(\theta, x) = \gamma x \sin(2\pi\theta) - \frac{\kappa}{(2\pi)^2}\cos(2\pi x)$ they numerically observerd three types of bifurcations which they called *smooth bifurcation*, *spiky breakdown* and *folding breakdown*. For the folding breakdown there is, for every κ in some interval, a critical value $\gamma_c = \gamma_c(\kappa)$ such that the system satisfies the following:

1. For $\gamma < \gamma_c$ there is a uniformly hyperbolic FHIM given as the graph of $K_\gamma \colon \mathbb{T} \to \mathbb{R}^2$ with $\dim(E^u) = \dim(E^s) = 1$.
2. For $\gamma = \gamma_c$ there is a $\theta_0 \in \mathbb{T}$ such that $\partial_\theta K_{\gamma_c}(\theta_0 + k\omega) = \infty$ for all $k \in \mathbb{Z}$.
3. For $\gamma > \gamma_c$ there is a bounded measurable invariant object with positive Lyapunov exponent, a so called *strange saddle*.

1.2.3 Symplectic twist maps

We will now introduce one of the central objects of this thesis, namely *symplectic twist maps*.

Definition 7. A *symplectic twist map* of $\mathbb{R}^2$ is a diffeomorphism $F \colon \mathbb{R}^2 \to \mathbb{R}^2$, $(x, y) \mapsto (X, Y)$ satisfying:

1. F is isotopic to the identity,
2. *the twist condition*: the map $(x, y) \mapsto (x, X(x, y))$ is a diffeomorphism of $\mathbb{R}^2$,
3. F is a *symplectomorphism*: $F^*(dx \wedge dy) = dx \wedge dy$.

Symplectomorphisms arise naturally in the study of Hamiltonian dynamics, for example as time-1 maps of the Hamiltonian flow or as return maps around periodic orbits of the Hamiltonian flow. If the fixed point of such a return map is elliptic then, generically, the return map is a symplectic twist map. Symplectic twist maps also appear in convex billiards. For more information on these and other ways that symplectic twist maps naturally appear, see [30].

Remark 8. Note that F being a symplectomorphism of $\mathbb{R}^2$ is equivalent to F being area-preserving. Since we are only concerned with this setting we will use the terms interchangeably.

Remark 9. A convenient reformulation of the twist condition is to require that the map $y \mapsto X(x, y)$ be monotonic for all x or equivalently that $\frac{\partial X}{\partial y} \neq 0$. We will call a twist map *positive* if $\frac{\partial X}{\partial y} > 0$ and *negative* if $\frac{\partial X}{\partial y} < 0$,

Since F is a symplectomorphism of $\mathbb{R}^2$ it follows from the Poincaré lemma[4] that F is in fact an exact symplectomorphism, i.e., there is a function S such that $F^*(ydx) - ydx = YdX - ydx = dS$. Using the twist condition we can write $dS = dS(x, X)$. It then follows that we can write

$$\begin{aligned} y &= -S_1(x, X) \\ Y &= S_2(x, X) \end{aligned}$$

where the subscripts denote partial derivatives, $S_i(x, X) = \partial_i S(x, X)$. The function S is called the *generating function* of the map F since it allows us to rewrite F as

$$\begin{pmatrix} x \\ -S_1(x, X) \end{pmatrix} \overset{F}{\mapsto} \begin{pmatrix} X \\ S_2(x, X) \end{pmatrix}.$$

Definition 10. A symplectic twist map satisfies the *ratchet phenomenon* in $U \subset \mathbb{R}^2$ if there are cones Θ_v and Θ_h, centered around the y-axis and the x-axis respectively, such that if p, p' are any two points in U such that $p' \in p + \Theta_v$ then $F(p') \in F(p) + \Theta_h$.

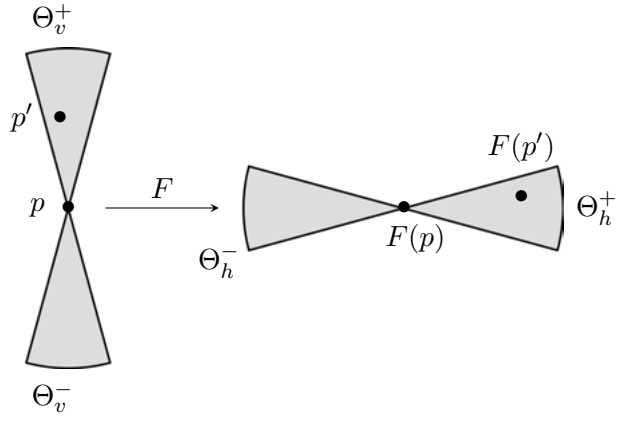

Figure 1.2. The ratchet phenomenon for a positive twist map. The signs are reversed for a negative twist map.

The ratchet phenomenon is satisfied by many twist maps. For example Lemma 12.1 of [30] states that if there is a constant $a > 0$ such that the twist map satisfies

$$\frac{\partial X}{\partial y} > a$$

[4]The Poincaré lemma states that any closed differential form defined on an open ball in $\mathbb{R}^n$ is in fact exact.

in some region U then it satisfies the ratchet phenomenon in U and the angles of the cones only depend on a. More importantly for us any twist map of $\mathbb{R}^2$ satisfies the ratchet phenomenon on a compact set.

If we consider the cones given by the ratchet phenomenon as being constant cone fields given by $\Theta_v(p) = p + \Theta_v$ and $\Theta_h(p) = p + \Theta_h$ it is also clear, by letting $p' \to p$, that $D_pF\left(\Theta_v(p)\right) \subset \Theta_h(F(p))$.

1.3 Renormalization

In this section we first give an informal introduction to the ideas of renormalization, using unimodal renormalization as a motivating example. We then give more detailed introductions to the particular renormalization schemes considered in this thesis.

The goal of using *renormalization* in dynamical systems is often to explain features of not just a single dynamical system but rather features that are universal to an entire class of dynamical systems constituting a Banach space, Banach manifold or a subset thereof. The most famous and succesful use of this technique is still the original introduction of renormalization into dynamical systems in order to explain the universality of period-doubling cascades seen in one-dimensional dynamics of *unimodal maps*. This work was started by Feigenbaum in [14] and [15] and independently by Coullet and Tresser in [48], culminating in the work by Lyubich [38].

Intuitively, a dynamical system $F\colon M \to M$ is called *renormalizable* if there is a subset $B \subset M$ such that $F^n(B) \subset B$ for some $n \geq 2$. For simplicity we will only consider the case $n = 2$ in this introduction. This case is called *period-doubling renormalization*.

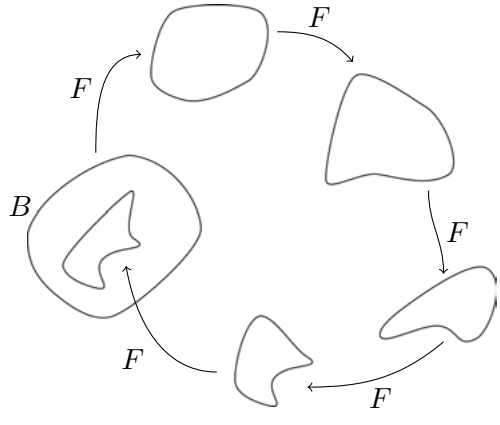

Figure 1.3. A renormalizable map.

For a renormalizable dynamical system F we can consider $F^2|_B$ as a new dynamical system. This new dynamical system can help us study the dynamics of F on the set B. To get a clearer picture of $F^2|_B$ we would like to introduce a rescaling ψ_F such that the $\psi_F(M) = B$, making the map $\tilde{F} = \psi_F^{-1} \circ F^2 \circ \psi_F$ a new dynamical system on M. The hope is that by choosing the map ψ_F cleverly the dynamical system $\tilde{F}$ will be of the same type as F since this would allow us to consider this procedure as the action of an operator $\mathcal{R}$ acting on that space of dynamical systems.

Having successfully defined a renormalization operator $\mathcal{R}\colon \mathcal{U} \to \mathcal{U}$ on a space $\mathcal{U}$ of dynamical systems we can then start analyzing the properties of $\mathcal{R}$ in order to better understand the structure of $\mathcal{U}$. Since $\mathcal{R}$ can be considered a dynamical system in its own right it is natural to analyze it in the same way that we would any dynamical system. Among the simplest structures of dynamical systems are fixed points. For $\mathcal{R}$ this would be a dynamical system $F_* \in \mathcal{U}$ such that $\mathcal{R}F_* = F_*$. Such a fixed point must have a number of interesting properties. Since $\mathcal{R}$ acts as an iteration and a rescaling the fixed point dynamical system must be equal to a rescaled iteration of itself. By iterating this rescaling we see that the initial subset B^0, invariant under F_*^2, must itself contain a subset B^1 which is invariant under $F_*^{2^2}$, the subset B^1 must contain a subset B^2, etc. As a result F_* will have periodic points of periods 2^n for all $n \geq 1$, the subsets B^n and their associated orbits will accumulate on an invariant subset $\mathcal{C}_{F_*}$ and there will be a high degree of self-similarity in both the structure of $\mathcal{C}_{F_*}$ and the dynamics of F_*. In general we call a dynamical system F *infinitely renormalizable* if $\mathcal{R}^n F$ is defined for all $n \geq 1$.

Remark 11. Throughout this introduction we will assume that the sets $\mathcal{C}_F$ are Cantor sets for simplicity. This is true for the motivating case of renormalization of unimodal maps as well as for renormalization of area-preserving maps, the main topic of this thesis. However this is not always the case and indeed for the setting of Paper IV the invariant sets are topological circles.

Furthermore the dynamics of $\mathcal{R}$ around the fixed point F_* can also be very important. For example one could ask if F_* is a hyperbolic fixed point of $\mathcal{R}$ or, more generally, what the spectrum of $D\mathcal{R}$ at F_* is. If so, then analyzing the assoicatied stable and unstable manifolds, $W^s(F_*)$ and $W^u(F_*)$ respectively, will also tell us a lot about the dynamical systems concerned. For example dynamical systems $F \in W^s(F_*)$ must, in some sense, behave similarly to the fixed point F_* on small scales since they converge towards F_* under the action of $\mathcal{R}$. In particular, they will also have invariant Cantor sets $\mathcal{C}_F$ since they are infinitely renormalizable. This leads us to make the following definitions.

Definition 12 (Rigidity)**.** We say that the invariant Cantor sets are *rigid* if for any two infinitely renormalizable dynamical systems F, G which are topo-

logically conjugate on their respective invariant Cantor sets there is a smooth conjugacy $h\colon \mathcal{C}_F \to \mathcal{C}_G$.

Note that for any infinitely renormalizable dynamical system F the sets $F^i\left(B^n\right)$ for $0 \leq i < 2^n$ form a *natural presentation* of the invariant Cantor set $\mathcal{C}_F$. We will denote these sets B_i^n. Note also that the sets B_i^n and $B_{i+2^{n-1}}^n$ are *neighboring sets* in the sense that both are subsets of B_i^{n-1} for $0 \leq i < 2^{n-1}$.

Definition 13 (Bounded geometry)**.** The invariant Cantor set $\mathcal{C}_F$ of an infinitely renormalizable dynamical system F is said to have *bounded geometry* if the distance between any two neighboring sets of its natural presentation is commensurable with the diameter of either of them, i.e., if there is some constant $D > 0$ such that

$$D^{-1} \leq \frac{\text{dist}\left(B_i^n, B_{i+2^{n-1}}^n\right)}{\text{diam}\left(B_k^n\right)} \leq D$$

for all $n \geq 1$ and $0 \leq i < 2^{n-1}$, where $k = i, i + 2^{n-1}$ and the $\text{dist}(X, Y) = \inf_{x \in X, y \in Y} \mathrm{d}(x, y)$.

Further questions regarding the invariant Cantor sets include whether they are contained in a smooth curve or, if not, a Lipschitz curve.

To summarize, having defined a renormalization operator we have the following list of properties that seem natural to examine[5].

1. Does $\mathcal{R}$ have a fixed point?
2. Is the fixed point hyperbolic?
3. What is the spectrum of $D\mathcal{R}$ at the fixed point?
4. Do the infinitely renormalizable maps have invariant Cantor sets?
5. Can the invariant Cantor sets be contained in smooth/Lipschitz curves?
6. Do the invariant Cantor sets have bounded geometry?
7. Are the dynamics on the invariant Cantor sets smoothly conjugated?

Returning to unimodal maps, the archetypical family undergoing a period-doubling cascade is the already mentioned *logistic family*, given by $F_r(x) = rx(1-x)$ where $x \in [0, 1]$ and $r \in [2, 4]$. One can show that for $r \in [2, 3)$ the map F_r has a globally attracting fixed point in $[0, 1]$. At $r = 3$ the family undergoes a period-doubling bifurcation in which the fixed point becomes repelling and an attracting orbit of period two appears. At $r = 1 + \sqrt{6}$ the family undergoes another period-doubling bifurcation in which the attracting orbit of period two becomes repelling and an attracting orbit of period four appears.

[5]There are of course many more interesting questions to consider. For example the renormalization considered in Paper IV admits not only a hyperbolic fixed point but an entire horseshoe attractor.

This continues until at $r \approx 3.59995$ the map becomes chaotic. This cascade can be seen in Figure 1.4 where the position of the attracting orbit is plotted for each $r \in [2, 4]$.

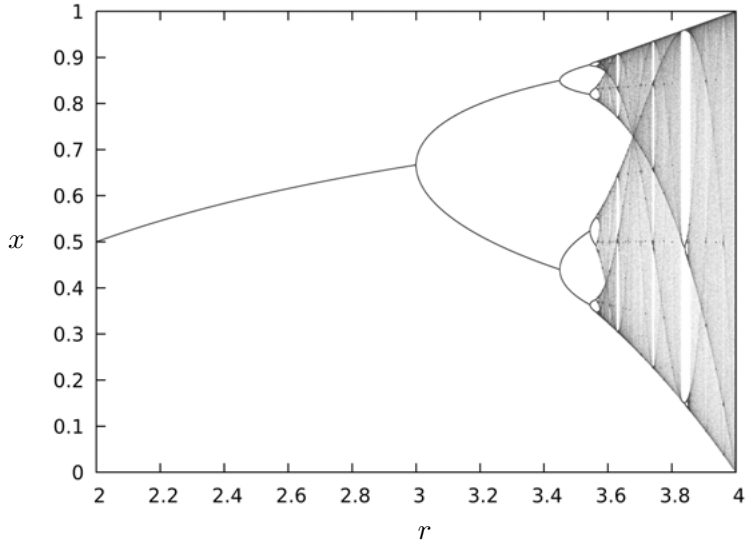

Figure 1.4. Period-doubling cascade in the logistic family.

Let r_n be the value at which an attracting orbit of period 2^n appears and let r_∞ be the limit point at which the map becomes chaotic. It can be shown that

$$\lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = \delta \approx 4.669\ldots \tag{1.1}$$

This value is called the *first Feigenbaum constant*. The surprising fact about of this number is its universality. While the particular numbers r_n and r_∞ depend on the unimodal family undergoing a period-doubling cascade the number δ does not: it is always the same. The explanation for this universality came via renormalization.

For $r = 3.4$ the logistic map $F_{3.4}$ has an attractive orbit of period two. The points of this orbit are therefore fixed points of $F_{3.4}^2$. In fact, looking at a graph of $F_{3.4}^2$ it is clear that a small copy of $F_{3.4}$ appears near these fixed points, see Figure 1.5.

It seems natural then to define a renormalization

$$\mathcal{R}F = \psi_F^{-1} \circ F^2 \circ \psi_F$$

acting on a suitable space $\mathcal{U}$ of unimodal maps where ψ_F are well-chosen affine contractions of $[0, 1]$. With careful analysis one can then prove the following properties of period-doubling unimodal renormalization.

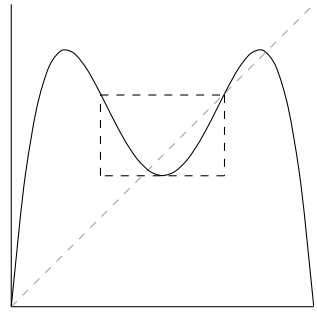

Figure 1.5. The graph of $F^2_{3.4}$.

1. $\mathcal{R}$ is a bounded C^2 operator on $\mathcal{U}$ with a hyperbolic fixed point F_*, called the *Feigenbaum fixed point*.
2. The derivative $D\mathcal{R}$ of $\mathcal{R}$ at the fixed point F_* is a compact operator. Its spectrum consists of a single eigenvalue $\delta = 4.669\ldots$ outside the unit circle and the rest are contained in the interior of the unit disk.
3. Every infinitely renormalizable map $F \in W^s(F_*)$ has an invariant Cantor set $\mathcal{C}_F$ with bounded geometry.
4. The dynamics of any two infinitely renormalizable maps are smoothly conjugate on their respective invariant Cantor sets.
5. Let Σ_n be the Banach submanifold of zero-entropy maps for which the critical point is periodic with period 2^n. Then Σ_n are codimension 1 submanifolds and for large enough n they intersect $W^u(F_*)$ transversally.

The combination of Properties 2 and 5 along with the fact that $\mathcal{R}(\Sigma_{n+1}) \subset \Sigma_n$ imply that the submanifolds Σ_n accumulate on $W^s(F_*)$ geometrically with rate δ^{-1}. This in turn explains the universality observed in period-doubling cascades: any family of unimodal maps crossing the unstable manifold $W^u(F_*)$ transversally will also cross the submanifolds Σ_n transversally for all large enough n. Since the distance between succsesive renormalizations is related to the distance between the submanifolds Σ_n and since these accumulate on the unstable manifold with rate δ^{-1} this determines that the ratio

$$\frac{r_n - r_{n-1}}{r_{n+1} - r_n}$$

must approach δ as $n \to \infty$.

The proofs of Properties 1 and 2 were first given by Lanford using rigorous numerical methods in [34]. Eckmann and Wittwer gave the first complete proof of 5 in [13]. Property 4 can be proven from Property 3. For a proof of this and of Property 3 itself see for example the book by de Melo and van Strien [10]. The full completely analytic proof of these properties was eventually completed by Lyubich in [38]. In fact this proof was done for all maps with bounded combinatorics and not just maps undergoing period-doubling.

1.3.1 Area-preserving renormalization

Shortly after Feigenbaum and Coullet and Tresser discovered the remarkable universality in one dimensional dynamical systems people started looking for similar universality phenomena in other areas and found them. For example they were detected numerically in [19], [18] and [2] as well as experimentally in [35]. As such there was plenty of motivation to generalize the renormalization technique.

Collet, Eckmann and Koch extended the results to higher dimensions in [5]. They did this by creating new period-doubling renormalization operators acting on highly dissipative dynamical systems which are higher dimensional perturbations of unimodal maps acting on $\mathbb{C}^n$. Following this they also formulated a period-doubling renormalization of area-preserving dynamical systems on $\mathbb{R}^2$ in [4]. At this time there were multiple competing renormalizations of such systems, but by carefully analyzing the different options they could show that their formulation was superior. This was formalized by Eckmann, Koch and Wittwer in [12] where they also proved the existence of a hyperbolic fixed point for this renormalization operator.

The domain of this renormalization operator $\mathcal{R}$ is the set of reversible area-preserving twist maps $F\colon \mathbb{R}^2 \to \mathbb{R}^2$. We will use the notation $F(x,y) = (X,Y)$. *Reversibility* means that $T \circ F \circ T = F^{-1}$ where $T(x,y) = (x,-y)$. Recall from Subsection 1.2.3 that an area-preserving twist map has a generating function $S(x,X)$. The reversibility implies that $S_1(X,x) = S_2(x,X)$, so that we can simplify the notation by defining $s(x,X) \coloneqq S_2(x,X)$. To see this write the reversibility condition as $F \circ T \circ F = T$. Then

$$\begin{aligned}(X, S_1(X,x)) &= F \circ T \circ F(X, -S_1(X,x)) \\ &= F \circ T(x, S_2(X,x)) \\ &= F(x, -S_2(X,x)), \\ (X, S_2(x,X)) &= F(x, -S_1(x,X)).\end{aligned}$$

Since F is a twist map it follows that we must have $S_1(X,x) = S_2(x,X)$ from the fact that for any fixed x_0 the map $y \mapsto X(x_0,y)$ is strictly monotone. We can now write the map F as

$$\begin{pmatrix} x \\ -s(X,x) \end{pmatrix} \overset{F}{\mapsto} \begin{pmatrix} X \\ s(x,X) \end{pmatrix}.$$

We now want to define the renormalization $\mathcal{R}F = \Lambda_F^{-1} \circ F \circ F \circ \Lambda_F^{-1}$. It was shown by Collet, Eckmann and Koch in [4] that the map Λ_F can be chosen as

$$\Lambda_F(x,y) = (\lambda_F x, \mu_F y)$$

where $\lambda_F, \mu_F \in (-1,1)$ and $\mu_F < \lambda_F^2$. With these definitions it is possible to derive a functional equation for the generating function $s_{\mathcal{R}F}$ of $\mathcal{R}F$, namely

$$s_{\mathcal{R}F}(x,X) = \mu^{-1} s(z(x,X), \lambda X) \tag{1.2}$$

where $z(x, X) = z(X, x)$ is a symmetric function solving the *midpoint equation*

$$s(\lambda x, z(x, X)) + s(\lambda X, z(x, X)) = 0. \tag{1.3}$$

Lastly, we apply the following normalization conditions:

1. $s(1, 0) = 0$,
2. $s_1(1, 0) = 1$,
3. $z(1, 0) = z(0, 1) = 1$,
4. $\mu = z_1(1, 0)$.

Note that with these normalizations we can also find λ from the midpoint equation by setting $(x, X) = (1, 0)$,

$$s(\lambda, 1) + s(0, 1) = 0.$$

A variation on this renormalization scheme was introduced recently by Gaidashev, Johnson and Martens in [23] which instead uses rescalings given by

$$\Lambda_F(x, y) = (\lambda_F x + p_F, \mu y).$$

This allows them to also require $F(0, 0) = (0, 0)$. When we need to distinguish between the two formulations we will use the notation $\mathcal{R}_{EKW}$ for the original formulation by Eckmann, Koch and Wittwer and $\mathcal{R}_{GJM}$ for the newer formulation by Gaidashev, Johnson and Martens.

Having defined a renormalization scheme for area-preserving maps the work of determining the properties of the renormalization operator could begin. The first results in this direction were provided by Eckmann, Koch and Wittwer in [12] where they proved the existence of a hyperbolic fixed point for their renormalization operator.

The analysis of area-preserving renormalization was continued by Gaidashev and Johnson who, in [21], proved the existence of stable invariant Cantor sets for infinitely renormalizable area-preserving maps. Further results were obtained by Gaidashev, Johnson and Martens in [23], proving rigidity in this setting.

Using [23] as a reference we summarize the most relevant results about area-preserving renormalization. $\text{Cons}_0(D)$ refers to the domain of definition of the renormalization operator and consists of area-preserving maps on a simply-connected open set D which is compactly contained in the real slice of a domain $\mathcal{D} \subset \mathbb{C}^2$ and satisfying all conditions above. Furthermore all maps in $\text{Cons}_0(D)$ have a unique orbit of period 2. One of the points in the periodic orbit is $(p_F, 0)$ and $p_F < 0$.

Theorem 14 (Theorem 2.1 of [23])**.** *There exists $F_* \in \text{Cons}_0(D)$ such that*

1. *$\text{Cons}_0(D)$ is a neighborhood of F_*,*

2. *F_* is a hyperbolic fixed point of $\mathcal{R}$ with a codimension* 1 *local stable manifold $W^s(F_*)$, called the* universality class of the Eckmann-Koch-Wittwer renormalization fixed point*, and a dimension* 1 *local unstable manifold $W^u(F_*)$,*
3. *The map Λ_F depends analytically on $F \in \mathrm{Cons}_0$.*

For any $F \in \mathrm{Cons}_0(D)$ we define the following maps:

$$\psi_0(F) = \Lambda_F \colon D \to \mathbb{R}^2,$$
$$\psi_1(F) = F \circ \Lambda_F \colon D \to \mathbb{R}^2.$$

These maps will be used to define the invariant Cantor set of an infinitely renormalizable area-preserving map using the following results.

Lemma 15 (Lemma 2.2 of [23])**.** *For $F \in \mathrm{Cons}_0(D)$ there are analytically defined simply connected domains $B_0(F) \subset D$ and $B_1(F) \subset D$ with $(p_F, 0) \in B_0(F)$ such that*

1. *$B_0(F) \cap B_1(F) = \emptyset$,*
2. *$F^2(B_0(F)) \cap B_0(F) \neq \emptyset$,*
3. *$\psi_w\left(B_0(F) \cup B_1(F)\right) \subset B_w(F)$*

where $w \in \{0, 1\}$.

For an infinitely renormalizable $F \in W^s(F_*)$ we will also consider compositions of the maps $\psi_w(F)$, namely

$$\Psi_w^n = \psi_{w_1}(F) \circ \psi_{w_2}(\mathcal{R}F) \circ \cdots \circ \psi_{w_n}(\mathcal{R}^{n-1}F).$$

Proposition 16 (Proposition 2.3 of [23])**.** *There exists $0 < \theta_1 < \theta_2 < 1$ such that for $F \in W^s_{loc}(F_*)$ we have*

$$\theta_1^4 |v| \leq \left|D\Psi_w^4(x, y)v\right| \leq \theta_2^4 |v|$$

for every $w \in \{0, 1\}^4$ and $(x, y) \in B_0(F) \cup B_1(F)$. Moreover

$$\frac{\theta_2 \nu}{\theta_1} < 1.$$

A corollary of this result is the following lemma.

Lemma 17 (Lemma 3.2 of [23])**.** *For every $F \in W^s(F_*)$ there exists a constant $C > 0$ such that for any word $w \in \{0, 1\}^n$ for $n \geq 1$ we have*

$$\|D\Psi_w^n\| \leq C\theta_2^n.$$

Estimates of θ_1, θ_2 and ν are given in [22] and are as follows:

- $\theta_1 \geq 0.061$,
- $\theta_2 \leq 0.249$,
- $\nu \leq 0.126$.

With these results we can start defining what will be the natural presentation of the invariant Cantor set for a map $F \in W^s(F_*)$. For each $w \in \{0,1\}^n$ and $i = 0, 1$ let

$$B^n_{wi}(F) = \Psi^n_w(B_i(\mathcal{R}^n F)).$$

These sets will be called the *pieces of level* n. The set of all pieces of level n will be denoted $\mathcal{B}^n$. We will endow the set $\{0,1\}^n$ with a group structure by identifying it with the additive group of residues $\mod 2^n$ using the map

$$w \mapsto \sum_{k=0}^{n-1} w_k 2^k.$$

We will also let $p\colon \{0,1\}^n \to \{0,1\}^n$ denote the operation of addition by 1 in this group. We then have the following lemma.

Lemma 18 (Lemmas 3.3 and 3.4 of [23])**.** *For every $F \in W^s(F_*)$, $w \in \{0,1\}^n$ and $n \geq 1$ we have*

1. *the pieces are nested according to $B^n_{wv} \subset B^{n-1}_w$ for all $v \in \{0,1\}$,*
2. *the pieces of each level are pairwise disjoint for $w \neq 1^n$,*
3. *the pieces are permuted by F according to $F(B^{n-1}_w) = B^{n-1}_{p(w)}$ for $w \neq 1^n$ and $F(B^{n-1}_{1^n}) \cap B^{n-1}_{0^n} \neq \emptyset$,*
4. *there exists a constant $C > 0$ such that* $\operatorname{diam}(B^{n-1}_w) \leq C\theta_2^{n-1}$.

This then naturally leads to the following theorem about the existence of the invariant Cantor set.

Theorem 19 (Theorem 3.5 of [23])**.** *For every $F \in W^s(F_*)$ the set*

$$\mathcal{C}_F = \bigcap_{n=1}^{\infty} \bigcup_{w \in \{0,1\}^n} B^{n-1}_w$$

is an invariant Cantor set and $F|_{\mathcal{C}_F}$ is topologically conjugate to the dyadic adding machine.

Since the adding machine is uniquely ergodic this means that we get an induced uniquely ergodic measure on $\mathcal{C}_F$, denoted μ, such that $\mu(B^{n-1}_w \cap \mathcal{C}_F) = 2^{-n}$.

The last result to mention is the rigidity of these invariant Cantor sets.

Theorem 20 (Theorem 4.1 of [23])**.** *The Cantor set $\mathcal{C}_F$ of any $F \in W^s_{\text{loc}}(F_*)$ is rigid, namely there is a $C^{1+\alpha}$-conjugacy $h\colon \mathcal{C}_F \to \mathcal{C}_{F_*}$ for every $0 < \alpha < \alpha_0$*

where

$$\alpha_0 = \frac{\ln(\theta_2\nu)}{\ln(\theta_1)} - 1 > 0.237.$$

Returning to the list of interesting properties of renormalization operators we can see that most questions have been answered yet the questions regarding the geometry of the invariant Cantor sets are still open. The main goal of this thesis is to address these questions.

Remark 21. It is worth pointing out that for renormalization of unimodal maps the rigidity result for the invariant Cantor sets were a natural consequence of the bounded geometry of said invariant Cantor sets.

For period-doubling renormalization of dissipative two-dimensional maps both rigidity and bounded geometry fail and this failure is measured by a topological invariant called the *average Jacobian* defined by

$$b_F = \exp \int \log \mathrm{Jac}(F) d\mu$$

where μ is again the ergodic measure induced on the invariant Cantor set from the topological conjugacy with the adding machine. However Lyubich and Martens have shown, in [37], that rigidity survives in a probabilistic sense, meaning that there is a conjugation $h\colon \mathcal{C}_F \to \mathcal{C}_{F_*}$ and a subset $X \subset \mathcal{C}_F$ of full measure such that $h|_X$ is a $(1+\alpha)$ diffeomorphism for some $\alpha > 0$.

This can be contrasted with the situation for area-preserving maps where, as we have seen, rigidity has been proven to hold yet very little is known about the geometry of the invariant Cantor sets. For example it is not known whether these sets are contained in smooth curves. This question is settled in Paper II of this thesis.

1.3.2 Renormalization of critical circle maps

Generalizing in a different direction, renormalization was applied to the study of critical circle maps by Feigenbaum, Kadanoff and Shenker in [16] and, in parallel, by Ostlund, Rand, Sethna and Siggia in [43]. These investigations were motivated by a desire to understand transitions to chaos in quasiperiodic flows, which they argued could be understood by analyzing the dynamics of an invariant curve of an appropriately chosen first return map. In this way the problem could be reduced to studying renormalizations of critical circle maps. A common way to study this type of renormalization, also used by [43], is to use *critical commuting pairs*, defined as follows.

Definition 22. A C^r-smooth *critical commuting pair*[6] is a pair $\zeta = (\eta, \xi)$ of C^r-smooth orientation preserving interval homeomorphisms, defined on I_η and I_ξ respectively, satisfying

1. $I_\eta = [0, \xi(0)]$ and $I_\xi = [\eta(0), 0]$,
2. both η and ξ have C^r-smooth homeomorphism extensions to interval neighborhoods of their respective domains such that $\eta \circ \xi = \xi \circ \eta$,
3. $\xi \circ \eta(0) \in I_\eta$,
4. both maps have a unique critical point at 0 which is cubic, i.e.,
$$\eta'(0) = \xi'(0) = \eta''(0) = \xi''(0) = 0$$
and $\eta'''(0) \neq 0 \neq \xi'''(0)$[7].

The connection to critical circle maps can be seen by considering the interval $I = [\eta(0), \xi \circ \eta(0)]$ and the map f_ζ given by

$$f_\zeta(x) = \begin{cases} \eta \circ \xi(x) & x \in [\eta(0), 0], \\ \eta(x) & x \in [0, \xi \circ \eta(0)]. \end{cases}$$

We can then create a circle by identifying the endpoints of I and by the properties of ζ the map f_ζ will then be a well-defined C^r-smooth circle map with a unique cubic critical point. Similarly, from a critical circle map we can construct a commuting pair.

Definition 23. The *height* $\chi(\zeta)$ of a critical commuting pair $\zeta = (\eta, \xi)$ is the natural number $\chi(\zeta) = r \geq 1$, if it exists, such that $0 \in [\eta^{r+1}(\xi(0)), \eta^r(\xi(0))]$. If no such r exists then we say that $\chi(\zeta) = \infty$.

Having defined the height of a commuting pair we can start giving an informal definition of the renormalization of a commuting pair. We will say that a critical commuting pair ζ is *renormalizable* if $\chi(\zeta) < \infty$. Given a renormalizable commuting pair $\zeta = (\eta, \xi)$ of height $\chi(\zeta) = r$ we can create a new commuting pair by first considering the *prerenormalization* $\bar{\zeta}_1 = \left(\eta^r \circ \xi|_{I_\xi}, \eta|_{[0, \eta^r(\xi(0))]}\right)$. We will denote $\bar{I}^1_\eta = I_\xi$ and $\bar{I}^1_\xi = [0, \eta^r(\xi(0))]$. To get back to the original scale we rescale both the maps and their domains by the linear factor $\lambda = \eta(0)$ giving us the pair $\zeta_1(z) = \lambda^{-1}\bar{\zeta}_1(\lambda z)$ of maps η_1 and ξ_1 defined on $I^1_\eta = \lambda^{-1}\bar{I}^1_\eta$ and $I^1_\xi = \lambda^{-1}\bar{I}^1_\xi$. We define the *renormalization* of the critical commuting pair ζ as

$$\mathcal{R}\zeta = \zeta_1.$$

The height of a commuting pair ζ is connected to the rotation number of the corresponding critical circle map f_ζ in the sense that if $\chi(\zeta) = r$ then the

[6] For simplicity of notation we will simply call these commuting pairs. Note however that usually a commuting pair is not required to have a critical point.

[7] Sometimes it is only required that the first derivative be zero.

rotaton number $\rho(f_\zeta) = [r, \dots]$. As such we also define the *rotation number of an infinitely renormalizable commuting pair* ζ to be

$$\rho(\zeta) = [r_0, r_1, \dots]$$

where $r_i = \rho(\mathcal{R}^i\zeta)$.

Unfortunately this setting presents a difficulty not encountered in our previously discussed renormalization operators, namely the set of critical commuting pairs does not form a Banach manifold in a natural way. As such it becomes impossible to show that the renormalization operator is analytic and furthermore hyperbolic. Despite this difficulty several important results were produced. Aside from the already mentioned papers [16] and [43] an important early result is by Mestel who in [41] used rigorous computer estimates to show the local existence and hyperbolicity of a fixed point for the goldean-mean rotation number.

Later developments followed by de Faria and de Melo in a series of papers [[6], [7], [8], [9]] where they proved the existence of an attracting horseshoe for critical circle maps of bounded type, that the convergence towards this attractor is geometric and rigidity of the scaling ratios. This was further improved upon by Yampolsky in [51] where he constructed the full renormalization horseshoe, including maps of unbounded type. We detail some of those results here, building towards the resolution of the question of hyperbolicity.

We first state the theorem about the existence of the full renormalization horsehoe from [51].

Theorem 24 (Theorem A from [51])**.** *There exists an $\mathcal{R}$-invariant set $\mathcal{X}$ of commuting pairs with irrational rotation numbers with the following properties. The action of $\mathcal{R}$ on $\mathcal{X}$ is bijective and topologically conjugate to the shift map on the set of bi-infinite sequences of natural numbers. The set $\mathcal{X}$ is precompact in the Carathéodory topology and its closure $\mathcal{A}$ is the attractor of the renormalization operator:*

$$\mathcal{R}^n\zeta \to \mathcal{A}, \text{ for all } \zeta \in \mathcal{E} \text{ with } \rho(\zeta) \in \mathbb{R} \setminus \mathbb{Q}$$

where $\mathcal{E}$ denotes the Epstein class. More precisely, for any pair $\zeta' \in \mathcal{A}$ with $\rho(\zeta') = \rho(\zeta)$ we have

$$dist(\mathcal{R}^n\zeta, \mathcal{R}^n\zeta') \to 0$$

in the C^0 distence between the analytic extensions of the renormalized pairs on an open neighborhood of the interval of definition.

In order to better study the dynamics of infinitely renormalizable commuting pairs we will introduce a set of subintervals of I_η and I_ξ, which form a partition of the circle modulo endpoints, called the *dynamical partition*. Let $\zeta = (\eta, \xi)$ be k-times renormalizable and let $\bar{\zeta}_k = (\bar{\eta}_k|_{I^k_\eta}, \bar{\xi}_k|_{I^k_\xi})$ denote its k-th

prerenormalization. Then $\bar{\eta}_k = \zeta^{\bar{s}_k}$ and $\bar{\xi}_k = \zeta^{\bar{t}_k}$ where $\bar{s}_k$ and $\bar{t}_k$ are some multiindices, i.e. both $\bar{\eta}_k$ and $\bar{\xi}_k$ are some combinations of compositions of η and ξ. More precisely, we consider multiindices

$$\bar{s} = (a_1, b_1, a_2, b_2, \ldots, a_n, b_n)$$

where we require $a_1 \in \mathbb{N}$, $a_j \in \mathbb{N} \setminus \{0\}$ for $2 \leq j$, $b_j \in \mathbb{N} \setminus \{0\}$ for $j \leq n-1$ and $b_n \in \mathbb{N}$. We then let write

$$\zeta^{\bar{s}} = \xi^{b_n} \circ \eta^{a_n} \circ \cdots \circ \xi^{b_1} \circ \eta^{a_1}.$$

We introduce an ordering $\prec$ on these multiindices according to $\bar{t} \prec \bar{s}$ for $\bar{s} = (a_1, b_1, \ldots, a_n, b_n)$ if $\bar{t} = (a_1, b_1, \ldots, a_m, b_m, c, d)$ where $m < n$ and either $c < a_{m+1}$ and $d = 0$ or $c = a_{m+1}$ and $d < b_{m+1}$. This ordering is meant to mimick the way that the unrescaled initial segments I_η^k and I_ξ^k of some prerenormalization $\bar{\zeta}_k$ move under the action of the original pair ζ in the sense that if $\bar{w} \prec \bar{s}_k$ then $\zeta^{\bar{s}_k} = \zeta^{\bar{v}} \circ \zeta^{\bar{w}}$ for some multiindex v. With this in mind we make the following definition.

Definition 25. Let ζ be a commuting pair. The *k-th dynamical partition of ζ* is the set of intervals

$$\mathcal{P}_k = \{\zeta^{\bar{w}}(I_\eta^k) \colon \bar{w} \prec \bar{s}_k\} \cup \{\zeta^{\bar{w}}(I_\xi^k) \colon \bar{w} \prec \bar{t}_k\}.$$

Example 26. Let $\zeta = (\eta, \xi)$ be an almost commuting pair with rotation number $\rho(\zeta) = [2, 1, \ldots]$. It's first prerenormalization is then given by $\bar{\zeta}_1 = (\bar{\eta}_1, \bar{\xi}_1)$ where $\bar{\eta}_1 = \eta^2 \circ \xi$ on $I_\eta^1 = I_\xi$ and $\bar{\xi}_1 = \eta$ on $I_\xi^1 = [0, \eta^2(\xi(0))]$. Thus we have $\bar{s}_1 = (0, 1, 2, 0)$ and $\bar{t}_1 = (1, 0)$. The $\bar{w} \prec \bar{s}_1$ are then $(0, 0)$, $(0, 1)$ and $(0, 1, 1, 0)$ and the only $\bar{w} \prec \bar{t}_1$ is $(0, 0)$. This gives us a first dynamical partition consisting of the intervals

$$\begin{aligned}
\zeta^{(0,0)}(I_\eta^1) &= I_\eta^1, \\
\zeta^{(0,1)}(I_\eta^1) &= \xi(I_\eta^1) = [\eta(\xi(0)), \xi(0)], \\
\zeta^{(0,1,1,0)}(I_\eta^1) &= \eta(\xi(I_\eta^1)) = [\eta^2(\xi(0)), \eta(\xi(0))], \\
\zeta^{(0,0)}(I_\xi^1) &= I_\xi^1.
\end{aligned}$$

The second dynamical partition can then be constructed by considering the first dynamical partition of the first prerenormalization $\bar{\zeta}_1$ and translating it through the first dynamical partition of ζ. See Figure 1.6 for an illustration of the first and second dynamical partition of this ζ.

With this setup it is possible to prove that for any $n \geq 1$ any two adjacent intervals of the level n dynamical partition are comparable. This is part of a result called *real a priori bounds*, first proven by Herman and Świątek in [32]

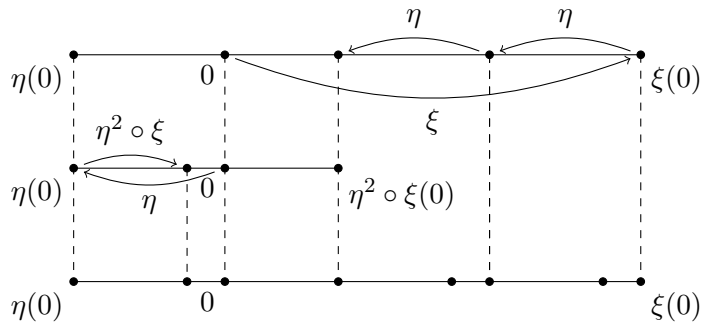

Figure 1.6. The first two dynamical partitions of a pair $\zeta = (\eta, \xi)$ with rotation number $\rho(\zeta) = [2, 1, 1, \dots]$. The first row shows the first dynamical partition of ζ, the second row shows the first dynamical partition of the prerenormalization $\bar{\zeta}_1$, the third row shows the second dynamical partition of ζ.

and [47] respectively. The following formulation, taken from [8], uses the language of critical circle maps and has been translated to our notation. We let $\mathcal{P}_k(x)$ denote the piece of the k-th dynamical partition containing the point x.

Theorem 27 (Theorem 3.1 of [8])**.** *Let f be a C^1 critical circle map with irrational rotation number. There exist constants $C_0 > 1$ and $0 < \mu_0 < \mu_1 < 1$ depending only on f such that:*

1. *If I and J are any two adjacent intervals of $\mathcal{P}_k$ then $C_0^{-1} |J| < |I| < C_0 |J|$.*
2. *For every point x we have $|\mathcal{P}_k(x)| < \mu_1 |\mathcal{P}_{k-1}(x)|$.*
3. *If the rotation number of f is of bounded type then $|\mathcal{P}_k(x)| > \mu_0^k C_0$.*
4. *If the rotation number of f is of bounded type then $|\mathcal{P}_k(x)| > C_0^{-1} |\mathcal{P}_k(x)|$.*
5. *If $0 < i \leq j \leq q_n$ then the distortion of the restriction of f^{j-i} to $f^i(I_\eta^{k-1})$ is bounded by C_0.*

In particular, the critical commuting pairs $\mathcal{R}^k \zeta_f$ form a bounded sequence in the C^1 topology.

In addition to this the constants in the above theorem are eventually universal, a property called *beau*. This is the content of the next theorem, also from [8].

Theorem 28 (Proposition 3.8 from [8])**.** *All bounds in Theorem 27 are beau. In other words there are universal constants $K_0 > 1$ and $0 < \lambda_0 < \lambda_1 < 1$ independent of f and some $N = N(f)$ such that for all $n \geq N$ the constants C_0, μ_0 and μ_1 can be replaced by K_0, λ_0 and λ_1 respectively.*

The first breakthrough in regards to hyperbolicity came from Yampolsky in [52]. In this paper Yampolsky defined a new renormalization operator acting on critical circle maps living on the cylinder $\mathbb{C}/\mathbb{Z}$ rather than on commuting pairs. Using this approach it is possible to create a natural Banach manifold of dynamical systems upon which the renormalization operator is analytic. This new approach to renormalization of critical circle maps enabled him to prove hyperbolicity of renormalization, first for periodic points in [52] and later for the entire renormalization horseshoe in [53].

Lastly, the most recent development in renormalization of critical circle maps and the approach we will be focusing on is that of *almost commuting maps*, first introduced by Stirnemann in [45] but seemingly abandoned until Gaidashev and Yampolsky were able to use it to prove hyperbolicity of renormalization for golden mean Siegel disks in [24].

Definition 29. An *almost commuting pair of order k* is a pair (η, ξ) of non-decreasing interval maps

$$\eta\colon [0, \xi(0)] \to [\eta(0), \eta(\xi(0))], \quad \xi\colon [\eta(0), 0] \to [\xi(\eta(0)), \xi(0)]$$

satisfying the following:

1. there is an open neighborhood of the interval $[0, \xi(0)]$ on which the map η is analytic with a single critical point of order k at $x = 0$,
2. there is an open neighborhood of the interval $[\eta(0), 0]$ on which the map ξ is analytic with a single critical point of order k at $x = 0$,
3. the commutator $[\eta, \xi](x) = \eta \circ \xi(x) - \xi \circ \eta(x) = o(x^k)$ at $x = 0$.

The set of almost commuting pairs of order 3 will be denoted by $\mathbf{B}$.

Let D, E be open topological disks in $\mathbb{C}$ containing $[0, \xi(0)]$ and $[\eta(0), 0]$ respectively. Let $C^\omega(D) \times C^\omega(E)$ denote the Banach space of pairs of bounded holomorphic functions equipped with the norm

$$\|(\eta, \xi)\| = \frac{1}{2}(\|\eta\| + \|\xi\|)$$

where $\|f\| = \sup_z |f(z)|$. We will then consider the subset $\mathbf{B}^{D,E} \subset C^\omega(D) \times C^\omega(E)$ consisting of almost commuting pairs $(\eta, \xi) \in \mathbf{B}$ of order 3 where the maps η and ξ have bounded analytic continuations to D and E respectively with the normalizing conditions $\xi(0) = 1$ and

$$\frac{1}{2C_0} < |\eta(0)| < 2C_0$$

where C_0 is the constant of commensurability for any two adjacent intervals of the dynamical partition for the almost commuting pair (η, ξ).

Using the formalism of almost commuting pairs, not only do the results above carry over but, in addition, it is possible to prove hyperbolicity of the

renormalization operator. Note, again, that hyperbolicity for renormalization of critical circle maps was first done by Yampolsky using cylinder renormalization.

Theorem 30 (Theorem 2.1 of [27])**.** *Let us fix a periodic point $\zeta_* \in \mathcal{A}$ of $\mathcal{R}$ of period k and let $\rho_* = \rho(\zeta_*)$. There exists a space $\mathbf{B}^{D,E}$ and $p = m \cdot k \in \mathbb{N}$ such that the following holds. The pair ζ_* is a fixed point of $\mathcal{R}^p$ in the space $\mathbf{B}^{D,E}$. The image*

$$\mathcal{R}^p(\zeta_*) \in \mathbf{B}^{D',E'} \text{ where } D \Subset D',\ E \Subset E'.$$

The linearization

$$\mathcal{L} \equiv D\mathcal{R}^p|_{\zeta_*}$$

in $\mathbf{B}^{D,E}$ is a compact operator with one simple unstable eigenvalue and the rest of the spectrum is compactly contained in $\mathbb{D}$. The stable manifold $W^s(\zeta_*)$ of ζ_* contains all pairs in $\mathbf{B}^{D,E}$ with rotation number ρ_*.

Using almost commuting pairs and their hyperbolicity it is then possbile to generalize to higher dimensional settings. This is the subject of Paper IV.

2. Summary of papers

2.1 Paper I

This paper does not deal with the main subject of this thesis, renormalization, but rather attemps to prove the existence of the folding breakdown type of bifurcation of FHIM observed in [17] for a more general class of systems.

In this paper we consider solutions $\mathbf{x} = \{x_k\}_{k\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ to the equation

$$\varepsilon Z(\theta_k, x_{k+1}, x_k, x_{k-1}) + V(\theta_k, x_k) = 0, \quad \forall k \in \mathbb{Z} \tag{2.1}$$

where $Z \in C^r(\mathcal{M} \times \mathbb{R}^3, \mathbb{R})$ and $V \in C^r(\mathcal{M} \times \mathbb{R}, \mathbb{R})$ where $r \geq 1$ and $\mathcal{M}$ is a compact, smooth manifold, $\{\theta_k\}_{k\in\mathbb{Z}} \in \mathcal{M}^{\mathbb{Z}}$ and, finally, ε is a real number such that $|\varepsilon|$ close to 0, meaning that we are considering the anti-integrable limit scenario.

To put this in perspective, the prototypical example of such a system is the quasiperiodically driven Chirikov standard map, given by

$$\begin{cases} \bar{\theta} &= \theta + \omega \mod 1 \\ \bar{x} &= x + \bar{y} \\ \bar{y} &= y - \frac{\partial W}{\partial x}(\theta, x) \end{cases} \tag{2.2}$$

where ω is an irrational number and $W \in C^2(\mathbb{T} \times \mathbb{R}, \mathbb{R})$. Considering a Lagrangian

$$L(\theta, \mathbf{x}) = \sum_{k\in\mathbb{Z}} \frac{1}{2}(x_{k+1} - x_k)^2 - W(\theta + k\omega, x_k) \tag{2.3}$$

we can recover the orbits of system (2.2) as the stationary solutions of the gradient flow of L,

$$\begin{aligned} \dot{x}_k &= -\frac{\partial L}{\partial x_k}(\theta_0, \mathbf{x}) \\ &= x_{k+1} - 2x_k + x_{k-1} + \frac{\partial W}{\partial x}(\theta_0 + k\omega, x_k) \\ &= 0, \end{aligned}$$

by letting $y_k = x_k - x_{k-1}$. From here we reach Equation (2.1) by generalizing $x_{k+1} - 2x_k + x_{k-1}$ to $Z(\theta_k, x_{k+1}, x_k, x_{k-1})$, generalizing $\frac{\partial W}{\partial x}(\theta_0 + k\omega, x_k)$ to $V(\theta_k, x_k)$ and finally taking the anti-integrable limit.

Note that, for simplicity, we will call V the *potential* even though in the above discussion it corresponds to the deriviative of the potential W. Note also

that while the equation we are considering is motivated by dynamical systems its solutions do not always define a dynamical system. Only for certain forms of Z do the solutions define a dynamical system, for example if $Z(\theta, a, b, c) = a + \tilde{Z}(\theta, b, c)$.

We also consider a simpler case, called the one-dimensional case, where we instead consider solutions to

$$\varepsilon Z(\theta_k, x_{k+1}, x_k) + V(\theta_k, x_k) = 0. \tag{2.4}$$

This can be considered a special case of Equation (2.1) where Z does not depend on x_{k+1}. For simplicity the results of this paper are presented for the situation where the sequence $\{\theta_k\}$ satisfies $\theta_k = h^k(\theta_0)$ for some homeomorphism h of $\mathcal{M}$ though only the first theorem actually requires it. The remaining theorems remain true without modification for arbitrary sequences $\{\theta_k\}$ and their proofs require no modification.

With this in mind, the main results of this paper are the following five theorems.

For the first theorem we assume that the sequence $\{\theta_k\}$ satisfies $\theta_{k+1} = h(\theta_k)$ for some homeomorphism h of $\mathcal{M}$.

Theorem 31. *Suppose that $Z \in C^r(\mathcal{M} \times \mathbb{R}^3, \mathbb{R})$, $V \in C^r(\mathcal{M} \times \mathbb{R}, \mathbb{R})$, $r \geq 1$, satisfies that $\partial_x V(\theta, x) \neq 0$ for all (θ, x) in a connected and bounded subset of the zero level set $V^{-1}(0)$. Then, there exist $\varepsilon_0 > 0$ and $K \in C^r((-\varepsilon_0, \varepsilon_0) \times \mathcal{M}, \mathbb{R})$ such that*

$$\varepsilon Z(\theta, K(\varepsilon, h(\theta)), K(\varepsilon, \theta), K(\varepsilon, h^{-1}(\theta))) + V(\theta, K(\varepsilon, \theta)) = 0 \tag{2.5}$$

holds for any $\theta \in \mathcal{M}$ and $|\varepsilon| < \varepsilon_0$. Moreover, in the case that Z defines a dynamical system it has positive Lyapunov exponents.

This is only a slight generalization of a result that appears in [17] and is included for completeness. It implies that there are solutions $\{x_k\}_{k\in\mathbb{Z}}$ lying on a smooth manifold and given by $x_k = K\left(\varepsilon, h^k\left(\theta_0\right)\right)$.

For the remaining theorems we will need some of the following extra conditions, where $I = [-1, 1]$ and $I_o = (-1, 1)$:

(C1) There is some $\varepsilon_0 > 0$ such that for every $|\varepsilon| \leq \varepsilon_0$ each connected component of the ε-level set of V is compactly contained in $\mathcal{M} \times I_o$ and projects surjectively onto $\mathcal{M}$,
(C2) $Z(\mathcal{M} \times I^3) \subset I_o$,
(C3) $\frac{\partial Z}{\partial x}(\theta, x, y, z) \neq 0 \neq \frac{\partial Z}{\partial z}(\theta, x, y, z)$ everywhere on $\mathcal{M} \times \mathbb{R}^3$.

Note that Condition (C2) is mostly included for convenience. Since $\mathcal{M} \times I^3$ is compact any continuous function f can be multiplied by a number ε such that $\varepsilon f(\mathcal{M} \times I^3) \subset I_o$ and since we are considering only the anti-integrable limit scenario this amounts to choosing a small enough ε_0 for the current Z.

Theorem 32. *Suppose that $Z(\mathcal{M} \times I^2) \subset I_o$ and $V \in C^1(\mathcal{M} \times \mathbb{R}, \mathbb{R})$ satisfies condition (C1). Let $h : \mathcal{M} \to \mathcal{M}$ be a map. Then for each $|\varepsilon| < \varepsilon_0$ and $\{\theta_k\}_{k\in\mathbb{Z}} \in \mathcal{M}^{\mathbb{Z}}$, $\theta_k = h(\theta_{k-1})$, there is $\{x_k\}_{k\in\mathbb{Z}} \in I^{\mathbb{Z}}$ satisfying Equation* (2.4)*.*

For the next theorem we will require an additional definition.

Definition 33. A connected component of $f_\theta^{-1}(0) \cap I^2$ is called *almost horizontal* if it is diffeomorphic to $[0, 1]$ with boundary points $p_1 \in \{-1\} \times I$ and $p_2 \in \{1\} \times I$ and if p_1, p_2 are its only points of intersection with the boundary of I^2.

Lastly, for a given $\theta_0 \in \mathcal{M}$ the *fiber* over θ_0 refers to the set $\{\theta_0\} \times \mathbb{R}^d$ where $d = 2$ for the one-dimensional case and $d = 3$ for the two-dimensional case.

Theorem 34. *Under the same assumptions as in Theorem 32 and, in addition, assuming that $\partial_x Z(\theta, x, y) \neq 0$ for every $(\theta, x, y) \in \mathcal{M} \times I^2$ and for $|\varepsilon| > 0$ and $0 < \delta < 1$ there is $\{\theta_k\}_{k\in\mathbb{Z}} \in \mathcal{M}^{\mathbb{Z}}$, $\theta_k = h(\theta_{k-1})$, satisfying the following:*

1. *The fiber over each θ_k contains an almost horizontal component with slope of absolute value at most $1 - \delta$ everywhere,*
2. *Infinitely many θ_k's have fibers containing at least two almost horizontal components with slope of absolute value at most $1 - \delta$ everywhere.*

Then for each $k \in \mathbb{Z}$ the coordinates x_k of all orbits of Equation (2.4) *contained in almost horizontal components form a Cantor set.*

Analogous theorems are true for the two-dimensional case.

Theorem 35. *Let $Z \in C^1(\mathcal{M} \times \mathbb{R}^3, \mathbb{R})$ and $V \in C^1(\mathcal{M} \times \mathbb{R}, \mathbb{R})$ satisfying Conditions (C1) and (C2), and let $h : \mathcal{M} \to \mathcal{M}$ be a homeomorphism. Then for each $|\varepsilon| < \varepsilon_0$ and $\theta_0 \in \mathcal{M}$ there exists a sequence $\{x_k\}_{k\in\mathbb{Z}} \in I^{\mathbb{Z}}$ satisfying Equation* (2.1)*.*

Theorem 36. *Under the same conditions as in Theorem 35 and, in addition, assuming Condition (C3) and that for $|\varepsilon| > 0$ and $0 < \delta < 1$ there is $\{\theta_k\}_{k\in\mathbb{Z}} \in \mathcal{M}^{\mathbb{Z}}$, $\theta_k = h(\theta_{k-1})$, satisfying the following:*

1. *The fiber over each θ_k contains an almost horizontal component such that each leaf of the natural foliations of the component has slope of absolute value at most $1 - \delta$ everywhere.*
2. *Infinitely many θ_k have fibers containing at least two almost horizontal components such that each leaf of the natural foliations of the components has slope of absolute value at most $1 - \delta$ everywhere.*

Then for each $k \in \mathbb{Z}$ the coordinates x_k of all orbits of Equation (2.1) *contained in almost horizontal components form a Cantor set.*

The meaning of *almost horizontal component* is the same though the definition is a bit more technical in this higher dimensional setting.

Definition 37. A connected component of $f_\theta^{-1}(0) \cap I^3$ is called *almost horizontal* if it projects surjectively onto $I^2 = \{(x, z)\}$, it is diffeomorphic to I^2 and if its boundary is entirely contained inside the boundary of I^3 and is the only intersection with the boundary of I^3.

The proofs of Theorems 32 and 35 are topological in nature while the proofs of 34 and 36 require differentiability.

The proof of Theorem 32 is most straight-forward. Begin by considering the sequence of sets

$$B_k = \left\{ \{x_l\} \in I^{\mathbb{Z}} \colon \varepsilon Z(\theta_k, x_{k+1}, x_k) + V(\theta_k, x_k) = 0 \right\}.$$

These sets are closed and non-empty since for any fixed x_{k+1} the equation has a solution by the Intermediate Value Theorem using Condition (C1). Furthermore, by induction and repeated use of the Intermediate Value Theorem it follows that any intersection of the form

$$\bigcap_{k_1 \leq k \leq k_2} B_k$$

is also non-empty and hence the sequence of sets B_k satisfy the finite intersection property. Since $I^{\mathbb{Z}}$ is compact by Tychonoff's theorem it follows that $\bigcap_{k \in \mathbb{Z}} B_k$ is also non-empty, completing the proof.

The proof of Theorem 35 uses a Brouwer degree argument[1]. The degree argument essentially replaces the intermediate-value theorem from the one-dimensional case, being used to prove existence of sequences that satisfy Equation (2.1) for a finite number of values of k and create a nested sequence of subsets of $I^{\mathbb{Z}}$. The proof then concludes by invoking Tychonoff's theorem.

To prove Theorem 34 we need more control on the solutions than the topological methods give us. Using the condition $\partial_x Z(\theta, x, y) \neq 0$ we can find a surjective function $x_{k+1}(x_k)$ defined on $O_1 \subset I$ for any given $k \in \mathbb{Z}$ such that

$$\varepsilon Z(\theta_k, x_{k+1}(x_k), x_k) + V(\theta_k, x_k) = 0.$$

Fixing a k we can, by composition, create surjective functions $x_{k+l}(x_k)$ for any l defined on a subset O_l. By the second assumption there are infinitely many l for which there is a choice of two or more almost horizontal components to map surjectively onto. Taking the union over all such choices we get a nested

[1] This version of the proof is due to the referee of Paper I.

sequence of sets $\bar{O}_l$ with increasing number of components. The condition on the slope of the almost horizontal components guarantees that the size of each component decreases geometrically. Finally, taking the intersection of all $\bar{O}_l$ yields the Cantor set we are looking for.

The idea for the proof of Theorem 36 is very similar. Using Condition (C3) we can find a surjective map $x_{k+1}(x_k, x_{k-1})$ satisfying Equation (2.1) with $x_{k+1} = x_{k+1}(x_k, x_{k-1})$ for $(x_k, x_{k-1}) \in O_{-1}$. Note that this set projects surjectively onto x_{k-1}. Again we can do this for each k and use pull-backs to create maps $x_{k+l}(x_k, x_{k-1})$ defined on O_l for each $l \geq 1$ and each of these sets O_l project surjectively onto x_{k-1}. Similarly we can find maps $x_{k-l}(x_{k+1}, x_k)$ defined on corresponding sets O_{-l} that all project surjectively onto x_{k+1}. Each time there are multiple almost horizontal components there are again multiple choices of surjective function, thus yielding more connected components in the same way as in the one-dimensional case. Let $\bar{O}_l$ and $\bar{O}_{-l}$ denote the union of all these components in the respective direction and $W_l^+ = \bigcap_{1 \leq i \leq l} \bar{O}_l$ and $W_l^- = \bigcap_{1 \leq i \leq l} \bar{O}_{-l}$. Recall that each component of W_l^+ projects surjectively onto x_{k-1} and each component of W_l^- projects surjectively onto x_{k+1}. Hence we consider their intersection W_l. Finally, $W = \bigcap_{l \geq 1} W_l$ is then the Cantor set we are looking for, nonempty since the sequence W_l is nested and using Tychonoff's theorem and the assumptions on the slope of the almost horizontal components guaranteeing that each component shrinks as required.

The paper ends with propositions giving conditions under which the horizontal components required by Theorems 34 and 36 exist and a corollary showing that if Z and V are appropriately periodic the solutions given by Theorems 32 and 35 attain any fibered rotation number, i.e. for any $\omega \in (0, 1)$ there is a sequence $\{x_k\}_{k \in \mathbb{Z}}$ satisfying Equation (2.1) such that

$$\omega = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} x_{k+1} - x_k.$$

2.2 Paper II

In this paper, motivated by the work of de Carvalho, Lyubich and Martens about the two-dimensional Hénon-like renormalization in [3], we begin the systematic study of the structure of the invariant Cantor sets of period-doubling type for infinitely renormalizable area-preserving maps. Here we deal with the question of whether the invariant Cantor sets can be covered by a Lipschitz curve or a smooth curve.

We have to work with both the original Eckmann, Koch and Wittwer (abbreviated EKW) renormalization operator and the more recent construction by Gaidashev, Johnson and Martens (abbreviated GJM). When it is necessary to distinguish them it is made with subscripts, for example F_{EKW} refers to the

fixed point of the EKW renormalization and F_{GJM} refers to the fixed point of the GJM renormalization.

The reason for using both renormalization schemes is that certain properties necessary for this paper are only proven for one of them and others are only proven for the other. Most notably rigidity is only proven for the GJM renormalization.

The following are the two main results of the paper.

Theorem 38. *Every map $F \in W^s(F_{GJM})$ admits a Lipschitz curve containing its invariant Cantor set $\mathcal{C}_F$. Furthermore, such a curve can be chosen without self intersections.*

Theorem 39. *There is no smooth curve containing $\mathcal{C}_F$ for any $F \in W^s_{loc}(F_{GJM})$.*

The main component of the proof of Theorem 38 consists of using a small enough upper bound on the rate of contraction provided by the renormalization microscope $\psi_i[F]$ for $F \in W^s(F_{GJM})$ and producing, by induction, an infinite sequence of piecewise smooth curves with uniformly bounded Lipschitz constants that come closer and closer to covering the invariant Cantor set. To finish off the proof the Arzelà-Ascoli theorem is then used to extract a convergent subsequence. The limit of this convergent subsequence is then our sought Lipschitz curve. This proof follows the proof of the corresponding result for the dissipative Hénon-like renormalization which can be found in [37]. It is noteworthy that the proof does not use any property specific to the maps being renormalized, it relies only on the renormalization microscope having strong enough contraction.

The proof of Theorem 39 however, though the basic idea is also the same as in the proof in [3] of the corresponding property, relies heavily on properties of the maps being renormalized as well as properties of the renormalization operator itself. It also uses the fact that rigidity has already been proven in [23].

The basic idea is as follows: If there were a smooth curve containing the invariant Cantor set $\mathcal{C}_{F_{GJM}}$ then this curve would induce a continuous invariant line field on $\mathcal{C}_{F_{GJM}}$. Thus if we can prove that such a continuous invariant line field cannot exist then we have also shown that the smooth curve cannot exist. In fact, due to properties of these invariant Cantor sets, it is enough to show that there are no continuous invariant direction fields. This is where the similarities with [3] end. Their proof that there are no continuous invariant line fields relies on the fact that the maps they are considering are strongly dissipative. Since our maps are area-preserving we therefore need to use a different property of our maps. This is also where we really need to use both the EKW and the GJM renormalizations. From the EKW renormalization we need some normalizations which allow us to prove the following lemma.

Lemma 40. *At the tip $\tau = \bigcap_{n\geq 0} B_0^n(F_{EKW}) = (0,0)$ of the EKW renormalization fixed point F_{EKW} we have*

$$\frac{\partial X}{\partial x}(\tau) < 0.$$

Note that this lemma is formulated only for the EKW renormalization. However since we also need the rigidity result for the GJM formulation we need to show that this is also true for F_{GJM}. Thus we need the next lemma.

Lemma 41. *The maps F_{EKW} and F_{GJM} are conjugate by a translation in the x-direction.*

With these lemmas we are now ready to prove the nonexistence of continuous invariant direction fields on $\mathcal{C}_{F_{GJM}}$. The idea is as follows:

1. Suppose there is an invariant direction field ℓ on $\mathcal{C}_{F_{GJM}}$.
2. If $\ell(\tau)$ is not horizontal then, using the fact that $F_{GJM}^{2^n}|_{B_0^n} = \psi_0^n \circ F \circ \psi_0^{-n}|_{B_0^n}$ and that ψ_0 is much more contracting in the vertical direction than the horizontal direction, we can pick n large enough so that $\psi_0^{-n}(\ell(\tau))$ is contained in either of the vertical cones given by the ratchet phenomenon. Since F_{GJM} is a twist map it follows that DF_{GJM} and DF_{GJM}^{-1} will map this direction to opposite sides of the vertical axis. Now applying ψ_0^n these directions end up close to the horizontal axis but on different sides of the vertical axis. By choosing n large enough we can get these points as close to the tip as we want and hence ℓ cannot be continuous.
3. If $\ell(\tau)$ is horizontal then by Lemma 40 it follows that $\ell\left(F_{GJM}(\tau)\right)$ must be on the other side of the vertical axis. Using the closing argument from the above step we again end up with two points arbitrarily close to the tip where ℓ must be close to horizontal but on opposite sides of the vertical axis and hence ℓ cannot be continuous.
4. This shows that ℓ cannot be continuous. By the above discussion it follows that there can be no smooth curve containing $\mathcal{O}_{F_{GJM}}$.

2.3 Paper III

Continuing the inquiry into the structure of the invariant Cantors sets of period-doubling type of infinitely renormalizable area-preserving maps this paper deals with the question of whether the geometry is bounded or unbounded. Motivating this question are the facts that for unimodal renormalization the invariant Cantor sets have bounded geometry but for dissipative Hénon-like renormalization in two dimensions the invariant Cantor sets have unbounded geometry. This unbounded geometry however only occurs on a subset of measure zero,

i.e., there is a subset of full measure of the invariant Cantor set which has bounded geometry.

It turns out that the situation for area-preserving maps is quite different. The main result of this paper is the following theorem.

Theorem 42. *Let $F \in W^s_{\text{loc}}(F_*)$. For every $1 < \gamma < |\lambda_*|\mu_*^{-1}$ there exist $N = N(F,\gamma) \in \mathbb{N}$, and $m = m(\gamma)$,*

$$m = \left\lceil 3 + \frac{\ln\left(8\ln|\lambda_*| - 8\ln\mu_*\right) - \ln\left(\ln|\lambda_*| - \ln\mu_* - \ln\gamma\right)}{\ln 2} \right\rceil,$$

such that the measure of the pieces in $\mathcal{B}^n$ satisfying

$$\operatorname{diam}(B^n_{wv}) > \gamma^{n-1}\operatorname{dist}(B^n_{w1}, B^n_{w0}),\ w \in \{0,1\}^n,\ v \in \{0,1\},$$

for all $n > N$ is at least 2^{-m}.

Note that, since $\gamma^n \to \infty$ as $n \to \infty$, Theorem 42 shows positive measure of unbounded geometry. This distinguishes infinitely renormalizable area-preserving maps from the previously mentioned renormalizations for unimodal and dissipative Hénon-like maps.

The idea behind the proof of positive measure of unbounded geometry is to first find two pieces of $\mathcal{B}^n$ which have unbounded geometry for all n. From there the proof then uses careful control over the maximum and minimum distortion of F to "spread" this unbounded geometry through a part of the orbit.

For our initial pieces of $\mathcal{B}_n$ we need to find pieces which are vertically separated, i.e., there is a vertical line intersecting both pieces. To do this we use the fact that all maps F are reversible, i.e. $T \circ F \circ T = F^{-1}$ where

$$T(x,y) = (x,-y).$$

Recall that the *tip* τ of the map F_* is given by $\tau = \lim_{n\to\infty}\psi_0^n(x,y) = (0,0) = (0,-s_*(1,0))$ and is the fixed-point of the map ψ_0. Then, using the normalizations on s_*,

$$\begin{aligned} F_*(\tau) = (1, s_*(0,1)) \ &\Rightarrow\ T(F_*(\tau)) \\ &= (1, s_*(\lambda_*, 1)) \\ &F_*\left(\lambda_*, -\mu_* s_*(0,1)\right) \\ &= F_* \circ \psi_0\left(1, -s_*(0,1)\right) \\ &= \psi_1\left(1, -s_*(0,1)\right), \end{aligned}$$

so $T \circ F(\tau)$ is in fact the fixed-point of ψ_1. We then have

$$\begin{aligned} F_*(\tau) &= (1, s_*(0,1)) \in B_{10}, \\ T(F_*(\tau)) &= (1, -s_*(0,1)) \in B_{11}, \end{aligned}$$

hence the two pieces B_{10} and B_{11} must be vertically separated, giving us our two candidates for proving positive measure of unbounded geometry.

Next we need bounds on the possible expansion and contraction by F in order to be able to spread the bounded and unbounded geometry around. The two lemmas giving these bounds are the following:

Lemma 43. *There is a constant $C > 0$ such that for any $D > 0$ satisfying*

$$D \geq 8\frac{n}{n+3}(\ln|\lambda_*| - \ln\mu_*) + \frac{8}{n+3}\ln\|DF_*|_{B_w}\| \tag{2.6}$$

for all $n \in \mathbb{N}$ and $w \in \{0,1\}$ the following bound holds for all $n \in \mathbb{N}$ and $0 \leq i < 2^{n-1}$:

$$\|DF_*^i|_{B_{0^{n-1}w}^{n-1}}\| \leq Ce^{D\frac{n}{2^n}i}. \tag{2.7}$$

Lemma 44. *There is a constant $C > 0$ such that for any $D > 0$ satisfying*

$$D \geq 8\frac{n}{n+3}(\ln|\lambda_*| - \ln\mu_*) - \frac{8}{n+3}\ln b \tag{2.8}$$

for all $n \in \mathbb{N}$ and where b is a lower bound on

$$\min_{w\in\{0,1\},x\in B_w,v\in\mathbb{R}^2\setminus\{0\}} \frac{\|DF(x)v\|}{\|v\|}$$

the following bound holds for all $n \in \mathbb{N}$, $0 \leq i < 2^{n-1}$ and all $v \in \mathbb{R}^2$:

$$\|DF_*^i|_{B_{0^{n-1}w}^{n-1}}v\| \geq C^{-1}e^{-D\frac{n}{2^n}i}\|v\|. \tag{2.9}$$

Using the Mean Value Theorem for vector valued functions it is now relatively straightforward to prove positive measure of unbounded geometry with the help of these bounds on a subset of length 2^{n-m} of the full orbit of length 2^n where m is a fixed but large enough constant.

However the bounds are not enough to prove positive measure of bounded geometry. Therefore, the rest of the paper instead explores ergodic properties of F_* and its renormalization microscope in order to identify properties that could help prove bounded geometry "on the average" in the future.

Let Ω denote the set of one-sided infinite sequences of 0 and 1, i.e., $\Omega = \{0,1\}^{\mathbb{N}}$. We also denote $\tau_w = \Psi_w^{F_*}(\tau)$. Consider the projectivization of the derivative DF on $\mathbb{R}P^1$

$$L(x)[u] = DF(x)u \mod Z(\mathbb{R}^2)$$

where $Z(\mathbb{R}^2)$ is the group of non-zero scalar transformations of $\mathbb{R}^2$. Note that if $k = \sum_{i=0}^{n-1} w_{i+1}2^i$ where $w \in \{0,1\}^n$ then we can write

$$L^k(x)[u] = \frac{D\Psi_w(x)\Lambda_{n,F}^{-1}u}{\left\|D\Psi_w(x)\Lambda_{n,F}^{-1}u\right\|} \mod I$$

where I is the reflection $I(u) = -u$.

With this we can define a dynamical system Φ on $\mathbb{R}P^1 \times \Omega$ by

$$\Phi([u], w) = (L(\tau_w)[u], p(w)) \tag{2.10}$$

where, we recall, $p(w)$ corresponds to addition of 1 to $\sum_{i=0}^{n-1} w_{i+1}2^i$. It turns out that this dynamical system admits an ergodic invariant measure which we will denote by η^*. Using this fact it is possible to show that both microscope branches in fact have the same average, $\ln \nu$. More precisely, for $v = 0, 1$

$$\int_{\mathbb{R}P^1 \times \Omega} \ln \frac{\left\| D\psi_v^{F_*}(\tau_w)u \right\|}{\|u\|} d\eta^* = \ln \nu.$$

From here we directly get that for almost all $([u], \rho) \in \mathbb{R}P^1 \times \Omega$

$$\begin{aligned} \ln \nu &= \lim_{n\to\infty} \frac{1}{2^n} \sum_{j=0}^{2^n-1} \ln \left\| D\psi_v^{F_*}(\tau_{p^j(\rho)}) L^j(\tau_\rho)[u] \right\| \\ &= \lim_{n\to\infty} \frac{1}{2^n} \sum_{w\in\{0,1\}^n} \ln \frac{\left\| D\psi_v^{F*}(\tau_{w\rho}) D\Psi_w^{F*}(\tau_\rho)\Lambda_*^{-n}u \right\|}{\left\| D\Psi_w^{F*}(\tau_\rho)\Lambda_*^{-n}u \right\|}. \end{aligned}$$

In particular, we can say that for any decreasing sequence $\varepsilon_k > 0$, $\lim_{k\to\infty} \varepsilon_k = 0$ we have that for almost all $([u], \rho)$ there is a sequence of integers n_k such that

$$\left| \frac{1}{2^{n_l}} \sum_{w\in\{0,1\}^{n_l}} \ln \frac{\left\| D\psi_v^{F_*}(\tau_{w\rho}) D\Psi_w^{F_*}(\tau_\rho)\Lambda_*^{-n_l}u \right\|}{\left\| D\Psi_w^{F_*}(\tau_\rho)\Lambda_*^{-n_l}u \right\|} - \ln \nu \right| < \varepsilon_k \left| \ln \nu \right| \tag{2.11}$$

for all $l \geq k$.

We are now ready to state the key results from this part of the paper.

Proposition 45. *Let $k(w) = \sum_{j=0}^{\infty} w_{j+1}2^j$, $|w| < \infty$, and $\rho \in \Omega$. Consider the orbits $\{v_{p^k(\rho)} = DF^k(\tau_\rho)v_\rho\}_{k=0}^{\infty}$ and $\{u_{p^k(\rho)} = DF^k(\tau_\rho)u_\rho\}_{k=0}^{\infty}$ of two non-zero vectors by DF. Let $\widehat{u_w v_w}$ denote the angles between these vectors. Then for a.a. ρ, v_ρ and u_ρ*

$$\lim_{k\to\infty} \frac{1}{k} \ln \frac{\sin \widehat{u_{p^k(\rho)} v_{p^k(\rho)}}}{\sin \widehat{u_\rho v_\rho}} = 0.$$

Proposition 46. *Let n_{k+1} and n_k be two integers such that $n_{k+1} > n_k$ and $l_k = n_{k+1} - n_k < n_k$ and $\varepsilon_k > 0$ be such that* (2.11) *is satisfied for all $l > n_k$. Let e be a unit vector, $x_{0^m} \in B_{0^m}^{m-1}$ and $w \in \{0,1\}^{n_k}$. Set*

$$u_w = D\Psi_w^{F_*}(x_{0^m})e. \tag{2.12}$$

Define the average log-contraction of a vector u_w by the renormalization microscope as

$$\delta_{k+1} = \frac{1}{2^{n_{k+1}}} \sum_{\rho\in\{0,1\}^{l_k}, w\in\{0,1\}^{n_k}} \ln \frac{\|u_{\rho w}\|}{\|u_w\|}. \tag{2.13}$$

Then there exist constants $B > 0$ and $N \in \mathbb{N}$ such that the following holds for any $m > N$ and a.a. unit vectors e:

$$l_k \ln \nu^{1+\varepsilon_k}(1 - B\theta_2^{m+n_k}) \leq \delta_{k+1} \leq l_k \ln \nu^{1-\varepsilon_k}(1 + B\theta_2^{m+n_k}). \tag{2.14}$$

2.4 Paper IV

The final paper of this thesis deals with two-dimensional complex renormalization, more precisely renormalization of two-dimensional perturbations of almost commuting pairs of holomorphic maps. This paper is motivated by two previous papers: [27] by Gaidashev and Yampolsky and [54] by Yang.

The goal of the paper is to unify the approaches to renormalization used by the above authors by generalizing the Hénon-style perturbation of one-dimensional almost commuting pairs used in [54] to any rotation number with periodic partial fraction expansion using [27] as a blueprint for dealing with general rotation numbers. We begin by making the necessary definitions.

Let $\zeta_* \in \mathbf{B}^{D,E}$ be the hyperbolic fixed point of $\mathcal{R}^p$ described in Theorem 30. We let $\mathbf{U}^{\Gamma_1,\Gamma_2}$ be the space of pairs of maps

$$A\colon \Gamma_1 \to \mathbb{C}^2, \quad B\colon \Gamma_2 \to \mathbb{C}^2$$

where $\Gamma_1 = D \times \tilde{D}$, $\Gamma_2 = E \times \tilde{E}$, $\tilde{D}$ and $\tilde{E}$ are domains in $\mathbb{C}$ containing 0 and A, B are both analytic and continuous up to the boundary. We equip $\mathbf{U}^{\Gamma_1,\Gamma_2}$ with the norm $\|(A,B)\|$ given by

$$\|(A,B)\| = \frac{1}{2}\left(\|A\| + \|B\|\right)$$

where $\|F\|$ denotes the uniform norm. We will also use the notation

$$\|F\|_y = \sup_{(x,y)\in W} \|\partial_y F(x,y)\|$$

for any smooth function $F\colon W \to \mathbb{C}^2$ where $W \subset \mathbb{C}^2$ is a domain. We set $\mathbf{A}^{\Gamma_1,\Gamma_2} := (\mathbf{U}^{\Gamma_1,\Gamma_2})^{\mathbb{R}}$, i.e., $\mathbf{A}^{\Gamma_1,\Gamma_2}$ consists of pairs of real-symmetric two-dimensional maps. For a map $A(x,y)$ we define the operator

$$\mathcal{L}(A)(x) := \pi_1(A)(x,0),$$

and, similarly, for a pair of maps $(A,B)(x,y)$ we let

$$\mathcal{L}(A,B)(x) := \left(\mathcal{L}(A)(x), \mathcal{L}(B)(x)\right).$$

Finally, the set of two-dimensional almost commuting pairs is defined as the subset $\mathbf{D}^{\Gamma_1,\Gamma_2} \subset \mathbf{A}^{\Gamma_1,\Gamma_2}$ consisting of all pairs (A, B) such that

$$A(x,y) = (a(x,y), h(x,y)) = (a_y(x), h_y(x))$$
$$B(x,y) = (b(x,y), x) = (b_y(x), x)$$

satisfying the conditions

(D1) $\mathcal{L}[A,B](x) = \mathcal{L}(A \circ B - B \circ A)(x) = o\left(|x|^3\right)$,
(D2) $\mathcal{L}B(0) = 1$.

The space $\mathbf{D}^{\Gamma_1,\Gamma_2}$ will serve as the Banach space containing the domain of definition for our renormalization operator. Since we want to create a perturbation of the renormalization theory of one-dimensional critical commuting pairs we also want to define an embedding of the space $\mathbf{B}^{D,E}$ into the space $\mathbf{D}^{\Gamma_1,\Gamma_2}$. The embedding is meant to mimic the way that the dissipative Hé-like renormalization was performed in [3] and also generalize the embedding used in [54].

Definition 47. Let $\zeta = (\eta, \xi) \in \mathbf{B}^{D,E}$ be a twice-renormalizable one-dimensional critical commuting pair such that

$$\mathcal{R}^2\zeta = \Lambda\left((\eta^r \circ \xi)^s \circ \eta, \eta^r \circ \xi\right)$$

where Λ denotes the rescaling, r is the height of ζ and s is the height of $\mathcal{R}\zeta$. The embedding $\iota\colon \mathbf{B}^{D,E} \to \mathbf{D}^{\Gamma_1,\Gamma_2}$ is then defined by

$$\iota(\eta,\xi)(x,y) \coloneqq \Lambda\left(\begin{pmatrix} (\eta^r \circ \xi)^s \circ \eta(x) \\ (\eta^r \circ \xi)^{s-1} \circ \eta(x) \end{pmatrix}, \begin{pmatrix} \eta^r \circ \xi(x) \\ x \end{pmatrix}\right) \tag{2.15}$$

where for a pair $Z = (A, B)$ the rescaling Λ is defined by

$$\Lambda(Z) = \left(S_Z^{-1} \circ A \circ S_Z, S_Z^{-1} \circ B \circ S_Z\right)$$

where $S_Z(x,y) = (\lambda_Z x, \lambda_Z y)$ and $\lambda_Z = \mathcal{L}(B)(0)$.

Note that the definition is defined in such a way that $\mathcal{L} \circ \iota = \mathcal{R}^2$. We can now define the subset $\mathbf{D}_\epsilon^{\Gamma_1,\Gamma_2}(Z_*)$ that will act as the domain of definition for the renormalization operator.

Definition 48. Let ζ_* be a one-dimensional almost commuting pair that is a periodic point of $\mathcal{R}$ and let $Z_* = \iota(\zeta_*)$. The subset $\mathbf{D}_\epsilon^{\Gamma_1,\Gamma_2} \subset \mathbf{D}^{\Gamma_1,\Gamma_2}$ is defined as the set of pairs $Z \in \mathbf{D}^{\Gamma_1,\Gamma_2}$ satisfying

(D3) $\mathcal{L}Z \in \mathbf{B}^{D,E}$ is twice-renormalizable,
(D4) $\|Z - Z_*\| \leq \epsilon$.

Having defined its domain the prerenormalization of $Z = (A, B) \in \mathbf{D}_\epsilon^{\Gamma_1,\Gamma_2}$ is now given by

$$p(Z) = ((B \circ A^r)^s \circ A, B \circ A^r) \tag{2.16}$$

where r is the height of the pair $\mathcal{L}Z$ and s is the height of the pair $\mathcal{R}(\mathcal{L}Z)$. Next we use a change of coordinates

$$H(x, y) = \left(\left((\pi_1 A^r(\bullet, y))^{-1}(x)\right), y\right)$$

to define a new prerenormalization given by

$$\tilde{p}(Z) = \left(H^{-1} \circ (B \circ A^r)^s \circ A \circ H, H^{-1} \circ B \circ A^r \circ H\right). \tag{2.17}$$

The first result of the paper shows that what we have done so far is in the right direction and well-defined.

Lemma 49. *There exists $\epsilon > 0$ such that any $Z \in \mathbf{D}_\epsilon^{\Gamma_1,\Gamma_2}$ is at least twice renormalizable and*

$$\|\Lambda(\tilde{p}Z) - \iota(\mathcal{L}Z)\| = O(\epsilon), \quad \|\tilde{p}Z\|_y = O(\epsilon^2).$$

The proof of this lemma is straight-forward using Cauchy estimates for the derivatives, the fact that $\zeta = \mathcal{L}Z$ will be close to $\mathcal{R}^2\zeta_*$ and precompactness properties of the sequence of renormalizations for such ζ.

Following this lemma we define a preliminary renormalization according to

$$\tilde{\mathcal{R}}Z = \Lambda(\tilde{p}Z).$$

While $\mathcal{R}$ maps commuting pairs in $\mathbf{D}_\epsilon^{\Gamma_1,\Gamma_2}(Z_*)$ to commuting pairs it is not guaranteed that an almost commuting pair will be mapped to an almost commuting pair. To solve this issue we introduce a projection Π acting as the identity on commuting pairs according to

$$\Pi(A, B)(x, y) = \left(\begin{pmatrix} a(x, y) \\ h(x, y) \end{pmatrix}, \begin{pmatrix} b(x, y) + cx + dx^2 + ex^3 + fx^4 \\ x \end{pmatrix}\right).$$

The next result of the paper validates the use of this projection.

Proposition 50. *There exists $\epsilon > 0$ such that the following holds. For every pair $(A, B) = Z \in \tilde{\mathcal{R}}\left(\mathbf{D}_\epsilon^{\Gamma_1,\Gamma_2}(Z_*)\right)$ there exists a unique tuple $(c, d, e, f) \in \mathbb{D}_{L\epsilon^2}(0)^{\otimes 4}$ so that*

$$\mathcal{L}[\tilde{A}, \tilde{B}](x) = o(|x|^3)$$

holds for the pair $(\tilde{A}, \tilde{B}) = \Pi Z$. Furthermore, the map

$$\Pi\colon Z \mapsto (c, d, e, f)$$

is analytic.

The proof of the above proposition is a straight forward application of the implicit function theorem but involves a lot of computation.

With this we can finally define the renormalization $\mathcal{R}$ by

$$\mathcal{R}Z \coloneqq \Pi(\tilde{\mathcal{R}}Z). \tag{2.18}$$

The main result of this part of the paper is the following.

Theorem 51 (2D renormalization hyperbolicity)**.** *The pair $Z_* = \iota(\zeta_*)$ is the unique fixed point of $\mathcal{R}^n$ in $K_n(Z_*) = \mathcal{R}^{-n}\left(D_\epsilon^{\Gamma_1,\Gamma_2}(Z_*)\right)$. The linearization $D|_{Z_*}\mathcal{R}$ is a compact operator whose nonzero eigenvalues correspond to the tangent space of the embedded one-dimensional pairs, $\iota\left(\mathbf{B}^{D,E}\right)$, where its spectrum equals that of the one-dimensional renormalization operator $D|_{\zeta_*}\mathcal{R}$.*

The essential part of the proof consists in proving that the convergence under iterated renormalizations of a pair Z to $\iota\left(\mathbf{B}^{D,E}\right)$ is fast enough that the nontrivial part of the linearization of $\mathcal{R}^n$ at Z_* is entirely contained in the tangent space to $\iota\left(\mathbf{B}^{D,E}\right)$. Together with showing that the two-dimensional renormalization operator restricted to $\iota\left(\mathbf{B}^{D,E}\right)$ matches the action of the one-dimensional renormalization operator, this finishes the proof.

The next step is defining the renormalization microscope and studying the attractors of infinitely renormalizable commuting pairs. To do this, first note that for a commuting pair Z we can write

$$\mathcal{R}^n Z = L_Z^{-1} \circ \left(Z^{\bar{t}_n^1}, Z^{\bar{t}_n^2}\right) \circ L_Z$$

where

$$L_Z = H_Z \circ S_Z \circ H_{\mathcal{R}Z} \circ S_{\mathcal{R}Z} \circ \cdots \circ H_{\mathcal{R}^{n-1}Z} \circ S_{\mathcal{R}^{n-1}Z}.$$

We also introduce the shorthand notation $\mathfrak{R} = \mathcal{R}^n$.

To construct the microscope we first define the maps

$$\psi_{\bar{w},Z}|_{\Gamma_i} = Z^{\bar{w}} \circ L_Z, \quad \bar{w} \prec \bar{t}_n^i$$

where $\prec$ is the same ordering as the one used for multiindices in the one-dimensional case. By composition we also define the maps

$$\Psi^k_{\hat{w}_{k-1},Z} = \psi^Z_{\bar{w}_0} \circ \psi^{\mathfrak{R}Z}_{\bar{w}_1} \circ \cdots \circ \psi^{\mathfrak{R}^{k-1}Z}_{\bar{w}_{k-1}}$$

where $\hat{w}_{k-1} = (\overline{w}_0, \overline{w}_1, \ldots, \overline{w}_{k-1})$.

We first need a lemma about the contraction properties of the microscope.

Lemma 52. *Let $Z \in W^s_{loc}(Z_*)$. Then there exists a constant $C > 0$ such that for any admissible $\hat{w}_{k-1}$*

$$\left\| \Psi^k_{\hat{w}_{k-1},Z}|_{\Gamma_i} \right\|_\infty < C\frac{1}{2^k}.$$

The proof is based on first showing that there is a small neighborhood of Z_* in which

$$\|\psi_{\bar{w},Z}|_{\Gamma_i}\| < \frac{1}{2}$$

and then using the fact that for any given $Z \in W^s_{loc}(Z_*)$ there is an $N \geq 0$ such that for all $k \geq N$ the renormalization $\mathfrak{R}^k Z$ is in this neighborhood.

With this lemma and the fact that the commuting pair

$$T_* := \left(T_{\rho_*}|_{[-1,0]}, T_{-1}|_{[0,\rho_*]}\right)$$

where ρ_* is the rotation number of Z_* and $T_a(x) = x + a$ is a fixed point of renormalization we are able to prove the main result of this part of the paper.

Theorem 53. *Let $\zeta_* = \mathcal{R}^n\zeta_*$ be a periodic point of the one-dimensional renormalization and let $Z_* = \iota(\zeta_*)$. Suppose that $Z \in W^s_{loc}(Z_*)$ is a commuting pair. Then Z has a minimal attractor Σ_Z in $((D \cup E) \cap \mathbb{R}) \times \mathbb{R}$. The attractor Σ_Z is a Jordan arc and the restriction $Z|_{\Sigma_Z}$ is topologically but not smoothly conjugate to T_*.*

Having defined the renormalization microscope the next part of the paper focuses on defining and studying the *average Jacobian*. Before giving the main results of this part we define the *dynamical partition* of a two-dimensional commuting pair using the renormalization microscope.

Definition 54. Let $\mathcal{R}^n\zeta_* = \zeta_*$ and $Z_* = \iota(\zeta_*)$ and let $Z \in W^s_{loc}(Z_*)$ be a commuting pair. The $2kn$-th *dynamical partition* of Z is the set

$$\mathcal{Q}_{2kn} = \left\{ \mathcal{Q}^i_{\hat{w}_{k-1}} := \Psi^k_{\hat{w}_{k-1},Z}(\Gamma_i) \colon \ \hat{w}_{k-1} \text{ is admissible on } \Gamma_i, i = 1, 2 \right\}.$$

Next note that by the fact that the rotation T_* is uniquely ergodic and the topological conjugacy $\phi \colon [-1, \rho_*] \to \Sigma_Z$ given by Theorem 53 it follows that $Z|_{\Sigma_Z}$ is also uniquely ergodic. We will denote this uniquely ergodic invariant measure by μ_Z.

Definition 55. The *average Jacobian* of a commuting pair Z with respect to the measure μ_Z is the number

$$b = \exp\left(\frac{1}{1+\rho_*}\int_{\Sigma_Z} \ln \operatorname{Jac} Z|_{\Sigma_Z} d\mu_Z\right).$$

The two main results about the average Jacobian are the following, where $A_{m,n}$ are constants depending only on the rotation number ρ_*,

Lemma 56. *Suppose $Z \in W^s_{loc}(Z_*)$. There exists $0 < \alpha < 1$ such that for any $l, m \in \mathbb{N}$, $l > m$,*

$$\begin{aligned}\operatorname{Jac} pA_{ln}|_{\mathcal{Q}^1_{\hat{0}_{l-1}}} &= b^{(1+O(|\lambda_{mn}|))(1+O(\alpha^{(l-m)n}))|\bar{u}_{ln}|}(1 + CA_{l-m,n}\alpha^{mn})),\\ \operatorname{Jac} pB_{ln}|_{\mathcal{Q}^2_{\hat{0}_{l-1}}} &= b^{(1+O(|\lambda_{mn}|))(1+O(\alpha^{(l-m)n}))|\bar{v}_{kn}|}(1 + CA_{l-m,n}\alpha^{mn})),\end{aligned}$$

and for any two points z_1 and z_2 in $\mathcal{Q}^i_{\hat{0}_{l-1}}$ we have

$$\ln \left| \frac{\operatorname{Jac} pZ_{ln}(z_1)}{\operatorname{Jac} pZ_{ln}(z_2)} \right| = O(\alpha^{ln}).$$

Note that the multiindices $\bar{u}$ and $\bar{v}$ are not the same as the multiindices $\bar{t}^1$ and $\bar{t}^2$. Rather they are multiindices giving the returns of the central intervals $I_{kn} = \lambda_{kn}(I)$ and $J_{kn} = \lambda_{kn}(J)$ of the pair T_*.

Lemma 57. *Suppose (Z, Σ_Z, μ_Z) is a commuting pair. Then it admits two characteristic exponents $\chi_0 = 0$ and $\chi_- = \ln b$.*

The next part of the paper focuses on proving universality. The first step is the following simple lemma.

Lemma 58. *The point $(1, 0)$ is an attracting fixed point of the zero branch of the renormalization microscope of Z_*.*

This lemma helps us to estimate the derivative of the zero branch of the renormalization microscope

$$\Phi^m_Z \equiv \Psi^m_{\hat{0}_{m-1},Z}.$$

To begin, let $t_1(x, y) = (x + 1, y)$ denote the translation by 1 in the x-coordinate and define $\ell_{Z,n}(x, y)$, $l_{Z,n}(x)$ and $\lambda_{Z,n}$ by

$$\begin{aligned}t_1^{-1} \circ L_Z \circ t_1(x, y) &= (\ell_{Z,n}(x, y), \lambda_{Z,n} y),\\ l_{z,n}(x) &= \ell_{Z,n}(x, 0)\end{aligned}$$

where

$$\lambda_{Z,n} = \lambda_Z \cdot \lambda_{\mathcal{R}Z} \cdot \ldots \cdot \lambda_{\mathcal{R}^{n-1}Z}. \tag{2.19}$$

By the above lemma the function $l_{Z,n}$ has an attracting fixed point at $x = 0$. Letting σ_Z denote the eigenvalue of $H_Z \circ \Lambda_Z$ corresponding to the first component (the other eigenvalue being simply the scaling factor λ_Z) we can then write

$$l'_{Z,n}(0) = \prod_{i=0}^{n-1} \sigma_{\mathcal{R}^i Z}.$$

In fact, it can be shown that $l'_{Z_*,n}(0) = \lambda^3_{Z_*,n}$. It follows that

$$\lim_{m\to\infty} \left\| \frac{l_{Z,n} \circ l_{\Re Z,n} \circ \ldots \circ l_{\Re^m Z,n}}{\prod_{k=1}^{(m+1)n} \sigma_{\mathcal{R}^{k-1}Z}} - \frac{l_{Z_*,n}^{\circ(m+1)}}{\prod_{k=0}^{n-1} \sigma_{\mathcal{R}^k Z_*}^{m+1}} \right\| = 0 \tag{2.20}$$

and the convergence is uniform on compact sets. In fact, both sequences converge to the linearizing map v_{Z_*} solving the equation

$$\prod_{k=0}^{n-1} \sigma_{k Z_*} v_{Z_*} = v_{Z_*} \circ l_{Z_*,n}. \tag{2.21}$$

With this preparation done it is possible to show that the derivative of the zero branch of the renormalization microscope is as follows, where $\alpha_1 < \alpha_2 < 1$ and $\gamma_1 < \gamma_2 < 1$:

$$D\Phi^m_{\Re^j Z} = \begin{bmatrix} G + E_{m,\Re^j Z} & \mathcal{E}_{m,\Re^j Z} \\ 0 & \prod_{l=0}^{m-1} \lambda_{\Re^{l+j} Z,n} \end{bmatrix} \tag{2.22}$$

where

$$G = \prod_{l=0}^{m-1} l'_{\Re^{l+j} Z n} \circ \prod_{i=l+1}^{m-1} \circ\, l_{\Re^{i+j} Z,n}$$

and the functions $E_{m,\Re^j Z}$ and $\mathcal{E}_{m,\Re^j Z}$ satisfy

$$c\epsilon^{2^{jn}} \alpha_1^m \le \|E_{m,\Re^j Z}\| \le C\epsilon^{2^{jn}} \alpha_2^m,$$
$$c\epsilon^{2^{jn}} \gamma_1^m \le \|\mathcal{E}_{m,\Re^j Z}\| \le C\epsilon^{2^{jn}} \gamma_2^m.$$

Finally universality can now be shown using the fact that $B_{kn} = (\Phi^k_Z)^{-1} \circ pB_{kn} \circ \Phi^k_Z$ and using Lemma 56. The result is the following theorem.

Theorem 59. *Let $\mathcal{R}^n Z_* = Z_*$ and $Z \in W^s_{loc}(Z_*)$ be a commuting pair. Then there is an $0 < \alpha < 1$ such that*

$$B_{kn}(x,y) = \begin{bmatrix} \xi_{kn}(x) + b^{(1+O(\alpha^n))|\bar{v}_{kn}|} f(x) y (1 + O(\alpha^{kn})) \\ x \end{bmatrix} \tag{2.23}$$

where $\xi_{kn} = \mathcal{L}(B_{kn})$ and $f(x)$ is a universal function which is uniformly bounded away from 0 and ∞ and has uniformly bounded derivative and distortion.

The final part of the paper deals with the question of rigidity. There are no special lemmas required for this, just the previous result about universality and a series of computations considering distances between points at three different levels of renormalization. The eventual result after a series of computaions is the following:

Theorem 60. *Let Z and $\tilde{Z}$ be two commuting pairs in $W^s_{loc}(Z_*)$ with average Jacobians $b > \tilde{b}$. Let ϕ be the homeomorphism that conjugates $Z|_{\Sigma_Z}$ to $Z|_{\Sigma_{\tilde{Z}}}$ normalized so that $\phi(\tau) = \tau$. If ϕ is κ-Hölder, then*

$$\kappa \leq \frac{1}{3} + \frac{2}{3}\frac{\ln b}{\ln \tilde{b}}.$$

3. Sammanfattning på svenska (Summary in Swedish)

Denna avhandling består av en inledning, i vilken denna sammanfattning ingår, samt fyra artiklar. Tre av de fyra artiklarna behandlar renormaliseringsteori inom dynamiska system.

3.1 Bakgrund

Inom dynamiska system används *renormalisering* för att studera egenskaper som är universella inom klasser av dynamiska system. Tekniken går ut på att skapa en operator som agerar på ett rum av dynamiska system och använda denna operators egenskaper för att härleda egenskaper hos de enskilda dynamiska systemen.

Denna teknik användes först till att studera endimensionella dynamiska system i diskret tid och hur dessa kan bli kaotiska när en parameter varieras genom en oändlig följd av bifurkationer, en så kallad *bifurkationskaskad*. Genom att konstruera en renormaliseringsoperator lyckades man förklara både varför dessa bifurkationskaskader är så vanliga och varför det finns flera likheter mellan varje sådan kaskad, t.ex. de så kallade *Feigenbaumkonstanterna*. Detta arbete påbörjades av Feigenbaum i [14] och oberoende av Tresser och Coullet i [48]. Det första fullständiga beviset genomfördes av Lanford ([34]) med hjälp av datorstödda metoder och det första beviset helt utan datorhjälp genomfördes av Lyubich ([38]).

Renormaliseringstekniken har sedan dess generaliserats till flera andra typer av dynamiska system, däribland areabevarande avbildningar i planet som är ämnet för två av artiklarna i denna avhandling och tvådimensionella komplexa perturbationer av kritiska cirkelavbildningar som behandlas i avhandlingens sista artikel.

3.2 Resultat

3.2.1 Artikel I

Figueras och Haro undersökte i [17] bifurkationer hos *fiberwise hyperbolic invariant manifold* (FHIM) i skew product system med numeriska metoder. De

bevisade också existensen av en typ av bifurkation, kallad *folding bifurcation*, för den antiintegrerbara gränsen.

I denna artikel bevisas existensen av en folding bifurcation i en allmännare situation och även vissa egenskaper hos det invarianta objekt som återstår efter bifurkationen.

Närmare bestämt betraktar vi lösningar $\{x_k\}_{k\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$ till ekvationen

$$\varepsilon Z(\theta_k, x_{k+1}, x_k, x_{k-1}) + V(\theta_k, x_k) = 0$$

där $\mathcal{M}$ är en mångfald, $Z\colon \mathcal{M}\times\mathbb{R}^3 \to \mathbb{R}$ är C^1, $V\colon \mathcal{M}\times\mathbb{R}\to\mathbb{R}$ är C^1 och $\theta_k \in \mathcal{M}^{\mathbb{Z}}$ genereras av en homeomorfism $h\colon \mathcal{M}\to\mathcal{M}$ enligt $\theta_k = h^k(\theta_0)$ för något $\theta_0 \in \mathcal{M}$.

Artikelns första resultat är följande sats som inte är ny men inkluderas för att göra artikeln komplett. Beviset sker via implicita funktionssatsen för Banachrum.

Sats 61 (Artikel I, Theorem 1)**.** *Antag att $\partial_x V(\theta, x) \neq 0$ för alla $(\theta, x) \in \mathcal{M}\times\mathbb{R}$ i en sammanhängande och begränsad omgivning av $V^{-1}(0)$. Då existerar ett $\varepsilon_0 > 0$ och ett $K \in C^1(\mathcal{M}\times(-\varepsilon_0,\varepsilon_0),\mathbb{R})$ sådana att*

$$\varepsilon Z(\theta, K(h(\theta),\varepsilon), K(\theta,\varepsilon), K(h^{-1}(\theta),\varepsilon)) + V(\theta, K(\theta,\varepsilon)) = 0$$

för alla $\theta \in \mathcal{M}$ och $|\varepsilon| < \varepsilon_0$.

För artikelns huvudresultat kräver vi dessutom följande antaganden:

(C1) Det finns ett $\varepsilon_0 > 0$ sådant att för varje $|\varepsilon| \le \varepsilon_0$ är varje sammanhängande komponent av nivåkurvan $V(\theta, x) = \varepsilon$ en kompakt delmängd av $\mathcal{M}\times I_o$ och dess projektion på $\mathcal{M}$ är surjektiv,
(C2) $Z(\mathcal{M}\times I^3) \subset I_o$,
(C3) $\frac{\partial Z}{\partial x}(\theta, x, y, z) \neq 0 \neq \frac{\partial Z}{\partial z}(\theta, x, y, z)$ överallt på $\mathcal{M}\times\mathbb{R}^3$.

Vi kan då ställa upp följande satser.

Sats 62 (Artikel I, Theorem 4)**.** *Låt $Z \in C^1(\mathcal{M}\times\mathbb{R}^3,\mathbb{R})$ och $V \in C^1(\mathcal{M}\times\mathbb{R},\mathbb{R})$ uppfylla Antagande (C1) och låt $h : \mathcal{M}\to\mathcal{M}$ vara en homeomorfism. För varje $|\varepsilon| < \varepsilon_0$ och $\theta_0 \in \mathcal{M}$ finns då en följd $\{x_k\}_{k\in\mathbb{Z}} \in I^{\mathbb{Z}}$ som uppfyller Ekvation* (2.1)*.*

Sats 63 (Artikel I, Theorem 5)**.** *Under samma antaganden som i föregående sats och dessutom Antagande (C2) och Antagande (C3) samt att för $|\varepsilon| > 0$ och $0 < \delta < 1$ finns det $\{\theta_k\}_{k\in\mathbb{Z}} \in \mathcal{M}^{\mathbb{Z}}$, $\theta_k = h(\theta_{k-1})$, sådant att:*

1. *Fibern över varje θ_k innehåller en nästan horisontell komponent sådant att varje löv i komponentens naturliga folieringar har lutning med absolutvärde som mest $1-\delta$ överallt.*

2. *Oändligt många θ_k har fibrer som innehåller minst två nästan horisontella komponenter sådana att varje löv i komponenternas naturliga folieringar har lutning med absolutvärde som mest $1 - \delta$ överallt.*

Då gäller att för varje $k \in \mathbb{Z}$ utgör koordinaterna x_k för alla lösningar till Ekvation (2.1) *som begränsas till nästan horisontella komponenter en Cantormängd.*

Tillsammans visar dessa tre satser på en stor kvalitativ skillnad i lösningsmängdens utseende beroende på potentialen V, vilket bevisar existensen av den sökta bifurkationen.

3.2.2 Artikel II

I artikel II påbörjas studierna av renormaliseringsteori för areabevarande dynamiska system i planet, vilket är avhandlingens huvudområde. I denna artikel undersöks om de invarianta Cantormängderna, som varje oändligt renormaliserbar avbildning har, kan täckas av en Lipschitzkurva respektive en slät kurva.

Renormaliseringen vi är intresserade av är definierad för en viss klass av areabevarande avbildningar av planet som har den så kallade *twistegenskapen* vilket betyder att $\partial_y F(x, y) \neq 0$ för alla (x, y). För detaljer om hur denna renormalisering definieras rekommenderas [23] eller inledningen till artikel II.

Följande satser bevisas i denna artikel:

Sats 64 (Artikel II, Theorem 9)**.** *För varje avbildning $F \in W^s(F_{GJM})$ finns en Lipschitzkurva som innehåller dess invarianta Cantormängd $\mathcal{O}_F$. Denna kurva kan dessutom väljas så att den är simpel.*

Sats 65 (Artikel II, Corollary 16)**.** *Det finns ingen slät kurva som innehåller $\mathcal{O}_F$ för något $F \in W^s_{loc}(F_{GJM})$.*

Bevisen av dessa resultat följer samma struktur som bevisen för motsvarande resultat för renormalisering av dissipativa Hénonlika avbildningar av planet från [3]. De stora skillnaderna är att vi här kan använda resultatet om rigiditet från [23] för att lösa vissa problem och beviset för att det inte existerar något kontinuerligt invariant riktningsfält på den invarianta Cantormängden använder en egenskap specifik för twistavbildningar för att ersätta ett argument som är specifikt för dissipativa avbildningar.

3.2.3 Artikel III

Undersökningen av renormaliseringsteori för areabevarande avbildningar fortsätter i artikel III där geometrin för de invarianta Cantormängderna för oändligt renormaliserbara areabevarande avbildningar studeras.

Artikelns huvudresultat är:

Sats 66 (Artikel III, Main Theorem)**.** *Låt $F \in W^s_{\text{loc}}(F_*)$. För varje $1 < \gamma < |\lambda_*|\mu_*^{-1}$ existerar ett $N = N(F,\gamma) \in \mathbb{N}$, och ett $m = m(\gamma)$,*

$$m = \left\lceil 3 + \frac{\ln\left(8\ln|\lambda_*| - 8\ln\mu_*\right) - \ln\left(\ln|\lambda_*| - \ln\mu_* - \ln\gamma\right)}{\ln 2} \right\rceil,$$

sådana att måttet av alla delar av $\mathcal{B}^n$ som uppfyller

$$\operatorname{diam}(B^n_{wv}) > \gamma^{n-1}\operatorname{dist}(B^n_{w1}, B^n_{w0}),\ w \in \{0,1\}^n,\ v \in \{0,1\},$$

för alla $n > N$, är minst 2^{-m}.

Beviset av Sats 66 bygger på att hitta ett par bitar i $\mathcal{B}^n$ med obegränsad geometri. Därefter används estimat av $\|DF^i_*|_{B^{n-1}_{0^{n-1}w}}\|$ för att visa att denna obegränsade geometri även gäller för bitar i $\mathcal{B}^n$ som är tillräckligt nära de ursprungliga delarna längs banan av F_*. Slutligen används rigiditeten för renormaliseringen för att visa att detta även gäller för andra $F \in W^s_{\text{loc}}(F_*)$.

3.2.4 Artikel IV

Artikel IV behandlar renormaliseringen av tvådimensionella perturbationer av *nästan kommutativa par*, vilka definierades i [45]. I denna artikel generaliserar vi renormaliseringsoperatorn och resultat från [54] om renormalisering för nästan kommutativa par med gyllene snittet som rotationstal till rotationstal med periodisk partialbråksuppdelning. Huvudresultaten är följande:

Sats 67 (Artikel IV, Theorem 2.3)**.** *Paret Z_* är den unika fixpunkten för $\mathcal{R}^n$ i $K_n(Z_*)$. Derivatan $D|_{Z_*}\mathcal{R}$ är en kompakt operator vars nollskilda egenvärden motsvarar tangentrummet till de inbäddade endimensionella paren, $\imath\left(\mathbf{B}^{D,E}\right)$, där dess spektrum dessutom sammanfaller med spektrumet för den endimensionella renormaliseringsoperatorn $D|_{\zeta_*}\mathcal{R}$.*

Sats 68 (Artikel IV, Theorem 3.4)**.** *Låt $\zeta_* = \mathcal{R}^n(\zeta_*)$ vara en periodisk punkt för den endimensionella renormaliseringsoperatorn och låt*

$$Z_* = \imath(\zeta_*) \in \mathbf{B}^{\Gamma_1\Gamma_2}_{\epsilon_n}.$$

Antag att $Z = (A,B) \in W^s_{loc}(Z_*) \subset \mathbf{K}_n$ och att avbildningarna A och B kommuterar, det vill säga $A \circ B = B \circ A$, överallt där båda sammansättningarna är definierade.

Då har Z en minimal attraktor Σ_Z inuti $((D\cup E)\cap\mathbb{R})\times\mathbb{R}$. Σ_Z är en Jordan kurvbåge och begränsningen $Z|_{\Sigma_Z}$ är topologiskt men inte slätt konjugerat med T_*.

Här är T_* ett endimensionellt kommuterande par definierat på $[-1,0]\cup[0,\rho_*]$ som motsvarar rotationen på cirkeln med rotationstal $\rho_*=\rho(\zeta_*)$.

Låt b vara medel-Jacobianen,

$$b=\exp\left(\frac{1}{1+\rho_*}\int_{\Sigma_Z}\ln\operatorname{Jac}Z|_{\Sigma_Z}d\mu_Z\right).$$

Lemma 69 (Artikel IV, Lemma 4.3)**.** *Låt $Z\in W^s_{loc}(Z_*)$. Det finns ett $0<\alpha<1$ sådant att för varje $l,m\in\mathbb{N}$, $l>m$,*

$$\begin{aligned}\operatorname{Jac}pA_{ln}|_{\mathcal{Q}^1_{\hat{0}_{l-1}}} &= b^{(1+O(|\lambda_{mn}|))(1+O(\alpha^{(l-m)n}))|\bar{u}_{ln}|}(1+CA_{l-m,n}\alpha^{mn})),\\ \operatorname{Jac}pB_{ln}|_{\mathcal{Q}^2_{\hat{0}_{l-1}}} &= b^{(1+O(|\lambda_{mn}|))(1+O(\alpha^{(l-m)n}))|\bar{v}_{kn}|}(1+CA_{l-m,n}\alpha^{mn})),\end{aligned}$$

och för varje par av punkter z_1 och z_2 i $\mathcal{Q}^i_{\hat{0}_{l-1}}$,

$$\ln\left|\frac{\operatorname{Jac}pZ_{ln}(z_1)}{\operatorname{Jac}pZ_{ln}(z_2)}\right|=O(\alpha^{ln}).$$

Lemma 70 (Artikel IV, Lemma 4.4)**.** *Låt (Z,Σ_Z,μ_Z) vara ett kommuterande par som ovan. Det har då två karaktäristiska exponenter $\chi_0=0$ och $\chi_-=\ln b$.*

Sats 71 (Artikel IV, Theorem 5.2)**.** *Låt $Z=(A,B)\in W^s(Z_*)$ vara ett kommuterande par, där $\mathcal{R}^nZ_*=Z_*$. Då finns ett $\alpha<1$ sådant att*

$$B_{kn}(x,y)=\begin{bmatrix}\xi_{kn}(x)+b^{(1+O(\alpha^n))|\bar{v}_{kn}|}f(x)y(1+O(\alpha^{kn}))\\ x\end{bmatrix}\tag{3.1}$$

där $\xi_{kn}=\mathcal{L}(B_{kn})$, b är medel-Jacobianen av Z, $\bar{v}_{kn}$ är ett multiindex relaterat till den dynamiska partitionen på nivå $2kn$ för T_* och $f(x)$ är en universell funktion som är likformigt begränsad utanför en omgivning av 0 och ∞ och som har likformigt begränsad derivata och distortion.

Sats 72 (Artikel IV, Theorem 6.1)**.** *Låt Z och $\tilde{Z}$ vara två diffeomorfismer i $W^s_{loc}(Z_*)$ med medel-Jacobianer $b>\tilde{b}$. Låt ϕ vara homeomorfismen som konjugerar $Z|_{\Sigma_Z}$ och $Z|_{\Sigma_{\tilde{Z}}}$ normaliserad så att $\phi(\tau)=\tau$. Om ϕ är α-Hölder måste*

$$\kappa\le\frac{1}{3}+\frac{2}{3}\frac{\ln b}{\ln\tilde{b}}.$$

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